Weighing Matrices

Generalizations and Related Configurations

Thomas Pender
***Joint work with Hadi Kharaghani, Sho Suda

Department of Mathematics and Computer Science
University of Lethbridge

June 22, 2022

University of Lethbridge

FIAT LUX

Summary

Preliminaries

Section Summary

- $J_{n\times m}$ the $n \times m$ matrix of all 1s.
- I_n the identity matrix of order n.

Definition: Balanced Incomplete Block Design

- A binary $v \times b$ $(0,1)$-matrix A such that:
  1. $AA^t = rI_v + \lambda(J_v - I_v)$, and
  2. $J_v A = kJ_v$.

  Write $BIBD(v, b, r, k, \lambda)$.
- The design is symmetric if $v = b$ (equiv. $k = r$).

- A $BIBD(13, 9, 6)$

$$A = \begin{pmatrix} 0&0&1&0&1&1&1&1&1&0&1&1&1 \\ 1&0&0&1&0&1&1&1&1&1&0&1&1 \\ 1&1&0&0&1&0&1&1&1&1&1&0&1 \\ 1&1&1&0&0&1&0&1&1&1&1&1&0 \\ 0&1&1&1&0&0&1&0&1&1&1&1&1 \\ 1&0&1&1&1&0&0&1&0&1&1&1&1 \\ 1&1&0&1&1&1&0&0&1&0&1&1&1 \\ 1&1&1&0&1&1&1&0&0&1&0&1&1 \\ 1&1&1&1&0&1&1&1&0&0&1&0&1 \\ 1&1&1&1&1&0&1&1&1&0&0&1&0 \\ 0&1&1&1&1&1&0&1&1&1&0&0&1 \\ 1&0&1&1&1&1&1&0&1&1&1&0&0 \\ 0&1&0&1&1&1&1&1&0&1&1&1&0 \end{pmatrix}.$$

$$AA^t = \begin{pmatrix} 9&6&6&6&6&6&6&6&6&6&6&6&6 \\ 6&9&6&6&6&6&6&6&6&6&6&6&6 \\ 6&6&9&6&6&6&6&6&6&6&6&6&6 \\ 6&6&6&9&6&6&6&6&6&6&6&6&6 \\ 6&6&6&6&9&6&6&6&6&6&6&6&6 \\ 6&6&6&6&6&9&6&6&6&6&6&6&6 \\ 6&6&6&6&6&6&9&6&6&6&6&6&6 \\ 6&6&6&6&6&6&6&9&6&6&6&6&6 \\ 6&6&6&6&6&6&6&6&9&6&6&6&6 \\ 6&6&6&6&6&6&6&6&6&9&6&6&6 \\ 6&6&6&6&6&6&6&6&6&6&9&6&6 \\ 6&6&6&6&6&6&6&6&6&6&6&9&6 \\ 6&6&6&6&6&6&6&6&6&6&6&6&9 \end{pmatrix}$$

Definition: Weighing Matrix

A $v \times v$ $(-1,0,1)$-matrix W such that

$$WW^t = kI_v.$$

Write $W(v,k)$.

- $W(v,v)$ is a *Hadamard matrix*
- $W(v,v-1)$ is a *conference matrix*

- A $W(13,9)$

$$W = \begin{pmatrix}
0 & 0 & - & 0 & - & 1 & - & - & 1 & 0 & - & - & - \\
1 & 0 & 0 & - & 0 & - & 1 & - & - & 1 & 0 & - & - \\
1 & 1 & 0 & 0 & - & 0 & - & 1 & - & - & 1 & 0 & - \\
1 & 1 & 1 & 0 & 0 & - & 0 & - & 1 & - & - & 1 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & - & 0 & - & 1 & - & - & 1 \\
- & 0 & 1 & 1 & 1 & 0 & 0 & - & 0 & - & 1 & - & - \\
1 & - & 0 & 1 & 1 & 1 & 0 & 0 & - & 0 & - & 1 & - \\
1 & 1 & - & 0 & 1 & 1 & 1 & 0 & 0 & - & 0 & - & 1 \\
- & 1 & 1 & - & 0 & 1 & 1 & 1 & 0 & 0 & - & 0 & - \\
1 & - & 1 & 1 & - & 0 & 1 & 1 & 1 & 0 & 0 & - & 0 \\
0 & 1 & - & 1 & 1 & - & 0 & 1 & 1 & 1 & 0 & 0 & - \\
1 & 0 & 1 & - & 1 & 1 & - & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & - & 1 & 1 & - & 0 & 1 & 1 & 1 & 0
\end{pmatrix}$$

$$WW^t = \begin{pmatrix} 9&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&9&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&9&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&9&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&9&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&9&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&9&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&9&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&9&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&9&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&9&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&9&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&9 \end{pmatrix}$$

Definition: Generalized Weighing Matrix

- Abelian group G not containg 0.
- A $v \times v$ $(0, G)$-matrix W such that

$$WW^* = (k \cdot 1_G)I_v \pmod{\mathbb{Z}G}.$$

Definition: Balanced Generalized Weighing Matrix

- Abelian group G not containing 0.
- A $v \times v$ $(0, G)$-matrix W such that

$$WW^* = (k \cdot 1_G)I_v + \frac{\lambda}{|G|}G(J_v - I_v).$$

- Write $BGW(v, k, \lambda; G)$.

- A $BGW(13, 9, 6; C_2)$

$$W = \begin{pmatrix}
0 & 0 & - & 0 & - & 1 & - & - & 1 & 0 & - & - & - \\
1 & 0 & 0 & - & 0 & - & 1 & - & - & 1 & 0 & - & - \\
1 & 1 & 0 & 0 & - & 0 & - & 1 & - & - & 1 & 0 & - \\
1 & 1 & 1 & 0 & 0 & - & 0 & - & 1 & - & - & 1 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & - & 0 & - & 1 & - & - & 1 \\
- & 0 & 1 & 1 & 1 & 0 & 0 & - & 0 & - & 1 & - & - \\
1 & - & 0 & 1 & 1 & 1 & 0 & 0 & - & 0 & - & 1 & - \\
1 & 1 & - & 0 & 1 & 1 & 1 & 0 & 0 & - & 0 & - & 1 \\
- & 1 & 1 & - & 0 & 1 & 1 & 1 & 0 & 0 & - & 0 & - \\
1 & - & 1 & 1 & - & 0 & 1 & 1 & 1 & 0 & 0 & - & 0 \\
0 & 1 & - & 1 & 1 & - & 0 & 1 & 1 & 1 & 0 & 0 & - \\
1 & 0 & 1 & - & 1 & 1 & - & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & - & 1 & 1 & - & 0 & 1 & 1 & 1 & 0
\end{pmatrix}$$

- Necessary conditions for BGWs:
  - weighing matrix
  - symmetric BIBD
- Sufficient in the case that $|G| =$ prime (Lam and Leung, 2000)

Definition: Orthogonal Design

- Real commuting variables $\{z_1, \ldots, z_u\}$.
- A $v \times v$ $(0, \pm z_1, \ldots, \pm z_u)$-matrix X such that

$$XX^t = \sigma I_v,$$

  where $\sigma = \sum_i s_i z_i^2$.
- Write $OD(v; s_1, \ldots, s_u)$.

- An $OD(4;1,1,1,1)$

$$X=\begin{pmatrix} a & b & c & d \\ \bar{b} & a & \bar{d} & c \\ \bar{c} & d & a & \bar{b} \\ \bar{d} & \bar{c} & b & a \end{pmatrix}.$$

$$XX^t = (a^2 + b^2 + c^2 + d^2)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Balancedly Splittable ODs

Section Summary

- Recall a Hadamard matrix is a $W(v, v)$.

Definition: Balancedly Splittable Hadamard Matrix (Kharaghani and Suda, 2019)

- A $W(v, v)$, say H.
- An $v \times \ell$ submatrix H_1 such that

$$H_1 H_1^t = \ell I_v + aA + b(J_v - I_v - A),$$

where A is a symmetric $(0, 1)$-matrix with zero diagonal.
- H is (v, ℓ, a, b) balancedly splittable.

- A Hadamard matrix of order 16

$$\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & - \\
1 & 1 & 1 & 1 & - & - & 1 & 1 & - & - & 1 & 1 & - & - & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & - & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & - & - & - & - & 1 & 1 & 1 & 1 & 1 & 1 & - & - \\
1 & - & 1 & - & 1 & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & - & 1 \\
1 & - & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - \\
1 & 1 & - & - & 1 & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - \\
1 & 1 & - & - & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & 1 \\
1 & - & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 \\
1 & - & - & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 \\
- & 1 & - & 1 & 1 & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - \\
- & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 \\
- & - & 1 & 1 & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - \\
- & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 \\
- & 1 & 1 & - & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & 1 \\
- & 1 & 1 & - & - & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & 1
\end{pmatrix}.$$

- A Hadamard matrix of order 16

$$
\left(\begin{array}{cccc|cccccc|cccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & - \\
1 & 1 & 1 & 1 & - & - & 1 & 1 & - & - & 1 & 1 & - & - & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & - & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & - & - & - & - & 1 & 1 & 1 & 1 & 1 & 1 & - & - \\
1 & - & 1 & - & 1 & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & - & 1 \\
1 & - & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - \\
1 & 1 & - & - & 1 & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - \\
1 & 1 & - & - & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & 1 \\
1 & - & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 \\
1 & - & - & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 \\
- & 1 & - & 1 & 1 & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - \\
- & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 \\
- & - & 1 & 1 & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - \\
- & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 \\
- & 1 & 1 & - & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & 1 \\
- & 1 & 1 & - & - & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & 1
\end{array}\right).
$$

- A balanced split.

Hadamard Matrices

$$\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
- & - & 1 & 1 & - & - \\
1 & 1 & - & - & - & - \\
- & - & - & - & 1 & 1 \\
1 & 1 & 1 & - & 1 & - \\
1 & 1 & - & 1 & - & 1 \\
1 & - & 1 & 1 & 1 & - \\
- & 1 & 1 & 1 & - & 1 \\
1 & - & 1 & - & 1 & 1 \\
- & 1 & - & 1 & 1 & 1 \\
1 & 1 & 1 & - & - & 1 \\
1 & 1 & - & 1 & 1 & - \\
- & 1 & 1 & 1 & 1 & - \\
1 & - & 1 & 1 & - & 1 \\
1 & - & - & 1 & 1 & 1 \\
- & 1 & 1 & - & 1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & - & 1 & - & 1 & 1 & 1 & - & 1 & - & 1 & 1 & - & 1 & 1 & - \\
1 & - & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & 1 & - & - & 1 \\
1 & 1 & - & - & 1 & - & 1 & 1 & 1 & - & 1 & - & 1 & 1 & - & 1 \\
1 & 1 & - & - & - & 1 & 1 & 1 & - & 1 & - & 1 & 1 & 1 & 1 & - \\
1 & - & - & 1 & 1 & - & 1 & - & 1 & 1 & - & 1 & 1 & - & 1 & 1 \\
1 & - & - & 1 & - & 1 & - & 1 & 1 & 1 & 1 & - & - & 1 & 1 & 1
\end{pmatrix}$$

Hadamard Matrices

$$
\begin{pmatrix}
6 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
\bar{2} & 6 & \bar{2} & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & \bar{2} \\
\bar{2} & \bar{2} & 6 & \bar{2} & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & \bar{2} \\
\bar{2} & \bar{2} & \bar{2} & 6 & \bar{2} & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & \bar{2} & 2 & 2 \\
2 & \bar{2} & 2 & \bar{2} & 6 & \bar{2} & 2 & \bar{2} & 2 & \bar{2} & 2 & 2 & 2 & \bar{2} & \bar{2} & 2 \\
2 & \bar{2} & 2 & \bar{2} & \bar{2} & 6 & \bar{2} & 2 & \bar{2} & 2 & 2 & 2 & \bar{2} & 2 & 2 & \bar{2} \\
2 & 2 & \bar{2} & \bar{2} & 2 & \bar{2} & 6 & \bar{2} & 2 & \bar{2} & \bar{2} & 2 & 2 & 2 & 2 & \bar{2} \\
2 & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & \bar{2} & 6 & \bar{2} & 2 & 2 & \bar{2} & 2 & 2 & \bar{2} & 2 \\
2 & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & 2 & \bar{2} & 6 & \bar{2} & 2 & \bar{2} & \bar{2} & 2 & 2 & 2 \\
2 & \bar{2} & \bar{2} & 2 & \bar{2} & 2 & \bar{2} & 2 & \bar{2} & 6 & \bar{2} & 2 & 2 & \bar{2} & 2 & 2 \\
2 & \bar{2} & 2 & \bar{2} & 2 & 2 & \bar{2} & 2 & 2 & \bar{2} & 6 & \bar{2} & \bar{2} & 2 & \bar{2} & 2 \\
2 & \bar{2} & 2 & \bar{2} & 2 & 2 & 2 & \bar{2} & \bar{2} & 2 & \bar{2} & 6 & 2 & \bar{2} & 2 & \bar{2} \\
2 & 2 & \bar{2} & \bar{2} & 2 & \bar{2} & 2 & 2 & \bar{2} & 2 & \bar{2} & 2 & 6 & \bar{2} & \bar{2} & 2 \\
2 & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & 2 & 2 & \bar{2} & 2 & \bar{2} & \bar{2} & 6 & 2 & \bar{2} \\
2 & \bar{2} & \bar{2} & 2 & \bar{2} & 2 & 2 & \bar{2} & 2 & 2 & \bar{2} & 2 & \bar{2} & 2 & 6 & \bar{2} \\
2 & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & \bar{2} & 2 & 2 & 2 & 2 & \bar{2} & 2 & \bar{2} & \bar{2} & 6
\end{pmatrix}
$$

$$\left(\begin{array}{cccc|cccccc|cccccc}
1&1&1&1&1&1&1&1&1&1&-&-&-&-&-&-\\
1&1&1&1&-&-&1&1&-&-&1&1&-&-&1&1\\
1&1&1&1&1&1&-&-&-&-&-&-&1&1&1&1\\
1&1&1&1&-&-&-&-&1&1&1&1&1&1&-&-\\
1&-&1&-&1&1&1&-&1&-&1&1&1&-&-&1\\
1&-&1&-&1&1&-&1&-&1&1&1&-&1&1&-\\
1&1&-&-&1&-&1&1&1&-&-&1&1&1&1&-\\
1&1&-&-&-&1&1&1&-&1&1&-&1&1&-&1\\
1&-&-&1&1&-&1&-&1&1&1&-&-&1&1&1\\
1&-&-&1&-&1&-&1&1&1&-&1&1&-&1&1\\
-&1&-&1&1&1&1&-&-&1&1&1&1&-&1&-\\
-&1&-&1&1&1&-&1&1&-&1&1&-&1&-&1\\
-&-&1&1&-&1&1&1&1&-&1&-&1&1&1&-\\
-&-&1&1&1&-&1&1&-&1&-&1&1&1&-&1\\
-&1&1&-&1&-&-&1&1&1&1&-&1&-&1&1\\
-&1&1&-&-&1&1&-&1&1&-&1&-&1&1&1
\end{array}\right)$$

$$
\left(\begin{array}{cccccccccc|cccccc}
1&1&1&1&1&1&1&1&1&1&-&-&-&-&-&-\\
1&1&1&1&-&-&1&1&-&-&1&1&-&-&1&1\\
1&1&1&1&1&1&-&-&-&-&-&-&1&1&1&1\\
1&1&1&1&-&-&-&-&1&1&1&1&1&1&-&-\\
1&-&1&-&1&1&1&-&1&-&1&1&1&-&-&1\\
1&-&1&-&1&1&-&1&-&1&1&1&-&1&1&-\\
1&1&-&-&1&-&1&1&1&-&-&1&1&1&1&-\\
1&1&-&-&-&1&1&1&-&1&1&-&1&1&-&1\\
1&-&-&1&1&-&1&-&1&1&1&-&-&1&1&1\\
1&-&-&1&-&1&-&1&1&1&-&1&1&-&1&1\\
-&1&-&1&1&1&1&-&-&1&1&1&1&-&1&-\\
-&1&-&1&1&1&-&1&1&-&1&1&-&1&-&1\\
-&-&1&1&-&1&1&1&1&-&1&-&1&1&1&-\\
-&-&1&1&1&-&1&1&-&1&-&1&1&1&-&1\\
-&1&1&-&1&-&-&1&1&1&1&-&1&-&1&1\\
-&1&1&-&-&1&1&-&1&1&-&1&-&1&1&1
\end{array}\right)
$$

$$\left(\begin{array}{cccccccccccccccc}
1&1&1&1&1&1&1&1&1&1&-&-&-&-&-&-\\
1&1&1&1&-&-&1&1&-&-&1&1&-&-&1&1\\
1&1&1&1&1&1&-&-&-&-&-&-&1&1&1&1\\
1&1&1&1&-&-&-&-&1&1&1&1&1&1&-&-\\
\hline
1&-&1&-&1&1&1&-&1&-&1&1&1&-&-&1\\
1&-&1&-&1&1&-&1&-&1&1&1&-&1&1&-\\
1&1&-&-&1&-&1&1&1&-&-&1&1&1&1&-\\
1&1&-&-&-&1&1&1&-&1&1&-&1&1&-&1\\
1&-&-&1&1&-&1&-&1&1&1&-&-&1&1&1\\
1&-&-&1&-&1&-&1&1&1&-&1&1&-&1&1\\
\hline
-&1&-&1&1&1&1&-&-&1&1&1&1&-&1&-\\
-&1&-&1&1&1&-&1&1&-&1&1&-&1&-&1\\
-&-&1&1&-&1&1&1&1&-&1&-&1&1&1&-\\
-&-&1&1&1&-&1&1&-&1&-&1&1&1&-&1\\
-&1&1&-&1&-&-&1&1&1&1&-&1&-&1&1\\
-&1&1&-&-&1&1&-&1&1&-&1&-&1&1&1
\end{array}\right)$$

$$\left(\begin{array}{cccccccccccccccc}
1&1&1&1&1&1&1&1&1&1&-&-&-&-&-&-\\
1&1&1&1&-&-&1&1&-&-&1&1&-&-&1&1\\
1&1&1&1&1&1&-&-&-&-&-&-&1&1&1&1\\
1&1&1&1&-&-&-&-&1&1&1&1&1&1&-&-\\
1&-&1&-&1&1&1&-&1&-&1&1&1&-&-&1\\
1&-&1&-&1&1&-&1&-&1&1&1&-&1&1&-\\
1&1&-&-&1&-&1&1&1&-&-&1&1&1&1&-\\
1&1&-&-&-&1&1&1&-&1&1&-&1&1&-&1\\
1&-&-&1&1&-&1&-&1&1&1&-&-&1&1&1\\
1&-&-&1&-&1&-&1&1&1&-&1&1&-&1&1\\
\hline
-&1&-&1&1&1&1&-&-&1&1&1&1&-&1&-\\
-&1&-&1&1&1&-&1&1&-&1&1&-&1&-&1\\
-&-&1&1&-&1&1&1&1&-&1&-&1&1&1&-\\
-&-&1&1&1&-&1&1&-&1&-&1&1&1&-&1\\
-&1&1&-&1&-&-&1&1&1&1&-&1&-&1&1\\
-&1&1&-&-&1&1&-&1&1&-&1&-&1&1&1
\end{array}\right)$$

- An $OD(16;8,8)$

$$
\begin{pmatrix}
a & b & a & b & a & b & a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} \\
a & b & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & \bar{b} & a & b \\
a & b & a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} & a & b & a & b \\
a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & a & b & a & b & a & b & \bar{a} & \bar{b} \\
b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a \\
b & \bar{a} & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} \\
a & b & \bar{a} & \bar{b} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} \\
a & b & \bar{a} & \bar{b} & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a \\
b & \bar{a} & \bar{b} & a & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b \\
b & \bar{a} & \bar{b} & a & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b \\
\bar{b} & a & \bar{b} & a & a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} \\
\bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a \\
\bar{a} & \bar{b} & a & b & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} \\
\bar{a} & \bar{b} & a & b & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a \\
\bar{b} & a & b & \bar{a} & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b \\
\bar{b} & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b
\end{pmatrix}
$$

$$
\left(\begin{array}{cccc|cccccc|cccccc}
a & b & a & b & a & b & a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} \\
a & b & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & \bar{b} & a & b \\
a & b & a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} & a & b & a & b \\
a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & a & b & a & b & a & b & \bar{a} & \bar{b} \\
b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a \\
b & \bar{a} & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} \\
a & b & \bar{a} & \bar{b} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} \\
a & b & \bar{a} & \bar{b} & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a \\
b & \bar{a} & \bar{b} & a & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b \\
b & \bar{a} & \bar{b} & a & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b \\
\bar{b} & a & \bar{b} & a & a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} \\
\bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a \\
\bar{a} & \bar{b} & a & b & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} \\
\bar{a} & \bar{b} & a & b & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a \\
\bar{b} & a & b & \bar{a} & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b \\
\bar{b} & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b
\end{array}\right)
$$

$$
(a^2+b^2)\begin{pmatrix}
3 & - & - & - & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\\
- & 3 & - & - & - & - & 1 & 1 & - & - & - & - & 1 & 1 & - & -\\
- & - & 3 & - & 1 & 1 & - & - & - & - & 1 & 1 & - & - & - & -\\
- & - & - & 3 & - & - & - & - & 1 & 1 & - & - & - & - & 1 & 1\\
1 & - & 1 & - & 3 & - & 1 & - & 1 & - & 1 & 1 & 1 & - & - & 1\\
1 & - & 1 & - & - & 3 & - & 1 & - & 1 & 1 & 1 & - & 1 & 1 & -\\
1 & 1 & - & - & 1 & - & 3 & - & 1 & - & - & 1 & 1 & 1 & 1 & -\\
1 & 1 & - & - & - & 1 & - & 3 & - & 1 & 1 & - & 1 & 1 & - & 1\\
1 & - & - & 1 & 1 & - & 1 & - & 3 & - & 1 & - & - & 1 & 1 & 1\\
1 & - & - & 1 & - & 1 & - & 1 & - & 3 & - & 1 & 1 & - & 1 & 1\\
1 & - & 1 & - & 1 & 1 & - & 1 & 1 & - & 3 & - & - & 1 & - & 1\\
1 & - & 1 & - & 1 & 1 & 1 & - & - & 1 & - & 3 & 1 & - & 1 & -\\
1 & 1 & - & - & 1 & - & 1 & 1 & - & 1 & - & 1 & 3 & - & - & 1\\
1 & 1 & - & - & - & 1 & 1 & 1 & 1 & - & 1 & - & - & 3 & 1 & -\\
1 & - & - & 1 & - & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 & 3 & -\\
1 & - & - & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - & - & 3
\end{pmatrix}
$$

$$\left(\begin{array}{cccccccccccccccc}
a & b & a & b & a & b & a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} \\
a & b & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & \bar{b} & a & b \\
a & b & a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} & a & b & a & b \\
a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & a & b & a & b & a & b & \bar{a} & \bar{b} \\
\hline
b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a \\
b & \bar{a} & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} \\
a & b & \bar{a} & \bar{b} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} \\
a & b & \bar{a} & \bar{b} & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a \\
b & \bar{a} & \bar{b} & a & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b \\
b & \bar{a} & \bar{b} & a & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b \\
\hline
\bar{b} & a & \bar{b} & a & a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} \\
\bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a \\
\bar{a} & \bar{b} & a & b & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} \\
\bar{a} & \bar{b} & a & b & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a \\
\bar{b} & a & b & \bar{a} & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b \\
\bar{b} & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b
\end{array}\right)$$

- Products in the set $\{\pm 2a^2, \pm 2b^2, \pm 2ab\}$.

Definition: Balancedly Splittable (Kharaghani et al., 2021)

- An OD X with $v \times \ell$ submatrix X_1 admits a stable (v, ℓ, a, b) split if
$$X_1 X_1^t = \sigma(cI_v + aA + b(J_v - I_v - A)).$$
- The split is unstable if it is not stable and
$$X_1 X_1^t = \ell I_v + aA + b(J_v - I_v - A)$$
after replacing each variable with 1.

Definition: Auxiliary Matrices

- X a full $OD(v; s_1, \ldots, s_u)$ and H the Hadamard matrix obtained by replacing each variable with 1.
- Label rows of X and H by $x_0, \ldots, x_{v-1}$, and $h_0, \ldots, h_{v-1}$.
- The auxiliary matrices of X are given by

$$c_i = h_i^t x_i, \qquad i = 0, \ldots, v-1.$$

- An $OD(2;1,1)$:

$$\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$$

- Auxiliary matrices:

$$c_0 = \begin{pmatrix} a & b \\ a & b \end{pmatrix}, \qquad c_1 = \begin{pmatrix} b & -a \\ -b & a \end{pmatrix}$$

- Form the sequences

$$\alpha = (c_0, c_1, c_1), \qquad \beta = (c_0, c_1, -c_1).$$

- The sum of the periodic autocorrelations of α and β satisfy

$$\sum_i \alpha_i \alpha_{i+j}^t + \sum_i \beta_i \beta_{i+j}^t = O, \qquad j = 1, 2.$$

- Sums of cross-correlations

$$\sum_i \alpha_i \beta_{i+j}^t + \sum_i \beta_i \alpha_{i+j}^t = O, \qquad j = 1, 2.$$

- Form the matrices

$$A = circ(\alpha), \qquad B = circ(\beta),$$

- Then the matrix

$$\Theta = \begin{pmatrix} A & B \\ B & A \end{pmatrix}.$$

$$
\Theta = \left(\begin{array}{cccccc|cccccc}
a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a \\
a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} \\
b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} \\
\bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a \\
b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b \\
\bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b \\
\hline
a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} \\
a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a \\
\bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} \\
b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a \\
b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b \\
\bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b
\end{array}\right)
$$

$$\Theta\Theta^t = \begin{pmatrix} AA^t + BB^t & AB^t + BA^t \\ AB^t + BA^t & AA^t + BB^t \end{pmatrix}.$$

$$\Theta\Theta^t = (a^2+b^2)\left(\begin{array}{cccccc|cccccc}
6 & \bar{2} & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 \\
\bar{2} & 6 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 6 & \bar{2} & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 \\
0 & 0 & \bar{2} & 6 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 6 & \bar{2} & 0 & 0 & 0 & 0 & 2 & 2 \\
0 & 0 & 0 & 0 & \bar{2} & 6 & 0 & 0 & 0 & 0 & 2 & 2 \\
\hline
2 & 2 & 0 & 0 & 0 & 0 & 6 & \bar{2} & 0 & 0 & 0 & 0 \\
2 & 2 & 0 & 0 & 0 & 0 & \bar{2} & 6 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 6 & \bar{2} & 0 & 0 \\
0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & \bar{2} & 6 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 6 & \bar{2} \\
0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & \bar{2} & 6
\end{array}\right)$$

$$
\left(\begin{array}{cccc|cccccc|cccccc}
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
\hline
* & * & * & * & a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a \\
* & * & * & * & a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} \\
* & * & * & * & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} \\
* & * & * & * & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a \\
* & * & * & * & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b \\
* & * & * & * & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b \\
\hline
* & * & * & * & a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} \\
* & * & * & * & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a \\
* & * & * & * & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} \\
* & * & * & * & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a \\
* & * & * & * & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b \\
* & * & * & * & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b
\end{array}\right)
$$

- Define:

$$\begin{aligned} Y &= \left(\begin{smallmatrix} 1 & 1 \\ 1 & - \end{smallmatrix}\right) \otimes X, \\ K &= \left(\begin{smallmatrix} 1 & 1 \\ 1 & - \end{smallmatrix}\right) \otimes H. \end{aligned}$$

- Index rows as $y_0, \ldots, y_{2n-1}$, and $k_0, \ldots, k_{2n-1}$.

$$Y = \left(\begin{smallmatrix} 1 & 1 \\ 1 & - \end{smallmatrix}\right) \otimes X \qquad K = \left(\begin{smallmatrix} 1 & 1 \\ 1 & - \end{smallmatrix}\right) \otimes H$$

Rows: $y_0, \ldots, y_{2n-1}$ Rows: $k_0, \ldots, k_{2n-1}$

- Form

$$F = \begin{pmatrix} k_1^t x_0 & k_2^t x_0 & k_3^t x_0 \end{pmatrix},$$

$$E = \begin{pmatrix} h_0^t y_1 \\ h_0^t y_2 \\ h_0^t y_3 \end{pmatrix},$$

$$G = k_0^t y_0.$$

$$\left(\begin{array}{c|cccccc|cccccc}
G & \multicolumn{6}{c|}{F} & \multicolumn{6}{c}{-F} \\ \hline
 & a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a \\
 & a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} \\
E & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} \\
 & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a \\
 & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b \\
 & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b \\ \hline
 & a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} \\
 & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a \\
-E & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} \\
 & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a \\
 & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b \\
 & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b
\end{array}\right)$$

$$
\left(\begin{array}{cccc|cccccc|cccccc}
a & b & a & b & a & b & a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} \\
a & b & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & \bar{b} & a & b \\
a & b & a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} & a & b & a & b \\
a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & a & b & a & b & a & b & \bar{a} & \bar{b} \\
\hline
b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a \\
b & \bar{a} & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} \\
a & b & \bar{a} & \bar{b} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} \\
a & b & \bar{a} & \bar{b} & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a \\
b & \bar{a} & \bar{b} & a & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b \\
b & \bar{a} & \bar{b} & a & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b \\
\hline
\bar{b} & a & \bar{b} & a & a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} \\
\bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a \\
\bar{a} & \bar{b} & a & b & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} \\
\bar{a} & \bar{b} & a & b & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a \\
\bar{b} & a & b & \bar{a} & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b \\
\bar{b} & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b
\end{array}\right)
$$

Theorem (Kharaghani et al., 2021)

Whenever there is a full $OD(v, s_1, \ldots, s_u)$, there is a full $OD(4v^2; 4vs_1, \ldots, 4vs_u)$ admitting two stable vertical splits and two unstable horizontal splits.

- The following is a $OD(2;1,1)$ of quaternions.

$$\begin{pmatrix} \bar{a} & bi \\ \bar{b}j & ak \end{pmatrix},$$

- where a and b are real variables.

$$
\left(\begin{array}{cccc|cccccc|cccccc}
\bar{a} & bi & \bar{a} & bi & \bar{a} & bi & \bar{a} & bi & \bar{a} & bi & a & \bar{b}i & a & \bar{b}i & a & \bar{b}i \\
\bar{a} & bi & \bar{a} & bi & a & \bar{b}i & \bar{a} & bi & a & \bar{b}i & \bar{a} & bi & a & \bar{b}i & \bar{a} & bi \\
\bar{a} & bi & \bar{a} & bi & \bar{a} & bi & a & \bar{b}i & a & \bar{b}i & a & \bar{b}i & \bar{a} & bi & \bar{a} & bi \\
\bar{a} & bi & \bar{a} & bi & a & \bar{b}i & a & \bar{b}i & \bar{a} & bi & \bar{a} & bi & \bar{a} & bi & a & \bar{b}i \\
\hline
\bar{b}j & ak & \bar{b}j & ak & \bar{a} & bi & \bar{b}j & ak & \bar{b}j & ak & \bar{a} & bi & \bar{b}j & ak & bj & \bar{a}k \\
\bar{b}j & ak & \bar{b}j & ak & \bar{a} & bi & bj & \bar{a}k & bj & \bar{a}k & \bar{a} & bi & bj & \bar{a}k & \bar{b}j & ak \\
\bar{a} & bi & a & \bar{b}i & \bar{b}j & ak & \bar{a} & bi & \bar{b}j & ak & bj & \bar{a}k & \bar{a} & bi & \bar{b}j & ak \\
\bar{a} & bi & a & \bar{b}i & bj & \bar{a}k & \bar{a} & bi & bj & \bar{a}k & \bar{b}j & ak & \bar{a} & bi & bj & \bar{a}k \\
\bar{b}j & ak & bj & \bar{a}k & \bar{b}j & ak & \bar{b}j & ak & \bar{a} & bi & \bar{b}j & ak & bj & \bar{a}k & \bar{a} & bi \\
\bar{b}j & ak & bj & \bar{a}k & bj & \bar{a}k & bj & \bar{a}k & \bar{a} & bi & bj & \bar{a}k & \bar{b}j & ak & \bar{a} & bi \\
\hline
bj & \bar{a}k & bj & \bar{a}k & \bar{a} & bi & \bar{b}j & ak & bj & \bar{a}k & \bar{a} & bi & \bar{b}j & ak & \bar{b}j & ak \\
bj & \bar{a}k & bj & \bar{a}k & \bar{a} & bi & bj & \bar{a}k & \bar{b}j & ak & \bar{a} & bi & bj & \bar{a}k & bj & \bar{a}k \\
a & \bar{b}i & \bar{a} & bi & bj & \bar{a}k & \bar{a} & bi & \bar{b}j & ak & \bar{b}j & ak & \bar{a} & bi & \bar{b}j & ak \\
a & \bar{b}i & \bar{a} & bi & \bar{b}j & ak & \bar{a} & bi & bj & \bar{a}k & bj & \bar{a}k & \bar{a} & bi & bj & \bar{a}k \\
bj & \bar{a}k & \bar{b}j & ak & \bar{b}j & ak & bj & \bar{a}k & \bar{a} & bi & \bar{b}j & ak & \bar{b}j & ak & \bar{a} & bi \\
bj & \bar{a}k & \bar{b}j & ak & bj & \bar{a}k & \bar{b}j & ak & \bar{a} & bi & bj & \bar{a}k & bj & \bar{a}k & \bar{a} & bi
\end{array}\right)
$$

Definition: Unbiased Hadamard Matrices

Hadamard matrices H and K of the same order v are unbiased if

$$HK^t = \sqrt{v}L$$

for some Hadamard matrix L.

Proposition (Kharaghani and Suda, 2019)

$H = \begin{pmatrix} H_1 & H_2 \end{pmatrix}$ a Hadamard matrix of order v with

$$H_1 H_1^t = \ell I_n + aS$$

where S is a $(-1, 0, 1)$-matrix. The following are equivalent:

1. $K = \begin{pmatrix} -H_1 & H_2 \end{pmatrix}$ is unbiased with H, and
2. $(\ell, a) = ((v \pm \sqrt{v})/2, \sqrt{v}/2)$.

- each split gives equiangular lines
- parametric conditions of the proposition are satisfied
- we can always construct unbiased Hadamard matrices

Definition: Unbiased ODs (Kharaghani and Suda, 2018)

- X_1 and X_2 be two $OD(v, s_1, \dots, s_u)$ in $\{\pm z_1, \dots, \pm z_u\}$.
- X_1 and X_2 are unbiased if there is a Hadamard matrix H such that

$$X_1 X_2^t = \left(\alpha^{-\frac{1}{2}} \sum_i s_i z_i^2 \right) H.$$

for some $\alpha \in \mathbb{R}^+$.

Definition: Unbiased ODs (Kharaghani and Suda, 2018)

- X_1 and X_2 be two $OD(v, s_1, \ldots, s_u)$ in $\{\pm z_1, \ldots, \pm z_u\}$.
- X_1 and X_2 are unbiased if there is a Hadamard matrix H such that

$$X_1 X_2^t = \left(\alpha^{-\frac{1}{2}} \sum_i s_i z_i^2 \right) H.$$

for some $\alpha \in \mathbb{R}^+$.

- $$\left\{ \begin{array}{l} \text{stable split} \\ \text{equiangular lines} \\ \text{param. cond.} \end{array} \right\} \Rightarrow \text{unbiased ODs}$$

- Recall:

$$X = \begin{pmatrix} G & F & -F \\ E & A & B \\ -E & B & A \end{pmatrix}.$$

- Take

$$Y = \begin{pmatrix} G & F & F \\ E & A & -B \\ -E & B & -A \end{pmatrix}.$$

- Then $XY^t = 2\left(\sum_i s_i z_i^2\right) H$, where H is Hadamard.

$$\begin{pmatrix}
a & b & a & b & a & b & a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} \\
a & b & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & \bar{b} & a & b \\
a & b & a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} & a & b & a & b \\
a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & a & b & a & b & a & b & \bar{a} & \bar{b} \\
b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a \\
b & \bar{a} & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} \\
a & b & \bar{a} & \bar{b} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} \\
a & b & \bar{a} & \bar{b} & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a \\
b & \bar{a} & \bar{b} & a & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b \\
b & \bar{a} & \bar{b} & a & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b \\
\bar{b} & a & \bar{b} & a & a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} \\
\bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a \\
\bar{a} & \bar{b} & a & b & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} \\
\bar{a} & \bar{b} & a & b & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a \\
\bar{b} & a & b & \bar{a} & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b \\
\bar{b} & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b
\end{pmatrix} \cdot \begin{pmatrix}
a & a & a & a & b & b & a & a & b & b & \bar{b} & \bar{b} & \bar{a} & \bar{a} & \bar{b} & \bar{b} \\
b & b & b & b & \bar{a} & \bar{a} & b & b & \bar{a} & \bar{a} & a & a & \bar{b} & \bar{b} & a & a \\
a & a & a & a & b & b & \bar{a} & \bar{a} & \bar{b} & \bar{b} & \bar{b} & \bar{b} & a & a & b & b \\
b & b & b & b & \bar{a} & \bar{a} & \bar{b} & \bar{b} & a & a & a & a & b & b & \bar{a} & \bar{a} \\
a & \bar{a} & a & \bar{a} & a & a & b & \bar{b} & b & \bar{b} & a & a & \bar{b} & b & b & \bar{b} \\
b & \bar{b} & b & \bar{b} & b & b & \bar{a} & a & \bar{a} & a & b & b & a & \bar{a} & \bar{a} & a \\
a & a & \bar{a} & \bar{a} & b & \bar{b} & a & a & b & \bar{b} & b & \bar{b} & a & a & \bar{b} & b \\
b & b & \bar{b} & \bar{b} & \bar{a} & a & b & b & \bar{a} & a & \bar{a} & a & b & b & a & \bar{a} \\
a & \bar{a} & \bar{a} & a & b & \bar{b} & b & \bar{b} & a & a & \bar{b} & b & b & \bar{b} & a & a \\
b & \bar{b} & \bar{b} & b & \bar{a} & a & \bar{a} & a & b & b & a & \bar{a} & \bar{a} & a & b & b \\
a & \bar{a} & a & \bar{a} & \bar{a} & \bar{a} & b & \bar{b} & \bar{b} & b & \bar{a} & \bar{a} & \bar{b} & b & \bar{b} & b \\
b & \bar{b} & b & \bar{b} & \bar{b} & \bar{b} & \bar{a} & a & a & \bar{a} & \bar{b} & \bar{b} & a & \bar{a} & a & \bar{a} \\
a & a & \bar{a} & \bar{a} & \bar{b} & b & \bar{a} & \bar{a} & b & \bar{b} & \bar{b} & b & \bar{a} & \bar{a} & \bar{b} & b \\
b & b & \bar{b} & \bar{b} & a & \bar{a} & \bar{b} & \bar{b} & \bar{a} & a & a & \bar{a} & \bar{b} & \bar{b} & a & \bar{a} \\
a & \bar{a} & \bar{a} & a & b & \bar{b} & \bar{b} & b & \bar{a} & \bar{a} & \bar{b} & b & \bar{b} & b & \bar{a} & \bar{a} \\
b & \bar{b} & \bar{b} & b & \bar{a} & a & a & \bar{a} & \bar{b} & \bar{b} & a & \bar{a} & a & \bar{a} & \bar{b} & \bar{b}
\end{pmatrix}$$

$$2(a^2+b^2)\begin{pmatrix}
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&-&-&1&1&-&-&-&-&1&1&-&-\\
1&1&1&1&1&1&-&-&-&-&1&1&-&-&-&-\\
1&1&1&1&-&-&-&-&1&1&-&-&-&-&1&1\\
1&-&1&-&1&1&1&-&1&-&-&-&1&-&-&1\\
1&-&1&-&1&1&-&1&-&1&-&-&-&1&1&-\\
1&1&-&-&1&-&1&1&1&-&-&1&-&-&1&-\\
1&1&-&-&-&1&1&1&-&1&1&-&-&-&-&1\\
1&-&-&1&1&-&1&-&1&1&1&-&-&1&-&-\\
1&-&-&1&-&1&-&1&1&1&-&1&1&-&-&-\\
1&-&1&-&-&-&-&1&1&-&1&1&-&1&-&1\\
1&-&1&-&-&-&1&-&-&1&1&1&1&-&1&-\\
1&1&-&-&1&-&-&-&-&1&-&1&1&1&-&1\\
1&1&-&-&-&1&-&-&1&-&1&-&1&1&1&-\\
1&-&-&1&-&1&1&-&-&-&-&1&-&1&1&1\\
1&-&-&1&1&-&-&1&-&-&1&-&1&-&1&1
\end{pmatrix}$$

- Further configurations:
  - quasi-symmetric designs
  - quaternion frames

A New Class of BGWs

Section Summary

Unifying Construction of Weighing Matrices

Section Summary

Equivalence to Association Schemes

Section Summary

Future Work

References

Kharaghani, H., Pender, T., and Suda, S. (2021). Balancedly splittable orthogonal designs and equiangular tight frames. *Des. Codes Cryptogr.*, 89(9):2033–2050.

Kharaghani, H. and Suda, S. (2018). Unbiased orthogonal designs. *Des. Codes Cryptogr.*, 86(7):1573–1588.

Kharaghani, H. and Suda, S. (2019). Balancedly splittable Hadamard matrices. *Discrete Math.*, 342(2):546–561.

Lam, T. Y. and Leung, K. H. (2000). On vanishing sums of roots of unity. *J. Algebra*, 224(1):91–109.