§1. Weighing Matrices

1.1. Definitions

Imagine, for a moment, the simple senario in which you need to weigh several objects, say, four objects (see **?**, Chapter 2). Imagine further that you are using a simple balance with two pans that makes an error ϵ everytime that it is used, where ϵ is random with mean 0 and variance σ^2.

Assume the actual weights are a, b, c, and d. If we weigh them seperately with measurements y_1, y_2, y_3, and y_4, and if the errors are $\epsilon_1, \epsilon_2, \epsilon_3$, and ϵ_4, then we obtain the four equations

$$a = y_1 + \epsilon_1, \qquad b = y_2 + \epsilon_2, \qquad c = y_3 + \epsilon_3, \qquad d = y_4 + \epsilon_4.$$

The estimates of the weights are then

$$\hat{a} = y_1 = a - \epsilon_1, \quad \hat{b} = y_2 = b - \epsilon_2, \quad \hat{c} = y_3 = c - \epsilon_3, \quad \hat{d} = y_4 = d - \epsilon_4,$$

each having variance σ^2.

Now, we will weigh the objects together in the following way.

$$\begin{aligned} a + b + c + d &= y_1 + \epsilon_1, \\ a - b + c - d &= y_2 + \epsilon_2, \\ a + b - d - d &= y_3 + \epsilon_3, \\ a - b - c + d &= y_4 + \epsilon_4, \end{aligned}$$

where we have used $+1$ to indicate being placed on the right pan, and we have used -1 to indicate being placed on the left pan.

One can see that the coefficient matrix for the weighing configuration is non-singular. As such, we can solve for the variables; for example, we show the estimate for a:

$$\hat{a} = \frac{y_1 + y_2 + y_3 + y_4}{4}.$$

From this, we can see that the variance of $\hat{a}$ is given by $\sigma^2/4$, a large improvement from the initial configuration in which each weight was weighed independently.

In general, if there are n weights, and if there is a non-singular ternary (-1,0,1)-matrix of order n with a constant number of non-zero entries in each row, then that matrix can be used as a weighing configuration for the collection of objects in which the variance of the errors can be reduced.

All this motivates the following definition.

1.1 Definition. Let W be a $(-1, 0, 1)$-matrix of order n. Then:

(1.1.a) If $WW^t = kI_n$, we say that W is a *weighing matrix* of *order* n and *weight* k.

(1.1.b) If $n = k$, then we say that W is a *Hadamard matrix*.

(1.1.c) If $n - 1 = k$, then we say that W is a *conference matrix*.

In any event, we write $\mathrm{W}(n, k)$ to denote this property.

1.2 Example. The following can be verified directly to be a $\mathrm{W}(7, 4)$ (NB: We have used $-$ in place of -1 and $+$ in place of $+1$)

$$
\text{(1.2.a)} \quad \begin{pmatrix} - & - & + & 0 & + & 0 & 0 \\ 0 & - & - & + & 0 & + & 0 \\ 0 & 0 & - & - & + & 0 & + \\ - & 0 & 0 & - & - & + & 0 \\ 0 & - & 0 & 0 & - & - & + \\ - & 0 & - & 0 & 0 & - & - \\ + & - & 0 & - & 0 & 0 & - \end{pmatrix}.
$$

The weighing matrix of the previous example is indicative of a more general construction that uses relative difference sets (see §§3.4). Note that upon changing the nonzero entries of (1.2.a), one obtains the incidence matrix of the complement design (**??**).

1.2. Necessary Conditions on Existence

The conditions placed on a matrix in order for it to be a weighing matrix are none too restrictive: One only needs orthogonality of the rows, a constant number of nonzero entries in every row, and the non-zero entries to have absolute value 1. Usually, in order to construct these objects one must assume some further combinatorial and/or algebraic properties. So-called cocyclic matrices, for example, are studied extensively in **?** and **?**. Nevertheless, we can say a few things at the outset.

By Cauchy's property of the determinant[1] $\det(W)^2 = k^n I_n$, for any $\mathrm{W}(n, k)$. In particular, if $n \equiv 1 \bmod 2$, then k is a square. This leaves the case that $n \equiv 0 \bmod 2$. It turns out, though we will not show it here, that if $n \equiv 2 \bmod 4$, then it must be the case that $k = a^2 + b^2$, for some $a, b \in \mathbf{Z}$.[a] Much is not known about the case $n \equiv 0 \bmod 4$, though **?** conjectures that a $\mathrm{W}(4n, k)$ exists for every n and $k \leq 4n$.

Considering the weights, if $k = n - 1$, then it is clear that $n \equiv 0 \bmod 2$; for otherwise, if n is odd, then there will be $n - 2$ instances of $\binom{1}{1}$, $\binom{1}{-}$, or their negatives, in the product between any two distinct rows. Therefore, the product will resolve to a sum of $n-2$ terms each consisting of ± 1. Since $n-2 \equiv 1 \bmod 2$, this can never be zero.

In the case that $n = k$, we can assume (see §§3.5) that the first row and column consist of all ones. Moreover, we can assume the first three rows have the following

[a]This is a consequence of the Hasse-Minkowski Theory, more specifically, the Bruck-Ryser-Chowla Theorem. There are a number of references that treat this result. The interested reader is asked to consult **?** and **?**.

form.

$$\overbrace{\begin{matrix}1 & \dots & 1\\ 1 & \dots & 1\\ 1 & \dots & 1\end{matrix}}^{a}\quad \overbrace{\begin{matrix}1 & \dots & 1\\ 1 & \dots & 1\\ - & \dots & -\end{matrix}}^{b}\quad \overbrace{\begin{matrix}1 & \dots & 1\\ - & \dots & -\\ 1 & \dots & 1\end{matrix}}^{c}\quad \overbrace{\begin{matrix}1 & \dots & 1\\ - & \dots & -\\ - & \dots & -\end{matrix}}^{d}$$

Evidently, this configuration yields the following linear system.

$$\begin{aligned}&a+b+c+d=n,\\ &a+b-c-d=0,\\ &a-b+c-d=0,\\ &a-b-c+d=0,\end{aligned}$$

whose solution is $a=b=c=d=n/4$. As these are integers, it must be the case, outside of the trivial cases $n=1$ or 2, that $n \equiv 0 \bmod 4$ whenever $n=k$.

Finally, it follows by the definition and the above remarks that W is non-singular with $W^{-1}=k^{-1}W^t$, hence $W^tW=kI_n$. We record these result below.

1.3 Proposition. If there exists a W(n,k), say, W, then the following must hold.

(i) If $n-1=k$, then n is even;

(ii) if $n=k$, then n is 1, 2, or a multiple of 4;

(iii) if n is odd, then k is a square;

(iv) if 2 exactly divides n, then k is the sum of two squares; and

(v) W is non-singular and $WW^t=W^tW=kI_n$.

1.3. Complex Weighing Matrices

There are many useful generalizations of weighing matrices. We note the following two special cases, where we have used A^* to denote the conjugate (Hermitian) transpose of a complex matrix A.

1.4 Definition. Let $G=\{\exp(\frac{2\pi i m}{p}) : 0 \le m < p\}$, and let W be a $(0,G)$-matrix of order n such that $WW^*=kI_n$. In this case, we say that W is a Butson weighing matrix of order n and weight k, and we write bw$(n,k;p)$ to denote this property.

1.5 Example. Let $\xi=\exp(\frac{2\pi i}{7})$, and let $H=[h_{ij}]$, for $0 \le i,j < 7$, be defined by $h_{ij}=\xi^{ij}$. It follows easily that H is a Butson weighing matrix (see **?**, §2.5); in fact, it is a Butson Hadamard matrix as $n=k$. To be concrete, the following is a bw$(7,7;7)$

$$\begin{pmatrix}0&0&0&0&0&0&0\\0&1&2&3&4&5&6\\0&2&4&6&1&3&5\\0&3&6&2&5&1&4\\0&4&1&5&2&6&3\\0&5&3&1&6&4&2\\0&6&5&4&3&2&1\end{pmatrix},$$

where the entries are taken to be the logarithms of ξ.

The next result, to be used later, is Lemma 2.8.5. of **?**.

1.6 Lemma. Let p be a prime, and let ξ be a complex primitive p-th root of unity. Then $\sum_{i=0}^{n} a_i \xi^i = 0$ for some $n < p$ and $a_i \in \mathbf{N}$ if and only if $n = p - 1$ and $a_0 = \cdots = a_n$.

Proof. Let $f(x) = \sum_{i=0}^{n} a_i \xi^i \in \mathbf{Z}[x]$. The minimal polynomial of ξ over $\mathbf{Q}$ is the p-th cyclotomic polynomial $h(x) = 1 + x + \cdots + x^{p-1}$. Therefore, since the degree of f is at most p, we conclude that if $f(\xi) = 0$, then it must be an integer multiple of h. ■

1.4. Difference Set Construction

1.5. Isomorphisms of Weighing Matrices

For the case in which $n = k$, we will need the following. Let W be some $\mathrm{W}(n, k)$, and let $P, Q \in \langle -1 \rangle \wr S_n$, i.e. P and Q are signed permutation matrices of order n.[b] It is then clear that PWQ is again a $\mathrm{W}(n, k)$. Let A be the collection of all $\mathrm{W}(n, k)$s. Then the so-called Hadamard equivalent matrices are found in all the intersections of the orbits of the left action and the orbits of the right action of $\langle -1 \rangle \wr S_n$ on A.

[b]Let H and $G \leq S_n$ be finite groups. Recall the wreath product $H \wr G$ is the semi-direct product $H^n \rtimes G$ where $g(h_0, \ldots, h_{n-1})g^{-1} = (h_{g^{-1}0}, \ldots, h_{g^{-1}n-1})$. See **?** and **?** for details.

Bibliography

§2. Balanced Generalized Weighing Matrices

In the previous section, we defined a weighing matrix as some matrix over $\{-1, 0, 1\}$. We then extended this definition to include those matrices over 0 together with the complex p-th roots of unity. More generally, we can have weighing matrices over any finite group.

Before we can do this, however, we need to extend the conjugate transpose to group matrices. To accomplish this, let $A = [a_{ij}]$ be some matrix over a finite group G. Take $\overline{A} = [a_{ij}^{-1}]$ to be the matrix obtained by taking the group inverse of the elements. Finally, define $A^* = \overline{A}^t$. We then have the following.

2.1 Definition. Let G be some finite group not containing the symbol 0, and let W be a $(0, G)$-matrix of order n. If $WW^* = kI_n$ modulo the ideal $\mathbf{Z}G$, then we say that W is a generalized weighing matrix of order n and weight k. We write $\mathrm{gw}(n, k; G)$ to denote this property.[c]

2.2 Example. A real $\mathrm{W}(n, k)$ is a $\mathrm{gw}(n, k; C_2)$, and a bwnkp is a $\mathrm{gw}(n, k; C_p)$, were C_p denotes the cyclic group of order p.

2.3 Example. The following is a $\mathrm{gw}(15, 7; C_3)$, where the nonzero elements are the logarithms of a generator of C_3.

$$\begin{bmatrix}
0&3&3&3&3&3&3&3&0&0&0&0&0&0&0\\
0&3&2&0&1&0&0&0&0&0&2&0&2&2&3\\
0&0&3&2&0&1&0&0&3&0&0&2&0&2&2\\
0&0&0&3&2&0&1&0&2&3&0&0&2&0&2\\
0&0&0&0&3&2&0&1&2&2&3&0&0&2&0\\
0&1&0&0&0&3&2&0&0&2&2&3&0&0&2\\
0&0&1&0&0&0&3&2&2&0&2&2&3&0&0\\
0&2&0&1&0&0&0&3&0&2&0&2&2&3&0\\
3&3&1&0&2&0&0&0&3&2&0&1&0&0&0\\
3&0&3&1&0&2&0&0&0&3&2&0&1&0&0\\
3&0&0&3&1&0&2&0&0&0&3&2&0&1&0\\
3&0&0&0&3&1&0&2&0&0&0&3&2&0&1\\
3&2&0&0&0&3&1&0&1&0&0&0&3&2&0\\
3&0&2&0&0&0&3&1&0&1&0&0&0&3&2\\
3&1&0&2&0&0&0&3&2&0&1&0&0&0&3
\end{bmatrix}$$

Our goal in introducing weighing matrices over arbitrary finite groups is to synthesis the ideas of weighing matrices and balanced incomplete block designs. We combine these concepts thus.

[c] Here, we are doing arithmetic over the group ring $\mathbf{Z}[G]$, and $\mathbf{Z}G$, as is customary, denotes integer multiples of $\sum_{g \in G} g$.

2.4 Definition. Let G be some finite group, and let A be a $v \times b$ $(0, G)$-matrix such that[d]

$$\text{(2.4.a)} \quad AA^* = rI_v + \frac{\lambda}{|G|}\left(\sum_{g \in G} g\right)(J_v - I_v),$$

for some positive integers r and λ, and such that there are k non-zero entries in every column. We then say that A is a generalized Bhaskar Rao design (henceforth GBRD), and we write GBRD$(v, k, \lambda; G)$ to denote this property. If we need to stress the remaining parameters, then we write GBRD$(v, b, r, k, \lambda; G)$.

Often it is helpful to give a combinatorial definition of GBRDs that is equivalent to the one just given. Again let $A = [a_{ij}]$ be a $v \times b$ $(0, G)$-matrix that has k nonzero entries in every column. If the multisets $\{a_{ik}a_{jk}^{-1} : a_{ik} \neq 0 \neq a_{jk}$ and $0 \leq k < b\}$, for $i, j \in \{0, \ldots, v-1\}$, $i \neq j$, have $\lambda/|G|$ copies of every group element in G, then we say that A is a GBRD$(v, k, \lambda; G)$.

A few things are rather immediate. If $\check{A}$ denotes the matrix obtained from A by changing each non-zero entry to 1, then condition (2.4.a) implies that $\check{A}$ is a also a BIBD. Conversely, a BIBD is a GBRD over the trivial group $\{1\}$.

Evidently, Fisher's inequality applies, hence $b \geq v$. Moreover, the necessary conditions of Proposition **??** also hold for the parameters of GBRDs. However, since we are now dealing with group matrices that are balanced with respect to the group, the next result is clear.

2.5 Proposition. Let $A = [a_{ij}]$ be a GBRD$(v, k, \lambda; G)$, and let $\phi : G \to H$ be some group epimorphism. Then $[\phi(a_{ij})]$ is a GBRD(v, k, λ) over H with the same parameters.

Again, the extremal case of Fisher's inequality presents interesting problems; futher, the case in which $v = k$ and $b = r$ is interesting as well.

2.6 Definition. A balanced generalized weighing matrix is a GBRD$(v, b, r, k, \lambda; G)$ in which $v = b$ (equiv. $k = r$). We use the denotation BGW$(v, k, \lambda; G)$. A BGW$(v, k, \lambda; G)$ in which $v = k = \lambda$ is called a generalized Hadamard matrix, and we denote this as GH(G, λ) where $\lambda = v/|G|$.

If $G = \text{EA}(q)$, the elementary abelian group of order q, in a GH(G, λ), then we write GH(q, λ) instead.[e]

A balanced weighing matrix is a BGW$(v, k, \lambda; \{-1, 1\})$.

2.7 Example. The generalized weighing matrix of Example 2.3 can be seen to be a BGW$(15, 7, 3, ; C_3)$.

[d]As before, we are computing over the ring $\mathbf{Z}[G]$.

[e]If $q = p^n$, for some prime p, then $\text{EA}(q) \simeq \underbrace{C_p \oplus \cdots \oplus C_p}_{n}$.

Our work from the previous section, namely, Lemma 1.6 yields the following.

2.8 Proposition. A Butson weighing matrix W is a BGW if and only if $\check{W}$ is a BIBD.

The remainder of this chapter will focus on BGW matrices and will follow Chapter 10 of **?** closely. We first present a few simple constructions, where we use GF(q) for the Galois field of order q. GF$(q)^+$ and GF$(q)^*$ denote, respectively, the additive and multiplicative groups of GF(q). Moreover, recall that GF$(q)^+ \simeq$ EA(q).

2.9 Proposition. Let GF$(q) = \{a_0, \ldots, a_{q-1}\}$, and define $H = [h_{ij}]$ of order q by $h_{ij} = a_i a_j$. Then H is a GH$(q, 1)$.

Proof. Let H be so defined, and let $i, j \in \{0, \ldots, q-1\}$, $i \neq j$. Observe $\sum_k (a_i a_k - a_j a_k) = (a_i - a_j) \sum_k a_k$, hence each group element appears precisely once in $\{h_{ik} h_{jk}^{-1} : 0 \leq k < q\}$. ■

2.10 Proposition. Let GF$(q) = \{a_0, \ldots, a_{q-1}\}$, and define $W = [w_{ij}]$ of order $q+1$ by

$$w_{ij} = \begin{cases} 0 & \text{if } i = j = 0; \\ 1 & \text{if } i = 0 \text{ or } j = 0\text{, but } i \neq j\text{; and} \\ a_{i-1} - a_{j-1} & \text{otherwise.} \end{cases}$$

Then W is a BGW$(q+1, q, q-1; \mathrm{GF}(q)^*)$.

Proof. Let $i, j \in \{1, \ldots, q\}$, $i \neq j$. Then, for $k \neq j$,

$$w_{ik} w_{jk}^{-1} = \frac{a_{i-1} - a_{k-1}}{a_{j-1} - a_{k-1}}^{-1} = \frac{a_{i-1} - a_{j-1}}{a_{j-1} - a_{k-1}} + 1.$$

As k ranges over $\{1, \ldots, q\} - \{j\}$, the difference $a_{j-1} - a_{k-1}$ ranges over GF$(q^*$. Since $w_{i0} = w_{j0} = 1$, the multiset $\{w_{ik} w_{jk}^{-1} : w_{ik} \neq 0 \neq w_{jk}$ and $0 \leq k < q+1\}$ contains each element of GF$(q)^*$ once. The remaining cases in which $i = 0$ or $j = 0$ are trivial. ■

2.11 Example. The matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \alpha & \beta & \alpha\beta \\ 1 & \beta & \alpha\beta & \alpha \\ 1 & \alpha\beta & \alpha & \beta \end{bmatrix}$$

is a GH$(G; 1)$, where $G = \langle \alpha, \beta : \alpha^2 = \beta^2 = 1 \rangle \simeq$ EA(4).

2.12 Example. A $\mathrm{BGW}(8,7,6;\mathrm{GF}(7)^*)$ formed from the proposition is given below

$$\begin{bmatrix} 0 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\ 3 & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 0 & 3 & 1 & 6 & 2 & 5 \\ 3 & 5 & 6 & 0 & 4 & 2 & 1 & 3 \\ 3 & 6 & 4 & 1 & 0 & 5 & 3 & 2 \\ 3 & 1 & 3 & 5 & 2 & 0 & 6 & 4 \\ 3 & 2 & 5 & 4 & 6 & 3 & 0 & 1 \\ 3 & 3 & 2 & 6 & 5 & 1 & 4 & 0 \end{bmatrix},$$

where the nonzero elements are the logarithms of some generator of $\mathrm{GF}(7)^*$. One can see that the matrix is skew-symmetric.

We present one final construction due to **?**, which we do not prove here, that yields what are called the classical family of BGWs; more than that, however, the matrices so constructed are what is termed ω-circulant, a simple generalization of cirulant matrices.

Let $G = \langle \omega \rangle$ be a finite cyclic group, and let $W = [w_{ij}]$ be a matrix over $\mathbf{Z}[G]$ with first row $(\alpha_0, \ldots, \alpha_{n-1})$. W is ω-circulant if and only if $w_{ij} = \alpha_{j-i}$ if $i \leq j$ and $w_{ij} = \omega\alpha_{j-i}$ if $i > j$, where the indices are calculated modulo n.

Finally, we are ready to present this very useful construction.

2.13 Proposition. Let q be a prime power, and let β be a primitive element of the extension of order d of the field GF(q). Further, take $m = (q^d - 1)/(q-1)$, and define $\omega = \beta^{-m} \in$ GF(q), i.e. the norm of β. Finally, we claim that the ω-circulant matrix with first row $(\mathrm{Tr}\beta^k)_{k=0}^{m-1}$ is a $\mathrm{BGW}(m, q^{d-1}, q^{d-1} - q^{d-2}; \mathrm{GF}(q)^*)$.[f]

2.14 Example. The matrix

$$\begin{bmatrix} 0&2&2&2&0&1&2&2&1&2&0&2&0 \\ 0&0&2&2&2&0&1&2&2&1&2&0&2 \\ 1&0&0&2&2&2&0&1&2&2&1&2&0 \\ 0&1&0&0&2&2&2&0&1&2&2&1&2 \\ 1&0&1&0&0&2&2&2&0&1&2&2&1 \\ 2&1&0&1&0&0&2&2&2&0&1&2&2 \\ 1&2&1&0&1&0&0&2&2&2&0&1&2 \\ 1&1&2&1&0&1&0&0&2&2&2&0&1 \\ 2&1&1&2&1&0&1&0&0&2&2&2&0 \\ 0&2&1&1&2&1&0&1&0&0&2&2&2 \\ 1&0&2&1&1&2&1&0&1&0&0&2&2 \\ 1&1&0&2&1&1&2&1&0&1&0&0&2 \\ 1&1&1&0&2&1&1&2&1&0&1&0&0 \end{bmatrix}$$

is an ω-circulant $\mathrm{BGW}(13,9,6;\mathrm{GF}(4)^*)$, where the nonzero elements are the logarithms of some generator of $\mathrm{GF}(4)^*$.

[f] Recall that for a finite field K with finite extension F of order d, the trace function is the linear epimorphism $F \to K$ defined by $\mathrm{Tr}_{F/K}(\alpha) = \alpha + \alpha^q + \cdots + \alpha^{q^{d-1}}$, for $\alpha \in F$. Furthermore, the norm function is the linear epimorphism $F \to K$ defined by $\mathrm{N}_{F/K}(\alpha) = \alpha^{\frac{q^d-1}{q-1}}$ for $\alpha \in F$. See **?** for details.

We present a few brief remarks on the conjugate transpose and similar operations on BGWs.

2.15 Proposition. If W is a BGW$(v,k,\lambda;G)$, then W^* is also a BGW$(v,k,\lambda;G)$.

Proof. Consider $W=[w_{ij}]$ as a matrix over the ring $\mathbf{Q}[G]$ so that it satisfies (2.4.a) over $\mathbf{Q}[G]$. Let $\pi:\mathbf{Q}[G]\to\mathbf{Q}[G]/\mathbf{Q}G$ be the natural ring epimorphism. If πW denotes the matrix $[\pi w_{ij}]$, then (2.4.a) becomes $(\pi W)(\pi W^*)=kI_v$, and hence $(\pi W)^{-1}=k^{-1}(\pi W^*)$. Therefore, $(\pi W^*)(\pi W)=kI_v$ so that $W^*W=kI_v+A$ for some $A=[a_{ij}\sum_{g\in G}g]$ over the ideal $\mathbf{Q}G$. Moreover, since $\check{W}^t$ is a BIBD, there exist integers a_g such that, for $i\neq j$, $\sum_k w_{ki}w_{kj}^{-1}=\sum_{g\in G}a_g g$ where $\sum_{g\in G}a_g=\lambda$. Evidently, then, $A=\frac{\lambda}{|G|}(\sum_{g\in G}g)(J_v-I_v)$, and the result follows. ■

2.16 Corollary. If W is a BGW$(v,k,\lambda;G)$ where G is abelian, then $\overline{W}$ and W^t are also BGW$(v,k,\lambda;G)$s.

Proof. Since the group is abelian, the map $g\mapsto g^{-1}$ is an automorphism; hence, by Proposition 2.5, $\overline{W}$ is also a BGW$(v,k,\lambda;G)$. Then, by the proposition, $(\overline{W})^*=W^t$ is also a BGW$(v,k,\lambda;G)$. ■

As before, we can impose an equivalence on the set A of all GBRD$(v,b,r,k,\lambda;G)$s. Specifically, the matrices that reside in the intersection of any orbit of the left action of $G\wr S_v$ on A with any orbit of the right action of $G\wr S_b$ on A are said to be monomially equivalent.

We conclude by altering somewhat Definition 2.4 as in (**?**, Part V).

2.17 Definition. let G be some finite group, and let A be a $v\times b$ $(0,G)$-matrix. If there is an element $c\in\mathbf{Z}[G]$ such that

(2.17.a) $AA^*=rI_v+c(J_v-I_v)$,

and if $\check{A}$ is a BIBD, then we say that A is a c-GBRD, or a c-GBRD if more precision required.

Of course, a c-GBRD is a GBRD precisely when $c=\frac{\lambda}{|G|}\left(\sum_{g\in G}g\right)$.

Bibliography

§3. Orthogonal Designs

Notes

1.

Bibliography