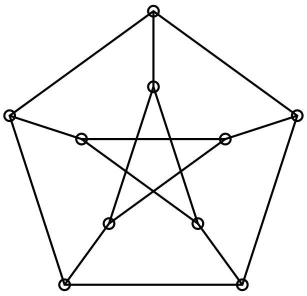
0.1. Definition and Elementary Properties. A *graph*, similar to a balanced incomplete block design, is simply an ordered pair $\Gamma = (V, E)$ where V is a finite set of v points, called *vertices*, and where $E \subset \binom{V}{2}$, called the *edges*.simple-graph

If $x, y \in V$ are distinct, and if $\{x, y\} \in E$, then we say that x and y are *adjacent*, and we write $x \sim y$. If there is a sequence of vertices $x = v_0, v_1, \ldots, v_{n-1} = y$ such that $v_{i-1} \sim v_i$, for $i \in \{1, \ldots, n-1\}$, then we say that x and y are *connected*. Connectedness is easily seen to be an equivalence relation of the vertex set V.

The *degree* of vertex x is defined as the number of vertices that are adjacent to x. If every vertex has the same degree, say, k, then we say that the graph is *regular* with degree k.

The *complement* of a graph $\Gamma = (V, E)$ is given by $\overline{\Gamma} = (V, \binom{V}{2} \backslash E)$.

0.1. Example. The following is a graph on 10 vertices called the *Petersen graph*, where the vertices are the nodes, and where the line segments represent adjacencies. One can see that each pair of distinct vertices are connected.

(0.1.a) 

This graph evinces further interesting properties, namely, the graph is regular with degree 3, each pair of adjacenct vertices have 0 common neighbours, and each pair of non-adjacenct vertices have 1 common neighbour.

The above example motivates the following definition.

0.2. Definition. Let $\Gamma = (V, E)$ be a graph with $|V| = v$. Γ is said to be *strongly regular* if the following are satisfied.

(0.2.a) The graph is regular with degree k,

(0.2.b) each pair of adjacenct vertices have λ common neighbours, and

(0.2.c) each pair of non-adjacenct vertices have μ common neighbours.

In such a case we write Γ is an SRG(v, k, λ, μ).

0.3. Example. The Petersen graph of Example 0.1 is an SRG$(10, 3, 0, 1)$.

0.4. Example. Let $[m] = \{0, \ldots, m-1\}$, for $m > 2$, and take $V = \binom{[m]}{2}$. Define adjacencies by $x \sim y$ iff $|x \cap y| = 1$. The ensuing structure is called the T(m) graph. Clearly, the graph is regular of degree $k = 2(m-2)$. The number of vertices adjacenct to both $\{a, c\}$ and $\{b, c\}$ is given by those $\{c, d\}$, where $d \neq a, b, c$, and $\{a, b\}$. Hence, $\lambda = m - 2$. Finally, the number of vertices adjacenct to the nonadjacent vertices $\{a, b\}$ and $\{c, d\}$ is given by $\{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}$; hence, $\mu = 4$. We have shown that $\mathrm{T}(m)$ is strongly regular with parameters $(m(m-1)/2, 2(m-2), m-2, 4)$.

0.5. Proposition. The parameters of an SRG(v, k, λ, μ) satisfy

(0.5.a) $k(k - \lambda - 1) = \mu(v - k - 1)$.

Proof. Let $\Gamma = (V, E)$ be an SRG(v, k, λ, μ), and fix a vertex $x \in V$. Count in two ways the edges $\{y, z\} \in E$ such that $x \sim y$, $x \not\sim z$, and $z \neq x$. First, there are k choices for y and $k - \lambda - 1$ choices for z. On the other hand, there are $v - k - 1$ choices for z and μ choices for y. ■

With graphs, as with BIBDs, we can represent these objects with matrices.

0.6. Definition. Let $\Gamma = (\{x_0, \ldots, x_{v-1}\}, E)$ be some graph. The *adjacency matrix* $A(\Gamma) = (a_{ij})$ of Γ, or simply A, is the $(0, 1)$-matrix of order v defined by

(0.6.a) $a_{ij} = \begin{cases} 1 & \text{if } x_i \sim x_j \text{, and} \\ 0 & \text{otherwise.} \end{cases}$

From the definition, we can se that $A(\Gamma)$ is symmetric and has zero diagonal. Furthermore, $A(\overline{\Gamma}) = J_v - I_v - A(\Gamma)$.

Using the adjacency matrix, it turns out that SRGs are completely characterized by a simple matrix equation.

0.7. Proposition. A graph Γ is strongly regular with parameters (v, k, λ, μ) and adjacency matrix $A = A(\Gamma)$ iff

(0.7.a) $A^2 = kI_v + \lambda A + \mu(J_v - I_v - A)$.

Proof. Restatement of Definition 0.2. ■

0.8. Corollary. The complement of an SRG(v, k, λ, μ) is an SRG$(v, v - k - 1, v - 2k + \mu - 2, v - 2k + \lambda)$.

We conclude this subsection with one further result. Its justification is rather immediate, and as such, it is omitted (see **?**, Proposition 7.1.6).

0.9. Proposition. Let Γ be an SRG(v, k, λ, μ). Then the following are equivalent.

(0.9.a) $\mu = 0$;

(0.9.b) $k = \lambda + 1$;

(0.9.c) Γ is not connected; and

(0.9.d) Γ is the disjoint union of v/k copies of K_k.

The above proposition shows that the interesting SRGs are those which are connected. We will assume this to be the case unless otherwise stated.

* * *

0.2. Eigenvalues. This section will closely follow §1.2 of **?**.

Let Γ be an SRG(v, k, λ, μ) with adjacency matrix $A = A(\Gamma)$. Since Γ is regular of degree k, it follows that $AJ = JA = kJ$. Since A and J are then symmetric matrices which commute, they are simultaneously orthogonally diagonalizable.sim-diag

The rank of J is one, and its eigenvalues are v and 0 with multiplicities 1 and $v-1$, respectively.

Since Γ is regular of degree k, it has eigenvalue k with multiplicity given by the number of connected components of the graph (see **?**, Theorem 1.5). Since we are assuming that Γ is connected, the eigenvalue k has multiplicity 1. Moreover, by (0.7.a), it follows that $k^2 = k + \lambda k + \mu(v - k - 1)$, which shows again (0.5.a).

Since Tr$(A) = 0$, it follows that there is at least one further eigenvalue. If there is only one further eigenvalue, say, ϱ, then $k + (v-1)\varrho = 0$; hence, $\varrho = -\frac{k}{v-1}$ is a rational integer. But this would imply that $\Gamma = K_v$.complete-graph-eigenvalues We therefore assume that Γ has more than two eigenvalues.

If ϱ is any other eigenvalue of A, then $\varrho^2 = k - \mu + (\lambda - \mu)\varrho$; hence, there are precisely three eigenvalues of A given by k, $r = (\lambda - \mu + \sqrt{(\lambda-\mu)^2 + 4(k-\mu)})/2$, and $s = (\lambda - \mu - \sqrt{(\lambda-\mu)^2 + 4(k-\mu)})/2$.

Furthermore, if the multiplicities of r and s are f and g, respectively, then $v = 1 + f + g$ and $k + fr + gs = 0$. Solving this system yields $f = (v - 1 + [(\mu-\lambda)(v-1) - 2k]/\sqrt{(\lambda-\mu)^2 + 4(k-\mu)})/2$ and $g = (v - 1 - [(\mu-\lambda)(v-1) - 2k]/\sqrt{(\lambda-\mu)^2 + 4(k-\mu)})/2$.

We record these results for future reference.

0.10. Theorem. Let Γ be a connected, noncomplete SRG(v, k, λ, μ) with adjacency matrix $A = A(\Gamma)$. Then A has eigenvalues

(0.10.a) k of multiplicity 1,

(0.10.b) $r = \frac{1}{2}(\lambda - \mu + \sqrt{(\lambda-\mu)^2 + 4(k-\mu)})$ of multiplicity $f = \frac{1}{2}\left(v - 1 + \frac{(\mu-\lambda)(v-1)-2k}{\sqrt{(\lambda-\mu)^2+4(k-\mu)}}\right)$, and

(0.10.c) $s = \frac{1}{2}(\lambda - \mu - \sqrt{(\lambda-\mu)^2 + 4(k-\mu)})$ of multiplicity $g = \frac{1}{2}\left(v - 1 - \frac{(\mu-\lambda)(v-1)-2k}{\sqrt{(\lambda-\mu)^2+4(k-\mu)}}\right)$.

Moreover, we have that

(0.10.d) $\lambda = k + r + s + rs$, and

(0.10.e) $\mu = k + rs$.

The following result gives necessary conditions on the existence of an SRG. We refer the reader to **?** for its derivation.

0.11. Proposition. Suppose that Γ is an SRG(v, k, λ, μ) which neither complete nor null. Then precisely one of the following holds.

(0.11.a) $k = v - k - 1$, $\mu = \lambda + 1 = k/2$, $f = g = k$.

(0.11.b) $D = (\lambda - \mu)^2 + 4(k - \mu)$ is a perfect square. Further,

(i) $\sqrt{D}$ divides $2k + (\lambda - \mu)(v - 1)$ if v is even, and

(ii) $2\sqrt{D}$ divides $2k + (\lambda - \mu)(v - 1)$ if v is odd.

* * *

0.3. Generated Matrix Algebra. Let Mat$_n(\mathbf{C})$ be the algebra of $n \times n$ matrices with entries from $\mathbf{C}$. Let Γ be an SRG, and let $A = A(\Gamma)$ and $\overline{A} = A(\overline{\Gamma})$. Since $A + \overline{A} = J - I$, it follows that I, A and $\overline{A}$ are $\mathbf{C}$-linearly independent and, therefore, span a 3-dimensional linear subspace of Mat$_n(\mathbf{C})$, say, $\mathfrak{U}$.

If we use $\cdot \circ \cdot$ to denote Schur multiplication,schur-mult then we see at once that $X \circ X = X$, for each $X \in \{I, A, \overline{A}\}$. Therefore, $\mathfrak{U}$ is closed under Schur multiplication. We denote this algebra as $\mathfrak{U}^0$ and call I, A, and $\overline{A}$ the *Schur idempotents* of $\mathfrak{U}^0$.

It turns out that $\mathfrak{U}$ also has an algebra structure with respect to matrix multiplication.

0.12. Proposition. Let Γ be an SRG, and let $A = A(\Gamma)$. Then $\mathfrak{U} = \langle I, A, \overline{A} \rangle$ is closed under standard matrix multiplication.

Proof. By (0.7.a), we have that $J - I = \mu^{-1}[A^2 + A(\mu - \lambda) - kI]$. Therefore, $\overline{A} = \mu^{-1}A^2 + A[\mu^{-1}(\mu\lambda) - 1] - \mu^{-1}kI$; so, $\overline{A}$ is a polynomial in A, whence it commutes with A. Then

$$\begin{aligned}\overline{A}A = A\overline{A} = A(J - I - A) &= kJ - A - A^2 \\ &= kJ - A - kI - \lambda A - \mu\overline{A} \\ &= k(I + A + \overline{A}) - A - kI - \lambda A - \mu\overline{A} \\ &= (k - \lambda - 1)A + (k - \mu)\overline{A},\end{aligned}$$

and $A\overline{A} = \overline{A}A \in \mathfrak{U}$. By (0.7.a), we see that $A^2, \overline{A}^2 \in \mathfrak{U}$. We therefore have that $\mathfrak{U}$ is closed under standard matrix multiplication. ■

Since I, A, and $\overline{A}$ are symmetric, pairwise commuting matrices, they are simultaneously diagonalizable; hence, $\mathfrak{U}$ is semisimple (see **?**). It follows that there are matrices $E_0, E_1, E_2 \in \mathfrak{U}$ such that (a) $\mathfrak{U} = \langle E_0, E_1, E_2 \rangle$, (b) $E_0 + E_1 + E_2 = I$, and (c) $E_i E_j = \delta_{i,j} E_j$.