Weighing Matrices Generalizations and Related Configurations

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June 22, 2022



Summary

- Preliminaries
- 2 Balancedly Splittable Orthogonal Designs
- 3 Unifying Construction of Weighing Matrices
- 4 A New Class of Balanced Weighing Matrices
- 5 Equivalence to Association Schemes

Preliminaries



Section Summary

- Preliminaries
 - Balanced Incomplete Block Designs
 - Weighing Matrices
 - Balanced Generalized Weighing Matrices
 - Orthogonal Designs



- $J_{n \times m}$ the $n \times m$ matrix of all 1s.
- I_n the identity matrix of order n.

Definition: Balanced Incomplete Block Design

- A binary $v \times b$ (0,1)-matrix A such that:

 - $2 J_v A = k J_v.$

Write $BIBD(v, b, r, k, \lambda)$.

• The design is symmetric if v = b (equiv. k = r).

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Definition: Weighing Matrix

A v imes v (-1,0,1)-matrix W such that

$$WW^t = kI_v.$$

Write W(v, k).

- W(v, v) is a Hadamard matrix
- W(v, v 1) is a conference matrix

• A W(13,9)

$$W = \begin{pmatrix} 0 & 0 & - & 0 & - & 1 & - & - & 1 & 0 & - & - & - \\ 1 & 0 & 0 & - & 0 & - & 1 & - & - & 1 & 0 & - & - \\ 1 & 1 & 0 & 0 & - & 0 & - & 1 & - & - & 1 & 0 & - \\ 1 & 1 & 1 & 0 & 0 & - & 0 & - & 1 & - & - & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & - & 0 & - & 1 & - & - & 1 \\ - & 0 & 1 & 1 & 1 & 0 & 0 & - & 0 & - & 1 & - & - \\ 1 & - & 0 & 1 & 1 & 1 & 0 & 0 & - & 0 & - & 1 \\ - & 1 & 1 & - & 0 & 1 & 1 & 1 & 0 & 0 & - & 0 \\ 1 & - & 1 & 1 & - & 0 & 1 & 1 & 1 & 0 & 0 & - \\ 0 & 1 & - & 1 & 1 & - & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & - & 1 & 1 & - & 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Definition: Generalized Weighing Matrix

- Abelian group G not containg 0.
- A $v \times v$ (0, G)-matrix W such that

$$WW^* = (k \cdot 1_G)I_V \pmod{\mathbb{Z}G}.$$

Definition: Balanced Generalized Weighing Matrix

- Abelian group G not containing 0.
- A $v \times v$ (0, G)-matrix W such that

$$WW^* = (k \cdot 1_G)I_v + \frac{\lambda}{|G|}G(J_v - I_v).$$

- Write $BGW(v, k, \lambda; G)$.
- A $BGW(v, k, \lambda; C_2)$ is a balanced weighing matrix.
- $\lambda = k(k-1)/(v-1)$

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Definition: Orthogonal Design

- Real commuting variables $\{z_1, \ldots, z_u\}$.
- A $v \times v$ $(0, \pm z_1, \dots, \pm z_u)$ -matrix X such that

$$XX^t = \sigma I_v,$$

where
$$\sigma = \sum_{i} s_{i} z_{i}^{2}$$
.

• Write $OD(v; s_1, \ldots, s_u)$.



• An *OD*(4; 1, 1, 1, 1)

$$X = \begin{pmatrix} a & b & c & d \\ \overline{b} & a & \overline{d} & c \\ \overline{c} & d & a & \overline{b} \\ \overline{d} & \overline{c} & b & a \end{pmatrix}.$$



Balancedly Splittable ODs

Section Summary

- 2 Balancedly Splittable Orthogonal Designs
 - Balancedly Splittable Hadamard Matrices
 - Balancedly Splittable Orthogonal Designs
 - Faithful Construction
 - Related Configurations

• Recall a Hadamard matrix is a W(v, v).

Definition: Balancedly Splittable Hadamard Matrix (Kharaghani and Suda, 2019)

- A W(v, v), say H.
- An $v \times \ell$ submatrix H_1 such that

$$H_1H_1^t = \ell I_v + aA + b(J_v - I_v - A),$$

where A is a symmetric (0,1)-matrix with zero diagonal.

• H is (v, ℓ, a, b) balancedly splittable.

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• A Hadamard matrix of order 16

• A Hadamard matrix of order 16

• A balanced split.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ - & - & 1 & 1 & - & - \\ 1 & 1 & - & - & - & - \\ - & - & - & - & 1 & 1 \\ 1 & 1 & 1 & - & 1 & - \\ 1 & 1 & - & 1 & - & 1 \\ 1 & 1 & - & 1 & 1 & - \\ - & 1 & 1 & 1 & - & 1 \\ 1 & 1 & - & - & 1 & 1 \\ 1 & 1 & - & 1 & 1 & - \\ - & 1 & 1 & 1 & - & 1 \\ 1 & - & - & 1 & 1 & 1 \\ - & 1 & 1 & - & 1 & 1 \\ - & 1 & 1 & - & 1 & 1 \end{pmatrix}$$

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• An *OD*(16; 8, 8)

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• Products in the set $\{\pm 2a^2, \pm 2b^2, \pm 2ab\}$.

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Definition: Balancedly Splittable (Kharaghani et al., 2021)

• An OD X with $v \times \ell$ submatrix X_1 admits a stable (v, ℓ, a, b) split if

$$X_1X_1^t = \sigma(cI_v + aA + b(J_v - I_v - A)).$$

The split is unstable if it is not stable and

$$X_1 X_1^t = \ell I_v + aA + b(J_v - I_v - A)$$

after replacing each variable with 1.

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Definition: Auxiliary Matrices

- X a full $OD(v; s_1, \ldots, s_u)$ and H the Hadamard matrix obtained by replacing each variable with 1.
- Label rows of X and H by $x_0, \ldots, x_{\nu-1}$, and $h_0, \ldots, h_{\nu-1}$.
- The auxiliary matrices of X are given by

$$c_i = h_i^t x_i, \qquad i = 0, \dots, v - 1.$$

• An *OD*(2; 1, 1):

$$\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$$

Auxiliary matrices:

$$c_0 = \begin{pmatrix} a & b \\ a & b \end{pmatrix}, \qquad c_1 = \begin{pmatrix} b & -a \\ -b & a \end{pmatrix}$$

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Form the sequences

$$\alpha = (c_0, c_1, c_1), \qquad \beta = (c_0, c_1, -c_1).$$

• The sum of the periodic autocorrelations of α and β satisfy

$$\sum_{i} \alpha_{i} \alpha_{i+j}^{t} + \sum_{i} \beta_{i} \beta_{i+j}^{t} = 0, \qquad j \not\equiv 0 \pmod{3}$$

Sums of cross-correlations

$$\sum_{i} \alpha_{i} \beta_{i+j}^{t} + \sum_{i} \beta_{i} \alpha_{i+j}^{t} = 0, \qquad j \not\equiv 0 \pmod{3}$$

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Form the matrices

$$A = circ(\alpha), \qquad B = circ(\beta),$$

• Then the matrix

$$\Theta = \begin{pmatrix} A & B \\ B & A \end{pmatrix}.$$

$$\Theta = \begin{pmatrix} a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a \\ a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} \\ b & \bar{a} & a & b & \bar{b} & \bar{a} & \bar{b} & a & a & b & b & \bar{a} \\ \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a \\ b & \bar{a} & b & \bar{a} & a & b & \bar{b} & a & a & b & \bar{b} & a & a \\ \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b \\ \bar{a} & b & \bar{b} & \bar{a} & \bar{b} & \bar{a} & a & b & \bar{b} & \bar{a} & \bar{b} & \bar{a} \\ \bar{a} & b & \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{a} & b & \bar{b} & \bar{a} & \bar{b} & \bar{a} \\ \bar{b} & a & a & b & \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{a} \\ \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{a} \\ \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{a} & \bar{b} & \bar{b} & \bar{a} & \bar{b} & \bar{a} \\ \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{a} & \bar{b} & \bar{b} & \bar{a} & \bar{b} & \bar{a} \\ \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{a} & \bar{b} & \bar{b} & \bar{a} & \bar{b} \\ \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{a} & \bar{b} & \bar{b} & \bar{a} & \bar{b} \\ \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{a} & \bar{b} & \bar{b} & \bar{a} & \bar{b} \\ \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{b} & \bar{a} & \bar{b} \\ \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{b} & \bar{a} & \bar{b} \\ \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{b} & \bar{a} & \bar{b} \\ \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{b} & \bar{a} & \bar{b} \\ \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{b} & \bar{a} & \bar{b} \\ \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{b} & \bar{a} & \bar{b} \\ \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{b} & \bar{a} & \bar{b} \\ \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{b} & \bar{a} & \bar{b} \\ \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{b} & \bar{a} & \bar{b} \\ \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{b} & \bar{a} & \bar{b} \\ \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{b} & \bar{a} & \bar{b} \\ \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{b} & \bar{a} & \bar{b} \\ \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{b} & \bar{a} & \bar{b} \\ \bar{b} & \bar{a} & \bar{b} & \bar{b} & \bar{a} & \bar{b} & \bar{b} \\ \bar{b} & \bar{a} & \bar{b} & \bar{b} & \bar{a} & \bar{b} & \bar{b} \\ \bar{b} & \bar{a} & \bar{b} & \bar{b} & \bar{a} & \bar{b} & \bar{b} \\ \bar{b} & \bar{a} & \bar{b} & \bar{b} & \bar{a} & \bar{b} \\ \bar{b} & \bar{a} & \bar{b} & \bar{b} & \bar{b} & \bar{b} & \bar{b} \\ \bar{b} & \bar{b} & \bar{b} & \bar{b} & \bar{b} & \bar{b} \\ \bar{b} & \bar{b} & \bar{b} & \bar{b} & \bar{b} & \bar$$

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Define:

$$Y = \begin{pmatrix} 1 & 1 \\ 1 & - \end{pmatrix} \otimes X,$$

$$K = \begin{pmatrix} 1 & 1 \\ 1 & - \end{pmatrix} \otimes H.$$

• Index rows as $y_0, ..., y_{2n-1}$, and $k_0, ..., k_{2n-1}$.

$$Y = \begin{pmatrix} 1 & 1 \\ 1 & - \end{pmatrix} \otimes X$$
 $K = \begin{pmatrix} 1 & 1 \\ 1 & - \end{pmatrix} \otimes H$
Rows: y_0, \dots, y_{2n-1} Rows: k_0, \dots, k_{2n-1}

Form

$$F = \begin{pmatrix} k_1^t x_0 & k_2^t x_0 & k_3^t x_0 \end{pmatrix},$$

$$E = \begin{pmatrix} h_0^t y_1 \\ h_0^t y_2 \\ h_0^t y_3 \end{pmatrix},$$

$$G = k_0^t y_0.$$

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Theorem (Kharaghani et al., 2021)

If there is a full $OD(v, s_1, ..., s_u)$, then there is a full $OD(4v^2; 4vs_1, ..., 4vs_u)$ for which:

- there are two stable verticle splits, and
- there are two unstable horizontal splits.

Proof

In the general case:

$$\alpha = (c_0, c_1, \dots, c_{n-1}, c_{n-1}, \dots, c_1)$$

$$\beta = (c_0, c_1, \dots, c_{n-1}, -c_{n-1}, \dots, -c_1).$$

• Tedious checking of products between block rows.

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- Related configurations:
 - equiangular lines
 - Unbiased orthogonal designs
 - quasi-symmetric designs
 - quaternion ODs and frames

Unifying Construction of Weighing Matrices

Section Summary

- 3 Unifying Construction of Weighing Matrices
 - Classical Parameter BGWs
 - Normalization
 - Generalized Simplex Codes
 - Construction

Proposition: Classical Parameter BGWs

- Let q be a prime power and d > 0.
- BGWs are known to exist with the following parameters:

$$\left(v = \frac{q^{d+1} - 1}{q - 1}, k = q^d, \lambda = q^d - q^{d-1}; G = C_{q-1}\right)$$

for every q and d > 0, and

$$\left(v = \frac{q^{d+1}-1}{q-1}, k = q^d, \lambda = q^d - q^{d-1}; G = C_{2q-2}\right)$$

for even q and d.

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T. Pender (U of L)

• Every weighing matrix is equivalent to one of the following form

$$\begin{pmatrix} \mathbf{0} & \mathbf{R} \\ \mathbf{1} & \mathbf{D} \end{pmatrix}$$

- R is the residual-part
- D is the derived-part

Definition: Simplex Code

- q a prime power and d > 0
- Form matrix G with columns given by reps. of 1-D subspaces of $GF(q^{d+1})$
- simplex code $S_{a,d} = row(G)$

Proposition

For $S_{a,d}$:

- $wt(x) = q^d$ for all $x \in S_{q,d}$
- $dist(x, y) = q^d$ for all $x, y \in S_{q,d}$ and $x \neq y$

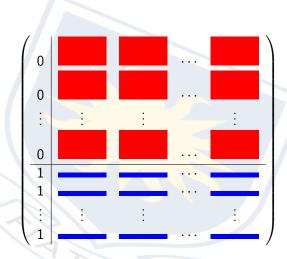
- Ingredients of unifying construction:
 - a normalized W(v,q) (seed matrix) with residual-part R and derived-part D
 - a $W((q^{d+1}-1)/(q-1), q^d)$, say W
 - \circ $\mathcal{S}_{q,d}$



- Recipie of unifying construction:
 - Form $A = W \otimes R$.
 - Form B by replacing elements of $S_{a,d}$ by rows of D.
 - Then

$$\begin{pmatrix} \mathbf{0} & A \\ \mathbf{1} & B \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{0} & A \\ \mathbf{1} & B \end{pmatrix}$$
 is a $W((v-1)(q^{d+1}-1)/(q-1)+1,q^{d+1})$.



• A seed W(8,5)

$$\begin{pmatrix} \mathbf{0} & R \\ \mathbf{1} & D \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & 0 & 0 & 0 & 1 & 1 & -- \\ 1 & 1 & 1 & 1 & -0 & 0 & 0 \\ 1 & 1 & --0 & 0 & 0 & 1 & 0 \\ 1 & --1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

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• A classical parameter W(6,5)

$$W = \begin{pmatrix} - & 1 & - & 0 & 1 & 1 \\ - & - & 1 & - & 0 & 1 \\ - & - & - & 1 & - & 0 \\ 0 & - & - & - & 1 & - \\ 1 & 0 & - & - & - & 1 \\ - & 1 & 0 & - & - & - \end{pmatrix}$$

• The simplex code $S_{5,1}$ (transposed)

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- Take $A = W \otimes R$.
- Take B to be the matrix formed after replacing the entries of $S_{5,1}$ by the rows of D.
- Then

$$\begin{pmatrix} 0 & A \\ 1 & B \end{pmatrix}$$

is a W(43, 25).

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Theorem (Kharaghani et al., 2022a)

If there is a W(v,q), then there is a weighing matrix with parameters

$$\left(\frac{(v-1)(q^{d+1}-1)}{q-1}+1,q^{d+1}\right)$$

whenever:

- $\mathbf{0}$ q is odd and every d > 0, and
- \bigcirc q and d are both even.

Seed (v, k)	Succident (v', k')	Seed (v, k)	Succident (v', k')
(6, 5):	(31, 25), (156, 125), (781, 625)	(16, 3):	(69, 9), (196, 27), (601, 81)
(8, 5):	(43, 25), (218, 125)	(16, 5):	(91, 25), (466, 125)
(8, 7):	(57, 49), (400, 343)	(16, 7):	(121, 49), (856, 343)
(10, 5):	(55, 25), (280, 125)	(16, 9):	(151, 81)
(10, 9):	(91, 81), (820, 729)	(16, 11):	(181, 121)
(12, 5):	(67, 25), (342, 12 <mark>5</mark>)	(16, 13):	(211, 169)
(12, 7):	(89, 49), (628, 343)	(18, 13):	(239, 169)
(12, 9):	(111, 81)	(19, 9):	(181, 81)
(13, 9):	(121, 81)	(20, 7):	(153, 49)
(14, 9):	(131, 81)	(20, 13):	(267, 169)
(14, 13):	(183, 169)		

A New Class of BWs

Section Summary

- 4 A New Class of Balanced Weighing Matrices
 - Seed Matrix
 - Construction

• Consider the following balanced W(19,9):

Computationally found in de Launey and Sarvate (1984).

T. Pender (U of L) Weighing Matrices 06/22/2022 60 / 82

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• Let Ξ be a BGW with parameters

$$\left(v = \frac{9^{d+1} - 1}{8}, k = 9^d, \lambda = 9^d - 9^{d-1}\right)$$

over the group $C_4 = \{1, g, g^2, g^3\}$.

Decompose ≡ as

$$\Xi = \Xi_1 + g\Xi_g + g^2\Xi_{g^2} + g^3\Xi_{g^3},$$

where the Ξ_i s are disjoint (0,1)-matrices.

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• Form:

$$\Xi \otimes R_1 = \Xi_1 \otimes R_1 - \Xi_g \otimes R_2 - \Xi_{g^2} \otimes R_1 + \Xi_{g^3} \otimes R_2$$

- Applying $R_1 \mapsto -R_2 \mapsto -R_1 \mapsto R_2 \mapsto R_1$.
- Form D by substituting for the elements of $S_{9,d}$ the rows of the derived part of W_{19} .

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The matrix

$$\begin{pmatrix} \mathbf{0} & \Xi \otimes R_1 \\ 1 & D \end{pmatrix}$$

is a balanced weighing matrix.

Theorem: (Kharaghani et al., 2022b)

For every d > 0, there is a balanced weighing matrix with parameters

$$\left(v = \frac{9(9^{d+1} - 1)}{4} + 1, k = 9^{d+1}\right).$$

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Equivalence to **Association Schemes**

Section Summary

- 5 Equivalence to Association Schemes
 - Association Schemes
 - Equivalence to Balanced Weighing Matrices
 - Equivalence to Balanced Generalized Weighing Matrices

Definition (Commutative) Association Scheme

A (commutative) association scheme is a pair $\mathfrak{X} = (X, \mathscr{A})$ where:

- X is a point set and $\mathscr{A} = \{A_0, \dots, A_d\}$ a collection of (0,1)-matrices indexed by elements of X.
- $A_0 = I$;
- $\bullet \sum_{i=0}^{\infty} A_i = J;$
- A is closed under transposition;
- there exist non-negative integers p_{ij}^k such that $A_iA_j = \sum_{k=0}^{u} p_{ij}^k A_k$; and
- $\bullet \ A_i A_j = A_j A_i.$
- *** \mathfrak{X} is symmetric if $A_i^t = A_i$, for every i.

- $\mathfrak{U} = \langle \mathscr{A} \rangle$ is the adjacency algebra of \mathfrak{X} .
- \mathfrak{U} has a dual basis of idempotents $\{E_0 = |X|^{-1}J, \ldots, E_d\}$.
- There are change of basis matrices P and Q such that

$$A_j = \sum_{i=0}^d P_{ij} E_i, \qquad E_j = |X|^{-1} \sum_{i=0}^d Q_{ij} A_i$$

- Let W be a $BGW(v, k, \lambda; C_2)$.
- Decompose...

$$W=W_1-W_2,$$

where W_1 and W_2 are disjoint (0,1)-matrices.

• Properties...

• Properties...

•
$$(W_1 - W_2)(W_1 - W_2)^t = (W_1 - W_2)^t(W_1 - W_2) = kI$$

• Properties...

•
$$(W_1 - W_2)(W_1 - W_2)^t = (W_1 - W_2)^t(W_1 - W_2) = kI$$

• $(W_1 + W_2)(W_1 + W_2)^t = (W_1 + W_2)^t(W_1 + W_2) = kI + \lambda(J - I)$

- Properties...
 - $(W_1 W_2)(W_1 W_2)^t = (W_1 W_2)^t(W_1 W_2) = kI$
 - $(W_1 + W_2)(W_1 + W_2)^t = (W_1 + W_2)^t(W_1 + W_2) = kI + \lambda(J I)$
 - $W_1W_1^t + W_2W_2^t = W_1^tW_1 + W_2^tW_2 = kI + \frac{\lambda}{2}(J-I)$

Properties...

•
$$(W_1 - W_2)(W_1 - W_2)^t = (W_1 - W_2)^t(W_1 - W_2) = kI$$

• $(W_1 + W_2)(W_1 + W_2)^t = (W_1 + W_2)^t(W_1 + W_2) = kI + \lambda(J - I)$
• $W_1W_1^t + W_2W_2^t = W_1^tW_1 + W_2^tW_2 = kI + \frac{\lambda}{2}(J - I)$

• $W_1W_2^t + W_2W_1^t = W_1^tW_2 + W_2^tW_1 = \frac{\lambda}{2}(J-I)$

• With $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, form the (0, 1)-matrices...

$$A_{0} = I_{2} \otimes I_{2} \otimes I_{v}$$

$$A_{1} = I_{2} \otimes P \otimes I_{v}$$

$$A_{2} = I_{2} \otimes J_{2} \otimes (J_{v} - I_{v})$$

$$A_{3} = \begin{pmatrix} O & I_{2} \otimes W_{1} + P \otimes W_{2} \\ I_{2} \otimes W_{1}^{t} + P \otimes W_{2}^{t} & O \end{pmatrix}$$

$$A_{4} = \begin{pmatrix} O & I_{2} \otimes W_{2} + P \otimes W_{1} \\ I_{2} \otimes W_{2}^{t} + P \otimes W_{1}^{t} & O \end{pmatrix}$$

$$A_{5} = \begin{pmatrix} O & J_{2} \otimes (J_{v} - W_{1} - W_{2}) \\ J_{2} \otimes (J_{v} - W_{1}^{t} - W_{2}^{t}) & O \end{pmatrix}$$

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• Character tables...

$$P = \begin{pmatrix} 1 & 1 & 2(v-1) & k & k & 2(v-k) \\ 1 & -1 & 0 & \sqrt{k} & -\sqrt{k} & 0 \\ 1 & -1 & 0 & -\sqrt{k} & \sqrt{k} & 0 \\ 1 & 1 & 2(v-1) & -k & -k & 2(k-v) \\ 1 & 1 & -2 & -\sqrt{\frac{k(v-k)}{v-1}} & -\sqrt{\frac{k(v-k)}{v-1}} & 2\sqrt{\frac{k(v-k)}{v-1}} \\ 1 & 1 & -2 & \sqrt{\frac{k(v-k)}{v-1}} & \sqrt{\frac{k(v-k)}{v-1}} & -2\sqrt{\frac{k(v-k)}{v-1}} \end{pmatrix}$$

$$Q = \begin{pmatrix} 1 & v & v & 1 & v-1 & v-1 \\ 1 & -v & -v & 1 & v-1 & v-1 \\ 1 & 0 & 0 & 1 & -1 & -1 \\ 1 & \frac{v}{\sqrt{k}} & -\frac{v}{\sqrt{k}} & -1 & -\sqrt{\frac{(v-1)(v-k)}{k}} & \sqrt{\frac{(v-1)(v-k)}{k}} \\ 1 & -\frac{v}{\sqrt{k}} & \frac{v}{\sqrt{k}} & -1 & -\sqrt{\frac{(v-1)(v-k)}{k}} & \sqrt{\frac{(v-1)(v-k)}{k}} \\ 1 & 0 & 0 & -1 & \sqrt{\frac{k(v-1)}{v-k}} & -\sqrt{\frac{k(v-1)}{v-k}} \end{pmatrix}$$

Theorem: Weighing Schemes (Kharaghani et al., 2022b)

There is a symmetric scheme with character tables P and Q given above if and only if there is a balanced W(v, k).

Proof

- We have sufficiency.
- To show necessity, use eigenvalues to infer structure of corresponding graphs/adjacency matrices.

- $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ form a matrix representation of C_2 .
- For a finite abelian group $G = \{g_0, \dots, g_{n-1}\}$ and $h \in G$...

$$U_h = (\delta(g_i^{-1}hg_j))$$

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$$A_{0,g} = I_2 \otimes U_g \otimes I_V, \text{ for } g \in G,$$

$$A_1 = I_2 \otimes J_n \otimes (J_V - I_V),$$

$$A_{2,g} = \begin{pmatrix} O & \sum_{h \in G} (U_h \otimes W_{gh}) \\ \sum_{h \in G} (U_h \otimes W_{g^{-1}h^{-1}}) & O \end{pmatrix}, \text{ for } g \in G, \text{ and }$$

$$A_2 = \begin{pmatrix} O & J_n \otimes (J_V - \sum_{h \in G} W_h) \\ J_n \otimes (J_V - \sum_{h \in G} W_h) & O \end{pmatrix}.$$

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Future Work



- More relations to BGWs!
 - error-correcting codes
 - more general objects (quasi-balanced weighing matrices)
- Can the theory of association schemes be used to say anything about BGWs?



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