Math 342 Tutorial
May 21, 2025

Question 1. Prove every integer n with $|n| > 1$ is either prime or can be factored into a product of prime numbers [Hint: use the principle of strong mathematical induction].

Question 2. Use Question 1 to show there are infinitely many prime numbers [Hint. use contradiction and consider the number $N = 1 + p_1 \cdots p_n$ where $p_1, \ldots, p_n$ are the assumed finite number of primes].

Question 3. The gcd of a multiset $\{a_1, \ldots, a_n\}$ of integers is defined inductively by $(a_1, a_2, \ldots, a_n) = (a_1, (a_2, \ldots, a_n))$. Show **(a)** the gcd of $\{a_1, \ldots, a_n\}$ is independent of the ordering chosen for the elements of the set, and **(b)** there exists integers $x_1, \ldots, x_n$ such that $(a_1, a_2, \ldots, a_n) = x_1a_1 + \cdots + x_na_n$.

Question 4. Let p be a prime. Show that if $p \mid ab$ then $p \mid a$ or $p \mid b$.

Question 5. If $(a, b) = 1$, then $(a + b, a - b) = 1$ or 2.

Question 6. If $(a, b) = 1$, then $(a + b, a^2 - ab + b^2) = 1$ or 3.

Question 7. If $(a, b) = 1$, then $(a^n, b^k) = 1$ for all $n, k \geqq 1$.

Question 8. If $2^n - 1$ is a prime, then n is a prime.

Question 9. If $2^n + 1$ is a prime, then n is a power of 2.

Question 10. **(a)** Suppose that $(a, b) = (c, d) = 1$ and $\frac{a}{b} + \frac{c}{d} = n$ is an integer. Show $|b| = |d|$. **(b)** Prove the sum $\sum_{k=1}^{n} \frac{1}{k}$ is not an integer for $n > 1$.

Question 11. Prove: **(a)** For every integer k the numbers $2k + 1$ and $9k + 4$ are relatively prime. **(b)** For every integer k, express the gcd of $2k - 1$ and $9k + 4$ as a function of k.

Question 12. Prove that for positive integers m and a we have

$$\left(\frac{a^m - 1}{a - 1}, a - 1\right) = (a - 1, m).$$