Weighing Matrices

Generalizations and Related Configurations

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Summary

Preliminaries

Section Summary

1. Preliminaries
   - Balanced Incomplete Block Designs
   - Weighing Matrices
   - Balanced Generalized Weighing Matrices
   - Orthogonal Designs

- $J_{n\times m}$ the $n \times m$ matrix of all 1s.
- I_n the identity matrix of order n.

Definition: Balanced Incomplete Block Design

- A binary $v \times b$ $(0,1)$-matrix A such that:
  1. $AA^t = rI_v + \lambda(J_v - I_v)$, and
  2. $J_v A = kJ_v$.

  Write $BIBD(v, b, r, k, \lambda)$.
- The design is symmetric if $v = b$ (equiv. $k = r$).

Definition: Weighing Matrix

A $v \times v$ $(-1, 0, 1)$-matrix W such that

$$WW^t = kI_v.$$

Write $W(v, k)$.

- $W(v, v)$ is a *Hadamard matrix*
- $W(v, v-1)$ is a *conference matrix*

- A $W(13,9)$

$$W = \begin{pmatrix}
0 & 0 & - & 0 & - & 1 & - & - & 1 & 0 & - & - & - \\
1 & 0 & 0 & - & 0 & - & 1 & - & - & 1 & 0 & - & - \\
1 & 1 & 0 & 0 & - & 0 & - & 1 & - & - & 1 & 0 & - \\
1 & 1 & 1 & 0 & 0 & - & 0 & - & 1 & - & - & 1 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & - & 0 & - & 1 & - & - & 1 \\
- & 0 & 1 & 1 & 1 & 0 & 0 & - & 0 & - & 1 & - & - \\
1 & - & 0 & 1 & 1 & 1 & 0 & 0 & - & 0 & - & 1 & - \\
1 & 1 & - & 0 & 1 & 1 & 1 & 0 & 0 & - & 0 & - & 1 \\
- & 1 & 1 & - & 0 & 1 & 1 & 1 & 0 & 0 & - & 0 & - \\
1 & - & 1 & 1 & - & 0 & 1 & 1 & 1 & 0 & 0 & - & 0 \\
0 & 1 & - & 1 & 1 & - & 0 & 1 & 1 & 1 & 0 & 0 & - \\
1 & 0 & 1 & - & 1 & 1 & - & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & - & 1 & 1 & - & 0 & 1 & 1 & 1 & 0
\end{pmatrix}$$

Definition: Generalized Weighing Matrix

- Abelian group G not containg 0.
- A $v \times v$ $(0, G)$-matrix W such that

$$WW^* = (k \cdot 1_G)I_v \pmod{\mathbb{Z}G}.$$

Definition: Balanced Generalized Weighing Matrix

- Abelian group G not containing 0.
- A $v \times v$ $(0, G)$-matrix W such that

$$WW^* = (k \cdot 1_G)I_v + \frac{\lambda}{|G|}G(J_v - I_v).$$

- Write $BGW(v, k, \lambda; G)$.

- A $BGW(v, k, \lambda; C_2)$ is a balanced weighing matrix.
- $\lambda = k(k-1)/(v-1)$

Definition: Orthogonal Design

- Real commuting variables $\{z_1, \ldots, z_u\}$.
- A $v \times v$ $(0, \pm z_1, \ldots, \pm z_u)$-matrix X such that

$$XX^t = \sigma I_v,$$

  where $\sigma = \sum_i s_i z_i^2$.
- Write $OD(v; s_1, \ldots, s_u)$.

- An $OD(4; 1, 1, 1, 1)$

$$X = \begin{pmatrix} a & b & c & d \\ \bar{b} & a & \bar{d} & c \\ \bar{c} & d & a & \bar{b} \\ \bar{d} & \bar{c} & b & a \end{pmatrix}.$$

Balancedly Splittable ODs

Section Summary

- Recall a Hadamard matrix is a $W(v, v)$.

Definition: Balancedly Splittable Hadamard Matrix (Kharaghani and Suda, 2019)

- A $W(v, v)$, say H.
- An $v \times \ell$ submatrix H_1 such that

$$H_1 H_1^t = \ell I_v + aA + b(J_v - I_v - A),$$

  where A is a symmetric $(0, 1)$-matrix with zero diagonal.
- H is (v, ℓ, a, b) balancedly splittable.

- A Hadamard matrix of order 16

$$\begin{pmatrix}
1&1&1&1&1&1&1&1&1&1&-&-&-&-&-&-\\
1&1&1&1&-&-&1&1&-&-&1&1&-&-&1&1\\
1&1&1&1&1&1&-&-&-&-&-&-&1&1&1&1\\
1&1&1&1&-&-&-&-&1&1&1&1&1&1&-&-\\
1&-&1&-&1&1&1&-&1&-&1&1&1&-&-&1\\
1&-&1&-&1&1&-&1&-&1&1&1&-&1&1&-\\
1&1&-&-&1&-&1&1&1&-&-&1&1&1&1&-\\
1&1&-&-&-&1&1&1&-&1&1&-&1&1&-&1\\
1&-&-&1&1&-&1&-&1&1&1&-&-&1&1&1\\
1&-&-&1&-&1&-&1&1&1&-&1&1&-&1&1\\
-&1&-&1&1&1&1&-&-&1&1&1&1&-&1&-\\
-&1&-&1&1&1&-&1&1&-&1&1&-&1&-&1\\
-&-&1&1&-&1&1&1&1&-&1&-&1&1&1&-\\
-&-&1&1&1&-&1&1&-&1&-&1&1&1&-&1\\
-&1&1&-&1&-&-&1&1&1&1&-&1&-&1&1\\
-&1&1&-&-&1&1&-&1&1&-&1&-&1&1&1
\end{pmatrix}.$$

- A Hadamard matrix of order 16

$$\left(\begin{array}{cccc|cccccc|cccccc}
1&1&1&1&1&1&1&1&1&1&-&-&-&-&-&-\\
1&1&1&1&-&-&1&1&-&-&1&1&-&-&1&1\\
1&1&1&1&1&1&-&-&-&-&-&-&1&1&1&1\\
1&1&1&1&-&-&-&-&1&1&1&1&1&1&-&-\\
1&-&1&-&1&1&1&-&1&-&1&1&1&-&-&1\\
1&-&1&-&1&1&-&1&-&1&1&1&-&1&1&-\\
1&1&-&-&1&-&1&1&1&-&-&1&1&1&1&-\\
1&1&-&-&-&1&1&1&-&1&1&-&1&1&-&1\\
1&-&-&1&1&-&1&-&1&1&1&-&-&1&1&1\\
1&-&-&1&-&1&-&1&1&1&-&1&1&-&1&1\\
-&1&-&1&1&1&1&-&-&1&1&1&1&-&1&-\\
-&1&-&1&1&1&-&1&1&-&1&1&-&1&-&1\\
-&-&1&1&-&1&1&1&1&-&1&-&1&1&1&-\\
-&-&1&1&1&-&1&1&-&1&-&1&1&1&-&1\\
-&1&1&-&1&-&-&1&1&1&1&-&1&-&1&1\\
-&1&1&-&-&1&1&-&1&1&-&1&-&1&1&1
\end{array}\right).$$

- A balanced split.

$$\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
- & - & 1 & 1 & - & - \\
1 & 1 & - & - & - & - \\
- & - & - & - & 1 & 1 \\
1 & 1 & 1 & - & 1 & - \\
1 & 1 & - & 1 & - & 1 \\
1 & - & 1 & 1 & 1 & - \\
- & 1 & 1 & 1 & - & 1 \\
1 & - & 1 & - & 1 & 1 \\
- & 1 & - & 1 & 1 & 1 \\
1 & 1 & 1 & - & - & 1 \\
1 & 1 & - & 1 & 1 & - \\
- & 1 & 1 & 1 & 1 & - \\
1 & - & 1 & 1 & - & 1 \\
1 & - & - & 1 & 1 & 1 \\
- & 1 & 1 & - & 1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & - & 1 & - & 1 & 1 & 1 & - & 1 & - & 1 & 1 & - & 1 & 1 & - \\
1 & - & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & 1 & - & - & 1 \\
1 & 1 & - & - & 1 & - & 1 & 1 & 1 & - & 1 & - & 1 & 1 & - & 1 \\
1 & 1 & - & - & - & 1 & 1 & 1 & - & 1 & - & 1 & 1 & 1 & 1 & - \\
1 & - & - & 1 & 1 & - & 1 & - & 1 & 1 & - & 1 & 1 & - & 1 & 1 \\
1 & - & - & 1 & - & 1 & - & 1 & 1 & 1 & 1 & - & - & 1 & 1 & 1
\end{pmatrix}$$

$$
\begin{pmatrix}
6 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
\bar{2} & 6 & \bar{2} & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & \bar{2} \\
\bar{2} & \bar{2} & 6 & \bar{2} & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & \bar{2} \\
\bar{2} & \bar{2} & \bar{2} & 6 & \bar{2} & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & \bar{2} & 2 & 2 \\
2 & \bar{2} & 2 & \bar{2} & 6 & \bar{2} & 2 & \bar{2} & 2 & \bar{2} & 2 & 2 & 2 & \bar{2} & \bar{2} & 2 \\
2 & \bar{2} & 2 & \bar{2} & \bar{2} & 6 & \bar{2} & 2 & \bar{2} & 2 & 2 & 2 & \bar{2} & 2 & 2 & \bar{2} \\
2 & 2 & \bar{2} & \bar{2} & 2 & \bar{2} & 6 & \bar{2} & 2 & \bar{2} & \bar{2} & 2 & 2 & 2 & 2 & \bar{2} \\
2 & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & \bar{2} & 6 & \bar{2} & 2 & 2 & \bar{2} & 2 & 2 & \bar{2} & 2 \\
2 & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & 2 & \bar{2} & 6 & \bar{2} & 2 & \bar{2} & \bar{2} & 2 & 2 & 2 \\
2 & \bar{2} & \bar{2} & 2 & \bar{2} & 2 & \bar{2} & 2 & \bar{2} & 6 & \bar{2} & 2 & 2 & \bar{2} & 2 & 2 \\
2 & \bar{2} & 2 & \bar{2} & 2 & 2 & \bar{2} & 2 & 2 & \bar{2} & 6 & \bar{2} & \bar{2} & 2 & \bar{2} & 2 \\
2 & \bar{2} & 2 & \bar{2} & 2 & 2 & 2 & \bar{2} & \bar{2} & 2 & \bar{2} & 6 & 2 & \bar{2} & 2 & \bar{2} \\
2 & 2 & \bar{2} & \bar{2} & 2 & \bar{2} & 2 & 2 & \bar{2} & 2 & \bar{2} & 2 & 6 & \bar{2} & \bar{2} & 2 \\
2 & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & 2 & 2 & \bar{2} & 2 & \bar{2} & \bar{2} & 6 & 2 & \bar{2} \\
2 & \bar{2} & \bar{2} & 2 & \bar{2} & 2 & 2 & \bar{2} & 2 & 2 & \bar{2} & 2 & \bar{2} & 2 & 6 & \bar{2} \\
2 & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & \bar{2} & 2 & 2 & 2 & 2 & \bar{2} & 2 & \bar{2} & \bar{2} & 6
\end{pmatrix}
$$

$$\left(\begin{array}{cccc|cccccc|cccccc}
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1 & 1 & 1 & 1 & - & - & 1 & 1 & - & - & 1 & 1 & - & - & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & - & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & - & - & - & - & 1 & 1 & 1 & 1 & 1 & 1 & - & - \\
1 & - & 1 & - & 1 & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & - & 1 \\
1 & - & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - \\
1 & 1 & - & - & 1 & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - \\
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1 & - & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 \\
1 & - & - & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 \\
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\end{array}\right)$$

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1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & - \\
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1 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & - & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & - & - & - & - & 1 & 1 & 1 & 1 & 1 & 1 & - & - \\
1 & - & 1 & - & 1 & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & - & 1 \\
1 & - & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - \\
1 & 1 & - & - & 1 & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - \\
1 & 1 & - & - & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & 1 \\
1 & - & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 \\
1 & - & - & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 \\
- & 1 & - & 1 & 1 & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - \\
- & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 \\
- & - & 1 & 1 & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - \\
- & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 \\
- & 1 & 1 & - & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & 1 \\
- & 1 & 1 & - & - & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & 1
\end{array}\right)
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$$\left(\begin{array}{cccccccccc|cccccc}
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1&1&1&1&-&-&1&1&-&-&1&1&-&-&1&1\\
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1&1&1&1&-&-&-&-&1&1&1&1&1&1&-&-\\
1&-&1&-&1&1&1&-&1&-&1&1&1&-&-&1\\
1&-&1&-&1&1&-&1&-&1&1&1&-&1&1&-\\
1&1&-&-&1&-&1&1&1&-&-&1&1&1&1&-\\
1&1&-&-&-&1&1&1&-&1&1&-&1&1&-&1\\
1&-&-&1&1&-&1&-&1&1&1&-&-&1&1&1\\
1&-&-&1&-&1&-&1&1&1&-&1&1&-&1&1\\
-&1&-&1&1&1&1&-&-&1&1&1&1&-&1&-\\
-&1&-&1&1&1&-&1&1&-&1&1&-&1&-&1\\
-&-&1&1&-&1&1&1&1&-&1&-&1&1&1&-\\
-&-&1&1&1&-&1&1&-&1&-&1&1&1&-&1\\
-&1&1&-&1&-&-&1&1&1&1&-&1&-&1&1\\
-&1&1&-&-&1&1&-&1&1&-&1&-&1&1&1
\end{array}\right)$$

$$\left(\begin{array}{cccccccccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & - \\
1 & 1 & 1 & 1 & - & - & 1 & 1 & - & - & 1 & 1 & - & - & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & - & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & - & - & - & - & 1 & 1 & 1 & 1 & 1 & 1 & - & - \\
\hline
\color{blue}1 & \color{blue}- & \color{blue}1 & \color{blue}- & \color{blue}1 & \color{blue}1 & \color{blue}1 & \color{blue}- & \color{blue}1 & \color{blue}- & \color{blue}1 & \color{blue}1 & \color{blue}1 & \color{blue}- & \color{blue}- & \color{blue}1 \\
\color{blue}1 & \color{blue}- & \color{blue}1 & \color{blue}- & \color{blue}1 & \color{blue}1 & \color{blue}- & \color{blue}1 & \color{blue}- & \color{blue}1 & \color{blue}1 & \color{blue}1 & \color{blue}- & \color{blue}1 & \color{blue}1 & \color{blue}- \\
\color{blue}1 & \color{blue}1 & \color{blue}- & \color{blue}- & \color{blue}1 & \color{blue}- & \color{blue}1 & \color{blue}1 & \color{blue}1 & \color{blue}- & \color{blue}- & \color{blue}1 & \color{blue}1 & \color{blue}1 & \color{blue}1 & \color{blue}- \\
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\color{blue}1 & \color{blue}- & \color{blue}- & \color{blue}1 & \color{blue}- & \color{blue}1 & \color{blue}- & \color{blue}1 & \color{blue}1 & \color{blue}1 & \color{blue}- & \color{blue}1 & \color{blue}1 & \color{blue}- & \color{blue}1 & \color{blue}1 \\
\hline
- & 1 & - & 1 & 1 & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - \\
- & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 \\
- & - & 1 & 1 & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - \\
- & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 \\
- & 1 & 1 & - & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & 1 \\
- & 1 & 1 & - & - & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & 1
\end{array}\right)$$

$$\left(\begin{array}{cccccccccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & - \\
1 & 1 & 1 & 1 & - & - & 1 & 1 & - & - & 1 & 1 & - & - & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & - & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & - & - & - & - & 1 & 1 & 1 & 1 & 1 & 1 & - & - \\
1 & - & 1 & - & 1 & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & - & 1 \\
1 & - & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - \\
1 & 1 & - & - & 1 & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - \\
1 & 1 & - & - & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & 1 \\
1 & - & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 \\
1 & - & - & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 \\
\hline
\color{blue}- & \color{blue}1 & \color{blue}- & \color{blue}1 & \color{blue}1 & \color{blue}1 & \color{blue}1 & \color{blue}- & \color{blue}- & \color{blue}1 & \color{blue}1 & \color{blue}1 & \color{blue}1 & \color{blue}- & \color{blue}1 & \color{blue}- \\
\color{blue}- & \color{blue}1 & \color{blue}- & \color{blue}1 & \color{blue}1 & \color{blue}1 & \color{blue}- & \color{blue}1 & \color{blue}1 & \color{blue}- & \color{blue}1 & \color{blue}1 & \color{blue}- & \color{blue}1 & \color{blue}- & \color{blue}1 \\
\color{blue}- & \color{blue}- & \color{blue}1 & \color{blue}1 & \color{blue}- & \color{blue}1 & \color{blue}1 & \color{blue}1 & \color{blue}1 & \color{blue}- & \color{blue}1 & \color{blue}- & \color{blue}1 & \color{blue}1 & \color{blue}1 & \color{blue}- \\
\color{blue}- & \color{blue}- & \color{blue}1 & \color{blue}1 & \color{blue}1 & \color{blue}- & \color{blue}1 & \color{blue}1 & \color{blue}- & \color{blue}1 & \color{blue}- & \color{blue}1 & \color{blue}1 & \color{blue}1 & \color{blue}- & \color{blue}1 \\
\color{blue}- & \color{blue}1 & \color{blue}1 & \color{blue}- & \color{blue}1 & \color{blue}- & \color{blue}- & \color{blue}1 & \color{blue}1 & \color{blue}1 & \color{blue}1 & \color{blue}- & \color{blue}1 & \color{blue}- & \color{blue}1 & \color{blue}1 \\
\color{blue}- & \color{blue}1 & \color{blue}1 & \color{blue}- & \color{blue}- & \color{blue}1 & \color{blue}1 & \color{blue}- & \color{blue}1 & \color{blue}1 & \color{blue}- & \color{blue}1 & \color{blue}- & \color{blue}1 & \color{blue}1 & \color{blue}1
\end{array}\right)$$

- An $OD(16;8,8)$

$$
\begin{pmatrix}
a & b & a & b & a & b & a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} \\
a & b & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & \bar{b} & a & b \\
a & b & a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} & a & b & a & b \\
a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & a & b & a & b & a & b & \bar{a} & \bar{b} \\
b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a \\
b & \bar{a} & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} \\
a & b & \bar{a} & \bar{b} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} \\
a & b & \bar{a} & \bar{b} & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a \\
b & \bar{a} & \bar{b} & a & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b \\
b & \bar{a} & \bar{b} & a & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b \\
\bar{b} & a & \bar{b} & a & a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} \\
\bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a \\
\bar{a} & \bar{b} & a & b & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} \\
\bar{a} & \bar{b} & a & b & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a \\
\bar{b} & a & b & \bar{a} & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b \\
\bar{b} & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b
\end{pmatrix}
$$

$$
\left(\begin{array}{cccc|cccccc|cccccc}
a & b & a & b & \color{blue}a & \color{blue}b & \color{blue}a & \color{blue}b & \color{blue}a & \color{blue}b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} \\
a & b & a & b & \color{blue}\bar{a} & \color{blue}\bar{b} & \color{blue}a & \color{blue}b & \color{blue}\bar{a} & \color{blue}\bar{b} & a & b & \bar{a} & \bar{b} & a & b \\
a & b & a & b & \color{blue}a & \color{blue}b & \color{blue}\bar{a} & \color{blue}\bar{b} & \color{blue}\bar{a} & \color{blue}\bar{b} & \bar{a} & \bar{b} & a & b & a & b \\
a & b & a & b & \color{blue}\bar{a} & \color{blue}\bar{b} & \color{blue}\bar{a} & \color{blue}\bar{b} & \color{blue}a & \color{blue}b & a & b & a & b & \bar{a} & \bar{b} \\
b & \bar{a} & b & \bar{a} & \color{blue}a & \color{blue}b & \color{blue}b & \color{blue}\bar{a} & \color{blue}b & \color{blue}\bar{a} & a & b & b & \bar{a} & \bar{b} & a \\
b & \bar{a} & b & \bar{a} & \color{blue}a & \color{blue}b & \color{blue}\bar{b} & \color{blue}a & \color{blue}\bar{b} & \color{blue}a & a & b & \bar{b} & a & b & \bar{a} \\
a & b & \bar{a} & \bar{b} & \color{blue}b & \color{blue}\bar{a} & \color{blue}a & \color{blue}b & \color{blue}b & \color{blue}\bar{a} & \bar{b} & a & a & b & b & \bar{a} \\
a & b & \bar{a} & \bar{b} & \color{blue}\bar{b} & \color{blue}a & \color{blue}a & \color{blue}b & \color{blue}\bar{b} & \color{blue}a & b & \bar{a} & a & b & \bar{b} & a \\
b & \bar{a} & \bar{b} & a & \color{blue}b & \color{blue}\bar{a} & \color{blue}b & \color{blue}\bar{a} & \color{blue}a & \color{blue}b & b & \bar{a} & \bar{b} & a & a & b \\
b & \bar{a} & \bar{b} & a & \color{blue}\bar{b} & \color{blue}a & \color{blue}\bar{b} & \color{blue}a & \color{blue}a & \color{blue}b & \bar{b} & a & b & \bar{a} & a & b \\
\bar{b} & a & \bar{b} & a & \color{blue}a & \color{blue}b & \color{blue}b & \color{blue}\bar{a} & \color{blue}\bar{b} & \color{blue}a & a & b & b & \bar{a} & b & \bar{a} \\
\bar{b} & a & \bar{b} & a & \color{blue}a & \color{blue}b & \color{blue}\bar{b} & \color{blue}a & \color{blue}b & \color{blue}\bar{a} & a & b & \bar{b} & a & \bar{b} & a \\
\bar{a} & \bar{b} & a & b & \color{blue}\bar{b} & \color{blue}a & \color{blue}a & \color{blue}b & \color{blue}b & \color{blue}\bar{a} & b & \bar{a} & a & b & b & \bar{a} \\
\bar{a} & \bar{b} & a & b & \color{blue}b & \color{blue}\bar{a} & \color{blue}a & \color{blue}b & \color{blue}\bar{b} & \color{blue}a & \bar{b} & a & a & b & \bar{b} & a \\
\bar{b} & a & b & \bar{a} & \color{blue}b & \color{blue}\bar{a} & \color{blue}\bar{b} & \color{blue}a & \color{blue}a & \color{blue}b & b & \bar{a} & b & \bar{a} & a & b \\
\bar{b} & a & b & \bar{a} & \color{blue}\bar{b} & \color{blue}a & \color{blue}b & \color{blue}\bar{a} & \color{blue}a & \color{blue}b & \bar{b} & a & \bar{b} & a & a & b
\end{array}\right)
$$

$$(a^2+b^2)\begin{pmatrix}
3 & - & - & - & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
- & 3 & - & - & - & - & 1 & 1 & - & - & - & - & 1 & 1 & - & - \\
- & - & 3 & - & 1 & 1 & - & - & - & - & 1 & 1 & - & - & - & - \\
- & - & - & 3 & - & - & - & - & 1 & 1 & - & - & - & - & 1 & 1 \\
1 & - & 1 & - & 3 & - & 1 & - & 1 & - & 1 & 1 & 1 & - & - & 1 \\
1 & - & 1 & - & - & 3 & - & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - \\
1 & 1 & - & - & 1 & - & 3 & - & 1 & - & - & 1 & 1 & 1 & 1 & - \\
1 & 1 & - & - & - & 1 & - & 3 & - & 1 & 1 & - & 1 & 1 & - & 1 \\
1 & - & - & 1 & 1 & - & 1 & - & 3 & - & 1 & - & - & 1 & 1 & 1 \\
1 & - & - & 1 & - & 1 & - & 1 & - & 3 & - & 1 & 1 & - & 1 & 1 \\
1 & - & 1 & - & 1 & 1 & - & 1 & 1 & - & 3 & - & - & 1 & - & 1 \\
1 & - & 1 & - & 1 & 1 & 1 & - & - & 1 & - & 3 & 1 & - & 1 & - \\
1 & 1 & - & - & 1 & - & 1 & 1 & - & 1 & - & 1 & 3 & - & - & 1 \\
1 & 1 & - & - & - & 1 & 1 & 1 & 1 & - & 1 & - & - & 3 & 1 & - \\
1 & - & - & 1 & - & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 & 3 & - \\
1 & - & - & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - & - & 3
\end{pmatrix}$$

$$
\left(\begin{array}{cccccccccccccccc}
a & b & a & b & a & b & a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} \\
a & b & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & \bar{b} & a & b \\
a & b & a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} & a & b & a & b \\
a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & a & b & a & b & a & b & \bar{a} & \bar{b} \\
\hline
b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a \\
b & \bar{a} & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} \\
a & b & \bar{a} & \bar{b} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} \\
a & b & \bar{a} & \bar{b} & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a \\
b & \bar{a} & \bar{b} & a & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b \\
b & \bar{a} & \bar{b} & a & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b \\
\hline
\bar{b} & a & \bar{b} & a & a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} \\
\bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a \\
\bar{a} & \bar{b} & a & b & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} \\
\bar{a} & \bar{b} & a & b & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a \\
\bar{b} & a & b & \bar{a} & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b \\
\bar{b} & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b
\end{array}\right)
$$

- Products in the set $\{\pm 2a^2, \pm 2b^2, \pm 2ab\}$.

Definition: Balancedly Splittable (Kharaghani et al., 2021)

- An OD X with $v \times \ell$ submatrix X_1 admits a stable (v, ℓ, a, b) split if

$$X_1 X_1^t = \sigma(cI_v + aA + b(J_v - I_v - A)).$$

- The split is unstable if it is not stable and

$$X_1 X_1^t = \ell I_v + aA + b(J_v - I_v - A)$$

after replacing each variable with 1.

Definition: Auxiliary Matrices

- X a full $OD(v; s_1, \ldots, s_u)$ and H the Hadamard matrix obtained by replacing each variable with 1.
- Label rows of X and H by $x_0, \ldots, x_{v-1}$, and $h_0, \ldots, h_{v-1}$.
- The auxiliary matrices of X are given by

$$c_i = h_i^t x_i, \qquad i = 0, \ldots, v-1.$$

- An $OD(2;1,1)$:

$$\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$$

- Auxiliary matrices:

$$c_0 = \begin{pmatrix} a & b \\ a & b \end{pmatrix}, \qquad c_1 = \begin{pmatrix} b & -a \\ -b & a \end{pmatrix}$$

- Form the sequences

$$\alpha = (c_0, c_1, c_1), \qquad \beta = (c_0, c_1, -c_1).$$

- The sum of the periodic autocorrelations of α and β satisfy

$$\sum_i \alpha_i \alpha_{i+j}^t + \sum_i \beta_i \beta_{i+j}^t = O, \qquad j \not\equiv 0 \pmod 3$$

- Sums of cross-correlations

$$\sum_i \alpha_i \beta_{i+j}^t + \sum_i \beta_i \alpha_{i+j}^t = O, \qquad j \not\equiv 0 \pmod 3$$

- Form the matrices

$$A = circ(\alpha), \qquad B = circ(\beta),$$

- Then the matrix

$$\Theta = \begin{pmatrix} A & B \\ B & A \end{pmatrix}.$$

$$\Theta = \left(\begin{array}{cccccc|cccccc}
a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a \\
a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} \\
b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} \\
\bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a \\
b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b \\
\bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b \\
\hline
a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} \\
a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a \\
\bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} \\
b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a \\
b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b \\
\bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b
\end{array}\right)$$

$$\Theta\Theta^t = \begin{pmatrix} AA^t + BB^t & AB^t + BA^t \\ AB^t + BA^t & AA^t + BB^t \end{pmatrix}.$$

$$\Theta\Theta^t = (a^2+b^2)\left(\begin{array}{cccccc|cccccc}
6 & \bar{2} & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 \\
\bar{2} & 6 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 6 & \bar{2} & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 \\
0 & 0 & \bar{2} & 6 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 6 & \bar{2} & 0 & 0 & 0 & 0 & 2 & 2 \\
0 & 0 & 0 & 0 & \bar{2} & 6 & 0 & 0 & 0 & 0 & 2 & 2 \\
\hline
2 & 2 & 0 & 0 & 0 & 0 & 6 & \bar{2} & 0 & 0 & 0 & 0 \\
2 & 2 & 0 & 0 & 0 & 0 & \bar{2} & 6 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 6 & \bar{2} & 0 & 0 \\
0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & \bar{2} & 6 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 6 & \bar{2} \\
0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & \bar{2} & 6
\end{array}\right)$$

$$
\left(\begin{array}{cccc|cccccc|cccccc}
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
\hline
* & * & * & * & a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a \\
* & * & * & * & a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} \\
* & * & * & * & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} \\
* & * & * & * & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a \\
* & * & * & * & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b \\
* & * & * & * & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b \\
\hline
* & * & * & * & a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} \\
* & * & * & * & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a \\
* & * & * & * & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} \\
* & * & * & * & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a \\
* & * & * & * & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b \\
* & * & * & * & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b
\end{array}\right)
$$

- Define:

$$Y = \left(\begin{smallmatrix} 1 & 1 \\ 1 & - \end{smallmatrix}\right) \otimes X,$$
$$K = \left(\begin{smallmatrix} 1 & 1 \\ 1 & - \end{smallmatrix}\right) \otimes H.$$

- Index rows as $y_0, \ldots, y_{2n-1}$, and $k_0, \ldots, k_{2n-1}$.

$$Y = \begin{pmatrix} 1 & 1 \\ 1 & - \end{pmatrix} \otimes X \qquad K = \begin{pmatrix} 1 & 1 \\ 1 & - \end{pmatrix} \otimes H$$

Rows: $y_0, \ldots, y_{2n-1}$ Rows: $k_0, \ldots, k_{2n-1}$

- Form

$$F = \begin{pmatrix} k_1^t x_0 & k_2^t x_0 & k_3^t x_0 \end{pmatrix},$$

$$E = \begin{pmatrix} h_0^t y_1 \\ h_0^t y_2 \\ h_0^t y_3 \end{pmatrix},$$

$$G = k_0^t y_0.$$

$$
\left(\begin{array}{c|cccccc|cccccc}
G & & & F & & & & & & -F & & & \\
\hline
 & a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a \\
 & a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} \\
E & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} \\
 & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a \\
 & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b \\
 & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b \\
\hline
 & a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} \\
 & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a \\
 & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} \\
-E & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a \\
 & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b \\
 & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b
\end{array}\right)
$$

$$
\left(\begin{array}{cccc|cccccc|cccccc}
a & b & a & b & a & b & a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} \\
a & b & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & \bar{b} & a & b \\
a & b & a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} & a & b & a & b \\
a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & a & b & a & b & a & b & \bar{a} & \bar{b} \\
\hline
b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a \\
b & \bar{a} & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} \\
a & b & \bar{a} & \bar{b} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} \\
a & b & \bar{a} & \bar{b} & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a \\
b & \bar{a} & \bar{b} & a & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b \\
b & \bar{a} & \bar{b} & a & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b \\
\hline
\bar{b} & a & \bar{b} & a & a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} \\
\bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a \\
\bar{a} & \bar{b} & a & b & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} \\
\bar{a} & \bar{b} & a & b & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a \\
\bar{b} & a & b & \bar{a} & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b \\
\bar{b} & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b
\end{array}\right)
$$

Theorem (Kharaghani et al., 2021)

If there is a full $OD(v, s_1, \ldots, s_u)$, then there is a full $OD(4v^2; 4vs_1, \ldots, 4vs_u)$ for which:

- there are two stable verticle splits, and
- there are two unstable horizontal splits.

Proof

- In the general case:

$$\alpha = (c_0, c_1, \ldots, c_{n-1}, c_{n-1}, \ldots, c_1)$$
$$\beta = (c_0, c_1, \ldots, c_{n-1}, -c_{n-1}, \ldots, -c_1).$$

- Tedious checking of products between block rows.

$$
\begin{pmatrix}
a & b & a & b & a & b & a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} \\
a & b & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & \bar{b} & a & b \\
a & b & a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} & a & b & a & b \\
a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & a & b & a & b & a & b & \bar{a} & \bar{b} \\
b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a \\
b & \bar{a} & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} \\
a & b & \bar{a} & \bar{b} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} \\
a & b & \bar{a} & b & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a \\
b & \bar{a} & \bar{b} & a & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b \\
b & \bar{a} & \bar{b} & a & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b \\
\bar{b} & a & \bar{b} & a & a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} \\
\bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a \\
\bar{a} & \bar{b} & a & b & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} \\
\bar{a} & \bar{b} & a & b & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a \\
\bar{b} & a & b & \bar{a} & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b \\
\bar{b} & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b
\end{pmatrix}
$$

$$
\begin{pmatrix}
a & b & a & b & a & b & a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} \\
a & b & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & \bar{b} & a & b \\
a & b & a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} & a & b & a & b \\
a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & a & b & a & b & a & b & \bar{a} & \bar{b} \\
b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a \\
b & \bar{a} & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} \\
a & b & \bar{a} & \bar{b} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} \\
a & b & \bar{a} & \bar{b} & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a \\
b & \bar{a} & \bar{b} & a & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b \\
b & \bar{a} & \bar{b} & a & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b \\
\bar{b} & a & \bar{b} & a & a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} \\
\bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a \\
\bar{a} & \bar{b} & a & b & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} \\
\bar{a} & \bar{b} & a & b & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a \\
\bar{b} & a & b & \bar{a} & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b \\
\bar{b} & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b
\end{pmatrix}
$$

- Resolves conjecture: Is there a balanced splittable Hadamard matrix of order $64n^2$ given a Hadamard matrix of order $4n$?
- Related configurations:
  - equiangular lines
  - Unbiased orthogonal designs
  - quasi-symmetric designs
  - quaternion ODs and frames

Unifying Construction of Weighing Matrices

Section Summary

3. Unifying Construction of Weighing Matrices
   - Classical Parameter BGWs
   - Normalization
   - Generalized Simplex Codes
   - Construction

Proposition: Classical Parameter BGWs

- Let q be a prime power and $d > 0$.
- BGWs are known to exist with the following parameters:
  - $$\left(v = \frac{q^{d+1}-1}{q-1}, k = q^d, \lambda = q^d - q^{d-1}; G = C_{q-1}\right)$$

    for every q and $d > 0$, and
  - $$\left(v = \frac{q^{d+1}-1}{q-1}, k = q^d, \lambda = q^d - q^{d-1}; G = C_{2q-2}\right)$$

    for even q and d.

- Every weighing matrix is equivalent to one of the following form

$$\begin{pmatrix} \mathbf{0} & R \\ \mathbf{1} & D \end{pmatrix}.$$

- R is the residual-part
- D is the derived-part

Definition: Simplex Code

- q a prime power and $d > 0$
- Form matrix G with columns given by reps. of 1-D subspaces of $GF(q^{d+1})$
- simplex code $\mathcal{S}_{q,d} = row(G)$

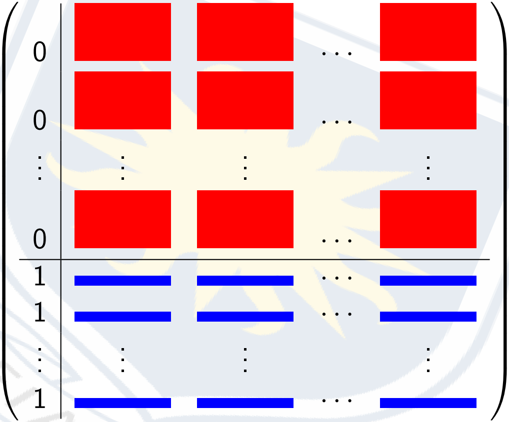
Proposition

For $\mathcal{S}_{q,d}$:

- $wt(x) = q^d$ for all $x \in \mathcal{S}_{q,d}$
- $dist(x, y) = q^d$ for all $x, y \in \mathcal{S}_{q,d}$ and $x \neq y$

- Ingredients of unifying construction:
  - a normalized $W(v, q)$ (seed matrix) with residual-part R and derived-part D
  - a $W((q^{d+1} - 1)/(q - 1), q^d)$, say W
  - $\mathcal{S}_{q,d}$

- Recipie of unifying construction:
  - Form $A = W \otimes R$.
  - Form B by replacing elements of $\mathcal{S}_{q,d}$ by rows of D.
  - Then
$$\begin{pmatrix} \mathbf{0} & A \\ \mathbf{1} & B \end{pmatrix}$$
is a $W((v-1)(q^{d+1}-1)/(q-1)+1, q^{d+1})$.


0
0
0
1
1
1

- A seed $W(8,5)$

$$\begin{pmatrix} \mathbf{0} & R \\ \mathbf{1} & D \end{pmatrix} = \left(\begin{array}{c|ccccccc} 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & - & - & 1 \\ 0 & 0 & 0 & 1 & 1 & - & 1 & - \\ \hline 1 & 0 & 0 & 0 & 1 & 1 & - & - \\ 1 & 1 & 1 & 1 & - & 0 & 0 & 0 \\ 1 & 1 & - & - & 0 & - & 0 & 0 \\ 1 & - & 1 & - & 0 & 0 & 1 & 0 \\ 1 & - & - & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

- A classical parameter $W(6,5)$

$$W = \begin{pmatrix} - & 1 & - & 0 & 1 & 1 \\ - & - & 1 & - & 0 & 1 \\ - & - & - & 1 & - & 0 \\ 0 & - & - & - & 1 & - \\ 1 & 0 & - & - & - & 1 \\ - & 1 & 0 & - & - & - \end{pmatrix}.$$

- The simplex code $\mathcal{S}_{5,1}$ (transposed)

$$\begin{pmatrix} 1&2&3&4&0&1&2&3&4&0&1&2&3&4&0&1&2&3&4&0&1&2&3&4&0 \\ 1&1&1&1&1&2&2&2&2&2&3&3&3&3&3&4&4&4&4&4&0&0&0&0&0 \\ 2&4&0&3&1&4&3&1&0&2&0&1&4&2&3&3&0&2&1&4&1&2&3&4&0 \\ 4&0&3&2&1&3&1&0&4&2&1&4&2&0&3&0&2&1&3&4&2&3&4&1&0 \\ 0&3&2&4&1&1&0&4&3&2&4&2&0&1&3&2&1&3&0&4&3&4&1&2&0 \\ 3&2&4&0&1&0&4&3&1&2&2&0&1&4&3&1&3&0&2&4&4&1&2&3&0 \end{pmatrix}$$

- Take $A = W \otimes R$.
- Take B to be the matrix formed after replacing the entries of $S_{5,1}$ by the rows of D.
- Then

$$\begin{pmatrix} \mathbf{0} & A \\ \mathbf{1} & B \end{pmatrix}$$

is a $W(43, 25)$.

$$
\left(\begin{array}{c|cccccccccccccccccccccccccccccccccccccccccc}
0 & - & 0 & 0 & - & - & - & - & 1 & 0 & 0 & 1 & 1 & 1 & 1 & - & 0 & 0 & - & - & - & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & - & 0 & - & 1 & 1 & - & 0 & 1 & 0 & 1 & - & - & 1 & 0 & - & 0 & - & 1 & 1 & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & - & - & 1 & 0 & 1 & 0 & 1 & - & - & 1 \\
0 & 0 & 0 & - & - & 1 & - & 1 & 0 & 0 & 1 & 1 & - & 1 & - & 0 & 0 & - & - & 1 & - & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & - & 1 & - & 0 & 0 & 1 & 1 & - & 1 & - \\
0 & - & 0 & 0 & - & - & - & - & - & 0 & 0 & - & - & - & - & 1 & 0 & 0 & 1 & 1 & 1 & 1 & - & 0 & 0 & - & - & - & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
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0 & 0 & 0 & - & - & 1 & - & 1 & 0 & 0 & - & - & 1 & - & 1 & 0 & 0 & 1 & 1 & - & 1 & - & 0 & 0 & - & - & 1 & - & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & - & 1 & - \\
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0 & 0 & 0 & - & - & 1 & - & 1 & 0 & 0 & - & - & 1 & - & 1 & 0 & 0 & - & - & 1 & - & 1 & 0 & 0 & 1 & 1 & - & 1 & - & 0 & 0 & - & - & 1 & - & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & - & 0 & 0 & - & - & - & - & - & 0 & 0 & - & - & - & - & - & 0 & 0 & - & - & - & - & 1 & 0 & 0 & 1 & 1 & 1 & 1 & - & 0 & 0 & - & - & - & - \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & - & 0 & - & 1 & 1 & - & 0 & - & 0 & - & 1 & 1 & - & 0 & - & 0 & - & 1 & 1 & - & 0 & 1 & 0 & 1 & - & - & 1 & 0 & - & 0 & - & 1 & 1 & - \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & - & - & 1 & - & 1 & 0 & 0 & - & - & 1 & - & 1 & 0 & 0 & - & - & 1 & - & 1 & 0 & 0 & 1 & 1 & - & 1 & - & 0 & 0 & - & - & 1 & - & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & - & 0 & 0 & - & - & - & - & - & 0 & 0 & - & - & - & - & - & 0 & 0 & - & - & - & - & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
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0 & 0 & 0 & - & - & 1 & - & 1 & 0 & 0 & 1 & 1 & - & 1 & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & - & - & 1 & - & 1 & 0 & 0 & - & - & 1 & - & 1 & 0 & 0 & - & - & 1 & - & 1 \\
\hline
1 & 1 & 1 & 1 & - & 0 & 0 & 0 & 1 & 1 & 1 & - & 0 & 0 & 0 & 1 & - & - & 0 & - & 0 & 0 & - & - & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & - & - & - & 1 & - & 0 & 0 & 1 & 0 \\
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1 & - & - & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & - & 0 & 0 & 0 & - & 1 & - & 0 & 0 & 1 & 0 & 1 & - & - & 0 & - & 0 & 0 & - & - & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & - & - \\
1 & 0 & 0 & 0 & 1 & 1 & - & - & 1 & 1 & 1 & - & 0 & 0 & 0 & 1 & 1 & 1 & - & 0 & 0 & 0 & 1 & 1 & 1 & - & 0 & 0 & 0 & 1 & 1 & 1 & - & 0 & 0 & 0 & 1 & 1 & 1 & - & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & - & 0 & 0 & 0 & 1 & - & - & 0 & - & 0 & 0 & - & - & 1 & 0 & 0 & 0 & 1 & - & 1 & - & 0 & 0 & 1 & 0 & 1 & 1 & 1 & - & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & - & - \\
1 & 1 & - & - & 0 & - & 0 & 0 & 1 & - & - & 0 & - & 0 & 0 & - & 1 & - & 0 & 0 & 1 & 0 & 1 & 1 & 1 & - & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & - & - & - & - & 1 & 0 & 0 & 0 & 1 \\
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1 & - & - & 1 & 0 & 0 & 0 & 1 & 1 & - & - & 0 & - & 0 & 0 & 0 & 0 & 0 & 1 & 1 & - & - & - & - & 1 & 0 & 0 & 0 & 1 & - & 1 & - & 0 & 0 & 1 & 0 & 1 & 1 & 1 & - & 0 & 0 & 0 \\
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1 & 1 & 1 & 1 & - & 0 & 0 & 0 & - & 1 & - & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & - & - & 1 & 1 & 1 & - & 0 & 0 & 0 & - & - & 1 & 0 & 0 & 0 & 1 & 1 & - & - & 0 & - & 0 & 0 \\
1 & 1 & - & - & 0 & - & 0 & 0 & - & 1 & - & 0 & 0 & 1 & 0 & 1 & 1 & 1 & - & 0 & 0 & 0 & - & - & 1 & 0 & 0 & 0 & 1 & 1 & - & - & 0 & - & 0 & 0 & 0 & 0 & 0 & 1 & 1 & - & - \\
1 & - & 1 & - & 0 & 0 & 1 & 0 & - & 1 & - & 0 & 0 & 1 & 0 & - & - & 1 & 0 & 0 & 0 & 1 & 1 & - & - & 0 & - & 0 & 0 & 0 & 0 & 0 & 1 & 1 & - & - & 1 & 1 & 1 & - & 0 & 0 & 0 \\
1 & - & - & 1 & 0 & 0 & 0 & 1 & - & 1 & - & 0 & 0 & 1 & 0 & 1 & - & - & 0 & - & 0 & 0 & 0 & 0 & 0 & 1 & 1 & - & - & 1 & 1 & 1 & - & 0 & 0 & 0 & - & - & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 1 & - & - & - & 1 & - & 0 & 0 & 1 & 0 & - & 1 & - & 0 & 0 & 1 & 0 & - & 1 & - & 0 & 0 & 1 & 0 & - & 1 & - & 0 & 0 & 1 & 0 & - & 1 & - & 0 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & - & 0 & 0 & 0 & - & - & 1 & 0 & 0 & 0 & 1 & - & 1 & - & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & - & - & 1 & - & - & 0 & - & 0 & 0 & 1 & 1 & 1 & - & 0 & 0 & 0 \\
1 & 1 & - & - & 0 & - & 0 & 0 & - & - & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & - & - & 1 & - & - & 0 & - & 0 & 0 & 1 & 1 & 1 & - & 0 & 0 & 0 & - & 1 & - & 0 & 0 & 1 & 0 \\
1 & - & 1 & - & 0 & 0 & 1 & 0 & - & - & 1 & 0 & 0 & 0 & 1 & 1 & - & - & 0 & - & 0 & 0 & 1 & 1 & 1 & - & 0 & 0 & 0 & - & 1 & - & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & - & - \\
1 & - & - & 1 & 0 & 0 & 0 & 1 & - & - & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & - & 0 & 0 & 0 & - & 1 & - & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & - & - & 1 & - & - & 0 & - & 0 & 0 \\
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1 & 1 & 1 & 1 & - & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & - & - & 1 & 1 & 1 & - & 0 & 0 & 0 & 1 & - & - & 0 & - & 0 & 0 & - & 1 & - & 0 & 0 & 1 & 0 & - & - & 1 & 0 & 0 & 0 & 1 \\
1 & 1 & - & - & 0 & - & 0 & 0 & 0 & 0 & 0 & 1 & 1 & - & - & 1 & - & - & 0 & - & 0 & 0 & - & 1 & - & 0 & 0 & 1 & 0 & - & - & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & - & 0 & 0 & 0 \\
1 & - & 1 & - & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & - & - & - & 1 & - & 0 & 0 & 1 & 0 & - & - & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & - & 0 & 0 & 0 & 1 & - & - & 0 & - & 0 & 0 \\
1 & - & - & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & - & - & - & - & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & - & 0 & 0 & 0 & 1 & - & - & 0 & - & 0 & 0 & - & 1 & - & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 & 1 & - & - & 0 & 0 & 0 & 1 & 1 & - & - & 0 & 0 & 0 & 1 & 1 & - & - & 0 & 0 & 0 & 1 & 1 & - & - & 0 & 0 & 0 & 1 & 1 & - & - & 0 & 0 & 0 & 1 & 1 & - & -
\end{array}\right)
$$

Theorem (Kharaghani et al., 2022a)

If there is a $W(v, q)$, then there is a weighing matrix with parameters

$$\left(\frac{(v-1)(q^{d+1}-1)}{q-1}+1, q^{d+1}\right)$$

whenever:

1. q is odd and every $d > 0$, and
2. q and d are both even.

Seed (v,k)	Succident (v',k')	Seed (v,k)	Succident (v',k')
$(6,5)$:	$(31,25)$, $(156,125)$, $(781,625)$	$(16,3)$:	$(69,9)$, $(196,27)$, $(601,81)$
$(8,5)$:	$(43,25)$, $(218,125)$	$(16,5)$:	$(91,25)$, $(466,125)$
$(8,7)$:	$(57,49)$, $(400,343)$	$(16,7)$:	$(121,49)$, $(856,343)$
$(10,5)$:	$(55,25)$, $(280,125)$	$(16,9)$:	$(151,81)$
$(10,9)$:	$(91,81)$, $(820,729)$	$(16,11)$:	$(181,121)$
$(12,5)$:	$(67,25)$, $(342,125)$	$(16,13)$:	$(211,169)$
$(12,7)$:	$(89,49)$, $(628,343)$	$(18,13)$:	$(239,169)$
$(12,9)$:	$(111,81)$	$(19,9)$:	$(181,81)$
$(13,9)$:	$(121,81)$	$(20,7)$:	$(153,49)$
$(14,9)$:	$(131,81)$	$(20,13)$:	$(267,169)$
$(14,13)$:	$(183,169)$		

A New Class of BWs

Section Summary

- Consider the following balanced $W(19,9)$:

$$W_{19} = \left(\begin{array}{c|cccccccccccccccccc}
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & - & - & 0 & 1 & 0 & 0 & 0 & 1 & - & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & - & 0 & - & 0 & 0 & 1 & 1 & 0 & 0 & 1 & - & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & - & - & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & - & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & - & - & 0 & 1 & 0 & 0 & 0 & 1 & - & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & - & 0 & - & 0 & 0 & 1 & 1 & 0 & 0 & 1 & - & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & - & - & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & - & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & - & - & 0 & 1 & 0 & 0 & 0 & 1 & - & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & - & 0 & - & 0 & 0 & 1 & 1 & 0 & 0 & 1 & - & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & - & - & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & - \\
\hline
1 & 0 & 0 & 0 & 1 & 0 & - & 1 & - & 0 & 0 & - & - & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & - & 1 & 0 & 0 & 1 & - & - & 0 & - & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & - & 1 & - & 0 & 1 & - & - & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & - & 0 & 0 & 0 & 0 & 1 & 0 & - & 0 & 0 & 1 & 0 & - & - & 0 & 1 & 0 \\
1 & 0 & 1 & - & 0 & 0 & 0 & - & 1 & 0 & 1 & 0 & 0 & - & 0 & - & 0 & 0 & 1 \\
1 & - & 0 & 1 & 0 & 0 & 0 & 0 & - & 1 & 0 & 1 & 0 & - & - & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & - & 1 & - & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & - & - \\
1 & - & 1 & 0 & 0 & 1 & - & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & - & 0 & - \\
1 & 0 & - & 1 & - & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & - & - & 0
\end{array}\right)$$

- Computationally found in de Launey and Sarvate (1984).

$$R_1 = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & - & - & 0 & 1 & 0 & 0 & 0 & 1 & - & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
- & 0 & - & 0 & 0 & 1 & 1 & 0 & 0 & 1 & - & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
- & - & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & - & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & - & - & 0 & 1 & 0 & 0 & 0 & 1 & - & 1 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & - & 0 & - & 0 & 0 & 1 & 1 & 0 & 0 & 1 & - & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & - & - & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & - & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & - & - & 0 & 1 & 0 & 0 & 0 & 1 & - & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & - & 0 & - & 0 & 0 & 1 & 1 & 0 & 0 & 1 & - & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & - & - & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & -
\end{pmatrix}$$

$$|R_1| = \begin{pmatrix} 1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0 \\ 0&1&1&0&1&0&0&0&1&1&1&1&0&1&0&0&0&1 \\ 1&0&1&0&0&1&1&0&0&1&1&1&0&0&1&1&0&0 \\ 1&1&0&1&0&0&0&1&0&1&1&1&1&0&0&0&1&0 \\ 0&0&1&0&1&1&0&1&0&0&0&1&1&1&1&0&1&0 \\ 1&0&0&1&0&1&0&0&1&1&0&0&1&1&1&0&0&1 \\ 0&1&0&1&1&0&1&0&0&0&1&0&1&1&1&1&0&0 \\ 0&1&0&0&0&1&0&1&1&0&1&0&0&0&1&1&1&1 \\ 0&0&1&1&0&0&1&0&1&0&0&1&1&0&0&1&1&1 \\ 1&0&0&0&1&0&1&1&0&1&0&0&0&1&0&1&1&1 \end{pmatrix}$$

$$|R_2| = J - |R_1| = \begin{pmatrix}
0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1\\
1&0&0&1&0&1&1&1&0&0&0&0&1&0&1&1&1&0\\
0&1&0&1&1&0&0&1&1&0&0&0&1&1&0&0&1&1\\
0&0&1&0&1&1&1&0&1&0&0&0&0&1&1&1&0&1\\
1&1&0&1&0&0&1&0&1&1&1&0&0&0&0&1&0&1\\
0&1&1&0&1&0&1&1&0&0&1&1&0&0&0&1&1&0\\
1&0&1&0&0&1&0&1&1&1&0&1&0&0&0&0&1&1\\
1&0&1&1&1&0&1&0&0&1&0&1&1&1&0&0&0&0\\
1&1&0&0&1&1&0&1&0&1&1&0&0&1&1&0&0&0\\
0&1&1&1&0&1&0&0&1&0&1&1&1&0&1&0&0&0
\end{pmatrix}$$

$$R_2 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
- & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & - & 1 & - & 0 \\
0 & - & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & - & 1 & 0 & 0 & 1 & - \\
0 & 0 & - & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & - & 1 & - & 0 & 1 \\
1 & 1 & 0 & - & 0 & 0 & 1 & 0 & 1 & 1 & - & 0 & 0 & 0 & 0 & 1 & 0 & - \\
0 & 1 & 1 & 0 & - & 0 & 1 & 1 & 0 & 0 & 1 & - & 0 & 0 & 0 & - & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & - & 0 & 1 & 1 & - & 0 & 1 & 0 & 0 & 0 & 0 & - & 1 \\
1 & 0 & 1 & 1 & 1 & 0 & - & 0 & 0 & 1 & 0 & - & 1 & - & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 & - & 0 & - & 1 & 0 & 0 & 1 & - & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & - & 0 & - & 1 & - & 0 & 1 & 0 & 0 & 0
\end{pmatrix}$$

$$
W'_{19} = \left(\begin{array}{c|cccccccccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & - & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & - & 1 & - & 0 \\
0 & 0 & - & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & - & 1 & 0 & 0 & 1 & - \\
0 & 0 & 0 & - & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & - & 1 & - & 0 & 1 \\
0 & 1 & 1 & 0 & - & 0 & 0 & 1 & 0 & 1 & 1 & - & 0 & 0 & 0 & 0 & 1 & 0 & - \\
0 & 0 & 1 & 1 & 0 & - & 0 & 1 & 1 & 0 & 0 & 1 & - & 0 & 0 & 0 & - & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & - & 0 & 1 & 1 & - & 0 & 1 & 0 & 0 & 0 & 0 & - & 1 \\
0 & 1 & 0 & 1 & 1 & 1 & 0 & - & 0 & 0 & 1 & 0 & - & 1 & - & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & - & 0 & - & 1 & 0 & 0 & 1 & - & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & - & 0 & - & 1 & - & 0 & 1 & 0 & 0 & 0 \\
\hline
1 & 0 & 0 & 0 & 1 & 0 & - & 1 & - & 0 & 0 & - & - & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & - & 1 & 0 & 0 & 1 & - & - & 0 & - & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & - & 1 & - & 0 & 1 & - & - & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & - & 0 & 0 & 0 & 0 & 1 & 0 & - & 0 & 0 & 1 & 0 & - & - & 0 & 1 & 0 \\
1 & 0 & 1 & - & 0 & 0 & 0 & - & 1 & 0 & 1 & 0 & 0 & - & 0 & - & 0 & 0 & 1 \\
1 & - & 0 & 1 & 0 & 0 & 0 & 0 & - & 1 & 0 & 1 & 0 & - & - & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & - & 1 & - & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & - & - \\
1 & - & 1 & 0 & 0 & 1 & - & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & - & 0 & - \\
1 & 0 & - & 1 & - & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & - & - & 0
\end{array}\right)
$$

- Let Ξ be a BGW with parameters

$$\left(v = \frac{9^{d+1} - 1}{8}, k = 9^d, \lambda = 9^d - 9^{d-1}\right)$$

over the group $C_4 = \{1, g, g^2, g^3\}$.

- Decompose Ξ as

$$\Xi = \Xi_1 + g\Xi_g + g^2\Xi_{g^2} + g^3\Xi_{g^3},$$

where the Ξ_is are disjoint $(0,1)$-matrices.

- Form:

$$\Xi \otimes R_1 = \Xi_1 \otimes R_1 - \Xi_g \otimes R_2 - \Xi_{g^2} \otimes R_1 + \Xi_{g^3} \otimes R_2$$

- Applying $R_1 \mapsto -R_2 \mapsto -R_1 \mapsto R_2 \mapsto R_1$.
- Form D by substituting for the elements of $S_{9,d}$ the rows of the derived part of W_{19}.

- The matrix

$$\begin{pmatrix} \mathbf{0} & \Xi \otimes R_1 \\ 1 & D \end{pmatrix}$$

is a balanced weighing matrix.

Theorem: (Kharaghani et al., 2022b)

For every $d > 0$, there is a balanced weighing matrix with parameters

$$\left(v = \frac{9(9^{d+1} - 1)}{4} + 1, k = 9^{d+1} \right).$$

Equivalence to Association Schemes

Section Summary

Definition (Commutative) Association Scheme

A (commutative) association scheme is a pair $\mathfrak{X} = (X, \mathscr{A})$ where:

- X is a point set and $\mathscr{A} = \{A_0, \ldots, A_d\}$ a collection of $(0,1)$-matrices indexed by elements of X.
- $A_0 = I$;
- $\sum_{i=0}^{d} A_i = J$;
- $\mathscr{A}$ is closed under transposition;
- there exist non-negative integers p_{ij}^k such that $A_i A_j = \sum_{k=0}^{d} p_{ij}^k A_k$; and
- $A_i A_j = A_j A_i$.

*** $\mathfrak{X}$ is symmetric if $A_i^t = A_i$, for every i.

- $\mathfrak{U} = \langle \mathscr{A} \rangle$ is the adjacency algebra of $\mathfrak{X}$.
- $\mathfrak{U}$ has a dual basis of idempotents $\{E_0 = |X|^{-1}J, \ldots, E_d\}$.
- There are change of basis matrices P and Q such that

$$A_j = \sum_{i=0}^{d} P_{ij} E_i, \qquad E_j = |X|^{-1} \sum_{i=0}^{d} Q_{ij} A_i$$

- Let W be a $BGW(v, k, \lambda; C_2)$.
- Decompose...

$$W = W_1 - W_2,$$

where W_1 and W_2 are disjoint $(0, 1)$-matrices.

- Properties...

- Properties...
  - $(W_1 - W_2)(W_1 - W_2)^t = (W_1 - W_2)^t(W_1 - W_2) = kI$

- Properties...
  - $(W_1 - W_2)(W_1 - W_2)^t = (W_1 - W_2)^t(W_1 - W_2) = kI$
  - $(W_1 + W_2)(W_1 + W_2)^t = (W_1 + W_2)^t(W_1 + W_2) = kI + \lambda(J - I)$

- Properties...
  - $(W_1 - W_2)(W_1 - W_2)^t = (W_1 - W_2)^t(W_1 - W_2) = kI$
  - $(W_1 + W_2)(W_1 + W_2)^t = (W_1 + W_2)^t(W_1 + W_2) = kI + \lambda(J - I)$
  - $W_1 W_1^t + W_2 W_2^t = W_1^t W_1 + W_2^t W_2 = kI + \frac{\lambda}{2}(J - I)$

- Properties...
  - $(W_1 - W_2)(W_1 - W_2)^t = (W_1 - W_2)^t(W_1 - W_2) = kI$
  - $(W_1 + W_2)(W_1 + W_2)^t = (W_1 + W_2)^t(W_1 + W_2) = kI + \lambda(J - I)$
  - $W_1 W_1^t + W_2 W_2^t = W_1^t W_1 + W_2^t W_2 = kI + \frac{\lambda}{2}(J - I)$
  - $W_1 W_2^t + W_2 W_1^t = W_1^t W_2 + W_2^t W_1 = \frac{\lambda}{2}(J - I)$

- With $P = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, form the $(0,1)$-matrices...

$$A_0 = I_2 \otimes I_2 \otimes I_v$$
$$A_1 = I_2 \otimes P \otimes I_v$$
$$A_2 = I_2 \otimes J_2 \otimes (J_v - I_v)$$
$$A_3 = \begin{pmatrix} O & I_2 \otimes W_1 + P \otimes W_2 \\ I_2 \otimes W_1^t + P \otimes W_2^t & O \end{pmatrix}$$
$$A_4 = \begin{pmatrix} O & I_2 \otimes W_2 + P \otimes W_1 \\ I_2 \otimes W_2^t + P \otimes W_1^t & O \end{pmatrix}$$
$$A_5 = \begin{pmatrix} O & J_2 \otimes (J_v - W_1 - W_2) \\ J_2 \otimes (J_v - W_1^t - W_2^t) & O \end{pmatrix}$$

- Character tables...

$$P = \begin{pmatrix} 1 & 1 & 2(v-1) & k & k & 2(v-k) \\ 1 & -1 & 0 & \sqrt{k} & -\sqrt{k} & 0 \\ 1 & -1 & 0 & -\sqrt{k} & \sqrt{k} & 0 \\ 1 & 1 & 2(v-1) & -k & -k & 2(k-v) \\ 1 & 1 & -2 & -\sqrt{\frac{k(v-k)}{v-1}} & -\sqrt{\frac{k(v-k)}{v-1}} & 2\sqrt{\frac{k(v-k)}{v-1}} \\ 1 & 1 & -2 & \sqrt{\frac{k(v-k)}{v-1}} & \sqrt{\frac{k(v-k)}{v-1}} & -2\sqrt{\frac{k(v-k)}{v-1}} \end{pmatrix}$$

$$Q = \begin{pmatrix} 1 & v & v & 1 & v-1 & v-1 \\ 1 & -v & -v & 1 & v-1 & v-1 \\ 1 & 0 & 0 & 1 & -1 & -1 \\ 1 & \frac{v}{\sqrt{k}} & -\frac{v}{\sqrt{k}} & -1 & -\sqrt{\frac{(v-1)(v-k)}{k}} & \sqrt{\frac{(v-1)(v-k)}{k}} \\ 1 & -\frac{v}{\sqrt{k}} & \frac{v}{\sqrt{k}} & -1 & -\sqrt{\frac{(v-1)(v-k)}{k}} & \sqrt{\frac{(v-1)(v-k)}{k}} \\ 1 & 0 & 0 & -1 & \sqrt{\frac{k(v-1)}{v-k}} & -\sqrt{\frac{k(v-1)}{v-k}} \end{pmatrix}$$

Theorem: Weighing Schemes (Kharaghani et al., 2022b)

There is a symmetric scheme with character tables P and Q given above if and only if there is a balanced $W(v,k)$.

Proof

- We have sufficiency.
- To show necessity, use eigenvalues to infer structure of corresponding graphs/adjacency matrices.

- $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ form a matrix representation of C_2.
- For a finite abelian group $G = \{g_0, \ldots, g_{n-1}\}$ and $h \in G$...

$$U_h = (\delta(g_i^{-1} h g_j))$$

$$A_{0,g} = I_2 \otimes U_g \otimes I_v, \text{ for } g \in G,$$
$$A_1 = I_2 \otimes J_n \otimes (J_v - I_V),$$
$$A_{2,g} = \begin{pmatrix} O & \sum_{h\in G}(U_h \otimes W_{gh}) \\ \sum_{h\in G}(U_h \otimes W^t_{g^{-1}h^{-1}}) & O \end{pmatrix}, \text{ for } g \in G, \text{ and}$$
$$A_2 = \begin{pmatrix} O & J_n \otimes (J_v - \sum_{h\in G} W_h) \\ J_n \otimes (J_v - \sum_{h\in G} W^t_h) & O \end{pmatrix}.$$

Future Work

- More relations to BGWs!
  - error-correcting codes
  - more general objects (quasi-balanced weighing matrices)
- Can the theory of association schemes be used to say anything about BGWs?

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