## Math 342 Tutorial May 28, 2025

**Question 1.** Prove that if a and b are different integers, then there exist infinitely many positive integers a such that a+n and b+n are coprime. [Hint: Consider linear combinations of a0 and a1 and a2 if a3 and a4.]

**Question 2.** Prove that every integer > 6 can be represented as a sum of two integers > 1 which are coprime. [Hint. Consider the three cases  $n = 4k \pm 1$ , n = 4k, and n = 4k + 2 separately, and write the summands interms of k].

**Question 3.** An integer n is *powerful* if, whenever a prime p divides n,  $p^2$  divides n. Show that every powerful integer n can be written as the product of a perfect square and a perfect cube.

**Question 4.** Show that  $(a, b) \mid [a, b]$ . When does (a, b) = [a, b]?

**Question 5.** Show that if a, b, c > 0, then

$$(a, b, c)[ab, ac, bc] = abc = (ab, ac, bc)[a, b, c].$$

Question 6. An arithmetic function  $f: \mathbf{N} \to \mathbf{C}$  is multiplicative if f(mn) = f(m)f(n) whenever (m, n) = 1. The summatory function F of an arithmetic function  $f: \mathbf{N} \to \mathbf{C}$  is defined as  $F(x) = \sum_{d|x} f(d)$ . The number of divisors functions is defined as  $\tau(x) = \#\{d: d \mid x\}$ . (a) Show that every summatory function is multiplicative. (b) Show the number of divisors function is multiplicative. (c) If  $n = p_1^{e_1} \cdots p_k^{e_k}$ , show that  $\tau(n) = (e_1 + 1) \cdots (e_k + 1)$ . (d) Prove that for every positive integer k, the set of all positive integers n whose number of positive integer divisors is divisible by k contains an infinite arithmetic progression. [Hint: Consider a progression defined by a linear combination of consecutive powers of 2, and use part (c).]

**Question 7.** Prove that there exists infinitely many triplets of positive integers x, y, z for which the numbers x(x+1), y(y+1), z(z+1) form an increasing arithmetic progression. [Hint: write y and z as increasing linear functions of x.]

**Question 8.** Prove that for every even n > 6 there exist primes p and q such that (n - p, n - q) = 1.

**Question 9.** (a) Prove that for every three consecutive odd integers, one must be divisible by 3. [Hint. Write n = 2k + 1 and consider the possible cases for  $k \pmod{3}$ .] (b) Find all primes which can be represented as both a sum and difference of primes.

**Question 10.** Find all integer solutions x, y of the equation  $2x^3 + xy - 7 = 0$  and prove that this equation has infinitely many solutions in positive rationals. [Hint: Use the possible values for x in the first part to infer a possible form for x in the second part.]

**Question 11.** An astronomer knows that a satellite orbits the Earth in a period that is an exact multiple of 1 hour that is less than 1 day. If the astronomer notes that the satellite completes 11 orbits in an interval that starts when a 24-hour clock reads 0 hours and ends when the clock reads 17 hours, how long is the orbital period of the satellite?

**Question 12.** (a) Let p be an odd prime. Show the congruence  $x^2 \equiv 1 \pmod{p^k}$  has exactly two incongruent solutions, namely,  $x \equiv \pm 1 \pmod{p^k}$ . (b) Show that the congruence  $x^2 \equiv 1 \pmod{2^k}$  has exactly four ingongruent solutions, namely,  $x \equiv \pm 1 \pm (1 - 2^{k-1}) \pmod{2^k}$ , when k > 2. Show there is one when k = 1 and two when k = 2.