§1. Balancedly Splittable Hadamard Matrices

Most of the results of this section are due to **?**. We include this material here to evince the fact that the concept of a balancedly splittable orthogonal design is a generalization of previous work.

* * *

1.1. Definition. Recall that a Hadamard matrix is a $\mathrm{W}(n, n)$ weighing matrix (see §3). These elusive objects have vexed combinatorialists for over a century now. Every more clever techniques from ever more branches of mathematics are needed in order to construct these objects.

We will consider one such construction in preparation for our study of orthogonal designs. The matrices we will study are the so-called balancedly splittable Hadamard matrices. First, an example.

1.1. Example. Consider a Hadamard matrix of order 4 shown below

$$\text{(1.1.a)}\quad \begin{pmatrix} + & + & + & + \\ + & + & - & - \\ + & - & + & - \\ + & - & - & + \end{pmatrix}.$$

Label the rows h_0, h_1, h_2, and h_3. We then form the block matrix M defined by $M_{ij} = h_j^t h_i$.

$$\text{(1.1.b)}\quad \left(\begin{array}{cccccccccccccccc}
\mathbf{+} & \mathbf{+} & \mathbf{+} & \mathbf{+} & \mathbf{+} & \mathbf{+} & \mathbf{+} & \mathbf{+} & \mathbf{+} & \mathbf{+} & \mathbf{+} & \mathbf{+} & \mathbf{+} & \mathbf{+} & \mathbf{+} & \mathbf{+} \\
\mathbf{+} & \mathbf{+} & \mathbf{+} & \mathbf{+} & \mathbf{+} & \mathbf{+} & \mathbf{+} & \mathbf{+} & \mathbf{-} & \mathbf{-} & \mathbf{-} & \mathbf{-} & \mathbf{-} & \mathbf{-} & \mathbf{-} & \mathbf{-} \\
\mathbf{+} & \mathbf{+} & \mathbf{+} & \mathbf{+} & \mathbf{-} & \mathbf{-} & \mathbf{-} & \mathbf{-} & \mathbf{+} & \mathbf{+} & \mathbf{+} & \mathbf{+} & \mathbf{-} & \mathbf{-} & \mathbf{-} & \mathbf{-} \\
\mathbf{+} & \mathbf{+} & \mathbf{+} & \mathbf{+} & \mathbf{-} & \mathbf{-} & \mathbf{-} & \mathbf{-} & \mathbf{-} & \mathbf{-} & \mathbf{-} & \mathbf{-} & \mathbf{+} & \mathbf{+} & \mathbf{+} & \mathbf{+} \\
+ & + & - & - & + & + & - & - & + & + & - & - & + & + & - & - \\
+ & + & - & - & + & + & - & - & - & - & + & + & - & - & + & + \\
+ & + & - & - & - & - & + & + & + & + & - & - & - & - & + & + \\
+ & + & - & - & - & - & + & + & - & - & + & + & + & + & - & - \\
+ & - & + & - & + & - & + & - & + & - & + & - & + & - & + & - \\
+ & - & + & - & + & - & + & - & - & + & - & + & - & + & - & + \\
+ & - & + & - & - & + & - & + & + & - & + & - & - & + & - & + \\
+ & - & + & - & - & + & - & + & - & + & - & + & + & - & + & - \\
+ & - & - & + & + & - & - & + & + & - & - & + & + & - & - & + \\
+ & - & - & + & + & - & - & + & - & + & + & - & - & + & + & - \\
+ & - & - & + & - & + & + & - & + & - & - & + & - & + & + & - \\
+ & - & - & + & - & + & + & - & - & + & + & - & + & - & - & +
\end{array}\right).$$

The matrix M has the submatrix M_1 which we have shown in bold above. This submatrix has the property that $M_1^t M_1 = 4I + 4A$, for some symmetric $(0, 1)$-matrix A with zero diagonal. In particular, there are only two angles that exist between the columns of M_1.

The above example motivates the following definition.

1.2. Definition. A Hadamard Matrix H of order n is *balancedly splittable* with parameters (n, ℓ, a, b) if H has an $\ell \times n$ submatrix H_1 such that

$$\text{(1.2.a)}\quad H_1^t H_1 = \ell I + aA + b(J - I - A).$$

This definition and the previous example are suggestive. A connection to strongly regular graphs has already been noted, and further, the definition given assumes the existence of a set of biangular lines. These connections will be taken up in the following subsections.

* * *

1.2. Equiangular Lines. In the previous subsection, it was intimated that balancedly splittable Hadamard matrices were related to collections of biangular lines. By a set of *lines*, we mean a collection of vectors in $\mathbf{R}^\ell$, for some ℓ. Given a collection $\mathscr{L}$ of lines in $\mathbf{R}^\ell$, define $\Xi = \{|\langle u, v\rangle| : u, v \in \mathscr{L} \text{ and } u \neq v\}$. If $|\Xi| = 2$, then we say that $\mathscr{L}$ is a set of *biangular lines*; while if $|\Xi| = 1$, then we say that $\mathscr{L}$ is a set of *equiangular lines*.****NOTE****

Clearly, if H is balancedly splittable with respect to the $\ell \times n$ submatrix H_1, then Definition 1.2 implies that the columns of H_1 are at least biangular. They are equiangular precisely in the case that $b = -a$.

We will require the following proposition due to **?**.

1.3. Proposition. Let $\mathscr{L} \subset \mathbf{R}^\ell$ be a set of lines (vectors) such that $|\langle u, v\rangle| = a$, for every pair of distinct lines u and v in $\mathscr{L}$. If $\ell < a^{-2}$, then

(1.3.a) $|\mathscr{L}| \leq \ell(1 - a^2)/(1 - \ell a^2)$.

Using balancedly splittable Hadamard matrices, we can construct optimal sets of equiangular lines.

1.4. Theorem. If there exists a balancedly splittable Hadamard matrix with parameters $(n, \ell, a, -a)$, then there is an optimal set of equiangular lines in $\mathbf{R}^\ell$.

Proof. Suppose that H is a balancedly splittable Hadamard matrix with respect to the $\ell \times n$ submatrix H_1 with parameters $(n, \ell, a, -a)$. Take $\mathscr{L}$ to be the collection of normalized columns of H_1. Note that $a^2\ell^2 = \ell(n - \ell)/(n - 1)$; then the absolute value of the inner product between distinct lines in $\mathscr{L}$ is given by $a = \sqrt{n - \ell}/\sqrt{\ell(n - 1)}$; moreover, $\ell \leq a$. The right-hand side of (1.3.a) reduces to n. We have, therefore, exhibited an optimal set of equiangular lines. ■

In §10, we will pursue this topic again in the more restricted setting of frames.

* * *

1.3. Constructions. For the sake of completeness, we make Example 1.1 general.

1.5. Proposition. If there exists a Hadamard matrix of order n, then there exists a balancedly splittable Hadamard matrix with parameters $(n^2, n, n, 0)$.

Proof. Let H be a Hadamard matrix of order n, and label the rows $h_0 = \mathbf{1}, \dots, h_{n-1}$. Take M to be the block matrix defined by $M_{ij} = h_j^t h_i$. Then $M_{ij} M_{kj}^t = (h_j^t h_i)(h_j^t h_k)^t = h_j^t (h_i h_k^t) h_j = 0$ whenever $i \neq k$, hence M is a Hadamard matrix of order n^2 ****CITE****. Take M_1 to be the first block row M. Then

$$M_1^t M_1 = \begin{pmatrix} J \\ \mathbf{1}^t h_1 \\ \vdots \\ \mathbf{1}^t h_{n-1} \end{pmatrix} \begin{pmatrix} J & h_2^t \mathbf{1} & \dots & h_{n-1}^t \mathbf{1} \end{pmatrix} = mI \otimes J,$$

and the proof is complete. ■

There are many more constructions presented in **?** that the interested reader may consult. For our purposes, however, we present a novel construction. Consider the following eaxmple.

1.6. Example. Consider the Hadamard matrix of order 2

$$\text{(1.6.a)} \quad \begin{pmatrix} + & + \\ + & - \end{pmatrix},$$

and label the rows h_0 and h_1. Form the matrices $c_i = h_i^t h_i$ shown in order below.

$$\text{(1.6.b)} \quad c_0 = \begin{pmatrix} + & + \\ + & + \end{pmatrix}, c_1 = \begin{pmatrix} + & - \\ - & + \end{pmatrix}.$$

Take $K = H \otimes H$, labelling the rows as k_1, k_2, k_3, and k_4. Form the block circulant matrices A and B with first rows (c_0, c_1, c_1) and $(c_0, c_1, -c_1)$, shown below.

$$\text{(1.6.c)} \quad A = \begin{pmatrix} + & + & + & - & + & - \\ + & + & - & + & - & + \\ + & - & + & + & + & - \\ - & + & + & + & - & + \\ + & - & + & - & + & + \\ - & + & - & + & + & + \end{pmatrix}, B = \begin{pmatrix} + & + & + & - & - & + \\ + & + & - & + & + & - \\ - & + & + & + & + & - \\ + & - & + & + & - & + \\ + & - & - & + & + & + \\ - & + & + & - & + & + \end{pmatrix}.$$

Now, form the block matrix $F = \left(\begin{smallmatrix} F_1 & \dots & F_7 \end{smallmatrix} \right)$ by defining $F_i = k_i^t h_0$

$$\text{(1.6.d)} \quad F = \begin{pmatrix} + & + & + & + & + & + \\ - & - & + & + & - & - \\ + & + & - & - & - & - \\ - & - & - & - & + & + \end{pmatrix}.$$

Finally, take $E = F^t$. We then form the block matrix $X = \left(\begin{smallmatrix} J & F & -F \\ E & A & B \\ -E & B & A \end{smallmatrix} \right)$

$$\text{(1.6.e)} \quad X = \begin{pmatrix}
+ & + & + & + & + & + & + & + & + & + & - & - & - & - & - & - \\
+ & + & + & + & - & - & + & + & - & - & + & + & - & - & + & + \\
+ & + & + & + & + & + & - & - & - & - & - & - & + & + & + & + \\
+ & + & + & + & - & - & - & - & + & + & + & + & + & + & - & - \\
+ & - & + & - & + & + & + & - & + & - & + & + & + & - & - & + \\
+ & - & + & - & + & + & - & + & - & + & + & + & - & + & + & - \\
+ & + & - & - & + & - & + & + & + & - & - & + & + & + & + & - \\
+ & + & - & - & - & + & + & + & - & + & + & - & + & + & - & + \\
+ & - & - & + & + & - & + & - & + & + & + & - & - & + & + & + \\
+ & - & - & + & - & + & - & + & + & + & - & + & + & - & + & + \\
- & + & - & + & + & + & + & - & - & + & + & + & + & - & + & - \\
- & + & - & + & + & + & - & + & + & - & + & + & - & + & - & + \\
- & - & + & + & - & + & + & + & + & - & + & - & + & + & + & - \\
- & - & + & + & + & - & + & + & - & + & - & + & + & + & - & + \\
- & + & + & - & + & - & - & + & + & + & + & - & + & - & + & + \\
- & + & + & - & - & + & + & - & + & + & - & + & - & + & + & +
\end{pmatrix}.$$

The matrix X is a Hadamard matrix of order 16, and evidently, it is balancedly splittable. Indeed, the submatrix shown in bold above can be used to form a balanced split. Due to the form of X, however, any one of the following can be used to form a balanced split

$$\text{(1.6.f)}\quad \begin{pmatrix} F \\ A \\ B \end{pmatrix}, \begin{pmatrix} -F \\ B \\ A \end{pmatrix}, \begin{pmatrix} E & A & B \end{pmatrix}, \begin{pmatrix} -E & B & A \end{pmatrix}.$$

In any event, the parameters of the splits are $(16, 6, 2, -2)$. Therefore, we have optimal sets of equiangular lines, and can form SRGs with parameters $(16, 6, 0, 2)$.

This construction can be made perfectly general. Since this result is ultimately a special case of Theorem ****CITE****, we will not show it explicitly.

1.7. Theorem. There is a balancedly splittable Hadamard matrix of order $4n^2$ with parameters $(4n^2, 2n^2 - n, n, -n)$ whenever there is a Hadamard matrix of order n.

1.8. Corollary. If there is a Hadamard matrix of order n, then there is an optimal set of equiangular lines in $\mathbf{R}^{2n^2-n}$.

We mention is passing that **?** showed that balancedly splittable Hadamard matrices can be used to construct various association schemes. As nothing essentially new about association schemes has been added in our study of orthogonal designs, we will not pursue tha topic here and simply refer the reader to the aforementioned article.

Bibliography

§2. Balancedly Splittable Orthogonal Designs

In this and the following section, the new results of **?** are presented. Here we define balanced splittability of orthogonal designs and give several constructions.

* * *

2.1. Definition. Generalizing balanced splittability to orthogonal designs presents several difficulties. The various cases are encapsulated in the next definition.

2.1. Definition. Let X be a full QOD$(n; s_1, \ldots, s_u)$. X is *balancedly splittable* if there is an $\ell \times n$ submatrix X_1 where one of the following conditions holds. In what follows $\alpha, \beta \in \{a + ib + jc + kd : a, b, c, d \in \mathbf{R}\}$.

(2.1.a) The off-diagonal entries of $X_1^* X_1$ are in the set

$$\{\pm\varepsilon c x_1^{m_1} \cdots x_u^{m_u} x_1^{*m_1'} \cdots x_u^{*m_u'} : m_i, m_i' \in \mathbf{N}, \varepsilon \in \{1, i, j, k\}, c \in \{\alpha, \alpha^*, \beta, \beta^*\}\}.$$

(2.1.b) The off-diagonal entries of $X_1^* X_1$ are in the set

$$S = \left\{ \sum_{i=1}^{u} t_i |x_i|^2 : t_i \in \mathbf{N}, \sum_{i=0}^{u} t_i = m \right\}$$

or in the set

$$\{\pm\varepsilon c x_1^{m_1} \cdots x_u^{m_u} x_1^{*m_1'} \cdots x_u^{*m_u'} : m_i, m_i' \in \mathbf{N}, \varepsilon \in \{1, i, j, k\}, c \in \{\alpha, \alpha^*, \beta, \beta^*\}\}.$$

(2.1.c) The off-diagonal entries of $X_1^* X_1$ are in the set

$$\{\pm\varepsilon c\sigma : \varepsilon \in \{1, i, j, k\}, c \in \{\alpha, \alpha^*, \beta, \beta^*\}\},$$

where $\sigma \sum_i s_i |x_i|^2$ (cf. **??**).

In the first case, the split is *unstable*; in the second, the split is *unfaithfully unstable*; and in the third, the split is *stable*. The term *faithful* is used to describe the first and third cases.

From the definition, we see that if α and β are the same in absolute value, the split corresponds to a set of equiangular lines. Interestingly, we will see that both conditions in (2.1.b) can hold simultaneously.

The next to subsections will present constructions for both the unfaithful, and the faithful case.

* * *

2.2. Unfaithful Construction. stuff here

* * *

2.3. Faithful Construction. stuff here

Bibliography

§3. Related Configurations

In this section, the balancedly splittable orthogonal designs of the previous section are applied in the construction of related objects. We will explore quasi-symmetric balanced incopmlete block designs, equiangular tight frames, and unbiased orthogonal designs.

* * *

3.1. Quasi-Symmetric BIBDs. stuff here

* * *

3.2. Equiangular Tight Frames. stuff here

* * *

3.3. Unbiased Orthogonal Designs. stuff here

Bibliography

Notes

Bibliography