Math 342 Tutorial

May 28, 2025

Question 1. Prove that if a and b are different integers, then there exist infinitely many positive integers n such that $a+n$ and $b+n$ are coprime. [Hint: Consider linear combinations of $b-a$ and $1-a$ if $a < b$.]

Question 2. Prove that every integer > 6 can be represented as a sum of two integers > 1 which are coprime. [Hint. Consider the three cases $n = 4k \pm 1$, $n = 4k$, and $n = 4k + 2$ seperately, and write the summands interms of k].

Question 3. An integer n is *powerful* if, whenever a prime p divides n, p^2 divides n. Show that every powerful integer n can be written as the product of a perfect square and a perfect cube.

Question 4. Show that $(a,\, b) \mid [a,\, b]$. When does $(a,\, b) = [a,\, b]$?

Question 5. Show that if $a, b, c > 0$, then

$$(a,\, b,\, c)[ab,\, ac,\, bc] = abc = (ab,\, ac,\, bc)[a,\, b,\, c].$$

Question 6. An arithmetic function $f : \mathbf{N} \to \mathbf{C}$ is *multiplicative* if $f(mn) = f(m)f(n)$ whenever $(m,\, n) = 1$. The summatory function F of an arithmetic function $f : \mathbf{N} \to \mathbf{C}$ is defined as $F(x) = \sum_{d|x} f(d)$. The number of divisors functions is defined as $\tau(x) = \#\{d : d \mid x\}$. **(a)** Show that every summatory function of a multiplicative function is multiplicative. **(b)** Show the number of divisors function is multiplicative. **(c)** If $n = p_1^{e_1} \cdots p_k^{e_k}$, show that $\tau(n) = (e_1 + 1) \cdots (e_k + 1)$. **(d)** Prove that for every positive integer k, the set of all positive integers n whose number of positive integer divisors is divisible by k contains an infinite arithmetic progression. [Hint: Consider a progression defined by a linear combination of consecutive powers of 2, and use part (c).]

Question 7. Prove that there exists infinitely many triplets of positive integers $x,\, y,\, z$ for which the numbers $x(x+1)$, $y(y+1)$, $z(z+1)$ form an increasing arithmetic progression. [Hint: write y and z as increasing linear functions of x.]

Question 8. Prove that for every even $n > 6$ there exist primes p and q such that $(n-p,\, n-q) = 1$.

Question 9. **(a)** Prove that for every three consecutive odd integers, one must be divisible by 3. [Hint. Write $n = 2k + 1$ and consider the possible cases for $k \pmod 3$.] **(b)** Find all primes which can be represented as both a sum and difference of primes.

Question 10. Find all integer solutions $x,\, y$ of the equation $2x^3 + xy - 7 = 0$ and prove that this equation has infinitely many solutions in positive rationals. [Hint: Use the possible values for x in the first part to infer a possible form for x in the second part.]

Question 11. An astronomer knows that a satellite orbits the Earth in a period that is an exact multiple of 1 hour that is less than 1 day. If the astronomer notes that the satellite completes 11 orbits in an interval that starts when a 24-hour clock reads 0 hours and ends when the clock reads 17 hours, how long is the orbital period of the satellite?

Question 12. **(a)** Let p be an odd prime. Show the congruence $x^2 \equiv 1 \pmod{p^k}$ has exactly two incongruent solutions, namely, $x \equiv \pm 1 \pmod{p^k}$. **(b)** Show that the congruence $x^2 \equiv 1 \pmod{2^k}$ has exactly four incongruent solutions, namely, $x \equiv \pm 1 \pm (1 - 2^{k-1}) \pmod{2^k}$, when $k > 2$. Show there is one when $k = 1$ and two when $k = 2$.