# Weighing Matrices Generalizations and Related Configurations

Thomas Pender
\*\*\*Joint work with Hadi Kharaghani, Sho Suda

Department of Mathematics and Computer Science University of Lethbridge

June 22, 2022



### Summary

- Preliminaries
- 2 Balancedly Splittable Orthogonal Designs
- 3 Unifying Construction of Weighing Matrices
- 4 A New Class of Balanced Weighing Matrices
- 5 Equivalence to Association Schemes

## **Preliminaries**



### Section Summary

- Preliminaries
  - Balanced Incomplete Block Designs
  - Weighing Matrices
  - Balanced Generalized Weighing Matrices
  - Orthogonal Designs



- $J_{n \times m}$  the  $n \times m$  matrix of all 1s.
- $I_n$  the identity matrix of order n.

#### **Definition:** Balanced Incomplete Block Design

- A binary  $v \times b$  (0,1)-matrix A such that:

  - $2 J_v A = k J_v.$

Write  $BIBD(v, b, r, k, \lambda)$ .

• The design is symmetric if v = b (equiv. k = r).

(ロ) (国) (国) (国) (国) (ロ)

5 / 83

• A BIBD(13, 9, 6)

4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶

#### **Definition:** Weighing Matrix

A v imes v (-1,0,1)-matrix W such that

$$WW^t = kI_v.$$

Write W(v, k).

- W(v, v) is a Hadamard matrix
- W(v, v 1) is a conference matrix

• A W(13,9)

$$W = \begin{pmatrix} 0 & 0 & - & 0 & - & 1 & - & - & 1 & 0 & - & - & - \\ 1 & 0 & 0 & - & 0 & - & 1 & - & - & 1 & 0 & - & - \\ 1 & 1 & 0 & 0 & - & 0 & - & 1 & - & - & 1 & 0 & - \\ 1 & 1 & 1 & 0 & 0 & - & 0 & - & 1 & - & - & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & - & 0 & - & 1 & - & - & 1 \\ - & 0 & 1 & 1 & 1 & 0 & 0 & - & 0 & - & 1 & - & - \\ 1 & - & 0 & 1 & 1 & 1 & 0 & 0 & - & 0 & - & 1 \\ - & 1 & 1 & - & 0 & 1 & 1 & 1 & 0 & 0 & - & 0 \\ 1 & - & 1 & 1 & - & 0 & 1 & 1 & 1 & 0 & 0 & - \\ 0 & 1 & - & 1 & 1 & - & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & - & 1 & 1 & - & 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

#### **Definition:** Generalized Weighing Matrix

- Abelian group G not containg 0.
- A  $v \times v$  (0, G)-matrix W such that

$$WW^* = (k \cdot 1_G)I_V \pmod{\mathbb{Z}G}.$$

#### **Definition:** Balanced Generalized Weighing Matrix

- Abelian group G not containing 0.
- A  $v \times v$  (0, G)-matrix W such that

$$WW^* = (k \cdot 1_G)I_v + \frac{\lambda}{|G|}G(J_v - I_v).$$

- Write  $BGW(v, k, \lambda; G)$ .
- A  $BGW(v, k, \lambda; C_2)$  is a balanced weighing matrix.
- $\lambda = k(k-1)/(v-1)$

4 D > 4 D > 4 E > 4 E > E 990

T. Pender (U of L)

- Necessary conditions for BGWs:
  - weighing matrix
  - symmetric BIBD
- Sufficient in the case that |G| = prime (Lam and Leung, 2000)

#### **Definition:** Orthogonal Design

- Real commuting variables  $\{z_1, \ldots, z_u\}$ .
- A  $v \times v$   $(0, \pm z_1, \dots, \pm z_u)$ -matrix X such that

$$XX^t = \sigma I_v,$$

where 
$$\sigma = \sum_{i} s_{i} z_{i}^{2}$$
.

• Write  $OD(v; s_1, \ldots, s_u)$ .



• An *OD*(4; 1, 1, 1, 1)

$$X = \begin{pmatrix} a & b & c & d \\ \overline{b} & a & \overline{d} & c \\ \overline{c} & d & a & \overline{b} \\ \overline{d} & \overline{c} & b & a \end{pmatrix}.$$



# **Balancedly Splittable ODs**

### Section Summary

- 2 Balancedly Splittable Orthogonal Designs
  - Balancedly Splittable Hadamard Matrices
  - Balancedly Splittable Orthogonal Designs
  - Faithful Construction
  - Related Configurations

• Recall a Hadamard matrix is a W(v, v).

# **Definition:** Balancedly Splittable Hadamard Matrix (Kharaghani and Suda, 2019)

- A W(v, v), say H.
- An  $v \times \ell$  submatrix  $H_1$  such that

$$H_1H_1^t = \ell I_v + aA + b(J_v - I_v - A),$$

where A is a symmetric (0,1)-matrix with zero diagonal.

• H is  $(v, \ell, a, b)$  balancedly splittable.

◆ロト ◆個ト ◆差ト ◆差ト 差 めるぐ

• A Hadamard matrix of order 16

• A Hadamard matrix of order 16

• A balanced split.

18 / 83

```
2 2 2 2 2 6 2 2 2 2 2 2 2 2 2 2
2 2 2 2 2 2 6 2 2 2 2 2 2 2 2 2
2 2 2 2 2 2 2 2 6 2 2 2 2 2 2 2
2 2 2 2 2 2 2 2 2 6 2 2 2 2 2 2
2 2 2 2 2 2 2 2 2 6 2 2 2 2 2
222222222226
```

#### • An *OD*(16; 8, 8)

ababababābābāb babā bābabā bābabābab bababā bābā bābabab babā bā bababā b ābāabbābāabbāba ābāabbabaabbabā bābbāabbābaabbā bābbaabbabāabba ābabābāabbābaab БаБ<mark>ававБа</mark>вав baabbabaabbaba Баа b Ба b ā a b Ба Ба abbaabbābāabbā babbāabbabaabba abābābaabbābāab abābabāabbabaab

a b a b a b a b a b ā b ā b ā b babābābababāb bābāabbābāabbāba ābāabbababā bābbāabbābaabbā bābbaabbabāabba ā b a b ā b ā a b b ā b a a b ābabaabbabāab baabbābaabbābā babbaabbabaabba  $\bar{b}$  a b  $\bar{b}$   $\bar{a}$  a b  $\bar{b}$  a  $\bar{b}$  a a b  $\bar{b}$  a a b ā b ā b a a b b ā b ā a b bābabāabbaab

'abababababā bābābāb' ababābabābabābab abababābābābabab ababābābababāb bābāabbābāabbāba bābāabbabaabbabā abābbāabbābaabbā abābbaabbabāabba bābabābāabbābaab bābabaabbabāab babaabbābaabbābā a b̄ a a b b̄ a b ā a b b̄ a b̄ a 

• Products in the set  $\{\pm 2a^2, \pm 2b^2, \pm 2ab\}$ .

4□ > 4□ > 4□ > 4□ > 4□ > 4□ > 4□

#### **Definition:** Balancedly Splittable (Kharaghani et al., 2021)

• An OD X with  $v \times \ell$  submatrix  $X_1$  admits a stable  $(v, \ell, a, b)$  split if

$$X_1X_1^t = \sigma(cI_v + aA + b(J_v - I_v - A)).$$

The split is unstable if it is not stable and

$$X_1 X_1^t = \ell I_v + aA + b(J_v - I_v - A)$$

after replacing each variable with 1.

< ロ ト → 個 ト → 置 ト → 置 ・ り Q (^)

#### **Definition:** Auxiliary Matrices

- X a full  $OD(v; s_1, ..., s_u)$  and H the Hadamard matrix obtained by replacing each variable with 1.
- Label rows of X and H by  $x_0, \ldots, x_{\nu-1}$ , and  $h_0, \ldots, h_{\nu-1}$ .
- The auxiliary matrices of X are given by

$$c_i = h_i^t x_i, \qquad i = 0, \dots, v - 1.$$

< ロ ト → 個 ト → 置 ト → 置 ・ り Q (^)

30 / 83

• An *OD*(2; 1, 1):

$$\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$$

Auxiliary matrices:

$$c_0 = \begin{pmatrix} a & b \\ a & b \end{pmatrix}, \qquad c_1 = \begin{pmatrix} b & -a \\ -b & a \end{pmatrix}$$

Form the sequences

$$\alpha = (c_0, c_1, c_1), \qquad \beta = (c_0, c_1, -c_1).$$

• The sum of the periodic autocorrelations of  $\alpha$  and  $\beta$  satisfy

$$\sum_{i} \alpha_{i} \alpha_{i+j}^{t} + \sum_{i} \beta_{i} \beta_{i+j}^{t} = 0, \qquad j \not\equiv 0 \pmod{3}$$

Sums of cross-correlations

$$\sum_{i} \alpha_{i} \beta_{i+j}^{t} + \sum_{i} \beta_{i} \alpha_{i+j}^{t} = 0, \quad j \not\equiv 0 \pmod{3}$$

< ロ ト → 個 ト → 重 ト → 重 → りへで

T. Pender (U of L)

Form the matrices

$$A = circ(\alpha), \qquad B = circ(\beta),$$

• Then the matrix

$$\Theta = \begin{pmatrix} A & B \\ B & A \end{pmatrix}.$$

$$\Theta = \begin{pmatrix} a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a \\ a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} \\ b & \bar{a} & a & b & \bar{b} & \bar{a} & \bar{b} & a & a & b & \bar{b} & \bar{a} \\ \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a \\ b & \bar{a} & b & \bar{a} & a & b & \bar{b} & a & b & \bar{a} & a & b \\ \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b \\ \bar{a} & b & \bar{b} & \bar{a} & \bar{b} & \bar{a} & a & b & \bar{b} & \bar{a} & \bar{b} & \bar{a} \\ \bar{a} & b & \bar{b} & \bar{a} & \bar{b} & \bar{a} & a & b & \bar{b} & \bar{a} & \bar{b} & \bar{a} \\ \bar{b} & a & a & b & \bar{b} & \bar{a} & \bar{b} & \bar{a} & a & b & \bar{b} & \bar{a} \\ \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{a} & \bar{b} & \bar{b} & \bar{a} & \bar{b} & \bar{a} \\ \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{a} & \bar{b} & \bar{b} & \bar{a} & \bar{b} & \bar{a} \\ \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{a} & \bar{b} & \bar{b} & \bar{a} & \bar{b} & \bar{a} \\ \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{a} & \bar{b} & \bar{b} & \bar{a} & \bar{b} \\ \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{a} & \bar{b} & \bar{b} & \bar{a} & \bar{b} \\ \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{a} & \bar{b} & \bar{b} & \bar{a} & \bar{b} \\ \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{a} & \bar{b} & \bar{b} & \bar{a} & \bar{b} \\ \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{a} & \bar{b} & \bar{b} & \bar{a} & \bar{b} \\ \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{a} & \bar{b} & \bar{b} & \bar{a} & \bar{b} \\ \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{a} & \bar{b} & \bar{b} & \bar{a} & \bar{b} \\ \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} \\ \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{b} & \bar{a} & \bar{b} \\ \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{b} & \bar{a} & \bar{b} \\ \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{b} & \bar{a} & \bar{b} \\ \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{b} & \bar{a} & \bar{b} \\ \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{b} & \bar{a} & \bar{b} \\ \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{b} & \bar{a} & \bar{b} \\ \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{b} & \bar{a} & \bar{b} \\ \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{b} & \bar{a} & \bar{b} \\ \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{b} & \bar{a} & \bar{b} \\ \bar{b} & \bar{a} & \bar{b} & \bar{b} & \bar{a} & \bar{b} & \bar{b} \\ \bar{b} & \bar{a} & \bar{b} & \bar{b} & \bar{a} & \bar{b} & \bar{b} \\ \bar{b} & \bar{a} & \bar{b} & \bar{b} & \bar{b} & \bar{b} & \bar{b} \\ \bar{b} & \bar{b} & \bar{b} & \bar{b} & \bar{b} & \bar{b} \\ \bar{b} & \bar{b} & \bar{b} & \bar{b} & \bar{b} & \bar{b} \\ \bar{b} & \bar{b} & \bar{b}$$

< □ > < □ > < □ > < = > < = > < > < (°

$$\Theta\Theta^{t} = \begin{pmatrix} AA^{t} + BB^{t} & AB^{t} + BA^{t} \\ AB^{t} + BA^{t} & AA^{t} + BB^{t} \end{pmatrix}.$$

4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶

$$\Theta\Theta^t = (a^2 + b^2)$$

```
ā
      b
        ā
          a
      b ā b
    b b a b a a b b a
 ābāabbāba
    aabbabā
    āba
 b b a b a a b b a b
    b b ā b ā a b b ā
   a b \bar{b} a \bar{b}
 ābaabbābā
babāabbabaa
```

◆ロト ◆個ト ◆差ト ◆差ト = 900

Define:

$$Y = \begin{pmatrix} 1 & 1 \\ 1 & - \end{pmatrix} \otimes X,$$
  
$$K = \begin{pmatrix} 1 & 1 \\ 1 & - \end{pmatrix} \otimes H.$$

• Index rows as  $y_0, ..., y_{2n-1}$ , and  $k_0, ..., k_{2n-1}$ .

$$Y = \begin{pmatrix} 1 & 1 \\ 1 & - \end{pmatrix} \otimes X$$
  $K = \begin{pmatrix} 1 & 1 \\ 1 & - \end{pmatrix} \otimes H$   
Rows:  $y_0, \dots, y_{2n-1}$  Rows:  $k_0, \dots, k_{2n-1}$ 

#### Form

$$F = \begin{pmatrix} k_1^t x_0 & k_2^t x_0 & k_3^t x_0 \end{pmatrix},$$

$$E = \begin{pmatrix} h_0^t y_1 \\ h_0^t y_2 \\ h_0^t y_3 \end{pmatrix},$$

$$G = k_0^t y_0.$$

		F	=					_	F		
а	b	b	ā	b	ā	a	b	b	ā	b	а
а	b	Б	a	Б	а	а	b	$\bar{b}$	a	b	ā
Ь	ā	a	b	b	ā	$\bar{b}$	a	a	b	b	ā
$\bar{b}$	a	a	b	$\bar{b}$	а	Ь	ā	a	b	$\bar{b}$	a
Ь	ā	b	ā	a	b	b	ā	$\bar{b}$	a	a	Ь
$\bar{b}$	a	$\bar{b}$	a	a	b	$\bar{b}$	a	b	ā	a	b
а	b	Ь	ā	b	а	а	b	b	ā	b	ā
a	b	$\bar{b}$	a	b	ā	a	b	$\bar{b}$	a	$\bar{b}$	a
$\bar{b}$	a	а	b	b	ā	b	ā	a	b	b	ā
b	ā	a	b	$\bar{b}$	а	$\bar{b}$	a	a	b	$\bar{b}$	a
Ь	ā	$\bar{b}$	a	а	b	Ь	ā	b	ā	a	b
$\bar{b}$	a	b	ā	а	b	$\bar{b}$	a	$\bar{b}$	a	a	b
	a b b b b a a b b b	a b a b a b a b b a b b a b b a b b a b b b a b b b b a b b a b b a b b a b b a b b a b b a b b a b b a b b a b b a b b a b b a b b a b b a b b a b b b a b b a b b a b b a b b b a b b a b b a b b b a b b b a b b b a b b b a b b b a b b b a b b b a b b b a b b b a b b b a b b b b a b b b b a b b b b b a b	a b b b a b b a b b a b b a b b b b b b	a       b       \bar{b}       a       b         b       \bar{a}       a       b       \bar{a}         b       \bar{a}       \bar{b}       \bar{a}         a       b       \bar{b}       a       b         b       \bar{a}       a       b       b         b       \bar{a}       a       b       b         b       \bar{a}       \bar{b}       a       b         b       \bar{a}       \bar{b}       a       b	a       b       \bar{b}       a       \bar{b}         b       \bar{a}       a       b       \bar{b}         b       \bar{a}       b       \bar{a}       a         b       \bar{a}       \bar{b}       \bar{a}       \bar{b}         a       b       \bar{a}       a       b       \bar{b}         b       \bar{a}       a       b       \bar{b}         b       \bar{a}       a       b       \bar{b}         b       \bar{a}       \bar{b}       a       a	a       b       \bar{b}       a       \bar{b}       a         b       \bar{a}       a       b       \bar{a}       a       b       \bar{a}       a       b       \bar{a}       a       b       \bar{a}       \bar{a}       b       \bar{a}       \bar{a}       b       \bar{a}       \bar{a}       b       \bar{a}       \bar{a}       b       \bar{a}       \b	a       b       \bar{b}       a       \bar{b}       a       a       b       \bar{a}       \bar{b}       a       b       \bar{a}       \bar{a}       b       \bar{a}       \bar{a}       b       \bar{a}       \bar{a}	a       b       \bar{b}       a       b       \bar{a}       b       \bar{a}       \bar{b}       a       b       \bar{a}       b       \bar	a       b       \bar{b}       a       b       \bar{b}       a       b       \bar{b}         b       \bar{a}       a       b       \bar{a}       a       b       \bar{a}       a       b       \bar{a}       a       b       \bar{a}       \bar{a}       b       \bar{a}       \bar{b}       a       b       \bar{a}       b       \bar{a}       b       \bar{a}       b       \bar{a}       b       \bar{a}       a       b       \bar{b}       a       a       b       \bar{a}       \bar{a}       b       \bar{a}       b       \bar{a}       \bar{a}	a       b       b       ā       b       ā       a       b       b       ā         a       b       b       ā       b       ā       a       b       ā	a       b       ā       b       ā       b       ā       b       ā       b       ā       b       ā       b       ā       b       ā       b       ā       b       ā       b       a       b       b       a       b       b       a       b       b       a       b       b       a       b       b       a       b       b       a       b       b       a       b       b       a       b       b       a       b       b       a       b       b       a       b       b       a       b       b       a       b

◄□▶◀圖▶◀불▶◀불▶ 불 쒸٩○

abababababābābāb ababā bā bababā b bābāabbābāabbāba ābāabbabaabā bābbāabbābaabbā bābbaabbabāabba ābabāaabbābaab ā b a b a b a a b b a b a a b a b a a b b ā b a a b b ā b ā baabbabāabbaba babbaabbabaabba babbaabbabaabba abābābaabbābāab abābabāabbabaab

◆□▶ ◆□▶ ◆壹▶ ◆壹▶ □ りへ○

#### **Theorem** (Kharaghani et al., 2021)

If there is a full  $OD(v, s_1, ..., s_u)$ , then there is a full  $OD(4v^2; 4vs_1, ..., 4vs_u)$  for which:

- there are two verticle splits, and
- there are two unstable horizontal splits.

#### **Proof**

• In the general case:

$$\alpha = (c_0, c_1, \dots, c_{n-1}, c_{n-1}, \dots, c_1)$$
  
$$\beta = (c_0, c_1, \dots, c_{n-1}, -c_{n-1}, \dots, -c_1).$$

Tedious checking of products between block rows.

- Related configurations:
  - equiangular lines
  - Unbiased orthogonal designs
  - quasi-symmetric designs
  - quaternion ODs and frames

# Unifying Construction of Weighing Matrices

## **Section Summary**

- 3 Unifying Construction of Weighing Matrices
  - Classical Parameter BGWs
  - Normalization
  - Generalized Simplex Codes
  - Construction

#### **Proposition**: Classical Paramaeter BGWs

- Let q be a prime power and d > 0.
- BGWs are known to exist with the following parameters:

$$\left(v = \frac{q^{d+1} - 1}{q - 1}, k = q^d, \lambda = q^d - q^{d-1}; G = C_{q-1}\right)$$

for every q and d > 0, and

$$\left(v = \frac{q^{d+1}-1}{q-1}, k = q^d, \lambda = q^d - q^{d-1}; G = C_{2q-2}\right)$$

for even q and d.

◆ロト ◆個ト ◆差ト ◆差ト 差 めるぐ

46 / 83

T. Pender (U of L) Weighing Matrices 06/22/2022

- Applying an epimorphism to the nonzero entries of a BGW gives a BGW with the same parameters.
- Equivalence operations for generalized weighing matrices:
  - permutation of rows
  - permutation of columns
  - post-multiplying column elements by a group element
  - pre-multiplying row elements by a group element
- Every weighing matrix is equivalent to one of the following form

$$\begin{pmatrix} \mathbf{0} & R \\ \mathbf{1} & D \end{pmatrix}$$

- R is the residual-part
- D is the derived-part

<ロト < 個 ト < 重 ト < 重 ト 三 重 の < で

#### **Definition:** Simplex Code

- q a prime power and d > 0
- Form matrix G with columns given by reps. of 1-D subspaces of  $GF(q^{d+1})$
- ullet simplex code  $\mathcal{S}_{q,d} = \mathit{row}(\mathcal{G})$

#### Proposition

For  $S_{q,d}$ :

- $wt(x) = q^d$  for all  $x \in \mathcal{S}_{q,d}$
- $dist(x, y) = q^d$  for all  $x, y \in \mathcal{S}_{q,d}$  and  $x \neq y$

4 D > 4 B > 4 B > 4 B > 9 Q @

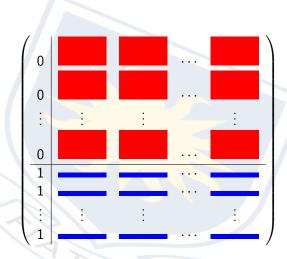
- Ingredients of unifying construction:
  - a normalized W(v,q) (seed matrix) with residual-part R and derived-part D
  - a  $W((q^{d+1}-1)/(q-1), q^d)$ , say W
  - ullet  $\mathcal{S}_{q,d}$



- Recipie of unifying construction:
  - Form  $A = W \otimes R$ .
  - Form B by replacing elements of  $S_{a,d}$  by rows of D.
  - Then

$$\begin{pmatrix} \mathbf{0} & A \\ \mathbf{1} & B \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{0} & A \\ \mathbf{1} & B \end{pmatrix}$$
 is a  $W((v-1)(q^{d+1}-1)/(q-1)+1,q^{d+1})$ .



• A seed *W*(8, 5)

$$\begin{pmatrix} \mathbf{0} & R \\ \mathbf{1} & D \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & 0 & 0 & 0 & 1 & 1 & -- \\ 1 & 1 & 1 & 1 & -0 & 0 & 0 \\ 1 & 1 & --0 & 0 & 0 & 1 & 0 \\ 1 & --1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

◆ロト ◆個 ト ◆ 差 ト ◆ 差 ・ 夕 Q (\*)

• A classical parameter W(6,5)

$$W = \begin{pmatrix} - & 1 & - & 0 & 1 & 1 \\ - & - & 1 & - & 0 & 1 \\ - & - & - & 1 & - & 0 \\ 0 & - & - & - & 1 & - \\ 1 & 0 & - & - & - & 1 \\ - & 1 & 0 & - & - & - \end{pmatrix}$$

• The simplex code  $S_{5,1}$  (transposed)

◆□▶ ◆□▶ ◆■▶ ◆■▶ ■ 釣魚@

53 / 83

- Take  $A = W \otimes R$ .
- Take B to be the matrix formed after replacing the entries of  $S_{5,1}$  by the rows of D.
- Then

$$\begin{pmatrix} 0 & A \\ 1 & B \end{pmatrix}$$

is a W(43, 25).

1000 Ŏ \_ ō 1 1  $-\dot{0}$ 0 0 0 ŏ i -0 - 11 ō Ĭ -10011-Ò 0 Ò 1 - 0 - 0 -Ò 0 1 --1-10-000--000--000Ó Ó 0  $1 \ 1 - 0$ -0-1-0-0-1 ŏ  $0 \ 0 \ --1 \ -1 \ 0$ Ŏ 0 ō  $\tilde{-}$  0 0 ----0.0ŏ ŏ ŏ ŏ ŏ  $0\ 0\ 0\ 0\ -0\ -1\ 1\ -0\ -0\ -1$ 001000 Ŏ Ĭ --11-001  $0 \ 1 \ - \ 0$ Ò 1 - 0Ô 0.0 0 0 1 1 - - - 1 - 0 0 1 -0.0~ ñ 1 0 Ò 0 Ò - 0 Ò \_\_\_ δŏ i Ŏ -00\_ ŏ 0 0 0 1 - 1 - 0 ŏ 0 1 0 0 1 1 0 0 ŏ ō -100 0 ŏ ō 1 0 0 0 -1 - 00 0 \_\_\_\_ 0 \_ 0 0 - - 0 -Ŏ - - 00 0 0 Ó 0 0 0 -00-1n 1 1 -\_ 1 0 n 0 -0000-1000 $\frac{1}{1}$   $\frac{1}{1}$   $\frac{1}{0}$   $\frac{1}{0}$ 0 1 1 - - 0 - 00 0  $0 \ 1 - - 0 - 0 \ 0$ 0 0 0 1 1 0 0 ĭ ŏ 0 0 0 -1-0.00 Ŏ 0  $\frac{1}{1} - \frac{1}{1} - \frac{1}{1} - \frac{1}{1} = 0$ 0 0 1 1 0 0 0 1 -1 0 0 0 1 0 0 0 1 0 0 1 0 0 0 0 1 -11-1\_ Ò - - 1 00 0 - ō ŏ ō Ŏ \_ ŏ \_ -0-00-1 -0 0 1 0 Ò Ò Õ 0 10 - 1000

#### Theorem (Kharaghani et al., 2022a)

If there is a W(v,q), then there is a weighing matrix with parameters

$$\left(\frac{(v-1)(q^{d+1}-1)}{q-1}+1,q^{d+1}\right)$$

#### whenever:

- $\mathbf{0}$  q is odd and every d > 0, and
- $\bigcirc$  q and d are both even.

T. Pender (U of L)

Seed $(v, k)$	Succident $(v', k')$	Seed $(v, k)$	Succident $(v', k')$
(6, 5):	(31, 25), (156, 125), (781, 625)	(16, 3):	(69, 9), (196, 27), (601, 81)
(8, 5):	(43, 25), (218, 125)	(16, 5):	(91, 25), (466, 125)
(8, 7):	(57, 49), (400, 343)	(16, 7):	(121, 49), (856, 343)
(10, 5):	(55, 25), (280, 125)	(16, 9):	(151, 81)
(10, 9):	(91, 81), (820, 729)	(16, 11):	(181, 121)
(12, 5):	(67, 25), (342, 12 <mark>5)</mark>	(16, 13):	(211, 169)
(12,7):	(89, 49), (628, 343)	(18, 13):	(239, 169)
(12, 9):	(111, 81)	(19, 9):	(181, 81)
(13, 9):	(121, 81)	(20, 7):	(153, 49)
(14, 9):	(131, 81)	(20, 13):	(267, 169)
(14, 13):	(183, 169)		

# A New Class of BWs

## **Section Summary**

- 4 A New Class of Balanced Weighing Matrices
  - Seed Matrix
  - Construction

• Consider the following balanced W(19,9):

Computationally found in de Launey and Sarvate (1984).

T. Pender (U of L) Weighing Matrices 06/22/2022 60 / 83

◄□▶◀圖▶◀불▶◀불▶ 불 ∽٩

◆ロト ◆団ト ◆恵ト ◆恵ト ・恵 ・ 夕へで

→ロト→団ト→豆ト→豆 りへ○

• Let  $\Xi$  be a BGW with parameters

$$\left(v = \frac{9^{d+1} - 1}{8}, k = 9^d, \lambda = 9^d - 9^{d-1}\right)$$

over the group  $C_4 = \{1, g, g^2, g^3\}$ .

Decompose ≡ as

$$\Xi = \Xi_1 + g\Xi_g + g^2\Xi_{g^2} + g^3\Xi_{g^3},$$

where the  $\Xi_i$ s are disjoint (0,1)-matrices.

4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & g & g^2 & g^3 & 1 & g & g^2 & g^3 & 1 \\ 1 & g & 0 & 1 & g^2 & g^2 & g & g^3 & 1 & g^3 \\ 1 & g^2 & 1 & 0 & g & g^3 & g^3 & g^3 & 1 & g^3 \\ 1 & g^2 & 1 & 0 & g & g^3 & g^3 & 1 & g^3 \\ 1 & g^3 & g^2 & g & 0 & g^2 & 1 & 1 & g^3 & g \\ 1 & 1 & g^2 & g^3 & g^2 & 0 & g^3 & g & g & 1 \\ 1 & g & g & g^3 & 1 & g^3 & 0 & 1 & g^2 & g^2 \\ 1 & g^3 & g^3 & g^3 & 1 & g^3 & g & g^2 & g & 0 & g^2 \\ 1 & 1 & g^3 & g & g & 1 & g^2 & g^3 & g^2 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&1&0\\ 0&0&0&0&1&1&0&0&0&0\\ 0&0&0&0&1&1&0&0&0&0\\ 0&0&1&0&0&0&0&1&0&0&0\\ 0&0&1&0&1&0&0&0&0&0&1\\ 0&1&0&1&0&0&0&0&0&0&1\\ 0&1&0&1&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1&0&0&1&0\\ 0&0&0&0&0&0&1&0&1&0&1\\ \end{pmatrix}$$

• Form:

$$\Xi \otimes R_1 = \Xi_1 \otimes R_1 - \Xi_g \otimes R_2 - \Xi_{g^2} \otimes R_1 + \Xi_{g^3} \otimes R_2$$

- Applying  $R_1 \mapsto -R_2 \mapsto -R_1 \mapsto R_2 \mapsto R_1$ .
- Form D by substituting for the elements of  $S_{9,d}$  the rows of the derived part of  $W_{19}$ .

◆ロト ◆部ト ◆差ト ◆差ト 差 める()

The matrix

$$\begin{pmatrix} \mathbf{0} & \Xi \otimes R_1 \\ 1 & D \end{pmatrix}$$

is a balanced weighing matrix.

Theorem: (Kharaghani et al., 2022b)

For every d > 0, there is a balanced weighing matrix with parameters

$$\left(v = \frac{9(9^{d+1} - 1)}{4} + 1, k = 9^{d+1}\right).$$

◆ロト ◆個 ト ◆ 差 ト ◆ 差 ・ 夕 Q (\*)

# **Equivalence** to **Association Schemes**

### Section Summary

- 5 Equivalence to Association Schemes
  - Association Schemes
  - Equivalence to Balanced Weighing Matrices
  - Equivalence to Balanced Generalized Weighing Matrices

#### **Definition** (Commutative) Association Scheme

A (commutative) association scheme is a pair  $\mathfrak{X} = (X, \mathscr{A})$  where:

- X is a point set and  $\mathscr{A} = \{A_0, \dots, A_d\}$  a collection of (0,1)-matrices indexed by elements of X.
- $A_0 = I$ ;
- $\bullet \sum_{i=0}^{\infty} A_i = J;$
- A is closed under transposition;
- there exist non-negative integers  $p_{ij}^k$  such that  $A_iA_j = \sum_{k=0}^{u} p_{ij}^k A_k$ ; and
- $\bullet \ A_i A_j = A_j A_i.$
- \*\*\*  $\mathfrak{X}$  is symmetric if  $A_i^t = A_i$ , for every i.

↓□▶ ↓□▶ ↓□▶ ↓□▶ ↓□ ♥ ♀○

- $\mathfrak{U} = \langle \mathscr{A} \rangle$  is the adjacency algebra of  $\mathfrak{X}$ .
- $\mathfrak{U}$  has a dual basis of idempotents  $\{E_0 = |X|^{-1}J, \ldots, E_d\}$ .
- There are change of basis matrices P and Q such that

$$A_j = \sum_{i=0}^d P_{ij} E_i, \qquad E_j = |X|^{-1} \sum_{i=0}^d Q_{ij} A_i$$

- Let W be a  $BGW(v, k, \lambda; C_2)$ .
- Decompose...

$$W=W_1-W_2,$$

where  $W_1$  and  $W_2$  are disjoint (0,1)-matrices.

• Properties...

• Properties...

• 
$$(W_1 - W_2)(W_1 - W_2)^t = (W_1 - W_2)^t(W_1 - W_2) = kI$$

Properties...

• 
$$(W_1 - W_2)(W_1 - W_2)^t = (W_1 - W_2)^t(W_1 - W_2) = kI$$
  
•  $(W_1 + W_2)(W_1 + W_2)^t = (W_1 + W_2)^t(W_1 + W_2) = kI + \lambda(J - I)$ 

- Properties...
  - $(W_1 W_2)(W_1 W_2)^t = (W_1 W_2)^t(W_1 W_2) = kI$
  - $(W_1 + W_2)(W_1 + W_2)^t = (W_1 + W_2)^t(W_1 + W_2) = kI + \lambda(J I)$
  - $W_1W_1^t + W_2W_2^t = W_1^tW_1 + W_2^tW_2 = kI + \frac{\lambda}{2}(J-I)$

### Properties...

• 
$$(W_1 - W_2)(W_1 - W_2)^t = (W_1 - W_2)^t(W_1 - W_2) = kI$$
  
•  $(W_1 + W_2)(W_1 + W_2)^t = (W_1 + W_2)^t(W_1 + W_2) = kI + \lambda(J - I)$   
•  $W_1W_1^t + W_2W_2^t = W_1^tW_1 + W_2^tW_2 = kI + \frac{\lambda}{2}(J - I)$ 

•  $W_1W_2^t + W_2W_1^t = W_1^tW_2 + W_2^tW_1 = \frac{\lambda}{2}(J-I)$ 

• With  $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ , form the (0, 1)-matrices...

$$A_{0} = I_{2} \otimes I_{2} \otimes I_{v}$$

$$A_{1} = I_{2} \otimes P \otimes I_{v}$$

$$A_{2} = I_{2} \otimes J_{2} \otimes (J_{v} - I_{v})$$

$$A_{3} = \begin{pmatrix} O & I_{2} \otimes W_{1} + P \otimes W_{2} \\ I_{2} \otimes W_{1}^{t} + P \otimes W_{2}^{t} & O \end{pmatrix}$$

$$A_{4} = \begin{pmatrix} O & I_{2} \otimes W_{2} + P \otimes W_{1} \\ I_{2} \otimes W_{2}^{t} + P \otimes W_{1}^{t} & O \end{pmatrix}$$

$$A_{5} = \begin{pmatrix} O & J_{2} \otimes (J_{v} - W_{1} - W_{2}) \\ J_{2} \otimes (J_{v} - W_{1}^{t} - W_{2}^{t}) & O \end{pmatrix}$$

◆ロト ◆個ト ◆差ト ◆差ト 差 めへぐ

• Character tables...

$$P = \begin{pmatrix} 1 & 1 & 2(v-1) & k & k & 2(v-k) \\ 1 & -1 & 0 & \sqrt{k} & -\sqrt{k} & 0 \\ 1 & -1 & 0 & -\sqrt{k} & \sqrt{k} & 0 \\ 1 & 1 & 2(v-1) & -k & -k & 2(k-v) \\ 1 & 1 & -2 & -\sqrt{\frac{k(v-k)}{v-1}} & -\sqrt{\frac{k(v-k)}{v-1}} & 2\sqrt{\frac{k(v-k)}{v-1}} \\ 1 & 1 & -2 & \sqrt{\frac{k(v-k)}{v-1}} & \sqrt{\frac{k(v-k)}{v-1}} & -2\sqrt{\frac{k(v-k)}{v-1}} \end{pmatrix}$$

$$Q = \begin{pmatrix} 1 & v & v & 1 & v-1 & v-1 \\ 1 & -v & -v & 1 & v-1 & v-1 \\ 1 & 0 & 0 & 1 & -1 & -1 \\ 1 & \frac{v}{\sqrt{k}} & -\frac{v}{\sqrt{k}} & -1 & -\sqrt{\frac{(v-1)(v-k)}{k}} & \sqrt{\frac{(v-1)(v-k)}{k}} \\ 1 & -\frac{v}{\sqrt{k}} & \frac{v}{\sqrt{k}} & -1 & -\sqrt{\frac{(v-1)(v-k)}{k}} & \sqrt{\frac{(v-1)(v-k)}{k}} \\ 1 & 0 & 0 & -1 & \sqrt{\frac{k(v-1)}{v-k}} & -\sqrt{\frac{k(v-1)}{v-k}} \end{pmatrix}$$

### Theorem: Weighing Schemes (Kharaghani et al., 2022b)

There is a symmetric scheme with character tables P and Q given above if and only if there is a balanced W(v, k).

#### **Proof**

- We have sufficiency.
- To show necessity, use eigenvalues to infer structure of corresponding graphs/adjacency matrices.

- $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$  and  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  form a matrix representation of  $C_2$ .
- For a finite abelian group  $G = \{g_0, \dots, g_{n-1}\}$  and  $h \in G$ ...

$$U_h = (\delta(g_i^{-1}hg_j))$$

4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶
4□▶

$$\begin{aligned} &A_{0,g} = I_2 \otimes U_g \otimes I_v, \text{ for } g \in G, \\ &A_1 = I_2 \otimes J_n \otimes (J_v - I_V), \\ &A_{2,g} = \begin{pmatrix} O & \sum_{h \in G} (U_h \otimes W_{gh}) \\ \sum_{h \in G} (U_h \otimes W_{g^{-1}h^{-1}}^t) & O \end{pmatrix}, \text{ for } g \in G, \text{ and} \\ &A_2 = \begin{pmatrix} O & J_n \otimes (J_v - \sum_{h \in G} W_h) \\ J_n \otimes (J_v - \sum_{h \in G} W_h) \end{pmatrix}. \end{aligned}$$

4□ > 4□ > 4 = > 4 = > = 9 < 0</p>

T. Pender (U of L)

# Future Work

- More relations to BGWs!
  - error-correcting codes
  - more general objects (quasi-balanced weighing matrices)
- Can the theory of association schemes be used to say anything about BGWs?



# References

- de Launey, W. and Sarvate, D. G. (1984). Nonexistence of certain GBRDs. *Ars Combin.*, 18:5–20.
- Kharaghani, H., Pender, T., and Suda, S. (2021). Balancedly splittable orthogonal designs and equiangular tight frames. *Des. Codes Cryptogr.*, 89(9):2033–2050.
- Kharaghani, H., Pender, T., and Suda, S. (2022a). Balanced weighing matrices. *J. Combin. Theory Ser. A*, 186:Paper No. 105552, 18.
- Kharaghani, H., Pender, T., and Suda, S. (2022b). A family of balanced generalized weighing matrices. *Combinatorica, to appear.*
- Kharaghani, H. and Suda, S. (2019). Balancedly splittable Hadamard matrices. *Discrete Math.*, 342(2):546–561.
- Lam, T. Y. and Leung, K. H. (2000). On vanishing sums of roots of unity. J. Algebra, 224(1):91–109.

4 L > 4 D > 4 E > 4 E > 9 C (\*