

0.1. Definition. Recall that a Hadamard matrix is a weighing matrix $W(n, n)$ (see §3). These elusive objects have vexed combinatorialists for over a century now. Ever more clever techniques from ever more branches of mathematics are needed in order to construct these objects.

We will consider one such construction in preparation for our study of orthogonal designs. The matrices we will study are the so-called balancedly splittable Hadamard matrices. First, an example.

0.1. Example. Consider a Hadamard matrix of order 4 shown below

$$(0.1.a) \begin{pmatrix} + & + & + & + \\ + & + & - & - \\ + & - & + & - \\ + & - & - & + \end{pmatrix}.$$

Label the rows h_0, h_1, h_2 , and h_3 . We then form the block matrix with (i, j) -th entry given by $h_j^t h_i$.

$$(0.1.b) \begin{pmatrix} + & + & + & + & + & + & + & + & + & + & + & + & + & + & + \\ + & + & + & + & + & + & + & + & - & - & - & - & - & - & - \\ + & + & + & + & - & - & - & - & + & + & + & + & - & - & - \\ + & + & - & - & + & + & - & - & + & + & - & - & + & + & - \\ + & + & - & - & + & + & - & - & - & - & + & + & - & - & + \\ + & + & - & - & - & - & + & + & + & + & - & - & - & - & + \\ + & - & + & - & + & - & + & - & + & - & + & - & + & - & + \\ + & - & + & - & + & - & + & - & - & - & + & - & + & - & + \\ + & - & + & - & - & + & - & + & + & - & + & - & - & + & - \\ + & - & - & + & + & - & - & + & + & - & - & + & + & - & - \\ + & - & - & + & + & - & - & + & - & + & - & - & + & + & - \\ + & - & - & + & - & + & + & - & + & - & - & + & - & + & - \\ + & - & - & + & - & + & + & - & - & + & - & - & + & - & + \end{pmatrix}.$$

The matrix ?? has the submatrix H_1 which we have shown in bold above. This submatrix has the property that $H_1^t H_1 = 4I + 4A$, for some symmetric $(0, 1)$ -matrix A with zero diagonal. In particular, there are only two angles that exist between the columns of H_1 .

The above example motivates the following definition.

0.2. Definition. A Hadamard Matrix H of order n is *balancedly splittable* with parameters (n, ℓ, a, b) if H has an $\ell \times n$ submatrix H_1 such that

$$(0.2.a) \quad H_1^t H_1 = \ell I + aA + b(J - I - A),$$

for some binary $(0, 1)$ -matrix A with zero diagonal.

This definition and the previous example are suggestive. Notably, a connection to sets of biangular lines is inherent in the definition. These connections will be taken up in the following subsections.

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0.2. Equiangular Lines. In the previous subsection, it was intimated that balancedly splittable Hadamard matrices were related to collections of biangular lines. By a set of *lines*, we mean a collection of vectors in \mathbf{R}^ℓ , for some ℓ . Given a collection \mathcal{L} of lines in \mathbf{R}^ℓ , define $\Xi = \{|\langle u, v \rangle| : u, v \in \mathcal{L} \text{ and } u \neq v\}$. If $|\Xi| = 2$, then we say that \mathcal{L} is a set of *biangular lines*; while if $|\Xi| = 1$, then we say that \mathcal{L} is a set of *equiangular lines* equiangular-lines-note.

Clearly, if H is balancedly splittable with respect to the $\ell \times n$ submatrix H_1 , then Definition 0.2 implies that the columns of H_1 are at most biangular. They are equiangular precisely in the case that $b = -a$.

We will require the following proposition due to ?.

0.3. Proposition. Let $\mathcal{L} \subset \mathbf{R}^\ell$ be a set of lines (vectors) such that $|\langle u, v \rangle| = a$, for every pair of distinct lines u and v in \mathcal{L} . If $\ell < a^{-2}$, then

$$(0.3.a) \quad |\mathcal{L}| \leq \ell(1 - a^2)/(1 - \ell a^2).$$

Using balancedly splittable Hadamard matrices, we can construct optimal sets of equiangular lines.

0.4. Theorem. If there exists a balancedly splittable Hadamard matrix with parameters $(n, \ell, a, -a)$, then there is an optimal set of equiangular lines in \mathbf{R}^ℓ .

Proof. Suppose that H is a balancedly splittable Hadamard matrix with respect to the $\ell \times n$ submatrix H_1 with parameters $(n, \ell, a, -a)$. Take \mathcal{L} to be the collection of normalized columns of H_1 . Note that $a^2 \ell^2 = \ell(n - \ell)/(n - 1)$; then the absolute value of the inner product between distinct lines in \mathcal{L} is given by $a = \sqrt{n - \ell}/\sqrt{\ell(n - 1)}$; moreover, $\ell \leq a$. The right-hand side of (0.3.a) reduces to n . We have, therefore, exhibited an optimal set of equiangular lines. ■

In §10, we will pursue this topic again in the more restricted setting of frames.

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0.3. Constructions. For the sake of completeness, we make Example 0.1 general.

0.5. Proposition. If there exists a Hadamard matrix of order n , then there exists a balancedly splittable Hadamard matrix with parameters $(n^2, n, n, 0)$.

Proof. Let H be a normalized Hadamard matrix of order n , and label the rows $h_0 = \mathbf{1}, \dots, h_{n-1}$. Take M to be the block matrix defined by $M_{ij} = h_j^t h_i$. Then $M_{ij} M_{kj}^t = (h_j^t h_i)(h_j^t h_k)^t = h_j^t (h_i h_k^t) h_j = O$ whenever $i \neq k$, hence M is a Hadamard matrix of order n^2 (cf ?). Take M_1 to be the first block row of M .

Then

$$M_1^t M_1 = \begin{pmatrix} J \\ \mathbf{1}^t h_1 \\ \vdots \\ \mathbf{1}^t h_{n-1} \end{pmatrix} (J \quad h_2^t \mathbf{1} \quad \dots \quad h_{n-1}^t \mathbf{1}) = mI \otimes J,$$

and the proof is complete. ■

There are many more constructions presented in ? that the interested reader may consult. For our purposes, however, we present a novel construction. Consider the following example.

0.6. Example. Consider the Hadamard matrix of order 2

$$(0.6.a) \begin{pmatrix} + & + \\ + & - \end{pmatrix},$$

and label the rows h_0 and h_1 . Form the matrices $c_i = h_i^t h_i$, the so-called *auxiliary matrices*, shown in order below.

$$(0.6.b) \quad c_0 = \begin{pmatrix} + & + \\ + & + \end{pmatrix}, c_1 = \begin{pmatrix} + & - \\ - & + \end{pmatrix}.$$

Take $K = H \otimes H$, labelling the rows as k_0, k_1, k_2 , and k_3 . Form the block circulant matrices A and B with first rows (c_0, c_1, c_1) and $(c_0, c_1, -c_1)$, shown below.

$$(0.6.c) \quad A = \begin{pmatrix} + & + & + & - & + & - \\ + & + & - & + & - & + \\ + & - & + & + & + & - \\ - & + & + & + & - & + \\ + & - & + & - & + & + \\ - & + & - & + & + & + \end{pmatrix}, B = \begin{pmatrix} + & + & + & - & - & + \\ + & + & - & + & + & - \\ - & + & + & + & + & - \\ + & - & + & + & - & + \\ + & - & - & + & + & + \\ - & + & + & - & + & + \end{pmatrix}.$$

Now, form the block matrix $F = (F_1 \dots F_7)$ by defining $F_i = k_i^t h_0$

$$(0.6.d) \quad F = \begin{pmatrix} + & + & + & + & + & + \\ - & - & + & + & - & - \\ + & + & - & - & - & - \\ - & - & - & - & + & + \end{pmatrix}.$$

Finally, take $E = F^t$. We then form the block matrix $X = \begin{pmatrix} J & F & -F \\ E & A & B \\ -E & B & A \end{pmatrix}$

$$(0.6.e) \quad X = \begin{pmatrix} + & + & + & + & + & + & + & + & - & - & - & - & - & - \\ + & + & + & + & - & - & + & + & - & - & + & + & - & - & + & + \\ + & + & + & + & + & + & - & - & - & - & - & + & + & + & + \\ + & + & + & + & - & - & - & - & + & + & + & + & - & - & - \\ + & - & + & - & + & + & + & - & + & + & + & - & - & + & - \\ + & - & + & - & + & + & - & + & + & + & - & + & + & - & - \\ + & + & - & - & + & - & + & + & + & - & - & + & + & + & - \\ + & + & - & - & - & + & + & + & - & + & + & - & + & - & + \\ + & - & - & + & + & - & + & - & + & + & + & - & - & + & + & + \\ + & - & - & + & - & + & - & + & + & + & - & + & + & + & + \\ - & + & - & + & + & + & - & - & + & + & + & - & + & - & - \\ - & + & - & + & + & + & - & + & + & - & + & + & - & + & - \\ - & - & + & + & - & + & + & + & + & - & + & - & + & + & - \\ - & - & + & + & + & - & + & + & + & - & + & + & + & - & + \\ - & + & + & - & + & - & - & + & + & + & + & - & + & - & + & + \\ - & + & + & - & - & + & + & - & + & + & - & + & - & + & + & + \end{pmatrix}.$$

The matrix X is a Hadamard matrix of order 16, and evidently, it is balancedly splittable. Indeed, the submatrix shown in bold above can be used to form a balanced split. Due to the form of X , however, any one of the following can be used to form a balanced split

$$(0.6.f) \quad \begin{pmatrix} F \\ A \\ B \end{pmatrix}, \begin{pmatrix} -F \\ B \\ A \end{pmatrix}, (E \quad A \quad B), (-E \quad B \quad A).$$

In any event, the parameters of the splits are $(16, 6, 2, -2)$. Therefore, we have optimal sets of equiangular lines.

0.7. Example. Beginning with the Hadamard matrix of order 4

$$(0.7.a) \quad \begin{pmatrix} + & + & + & + \\ - & + & - & + \\ - & + & + & - \\ - & - & + & + \end{pmatrix},$$

we obtain

This construction can be made perfectly general. Since this result is ultimately a special case of Theorem ??, we will not show it explicitly.

0.8. Theorem. There is a balancedly splittable Hadamard matrix of order $4n^2$ with parameters $(4n^2, 2n^2 - n, n, -n)$ whenever there is a Hadamard matrix of order n .

0.9. Corollary. If there is a Hadamard matrix of order n , then there is an optimal set of equiangular lines in \mathbf{R}^{2n^2-n} .

We mention in passing that ? showed that balancedly splittable Hadamard matrices can be used to construct various association schemes. As nothing essentially new about association schemes has been added in our study of orthogonal designs, we will not pursue this topic here and simply refer the reader to the aforementioned article.