WEIGHING MATRICES: GENERALIZATIONS, RELATED CONFIGURATIONS

THOMAS PENDER
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THOMAS PENDER

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Dr. H. Kharaghani Supervisor	Professor	Ph.D.
Dr. W. Holzmann Examination Committee Member	Associate Professor Emeritus	Ph.D.
Dr. M. Walton Examination Committee Member	Professor	Ph.D.
Dr. J. Sheriff Chair, Examination Committee	Assistant Professor Associate Chair	Ph.D.

DEDICATION

To my wife and son who changed everything.

ABSTRACT

It is the purpose of this thesis to explore the relationships that exist between weighing matrices, including their generalizations, and various other combinatorial configurations.

Principally, it will be shown that any balanced generalized weighing matrix with entries from a finite abelian group is equivalent to the existence of families of commutative association schemes.

Additionally, novel constructions of related combinatorial configurations are presented such as balancedly splittable orthogonal designs and new families of balanced weighing matrices.

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Contents

LIST OF SYMBOLS

$\mathbf{N}$	The additive semigroup of natural numbers $\{0, 1, 2 \dots\}$.
$\mathbf{Z}$	The ring of integers $\{0, \pm 1, \pm 2, \dots\}$.
$\mathbf{Q}$	The rational number field $\{\frac{n}{m} : n, m \in \mathbf{Z}\}$.
$\mathbf{R}$	The real number field.
$\mathbf{C}$	The complex number field $\{x + y\sqrt{-1} : x, y \in \mathbf{R}\}$.
$\mathbf{Z}/n\mathbf{Z}$	The integers modulo n $\{\overline{0}, \dots, \overline{n-1}\}$.
GF(q)	The Galois field of q elements.
$\binom{X}{k}$	The collection of k-subsets of the set X.
$\mathbf{D} = (X, \mathscr{B})$	A balanced incomplete block design with point set X and block set $\mathscr{B}$.
BIBD(v, k, λ)	A balanced incomplete block design with associated parameters.
$\mathfrak{C}(\mathbf{D})$	Complementary design of $\mathbf{D}$.
$\mathfrak{R}(\mathbf{D})$	Residual design of $\mathbf{D}$.
$\mathfrak{D}(\mathbf{D})$	Derived design of $\mathbf{D}$.
$A(\mathbf{D})$	Incidence matrix of the design $\mathbf{D}$.
$[n, k, d, w]_q$-code	A linear code in GF(q^n) of dimension k, minimum distance d, and minimum weight w.
$\mathrm{A}_q(n, d, w)$	The maximum size of any $(n, M, d, w)_q$-code, i.e. $M \leq \mathrm{A}_q(n, d, w)$.
W(v, k)	A weighing matrix of order v and weight k.
BW$(v, k; n)$	A Butson weighing matrix of order v and weight k over the n-th roots of unity.
GBRD$(v, k, \lambda; G)$	A generalized Bhaskar Rao design over the finite group G with associated parameters.
BGW$(v, k, \lambda; G)$	A balanced generalized weighing matrix of order v, weight k, and index $\lambda/\lvert G\rvert$, over the finite group G.
$\mathfrak{X} = (X, \mathscr{R})$	An association scheme with point set X and relations $\mathscr{R}$.
$\mathscr{S}_{q,n}$	A generalized simplex code.
$\mathscr{H}_{q,n}$	A generalized Hamming code.
$A \otimes B$	The Kronecker product of A by B.

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Introduction

This preliminary chapter introduces the objects with which we will be working. We intend to move briskly, so only references to proofs and more in-depth discussions will be given. The following sections will define in turn: balanced incomplete block designs, error-correcting codes, weighing matrices and their generalizations, and finally association schemes.

§1. Balanced Incomplete Block Designs

This section presents some basic definitions and results about block designs that will be used throughout this work. Particular emphasis will be placed on matrix representations of such objects.

* * *

1.1. Definition and Necessary Conditions. A block design is simply a collection of subsets of a finite point set such that the points are regularly distributed amongst the subsets in question. There are numerous directions to take with this vague understanding, but we only require the following.

1.1. Definition. Let X be a set of cardinality v, called the set of varieties; and let $\mathscr{B} \subset \binom{X}{k}$, called the set of blocks, have order b. The ordered pair $\mathbf{D} = (X, \mathscr{B})$ is a *balanced incomplete* *block design* (henceforth BIBD) if there is a positive integer λ such that each 2-subset of varieties appears in λ blocks of $\mathscr{B}$.

The conditions placed on a finite set and a collection of its subsets in order to form a BIBD are quite strong, and we have at once the following result which can be shown using the elementary techniques of double counting (see Cameron, 1994, Chapter 2 for an abstract discussion).

1.2. Proposition. Let $\mathbf{D} = (X, \mathscr{B})$ be a BIBD with $|X| = v$, and $\mathscr{B} \subset \binom{X}{k}$ for which $|\mathscr{B}| = b$. Then:

(1.2.a) Every point of X occurs in $r = \frac{\lambda(v-1)}{k-1}$ blocks, and

(1.2.b) there are $b = \frac{vr}{k} = \frac{\lambda v(v-1)}{k(k-1)}$ blocks in $\mathscr{B}$.

1.3. Corollary. For the parameters v, k, λ of a BIBD, it must hold that

(1.3.a) $\lambda(v-1) \equiv 0 \bmod (k-1)$, and

(1.3.b) $\lambda v(v-1) \equiv 0 \bmod k(k-1)$.

If $\mathbf{D} = (X, \mathscr{B})$ is a BIBD with the parameters shown above, then we denote this property as BIBD(v, b, r, k, λ). As we have seen, however, the parameters b and r are expressible in terms of v, k, and λ; hence, we will usually shorten the denotation to BIBD(v, k, λ) whenever no confusion will arise.

Corollary 1.3 imposes some necessary conditions on the parameters of a BIBD. Our next result, due to Fisher (1940), is a strong necessary condition relating the number of points to the number of blocks of a BIBD, and it has far reaching consequences in the applications of designs to fields like statistics.

This most important result admits several interesting derivations employing techniques ranging from determinants to variance counting.[1]

1.4. Fisher's Inequality. Let $\mathbf{D} = (X, \mathscr{B})$ be a BIBD(v, b, r, k, λ). It follows that

(1.4.a) $b \geq v$.

The extremal case of Fisher's inequality is naturally very interesting and important. We single this case out thus.

1.5. Definition. Let $\mathbf{D} = (X, \mathscr{B})$ be some BIBD(v, b, r, k, λ). If $v = b$ (equiv. $k = r$), then we say that $\mathbf{D}$ is a *symmetric* balanced incomplete block design, or simply a symmetric design.

* * *

1.2. Related Configurations. Thus far, we have been thinking of designs strictly as subsets of some finite set. We can, however, broaden our view to include the following tool, and in so doing the theory of linear algebra can be brought to bear on the subject.

1.6. Definition. Let $\mathbf{D} = (\{x_0, \ldots, x_{v-1}\}, \{B_0, \ldots, B_{b-1}\})$ be a BIBD(v, b, r, k, λ), and let A be a $v \times b$ $(0, 1)$-matrix defined by

$$A_{ij} = \begin{cases} 1 & \text{if } x_i \in B_j \text{, and} \\ 0 & \text{if } x_i \notin B_j. \end{cases} \tag{1.6.a}$$

We call A the *incidence matrix* of the design.

1.7. Example. The following is the incidence matrix of a BIBD$(13, 9, 6)$

$$\begin{pmatrix}
0&0&1&0&1&1&1&1&1&0&1&1&1\\
1&0&0&1&0&1&1&1&1&1&0&1&1\\
1&1&0&0&1&0&1&1&1&1&1&0&1\\
1&1&1&0&0&1&0&1&1&1&1&1&0\\
0&1&1&1&0&0&1&0&1&1&1&1&1\\
1&0&1&1&1&0&0&1&0&1&1&1&1\\
1&1&0&1&1&1&0&0&1&0&1&1&1\\
1&1&1&0&1&1&1&0&0&1&0&1&1\\
1&1&1&1&0&1&1&1&0&0&1&0&1\\
1&1&1&1&1&0&1&1&1&0&0&1&0\\
0&1&1&1&1&1&0&1&1&1&0&0&1\\
1&0&1&1&1&1&1&0&1&1&1&0&0\\
0&1&0&1&1&1&1&1&0&1&1&1&0
\end{pmatrix}. \tag{1.7.a}$$

The next result is immediate. Note that we use I_n and J_n to denote the identity matrix and the all ones matrix, respectively, of n rows and n columns. Similarly, $\mathbf{1}_n$ and $\mathbf{0}_n$ will denote the column with n ones and the column with n zeros. For simplicity, the indices will at times be omitted.

1.8. Proposition. Let $\mathbf{D} = (X, \mathscr{B})$ be a BIBD(v, b, r, k, λ), and let A be a $v \times b$ $(0, 1)$-matrix. Then A is the incidence matrix of the design if and only if the following hold.

$$AA^t = rI_v + \lambda(J_v - I_v) \text{, and} \tag{1.8.a}$$

(1.8.b) $\mathbf{1}_v^t A = k\mathbf{1}_b^t$.

As balanced incomplete block designs can more generally be regarded as finite incidence structures,[2] there can be related a number of further such objects. For our purposes, we will be interested in the following.

1.9. Definition. Let $\mathbf{D} = (X, \mathscr{B})$ be a BIBD, and let $B \in \mathscr{B}$. Then the *complement design* $\mathfrak{C}(\mathbf{D})$ is the pair $(X, \binom{X}{k} \backslash \mathscr{B})$. If $\mathbf{D}$ is symmetric, we have that the *derived design* $\mathfrak{D}(\mathbf{D})$ is the pair $(B_0, \{B \cap B_0 : B \in \mathscr{B} \text{ and } B \neq B_0\})$, and the *residual design* $\mathfrak{R}(\mathbf{D})$ is the pair $(X \backslash B_0, \{B - B_0 : B \in \mathscr{B} \text{ and } B \neq B_0\})$. When convenient, we will simply denote the complement, derived, and residual designs as $\mathfrak{C}$, $\mathfrak{D}$, and $\mathfrak{R}$, respectively.

The following result shows why, in part, these substructures are interesting.

1.10. Proposition. Let $\mathbf{D} = (X, \mathscr{B})$ be a BIBD(v, b, r, k, λ). Then

(1.10.a) $\mathfrak{C}$ is a BIBD$(v, b, b - r, v - k, b - 2r + \lambda)$.

If $\mathbf{D}$ is symmetric, then we further have that

(1.10.b) $\mathfrak{D}$ is a BIBD$(k, b - 1, k - 1, \lambda, \lambda - 1)$, and

(1.10.c) $\mathfrak{R}$ is a BIBD$(v - k, b - 1, k, k - \lambda, \lambda)$.

1.11. Example. The residual design of (1.7.a) is given by

$$\text{(1.11.a)} \quad \begin{pmatrix} 0&1&0&1&1&1&1&1&0&1&1&1 \\ 1&1&1&0&0&1&0&1&1&1&1&1 \\ 1&1&1&1&1&0&1&1&1&0&0&1 \\ 1&0&1&1&1&1&1&0&1&1&1&0 \end{pmatrix};$$

while the derived design is given by

$$\text{(1.11.b)} \quad \begin{pmatrix} 0&0&1&0&1&1&1&1&1&0&1&1 \\ 1&0&0&1&0&1&1&1&1&1&0&1 \\ 1&1&0&0&1&0&1&1&1&1&1&0 \\ 0&1&1&1&0&0&1&0&1&1&1&1 \\ 1&0&1&1&1&0&0&1&0&1&1&1 \\ 1&1&0&1&1&1&0&0&1&0&1&1 \\ 1&1&1&0&1&1&1&0&0&1&0&1 \\ 1&1&1&1&0&1&1&1&0&0&1&0 \\ 0&1&1&1&1&1&0&1&1&1&0&0 \end{pmatrix}$$

* * *

1.3. Isomorphisms of Designs. We conclude this section by briefly discussing isomorphisms of designs.

1.12. Definition. Let $\mathbf{D}_1 = (X_1, \mathscr{B}_1)$ and $\mathbf{D}_2 = (X_2, \mathscr{B}_2)$ be two BIBDs with the same parameters, and let $f : X_1 \to X_2$ be some bijection. If $f(\mathscr{B}_1) = B_2$, then we say that f is an *isomorphism* and that the two designs are *isomporhic*. For the case in which $\mathbf{D}_1 = \mathbf{D}_2$, we say that f is an *automorphism*. The collection of all automorphisms of a design $\mathbf{D}$ forms a group under composition called the *automorphism group* of the design.

In practice, one is usually concerned with the actions of isomorphisms on the incidence matrices of designs. In particular, two BIBD(v, b, r, k, λ)s with incidence matrices A_1 and A_2 are isomorphic if and only if there is a permutation matrix P of order v and a permutation matrix Q of order b such that

(1.12.a) $PA_1Q = A_2$.[3]

1.13. Definition. As nothing essential is changed under the action of an isomorphism, one can then assume that the incidence matrix of a symmetric (v, k, λ) design has the following form

(1.13.a) $$\begin{pmatrix} \mathbf{0}_{v-k} & \mathfrak{R} \\ \mathbf{1}_k & \mathfrak{D} \end{pmatrix}.$$

We will say that such an incidence matrix is in *normal form*.

* * *

§2. Error-Correcting Codes

In this section, the definitions of linear error-correcting codes will be given. We then move on to consider the famous generalized Hamming and simplex codes. As these are the only family of codes that we require, this section will be brief. The interested reader is referred to the standard references of Huffman and Pless (2003) and MacWilliams and Sloane (1977) for a greater exposition of the subject.

* * *

2.1. Definitions. In essence a code is simply a finite collection of words over a given finite alphabet. There are many ways to formalize such a concept; for instance, we may consider a code to be a subset of functions from one finite set into another.

For the purposes at hand, however, we are interested in the case the alphabet is endowed with arithmetic sufficient to form a field in which case we may consider the code to be a linear subspace of an extension of the alphabet.

2.1. Definition. Let $H = (A \mid I_{n-k})$ be an $(n-k) \times n$ matrix over GF(q). The *linear code* $\mathscr{C}$ with parity check matrix H is given by $\mathscr{C} = \text{Null}(H) = \{x \in \text{GF}(q^n) : Hx^t = 0\}$, where the extension GF$(q^n)$ is interpreted here as a linear space over GF(q). We say that $\mathscr{C}$ is a linear $[n,k]_q$-code, where clearly $\dim(\mathscr{C}) = k$.

We have defined a linear code by its parity check matrix. An equivalent definition is to use the so-called *generator matrix* G. If $H = (A \mid I_{n-k})$ is the parity check matrix of the code, then $G = (I_k \mid -A^t)$. The code is then given by the linear span of the rows of G. Note that by construction, it follows that $GH^t = HG^t = O$.

The dual of a linear code $\mathscr{C}$ is given in the usual way by $\mathscr{C}^\perp = \{x \in \text{GF}(q^n) : xy^t = 0, \text{ for every } y \in \mathscr{C}\}$. If H and G are the parity check and generator matrices of $\mathscr{C}$, then G and H are the parity check and generator matrices of $\mathscr{C}^\perp$, respectively.

In Definition 2.1, there are two parameters explicitly given of a code, namely, the length n of the codewords and the dimension k of the linear space consisting of the codewords. Two more fundamental parameters for us are the minimum distance and the minimum weight defined thus.

2.2. Definition. Let $\mathscr{C}$ be a linear $[n,k]_q$-code, and let $x = x_0 \cdots x_{n-1}$ and $y = y_0 \cdots y_{n-1}$ be any two codewords of $\mathscr{C}$. Then:

(2.2.a) The *Hamming weight* of x is $\text{wt}(x) = \#\{i : x_i \neq 0\}$;

(2.2.b) the *Hamming distance* between x and y is $\text{dist}(x,y) = \#\{i : x_i \neq y_i\}$;

(2.2.c) the *minimum weight* of the code is $\text{wt}(\mathscr{C}) = \min_{x \in \mathscr{C} \setminus \{0\}} \text{wt}(x)$; and

(2.2.d) the *minimum distance* of the code is $\text{dist}(\mathscr{C}) = \min_{\substack{x,y \in \mathscr{C} \\ x \neq y}} \text{dist}(x,y)$.

If we wish to emphasize the distance $d = \mathrm{dist}(\mathscr{C})$ of a code, we write $[n, k, d]_q$-code.

The Hamming distance can easily be seen to form a metric[4] on GF(q^n). The next result is then clear (see Hill, 1986, Theorem 1.9).

2.3. Proposition. If $\mathscr{C}$ is any $[n, k, d]_q$-code, then

(2.3.a) for every $x, y \in \mathscr{C}$, it holds that $\mathrm{dist}(x, y) = \mathrm{wt}(x - y)$, hence $\mathrm{dist}(\mathscr{C}) = \mathrm{wt}(\mathscr{C})$.

* * *

2.2. The Hamming and Simplex Codes. The theory of error-correcting codes began with the seminal paper of Richard W. Hamming (1950) who introduced redundancy for the purpose of error correction. In this paper, he also constructed a most useful family of codes aptly called the *Hamming codes*. The codes constructed by Hamming were binary; however, it is a simple matter to extend the construction to an arbitrary finite field.

2.4. Definition. Let H be the $n \times (q^n - 1)/(q - 1)$ matrix over GF(q) whose columns are representatives of the nonzero 1-dimensional subspaces of the extension field GF(q^n) over GF(q). The linear code $\mathscr{H}_{q,n}$ whose parity check matrix is H is called a *Hamming code*. The linear code $\mathscr{S}_{q,n}$ whose generator matrix is H is called a *simplex code*.

2.5. Example. The following is the $\mathscr{S}_{3,2}$ code (transposed)

$$\text{(2.5.a)} \quad \begin{pmatrix} 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 \\ 1 & 1 & 1 & 2 & 2 & 2 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 1 & 2 & 1 & 2 & 0 \\ 0 & 2 & 1 & 1 & 0 & 2 & 2 & 1 & 0 \end{pmatrix}.$$

The following result is immediate.

2.6. Proposition. Let q be a prime power, $n > 1$, and let $v = (q^n - 1)/(q - 1)$. Then:

(2.6.a) $\mathscr{S}_{q,n}$ is a linear $[v, n]_q$-code, and

(2.6.b) $\mathscr{H}_{q,n}$ is a linear $[v, v - n]_q$-code.

It can be shown that $\mathscr{H}_{q,n}$ is a linear single-error correcting code. In what follows, however, the simplex code will play a central role. The reader may consult Theorem 3.9.27 of Ionin and Shrikhande (2006) for a derivation of the next result.

2.7. Theorem. Let q be a prime power, $n > 1$, and let $v = (q^n - 1)/(q - 1)$. Then a linear $[v, n]_q$-code $\mathscr{C}$ is a $\mathscr{S}_{q,n}$ code if and only if $\mathrm{wt}(x) = q^{n-1}$, for every $x \in \mathscr{C}$.

§3. Weighing Matrices and Their Generalizations

This section focuses on a third combinatorial configuration, namely, weighing matrices together with their generalizations. We begin with a brief look at weighing matrices themselves before moving onto to consider two generalizations. The first generalization is allowing the entries of a weighing matrix to be chosen from a finite group. This idea is then synthesized with the ideas of §1. Finally, we allow the entries of a weighing matrix to be taken from sets of indeterminates and consider the utility of such an approach.

* * *

3.1. Weighing Matrices. We begin with the following definition.

3.1. Definition. Let W be a $(-1, 0, 1)$-matrix of order n. W is a *weighing matrix* of *order* n and *weight* k if

(3.1.a) $WW^t = kI_n$.

If $n = k$, then we say that W is a *Hadamard matrix*. If $n - 1 = k$, then we say that W is a *conference matrix*. In any event, we write $\mathrm{W}(n, k)$ to denote this property.

3.2. Example. A $\mathrm{W}(13, 9)$

$$
(3.2.a)\quad \begin{pmatrix}
0&0&-&0&-&1&-&-&1&0&-&-&- \\
1&0&0&-&0&-&1&-&-&1&0&-&- \\
1&1&0&0&-&0&-&1&-&-&1&0&- \\
1&1&1&0&0&-&0&-&1&-&-&1&0 \\
0&1&1&1&0&0&-&0&-&1&-&-&1 \\
-&0&1&1&1&0&0&-&0&-&1&-&- \\
1&-&0&1&1&1&0&0&-&0&-&1&- \\
1&1&-&0&1&1&1&0&0&-&0&-&1 \\
-&1&1&-&0&1&1&1&0&0&-&0&- \\
1&-&1&1&-&0&1&1&1&0&0&-&0 \\
0&1&-&1&1&-&0&1&1&1&0&0&- \\
1&0&1&-&1&1&-&0&1&1&1&0&0 \\
0&1&0&1&-&1&1&-&0&1&1&1&0
\end{pmatrix}.
$$

There are many useful generalizations of weighing matrices. We will take this up generally in the following sections, but for now we note the following special case. Note that we use A^* to denote the conjugate (Hermitian) transpose of a complex matrix A.

3.3. Definition. Let $G = \{\exp(\frac{2\pi i m}{p}) : 0 \leq m < p\}$, and let W be a $(0, G)$-matrix of order v. We say that W is a *Butson weighing matrix* of *order* v and *weight* k if

(3.3.a) $WW^* = kI_n$,

and we write $\mathrm{BW}(v, k; p)$ to denote this property.

A result that is useful in studying properties of complex weighing matrices is the following that can be found in Lam and Leung (2000).

3.4. Lemma. Let p be a prime, and let ξ be a primitive complex p-th root of unity. Then $\displaystyle\sum_{i=0}^{n} a_i \xi^i = 0$ for some $n < p$ and $a_i \in \mathbf{N}$ if and only if $n = p - 1$ and $a_0 = \cdots = a_n$.

We now move on to consider some further generalizations of a weighing matrix.

* * *

3.2. Generalized Bhaskar Rao Designs. In the previous section, we defined a weighing matrix as a square matrix over $\{-1, 0, 1\}$. We then extended this definition to include those matrices over $\{0\}$ together with the complex p-th roots of unity. More generally, we can have weighing matrices over any finite group.

Before we can do this, however, we need to extend the conjugate transpose to group matrices. To accomplish this, let A be some matrix over a finite group G, and define $\overline{A}$ by $\overline{A}_{ij} = A_{ij}^{-1}$, that is, the matrix obtained by taking the group inverse of the nonzero entries of A. Finally, define $A^* = \overline{A}^t$. We then have the following.

3.5. Definition. Let G be some finite group, and let A be a $v \times b$ $(0, G)$-matrix considered as a matrix over $\mathbf{Z}[G]$ and such that

(3.5.a) $$AA^* = rI_v + \frac{\lambda}{|G|}\left(\sum_{g \in G} g\right)(J_v - I_v),$$

for some positive integers r and λ, and such that there are k nonzero entries in every column. We then say that A is a *generalized Bhaskar Rao design* (henceforth GBRD), and we write $\mathrm{GBRD}(v, k, \lambda; G)$ to denote this property. If we need to stress the remaining parameters, then we write $\mathrm{GBRD}(v, b, r, k, \lambda; G)$.

It is clear that replacing the nonzero entries of a GBRD with unity furnishes a BIBD. Hence, we see that Fisher's Inequality applies. Again we single out the extremal case of the inequality.

3.6. Definition. A *balanced generalized weighing matrix* is a $\mathrm{GBRD}(v, b, r, k, \lambda; G)$ in which $v = b$ (equiv. $k = r$). We use the denotation $\mathrm{BGW}(v, k, \lambda; G)$. A $\mathrm{BGW}(v, k, \lambda; G)$ in which $v = k$ is called a *generalized Hadamard matrix*, and we denote this as $\mathrm{GH}(G, \lambda)$ where $\lambda = v/|G|$. If $G = \mathrm{EA}(q)$, the elementary abelian group[5] of order q, then we write $\mathrm{GH}(q, \lambda)$ instead.

Using Lemma 3.4, we see that a Butson weighing matrix over prime complex roots of unity forms a BGW precisely in the case that the nonzero entries yield the incidence matrix of a symmetric design.

3.7. Example. Taking the absolute values of the entries of (3.2.a) yields the symmetric design (1.7.a).

BGW matrices are quite useful in the construction of other combinatorial designs (see, for example, Ionin and Kharaghani 2003a and 2003b), and satisfy a number of properties. For a more detailed discussion of BGW matrices not considered in this thesis, the interested reader may consult Ionin and Shrikhande (2006).

* * *

3.3. Isomorphisms of BGW matrices. As before, we can impose an equivalence on the set of all $v \times b$ $(0, G)$-matrices, which will play an important part in what is to come.

3.8. Definition. Two $v \times b$ $(0, G)$-matrices A_1 and A_2 are said to be *monomially equivalent* if there are monomial $(0, G)$-matrices P and Q of orders v and b, respectively, such that

(3.8.a) $PA_1Q = A_2$.[6]

In order to extend normality to BGW matrices, we begin by altering somewhat Definition 3.5 as in Part V of Colbourn and Dinitz (2007).

3.9. Definition. let G be some finite group, and let A be a $v \times b$ $(0, G)$-matrix. If A has k non-zero entries in every column, and if there is an element c in the integral group ring $\mathbf{Z}[G]$ such that

(3.9.a) $AA^* = rI_v + c(J_v - I_v)$,

then we say that A is a c-GBRD. In what follows, we will always take $c = \frac{\lambda}{|G|}G - 1$ and denote this property as c-GBRD$(v, b, r, k, \lambda - 1; G)$.

We can now properly extend the idea of normality to BGW matrices.

3.10. Definition. A BGW$(v, k, \lambda; G)$ is said to be in *normal form* if it has the form

$$\text{(3.10.a)} \quad \begin{pmatrix} \mathbf{0}_{v-k} & \mathfrak{R} \\ \mathbf{1}_k & \mathfrak{D} \end{pmatrix}.$$

Note that of necessity, $\mathfrak{R}$ is a GBRD$(v-k, v-1, k, k-\lambda, \lambda; G)$ and $\mathfrak{D}$ is a c-GBRD$(k, v-1, k-1, \lambda, \lambda-1; G)$. These have the parameters of the residual and derived designs of a symmetric BIBD(v, k, λ), hence we call them, respectively, a residual GBRD and a derived c-GBRD.

* * *

3.4. Orthogonal Designs. In the previous subsection, we covered one generalization of weighing matrices, namely, we allowed the nonzero entries to come from a finite group. In this subsection, we pursue another generalization in a different direction. In particular, instead of allowing the nonzero entries to come from some group, we will take the nonzero entries to be real, complex, or quaternion[7] indeterminates.

We begin with the case of real indeterminates.

3.11. Definition. Let $x_1, \ldots, x_u$ be real, commuting indeterminates, and let X be an $n \times n$ matrix with entries from $\{0, \pm x_1, \ldots, \pm x_u\}$. We say that X is an *orthogonal design* if

$$\text{(3.11.a)} \quad XX^t = \left(\sum_i s_i x_i^2\right) I_n.$$

We say that the orthogonal design is of order n and type $(s_1, \ldots, s_u)$, and we write X is an OD$(n; s_1, \ldots, s_u)$. If $\sum_i s_i = n$, then we say that the OD is *full*.

3.12. Example. The following are an OD$(2; 1, 1)$ and an OD$(4; 1, 1, 1, 1)$, respectively, (3.12.a) $\begin{pmatrix} a & b \\ \overline{b} & a \end{pmatrix}, \begin{pmatrix} a & b & c & d \\ \overline{b} & a & \overline{d} & c \\ \overline{c} & d & a & \overline{b} \\ \overline{d} & \overline{c} & b & a \end{pmatrix}.$

We now extend Definition 3.11 to the case of complex and quaternion elements where we use $\cdot^*$ to denote conjugation over the complex or quaternion elements and also to denote conjugate transposition of matrices.

3.13. Definition. Let $z_1, \ldots, z_u$ be complex, commuting indeterminates, and let X be a matrix of order n with entries from $\{0, \varepsilon_1 z_1, \ldots, \varepsilon_u z_u, \eta_1 z_1^*, \ldots, \eta_u z_u^*\}$, where each $\varepsilon_\ell, \eta_\ell \in \{\pm 1, \pm i\}$ (resp. $\varepsilon_\ell, \eta_\ell \in \{\pm 1, \pm i, \pm j, \pm k\}$). In the event that

(3.13.a) $X^* X = X X^* = \left(\sum_\ell s_\ell |z_\ell|^2 \right) I_n,$

then we say that X is a *complex* (resp. *quaternion*) orthogonal design of type $(s_1, \ldots, s_u)$. We write X is a COD$(n; s_1, \ldots, s_u)$ (resp. QOD$(n; s_1, \ldots, s_u)$).

Equivalence of orthogonal designs is the same as for BGW matrices where the monimal matrices have nonzero entries in set $\{\pm 1, \pm i, \pm j, \pm k\}$, and where we also allow a permutation of symbols.

* * *

3.5. Sequences and Circulants. Let A be an $n \times n$ matrix over a ring with an involution $\cdot^*$ which can be extended to define the conjugate transpose of a matrix and with first row $(a_0, \ldots, a_{n-1})$. Recall that A is *circulant* if $A_{ij} = a_{j-i}$, where the indices are calculated modulo n. In this way, the entire matrix is determined by its first row; moreover, if A and B are two circulants of the same dimension, then A^*, $A + B$, and AB are also circulant matrices.

What we are particularly interested with here is the following.

3.14. Definition. Let $\mathcal{A} = \{A_i\}$ be a finite collection of circulant matrices of the same dimension over a ring R endowed with an involution $\cdot^*$. The collection $\mathcal{A}$ is said to be *complementary* if

(3.14.a) $\sum_i A_i A_i^* = aI$, for some $a \in R$,

where, as usual, $(m_{ij})^* = (m_{ji}^*)$.

Note, however, that since circulant matrices are entirely characterized by their first rows, we can state this in terms of sequences. But first, a definition.

3.15. Definition. Let $a_0 = (a_{0,0}, \ldots, a_{0,n-1})$ be a sequence in a ring R with the involution $\cdot^*$. The j*-th aperiodic* and j*-th periodic autocorrelations* of the sequence a are given respectively by

(3.15.a) $N_j(a) = \sum_{i=0}^{n-j-1} a_{0,i}a^*_{0,i+j}$, and

(3.15.b) $P_j(a) = \sum_{i=0}^{n-1} a_{0,i}a^*_{0,i+j}$, indices calculated modulo n.

If $a_1 = (a_{1,0}, \ldots, a_{1,n-1}), \ldots, a_m = (a_{m,1}, \ldots, a_{m,n-1})$ are any other sequence in R, then $a_0, \ldots, a_m$ are *complementary* if

(3.15.c) $\sum_i P_j(a_i) = 0$, for every $j \in \{1, \ldots, n-1\}$.

We see immediately that $P_j(a) = N_j(a) + N_{n-j}(a)^*$; hence, if $N_j(a) + N_j(b) = 0$, for every $j \in \{1, \ldots, n-1\}$, then a and b are complementary. However, vanishing periodic autocorrelations do not in general imply vanishing aperiodic autocorrelations.

The importance of the periodic correlation is given by the fact that if the first row of the circulant A continues to be $a = (a_0, \ldots, a_{n-1})$, then the first row of AA^* is given by $(\sum_i |a_i|^2, P_{n-1}(a), \ldots, P_1(a))$. So we see that complementary circulants and complementary sequences are one and the same.

Complementary sequences and orthogonal designs are connected in an intimate way; in fact, complementary sequences offer many elegant constructions of ODs. To make the connection precise, we need to allow sequence elements to be indeterminates—whether real or complex—and where the involution is taken to be conjugation.

3.16. Proposition. Let $\{z_1, \ldots, z_u\}$ be commuting real or complex indeterminates, and let $a = (a_0, \ldots, a_{n-1})$ and $b = (b_0, \ldots, b_{n-1})$ be complementary sequences with entries from $\{0, \varepsilon_0 z_0, \ldots, \varepsilon_u z_u\}$, where each $\varepsilon_\ell \in \{\pm 1, \pm i, \pm j, \pm k\}$, such that $\sum_i (|a_i|^2 + |b_i|^2) = \sum_i s_i |z_i|^2$. Then

(3.16.a) $\begin{pmatrix} A & B \\ -B^* & A^* \end{pmatrix}$

is a QOD$(2n; s_1, \ldots, s_u)$, where A and B are the circulants with first rows a and b, respectively.

A matrix similar to (3.16.a) will feature as a submatrix in our later work where the sequences will be composed of matrices. We will need one further idea before we proceed.

3.17. Definition. Let $a = (a_0, \ldots, a_{n-1})$ and $b = (b_0, \ldots, b_{n-1})$ be two sequences over a ring with involution $\cdot^*$. The *j-th cross-correlation* of a by b is given by

(3.17.a) $C_j(a, b) = \sum_{i=0}^{n-1} a_i b^*_{i+j}$, for each $j \in \{1, \ldots, n-1\}$.

If A has first row $a = (a_0, \ldots, a_{n-1})$, and if B has first row $b = (b_0, \ldots, b_{n-1})$, then the first row of the circulant AB^* is $(\sum_i a_i b^*_i, C_{n-1}(a, b), \ldots, C_1(a, b))$. Note, however, that $C_j(a, b)$ is not in general equal to $C_j(b, a)$. So, care must be taken in extending Proposition 3.16 since amicability, i.e. $AB^t = BA^t$, is required in maintaining orthogonality when substituting into an OD.

Much more can be said about this most useful topic. The interested reader may consult Seberry (2017) and Seberry and Yamada (1992) for a wealth of material.

§4. Association Schemes

This fourth and final preliminary section briefly touches on association schemes, a fundamental abstract object used as a unifying tool across the otherwise disparate fields of combinatorics.

* * *

4.1. Definition. We begin with the following definition.

4.1. Definition. Let X be a finite set of v elements, and let $\mathscr{R} = \{R_0, \ldots, R_d\}$ be a collection of relations on X. We say that the ordered pair $\mathfrak{X} = (X, \mathscr{R})$ is an *association scheme* with d *classes* whenever the following are satisfied.

(4.1.a) $R_0 = \{(x, x) : x \in X\}$;

(4.1.b) $R_i \cap R_j = \emptyset$, for $i \neq j$, and $X \times X$ is the disjoint union of $R_0, \ldots, R_d$;

(4.1.c) $R^t_i = R_{i'}$ for some $i' \in \{0, \ldots, d\}$, where $R^t_i = \{(x, y) : (y, x) \in R_i\}$; and

(4.1.d) Given $(x, y) \in R_k$, the number of $z \in X$ such that $(x, z) \in R_i$ and $(z, y) \in R_j$ is a constant p^k_{ij}. We call the p^k_{ij}, the *intersection numbers* of the scheme.

The scheme $\mathfrak{X}$ is *commutative* if

(4.1.e) $p^k_{ij} = p^k_{ji}$, for all $i, j, k \in \{0, \dots, d\}$.

The scheme is *symmetric* if

(4.1.f) $i = i'$, for every $i \in \{0, \dots, d\}$, where i' is given by (4.1.c).

We denote p^0_{ii} as k_i, the *valency* of the relation R_i.

In what follows, we will assume that the association schemes we are working with are always commutative.

* * *

4.2. Adjacency Algebras. The importance of association schemes resides in the following equivalent definition.

4.2. Definition. Let $\mathfrak{X} = (X, \mathscr{R})$ be a d-class association scheme. For $i \in \{0, \dots, d\}$, define the $v \times v$ $(0, 1)$-matrix A_i with rows and columns indexed by elements of X by $(A_i)_{xy} = 1$ if and only if $(x, y) \in R_i$. We call A_i the *adjacency matrix* of the relation R_i. Then Definition 4.1 is equivalent to the following.

(4.2.a) $A_0 = I$;

(4.2.b) $A_0 + \cdots + A_d = J$;

(4.2.c) $A^t_i = A_{i'}$ for some $i' \in \{0, \dots, d\}$; and

(4.2.d) $A_i A_j = \sum_k p^k_{ij} A_k$, for every $i, j \in \{0, \dots d\}$.

If $\mathfrak{X}$ is commutative, then

(4.2.e) $A_i A_j = A_j A_i$, for each $i, j \in \{0, \dots, d\}$.

If $\mathfrak{X}$ is symmetric, then

(4.2.f) $A^t_i = A_i$, for every $i \in \{0, \dots, d\}$.

By (4.2.b), the adjacency matrices of the scheme are $\mathbf{C}$-linearly independent and generate a subspace $\mathfrak{U}$ of the matrix space $\mathrm{Mat}_v(\mathbf{C})$ of dimension $d+1$. By (4.2.d), $\mathfrak{U}$ is closed under standard matrix multiplication. We call $\mathfrak{U}$ the *adjacency algebra* of the association scheme $\mathfrak{X}$.

We further have that $A_i \circ A_j = \delta_{ij} A_i$ where $\cdot \circ \cdot$ denotes Schur, i.e. componentwise multiplication. Therefore, $A_0, \ldots, A_d$ also generate a comutative algebra $\hat{\mathfrak{U}}$ with Schur multiplication for which they are primitive idempotents.[8] We therefore also call the adjacency matrices the *Schur idempotents* of the scheme.

Assume an ordering of $X = \{x_0, \ldots, x_{v-1}\}$, and take e_{x_i} to be the standard vector with i-th position equal to 1 and 0 elsewhere. Take V to be the Hermitian space with the standard orthonormal basis $\{e_x : x \in X\}$.

Since we are assuming commutativity, the adjacency matrices $A_0, \ldots, A_d$ are pairwise commuting, normal matrices. So, they share an eigenbasis, and by the Spectral Theorem for normal matrices, $V = \bigoplus_{i=1}^r V_i$, where the V_i are maximal common eigenspaces.

Since $J = \sum_i A_i$, we find the eigenspace corresponding to the eigenvalue v is spanned by $\mathbf{1}_v$, i.e. it is 1-dimensional and hence maximal. It follows that this space is equal to V_i for some i. We can assume, then, that $i = 0$.

If we take E_i to be the orthogonal projection $V \to V_i$ with respect to the basis $\{e_x : x \in X\}$, then we can assume that $E_0 = |X|^{-1} J$, and we have $E_0 + \cdots + E_d = I$. Moreover, there is a unitary matrix Λ such that $\Lambda^* E_i \Lambda = \mathrm{diag}(0, \ldots, 0, \underbrace{1, \ldots, 1}_{m_i}, 0, \ldots, 0)$, where we have used m_i to denote $\dim(V_i)$. The numbers m_i, for $i \in \{0, \ldots, d\}$, are called the *multiplicities* of the scheme.

It can be shown (see Bannai and Ito, 1984, Theorem 3.1) that the projection matrices $E_0, \ldots, E_d$ are primitive idempotents of $\mathfrak{U}$ and form a dual basis of $\mathfrak{U}$. It follows that there are constants P_{ij} and Q_{ij}, for $i, j \in \{0, \ldots, d\}$, called the *eigenvalues* and *dual-eigenvalues* of the scheme, such that $A_j = \sum_i P_{ij} E_i$ and $E_j = |X|^{-1} \sum_i Q_{ij} A_i$. Using these constants, we form the matrices P and Q with (i, j)-th entry given by P_{ij} and Q_{ij}, respectively, and we call these matrices the *first* and *second character tables* of the scheme. By what has been said, we have at once that $PQ = QP = vI$.

* * *

4.3. Intersection Matrices. If we regard left multiplication of $\mathfrak{U}$ as linear transformations of $\mathfrak{U}$ and express them in terms of the basis $\{A_0, \ldots, A_d\}$, then we have an algebra homomorphism $\mathfrak{U} \to \mathrm{Mat}_v(\mathbf{C})$ called the *left regular representation of* $\mathfrak{U}$ with respect to $\{A_0, \ldots, A_d\}$.

For $i \in \{0, \ldots, d\}$, define the matrix B_i by $(B_i)_{jk} = p_{ij}^k$, called the *i-th intersection matrix*. It then follows by (4.2.d) that the image of A_i under the above homomorphism is B_i^t, whence $A_i \leftrightarrow B_i^t$ is an isomorphism. Since, however, we are assuming that our schemes are commutative, transposition is an isomorphism; thus, $A_i \leftrightarrow B_i$ is an isomorphism. If we denote $\mathfrak{B} = \langle B_0, \ldots, B_d \rangle$, then we have shown $\mathfrak{U} \simeq \mathfrak{B}$.

That the intersection matrices and the isomorphism given above are important for us is shown in the next result. First, however, we say that a vector is in *standard form* if the first entry is 1.

4.3. Theorem. Let $\mathfrak{X} = (X, \mathscr{R})$ be a commutative association scheme with d classes. Let $v_i = (P_{i0}, \ldots, P_{id})$ and $u_i = (\overline{P_{i0}}/k_0, \ldots, \overline{P_{id}}/k_d)$ be the row vectors obtained by standardizing the first row and column of P and Q, respectively. Then u_i is the unique standardized common left eigenvector u of the matrices B_j such that $uB_j = P_{ij}u$. Similarly, v_i^t is the common right eigenvector v^t of the matrices B_j such that $B_j v^t = P_{ij} v^t$.

As stated in Bannai and Ito (1984), by virture of this result, one can conceivably derive the character tables P and Q from the intersection matrices. In particular, if the algebra $\mathfrak{B}$ is generated by a single B_j, we can use the single matrix B_j to calculate the character tables. Following Lemma 2.2.1 of Brouwer et al. (1989), the authors go on to point out that if some B_j has $d+1$ distinct eigenvalues, then we can similarly use this intersection matrix. Of course, if $\mathfrak{B}$ is cyclic, then this property is satisfied by the generator matrix.

Notes

1. This technique is famously used in the justification of Hoffman's co-clique bound (see Brouwer et al., 1989, Proposition 1.3.2). In Cameron (1994) and Cameron and van Lint (1991), it is used to great effect in the context of designs.

2. By an incidence structure is meant a triple $S = (X, \mathscr{B}, I)$, where $I \subseteq X \times \mathscr{B}$. We say that X is the point set, $\mathscr{B}$ the block set, and I the set of flags. If $(x, B) \in I$, then the point x and block B are said to be incident. See Batten (1997) and Dembowski (1997) for a standard treatment of incidence. See Beth et al. (1999) for study of incidence in the context of design theory.

3. This is more generally given as the orbits of the action of $S_v \times S_b$ on the set of binary $v \times b$ matrices defined by $A \cdot (P, Q) = P^t AQ$, where the transposition is necessary in order to properly define a right action. Note that we are using the notation given in the seminal work Cameron (1999).

4. Recall that a metric space is a pair (X, ϱ), where X is a nonempty set, and where $\varrho : X \to \mathbf{R}_{\geq 0}$ is a map satisfying, for all $x, y, z \in X$, (a) $\varrho(x, y) \geq 0$ with equality iff $x = y$; (b) $\varrho(x, y) = \varrho(y, x)$; and (c) $\varrho(x, y) \leq \varrho(x, z) + \varrho(z, y)$. We then say that ϱ is a metric.

5. If $q = p^n$, for some prime p, then $\mathrm{EA}(q) \simeq \underbrace{C_p \times \cdots \times C_p}_{n}$ where C_p is the cyclic group of order p.

6. Let H and $K \leq S_v$ be finite groups. Recall that the wreath product $H \wr K$ is the semi direct product $H^v \rtimes K$ where $k(h_0, \ldots, h_{v-1})k^{-1} = (h_{k^{-1}0}, \ldots, h_{k^{-1}(v-1)})$. Let $H_1 = G \wr S_v$, let $H_2 = G \wr S_b$, and let $H = H_1 \times H_2$. We define an action of H on the set of all $v \times b$ $(0, G)$-matrices by $A \cdot (P, Q) = P^* AQ$. The monomially equivalent matrices are composed precisely by the orbits of this action.

7. Recall that the quaternion group $\{\pm 1, \pm i, \pm j, \pm k\}$ is defined with anticommutative multiplication given by $i^2 = j^2 = k^2 = -1$, $ij = k, jk = i, ki = j$. We then take the quaternion algebra to be the $\mathbf{R}$-algebra $\{a_1 + a_i i + a_j j + a_k k : a_1, a_i, a_j, a_k \in \mathbf{R}\}$. Conjugation is extended to the quaternion algebra by $(a_1 + a_i i + a_j j + a_k k)^* = a_1 - a_i i - a_j j - a_k k$.

8. The algebra $\hat{\mathfrak{U}}$ is in many senses the dual of the algebra $\mathfrak{U}$. This is explored in Bannai and Ito (1984), Bannai et al. (2021), and Delsarte (1973).

2

Balancedly Splittable Orthogonal Designs

This is the first chapter constituting the novel results of this work. Here balancedly splittable orthogonal designs are defined and various constructions are presented. These most interesting objects are connected to several objects such as frames and pairs of unbiased orthogonal designs. Aside from §5, the results presented here are shown in Kharaghani et al. (2021).

§5. Balancedly Splittable Hadamard Matrices

The results of this section are due to Kharaghani and Suda (2019). We include this material here to evince the fact that the concept of a balancedly splittable orthogonal design is a generalization of previous work and to provide a more intuitive introduction to the concept.

* * *

5.1. Definition. Recall that a Hadamard matrix is a weighing matrix $\mathrm{W}(n, n)$ (see §3). These elusive objects have vexed combinatorialists for over a century now. Ever more clever techniques from ever more branches of mathematics are needed in order to construct these objects.

We will consider one such construction in preparation for our study of orthogonal designs. The matrices we will study are the so-called balancedly splittable Hadamard matrices. First, an example.

5.1. Example. Consider a Hadamard matrix of order 4 shown below

$$\text{(5.1.a)} \quad \begin{pmatrix} + & + & + & + \\ + & + & - & - \\ + & - & + & - \\ + & - & - & + \end{pmatrix}.$$

Label the rows h_0, h_1, h_2, and h_3. We then form the block matrix with (i, j)-th entry given by $h_j^t h_i$.

$$\text{(5.1.b)} \quad \begin{pmatrix}
\mathbf{+} & \mathbf{+} & \mathbf{+} & \mathbf{+} & \mathbf{+} & \mathbf{+} & \mathbf{+} & \mathbf{+} & \mathbf{+} & \mathbf{+} & \mathbf{+} & \mathbf{+} & \mathbf{+} & \mathbf{+} & \mathbf{+} & \mathbf{+} \\
\mathbf{+} & \mathbf{+} & \mathbf{+} & \mathbf{+} & \mathbf{+} & \mathbf{+} & \mathbf{+} & \mathbf{+} & \mathbf{-} & \mathbf{-} & \mathbf{-} & \mathbf{-} & \mathbf{-} & \mathbf{-} & \mathbf{-} & \mathbf{-} \\
\mathbf{+} & \mathbf{+} & \mathbf{+} & \mathbf{+} & \mathbf{-} & \mathbf{-} & \mathbf{-} & \mathbf{-} & \mathbf{+} & \mathbf{+} & \mathbf{+} & \mathbf{+} & \mathbf{-} & \mathbf{-} & \mathbf{-} & \mathbf{-} \\
\mathbf{+} & \mathbf{+} & \mathbf{+} & \mathbf{+} & \mathbf{-} & \mathbf{-} & \mathbf{-} & \mathbf{-} & \mathbf{-} & \mathbf{-} & \mathbf{-} & \mathbf{-} & \mathbf{+} & \mathbf{+} & \mathbf{+} & \mathbf{+} \\
+ & + & - & - & + & + & - & - & + & + & - & - & + & + & - & - \\
+ & + & - & - & + & + & - & - & - & - & + & + & - & - & + & + \\
+ & + & - & - & - & - & + & + & + & + & - & - & - & - & + & + \\
+ & + & - & - & - & - & + & + & - & - & + & + & + & + & - & - \\
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+ & - & - & + & + & - & - & + & - & + & + & - & - & + & + & - \\
+ & - & - & + & - & + & + & - & + & - & - & + & - & + & + & - \\
+ & - & - & + & - & + & + & - & - & + & + & - & + & - & - & +
\end{pmatrix}.$$

The matrix (5.1.b) has the submatrix H_1 which we have shown in bold above. This submatrix has the property that $H_1^t H_1 = 4I + 4A$, for some symmetric $(0, 1)$-matrix A with zero diagonal, namely, $A = I_4 \otimes (J_4 - I_4)$. In particular, there is only a single angle that exist between the columns of H_1.

The above example motivates the following definition.

5.2. Definition. A Hadamard Matrix H of order n is *balancedly splittable* with parameters (n, ℓ, a, b), where $a < b$, if H has an $\ell \times n$ submatrix H_1 such that

$$\text{(5.2.a)} \quad H_1^t H_1 = \ell I + aA + b(J - I - A),$$

for some symmetric $(0, 1)$-matrix A with zero diagonal. Additionally, balanced splittability can equally well be defined with respect to submatrices formed by subsets of the rows.

This definition and the previous example are suggestive. Notably, a connection to sets of biangular lines is inherent in the definition. These connections will be taken up in the following subsections.

* * *

5.2. Equiangular Lines. In the previous subsection, it was intimated that balancedly splittable Hadamard matrices were related to collections of biangular lines. By a set of *lines*, we mean a collection of vectors in $\mathbf{R}^\ell$, for some ℓ. Given a collection $\mathscr{L}$ of lines in $\mathbf{R}^\ell$, define $\Xi = \{|\langle u, v\rangle| : u, v \in \mathscr{L} \text{ and } u \neq v\}$. If $|\Xi| = 2$, then we say that $\mathscr{L}$ is a set of *biangular lines*; while if $|\Xi| = 1$, then we say that $\mathscr{L}$ is a set of *equiangular lines*.[9]

Clearly, if H is balancedly splittable with respect to the $\ell \times n$ submatrix H_1, then Definition 5.2 implies that the columns of H_1 are at most biangular. They are equiangular precisely in the case that $b = -a$.

We will require the following proposition due to Delsarte et al. (1977).

5.3. Proposition. Let $\mathscr{L} \subset \mathbf{R}^\ell$ be a set of lines (vectors) such that $|\langle u, v\rangle| = a$, for every pair of distinct lines u and v in $\mathscr{L}$. If $\ell < a^{-2}$, then

(5.3.a) $|\mathscr{L}| \leq \ell(1 - a^2)/(1 - \ell a^2)$.

Using balancedly splittable Hadamard matrices, we can construct optimal sets of equiangular lines.

5.4. Theorem. If there exists a balancedly splittable Hadamard matrix with parameters $(n, \ell, -a, a)$, then there is an optimal set of equiangular lines in $\mathbf{R}^\ell$.

Proof. Suppose that H is a balancedly splittable Hadamard matrix with respect to the $\ell \times n$ submatrix H_1 with parameters $(n, \ell, a, -a)$. Take $\mathscr{L}$ to be the collection of normalized columns of H_1. Note that $a^2\ell^2 = \ell(n - \ell)/(n - 1)$; then the absolute value of the inner product between distinct lines in $\mathscr{L}$ is given by $a = \sqrt{n - \ell}/\sqrt{\ell(n - 1)}$; moreover, $\ell \leq a$. The right-hand side of (5.3.a) reduces to n. We have, therefore, exhibited an optimal set of equiangular lines. ∎

In §10, we will pursue this topic again in the more restricted setting of frames.

* * *

5.3. Constructions. For the sake of completeness, we make Example 5.1 general.

5.5. Proposition. If there exists a Hadamard matrix of order n, then there exists a balancedly splittable Hadamard matrix with parameters $(n^2, n, n, 0)$.

Proof. Let H be a normalized Hadamard matrix of order n, and label the rows $h_0 = \mathbf{1}, \dots, h_{n-1}$. Take M to be the block matrix defined by $M_{ij} = h_j^t h_i$. Then $M_{ij} M_{kj}^t = (h_j^t h_i)(h_j^t h_k)^t = h_j^t (h_i h_k^t) h_j = O$ whenever $i \neq k$, hence M is a Hadamard matrix of order n^2 (cf Kharaghani, 1985). Take M_1 to be the first block row of M. Then

$$M_1^t M_1 = \begin{pmatrix} J \\ \mathbf{1}^t h_1 \\ \vdots \\ \mathbf{1}^t h_{n-1} \end{pmatrix} \begin{pmatrix} J & h_2^t \mathbf{1} & \dots & h_{n-1}^t \mathbf{1} \end{pmatrix} = mI \otimes J,$$

and the proof is complete. ∎

There are many more constructions presented in Kharaghani and Suda (2019) that the interested reader may consult. For our purposes, however, we present a novel construction. Consider the following example.

5.6. Example. Consider the Hadamard matrix of order 2

(5.6.a) $\begin{pmatrix} + & + \\ + & - \end{pmatrix}$,

and label the rows h_0 and h_1. Form the matrices $c_i = h_i^t h_i$, the so-called *auxiliary matrices*, shown in order below.

(5.6.b) $c_0 = \begin{pmatrix} + & + \\ + & + \end{pmatrix}, c_1 = \begin{pmatrix} + & - \\ - & + \end{pmatrix}$.

Take $K = H \otimes H$, labelling the rows as k_0, k_1, k_2, and k_3. Form the block circulant matrices A and B with first rows (c_0, c_1, c_1) and $(c_0, c_1, -c_1)$, shown below.

(5.6.c) $A = \begin{pmatrix} + & + & + & - & + & - \\ + & + & - & + & - & + \\ + & - & + & + & + & - \\ - & + & + & + & - & + \\ + & - & + & - & + & + \\ - & + & - & + & + & + \end{pmatrix}, B = \begin{pmatrix} + & + & + & - & - & + \\ + & + & - & + & + & - \\ - & + & + & + & + & - \\ + & - & + & + & - & + \\ + & - & - & + & + & + \\ - & + & + & - & + & + \end{pmatrix}$.

Now, form the block matrix $F = \begin{pmatrix} F_1 & \dots & F_7 \end{pmatrix}$ by defining $F_i = k_i^t h_0$

(5.6.d) $F = \begin{pmatrix} + & + & + & + & + & + \\ - & - & + & + & - & - \\ + & + & - & - & - & - \\ - & - & - & - & + & + \end{pmatrix}$.

Finally, take $E = F^t$. We then form the block matrix

$$
\text{(5.6.e)} \quad \begin{pmatrix} J & F & -F \\ E & A & B \\ -E & B & A \end{pmatrix} = \left(\begin{array}{cccccccccccccccc}
+&+&+&+&\mathbf{+}&\mathbf{+}&\mathbf{+}&\mathbf{+}&\mathbf{+}&\mathbf{+}&-&-&-&-&-&- \\
+&+&+&+&\mathbf{-}&\mathbf{-}&\mathbf{+}&\mathbf{+}&\mathbf{-}&\mathbf{-}&+&+&-&-&+&+ \\
+&+&+&+&\mathbf{+}&\mathbf{+}&\mathbf{-}&\mathbf{-}&\mathbf{-}&\mathbf{-}&-&-&+&+&+&+ \\
+&+&+&+&\mathbf{-}&\mathbf{-}&\mathbf{-}&\mathbf{-}&\mathbf{+}&\mathbf{+}&+&+&+&+&-&- \\
+&-&+&-&\mathbf{+}&\mathbf{+}&\mathbf{+}&\mathbf{-}&\mathbf{+}&\mathbf{-}&+&+&+&-&-&+ \\
+&-&+&-&\mathbf{+}&\mathbf{+}&\mathbf{-}&\mathbf{+}&\mathbf{-}&\mathbf{+}&+&+&-&+&+&- \\
+&+&-&-&\mathbf{+}&\mathbf{-}&\mathbf{+}&\mathbf{+}&\mathbf{+}&\mathbf{-}&-&+&+&+&+&- \\
+&+&-&-&\mathbf{-}&\mathbf{+}&\mathbf{+}&\mathbf{+}&\mathbf{-}&\mathbf{+}&+&-&+&+&-&+ \\
+&-&-&+&\mathbf{+}&\mathbf{-}&\mathbf{+}&\mathbf{-}&\mathbf{+}&\mathbf{+}&+&-&-&+&+&+ \\
+&-&-&+&\mathbf{-}&\mathbf{+}&\mathbf{-}&\mathbf{+}&\mathbf{+}&\mathbf{+}&-&+&+&-&+&+ \\
-&+&-&+&\mathbf{+}&\mathbf{+}&\mathbf{+}&\mathbf{-}&\mathbf{-}&\mathbf{+}&+&+&+&-&+&- \\
-&+&-&+&\mathbf{+}&\mathbf{+}&\mathbf{-}&\mathbf{+}&\mathbf{+}&\mathbf{-}&+&+&-&+&-&+ \\
-&-&+&+&\mathbf{-}&\mathbf{+}&\mathbf{+}&\mathbf{+}&\mathbf{+}&\mathbf{-}&+&-&+&+&+&- \\
-&-&+&+&\mathbf{+}&\mathbf{-}&\mathbf{+}&\mathbf{+}&\mathbf{-}&\mathbf{+}&-&+&+&+&-&+ \\
-&+&+&-&\mathbf{+}&\mathbf{-}&\mathbf{-}&\mathbf{+}&\mathbf{+}&\mathbf{+}&+&-&+&-&+&+ \\
-&+&+&-&\mathbf{-}&\mathbf{+}&\mathbf{+}&\mathbf{-}&\mathbf{+}&\mathbf{+}&-&+&-&+&+&+
\end{array}\right) .
$$

This matrix is a Hadamard matrix of order 16, and evidently, it is balancedly splittable. Indeed, the submatrix shown in bold above can be used to form a balanced split. Due to the form of (5.6.e), however, any one of the following can be used to form a balanced split

$$
\text{(5.6.f)} \quad \begin{pmatrix} F \\ A \\ B \end{pmatrix}, \begin{pmatrix} -F \\ B \\ A \end{pmatrix}, \begin{pmatrix} E & A & B \end{pmatrix}, \begin{pmatrix} -E & B & A \end{pmatrix}.
$$

In any event, the parameters of the splits are $(16, 6, -2, 2)$. Therefore, we have optimal sets of equiangular lines.

5.7. Example. Beginning with the Hadamard matrix of order 4

$$
\text{(5.7.a)} \quad \begin{pmatrix} +&+&+&+ \\ -&+&-&+ \\ -&+&+&- \\ -&-&+&+ \end{pmatrix}
$$

we obtain the balancedly splittable Hadamard matrix of order 64 as constructed in appendix A1.

5.8. Example. Of course, we are not limited to real matrices. Beginning with the $\mathrm{BW}(3, 3; 3)$

$$
\text{(5.8.a)} \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix},
$$

where $\omega = e^{\frac{2\pi\sqrt{-1}}{3}}$, we can construct a balancedly splittable $\mathrm{BW}(36, 36; 6)$ as shown in appendix A2.

This construction can be made perfectly general. Since this result is ultimately a special case of Theorem 6.13, we will not show it explicitly.

5.9. Theorem. There is a balancedly splittable Hadamard matrix of order $4n^2$ with parameters $(4n^2, 2n^2 - n, n, -n)$ whenever there is a Hadamard matrix of order n.

5.10. Corollary. If there is a Hadamard matrix of order n, then there is an optimal set of equiangular lines in $\mathbf{R}^{2n^2-n}$.

We mention in passing that Kharaghani and Suda (2019) showed that balancedly splittable Hadamard matrices can be used to construct various association schemes. As nothing essentially new about association schemes has been added in our study of orthogonal designs, we will not pursue this topic here and simply refer the reader to the aforementioned article.

§6. Balancedly Splittable Orthogonal Designs

In this and the following section, the new results of Kharaghani et al. (2021) are presented. Here we define balanced splittability of orthogonal designs and give several constructions. To avoid obfuscation, we given the constructions in terms of real and complex ODs; however, the results are perfectly valid for the more general QODs.

* * *

6.1. Definition. Generalizing balanced splittability to orthogonal designs presents several difficulties. The various cases are encapsulated in the next definition.

6.1. Definition. Let X be a full $\mathrm{QOD}(n; s_1, \ldots, s_u)$. X is *balancedly splittable* if there is an $n \times \ell$ submatrix X_1 and if there are quaternion numbers a and b where one of the following conditions holds.

(6.1.a) The off-diagonal entries of $X_1^* X_1$ have the form $\varepsilon c \sigma$, where ε is a free unimodular quaternion number, c is in the set $\{a, b, a^*, b^*\}$, and where σ is given by (3.11.a).

(6.1.b) If H_1 is the matrix obtained from X_1 by replacing each indeterminate with unity, then $H_1^*H_1$ has off-diagonal entries of the form εc, where ε and c are as stated previously.

In the former case, the split is *stable*. If the latter case holds while the former does not, then the split is termed *unstable*.

The next two subsections will present constructions for both the unstable and the stable cases.

* * *

6.2. Unstable Constructions. The constructions of this section are similar to those presented in Fender et al. (2018) and Pender (2020), and are applicable to real and complex orthogonal designs.

To begin, if W is a skew-symmetric W$(q+1,q)$, then we take Q to be its core, i.e. the submatrix obtaind by deleting the first row and column. Further, we can assume that $W = \left(\begin{smallmatrix} 0 & \mathbf{1}^t \\ -\mathbf{1} & Q \end{smallmatrix}\right)$, hence $JQ = QJ = O$ and $Q^2 = J - qI$. We recursively define the following family of matrices.

$$\mathcal{J}_m = \begin{cases} aJ_1 & \text{if } m = 0, \text{ and} \\ J_q \otimes \mathcal{A}_{m-1} & \text{if } m > 0; \end{cases}$$

$$\mathcal{A}_m = \begin{cases} bJ_1 & \text{if } m = 0, \text{ and} \\ I_q \otimes \mathcal{J}_{m-1} + Q \otimes \mathcal{A}_{m-1} & \text{if } m > 0; \end{cases}$$

where a and b are commuting indeterminates.

We require the following lemma.

6.2. Lemma.

(6.2.a) $\mathcal{J}_m\mathcal{A}_m^t = \mathcal{A}_m\mathcal{J}_m^t$;

(6.2.b) $\mathcal{J}_m\mathcal{J}_m^t + q\mathcal{A}_m\mathcal{A}_m^t = (q^m a^2 + q^{m+1}b^2)I$; and

(6.2.c) $\mathcal{J}_1^t\mathcal{J}_1 = qa^2J, \mathcal{A}_1^t\mathcal{A}_1 = a^2I + b^2(qI - J)$, and $\mathcal{A}_1^t\mathcal{J}_1 = \mathcal{J}_1^t\mathcal{A}_1 = abJ$.

Proof. We have $\mathcal{J}_0\mathcal{A}_0^t = ab = ba = \mathcal{A}_0\mathcal{J}_0^t$. Assume $\mathcal{J}_{m-1}\mathcal{A}_{m-1}^t = \mathcal{A}_{m-1}\mathcal{J}_{m-1}^t$. Then

$$\begin{aligned}
\mathcal{J}_m\mathcal{A}_m^t &= (J \otimes \mathcal{A}_{m-1})(I \otimes \mathcal{J}_{m-1} + Q \otimes \mathcal{A}_{m-1})^t \\
&= J \otimes \mathcal{A}_{m-1}\mathcal{J}_{m-1}^t + JQ^t \otimes \mathcal{A}_{m-1}\mathcal{A}_{m-1}^t \\
&= J \otimes \mathcal{J}_{m-1}\mathcal{A}_{m-1}^t + QJ \otimes \mathcal{A}_{m-1}\mathcal{A}_{m-1}^t \\
&= (I \otimes \mathcal{J}_{m-1} + Q \otimes \mathcal{A}_{m-1})(J \otimes \mathcal{A}_{m-1})^t,
\end{aligned}$$

and (6.2.a) has been shown.

Clearly, $\mathcal{J}_0\mathcal{J}_0^t + q\mathcal{A}_0\mathcal{A}_0^t = a^2 + qb^2$; so, assume $\mathcal{J}_{m-1}\mathcal{J}_{m-1}^t + q\mathcal{A}_{m-1}\mathcal{A}_{m-1}^t = (a^2 + qb^2)I$. Then

$$\begin{aligned}
\mathcal{J}_m\mathcal{J}_m^t + q\mathcal{A}_m\mathcal{A}_m^t &= qJ \otimes \mathcal{A}_{m-1}\mathcal{A}_{m-1}^t + q(I \otimes \mathcal{J}_{m-1}\mathcal{J}_{m-1}^t - Q^2 \otimes \mathcal{A}_{m-1}\mathcal{A}_{m-1}^t) \\
&= qI \otimes (\mathcal{J}_{m-1}\mathcal{J}_{m-1}^t + q\mathcal{A}_{m-1}\mathcal{A}_{m-1}^t) \\
&= qI \otimes (q^{m-1}q^2 + q^m b^2)I \\
&= (q^m a^2 + q^{m+1}b^2)I,
\end{aligned}$$

and (6.2.b) is proven.

Finally, (6.2.c) is simply a restatement of the definitions of $\mathcal{J}_m$ and $\mathcal{A}_m$. ∎

We can now present the first construction of the novel balancedly splittable ODs.

6.3. Theorem. Let $W, \mathcal{A}_m$, and $\mathcal{J}_m$ be as above. Define $X_m = I \otimes \mathcal{J}_m + W \otimes \mathcal{A}_m$. Then:

(6.3.a) X_m is an OD$(q^m(q+1); q^m, q^{m+1})$, and

(6.3.b) The matrix X_1 admits an unstable balanced split.

Proof. X_m has entries from $\{\pm a, \pm b\}$. Observe:

$$\begin{aligned}
X_mX_m^t &= (I \otimes \mathcal{J}_m + W \otimes \mathcal{A}_m)(I \otimes \mathcal{J}_m + W \otimes \mathcal{A}_m)^t \\
&= I \otimes \mathcal{J}_m\mathcal{J}_m^t + WW^t \otimes \mathcal{A}_m\mathcal{A}_m^t \\
&= I \otimes \mathcal{J}_m\mathcal{J}_m^t + qI \otimes \mathcal{A}_m\mathcal{A}_m^t \\
&= I \otimes (\mathcal{J}_m\mathcal{J}_m^t + q\mathcal{A}_m\mathcal{A}_m^t) \\
&= I \otimes (q^m a^2 + q^{m+1}b^2)I
\end{aligned}$$

$$= (q^m a^2 + q^{m+1} b^2) I,$$

which shows that X_m is an OD$(q^m(q+1); q^m; q^{m+1})$. It remains to prove the balanced splittability of the base case.

Take $Y = (\, \mathcal{J}_1\ \mathcal{A}_1\ \ldots\ \mathcal{A}_1 \,)$, the first block row of X_1. Then

$$\begin{aligned} Y^t Y &= \begin{pmatrix} \mathcal{J}_1^t \mathcal{J}_1 & \mathbf{1}^t \otimes \mathcal{J}_1^t \mathcal{A}_1 \\ \mathbf{1} \otimes \mathcal{A}_1^t \mathcal{J}_1 & J \otimes \mathcal{A}_1^t \mathcal{A}_1 \end{pmatrix} \\ &= \begin{pmatrix} q b^2 J & ab\mathbf{1}^t \otimes J \\ ab\mathbf{1} \otimes J & J \otimes [(a^2 - qb^2) I + b^2 J] \end{pmatrix}. \end{aligned}$$

Hence, X_1 admits an unstable balanced split. ∎

6.4. Corollary. For every prime power $q \equiv -1 \pmod 4$, and for every integer $m > 0$, there is an OD$(q^m(q+1); q^m, q^{m+1})$

Proof. It is well known (see Ionin and Shrikhande, 2006) that there is a skew-symmetric W$(q+1, q)$. Apply the theorem to this matrix. ∎

6.5. Corollary. For every prime power $q \equiv -1 \pmod 4$, there is an unstable balancedly splittable OD$(q(q+1); q, q^2)$.

6.6. Example. Using the skew-symmetric Paley weighing matrix[10] $W(4,3)$ given by

$$\text{(6.6.a)} \quad \begin{pmatrix} 0 & + & + & + \\ - & 0 & + & - \\ - & - & 0 & + \\ - & + & - & 0 \end{pmatrix},$$

we construct the smallest case of an OD$(12; 3, 9)$ given by the theorem

$$
\text{(6.6.b)}\quad
\begin{pmatrix}
\mathbf{b} & \mathbf{b} & \mathbf{b} & \mathbf{a} & \mathbf{b} & \overline{\mathbf{b}} & \mathbf{a} & \mathbf{b} & \overline{\mathbf{b}} & \mathbf{a} & \mathbf{b} & \overline{\mathbf{b}} \\
\mathbf{b} & \mathbf{b} & \mathbf{b} & \overline{\mathbf{b}} & \mathbf{a} & \mathbf{b} & \overline{\mathbf{b}} & \mathbf{a} & \mathbf{b} & \overline{\mathbf{b}} & \mathbf{a} & \mathbf{b} \\
\mathbf{b} & \mathbf{b} & \mathbf{b} & \mathbf{b} & \overline{\mathbf{b}} & \mathbf{a} & \mathbf{b} & \overline{\mathbf{b}} & \mathbf{a} & \mathbf{b} & \overline{\mathbf{b}} & \mathbf{a} \\
\overline{a} & \overline{b} & b & b & b & b & a & b & \overline{b} & \overline{a} & \overline{b} & b \\
b & \overline{a} & \overline{b} & b & b & b & \overline{b} & a & b & b & \overline{a} & \overline{b} \\
\overline{b} & b & \overline{a} & b & b & b & b & \overline{b} & a & \overline{b} & b & \overline{a} \\
\overline{a} & \overline{b} & b & \overline{a} & \overline{b} & b & b & b & b & a & b & \overline{b} \\
b & \overline{a} & \overline{b} & b & \overline{a} & \overline{b} & b & b & b & \overline{b} & a & b \\
\overline{b} & b & \overline{a} & \overline{b} & b & \overline{a} & b & b & b & b & \overline{b} & a \\
\overline{a} & \overline{b} & b & a & b & \overline{b} & \overline{a} & \overline{b} & b & b & b & b \\
b & \overline{a} & \overline{b} & \overline{b} & a & b & b & \overline{a} & \overline{b} & b & b & b \\
\overline{b} & b & \overline{a} & b & \overline{b} & a & \overline{b} & b & \overline{a} & b & b & b
\end{pmatrix},
$$

where the unstable split is shown in bold.

Our first construction yields real ODs and is applicable in the case that we have a prime power $q \equiv -1 \pmod 4$. Of course, since $(q-1)/2$ is odd, it is known (see Ionin and Shrikhande, 2006) that there is a weighing matrix that is skew-symmetric, a property essential to the construction. If $q \equiv 1 \pmod 4$, then $(q-1)/2$ is even and the ensuing weighing matrix is symmetric. In this event, we need to appeal to complex ODs in order apply the construction.

To apply the complex units, we make the following recursive definitions where $W = \left(\begin{smallmatrix} 0 & \mathbf{1}^t \\ \mathbf{1} & Q \end{smallmatrix}\right)$ is a W$(q+1, q)$ with $Q^t = Q$.

$$
\mathcal{C}_m = \begin{cases} aJ_1 & \text{if } m = 0, \text{ and} \\ J_q \otimes \mathcal{D}_{m-1} & \text{if } m > 0; \end{cases}
$$

$$
\mathcal{D}_m = \begin{cases} bJ_1 & \text{if } m = 0, \text{ and} \\ I_q \otimes \mathcal{C}_{m-1} + iQ \otimes \mathcal{D}_{m-1} & \text{if } m > 0; \end{cases}
$$

where again a and b are real commuting indeterminates. As above, we have the following lemma that is shown in precisely the same way as before, save one replaces transposition with conjugate transposition.

6.7. Lemma.

(6.7.a) $\mathcal{C}_m \mathcal{D}_m^* = \mathcal{D}_m \mathcal{C}_m^*$;

(6.7.b) $\mathcal{C}_m \mathcal{C}_m^* + q\mathcal{D}_m \mathcal{D}_m^* = (q^m a^2 + q^{m+1} b^2)I$; and

(6.7.c) $\mathcal{C}_1^* \mathcal{C}_1 = qa^2 J, \mathcal{D}_1^* \mathcal{D}_1 = a^2 I + b^2(qI - J)$, and $\mathcal{D}_1^* \mathcal{C}_1 = \mathcal{C}_1^* \mathcal{D}_1 = abJ$.

6.8. Theorem. Let W, $\mathcal{C}_m$, and $\mathcal{D}_m$ be as above, and define $Y_m = iI \otimes \mathcal{C}_m + W \otimes \mathcal{D}_m$. Then:

(6.8.a) The matrix Y_m is a $\mathrm{COD}(q^m(q+1); q^m, q^{m+1})$, and

(6.8.b) Y_1 admits an unstable balanced split.

6.9. Corollary. For every prime power $q \equiv 1 \pmod 4$, and for every integer $m > 0$, there is a $\mathrm{COD}(q^m(q+1); q^m, q^{m+1})$.

6.10. Corollary. For every prime power $q \equiv 1 \pmod 4$, there is an unfaithful balancedly splittable $\mathrm{COD}(q(q+1); q, q^2)$.

We again explore the smallest case.

6.11. Example. Take $q = 5$ and consider the symmetric Paley weighing matrix $\mathrm{W}(6,5)$ given by

$$\begin{pmatrix} 0 & + & + & + & + & + \\ + & 0 & + & - & - & + \\ + & + & 0 & + & - & - \\ + & - & + & 0 & + & - \\ + & - & - & + & 0 & + \\ + & + & - & - & + & 0 \end{pmatrix}. \tag{6.11.a}$$

Applying the construction, we obtain

$$\left(\begin{array}{cccccccccccccccccccccccccccccc}
\mathbf{ib} & \mathbf{ib} & \mathbf{ib} & \mathbf{ib} & \mathbf{ib} & \mathbf{a} & \mathbf{ib} & \mathbf{i\overline{b}} & \mathbf{i\overline{b}} & \mathbf{ib} & \mathbf{a} & \mathbf{ib} & \mathbf{i\overline{b}} & \mathbf{i\overline{b}} & \mathbf{ib} & \mathbf{a} & \mathbf{ib} & \mathbf{i\overline{b}} & \mathbf{i\overline{b}} & \mathbf{ib} & \mathbf{a} & \mathbf{ib} & \mathbf{i\overline{b}} & \mathbf{i\overline{b}} & \mathbf{ib} & \mathbf{a} & \mathbf{ib} & \mathbf{i\overline{b}} & \mathbf{i\overline{b}} & \mathbf{ib} \\
\mathbf{ib} & \mathbf{ib} & \mathbf{ib} & \mathbf{ib} & \mathbf{ib} & \mathbf{ib} & \mathbf{a} & \mathbf{ib} & \mathbf{i\overline{b}} & \mathbf{i\overline{b}} & \mathbf{ib} & \mathbf{a} & \mathbf{ib} & \mathbf{i\overline{b}} & \mathbf{i\overline{b}} & \mathbf{ib} & \mathbf{a} & \mathbf{ib} & \mathbf{i\overline{b}} & \mathbf{i\overline{b}} & \mathbf{ib} & \mathbf{a} & \mathbf{ib} & \mathbf{i\overline{b}} & \mathbf{i\overline{b}} & \mathbf{ib} & \mathbf{a} & \mathbf{ib} & \mathbf{i\overline{b}} & \mathbf{i\overline{b}} \\
\mathbf{ib} & \mathbf{ib} & \mathbf{ib} & \mathbf{ib} & \mathbf{ib} & \mathbf{i\overline{b}} & \mathbf{ib} & \mathbf{a} & \mathbf{ib} & \mathbf{i\overline{b}} & \mathbf{i\overline{b}} & \mathbf{ib} & \mathbf{a} & \mathbf{ib} & \mathbf{i\overline{b}} & \mathbf{i\overline{b}} & \mathbf{ib} & \mathbf{a} & \mathbf{ib} & \mathbf{i\overline{b}} & \mathbf{i\overline{b}} & \mathbf{ib} & \mathbf{a} & \mathbf{ib} & \mathbf{i\overline{b}} & \mathbf{i\overline{b}} & \mathbf{ib} & \mathbf{a} & \mathbf{ib} & \mathbf{i\overline{b}} \\
\mathbf{ib} & \mathbf{ib} & \mathbf{ib} & \mathbf{ib} & \mathbf{ib} & \mathbf{i\overline{b}} & \mathbf{i\overline{b}} & \mathbf{ib} & \mathbf{a} & \mathbf{ib} & \mathbf{i\overline{b}} & \mathbf{i\overline{b}} & \mathbf{ib} & \mathbf{a} & \mathbf{ib} & \mathbf{i\overline{b}} & \mathbf{i\overline{b}} & \mathbf{ib} & \mathbf{a} & \mathbf{ib} & \mathbf{i\overline{b}} & \mathbf{i\overline{b}} & \mathbf{ib} & \mathbf{a} & \mathbf{ib} & \mathbf{i\overline{b}} & \mathbf{i\overline{b}} & \mathbf{ib} & \mathbf{a} & \mathbf{ib} \\
\mathbf{ib} & \mathbf{ib} & \mathbf{ib} & \mathbf{ib} & \mathbf{ib} & \mathbf{ib} & \mathbf{i\overline{b}} & \mathbf{i\overline{b}} & \mathbf{ib} & \mathbf{a} & \mathbf{ib} & \mathbf{i\overline{b}} & \mathbf{i\overline{b}} & \mathbf{ib} & \mathbf{a} & \mathbf{ib} & \mathbf{i\overline{b}} & \mathbf{i\overline{b}} & \mathbf{ib} & \mathbf{a} & \mathbf{ib} & \mathbf{i\overline{b}} & \mathbf{i\overline{b}} & \mathbf{ib} & \mathbf{a} & \mathbf{ib} & \mathbf{i\overline{b}} & \mathbf{i\overline{b}} & \mathbf{ib} & \mathbf{a} \\
a & ib & i\overline{b} & i\overline{b} & ib & ib & ib & ib & ib & ib & a & ib & i\overline{b} & i\overline{b} & ib & \overline{a} & i\overline{b} & ib & ib & i\overline{b} & \overline{a} & i\overline{b} & ib & ib & i\overline{b} & a & ib & i\overline{b} & i\overline{b} & ib \\
ib & a & ib & i\overline{b} & i\overline{b} & ib & ib & ib & ib & ib & ib & a & ib & i\overline{b} & i\overline{b} & i\overline{b} & \overline{a} & i\overline{b} & ib & ib & i\overline{b} & \overline{a} & i\overline{b} & ib & ib & ib & a & ib & i\overline{b} & i\overline{b} \\
i\overline{b} & ib & a & ib & i\overline{b} & ib & ib & ib & ib & ib & i\overline{b} & ib & a & ib & i\overline{b} & ib & i\overline{b} & \overline{a} & i\overline{b} & ib & ib & i\overline{b} & \overline{a} & i\overline{b} & ib & i\overline{b} & ib & a & ib & i\overline{b} \\
i\overline{b} & i\overline{b} & ib & a & ib & ib & ib & ib & ib & ib & i\overline{b} & i\overline{b} & ib & a & ib & ib & ib & i\overline{b} & \overline{a} & i\overline{b} & ib & ib & i\overline{b} & \overline{a} & i\overline{b} & i\overline{b} & i\overline{b} & ib & a & ib \\
ib & i\overline{b} & i\overline{b} & ib & a & ib & ib & ib & ib & ib & ib & i\overline{b} & i\overline{b} & ib & a & i\overline{b} & ib & ib & i\overline{b} & \overline{a} & i\overline{b} & ib & ib & i\overline{b} & \overline{a} & ib & i\overline{b} & i\overline{b} & ib & a \\
a & ib & i\overline{b} & i\overline{b} & ib & a & ib & i\overline{b} & i\overline{b} & ib & ib & ib & ib & ib & ib & a & ib & i\overline{b} & i\overline{b} & ib & \overline{a} & i\overline{b} & ib & ib & i\overline{b} & \overline{a} & i\overline{b} & ib & ib & i\overline{b} \\
ib & a & ib & i\overline{b} & i\overline{b} & ib & a & ib & i\overline{b} & i\overline{b} & ib & ib & ib & ib & ib & ib & a & ib & i\overline{b} & i\overline{b} & i\overline{b} & \overline{a} & i\overline{b} & ib & ib & i\overline{b} & \overline{a} & i\overline{b} & ib & ib \\
i\overline{b} & ib & a & ib & i\overline{b} & i\overline{b} & ib & a & ib & i\overline{b} & ib & ib & ib & ib & ib & i\overline{b} & ib & a & ib & i\overline{b} & ib & i\overline{b} & \overline{a} & i\overline{b} & ib & ib & i\overline{b} & \overline{a} & i\overline{b} & ib \\
i\overline{b} & i\overline{b} & ib & a & ib & i\overline{b} & i\overline{b} & ib & a & ib & ib & ib & ib & ib & ib & i\overline{b} & i\overline{b} & ib & a & ib & ib & ib & i\overline{b} & \overline{a} & i\overline{b} & ib & ib & i\overline{b} & \overline{a} & i\overline{b} \\
ib & i\overline{b} & i\overline{b} & ib & a & ib & i\overline{b} & i\overline{b} & ib & a & ib & ib & ib & ib & ib & ib & i\overline{b} & i\overline{b} & ib & a & i\overline{b} & ib & ib & i\overline{b} & \overline{a} & i\overline{b} & ib & ib & i\overline{b} & \overline{a} \\
a & ib & i\overline{b} & i\overline{b} & ib & \overline{a} & i\overline{b} & ib & ib & i\overline{b} & a & ib & i\overline{b} & i\overline{b} & ib & ib & ib & ib & ib & ib & a & ib & i\overline{b} & i\overline{b} & ib & \overline{a} & i\overline{b} & ib & ib & i\overline{b} \\
ib & a & ib & i\overline{b} & i\overline{b} & i\overline{b} & \overline{a} & i\overline{b} & ib & ib & ib & a & ib & i\overline{b} & i\overline{b} & ib & ib & ib & ib & ib & ib & a & ib & i\overline{b} & i\overline{b} & i\overline{b} & \overline{a} & i\overline{b} & ib & ib \\
i\overline{b} & ib & a & ib & i\overline{b} & ib & i\overline{b} & \overline{a} & i\overline{b} & ib & i\overline{b} & ib & a & ib & i\overline{b} & ib & ib & ib & ib & ib & i\overline{b} & ib & a & ib & i\overline{b} & ib & i\overline{b} & \overline{a} & i\overline{b} & ib \\
i\overline{b} & i\overline{b} & ib & a & ib & ib & ib & i\overline{b} & \overline{a} & i\overline{b} & i\overline{b} & i\overline{b} & ib & a & ib & ib & ib & ib & ib & ib & i\overline{b} & i\overline{b} & ib & a & ib & ib & ib & i\overline{b} & \overline{a} & i\overline{b} \\
ib & i\overline{b} & i\overline{b} & ib & a & i\overline{b} & ib & ib & i\overline{b} & \overline{a} & ib & i\overline{b} & i\overline{b} & ib & a & ib & ib & ib & ib & ib & ib & i\overline{b} & i\overline{b} & ib & a & i\overline{b} & ib & ib & i\overline{b} & \overline{a} \\
a & ib & i\overline{b} & i\overline{b} & ib & \overline{a} & i\overline{b} & ib & ib & i\overline{b} & \overline{a} & i\overline{b} & ib & ib & i\overline{b} & a & ib & i\overline{b} & i\overline{b} & ib & ib & ib & ib & ib & ib & a & ib & i\overline{b} & i\overline{b} & ib \\
ib & a & ib & i\overline{b} & i\overline{b} & i\overline{b} & \overline{a} & i\overline{b} & ib & ib & i\overline{b} & \overline{a} & i\overline{b} & ib & ib & ib & a & ib & i\overline{b} & i\overline{b} & ib & ib & ib & ib & ib & ib & a & ib & i\overline{b} & i\overline{b} \\
i\overline{b} & ib & a & ib & i\overline{b} & ib & i\overline{b} & \overline{a} & i\overline{b} & ib & ib & i\overline{b} & \overline{a} & i\overline{b} & ib & i\overline{b} & ib & a & ib & i\overline{b} & ib & ib & ib & ib & ib & i\overline{b} & ib & a & ib & i\overline{b} \\
i\overline{b} & i\overline{b} & ib & a & ib & ib & ib & i\overline{b} & \overline{a} & i\overline{b} & ib & ib & i\overline{b} & \overline{a} & i\overline{b} & i\overline{b} & i\overline{b} & ib & a & ib & ib & ib & ib & ib & ib & i\overline{b} & i\overline{b} & ib & a & ib \\
ib & i\overline{b} & i\overline{b} & ib & a & i\overline{b} & ib & ib & i\overline{b} & \overline{a} & i\overline{b} & ib & ib & i\overline{b} & \overline{a} & ib & i\overline{b} & i\overline{b} & ib & a & ib & ib & ib & ib & ib & ib & i\overline{b} & i\overline{b} & ib & a \\
a & ib & i\overline{b} & i\overline{b} & ib & a & ib & i\overline{b} & i\overline{b} & ib & \overline{a} & i\overline{b} & ib & ib & i\overline{b} & \overline{a} & i\overline{b} & ib & ib & i\overline{b} & a & ib & i\overline{b} & i\overline{b} & ib & ib & ib & ib & ib & ib \\
ib & a & ib & i\overline{b} & i\overline{b} & ib & a & ib & i\overline{b} & i\overline{b} & i\overline{b} & \overline{a} & i\overline{b} & ib & ib & i\overline{b} & \overline{a} & i\overline{b} & ib & ib & ib & a & ib & i\overline{b} & i\overline{b} & ib & ib & ib & ib & ib \\
i\overline{b} & ib & a & ib & i\overline{b} & i\overline{b} & ib & a & ib & i\overline{b} & ib & i\overline{b} & \overline{a} & i\overline{b} & ib & ib & i\overline{b} & \overline{a} & i\overline{b} & ib & i\overline{b} & ib & a & ib & i\overline{b} & ib & ib & ib & ib & ib \\
i\overline{b} & i\overline{b} & ib & a & ib & i\overline{b} & i\overline{b} & ib & a & ib & ib & ib & i\overline{b} & \overline{a} & i\overline{b} & ib & ib & i\overline{b} & \overline{a} & i\overline{b} & i\overline{b} & i\overline{b} & ib & a & ib & ib & ib & ib & ib & ib \\
ib & i\overline{b} & i\overline{b} & ib & a & ib & i\overline{b} & i\overline{b} & ib & a & i\overline{b} & ib & ib & i\overline{b} & \overline{a} & i\overline{b} & ib & ib & i\overline{b} & \overline{a} & ib & i\overline{b} & i\overline{b} & ib & a & ib & ib & ib & ib & ib
\end{array}\right), \tag{6.11.b}$$

a $\mathrm{COD}(30; 5, 25)$, where the unstable split is shown in bold.

* * *

6.3. Stable Construction. Here we will introduce a most useful construction of orthogonal designs admitting a stable split. We will see later how we can use these constructed matrices in developing other objects.

To begin, we assume the existence of a full OD$(n; s_1, \ldots, s_u)$, say X, and label the rows of X as $x_0, \ldots, x_{n-1}$. Further, assume that the coefficients of the indeterminates of the first row and column are $+1$. We need to extend the idea of an auxiliary matrix given in Example 5.6. To do this, we will follow Kharaghani and Suda (2018) in defining the auxiliary matrix of an OD thus: Let H be the Hadamard matrix obtained by setting each indeterminate of X to $+1$, and label the rows of H as $h_0, \ldots, h_{n-1}$. Then the auxiliary matrices[11] of X are given by $c_i = h_i^t x_i$. We have the following result.

6.12. Lemma. Let $c_i = h_i^t x_i$, for $i \in \{0, \ldots, n-1\}$, be the auxiliary matrices of an OD$(n; s_1, \ldots, s_u)$ X where $XX^t = \sigma I$. Then:

(6.12.a) $\sum\limits_i c_i c_i^t = n\sigma I_n$, and

(6.12.b) $c_i c_j^t = \delta_{ij} \sigma h_i^t h_i$.

We need the simple fact that if (a, b) denotes the concatenation of sequences a and b, then (a, b) and $(a, -b)$ is a Golay pair (see §3). Continuing to let $c_0, \ldots, c_{n-1}$ be the auxiliary matrices of the OD$(n; s_1, \ldots, s_u)$ X, then $a = (c_0, c_1, \ldots, c_{n-1}, c_{n-1}, \ldots, c_1)$ and $b = (c_0, c_1, \ldots, c_{n-1}, -c_{n-1}, \ldots, -c_1)$ form a complementary pair. Let A and B be the block-circulant matrices with first rows a and b, respectively.

Now, take $\tilde{X} = \left(\begin{smallmatrix} + & + \\ + & - \end{smallmatrix}\right) \otimes X$ and $\tilde{H} = \left(\begin{smallmatrix} + & + \\ + & - \end{smallmatrix}\right) \otimes H$, and label the block rows $\tilde{x}_0, \ldots, \tilde{x}_{2n-1}$ and $\tilde{h}_0, \ldots, \tilde{h}_{2n-1}$. Define $G = \tilde{h}_0^t \tilde{x}_0$, and define the block matrices $E^t = \left(\begin{smallmatrix} E_1^t & \cdots & E_{2n-1}^t \end{smallmatrix}\right)^t$ and $F = \left(\begin{smallmatrix} F_1 & \cdots & F_{2n-1} \end{smallmatrix}\right)$ by $E_i = h_0^t \tilde{x}_i$ and $F_i = \tilde{h}_i^t x_0$.

As before, we then take $Z = \left(\begin{smallmatrix} G & F & -F \\ E & A & B \\ -E & B & A \end{smallmatrix}\right)$. The next result then follows.

6.13. Theorem. If there is an OD$(n; s_1, \ldots, s_u)$, then the block matrix Z is an OD$(4n^2; 4ns_1, \ldots, 4ns_u)$.

Proof. The proof amounts to checking the block entries of ZZ^t.

To begin, $GE_i^t = (\tilde{h}_0^t \tilde{x}_0)(h_0^t \tilde{x}_i)^t = \tilde{h}_0^t (\tilde{x}_0 \tilde{x}_i^t) h_0 = O$, hence $GE^t = EG^t = O$. Then

$F_i c_j^t = (\tilde{h}_i^t x_0)(h_j^t x_j)^t = \tilde{h}_i^t (x_0 x_j^t) h_j = \delta_{0j} \sigma \tilde{h}_i^t h_0$ so that

$$FA^t = FB^t = \sigma \begin{pmatrix} \tilde{h}_1 \\ \vdots \\ \tilde{h}_{2n-1} \end{pmatrix}^t (\mathbf{1}_{2n-1} \otimes h_0).$$

We have, therefore, that $FA^t - FB^t = O$. Then the inner product between the first and second, and the first and third, block rows of Z vanish.

Next, $E_i E_j^t = (h_0^t \tilde{x}_i)(h_0^t \tilde{x}_j)^t = h_0^t (\tilde{x}_i \tilde{x}_j^t) h_0 = \delta_{ij} 2\sigma J_n$. It follows that $EE^t = (E_i E_j^t) = (\delta_{ij} 2\sigma J_n) = 2\sigma (I_{2n-1} \otimes J_n)$.

We need to examine the product AB^t and in order to do that, we need to examine the cross-product correlations (see §3). To begin, the product between the first block row and column of A and B^t is given by $c_0 c_0^t + \sum_{i=1}^{n-1} c_i c_i^t - \sum_{i=1}^{n-1} c_i c_i^t = \sigma J_n$. Next, let a and b be two sequences of length $2n-1$ defined by

$$a_i = \begin{cases} c_i & \text{if } 0 \leq i < n, \text{ and} \\ c_{2n-i-1} & \text{if } n \leq i < 2n-1, \end{cases}$$

$$b_i = \begin{cases} c_0 & \text{if } i = 0, \\ -c_i & \text{if } 0 < i < n, \text{ and} \\ c_{2n-i-1} & \text{if } n \leq i < 2n-1. \end{cases}$$

For $j \in \{1, \dots 2n-2\}$, we have $C_j(a,b) = \sum_{i=0}^{2n-2} a_i b_{i+j}^t = \sum_{i=0}^{n-1} a_i b_{i+j}^t + \sum_{i=n}^{2n-2} a_i b_{i+j}^t$, where precisely one of the right-hand sums is nonzero by (6.12.b). For $j \in \{1, \dots, n-1\}$, we have that

$$\begin{aligned} \sum_{i=0}^{n-1} a_i b_{i+j}^t &= a_0 b_j^t + \sum_{i=1}^{n-j-1} a_i b_{i+j}^t + \sum_{i=n-j}^{n-1} a_i b_{i+j}^t \\ &= c_0 c_j^t - \sum_{i=1}^{n-j-1} c_i c_{i+j}^t + \sum_{i=n-j}^{n-1} c_i c_{2n-j-i-1}^t \\ &= \sum_{i=0}^{j-1} c_{n-j+i} c_{n-i-1}^t, \end{aligned}$$

and

$$\begin{aligned}\sum_{i=n}^{2n-2} a_i b_{i+j}^t &= \sum_{i=n}^{2n-j-2} a_i b_{i+j}^t + \sum_{i=2n-j-1}^{2n-2} a_i b_{i+j}^t \\ &= \sum_{i=0}^{n-j-2} c_{n-i-1} c_{n-j-i-1}^t + c_j c_0^t - \sum_{i=1}^{j-1} c_{j-i} c_i^t \\ &= -\sum_{i=1}^{j-1} c_{j-i} c_i^t.\end{aligned}$$

We have shown that, for $j \in \{1, \ldots, n-1\}$,

$$\begin{aligned}C_j(a,b) &= \sum_{i=0}^{j-1} c_{n-j+i} c_{n-i-1}^t - \sum_{i=1}^{j-1} c_{j-i} c_i^t \\ &= \begin{cases} \sigma h_{n-i-1}^t h_{n-i-1} & \text{if } j-1 = 2i \text{, for some } i\text{; and} \\ -\sigma h_i^t h_i & \text{if } j = 2i \text{, for some } i \end{cases}\end{aligned}$$

Similarly, we find that $C_{2n-j-1}(a,b) = -C_j(a,b)$, for $j \in \{1, \ldots, n-1\}$, and $C_j(b,a) = -C_j(a,b)$, for all $j \in \{1, \ldots, 2n-2\}$.

Putting things together, we have shown that

$$\begin{aligned}AB^t &= \mathrm{circ}(\sigma J, C_{2n-2}(a,b), \ldots, C_1(a,b)) \\ &= \mathrm{circ}(\sigma J, -C_1(a,b), \ldots, -C_{n-1}(a,b), C_{n-1}(a,b), \ldots, C_1(a,b)) \\ &= \sigma \mathrm{circ}(J, -h_{n-1}^t h_{n-1}, \ldots, -h_{n/2}^t h_{n/2}, \\ &\qquad\qquad h_{n/2}^t h_{n/2}, \ldots, h_{n-1}^t h_{n-1}), \text{ and} \\ BA^t &= \sigma \mathrm{circ}(J, h_{n-1}^t h_{n-1}, \ldots, h_{n/2}^t h_{n/2}, \\ &\qquad\qquad - h_{n/2}^t h_{n/2}, \ldots, -h_{n-1}^t h_{n-1}).\end{aligned}$$

It follows that the product between the second and third block rows vanishes, i.e. $-EE^t + AB^t + BA^t = O$.

It remains to evaluate the block diagonal entries of ZZ^t. The first block row gives

$$GG^t + 2\sum_{i=1}^{2n-1} F_i F_i^t = 2\sigma J + 2\sigma \sum_{i=1}^{2n-1} \tilde{h}_i^t \tilde{h}_i$$

$$
\begin{aligned}
&= 2\sigma J + 2\sigma \left(\sum_{i=0}^{2n-1} \tilde{h}_i^t \tilde{h}_i - \tilde{h}_0^t \tilde{h}_0 \right) \\
&= 2\sigma J + 2\sigma \left(2nI - J \right) \\
&= 4n\sigma I.
\end{aligned}
$$

By a similar argument about the sequences a and b given above, we find that

$$
\begin{aligned}
AA^t &= \sigma \mathrm{circ}(2nI - J, h_{n-1}^t h_{n-1}, \ldots, h_{n/2}^t h_{n/2}, \\
&\qquad\qquad h_{n/2}^t h_{n/2}, \ldots, h_{n-1}^t h_{n-1}), \text{ and} \\
BB^t &= \sigma \mathrm{circ}(2nI - J, -h_{n-1}^t h_{n-1}, \ldots, -h_{n/2}^t h_{n/2}, \\
&\qquad\qquad - h_{n/2}^t h_{n/2}, \ldots, -h_{n-1}^t h_{n-1}).
\end{aligned}
$$

Thus, $EE^t + AA^t + BB^t = 4\sigma I$. The proof is complete. ∎

We have shown that Z is an OD. It remains to show that Z admits a stable split. This is accomplished with the next result.

6.14. Theorem. Assume the OD Z of the previous theorem. Then:

(6.14.a) The submatrices $\left(\begin{smallmatrix} F \\ A \\ B \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} -F \\ B \\ A \end{smallmatrix}\right)$ yield stable splits, and

(6.14.b) The submatrices $\left(\begin{smallmatrix} E & A & B \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} -E & B & A \end{smallmatrix}\right)$ yield unstable splits.

Proof. It suffices to show the result for one matrix of each class. Note that

$$
\begin{pmatrix} F \\ A \\ B \end{pmatrix} \begin{pmatrix} F^t & A^t & B^t \end{pmatrix} = \begin{pmatrix} FF^t & FA^t & FB^t \\ AF^t & AA^t & AB^t \\ BF^t & BA^t & BB^t \end{pmatrix}.
$$

It follows by the proof of the previous theorem that

$$
\begin{aligned}
FF^t &= 2n\sigma I - \sigma J, \\
FA^t &= FB^t = \sigma \sum_{i=1}^{2n-1} \tilde{h}_i^t h_0, \\
AA^t &= \sigma \mathrm{circ}(2nI - J, h_{n-1}^t h_{n-1}, \ldots, h_{n/2}^t h_{n/2}, \\
&\qquad\qquad h_{n/2}^t h_{n/2}, \ldots, h_{n-1}^t h_{n-1}), \text{ and}
\end{aligned}
$$

$$
\begin{aligned}
BB^t = \sigma \mathrm{circ}(2nI - J, -h_{n-1}^t h_{n-1}, \ldots, -h_{n/2}^t h_{n/2},\\
- h_{n/2}^t h_{n/2}, \ldots, -h_{n-1}^t h_{n-1}).
\end{aligned}
$$

Therefore, the product between distinct rows is $\pm\sigma$, which shows (6.14.a).

Next,

$$
\begin{pmatrix} E^t \\ A^t \\ B^t \end{pmatrix} \begin{pmatrix} E\ A\ B \end{pmatrix} = \begin{pmatrix} E^tE\ E^tA\ E^tB \\ A^tE\ A^tA\ A^tB \\ B^tE\ B^tA\ B^tB \end{pmatrix}.
$$

However, $E^tE = n\sum_{i=1}^{2n-1} \tilde{x}_i^t \tilde{x}_0$ has off-diagonal entries in the set

$$
\{\pm x_1^{m_1} \cdots x_u^{m_u} : m_i \in \mathbf{N}\},
$$

which shows (6.14.b). ∎

6.15. Example. Applying the construction to the $\mathrm{OD}(2; 1, 1)$

(6.15.a) $\begin{pmatrix} a & b \\ b & \overline{a} \end{pmatrix},$

we obtain the balancedly splittable $\mathrm{OD}(16; 8, 8)$

(6.15.b)
$$
\left(\begin{array}{cccccccccccccccc}
a & b & a & b & \mathbf{a} & \mathbf{b} & \mathbf{a} & \mathbf{b} & \mathbf{a} & \mathbf{b} & \overline{a} & \overline{b} & \overline{a} & \overline{b} & \overline{a} & \overline{b} \\
a & b & a & b & \overline{\mathbf{a}} & \overline{\mathbf{b}} & \mathbf{a} & \mathbf{b} & \overline{\mathbf{a}} & \overline{\mathbf{b}} & a & b & \overline{a} & \overline{b} & a & b \\
a & b & a & b & \mathbf{a} & \mathbf{b} & \overline{\mathbf{a}} & \overline{\mathbf{b}} & \overline{\mathbf{a}} & \overline{\mathbf{b}} & \overline{a} & \overline{b} & a & b & a & b \\
a & b & a & b & \overline{\mathbf{a}} & \overline{\mathbf{b}} & \overline{\mathbf{a}} & \overline{\mathbf{b}} & \mathbf{a} & \mathbf{b} & a & b & a & b & \overline{a} & \overline{b} \\
b & \overline{a} & b & \overline{a} & \mathbf{a} & \mathbf{b} & \mathbf{b} & \overline{\mathbf{a}} & \mathbf{b} & \overline{\mathbf{a}} & a & b & b & \overline{a} & \overline{b} & a \\
b & \overline{a} & b & \overline{a} & \mathbf{a} & \mathbf{b} & \overline{\mathbf{b}} & \mathbf{a} & \overline{\mathbf{b}} & \mathbf{a} & a & b & \overline{b} & a & b & \overline{a} \\
a & b & \overline{a} & \overline{b} & \mathbf{b} & \overline{\mathbf{a}} & \mathbf{a} & \mathbf{b} & \mathbf{b} & \overline{\mathbf{a}} & \overline{b} & a & a & b & b & \overline{a} \\
a & b & \overline{a} & \overline{b} & \overline{\mathbf{b}} & \mathbf{a} & \mathbf{a} & \mathbf{b} & \overline{\mathbf{b}} & \mathbf{a} & b & \overline{a} & a & b & \overline{b} & a \\
b & \overline{a} & \overline{b} & a & \mathbf{b} & \overline{\mathbf{a}} & \mathbf{b} & \overline{\mathbf{a}} & \mathbf{a} & \mathbf{b} & b & \overline{a} & \overline{b} & a & a & b \\
b & \overline{a} & \overline{b} & a & \overline{\mathbf{b}} & \mathbf{a} & \overline{\mathbf{b}} & \mathbf{a} & \mathbf{a} & \mathbf{b} & \overline{b} & a & b & \overline{a} & a & b \\
\overline{b} & a & \overline{b} & a & \mathbf{a} & \mathbf{b} & \mathbf{b} & \overline{\mathbf{a}} & \overline{\mathbf{b}} & \mathbf{a} & a & b & b & \overline{a} & b & \overline{a} \\
\overline{b} & a & \overline{b} & a & \mathbf{a} & \mathbf{b} & \overline{\mathbf{b}} & \mathbf{a} & \mathbf{b} & \overline{\mathbf{a}} & a & b & \overline{b} & a & \overline{b} & a \\
\overline{a} & \overline{b} & a & b & \overline{\mathbf{b}} & \mathbf{a} & \mathbf{a} & \mathbf{b} & \mathbf{b} & \overline{\mathbf{a}} & b & \overline{a} & a & b & b & \overline{a} \\
\overline{a} & \overline{b} & a & b & \mathbf{b} & \overline{\mathbf{a}} & \mathbf{a} & \mathbf{b} & \overline{\mathbf{b}} & \mathbf{a} & \overline{b} & a & a & b & \overline{b} & a \\
\overline{b} & a & b & \overline{a} & \mathbf{b} & \overline{\mathbf{a}} & \overline{\mathbf{b}} & \mathbf{a} & \mathbf{a} & \mathbf{b} & b & \overline{a} & b & \overline{a} & a & b \\
\overline{b} & a & b & \overline{a} & \overline{\mathbf{b}} & \mathbf{a} & \mathbf{b} & \overline{\mathbf{a}} & \mathbf{a} & \mathbf{b} & \overline{b} & a & \overline{b} & a & a & b
\end{array}\right),
$$

where a stable vertical split is shown in bold.

Finally, we see at once how Theorem 5.9 is a consequence of Theorem 6.13 as we can simply set the indeterminates to $+1$ to obtain the result.

6.16. Example. Beginning with the OD$(4;1,1,1,1)$

$$
\text{(6.16.a)} \quad \begin{pmatrix} a & b & c & d \\ \overline{b} & a & \overline{d} & c \\ \overline{c} & d & a & \overline{b} \\ \overline{d} & \overline{c} & b & a \end{pmatrix},
$$

we obtain the OD$(64;16,16,16,16)$ constructed in appendix A3.

6.17. Example. As before, we are not limited to real matrices. Consider the COD$(6;1,5)$

$$
\text{(6.17.a)} \quad \begin{pmatrix} ia & b & b & b & b & b \\ b & ia & b & \overline{b} & \overline{b} & b \\ b & b & ia & b & \overline{b} & \overline{b} \\ b & \overline{b} & b & ia & b & \overline{b} \\ b & \overline{b} & \overline{b} & b & ia & b \\ b & b & \overline{b} & \overline{b} & b & ia \end{pmatrix},
$$

where a and b are real, commuting indeterminates. Using the method, one can construct a COD$(144;24,120)$, which we do not show in this thesis due to its size.

§7. Related Configurations

In this section, the balancedly splittable orthogonal designs of the previous section are applied in the construction of related objects. We will explore quasi-symmetric balanced incomplete block designs, equiangular tight frames, and unbiased orthogonal designs.

* * *

7.1. Quasi-Symmetric BIBDs. Symmetric designs are characterized by the fact that the blocks of the design intersect in a constant number of points. The “next best” designs are those which have two cardinalities for the intersections between distinct blocks.

7.1. Definition. A balanced incomplete block design is *quasi-symmetric* if there exists at most two cardinalities that exist for the intersections between pairs of distinct blocks. The cardinalities of the intersections are called the *intersection numbers* of the design.[12]

These beautiful objects are studied extensively in Shrikhande and Sane (1991). Ionin and Shrikhande (2006) also contains a study of these objects, especially as it pertains to SRGs and association schemes.

7.2. Example. The following is the incidence matrix of a quasi-symmetric BIBD$(6, 15, 5, 2, 1)$

$$\begin{pmatrix} 0&1&0&0&0&1&0&0&1&0&0&0&1&0&1 \\ 0&1&0&0&0&0&1&1&0&0&0&1&0&1&0 \\ 1&0&0&0&1&0&0&1&0&0&1&0&0&0&1 \\ 1&0&0&1&0&0&0&0&1&1&0&0&0&1&0 \\ 0&0&1&1&0&0&1&0&0&0&1&0&1&0&0 \\ 0&0&1&0&1&1&0&0&0&1&0&1&0&0&0 \end{pmatrix} \tag{7.2.a}$$

with block intersection numbers 0 and 1.

Using the balancedly splittable Hadamard matrices constructed in Theorem 6.13, we can construct quasi-symmetric BIBDs. For the proposition, we assume the notations of $\S 5.3$.

7.3. Proposition. If there is a Hadamard matrix of order n, there is a quasi-symmetric BIBD$(2n^2-n, 4n^2-1, 2n^2-n-1, n^2-n, n^2-n-1)$ with intersection numbers $(n^2-n)/2$ and $(n^2-2n)/2$.

Proof. If there exists a normalized Hadamard matrix of order n, there exists a balancedly splittable Hadamard matrix of order $4n^2$ by Theorem 5.9 (or by Theorem 6.13 upon setting the indeterminates to unity if there is a full OD of order n). It suffices to show the result for one of the four splits obtained by the theorem; in particular, we will show the result for the submatrix $X = \begin{pmatrix} -F \\ B \\ A \end{pmatrix}$. Form the matrix $\tilde{Y} = (1/2)(J - X)$, and take Y to be the matrix formed by omitting the first row of $\tilde{Y}$ consisting of all zeros.

Since H is assumed to be normalized, each row of H and $\begin{pmatrix} + & + \\ + & - \end{pmatrix} \otimes H$, other than the first, of course, contain $n/2$ and $n^2/2$ minus 1s, respectively. By construction, then, the first row of F consists of all ones, while the remaining rows have $n^2 - n$ $+1$s; thus, the same rows of $-F$ have $n^2 - n$ -1s. Similarly, A and B have $n(2n-2)/2 = n^2 - n$ -1s in each row. It also follows that there must be $2n^2 - 2n + n - 1 = 2n^2 - n - 1$ -1s in each column of X.

Finally, the index of the design follows by considering the Menon design[13] induced by the regular Hadamard matrix of the construction.

It remains to consider the block intersection numbers of the design. Consider the first row of X along with any other two distinct rows. Without loss of generality, we can assume the following situation.

$$\begin{array}{cccc}
\overbrace{\begin{matrix} - & \dots & - \\ + & \dots & + \\ + & \dots & + \end{matrix}}^{a} &
\overbrace{\begin{matrix} - & \dots & - \\ + & \dots & + \\ - & \dots & - \end{matrix}}^{b} &
\overbrace{\begin{matrix} - & \dots & - \\ - & \dots & - \\ + & \dots & + \end{matrix}}^{c} &
\overbrace{\begin{matrix} - & \dots & - \\ - & \dots & - \\ - & \dots & - \end{matrix}}^{d}
\end{array}$$

This yields the two linear systems of equations

$$\begin{aligned}
a + b + c + d &= 2n^2 - n, \\
-a - b + c + d &= -n, \\
-a + b - c + d &= -n, \text{ and} \\
a - b - c + d &= \pm n.
\end{aligned}$$

Solving for d gives $2d = n^2 - n$ or $n^2 - 2n$, which proves the result. ∎

7.4. Example. The quasi-symmetric BIBD$(6, 15, 5, 2, 1)$ given by (7.2.a) is obtained by the splittable Hadamard matrix (5.6.e).

The parameters of the quasi-symmetric designs in the previous result are not new, however. They represent a subset of the designs obtained in Bracken et al. (2006) viz. McGuire (1997).

* * *

7.2. Equiangular Tight Frames. In our earliest collegiate education, we learn that given some linear space, the "nice" bases are those which are orthonormal. As Han et al. (2007) points out, however, the property of orthonormality can be too restrictive in many applications. For instance, if a signal is interfered with in transmission, then the lost information cannot be recovered.

By contrast, a frame is an overdetermined spanning set of vectors. In this way, calculations and applications involving these objects can at times be simplified and data loss deterred. Frames have theoretical applications as well, as shown in the seminal paper of Paley and Wiener (1987) using different language, however.

We will not have the opportunity to explore the theory of frames in any depth; rather, we direct the reader to the standard references of Christensen (2016), Waldron (2018), and Young

(2001).

7.5. Definition. Let H be a Hilbert space[14] with inner product $\langle\cdot,\cdot\rangle$, and let $\{f_i\} \subset H$. Then $\{f_i\}$ is a *frame* if there exists two constants A and B such that

(7.5.a) $A\|f\|^2 \leq \sum_i |\langle f, f_i\rangle|^2 \leq B\|f\|^2$, for all $f \in H$.

A and B are called frame bounds. If $\#\{f_i\} < \infty$, it is a *finite frame*. If $A = B$, the frame is said to be *tight*. If $\sum_i |\langle f, f_i\rangle|^2 = \|f\|^2$, then $\{f_i\}$ is said to be a *Parseval frame*.

The frame bounds of a frame are not unique. The *optimal upper frame bound* is the infimum of the collection of all upper bounds of the frame. Similarly, the *optimal lower frame bound* is the supremum of the collection of all lower bounds of the frame.

It also follows from the definition that $\overline{\text{span}}\{f_i\} = H$.

7.6. Example. If $\{e_i\}$ is an orthonormal basis, then $\{e_0, e_0, e_1, e_1, \dots\}$, where each element is repeated twice, is a tight frame with $A = 2$.

7.7. Example. If $\{e_i\}$ is an orthonormal basis, then $\{e_0, 2^{-\frac{1}{2}}e_1, 2^{-\frac{1}{2}}e_1, 3^{-\frac{1}{2}}e_2, 3^{-\frac{1}{2}}e_2, 3^{-\frac{1}{2}}e_2\}$, where the element $(n+1)^{-\frac{1}{2}}e_n$ is repeated $n+1$ times, is a frame with $A = 1$.

From this point on, we will assume that we are working with a finite dimensional Hilbert space of dimension k. Our frames will, therefore, always be finite. Just as for the standard bases of a space, we can relate matrices to finite frames.

7.8. Definition. Let $\{f_0, \dots, f_{n-1}\}$ be a finite frame in a finite dimensional Hilbert space H. The matrix $\Theta = \begin{pmatrix} f_0^* \\ \vdots \\ f_{n-1}^* \end{pmatrix}$ is the *analysis operator* of the frame, and Θ^* is called the *synthesis operator* of the frame.

The following is then a restatement of the definitions.

7.9. Proposition. Let $x_0, \dots, x_{n-1}$ be vectors in a Hilbert space H of dimension k, and define the $k \times n$ matrix $T = (x_0, \dots, x_{n-1})$ with columns given by the vectors. Then:

(7.9.a) $\{x_0, \dots, x_{n-1}\}$ is a frame if and only if $\text{rank}(T) = k$,

(7.9.b) $\{x_0, \dots, x_{n-1}\}$ is a tight frame with bound A if and only if $TT^* = \sqrt{A}I$.

In what follows, we will take $\mathbf{H} = \{a_1 + a_i i + a_j j + a_k k : a_1, a_i, a_j, a_k \in \mathbf{R}\}$, i.e. the $\mathbf{R}$-algebra of quaternions [15], and we will consider frames over $\mathbf{H}^k$. Furthermore, for $q = z + wj$ where $z, w \in \mathbf{C}$, we define $\mathrm{Co}_1(q) = z$ and $\mathrm{Co}_2(q) = w^*$. Finally, define $[\cdot]_{\mathbf{C}} : \mathbf{H}^d \to \mathbf{C}^{2d}$ by $[q]_{\mathbf{C}} = \begin{pmatrix} \mathrm{Co}_1(q) \\ \mathrm{Co}_2(q) \end{pmatrix} = \left(\begin{smallmatrix} z \\ w^* \end{smallmatrix} \right)$.

The following is Theorem 3.2 of Waldron (2020).

7.10. Theorem. (7.10.a) Tight frames for $\mathbf{C}^{2k}$ correspond to tight frames for $\mathbf{H}^k$, and

(7.10.b) Tight frames for $\mathbf{H}^k$ correspond to tight frames for $\mathbf{C}^{2k}$ if and only if $\sum_{i,j} |\mathrm{Co}_1(\langle f_i, f_j \rangle)|^2 = \sum_{i,j} |\mathrm{Co}_2(\langle f_i, f_j \rangle)|^2$.

In light of this result, it is of interest to find frames over $\mathbf{H}^k$ for which they are not equivalent to frames over $\mathbf{C}^{2k}$, i.e. there can be no reduction in the ring of scalars. We present two examples of such frames.

7.11. Example. The following is a $\mathrm{QOD}(2; 1, 1)$

(7.11.a) $$\begin{pmatrix} \overline{a} & bi \\ \overline{b}j & ak \end{pmatrix},$$

where a and b are real variables. We then arrive at the following $\mathrm{QOD}(16; 8, 8)$

(7.11.b) $$\begin{pmatrix}
\overline{a} & bi & \overline{a} & bi & \overline{a} & bi & \overline{a} & bi & \overline{a} & bi & a & \overline{b}i & a & \overline{b}i & a & \overline{b}i \\
\overline{a} & bi & \overline{a} & bi & a & \overline{b}i & \overline{a} & bi & a & \overline{b}i & \overline{a} & bi & a & \overline{b}i & \overline{a} & bi \\
\overline{a} & bi & \overline{a} & bi & \overline{a} & bi & a & \overline{b}i & a & \overline{b}i & a & \overline{b}i & \overline{a} & bi & \overline{a} & bi \\
\overline{a} & bi & \overline{a} & bi & a & \overline{b}i & a & \overline{b}i & \overline{a} & bi & \overline{a} & bi & \overline{a} & bi & a & \overline{b}i \\
\bar{\mathbf{b}}\mathbf{j} & \mathbf{ak} & \bar{\mathbf{b}}\mathbf{j} & \mathbf{ak} & \bar{\mathbf{a}} & \mathbf{bi} & \bar{\mathbf{b}}\mathbf{j} & \mathbf{ak} & \bar{\mathbf{b}}\mathbf{j} & \mathbf{ak} & \bar{\mathbf{a}} & \mathbf{bi} & \bar{\mathbf{b}}\mathbf{j} & \mathbf{ak} & \mathbf{bj} & \bar{\mathbf{a}}\mathbf{k} \\
\bar{\mathbf{b}}\mathbf{j} & \mathbf{ak} & \bar{\mathbf{b}}\mathbf{j} & \mathbf{ak} & \bar{\mathbf{a}} & \mathbf{bi} & \mathbf{bj} & \bar{\mathbf{a}}\mathbf{k} & \mathbf{bj} & \bar{\mathbf{a}}\mathbf{k} & \bar{\mathbf{a}} & \mathbf{bi} & \mathbf{bj} & \bar{\mathbf{a}}\mathbf{k} & \bar{\mathbf{b}}\mathbf{j} & \mathbf{ak} \\
\bar{\mathbf{a}} & \mathbf{bi} & \mathbf{a} & \bar{\mathbf{b}}\mathbf{i} & \bar{\mathbf{b}}\mathbf{j} & \mathbf{ak} & \bar{\mathbf{a}} & \mathbf{bi} & \bar{\mathbf{b}}\mathbf{j} & \mathbf{ak} & \mathbf{bj} & \bar{\mathbf{a}}\mathbf{k} & \bar{\mathbf{a}} & \mathbf{bi} & \bar{\mathbf{b}}\mathbf{j} & \mathbf{ak} \\
\bar{\mathbf{a}} & \mathbf{bi} & \mathbf{a} & \bar{\mathbf{b}}\mathbf{i} & \mathbf{bj} & \bar{\mathbf{a}}\mathbf{k} & \bar{\mathbf{a}} & \mathbf{bi} & \mathbf{bj} & \bar{\mathbf{a}}\mathbf{k} & \bar{\mathbf{b}}\mathbf{j} & \mathbf{ak} & \bar{\mathbf{a}} & \mathbf{bi} & \mathbf{bj} & \bar{\mathbf{a}}\mathbf{k} \\
\bar{\mathbf{b}}\mathbf{j} & \mathbf{ak} & \mathbf{bj} & \bar{\mathbf{a}}\mathbf{k} & \bar{\mathbf{b}}\mathbf{j} & \mathbf{ak} & \bar{\mathbf{b}}\mathbf{j} & \mathbf{ak} & \bar{\mathbf{a}} & \mathbf{bi} & \bar{\mathbf{b}}\mathbf{j} & \mathbf{ak} & \mathbf{bj} & \bar{\mathbf{a}}\mathbf{k} & \bar{\mathbf{a}} & \mathbf{bi} \\
\bar{\mathbf{b}}\mathbf{j} & \mathbf{ak} & \mathbf{bj} & \bar{\mathbf{a}}\mathbf{k} & \mathbf{bj} & \bar{\mathbf{a}}\mathbf{k} & \mathbf{bj} & \bar{\mathbf{a}}\mathbf{k} & \bar{\mathbf{a}} & \mathbf{bi} & \mathbf{bj} & \bar{\mathbf{a}}\mathbf{k} & \bar{\mathbf{b}}\mathbf{j} & \mathbf{ak} & \bar{\mathbf{a}} & \mathbf{bi} \\
bj & \overline{a}k & bj & \overline{a}k & \overline{a} & bi & \overline{b}j & ak & bj & \overline{a}k & \overline{a} & bi & \overline{b}j & ak & \overline{b}j & ak \\
bj & \overline{a}k & bj & \overline{a}k & \overline{a} & bi & bj & \overline{a}k & \overline{b}j & ak & \overline{a} & bi & bj & \overline{a}k & bj & \overline{a}k \\
a & \overline{b}i & \overline{a} & bi & bj & \overline{a}k & \overline{a} & bi & \overline{b}j & ak & \overline{b}j & ak & \overline{a} & bi & \overline{b}j & ak \\
a & \overline{b}i & \overline{a} & bi & \overline{b}j & ak & \overline{a} & bi & bj & \overline{a}k & bj & \overline{a}k & \overline{a} & bi & bj & \overline{a}k \\
bj & \overline{a}k & \overline{b}j & ak & \overline{b}j & ak & bj & \overline{a}k & \overline{a} & bi & \overline{b}j & ak & \overline{b}j & ak & \overline{a} & bi \\
bj & \overline{a}k & \overline{b}j & ak & bj & \overline{a}k & \overline{b}j & ak & \overline{a} & bi & bj & \overline{a}k & bj & \overline{a}k & \overline{a} & bi
\end{pmatrix}.$$

Taking the horizontal frame shown in bold as the synthesis operator of a frame over $\mathbf{H}^6$ after setting $a = b = 1$, we find that $(1/2)H^*H$ is given by

$$
\left(\begin{array}{cccccccccccccccc}
3 & i & - & i & \overline{j} & \overline{k} & 1 & \overline{i} & \overline{j} & \overline{k} & \overline{j} & \overline{k} & 1 & \overline{i} & \overline{j} & \overline{k} \\
\overline{i} & 3 & \overline{i} & - & k & \overline{j} & i & 1 & k & \overline{j} & k & \overline{j} & i & 1 & k & \overline{j} \\
- & i & 3 & i & \overline{j} & \overline{k} & - & i & j & k & \overline{j} & \overline{k} & - & i & j & k \\
\overline{i} & - & \overline{i} & 3 & k & \overline{j} & \overline{i} & - & \overline{k} & j & k & \overline{j} & \overline{i} & - & \overline{k} & j \\
j & \overline{k} & j & \overline{k} & 3 & i & 1 & i & 1 & i & 1 & \overline{i} & - & \overline{i} & 1 & i \\
k & j & k & j & \overline{i} & 3 & \overline{i} & 1 & \overline{i} & 1 & i & 1 & i & - & \overline{i} & 1 \\
1 & \overline{i} & - & i & 1 & i & 3 & i & 1 & i & 1 & i & 1 & \overline{i} & - & \overline{i} \\
i & 1 & \overline{i} & - & \overline{i} & 1 & \overline{i} & 3 & \overline{i} & 1 & \overline{i} & 1 & i & 1 & i & - \\
j & \overline{k} & \overline{j} & k & 1 & i & 1 & i & 3 & i & - & \overline{i} & 1 & i & 1 & \overline{i} \\
k & j & \overline{k} & \overline{j} & \overline{i} & 1 & \overline{i} & 1 & \overline{i} & 3 & i & - & \overline{i} & 1 & i & 1 \\
j & \overline{k} & j & \overline{k} & 1 & \overline{i} & 1 & i & - & \overline{i} & 3 & i & - & \overline{i} & - & \overline{i} \\
k & j & k & j & i & 1 & \overline{i} & 1 & i & - & \overline{i} & 3 & i & - & i & - \\
1 & \overline{i} & - & i & - & \overline{i} & 1 & \overline{i} & 1 & i & - & \overline{i} & 3 & i & - & \overline{i} \\
i & 1 & \overline{i} & - & i & - & i & 1 & \overline{i} & 1 & i & - & \overline{i} & 3 & i & - \\
j & \overline{k} & \overline{j} & k & 1 & i & - & \overline{i} & 1 & \overline{i} & - & \overline{i} & - & \overline{i} & 3 & i \\
k & j & \overline{k} & \overline{j} & \overline{i} & 1 & i & - & i & 1 & i & - & i & - & \overline{i} & 3
\end{array}\right). \tag{7.11.c}
$$

It follows that $320 = \sum_{i,j} |\mathrm{Co}_1(\langle f_i, f_j\rangle)|^2 \neq \sum_{i,j} |\mathrm{Co}_2(\langle f_i, f_j\rangle)|^2 = 64$ so that the frame is not reducible to a frame over $\mathbf{C}^{12}$.

7.12. Example. Beginning with the $\mathrm{QOD}(6; 1, 5)$

$$
\left(\begin{array}{cccccc}
a & b & b & b & b & b \\
b & \overline{a} & bk & \overline{bk} & \overline{bk} & bk \\
b & bk & \overline{a} & bk & \overline{bk} & \overline{bk} \\
b & \overline{bk} & bk & \overline{a} & bk & \overline{bk} \\
b & \overline{bk} & \overline{bk} & bk & \overline{a} & bk \\
b & bk & \overline{bk} & \overline{bk} & bk & \overline{a}
\end{array}\right), \tag{7.12.a}
$$

we apply the construction to obtain a $\mathrm{QOD}(144; 24, 120)$. Taking H to again be the first horizontal frame of the splittable QOD, we find that $29056 = \sum_{i,j} |\mathrm{Co}_1(\langle f_i, f_j\rangle)|^2 \neq \sum_{i,j} |\mathrm{Co}_2(\langle f_i, f_j\rangle)|^2 = 8960$. It follows that we have constructed another quaternion frame not reducible to a complex frame.

* * *

7.3. Unbiased Orthogonal Designs. Let A and B be two orthonormal bases of the Hilbert space $\mathbf{C}^k$. The bases are said to be *mutually unbiased* in the event that $|\langle a, b\rangle| = k^{-\frac{1}{2}}$, for every $a \in A$ and $b \in B$, and where $\langle a, b\rangle$ denotes the usual sesquilinear inner product between a and b.

The reader will recognize unbiased bases as a subtype of equiangular lines. These objects are fundamental in many applications such as quantum key distribution (see Boykin et al., 2005).

Unit Hadamard matrices, i.e. a matrices with entries from the unit circle in the complex plane whose rows are pairwise orthogonal, have received much attention with their connection to unbiased bases. Two unit Hadamard matrices H and K of order n are *unbiased* if $HK^* = \sqrt{n}L$, for some unit Hadamard matrix L. Recently, these ideas where studied for the case of unit weighing matrices[16] in Best et al. (2015). The case of unbiased real Hadamard matrices were studied in Holzmann et al. (2010), and the case of unbiased quaternary complex Hadamard in Best and Kharaghani (2010).

We have seen that extending concepts from matrices of concrete values to matrices of indeterminates usually presents us with subtle difficulties, and extending unbiasedness is no different. In Kharaghani and Suda (2018), unbiased orthogonal designs are presented thus.

7.13. Definition. Let X_1 and X_2 be two instances of an OD$(n; s_1, \dots, s_u)$ in the indeterminates $x_1, \dots, x_u$. Then X_1 and X_1 are *unbiased* with parameter $\alpha \in \mathbf{R}_+$ if there is a $(-1, 0, 1)$-matrix W such that

$$\text{(7.13.a)}\quad X_1 X_2^t = \left(\alpha^{-\frac{1}{2}} \sum_i s_i x_i^t\right) W.$$

We use the stable construction presented in the previous section to construct pairs of unbiased orthogonal designs. Note that this is essentially a generalization of a method presented in Kharaghani and Suda (2019). In what follows, we take Z to be as in §9.3.

7.14. Proposition. Let

$$V = \begin{pmatrix} -F \\ B \\ A \end{pmatrix}, \text{ and } U = \begin{pmatrix} G & F \\ E & A \\ -E & B \end{pmatrix}$$

so that $Z = \left(\begin{smallmatrix} U & V \end{smallmatrix}\right)$, and take $Y = \left(\begin{smallmatrix} U & -V \end{smallmatrix}\right)$. Then Y is an OD of the same order and type as Z. In particular, $ZY^t = 2\sigma K$, where K is a Hadamard matrix.

Proof. That Y is also an OD is clear. Then $ZY^t = (UU^t - VV^t)$. We claim that $K = (1/2\sigma)(UU^t - VV^t)$ is a Hadamard matrix. Indeed, since the vertical frames of Z constitute a stable split, and since Z is an OD, we find that K has entries from the set $\{-1, 1\}$. Furthermore,

$$KK^t = \frac{1}{4\sigma^2}(UU^tUU^t + VV^tVV^t)$$

$$= \frac{n}{\sigma}(UU^t + VV^t)$$
$$= 4n^2 I,$$

and the proof is complete. ∎

Notes

9. Lines are usually represented by vectors in a space such as $\mathbf{R}^\ell$; however, there is an essential difference that exists between the two, namely, there is a unique angle that exists between two vectors and a pair of complementary angles that exist between two lines. Therefore, in defining equiangularity and related concepts between lines, it is customary to only consider the absolute values of the inner products between the vectors representing the lines.

10. The classic construction due to Paley goes as follows. Let q be odd, and take GF$(q) = \{a_0 = 0, a_1, \ldots, a_{q-1}\}$. If η is the quadratic character on GF(q), define Q by $Q_{ij} = \eta(a_j - a_i)$. Finally, form the matrix $W = \begin{pmatrix} 0 & \mathbf{1}^t \\ (-1)^{(q-1)/2}\mathbf{1} & Q \end{pmatrix}$. Then W is a W$(q+1, q)$. If $q \equiv -1 \pmod 4$, then $W^t = -W$; while if $q \equiv 1 \pmod 4$, then $W^t = W$. See Hall (1986) for more detail.

11. In defining the auxiliary matrix of a full OD, we limit ourselves to using the Hadamard matrix obtained from the OD by setting the indeterminates equal to $+1$. This is done that the formation of the auxiliary matrices is unique.

12. Quasi-symmetric designs are often studied in the more general context of t-designs (see §1.1). This is the approach taken in Ionin and Shrikhande (2006) and Shrikhande and Sane (1991).

13. It isn't difficult to see that a regular Hadamard matrix H must have square order $4u^2$. Moreover, it follows that $(1/2)(J - H)$ is a symmetric BIBD$(4u^2, 2h^2 - h, h^2 - h)$. The converse can also seen to be true. The reader may consult Stinson (2004) for a standard treatment of these objects. Ionin and Shrikhande (2006) studies these objects and their applications in constructing other configurations.

14. Recall that a Hilbert space is a complete normed linear space where the norm is induced by an inner product. Halmos (1982) and Halmos (1998) are standard references on the subject of Hilbert spaces.

15. There is some ambiguity in defining frames over $\mathbf{H}^k$. As $\mathbf{H}$ is a skew-field, we can take $\mathbf{H}^k$ to be either a left- or right-module. This discussion is taken up in detail in Waldron (2020), where the right-module is always used.

16. There are different ways of defining unbiased weighing matrices. Classically, two unitary weighing matrices W(n, k), say H and K, are *unbiased* if $HK^* = \sqrt{k}L$, for some unitary weighing matrix L of weight k. In Nozaki and Suda (2015), the weighing matrices H and K above are *quasi-unbiased* with parameters (n, k, ℓ, a) if $a^{-\frac{1}{2}}HK^*$ is a W(n, ℓ). It is this idea of quasi-unbiasedness that is most easily extended to orthogonal designs.

3

A New Family of Balanced Weighing Matrices and Association Schemes

This chapter serves as a particular application of a more general construction to be given in the following chapter. Here a new family of balanced weighing matrices are constructed, and an equivalence to certain association schemes is developed. The work shown here is a modified version of Kharaghani et al. (2022b).

§8. Generalized Kronecker Product

In this section, we briefly review the Kronecker product and a few of its properties and applications, namely, we will see its applications to regular Hadamard matrices. Following this, we will consider a simple generalization of the Kronecker product in order to allow particular bijections to act on a matrix.

* * *

8.1. Definitions. Let A be an $n \times m$ matrix, and let B be an $s \times t$ matrix, each with entries from some field. The *Kronecker product* of A by B is the $ns \times mt$ matrix given by $A \otimes B = (A_{ij}B)$. This noncommutative matrix product has the following immediate properties wherever the sizes of the matrices make sense.

8.1. Lemma. Let A, B, C and D be matrices over some field F, and let $\lambda \in F$. Then:

(8.1.a) $\lambda A \otimes B = A \otimes \lambda B = \lambda(A \otimes B)$,

(8.1.b) $A \otimes B = (A \times I)(I \otimes B) = (I \otimes B)(A \otimes I)$,

(8.1.c) $(A + B) \otimes C = (A \otimes C) + (B \otimes C)$,

(8.1.d) $A \otimes (B + C) = (A \otimes B) + (A \otimes C)$,

(8.1.e) $(A \otimes B)(C \otimes D) = AC \otimes BD$,

(8.1.f) $(A \otimes B)^t = A^t \otimes B^t$, and

(8.1.g) if A is $n \times n$ and B is $m \times m$, then $\det(A \otimes B) = [\det(A)]^m[\det(B)]^n$.

There are a number of immediate applications of the Kronecker product to what we have done so far. For instance, if H is a $\mathrm{W}(n_1, k_1)$ and K a $\mathrm{W}(n_2, k_2)$, then $H \otimes K$ is a $\mathrm{W}(n_1 n_2, k_1 k_2)$.

8.2. Example. Let H be the Hadamard matrix of order 4 given by

(8.2.a) $$\begin{pmatrix} - & + & + & + \\ + & - & + & + \\ + & + & - & + \\ + & + & + & - \end{pmatrix}.$$

Then $H \otimes H$ is the Hadamard matrix of 16 given by

(8.2.b) $$\left(\begin{array}{cccccccccccccccc}
+ & - & - & - & - & + & + & + & - & + & + & + & - & + & + & + \\
- & + & - & - & + & - & + & + & + & - & + & + & + & - & + & + \\
- & - & + & - & + & + & - & + & + & + & - & + & + & + & - & + \\
- & - & - & + & + & + & + & - & + & + & + & - & + & + & + & - \\
- & + & + & + & + & - & - & - & - & + & + & + & - & + & + & + \\
+ & - & + & + & - & + & - & - & + & - & + & + & + & - & + & + \\
+ & + & - & + & - & - & + & - & + & + & - & + & + & + & - & + \\
+ & + & + & - & - & - & - & + & + & + & + & - & + & + & + & - \\
- & + & + & + & - & + & + & + & + & - & - & - & - & + & + & + \\
+ & - & + & + & + & - & + & + & - & + & - & - & + & - & + & + \\
+ & + & - & + & + & + & - & + & - & - & + & - & + & + & - & + \\
+ & + & + & - & + & + & + & - & - & - & - & + & + & + & + & - \\
- & + & + & + & - & + & + & + & - & + & + & + & + & - & - & - \\
+ & - & + & + & + & - & + & + & + & - & + & + & - & + & - & - \\
+ & + & - & + & + & + & - & + & + & + & - & + & - & - & + & - \\
+ & + & + & - & + & + & + & - & + & + & + & - & - & - & - & +
\end{array}\right),$$

where the block structure is evident.[17]

As one further application of the standard Kronecker product before we move on, we will consider regular Hadamard matrices.

8.3. Definition. A Hadamard matrix is *row regular* (or *column regular*) if its rows (resp. columns) have a constant sum. A Hadamard matrix is *regular* if it is both row and column regular.

It is not difficult to see that a Hadamard matrix is row regular or column regular if and only if it is regular. Moreover, if the constant row (column) sum is s, then the order of the matrix is s^2 (see Stinson, 2004, Chapter 4). By the definition of the standard Kronecker product, we also see that if H and K are regular Hadamard matrices, then so is $H \otimes K$.

Example 8.2 has evinced the existence of a regular $\mathrm{W}(4,4)$. It is also known that there is a regular $\mathrm{W}(36,36)$ (see Stinson, 2004, Chapter 4). Also, if there is a Hadamard matrix of order n, then there is a symmetric, regular, Hadamard matrix with constant block diagonal of order n^2 (see Colbourn and Dinitz, 2007, Part V). We then have the following result.

8.4. Proposition. If there exist Hadamard matrices of orders $n_1, \ldots, n_k$, then there is a regular Hadamard matrix of order $4^a 9^b n_1^2 \cdots n_k^2$, for $a, b \in \mathbf{Z}_+$ and $a \geq b$.

Having convinced ourselves of the utility of the standard Kronecker product, we generalize it in the following way. Let $\mathcal{M}$ be a collection of $n \times m$ matrices with entries from some commutative ring R, and let Ξ be an $s \times t$ matrix over the collection of endofunctions of $\mathcal{M}$. For each $A \in \mathcal{M}$, the $ns \times mt$ matrix $\Xi \otimes A$ over R is defined as $(\Xi_{ij}(A))$. If certain properties of A are left invariant under the elements of Ξ, then these properties will be reflected in the block structure of $\Xi \otimes A$.

With the above ideas in mind, if we have a collection $\mathcal{M}$ of objects, we desire a subset of the collection of endofunctions of $\mathcal{M}$ that are property preserving. If $\mathcal{M}$ is a collection of incidence structures, then a set of bijections of $\mathcal{M}$ that preserve incidence is called a *group of symmetries* of $\mathcal{M}$. If Ξ is a BGW over a group of symmetries, then $\Xi \otimes A$ has both inter-block regularites and intra-block regularities.

* * *

8.2. A First Application: Block designs. In the previous subsection, we began a brief discussion of groups of symmetries of incidence structures. As an example, let $\mathbf{D} = (X, \mathscr{B})$ be a $\mathrm{BIBD}(v, b, r, k, \lambda)$, and let $\mathscr{B} = \bigcup_{i=1}^{m} \mathscr{B}_i$ be a partition of the blocks of $\mathbf{D}$. If G_i acts on $\mathscr{B}_i$, then, $(X, \bigcup_{i=1}^{m} \mathscr{B}_i g_i)$, for $g_i \in G_i$, is again a $\mathrm{BIBD}(v, b, r, k, \lambda)$. Let $A = A(\mathbf{D})$ be the incidence matrix of $\mathbf{D}$, and assume that $A = \left(A(X,\mathscr{B}_1) \cdots A(X,\mathscr{B}_m) \right)$. Then $G = \prod_i G_i$ acts naturally on the columns of A. If in addition G_i is sharply transitive on $\mathscr{B}_i$, and if each point $x \in X$ appears r_i times in $\mathscr{B}_i$, then

$$|G|^{-1} \sum_{g \in G} Ag = \left(r_1 b_1^{-1} J_{v \times b_1} \cdots r_m b_m^{-1} J_{v \times b_m} \right),$$

where $b_i = |\mathscr{B}_i|$. Therefore, if $r_i b_j = r_j b_i$, for all $i, j \in \{1, \ldots, m\}$, then $\sum_{g \in G} Ag$ is an integer multiple of $J_{v \times b}$ since $b_i \mid |G|$. Since G acts on the columns of A, we also have that $Xg(Yg)^t = XY^t$, for each $X, Y \in AG$.

This motivates the following. If G is a group of symmetries of a collection $\mathcal{M}$ of BIBD(v, b, r, k, λ)s such that $\sum_{g \in G} Xg = \alpha_X J$, for some $\alpha_X \in \mathbf{N}$ and all $X \in \mathcal{M}$; and if $Xg(Yg)^t = XY^t$, for all $X, Y \in \mathcal{M}$; then we say that G is an *admissible* group of symmetries of $\mathbf{D}$. We then have the following (see Ionin, 2001).

8.5. Theorem. Let $\mathcal{M}$ be a collection of BIBD(v, b, r, k, λ)s, and let G be an admissible group of symmetries of $\mathcal{M}$. If Ξ is a BGW$(w, \ell, \mu; G)$ such that $kr\mu = v\lambda\ell$, then $\Xi \otimes X$ is a BIBD$(vw, bw, r\ell, k\ell, \lambda\ell)$, for any $X \in \mathcal{M}$.

Proof. Straightforward calculation. See Theorem 2.4 of Ionin (2001) for details. ∎

8.6. Corollary. If X is quasi-residual, then so is $\Xi \otimes X$.

8.7. Example. Let $q = p^n$ be a prime power, and let H be a GH$(q, 1)$ over EA(q) where the elements have the usual representation of $(0, 1)$-matrices. Then it isn't difficult to see that $A = \left(I_q \otimes \mathbf{1}_q \; H \right)$ is a BIBD$(q^2, q + q^2, 1 + q, q, 1)$. Importantly, the columns are placed into $1 + q$ consecutive disjoint groups of q columns each such that each point appears precisely once in every group.[18] Let G be the group which cyclically permutes the blocks in each group so that $|G| = q$.

If $1 + q$ is a prime power, then there is a BGW$(2 + q, 1 + q, q; C_q)$, say Ξ, where the group elements cyclically permute the blocks of each partition class in the above design. The parametric conditions of the theorem are met, hence $\Xi \otimes A$ is a BIBD$(q^2(2 + q), (q + q^2)(2 + q), (1 + q)^2, q(1 + q), 1 + q)$. More generally, we can take Ξ to be any BGW$((p^{n+1} - 1)/(p - 1), p^n, p^n - p^{n-1}; C_q)$, where $p = 1 + q$.

* * *

8.3. A Second Application: Bkaskar Rao designs. It isn't difficult to extend the ideas of Theorem 8.5 to the more general Bhaskar Rao designs. In order to accomplish this, however, we need to extend the idea of admissible groups of symmetries.

Let $\mathcal{M}$ be a collection of GBRD$(v, b, r, k, \lambda; H)$s. Then a group of bijections G of $\mathcal{M}$ is an admissible group of symmetries if (a) $\sum_{g \in G} Xg = \alpha_X HJ$, for some $\alpha_X \in \mathbf{N}$ and every $X \in \mathcal{M}$, and (b) $Xg(Yg)^* = XY^*$, for every $X, Y \in \mathcal{M}$.

The following is then a simple generalization of Theorem 8.5

8.8. Theorem. Let $\mathcal{M}$ be a collection of GBRD$(v, b, r, k, \lambda; H)$s, and let G be an admissible group of symmetries of $\mathcal{M}$. If Ξ is a BGW$(w, \ell, \mu; G)$ such that $kr\mu = v\lambda\ell$, then $\Xi \otimes X$ is a GBRD$(vw, bw, r\ell, k\ell, \lambda\ell; H)$, for any $X \in \mathcal{M}$.

8.9. Example. The following example was noted in Pender (2020). The following BGW$(15, 7, 3; C_3)$ was found by computational means in Gibbons and Mathon (1987)

$$
\begin{pmatrix}
0&3&3&3&3&3&3&3&0&0&0&0&0&0&0\\
0&3&2&0&1&0&0&0&0&0&2&0&2&2&3\\
0&0&3&2&0&1&0&0&3&0&0&2&0&2&2\\
0&0&0&3&2&0&1&0&2&3&0&0&2&0&2\\
0&0&0&0&3&2&0&1&2&2&3&0&0&2&0\\
0&1&0&0&0&3&2&0&0&2&2&3&0&0&2\\
0&0&1&0&0&0&3&2&2&0&2&2&3&0&0\\
0&2&0&1&0&0&0&3&0&2&0&2&2&3&0\\
3&3&1&0&2&0&0&0&3&2&0&1&0&0&0\\
3&0&3&1&0&2&0&0&0&3&2&0&1&0&0\\
3&0&0&3&1&0&2&0&0&0&3&2&0&1&0\\
3&0&0&0&3&1&0&2&0&0&0&3&2&0&1\\
3&2&0&0&0&3&1&0&1&0&0&0&3&2&0\\
3&0&2&0&0&0&3&1&0&1&0&0&0&3&2\\
3&1&0&2&0&0&0&3&2&0&1&0&0&0&3
\end{pmatrix},
\tag{8.9.a}
$$

where the nonzero entries are logarithms of some generator of C_3. Note that the core of (8.9.a) is composed of 4 circulant matrices. Taking R to be the residual part of (8.9.a), and letting $g = \begin{pmatrix} O & I_7 \\ \omega I_7 & O \end{pmatrix}$, we see that $G = \langle g \rangle$ forms an admissible group of symmetries for R. For any $n \in N$, there is a BGW$((7^{n+1} - 1)/6, 7^n, 6 \cdot 7^{n-1}; G)$ which satisfy the parametric conditions of Theorem 8.8. Therefore, for any $n > 0$, there is a GBRD$(4(7^{n+1} - 1)/3, 2 \cdot 7^{n+1}, 7^{n+1}, 4 \cdot 7^n, 3 \cdot 7^n; C_3)$.

This concludes our introduction to the generalized Kronecker product.

§9. A New Family of Balanced Weighing Matrices

Here the generalized Kronecker product of the previous subsection will be put to use in constructing a new family of balanced weighing matrices. Additionally, the simplex codes appear again and are applied in the construction. The construction presented here is indicative of a general method to be presented in the following chapter.

* * *

9.1. Lemmata. In §2.2, we introduced the linear simplex code $\mathscr{S}_{q,n}$. There it was shown that the code had constant weight q^{n-1}; in particular, it follows that it is equidistant with constant Hamming distance q^{n-1} since the code is linear.

In Rajkundlia (1983), and later reproduced in Ionin (2001) using the language of BGW matrices, generalized Hadamard matrices GH(q, q^{n-1}) were used recursively in conjunction with the classical parameter BGW$((q^n-1)/(q-1), q^{n-1}, q^{n-1}-q^{n-2}; \mathrm{GF}(q)^*)$s in order to construct certain designs. It turns out out that the GH(q, q^{n-1}) used in the construction can be replaced by $\mathscr{S}_{q,n}$, and so simplify the construction.

In order to apply the linear code $\mathscr{S}_{q,n}$, we will require the following lemma.

9.1. Lemma. Let GF$(q) = \{a_0 = 0, a_1, \ldots, a_{q-1}\}$, and let $n > 1$. Then there exist disjoint $(0,1)$-matrices $A_{a_1}, \ldots, A_{a_{q-1}}$ of dimensions $q^n \times (q^n-1)/(q-1)$ such that $\mathscr{S}_{q,n} = \sum_{\alpha \in \mathrm{GF}(q)^*} \alpha A_\alpha$. If we define $A_0 = J - \sum_{\alpha \in \mathrm{GF}(q)^*} A_\alpha$, then the following hold.

(9.1.a) $\sum_{\alpha \in \mathrm{GF}(q)} A_\alpha A_\alpha^t = \frac{q^{n-1}-1}{q-1} J + q^{n-1} I$, and

(9.1.b) $\sum_{\substack{\alpha,\beta \in \mathrm{GF}(q) \\ \alpha \neq \beta}} A_\alpha A_\beta^t = q^{n-1}(J-I)$.

Proof. Labeling the rows of $\mathscr{S}_{q,n}$ by $r_0, \ldots, r_{q^n-1}$, and taking $A = \mathscr{S}_{q,n}$, we then have that

$$\begin{aligned}
\Big(\sum_{\alpha \in \mathrm{GF}(q)} A_\alpha A_\alpha^t\Big)_{ij} &= \sum_{\alpha \in \mathrm{GF}(q)} (A_\alpha A_\alpha^t)_{ij} \\
&= \sum_{\alpha \in \mathrm{GF}(q)} \sum_{\ell=0}^{\frac{q(q^{n-1}-1)}{q-1}} (A_\alpha)_{i\ell}(A_\alpha)_{j\ell}
\end{aligned}$$

$$= \sum_{\alpha \in \mathrm{GF}(q)} \#\{\ell \in \{0, \ldots, \frac{q(q^{n-1}-1)}{q-1}\} : A_{i\ell} = A_{j\ell} = \alpha\}$$
$$= \#\{\ell \in \{0, \ldots, \frac{q(q^{n-1}-1)}{q-1}\} : A_{i\ell} = A_{j\ell}\}$$
$$= \frac{q^n - 1}{q-1} - \mathrm{dist}(r_i, r_j),$$

which shows (9.1.a).

Since $\sum_{\alpha \in \mathrm{GF}(q)} A_\alpha = J$, it follows that $\sum_{\alpha,\beta} A_\alpha A_\beta^t = (\sum_\alpha A_\alpha)(\sum_\beta A_\beta)^t = \frac{q^n-1}{q-1} J$, and (9.1.b) has been proven. ∎

If W is a $\mathrm{BGW}(v, k, \lambda; C_n)$ over some cyclic group $C_n = \{1, g, \ldots, g^{n-1}\}$ of order n, then there are n disjoint $(0,1)$-matrices $W_0, \ldots, W_{n-1}$ such that $W = W_0 + gW_1 + \cdots + g^{n-1}W_{n-1}$. We call W_0 and W_1 the *decomposition matrices* of the weighing matrix. Because W is a BGW matrix, we have the following lemma.

9.2. Lemma.

(9.2.a) $\sum_{i,j} g^{i-j} W_i W_j^t = \sum_{i,j} g^{i-j} W_j^t W_i = kI + \frac{\lambda}{n}(\sum_i g_i)(J - I)$,

(9.2.b) $\sum_i W_i W_i^t = \sum_i W_i^t W_i = kI + \frac{\lambda}{n}(J - I)$, and

(9.2.c) $\sum_i W_i W_{i+j}^t = \sum_i W_{i+j}^t W_i = \frac{\lambda}{n}(J - I)$, for $j \in \{1, \ldots, n-1\}$.

Proof. (9.2.a) is simply a restatement of the fact that both W and W^* are $\mathrm{BGW}(v, k, \lambda; C_n)$s (see Ionin and Shrikhande, 2006). (9.2.b) follows by noting that there are k nonzero entries in every row of W, and that 1 appears λ/n times in the conjugate inner product between distinct rows. Similarly, (9.2.c) follows by noting that each nonidentity element of the group appears λ/n times in the conjugate inner product between distinct rows of W, and that $i \neq i + j$, for each i whenever $j \not\equiv 0 \pmod{n}$. ∎

Now, consider the balanced $\mathrm{W}(19, 9)$ first appearing in de Launey and Sarvate (1984) where it was presented to the authors by Mathon.

$$W_{19} = \begin{pmatrix}
0 & + & + & + & + & + & + & + & + & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & - & - & 0 & + & 0 & 0 & 0 & + & - & + & + & 0 & + & 0 & 0 & 0 & + \\
0 & - & 0 & - & 0 & 0 & + & + & 0 & 0 & + & - & + & 0 & 0 & + & + & 0 & 0 \\
0 & - & - & 0 & + & 0 & 0 & 0 & + & 0 & + & + & - & + & 0 & 0 & 0 & + & 0 \\
0 & 0 & 0 & + & 0 & - & - & 0 & + & 0 & 0 & 0 & + & - & + & + & 0 & + & 0 \\
0 & + & 0 & 0 & - & 0 & - & 0 & 0 & + & + & 0 & 0 & + & - & + & 0 & 0 & + \\
0 & 0 & + & 0 & - & - & 0 & + & 0 & 0 & 0 & + & 0 & + & + & - & + & 0 & 0 \\
0 & 0 & + & 0 & 0 & 0 & + & 0 & - & - & 0 & + & 0 & 0 & 0 & + & - & + & + \\
0 & 0 & 0 & + & + & 0 & 0 & - & 0 & - & 0 & 0 & + & + & 0 & 0 & + & - & + \\
0 & + & 0 & 0 & 0 & + & 0 & - & - & 0 & + & 0 & 0 & 0 & + & 0 & + & + & - \\
+ & 0 & 0 & 0 & + & 0 & - & + & - & 0 & 0 & - & - & 0 & + & 0 & 0 & 0 & + \\
+ & 0 & 0 & 0 & - & + & 0 & 0 & + & - & - & 0 & - & 0 & 0 & + & + & 0 & 0 \\
+ & 0 & 0 & 0 & 0 & - & + & - & 0 & + & - & - & 0 & + & 0 & 0 & 0 & + & 0 \\
+ & + & - & 0 & 0 & 0 & 0 & + & 0 & - & 0 & 0 & + & 0 & - & - & 0 & + & 0 \\
+ & 0 & + & - & 0 & 0 & 0 & - & + & 0 & + & 0 & 0 & - & 0 & - & 0 & 0 & + \\
+ & - & 0 & + & 0 & 0 & 0 & 0 & - & + & 0 & + & 0 & - & - & 0 & + & 0 & 0 \\
+ & + & 0 & - & + & - & 0 & 0 & 0 & 0 & 0 & + & 0 & 0 & 0 & + & 0 & - & - \\
+ & - & + & 0 & 0 & + & - & 0 & 0 & 0 & 0 & 0 & + & + & 0 & 0 & - & 0 & - \\
+ & 0 & - & + & - & 0 & + & 0 & 0 & 0 & + & 0 & 0 & 0 & + & 0 & - & - & 0
\end{pmatrix}.$$

Take R_1 and D to be the residual and derived parts, respectively, of W_{19}. Define the matrix $|R_1|$ by $|R_1|_{ij} = |R_{1_{ij}}|$. Then $|R_1|$ is the incidence matrix of a residual BIBD$(10, 18, 9, 5, 4)$, hence $|R_2| = J - |R_1|$ is a BIBD with the same parameters. Moreover, $|R_2|$ is residual since $|R_2|$ together with $|D|$ also forms a symmetric design. We therefore seek a signing of $|R_2|$ over $\{-1, 1\}$. Such a signing is given by

$$R_2 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & + & + & + & + & + & + & + & + \\
- & 0 & 0 & + & 0 & + & + & + & 0 & 0 & 0 & 0 & + & 0 & - & + & - & 0 \\
0 & - & 0 & + & + & 0 & 0 & + & + & 0 & 0 & 0 & - & + & 0 & 0 & + & - \\
0 & 0 & - & 0 & + & + & + & 0 & + & 0 & 0 & 0 & 0 & - & + & - & 0 & + \\
+ & + & 0 & - & 0 & 0 & + & 0 & + & + & - & 0 & 0 & 0 & 0 & + & 0 & - \\
0 & + & + & 0 & - & 0 & + & + & 0 & 0 & + & - & 0 & 0 & 0 & - & + & 0 \\
+ & 0 & + & 0 & 0 & - & 0 & + & + & - & 0 & + & 0 & 0 & 0 & 0 & - & + \\
+ & 0 & + & + & + & 0 & - & 0 & 0 & + & 0 & - & + & - & 0 & 0 & 0 & 0 \\
+ & + & 0 & 0 & + & + & 0 & - & 0 & - & + & 0 & 0 & + & - & 0 & 0 & 0 \\
0 & + & + & + & 0 & + & 0 & 0 & - & 0 & - & + & - & 0 & + & 0 & 0 & 0
\end{pmatrix}.$$

Remarkably, R_2 together with D also forms a balanced W$(19, 9)$. The matrices R_1, R_2, and D then satisfy several properties.

9.3. Lemma.

(9.3.a) $R_1 R_1^t = R_2 R_2^t = I$, $R_1 R_2^t = R_2 R_1^t$;

(9.3.b) $DD^t = 9I - J$;

(9.3.c) $R_1D^t = R_2D^t = O$;

(9.3.d) $|R_1||R_1|^t = |R_2||R_2|^t = 5I + 4J$, $|R_1||R_2|^t = |R_2||R_1|^t = 5(J - I)$; and

(9.3.e) $|D||D|^t = 5I + 3J$, $|R_1||D|^t = |R_2||D^t| = 4J$.

Proof. Restatement of the fact that each of R_1, R_2, together with D form a balanced $\mathrm{W}(19, 4)$ weighing matrix. ∎

* * *

9.2. Construction. Having the lemmata of the previous subsection at our disposal, we are ready to present the construction of a new family of balanced weighing matrices. We desire to apply BGWs in the construction of these matrices, so we need an admissible group of symmetries. Take $\mathcal{M} = \{R_1, R_2\}$; then it isn't difficult to see that $-R_2 \mapsto -R_1 \mapsto R_2 \mapsto R_1 \mapsto -R_2$ is an admissible cyclic group of symmetries of order 4 for $\mathcal{M}$—though, this will be derived explicitly below.

Let $n > 1$, and take Ξ to be a $\mathrm{BGW}((9^n - 1)/8, 9^{n-1}, 9^{n-1} - 9^{n-2}; C_4)$. We claim that $\Xi \otimes R_1$ is the residual part of a balanced $\mathrm{W}([9(9^n - 1)/4] + 1, 9^n)$. Note there are disjoint $(0, 1)$-matrices $\Xi_0, \Xi_1, \Xi_2, \Xi_3$ such that $\Xi = \Xi_0 + g\Xi_1 + g^2\Xi_2 + g^3\Xi_3$ if $C_4 = \{e, g, g^2, g^3\}$. Then $\Xi \otimes R_1 = \Xi_0 \otimes R_1 - \Xi_1 \otimes R_2 - \Xi_2 \otimes R_1 + \Xi_3 \otimes R_2$.

Next, let $\mathscr{S}_{9,n} = \sum_{\alpha \in \mathrm{GF}(9)^*} \alpha A_\alpha$, and define $A_0 = J - \sum_{\alpha \in \mathrm{GF}(9)^*} A_\alpha$. Finally, take $\Theta = \sum_{\alpha \in \mathrm{GF}(9)} A_\alpha \otimes D$. It will be shown that Θ is the derived part of a balanced $W([9(9^n-1)/4]+1, 9^n)$.

We require the following lemma.

9.4. Lemma.

(9.4.a) $(\Xi \otimes R_1)(\Xi \otimes R_1)^t = 9^n I$;

(9.4.b) $\Theta\Theta^* = 9^n I - J$;

(9.4.c) $(\Xi \otimes R_1)\Theta^t = \Theta(\Xi \otimes R_1)^t = O$;

(9.4.d) $|\Xi \otimes R_1||\Xi \otimes R_1|^t = 5 \cdot 9^n I + 4 \cdot 9^n J$;

(9.4.e) $|\Theta||\Theta|^t = 5 \cdot 9^n I + (4 \cdot 9^n - 1)J$; and

(9.4.f) $|\Xi \otimes R_1||\Theta|^t = |\Theta||\Xi \otimes R_1|^t = 4 \cdot 9^n J$.

Proof. By Lemma 9.2 and (9.3.a),

$$\begin{aligned}(\Xi \otimes R_1)(\Xi \otimes R_1) &= 9\Xi_0\Xi_0^t \otimes I - \Xi_0\Xi_1^t \otimes R_1R_2^t - 9\Xi_0\Xi_2^t \otimes I + \Xi_0\Xi_3^t \otimes R_1R_2^t \\ &- \Xi_1\Xi_0^t \otimes R_2R_1^t + 9\Xi_1\Xi_1^t \otimes I + \Xi_1\Xi_2^t \otimes R_2R_1^t - 9\Xi_1\Xi_3^t \otimes I \\ &- 9\Xi_2\Xi_0^t \otimes I + \Xi_2\Xi_1^t \otimes R_1R_2^t + 9\Xi_2\Xi_2^t \otimes I - \Xi_2\Xi_3^t \otimes R_1R_2^t \\ &+ \Xi_3\Xi_0^t \otimes R_2R_1^t - 9\Xi_3\Xi_1^t \otimes I - \Xi_3\Xi_2^t \otimes R_2R_1^t + 9\Xi_3\Xi_3^t \otimes I \\ &= 9\sum_i(\Xi_i\Xi_i^t - \Xi_i\Xi_{i+2}^t) \otimes I - \sum_i(\Xi_i\Xi_{i+1} - \Xi_i\Xi_{i+3}) \otimes R_1R_2^t \\ &= 9^nI,\end{aligned}$$

and (9.4.a) is shown.

Next, by Lemma 9.1 and (9.3.b), and upon indexing the rows of D by elements of GF(9),

$$\begin{aligned}\Theta\Theta^t &= \sum_{\alpha,\beta\in \mathrm{GF}(9)} A_\alpha A_\beta^t \otimes r_\alpha r_\beta^t \\ &= \sum_{\alpha\in \mathrm{GF}(9)} A_\alpha A_\alpha^t \otimes r_\alpha r_\alpha^t + \sum_{\alpha\neq\beta} A_\alpha A_\beta^t \otimes r_\alpha r_\beta^t \\ &= 8\sum_{\alpha\in \mathrm{GF}(9)} A_\alpha A_\alpha^t - \sum_{\alpha\neq\beta} A_\alpha A_\beta^t \\ &= (9^{n-1}-1)J + 8\cdot 9^{n-1}I - 9^{n-1}(J-I) \\ &= 9^nI - J,\end{aligned}$$

which shows (9.4.b).

By Lemma (9.3.c),

$$\begin{aligned}(\Xi \otimes R_1)\Theta^t &= \sum_{\alpha\in \mathrm{GF}(9)} (\Xi_0A_\alpha^t \otimes R_1r_\alpha^t - \Xi_1A_\alpha^t \otimes R_2r_\alpha^t - \Xi_2A_\alpha^t \otimes R_2r_\alpha^t + \Xi_3A_\alpha^t \otimes R_2r_\alpha^t) \\ &= O,\end{aligned}$$

and (9.4.c) has been shown.

Since $|\Xi \otimes R_1| = (\Xi_0 + \Xi_2) \otimes |R_1| + (\Xi_1 + \Xi_3) \otimes |R_2|$, (9.4.d) is shown similarly to (9.4.a).

(9.4.e) is shown just as (9.4.b) after noting that $|\Theta| = \sum_{\alpha\in \mathrm{GF}(9)} A_\alpha \otimes |r_\alpha|$.

Finally, (9.4.f) is shown precisely as in (9.4.c). ∎

We are now ready to present the main construction.

9.5. Theorem. Given $\Xi \otimes R_1$ and Θ defined above,

$$(9.5.a) \quad \begin{pmatrix} \mathbf{0} & \Xi \otimes R_1 \\ \mathbf{1} & \Theta \end{pmatrix}$$

is a balanced $\mathrm{W}([9(9^n - 1)/4] + 1, 9^n)$.

Proof. By the lemma, $(\Xi \otimes R_1)(\Xi \otimes R_1)^t = 9^n I$, $\Theta\Theta^t = 9^n I - J$, and $(\Xi \otimes R_1)\Theta^t = O$; thus, (9.5.a) is a weighing matrix with the appropriate parameters. It remains to show it is balanced. But the lemma again gives $|\Xi \otimes R_1||\Xi \otimes R_1|^t = 5 \cdot 9^n I + 4 \cdot 9^n J$, $|\Theta||\Theta|^t = 5 \cdot 9^n I + (4 \cdot 9^n - 1)J$, and $|\Xi \otimes R_1|\Theta^t = 4 \cdot 9^n J$. We have then shown that (9.5.a) is balanced, and the proof is complete. ∎

§10. Weighing Matrices and Association Schemes

In Brouwer et al. (1989) it is shown that symmetric designs are equivalent to certain 3-class association schemes. It is a natural question to ask whether or not a balanced weighing matrix could be related to this scheme, particularly an augmentation. This is the goal of this subsection.

* * *

10.1. Adjacency Matrices. Assume the existence of a balanced $\mathrm{W}(v, k)$, say $W = W_0 - W_1$, where W_0 and W_1 are disjoint $(0,1)$-matrices, and define the following matrices where $P = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

$$A_0 = I_{4v},$$

$$A_1 = I_2 \otimes P \otimes I_v,$$

$$A_2 = I_2 \otimes J_2 \otimes (J_v - I_v),$$

$$A_3 = \begin{pmatrix} O & I_2 \otimes W_0 + P \otimes W_1 \\ I_2 \otimes W_0^t + P \otimes W_1^t & O \end{pmatrix},$$

$$A_4 = \begin{pmatrix} O & I_2 \otimes W_1 + P \otimes W_0 \\ I_2 \otimes W_1^t + P \otimes W_0^t & O \end{pmatrix},$$

$$A_5 = \begin{pmatrix} O & J_2 \otimes (J_v - W_0 - W_1) \\ J_2 \otimes (J_v - W_0^t - W_1^t) & O \end{pmatrix}.$$

We claim that $\{A_0, A_1, A_2, A_3, A_4, A_5\}$ form a 5-class symmetric association scheme. Clearly, (4.2.a), (4.2.b), and (4.2.f) are satisfied. It remains to show closure under multiplication.

First, $A_1^2 = (I_2 \otimes P \otimes I_v)^2 = I_2 \otimes P^2 \otimes I_v = I = A_0$. Next, since

$$\begin{aligned}[J_2 \otimes (J_v - I_v)]^2 &= 2J_2 \otimes [(v-1)I_v + (v-2)(J_v - I_v)] \\ &= 2(v-1)J_2 \otimes I_v + 2(v-2)J_2 \otimes (J_v - I_v) \\ &= 2(v-1)(I_2 + P) \otimes I_v + 2(v-2)J_2 \otimes (J_v - I_v),\end{aligned}$$

it follows that $A_2^2 = 2(v-1)(A_0 + A_1) + 2(v-2)A_2$. By (9.2.b) and (9.2.c), we find

$$\begin{aligned}(I_2 \otimes W_0 + P \otimes W_1)(I_2 \otimes W_0^t + P \otimes W_1^t) &= I_2 \otimes (W_0 W_0^t + W_1 W_1^t) \\ &\quad + P \otimes (W_0 W_1^t + W_1 W_0^t) \\ &= I_2 \otimes [kI_v + \frac{\lambda}{2}(J_v - I_v)] + P \otimes \frac{\lambda}{2}(J_v - I_v) \\ &= kI_{2v} + \frac{\lambda}{2} J_2 \otimes (J_v - I_v),\end{aligned}$$

where $\lambda = k(k-1)/(v-1)$. Therefore, $A_3^3 = A_4^2 = kA_0 + \frac{\lambda}{2}A_2$. Finally,

$$\begin{aligned}J_2^2 \otimes [J_v - W_0 - W_1)(J_v - W_0^t - W_1^t) &= J_2^2 \otimes (J_v^2 - 2J_v(W_0 + W_1) \\ &\quad + W_0W_0^t + W_1W_1^t + W_0W_1^t + W_1W_0^t] \\ &= J_2 \otimes [2(v-k)I_v + 2(v-2k+\lambda)(J_v - I_v)],\end{aligned}$$

hence $A_5^2 = 2(v-k)(A_0 + A_1) + 2(v-2k+\lambda)A_1$. We next show that $A_iA_j \in \langle A_0, \dots, A_5 \rangle$ whenever $i \neq j$.

Note $(P \otimes I_v)[J_2 \otimes (J_v - I_v)] = J_2 \otimes (J_v - I_v)$ so that $A_1A_2 = A_2A_1 = A_2$. Then

$$\begin{aligned}(I_2 \otimes W_0 + P \otimes W_1)(I_2 \otimes W_1^t + P \otimes W_0^t) &= I_2 \otimes (W_0W_1^t + W_1W_0^t) + P \otimes (W_0W_0^t + W_1W_1^t) \\ &= I_2 \otimes \frac{\lambda}{2}(J_v - I_v) + P \otimes [kI_v + \frac{\lambda}{2}(J_v - I_v)] \\ &= \frac{\lambda}{2} J_2 \otimes (J_v - I_v) + kP \otimes I_v\end{aligned}$$

so that $A_3A_4 = A_4A_3 = kA_1 + \frac{\lambda}{2}A_2$. Next, $(P \otimes I_v)(I_2 \otimes W_0 + P \otimes W_1) = I_2 \otimes W_1 + P \otimes W_0$, hence $A_1A_3 = A_3A_1 = A_4$; and similary, $A_1A_4 = A_4A_1 = A_3$. Then $(P \otimes I_v)[J_2 \otimes (J_v - W_0 - W_1)] = J_2 \otimes (J_v - W_0 - W_1)$ and $A_1A_5 = A_5A_1 = A_5$. Now,

$$\begin{aligned}(I_2 \otimes W_0 + P \otimes W_1)[J_2 \otimes (J_v - W_0^t - W_1^t)] &= J_2 \otimes [(W_0 + W_1)(J_v - (W_0 + W_1)^t)] \\ &= (k - \lambda)J_2 \otimes (J_v - I_v)\end{aligned}$$

so that $A_3A_5 = A_5A_3 = (k - \lambda)A_2$. Similarly, $A_4A_5 = A_5A_4 = (k - \lambda)A_2$. Since

$$\begin{aligned}[J_2 \otimes (J_v - I_v)](I_2 \otimes W_0 + P \otimes W_1 &= J_2 \otimes (J_v - I_v)(W_0 + W_1) \\ &= (k - 1)J_2 \otimes J_v + J_2 \otimes (J_v - W_0 - W_1),\end{aligned}$$

it follows that $A_2A_3 = A_3A_2 = (k - 1)(A_3 + A_4 + A_5) + kA_5$. Similarly, $A_2A_4 = A_4A_2 = (k - 1)(A_3 + A_4 + A_5) + kA_5$. Finally,

$$[J_2 \otimes (J_v - I_v)][J_2 \otimes (J_v - W_0 - W_1)] = J_2 \otimes [2(v - k)J_v - 2(J_v - W_0 - W_1)],$$

and $A_2A_5 = A_5A_2 = 2(v - k)(A_3 + A_4 + A_5) - 2A_5$.

We have shown the following result.

10.1. Theorem. If there is a balanced W(v, k), then there is a 5-class symmetric association scheme.

* * *

10.2. Character Tables. Our work from the previous subsection shows that the third intersection matrix is given by

$$B_3 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & k-1 & k-1 & k \\ k & 0 & \frac{k(k-1)}{2(v-1)} & 0 & 0 & 0 \\ 0 & k & \frac{k(k-1)}{2(v-1)} & 0 & 0 & 0 \\ 0 & 0 & \frac{k(v-k)}{v-1} & 0 & 0 & 0 \end{pmatrix}.$$

It can be shown that B_3^t has the six distinct eigenvalues $\pm k, \pm\sqrt{k}$, and $\pm\sqrt{k(v-k)/(v-1)}$ with corresponding eigenvectors

$$\mathbf{1}_6, \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ \frac{1}{\sqrt{k}} \\ -\frac{1}{\sqrt{k}} \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ -\frac{1}{\sqrt{k}} \\ \frac{1}{\sqrt{k}} \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ \frac{-1}{v-1} \\ \frac{v-k}{\sqrt{k(v-1)(v-k)}} \\ \sqrt{\frac{v-k}{k(v-1)}} \\ \frac{-k}{\sqrt{k(v-1)(v-k)}} \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ \frac{-1}{v-1} \\ \frac{k-v}{\sqrt{k(v-1)(v-k)}} \\ -\sqrt{\frac{v-k}{k(v-1)}} \\ \frac{k}{\sqrt{k(v-1)(v-k)}} \end{pmatrix}.$$

The valencies of the scheme are $k_0 = k_1 = 1$, $k_2 = 2(v-1)$, $k_3 = k_4 = k$, and $k_5 = 2(v-k)$. Define $\Delta_k = \mathrm{diag}(k_0, \ldots, k_5)$. Then $v_i^t = (\Delta_k u_i)^t$ are the standardized left eigenvectors of B_3^t. The vectors v_i^t form the rows of the first character table P.

Next, the multiplicities of the scheme are given by $m_i = 4v/\langle u_i, v_i\rangle$ and evaluate to $1, 1, v, v, v-1$, and $v-1$. Then $m_i u_i$ are the columns of the second character table Q.

Summing up, we have the following result.

10.2. Theorem. The 5-class symmetric association scheme of Theorem 10.1 has the character tables

$$P = \begin{array}{c} \\ E_0 \\ E_1 \\ E_2 \\ E_3 \\ E_4 \\ E_5 \end{array} \begin{array}{c} \begin{array}{cccccc} A_0 & A_1 & A_2 & A_3 & A_4 & A_5 \end{array} \\ \begin{pmatrix} 1 & 1 & 2(v-1) & k & k & 2(v-k) \\ 1 & -1 & 0 & \sqrt{k} & -\sqrt{k} & 0 \\ 1 & -1 & 0 & -\sqrt{k} & \sqrt{k} & 0 \\ 1 & 1 & 2(v-1) & -k & -k & 2(k-v) \\ 1 & 1 & -2 & -\sqrt{\frac{k(v-k)}{v-1}} & -\sqrt{\frac{k(v-k)}{v-1}} & 2\sqrt{\frac{k(v-k)}{v-1}} \\ 1 & 1 & -2 & \sqrt{\frac{k(v-k)}{v-1}} & \sqrt{\frac{k(v-k)}{v-1}} & -2\sqrt{\frac{k(v-k)}{v-1}} \end{pmatrix} \end{array},$$

$$Q = \begin{array}{c} \\ A_0 \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array} \begin{array}{c} \begin{array}{cccccc} E_0 & E_1 & E_2 & E_3 & E_4 & E_5 \end{array} \\ \begin{pmatrix} 1 & v & v & 1 & v-1 & v-1 \\ 1 & -v & -v & 1 & v-1 & v-1 \\ 1 & 0 & 0 & 1 & -1 & -1 \\ 1 & \frac{v}{\sqrt{k}} & -\frac{v}{\sqrt{k}} & -1 & -\sqrt{\frac{(v-1)(v-k)}{k}} & \sqrt{\frac{(v-1)(v-k)}{k}} \\ 1 & -\frac{v}{\sqrt{k}} & \frac{v}{\sqrt{k}} & -1 & -\sqrt{\frac{(v-1)(v-k)}{k}} & \sqrt{\frac{(v-1)(v-k)}{k}} \\ 1 & 0 & 0 & -1 & \sqrt{\frac{k(v-1)}{v-k}} & -\sqrt{\frac{k(v-1)}{v-k}} \end{pmatrix} \end{array}.$$

Interestingly, the converse holds as well.

10.3. Theorem. If there is a 5-class symmetric association scheme with the character tables given in the statement of Theorem 10.2, then there is a balanced $\mathrm{W}(v,k)$.

Proof. Let $\tilde{A}_0,\ldots,\tilde{A}_5$ be the adjacency matrices of the scheme. Then $\tilde{A}_1+\tilde{A}_2$ has eigenvalues $2v-1$ and -1 with multiplicities 2 and $4v-2$, respectively. It follows that $\tilde{A}_1+\tilde{A}_2 \sim I_2\otimes A(K_{2v})$. By the eigenvalues of $\tilde{A}_1$, it is the adjacency matrix of $2v$ disjoint 2-cliques, hence $\tilde{A}_1 \sim I_2\otimes P\otimes I_v$ and $\tilde{A}_2 \sim I_2\otimes J_2\otimes (J_v-I_v)$.

It follows that

$$A_3=\begin{pmatrix} O & O & X_1 & X_2\\ O & O & X_3 & X_4\\ X_1^t & X_3^t & O & O\\ X_2^t & X_4^t & O & O\end{pmatrix} \text{ and } A_4=\begin{pmatrix} O & O & Y_1 & Y_2\\ O & O & Y_3 & Y_4\\ Y_1^t & Y_3^t & O & O\\ Y_2^t & Y_4^t & O & O\end{pmatrix}.$$

It can be shown (see Bannai and Ito, 1984, Theorem 3.6.2, for example) that the eigenvalues of the scheme imply that $A_1A_3=A_4$ so that

$$\begin{pmatrix} O & O & X_3 & X_4\\ O & O & X_1 & X_2\\ X_2^t & X_4^t & O & O\\ X_1^t & X_3^t & O & O\end{pmatrix} = \begin{pmatrix} O & O & Y_1 & Y_2\\ O & O & Y_3 & Y_4\\ Y_1^t & Y_3^t & O & O\\ Y_2^t & Y_4^t & O & O\end{pmatrix}.$$

Therefore, there are $(0,1)$-matrices W_0 and W_1 such that

$$A_3=\begin{pmatrix} O & O & W_0 & W_1\\ O & O & W_1 & W_0\\ W_0^t & W_1^t & O & O\\ W_1^t & W_0^t & O & O\end{pmatrix} \text{ and } A_4=\begin{pmatrix} O & O & W_1 & W_0\\ O & O & W_0 & W_1\\ W_1^t & W_0^t & O & O\\ W_0^t & W_1^t & O & O\end{pmatrix}.$$

Appealing to the character tables again, it can be shown that

$$\begin{aligned} A_3^2 &= kA_0+\frac{k(k-1)}{2(v-1)}A_2,\\ A_3A_4=A_4A_3 &= kA_1+\frac{k(k-1)}{2(v-1)}A_2, \text{ and}\\ A_4^2 &= kA_0+\frac{k(k-1)}{2(v-1)}A_2. \end{aligned}$$

Therefore, $(A_3-A_4)^2=2k(A_0-A_1)$ and $(A_3+A_4)^2=2k(A_0+A_1)+\frac{2k(k-1)}{v-1}A_2$, from which it follows that

$$(W_0 - W_1)(W_0 - W_1)^t = kI, \text{ and}$$
$$(W_0 + W_1)(W_0 + W_1)^t = kI + \lambda(J - I),$$

where $\lambda = k(k-1)/(v-1)$.

We then have that $W_0 - W_1$ is the required matrix. ∎

In light of the equivalence, we make the following definition.

10.4. Definition. A symmetric 5-class association scheme is a *weighing scheme* if it has the character tables given in Theorem 10.2.

Notes

17. Seberry (2017), Seberry and Yamada (1992), and Wallis et al. (1972) all contain a wealth of constructions for weighing matrices and orthogonal designs using the Kronecker product.

18. Such a decomposition of the block set is called a *parallelism*. Beth et al. (1999) studies partitions of the blocks of an incidence structure in great detail.

4

A Unified Construction of Weighing Matrices

In this final chapter of results, we present a general and novel method of constructing certain configurations using the classical parameter BGW matrices and the generalized simplex codes (see Kharaghani et al., 2022a). The procedure is new and does not appear to be previously contained in the literature of the field. Parts of the method are similar to that given by Rajkundlia (1983) and Ionin (2001), but it can be seen that it is much simpler to apply than the recursive methods of the aforementioned seminal articles. Following the presentation of this construction technique, we find an equivalence between arbitrary BGW matrices over a finite abelian group and certain commutative association schemes. This result generalizes those obtained over the course of the previous chapter. The results presented here are a modified version of Kharaghani et al. (2022a).

§11. A General Method

In this section, the general method intimated above is derived and used in the construction of weighing matrices and symmetric designs. Resulting parameters of configurations so constructible are then tabulated.

* * *

11.1. A First Application: Weighing matrices. It is a peculiarity of the constuction that the application of the method to weighing matrices is predicated upon the weight of the matrix being a prime power. Indeed, as in the previous chapter, the generalized simplex codes are applied to the derived part of the matrix by substituting the rows of the derived part for the letters of the code. The simplex code, as the reader may remember, has a prime power number of letters.

To begin, let $W = \left(\begin{smallmatrix} \mathbf{0} & R \\ \mathbf{1} & D \end{smallmatrix}\right)$ be a $\mathrm{W}(v, q)$ weighing matrix in normal form, where q is some prime power. Then it is easy to see that $RR^t = kI$ and $DD^t = kI - J$. We say that W is the *seed matrix* of the construction.

Let $\mathscr{S}_{q,n} = \sum_{\alpha \in \mathrm{GF}(q)^*} \alpha A_\alpha$, and define $A_0 = J - \sum_{\alpha \in \mathrm{GF}(q)^*} A_\alpha$. Index the rows of D by the elements of $\mathrm{GF}(q)$, and take $\mathcal{D} = \sum_{\alpha \in \mathrm{GF}(q)} A_\alpha \otimes r_\alpha$. Next, by our previous work, there is a $\mathrm{W}((q^n - 1)/(q - 1), q^{n-1})$, say H, for every $n > 1$. We define $\mathcal{R} = H \otimes R$.

We claim that $\mathcal{W} = \left(\begin{smallmatrix} \mathbf{0} & \mathcal{R} \\ \mathbf{1} & \mathcal{D} \end{smallmatrix}\right)$ is again a weighing matrix. Indeed, $\mathcal{R}\mathcal{R}^t = (H \otimes R)(H \otimes R)^t = HH^t \otimes RR^t = q^n I$. Further, by Lemma 9.1,

$$\begin{aligned}
\mathcal{D}\mathcal{D}^t &= \left(\sum_\alpha A_\alpha \otimes r_\alpha\right)\left(\sum_\alpha A_\alpha \otimes r_\alpha\right) \\
&= \sum_\alpha A_\alpha A_\alpha^t \otimes r_\alpha r_\alpha^t + \sum_{\alpha \neq \beta} A_\alpha A_\beta^t \otimes r_\alpha r_\beta^t \\
&= (q-1)\sum_\alpha A_\alpha A_\alpha^t - \sum_{\alpha \neq \beta} A_\alpha A_\beta^t \\
&= (q-1)\left(\frac{q^{n-1}-1}{q-1} J + q^{n-1} I\right) - q^{n-1}(J - I) \\
&= q^n I - J.
\end{aligned}$$

Finally,

$$\begin{aligned}
\mathcal{R}\mathcal{D}^t &= (H \otimes R)\left(\sum_\alpha A_\alpha \otimes r_\alpha\right)^t \\
&= \sum_\alpha H A_\alpha^t \otimes R r_\alpha^t \\
&= \sum_\alpha H A_\alpha^t \otimes O \\
&= O.
\end{aligned}$$

It follows that $\mathcal{W}$ is a weighing matrix. We record this result below.

11.1. Theorem. If there exists a $\mathrm{W}(v, q)$ weighing matrix of odd prime power weight, then there is a weighing matrix with parameters

$$\left(\frac{(v-1)(q^n-1)}{q-1}+1, q^n\right). \tag{11.1.a}$$

11.2. Example. Take as a seed matrix the $\mathrm{W}(8,5)$ shown below

$$\begin{pmatrix}
0 & + & 0 & 0 & - & - & + & + \\
0 & 0 & + & 0 & - & + & - & + \\
0 & 0 & 0 & + & - & + & + & - \\
+ & + & + & + & + & 0 & 0 & 0 \\
+ & + & - & - & 0 & + & 0 & 0 \\
+ & - & + & - & 0 & 0 & + & 0 \\
+ & - & - & + & 0 & 0 & 0 & + \\
+ & 0 & 0 & 0 & - & - & - & -
\end{pmatrix}. \tag{11.2.a}$$

Applying the construction, we find a $\mathrm{W}(43, 25)$

$$\setcounter{MaxMatrixCols}{43}\begin{pmatrix}
0&0&0&0&0&0&0&0&+&0&0&-&-&+&+&+&0&0&-&-&+&+&+&0&0&-&-&+&+&+&0&0&-&-&+&+&+&0&0&-&-&+&+ \\
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+&-&-&+&0&0&0&+&-&-&+&0&0&0&+&0&0&0&-&-&-&-&-&+&-&0&0&+&0&+&+&+&+&0&0&0&+&-&-&0&+&0&0 \\
+&0&0&0&-&-&-&-&-&-&+&0&0&0&+&-&+&-&0&0&+&0&+&+&+&+&0&0&0&+&-&-&0&+&0&0&0&0&0&-&-&-&- \\
+&+&+&+&+&0&0&0&-&-&+&0&0&0&+&-&-&+&0&0&0&+&-&-&+&0&0&0&+&-&-&+&0&0&0&+&-&-&+&0&0&0&+ \\
+&+&-&-&0&+&0&0&0&0&0&-&-&-&-&-&-&+&0&0&0&+&+&+&+&+&0&0&0&-&+&-&0&0&+&0&+&-&-&0&+&0&0 \\
+&-&+&-&0&0&+&0&0&0&0&-&-&-&-&+&+&+&+&0&0&0&-&+&-&0&0&+&0&+&-&-&0&+&0&0&-&-&+&0&0&0&+ \\
+&-&-&+&0&0&0&+&0&0&0&-&-&-&-&-&+&-&0&0&+&0&+&-&-&0&+&0&0&-&-&+&0&0&0&+&+&+&+&+&0&0&0 \\
+&0&0&0&-&-&-&-&0&0&0&-&-&-&-&+&-&-&0&+&0&0&-&-&+&0&0&0&+&+&+&+&+&0&0&0&-&+&-&0&0&+&0 \\
+&+&+&+&+&0&0&0&0&0&0&-&-&-&-&0&0&0&-&-&-&-&0&0&0&-&-&-&-&0&0&0&-&-&-&-&0&0&0&-&-&-&- \\
+&+&-&-&0&+&0&0&+&+&+&+&0&0&0&+&-&-&0&+&0&0&-&+&-&0&0&+&0&-&-&+&0&0&0&+&0&0&0&-&-&-&- \\
+&-&+&-&0&0&+&0&+&+&+&+&0&0&0&-&+&-&0&0&+&0&-&-&+&0&0&0&+&0&0&0&-&-&-&-&+&-&-&0&+&0&0 \\
+&-&-&+&0&0&0&+&+&+&+&+&0&0&0&-&-&+&0&0&0&+&0&0&0&-&-&-&-&+&-&-&0&+&0&0&-&+&-&0&0&+&0 \\
+&0&0&0&-&-&-&-&+&+&+&+&0&0&0&0&0&0&-&-&-&-&+&-&-&0&+&0&0&-&+&-&0&0&+&0&-&-&+&0&0&0&+ \\
+&+&+&+&+&0&0&0&+&+&+&+&0&0&0&+&+&+&+&0&0&0&+&+&+&+&0&0&0&+&+&+&+&0&0&0&+&+&+&+&0&0&0
\end{pmatrix}. \tag{11.2.b}$$

Evidently, this resolves the question of existence of a $\mathrm{W}(43, 25)$ given in Part V of Colbourn and Dinitz (2007).

To further evince the utility of the method, we include a table of constructible parameters given a known seed weighing matrix.

Table 4.1: Small parameter consequential order/weight pairs.

Seed (v,k)	*Succident* (v',k')	*Seed* (v,k)	*Succident* (v',k')
$(6,5)^{\ddagger}$:	$(31,25)$‡, $(156,125)$‡, $(781,625)$‡	$(16,3)$:	$(69,9)$, $(196,27)$, $(601,81)$
$(8,5)$:	$(43,25)^{\dagger}$, $(218,125)$	$(16,5)$:	$(91,25)$, $(466,125)$
$(8,7)^{\ddagger}$:	$(57,49)^{\ddagger}$, $(400,343)^{\ddagger}$	$(16,7)$:	$(121,49)$, $(856,343)$
$(10,5)$:	$(55,25)$, $(280,125)$	$(16,9)$:	$(151,81)$
$(10,9)^{\ddagger}$:	$(91,81)^{\ddagger}$, $(820,729)^{\ddagger}$	$(16,11)$:	$(181,121)^{\dagger}$
$(12,5)$:	$(67,25)$, $(342,125)$	$(16,13)$:	$(211,169)$
$(12,7)$:	$(89,49)^{\dagger}$, $(628,343)$	$(18,13)$:	$(239,169)$
$(12,9)$:	$(111,81)^{\dagger}$	$(19,9)^{\ddagger}$:	$(181,81)^{*}$
$(13,9)^{\ddagger}$:	$(121,81)$	$(20,7)$:	$(153,49)$
$(14,9)$:	$(131,81)$	$(20,13)$:	$(267,169)^{\dagger}$
$(14,13)^{\ddagger}$:	$(183,169)^{\ddagger}$		

* Note that the $\mathrm{W}(181,81)$ constructible from the balanced seed $\mathrm{W}(19,9)$ can be made to be balanced as shown in the previous chapter. It is not, however, a consequence of the construction of this chapter that the succident matrix is balanced.

† Denotes previously unknown order weight pairs.

‡ Denotes a balanced weighing matrix.

* * *

11.2. A Second Application: Block designs. The construction is perfectly amenable to certain symmetric designs. For precisely the same reasoning given in the previous subsection, we require that the design have a prime power block size. Furthermore, we require that the parameters of the residual of the design be invariant under complementation. Explicitly, if $A = \left(\begin{smallmatrix} \mathbf{0} & R \\ \mathbf{1} & D \end{smallmatrix}\right)$ is the incidence matrix of the given design, then $J - R$ must have the same parameters as R.

As shown in Hall (1986), there is a $\mathrm{W}(2q+2, 2q+2)$ if and only if there is a symmetric $\mathrm{BIBD}(2q+1, q, (q-1)/2)$, called a *Hadamard design*.[19] Accordingly, the residual of this design is a $\mathrm{BIBD}(q+1, 2q, q, (q+1)/2, (q-1)/2)$, the parameters of which are invariant under complementation. Also, the derived design is a $\mathrm{BIBD}(q, 2q, q-1, (q-1)/2, (q-3)/2)$.

Let $H = H_0 - H_1$ be a balanced $\mathrm{W}((q^n-1)/(q-1), q^{n-1})$, and let $\{A_\alpha\}_{\alpha\in\mathrm{GF}(q)}$ be as above. If $A = \left(\begin{smallmatrix} \mathbf{0} & R \\ \mathbf{1} & D \end{smallmatrix}\right)$ is a Hadamard design with parameters $(2q+1, q, (q-1)/2)$, then we form $\mathcal{R} = H_0 \otimes R + H_1 \otimes (J - R)$. Indexing the rows of D by the elements of GF(q), we then form $\mathcal{D} = \sum_{\alpha \in \mathrm{GF}(q)} A_\alpha \otimes r_\alpha$.

We claim that $\mathcal{A} = \left(\begin{smallmatrix} \mathbf{0} & \mathcal{R} \\ \mathbf{1} & \mathcal{D} \end{smallmatrix}\right)$ is a symmetric design. To show this, we proceed as before. Using Lemma 9.2, we find

$$\begin{aligned}
\mathcal{R}\mathcal{R}^t &= [H_0 \otimes R + H_1 \otimes (J_{q+1,2q} - R)][H_0 \otimes R + H_1 \otimes (J_{q+1,2q} - R)]^t \\
&= (H_0 H_0^t + H_1 H_1^t) \otimes \left(\frac{q+1}{2} I_{q+1} + \frac{q-1}{2} J_{q+1} \right) \\
&\qquad\qquad + \frac{q+1}{2} (H_0 H_1^t + H_1 H_0^t) \otimes (J_{q+1} - I_{q+1}) \\
&= \frac{q^{n-2}}{4} [(q+1) I_{\frac{q^{n-1}}{q-1}} + (q-1) J_{\frac{q^{n-1}}{q-1}}] \otimes [(q+1) I_{q+1} + (q-1) J_{q+1}] \\
&\qquad\qquad + \frac{q^{n-2}(q^2-1)}{4} (J_{\frac{q^{n-1}}{q-1}} - I_{\frac{q^{n-1}}{q-1}}) \otimes (J_{q+1} - I_{q+1}) \\
&= \frac{q^{n-2}}{4} \left[2q(q+1) I_{\frac{(q+1)(q^n-1)}{q-1}} + 2q(q-1) J_{\frac{(q+1)(q^n-1)}{q-1}} \right] \\
&= q^n I_{\frac{(q+1)(q^n-1)}{q-1}} + \frac{q^n - q^{n-1}}{2} (J_{\frac{(q+1)(q^n-1)}{q-1}} - I_{\frac{(q+1)(q^n-1)}{q-1}}),
\end{aligned}$$

whence $\mathcal{R}$ is a quasi-residual BIBD with parameters

$$\left(\frac{(q+1)(q^n-1)}{q-1}, \frac{2q^{n+1}-2q}{q-1}, q^n, \frac{q^n+q^{n-1}}{2}, \frac{q^n-q^{n-1}}{2} \right).$$

Next, by Lemma 9.1

$$\begin{aligned}
\mathcal{D}\mathcal{D}^t &= \left(\sum_\alpha A_\alpha \otimes r_\alpha \right) \left(\sum_\alpha A_\alpha \otimes r_\alpha \right)^t \\
&= \sum_\alpha A_\alpha A_\alpha^t \otimes r_\alpha r_\alpha^t + \sum_{\alpha \neq \beta} A_\alpha A_\beta^t \otimes r_\alpha r_\beta^t \\
&= (q-1) \sum_\alpha A_\alpha A_\alpha^t + \frac{q-3}{2} \sum_{\alpha \neq \beta} A_\alpha A_\beta^t \\
&= (q-1) \left(\frac{q^{n-1}-1}{q-1} J - q^{n-1} I \right) + \frac{q^{n-1}(q-3)}{2} (J - I) \\
&= (q^n - 1) I + \frac{q^n - q^{n-1} - 2}{2} (J - I),
\end{aligned}$$

and $\mathcal{D}$ is a quasi-derived BIBD with parameters

$$\left(q^{n+1}, \frac{2q^{n+1}-2q}{q-1}, q^n - 1, \frac{q^n - q^{n-1}}{2}, \frac{q^n - q^{n-1} - 2}{2} \right).$$

Finally,

$$
\begin{aligned}
\mathcal{R}\mathcal{D}^t &= [H_0 \otimes R + H_1 \otimes (J-R)] \left(\sum_\alpha A_\alpha \otimes r_\alpha\right)^t \\
&= \sum_\alpha H_0 A_\alpha^t \otimes R r_\alpha^t + \sum_\alpha H_1 A_\alpha^t \otimes (J-R) r_\alpha^t \\
&= \frac{q-1}{2} \sum_\alpha H_0 A_\alpha^t \otimes \mathbf{1} + \frac{q-1}{2} \sum_\alpha H_1 A_\alpha^t \otimes \mathbf{1} \\
&= \frac{q-1}{2} (H_0 + H_1) \sum_\alpha A_\alpha^t \otimes \mathbf{1} \\
&= \frac{q-1}{2} (H_0 + H_1) J \\
&= \frac{q^n - q^{n-1}}{2} J.
\end{aligned}
$$

We have shown that $\mathcal{A} = \left(\begin{smallmatrix} \mathbf{0} & \mathcal{R} \\ \mathbf{1} & \mathcal{D} \end{smallmatrix}\right)$ is a symmetric BIBD. We record this result below.

11.3. Theorem. If there is a symmetric BIBD$(2q+1, q, (q-1)/2)$, then there is a symmetric BIBD with parameters

$$
\text{(11.3.a)} \quad \left(\frac{2q^{n+1} - 2q}{q-1} + 1, q^n, \frac{q^n - q^{n-1}}{2}\right).
$$

11.4. Example. Consider the Hadamard design

$$
\text{(11.4.a)} \quad \begin{pmatrix}
0&1&1&0&1&0&0 \\
0&0&1&1&0&1&0 \\
0&0&0&1&1&0&1 \\
1&0&0&0&1&1&0 \\
0&1&0&0&0&1&1 \\
1&0&1&0&0&0&1 \\
1&1&0&1&0&0&0
\end{pmatrix}
$$

with parameters $(7, 3, 1)$. The result of the first iteration of the construction is the symmetric BIBD$(25, 9, 3)$

$$
\begin{pmatrix}
0&0&0&0&0&0&0&1&1&0&1&0&0&1&1&0&1&0&0&1&1&0&1&0&0\\
0&0&0&0&0&0&0&0&1&1&0&1&0&0&1&1&0&1&0&0&1&1&0&1&0\\
0&0&0&0&0&0&0&0&0&1&1&0&1&0&0&1&1&0&1&0&0&1&1&0&1\\
0&0&0&0&0&0&0&1&0&0&0&1&1&1&0&0&0&1&1&1&0&0&0&1&1\\
0&0&0&1&0&1&1&0&0&0&0&0&0&0&0&1&0&1&1&1&1&0&1&0&0\\
0&1&0&0&1&0&1&0&0&0&0&0&0&1&0&0&1&0&1&0&1&1&0&1&0\\
0&1&1&0&0&1&0&0&0&0&0&0&0&1&1&0&0&1&0&0&0&1&1&0&1\\
0&0&1&1&1&0&0&0&0&0&0&0&0&0&1&1&1&0&0&1&0&0&0&1&1\\
0&0&0&1&0&1&1&1&1&0&1&0&0&0&0&0&0&0&0&0&0&1&0&1&1\\
0&1&0&0&1&0&1&0&1&1&0&1&0&0&0&0&0&0&0&1&0&0&1&0&1\\
0&1&1&0&0&1&0&0&0&1&1&0&1&0&0&0&0&0&0&1&1&0&0&1&0\\
0&0&1&1&1&0&0&1&0&0&0&1&1&0&0&0&0&0&0&0&1&1&1&0&0\\
0&0&0&1&0&1&1&0&0&1&0&1&1&1&1&0&1&0&0&0&0&0&0&0&0\\
0&1&0&0&1&0&1&1&0&0&1&0&1&0&1&1&0&1&0&0&0&0&0&0&0\\
0&1&1&0&0&1&0&1&1&0&0&1&0&0&0&1&1&0&1&0&0&0&0&0&0\\
0&0&1&1&1&0&0&0&1&1&1&0&0&1&0&0&0&1&1&0&0&0&0&0&0\\
1&0&1&0&0&0&1&0&1&0&0&0&1&1&0&1&0&0&0&0&0&0&1&1&0\\
1&1&0&1&0&0&0&0&1&0&0&0&1&0&0&0&1&1&0&1&0&1&0&0&0\\
1&0&0&0&1&1&0&0&1&0&0&0&1&0&1&0&0&0&1&0&1&0&0&0&1\\
1&0&1&0&0&0&1&1&0&1&0&0&0&0&0&0&1&1&0&0&1&0&0&0&1\\
1&1&0&1&0&0&0&1&0&1&0&0&0&0&1&0&0&0&1&0&0&0&1&1&0\\
1&0&0&0&1&1&0&1&0&1&0&0&0&1&0&1&0&0&0&1&0&1&0&0&0\\
1&0&1&0&0&0&1&0&0&0&1&1&0&0&1&0&0&0&1&1&0&1&0&0&0\\
1&1&0&1&0&0&0&0&0&0&1&1&0&1&0&1&0&0&0&0&1&0&0&0&1\\
1&0&0&0&1&1&0&0&0&0&1&1&0&0&0&0&1&1&0&0&0&0&1&1&0
\end{pmatrix}. \tag{11.4.b}
$$

§12. BGW Matrices and Association Schemes

In this section we present perhaps the most striking result of this thesis. Here the equivalence between BGW matrices and commutative association schemes is described in detail.

* * *

12.1. Adjacency Matrices. We must first extend Lemma 9.2 to the case of arbitrary BGW matrices. Let $G = \{g_0 = 1, g_1, \ldots, g_{n-1}\}$ be an abelian group, and let $W = \sum_{g\in G} gW_g$ be a BGW$(v, k, \lambda; G)$. We then have the following.

12.1. Lemma.

$$\sum_{g,h} gh^{-1}W_gW_h^t = \sum_{g,h} h^{-1}gW_h^tW_g = kI + \tfrac{\lambda}{n}\left(\sum_g g\right)(J - I), \tag{12.1.a}$$

$$\sum_g W_gW_g^t = \sum_g W_g^tW_g = kI + \tfrac{\lambda}{n}(J - I), \text{ and} \tag{12.1.b}$$

$$\sum_g W_gW_{gh^{-1}}^t = \sum_g W_{gh^{-1}}^tW_g = \tfrac{\lambda}{n}(J - I) \text{ whenever } h \neq 1. \tag{12.1.c}$$

Proof. Restatement of the fact that $W = \sum_g gW_g$ is a BGW$(v, k, \lambda; G)$. ∎

Let $\{U_g : g \in G\}$ be the usual regular linear representation of G, that is, $U_g = (\delta(g_i^{-1}gg_j))$ where $\delta(h) = 1$ if $h = 1$ and 0 otherwise. Consider the following family of matrices.

$$\begin{aligned}
A_{0,g} &= I_2 \otimes U_g \otimes I_v, \text{ for } g \in G,\\
A_1 &= I_2 \otimes J_n \otimes (J_v - I_v),\\
A_{2,g} &= \begin{pmatrix} O & \sum_{h\in G}(U_h \otimes W_{gh}) \\ \sum_{h\in G}(U_h \otimes W^t_{g^{-1}h^{-1}}) & O \end{pmatrix}, \text{ for } g \in G, \text{ and}\\
A_3 &= \begin{pmatrix} O & J_n \otimes (J_v - \sum_{h\in G} W_h) \\ J_n \otimes (J_v - \sum_{h\in G} W^t_h) & O \end{pmatrix}.
\end{aligned}$$

In showing that $\{A_1, A_3\} \cup \{A_{0,g}, A_{2,g} : g \in G\}$ is an association scheme, we proceed precisely as before. First, $A_1 + A_3 + \sum_{g\in G}(A_{0,g} + A_{2,g}) = J$. Second, $A^t_{0,g} = A_{0,g^{-1}}$, $A^t_1 = A_1$, $A^t_{2,g} = A_{2,g^{-1}}$, and $A^t_3 = A_3$.

We next show closure under multiplication. We have $A_{0,g}A_{0,h} = A_{0,gh} = A_{0,hg} = A_{0,h}A_{0,g}$. Then $A_{0,g}A_1 = A_1A_{0,g} = A_1$ and $A_{0,g}A_3 = A_3A_{0,g} = A_3$. Since $\sum_j(U_{gj} \otimes W_{hj}) = \sum_j(U_j \otimes W_{g^{-1}hj})$, it follows that $A_{0,g}A_{2,h} = A_{2,h}A_{0,g} = A_{2,g^{-1}h}$. As $[J_n \otimes (J_v - I_v)]^2 = n(v-1)\sum_g(U_g \otimes I_v) + n(v-2)J_n \otimes (J_v - I_v)$, we have $A_1^2 = n(v-1)\sum_g A_{0,g} + n(v-2)A_1$. Next,

$$\begin{aligned}
[J_n \otimes (J_v - I_v)] \sum_h (U_h \otimes W_{gh}) &= J_n \otimes (J_v - I_v) \sum_h W_h\\
&= (k-1)J_n \otimes J_v + J_n \otimes (J_v - \sum_h W_h),
\end{aligned}$$

hence $A_1A_{2,g} = A_{2,g}A_1 = (k-1)\sum_{g\in G} A_{2,g} + kA_3$. We have

$$\begin{aligned}
[J_n \otimes (J_v - I_v)] \left[J_n \otimes (J_v - \sum_h W_h)\right] &= nJ_n \otimes (J_v - I_v)\left(J_v - \sum_h W_h\right)\\
&= n(v-k)J_n \otimes J_v - nJ_n \otimes \left(J_v - \sum_h W_h\right)
\end{aligned}$$

so that $A_1A_3 = A_3A_1 = n(v-k)\sum_{g\in G} A_{2,g} + n(v-k-1)A_3$. Then

$$\sum_h (U_h \otimes W_{gh}) \left[J_n \otimes (J_v - \sum_j W_j^t) \right] = J_n \otimes \left(\sum_h W_h J_v - \sum_{h,j} W_h W_j^t \right)$$
$$= (k-\lambda) J_n \otimes (J_v - I_v),$$

and $A_{2,g}A_3 = A_3A_{2,g} = (k-\lambda)A_1$. Since

$$\left(J_v - \sum_{h\in G} W_h \right) \left(J_v - \sum_{g\in G} W_g^t \right) = vJ_v - 2kJ_v + kI_v + \lambda(J_v - I_v)$$
$$= (v - 2k + \lambda)(J_v - I_v) + (v-k)I_v,$$

one finds that $A_3^2 = n(v-2k+\lambda)A_1 + n(v-k)\sum_{g\in G} A_{0,g}$. Finally, for fixed $g, h \in G$, we note that

$$\begin{aligned}
\sum_{\alpha,\beta} (U_{\alpha\beta} \otimes W_{g\alpha} W^t_{h^{-1}\beta^{-1}}) &= \sum_\gamma \left(U_\gamma \otimes \sum_{\alpha\beta=\gamma} W_{g\alpha} W^t_{h^{-1}\beta^{-1}} \right) \\
&= \sum_{\gamma \neq g^{-1}h^{-1}} \left(U_\gamma \otimes \sum_{\alpha\beta=\gamma} W_{g\alpha} W^t_{h^{-1}\beta^{-1}} \right) \\
&\qquad + U_{g^{-1}h^{-1}} \otimes \sum_\gamma W_\gamma W_\gamma^t \\
&= \sum_{\gamma \neq g^{-1}h^{-1}} \left[U_\gamma \otimes \frac{\lambda}{n}(J_v - I_v) \right] \\
&\qquad + U_{g^{-1}h^{-1}} \otimes \left[kI_v + \frac{\lambda}{n}(J_v - I_v) \right] \\
&= \frac{\lambda}{n} J_n \otimes (J_v - I_v) + kU_{g^{-1}h^{-1}} \otimes I_v
\end{aligned}$$

so that $A_{2,g}A_{2,h} = A_{2,h}A_{2,g} = \frac{\lambda}{n}A_1 + kA_{0,g^{-1}h^{-1}}$.

We have shown the following result.

12.2. Theorem. If G is a finite abelian group, and if there is a BGW$(v, k, \lambda; G)$, then there is either a commutative $2n$- or $(2n+1)$-class association scheme predicated upon whether or not $v = k$.

* * *

12.2. Character Tables. As before, we desire to give explicitly the character tables of the scheme. The difference between the schemes arising from an arbitrary BGW vs. a balanced weighing matrix, is that the number of classes is no longer fixed. Therefore, we desire to exhibit a general form for the primitive idempotents of the scheme in terms of the adjacency matrices.

To this end, let $\hat{G} = \{\chi_g : g \in G\}$ be the collection of irreducible characters[20] of the group G, i.e. the dual group of G. Following Kharaghani and Suda (2021), we make the following definitions for each $g \in G$:

$$F_{0,g} = \sum_{h \in G} \chi_g(h) A_{0,h}, \qquad \text{and} \qquad F_{2,g} = \sum_{h \in G} \chi_g(h) A_{2,h}.$$

Using the intersection numbers derived above, and using the generalized orthogonality relation of characters (see Isaacs, 2006, Theorem 2.13), we have the following lemma.

12.3. Lemma.

(12.3.a) $F_{0,g}F_{0,h} = n\delta_{gh}F_{0,g}$,

(12.3.b) $F_{0,g}F_{2,h} = F_{2,h}F_{0,g} = n\delta_{g^{-1}h}F_{2,h}$, and

(12.3.c) $F_{2,g}F_{2,h} = n\lambda\delta_{gh}\delta_{g1}A_1 + nk\delta_{gh}F_{0,g^{-1}}$.

Proof. We have

$$\begin{aligned}
F_{0,g}F_{0,h} &= \sum_{\alpha,\beta} \chi_g(\alpha)\chi_h(\beta)A_{0,\alpha\beta} \\
&= n\sum_{\gamma}\left(\sum_{\alpha\beta=\gamma}\chi_g(\alpha)\chi_h(\beta)\right)A_{0,\gamma} \\
&= n\sum_{\gamma}\left(\sum_{\beta}\chi_g(\beta^{-1}\gamma)\chi_h(\beta)\right)A_{0,\gamma} \\
&= n\delta_{gh}\sum_{\gamma}\chi_g(\gamma)A_{0,\gamma} \\
&= n\delta_{gh}F_{0,g},
\end{aligned}$$

which shows (12.3.a). Next,

$$\begin{aligned}
F_{0,g}F_{2,h} &= \sum_{\alpha,\beta} \chi_g(\alpha)\chi_h(\beta)A_{0,g}A_{2,h} \\
&= n\sum_{\gamma}\left(\sum_{\alpha\beta=\gamma}\chi_{g^{-1}}(\alpha)\chi_h(\beta)\right)A_{2,\gamma} \\
&= n\delta_{g^{-1}h}\sum_{\gamma}\chi_h(\gamma)A_{2,\gamma} \\
&= n\delta_{g^{-1}h}F_{2,h}.
\end{aligned}$$

Since the scheme is commutative, $F_{2,h}F_{0,g} = F_{0,g}F_{2,h}$, which shows (12.3.b). Finally,

$$\begin{aligned}
F_{2,g}F_{2,h} &= \sum_{\alpha,\beta}\chi_g(\alpha)\chi_h(\beta)\left(\frac{\lambda}{n}A_1 + kA_{0,\alpha^{-1}\beta^{-1}}\right) \\
&= n\sum_{\gamma}\left(\sum_{\alpha\beta=\gamma}\chi_{g^{-1}}(\alpha)\chi_{h^{-1}}(\beta)\right)\left(\frac{\lambda}{n}A_1 + kA_{0,\gamma}\right) \\
&= n\sum_{\gamma}\left(\sum_{\beta}\chi_{g^{-1}}(\beta^{-1}\gamma)\chi_{h^{-1}}(\beta)\right)\left(\frac{\lambda}{n}A_1 + kA_{0,\gamma}\right) \\
&= n\delta_{gh}\sum_{\gamma}\chi_{g^{-1}}(\gamma)\left(\frac{\lambda}{n}A_1 + kA_{0,\gamma}\right) \\
&= n\lambda\delta_{gh}\delta_{g1}A_1 + nk\delta_{gh}F_{0,g^{-1}},
\end{aligned}$$

showing (12.3.c). This completes the proof. ∎

We are now ready to give the idempotents of the scheme. There are two cases two consider, namely, whether or not $v = k$. In the latter case, we have

$$\begin{aligned}
E_0 &= \frac{1}{2nv}(F_{0,0} + F_{2,0} + A_1 + A_3), \\
E_1 &= \frac{1}{2nv}\left(F_{0,0} - F_{2,0} + A_1 - A_3\right), \\
E_{2,1} &= \frac{1}{2nv}\left((v-1)F_{0,0} + \sqrt{\frac{(v-1)(v-k)}{k}}F_{2,0} - A_1 - \sqrt{\frac{k(v-1)}{v-k}}A_3\right), \\
E_{2,2} &= \frac{1}{2nv}\left((v-1)F_{0,0} - \sqrt{\frac{(v-1)(v-k)}{k}}F_{2,0} - A_1 + \sqrt{\frac{k(v-1)}{v-k}}A_3\right), \\
E_{3,g} &= \frac{1}{2nv}\left(vF_{0,g} + \frac{v}{\sqrt{k}}F_{2,g}\right), \text{ for } g \in G/\{1\}, \text{ and}
\end{aligned}$$

$$E_{4,g} = \frac{1}{2nv}\left(vF_{0,g} - \frac{v}{\sqrt{k}}F_{2,g}\right), \text{ for } g \in G/\{1\}.$$

In the case that $v = k$, then $A_3 = O$ and we replace $E_{2,1}$ and $E_{2,2}$ with $E_2 = E_{2,1} + E_{2,2}$. The following lemma is immediate.

12.4. Lemma. Let $\mathcal{E} = \{E_0, E_1, E_{2,1}, E_{2,2}, E_{3,g}, E_{4,g} : g \in G/\{1\}\}$. Then:

(12.4.a) $EF = \delta_{EF}E$, for all $E, F \in \mathcal{E}$;

(12.4.b) $I = \sum_{E\in\mathcal{E}} E$; and

(12.4.c) $E^* \in \mathcal{E}$, for each $E \in \mathcal{E}$.

Proof. Straightforward but tedious calculation. ∎

Using these lemmata, we have the following result.

12.5. Theorem. The commutative association scheme given in Theorem 12.2 has the first and second character tables

$$P = \begin{array}{c} \\ E_0 \\ E_1 \\ E_2 \\ E_{3,g} \\ E_{4,g} \end{array} \begin{array}{c} \begin{array}{ccc} A_{0,h} & A_1 & A_{2,h} \end{array} \\ \begin{pmatrix} 1 & n(v-1) & k \\ 1 & n(v-1) & -k \\ -n & 0 & \\ \chi_{g^{-1}}(h) & 0 & \sqrt{k}\chi_{g^{-1}}(h) \\ \chi_{g^{-1}}(h) & 0 & \chi_{g^{-1}}(h) \end{pmatrix} \end{array},$$

$$Q = \begin{array}{c} \\ A_{0,h} \\ A_1 \\ A_{2,h} \end{array} \begin{array}{c} \begin{array}{ccccc} E_0 & E_1 & E_2 & E_{3,g} & E_{4,g} \end{array} \\ \begin{pmatrix} 1 & 1 & 2(v-1) & v\chi_g(h) & v\chi_g(h) \\ 1 & 1 & -2 & 0 & 0 \\ 1 & -1 & 0 & \frac{v}{\sqrt{k}}\chi_g(h) & -\frac{v}{\sqrt{k}}\chi_g(h) \end{pmatrix} \end{array}$$

in the case that $v = k$ and

$$P = \begin{array}{c} \\ E_0 \\ E_1 \\ E_{2,1} \\ E_{2,2} \\ E_{3,g} \\ E_{3,g} \end{array} \begin{array}{c} \begin{array}{cccc} A_{0,h} & A_1 & A_{2,h} & A_3 \end{array} \\ \begin{pmatrix} 1 & n(v-1) & k & n(v-k) \\ 1 & n(v-1) & -k & n(k-v) \\ 1 & -n & \sqrt{\frac{k(v-k)}{v-1}} & -n\sqrt{\frac{k(v-k)}{v-1}} \\ 1 & -n & -\sqrt{\frac{k(v-k)}{v-1}} & n\sqrt{\frac{k(v-k)}{v-1}} \\ -\chi_{g^{-1}}(h) & 0 & \sqrt{k}\chi_{g^{-1}}(h) & 0 \\ \chi_{g^{-1}}(h) & 0 & -\sqrt{k}\chi_{g^{-1}}(h) & 0 \end{pmatrix} \end{array},$$

$$Q = \begin{array}{c} \\ A_{0,h} \\ A_1 \\ A_{2,h} \\ A_3 \end{array} \begin{array}{c} \begin{array}{cccccc} E_0 & E_1 & E_{2,1} & E_{2,2} & E_{3,g} & E_{4,g} \end{array} \\ \begin{pmatrix} 1 & 1 & v-1 & v-1 & v\chi_g(h) & v\chi_g(h) \\ 1 & 1 & -1 & -1 & 0 & 0 \\ 1 & -1 & \sqrt{\frac{(v-1)(v-k)}{v-k}} & -\sqrt{\frac{(v-1)(v-k)}{v-k}} & \frac{v}{\sqrt{k}}\chi_g(h) & -\frac{v}{\sqrt{k}}\chi_g(h) \\ 1 & 1 & -\sqrt{\frac{k(v-1)}{v-k}} & \sqrt{\frac{k(v-1)}{v-k}} & 0 & 0 \end{pmatrix} \end{array}$$

in the case that $v > k$.

As before, the converse also holds.

12.6. Theorem. If there is a commutative scheme with the character tables given in Theorem 12.5, then there is a BGW$(v, k, \lambda; G)$, where G is an abelian group isomorphic to $\{\chi_g\}$.

Proof. The derivation of this fact is the same mutatis mutandis as that for Theorem 10.3 and is, therefore, omitted. ∎

In light of this equivalence, we make the following final definition.

12.7. Definition. The commutative association schemes with character tables given in Theorem 12.5 are called *generalized weighing schemes*.

Since the largest—indeed, the most important—families of generalized matrices are those over a group which is either cyclic or elementary abelian, it would be beneficial to conclude with a brief discussion of the irreducible characters of these groups in an effort to make the preceding results more concrete.

To this end, recall the irreducible characters of the cyclic group $C_p \simeq \{1, g, \ldots, g^{p-1}\}$ of order p are given by $\chi_{g^i}(g^j) = e^{\frac{2\pi\sqrt{-1}ij}{p}}$. This suffices for the case in which the group is cyclic.

Let p be a prime, and let C_p be as above. If $q = p^n$, then $\mathrm{EA}(q) \simeq \underbrace{C_p \otimes \cdots \otimes C_p}_{n} = \{g^{m_0} \otimes \cdots \otimes g^{m_{n-1}} : 0 \leq m_0, \ldots, m_{n-1} < p\}$. By Theorem 4.21 of Isaacs (2006), it follows that

$$\begin{aligned}
\chi_{g^{m_0} \otimes \cdots \otimes g^{m_{n-1}}}(g^{k_0} \otimes \cdots \otimes g^{k_{n-1}}) &= \prod_i \chi_{g^{m_i}}(g^{k_i}) \\
&= \prod_i e^{\frac{2\pi\sqrt{-1}m_i k_i}{p}} \\
&= e^{\frac{2\pi\sqrt{-1}}{p}\sum_i m_i k_i}.
\end{aligned}$$

This concludes our study of weighing matrices and related configurations.

Notes

19. Let C be the matrix obtained upon deleting the first row and column of a normalized Hadamard matrix of order $4n$. Then $(1/2)(J + C)$ is a symmetric $\mathrm{BIBD}(4n - 1, 2n - 1, n - 1)$. See Hall (1986) for a proof of this resut and related discussions.

20. Recall that a linear representation of a group G is a homomorphism $\varrho : G \to \mathrm{GL}(V)$, where V is some linear space. The character of the representation is then $\chi(g) = \mathrm{Tr}(\varrho(g))$, for each $g \in G$. Isaacs (2006) is the classical reference for the character theory of finite groups.

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Appendix

A1. A Balancedly Splittable $\mathbf{W}(64, 64)$

Beginning with (5.7.a), a Hadamard matrix of order 4, we first construct the auxiliary matrices as shown.

$$c_0 = \begin{pmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{pmatrix}, c_1 = \begin{pmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{pmatrix}, c_2 = \begin{pmatrix} + & - & - & + \\ - & + & + & - \\ - & + & + & - \\ + & - & - & + \end{pmatrix}, c_3 = \begin{pmatrix} + & + & - & - \\ + & + & - & - \\ - & - & + & + \\ - & - & + & + \end{pmatrix}.$$

The block-circulant matrix A has first row $(c_0, c_1, c_2, c_3, c_3, c_2, c_1)$ and is given by

$$A = \left(\begin{array}{cccc|cccc|cccc|cccc|cccc|cccc|cccc}
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\end{array}\right)$$

Similarly, the block-circulant matrix B has first row $(c_0, c_1, c_2, c_3, -c_3, -c_2, -c_1)$ and is given by

$$B = \left(\begin{array}{cccc|cccc|cccc|cccc|cccc|cccc|cccc}
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\end{array}\right)$$

It remains to construct the block matrix F. Label the 8 rows of the Hadamard matrix formed by the Kronecker product of $\left(\begin{smallmatrix} + & + \\ + & - \end{smallmatrix}\right)$ with (5.7.a) by $k_0, \ldots, k_7$. We then form the following

$F_i = k_i^t h_0$, for $i \in \{1, 2, 3, 4, 5, 6, 7\}$.

$$F_1 = \begin{pmatrix} - & - & - & - \\ + & + & + & + \\ - & - & - & - \\ + & + & + & + \\ - & - & - & - \\ + & + & + & + \\ - & - & - & - \\ + & + & + & + \end{pmatrix}, \quad F_2 = \begin{pmatrix} - & - & - & - \\ + & + & + & + \\ + & + & + & + \\ - & - & - & - \\ - & - & - & - \\ + & + & + & + \\ + & + & + & + \\ - & - & - & - \end{pmatrix}, \quad F_3 = \begin{pmatrix} - & - & - & - \\ - & - & - & - \\ + & + & + & + \\ + & + & + & + \\ - & - & - & - \\ - & - & - & - \\ + & + & + & + \\ + & + & + & + \end{pmatrix}, \quad F_4 = \begin{pmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ - & - & - & - \\ - & - & - & - \\ - & - & - & - \\ - & - & - & - \end{pmatrix},$$

$$F_5 = \begin{pmatrix} - & - & - & - \\ + & + & + & + \\ - & - & - & - \\ + & + & + & + \\ + & + & + & + \\ - & - & - & - \\ + & + & + & + \\ - & - & - & - \end{pmatrix}, \quad F_6 = \begin{pmatrix} - & - & - & - \\ + & + & + & + \\ + & + & + & + \\ - & - & - & - \\ + & + & + & + \\ - & - & - & - \\ - & - & - & - \\ + & + & + & + \end{pmatrix}, \quad F_7 = \begin{pmatrix} - & - & - & - \\ - & - & - & - \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ - & - & - & - \\ - & - & - & - \end{pmatrix}.$$

We then use these to form

$$F = \left(\begin{array}{cccc|cccc|cccc|cccc|cccc|cccc|cccc}
- & - & - & - & - & - & - & - & - & - & - & - & + & + & + & + & - & - & - & - & - & - & - & - & - & - & - & - \\
+ & + & + & + & + & + & + & + & - & - & - & - & + & + & + & + & + & + & + & + & + & + & + & + & - & - & - & - \\
- & - & - & - & + & + & + & + & + & + & + & + & + & + & + & + & - & - & - & - & + & + & + & + & + & + & + & + \\
+ & + & + & + & - & - & - & - & + & + & + & + & + & + & + & + & + & + & + & + & - & - & - & - & + & + & + & + \\
- & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & + & + & + & + & + & + & + & + & + & + & + & + \\
+ & + & + & + & + & + & + & + & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & + & + & + & + \\
- & - & - & - & + & + & + & + & + & + & + & + & - & - & - & - & + & + & + & + & - & - & - & - & - & - & - & - \\
+ & + & + & + & - & - & - & - & + & + & + & + & - & - & - & - & - & - & - & - & + & + & + & + & - & - & - & -
\end{array}\right).$$

Taking $E = F^t$, we find that $\begin{pmatrix} J_8 & F & -F \\ E & A & B \\ -E & B & A \end{pmatrix}$ is the required balancedly splittable Hadamard matrix of order 64, which is too large to display here.

A2. A Balancedly Splittable BW$(36, 36; 6)$

Beginning with (5.8.a), a Butson Hadamard matrix of order 3 over the complex 3-rd roots of unity, we can construct a balancedly splittable BW$(36, 36; 6)$.

First, we rewrite (5.8.a) as

$$H = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & 2 \\ 0 & 2 & 4 \end{pmatrix},$$

where the entries of the matrix are the logarithms of $-e^{\frac{2\pi\sqrt{-1}}{3}}$.

The auxiliary matrices of H are then given by

$$c_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, c_1 = \begin{pmatrix} 0 & 4 & 2 \\ 2 & 0 & 4 \\ 4 & 2 & 0 \end{pmatrix}, c_2 = \begin{pmatrix} 0 & 2 & 4 \\ 4 & 0 & 2 \\ 2 & 4 & 0 \end{pmatrix}.$$

The block-circulant matrix A has first row $(c_0, c_1, c_2, c_2, c_1)$ and is given by

$$A = \left(\begin{array}{ccc|ccc|ccc|ccc|ccc}
0&0&0&0&4&2&0&2&4&0&2&4&0&4&2\\
0&0&0&2&0&4&4&0&2&4&0&2&2&0&4\\
0&0&0&4&2&0&2&4&0&2&4&0&4&2&0\\ \hline
0&4&2&0&0&0&0&4&2&0&2&4&0&2&4\\
2&0&4&0&0&0&2&0&4&4&0&2&4&0&2\\
4&2&0&0&0&0&4&2&0&2&4&0&2&4&0\\ \hline
0&2&4&0&4&2&0&0&0&0&4&2&0&2&4\\
4&0&2&2&0&4&0&0&0&2&0&4&4&0&2\\
2&4&0&4&2&0&0&0&0&4&2&0&2&4&0\\ \hline
0&2&4&0&2&4&0&4&2&0&0&0&0&4&2\\
4&0&2&4&0&2&2&0&4&0&0&0&2&0&4\\
2&4&0&2&4&0&4&2&0&0&0&0&4&2&0\\ \hline
0&4&2&0&2&4&0&2&4&0&4&2&0&0&0\\
2&0&4&4&0&2&4&0&2&2&0&4&0&0&0\\
4&2&0&2&4&0&2&4&0&4&2&0&0&0&0
\end{array}\right).$$

The block-circulant matrix B has first row $(c_0, c_1, c_2, -c_2, -c_1)$ and is given by

$$B = \left(\begin{array}{ccc|ccc|ccc|ccc|ccc}
0&0&0&0&4&2&0&2&4&3&5&1&3&1&5\\
0&0&0&2&0&4&4&0&2&1&3&5&5&3&1\\
0&0&0&4&2&0&2&4&0&5&1&3&1&5&3\\ \hline
3&1&5&0&0&0&0&4&2&0&2&4&3&5&1\\
5&3&1&0&0&0&2&0&4&4&0&2&1&3&5\\
1&5&3&0&0&0&4&2&0&2&4&0&5&1&3\\ \hline
3&5&1&3&1&5&0&0&0&0&4&2&0&2&4\\
1&3&5&5&3&1&0&0&0&2&0&4&4&0&2\\
5&1&3&1&5&3&0&0&0&4&2&0&2&4&0\\ \hline
0&2&4&3&5&1&3&1&5&0&0&0&0&4&2\\
4&0&2&1&3&5&5&3&1&0&0&0&2&0&4\\
2&4&0&5&1&3&1&5&3&0&0&0&4&2&0\\ \hline
0&4&2&0&2&4&3&5&1&3&1&5&0&0&0\\
2&0&4&4&0&2&1&3&5&5&3&1&0&0&0\\
4&2&0&2&4&0&5&1&3&1&5&3&0&0&0
\end{array}\right).$$

As before, we take $K = \left(\begin{smallmatrix} + & + \\ + & - \end{smallmatrix}\right) \otimes H$ and form $F_i = k_i^* h_0$, for $i \in \{1, 2, 3, 4, 5\}$.

$$F_1 = \begin{pmatrix} 0&0&0\\2&2&2\\4&4&4\\0&0&0\\2&2&2\\4&4&4 \end{pmatrix}, \qquad F_2 = \begin{pmatrix} 0&0&0\\4&4&4\\2&2&2\\0&0&0\\4&4&4\\2&2&2 \end{pmatrix}, \qquad F_3 = \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0\\3&3&3\\3&3&3\\3&3&3 \end{pmatrix},$$

$$F_4 = \begin{pmatrix} 0&0&0\\2&2&2\\4&4&4\\3&3&3\\5&5&5\\1&1&1 \end{pmatrix}, \qquad F_5 = \begin{pmatrix} 0&0&0\\4&4&4\\2&2&2\\3&3&3\\1&1&1\\5&5&5 \end{pmatrix}.$$

Then

$$F = \left(\begin{array}{ccc|ccc|ccc|ccc|ccc}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
2&2&2&4&4&4&0&0&0&2&2&2&4&4&4\\
4&4&4&2&2&2&0&0&0&4&4&4&2&2&2\\
0&0&0&0&0&0&3&3&3&3&3&3&3&3&3\\
2&2&2&4&4&4&3&3&3&5&5&5&1&1&1\\
4&4&4&2&2&2&3&3&3&1&1&1&5&5&5
\end{array}\right),$$

and $E = F^*$.

Finally, we form the matrix $\begin{pmatrix} J_6 & F & -F \\ E & A & B \\ -E & B & A \end{pmatrix}$ given by

$$
\left(\begin{array}{cccccc|ccccccccccccccc|ccccccccccccccc}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&3&3&3&3&3&3&3&3&3&3&3&3&3&3&3\\
0&0&0&0&0&0&2&2&2&4&4&4&0&0&0&2&2&2&4&4&4&5&5&5&1&1&1&3&3&3&5&5&5&1&1&1\\
0&0&0&0&0&0&4&4&4&2&2&2&0&0&0&4&4&4&2&2&2&1&1&1&5&5&5&3&3&3&1&1&1&5&5&5\\
0&0&0&0&0&0&0&0&0&0&0&0&3&3&3&3&3&3&3&3&3&3&3&3&3&3&3&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&2&2&2&4&4&4&3&3&3&5&5&5&1&1&1&5&5&5&1&1&1&0&0&0&2&2&2&4&4&4\\
0&0&0&0&0&0&4&4&4&2&2&2&3&3&3&1&1&1&5&5&5&1&1&1&5&5&5&0&0&0&4&4&4&2&2&2\\
\hline
0&4&2&0&4&2&0&0&0&0&4&2&0&2&4&0&2&4&0&4&2&0&0&0&0&4&2&0&2&4&3&5&1&3&1&5\\
0&4&2&0&4&2&0&0&0&2&0&4&4&0&2&4&0&2&2&0&4&0&0&0&2&0&4&4&0&2&1&3&5&5&3&1\\
0&4&2&0&4&2&0&0&0&4&2&0&2&4&0&2&4&0&4&2&0&0&0&0&4&2&0&2&4&0&5&1&3&1&5&3\\
0&2&4&0&2&4&0&4&2&0&0&0&0&4&2&0&2&4&0&2&4&3&1&5&0&0&0&0&4&2&0&2&4&3&5&1\\
0&2&4&0&2&4&2&0&4&0&0&0&2&0&4&4&0&2&4&0&2&5&3&1&0&0&0&2&0&4&4&0&2&1&3&5\\
0&2&4&0&2&4&4&2&0&0&0&0&4&2&0&2&4&0&2&4&0&1&5&3&0&0&0&4&2&0&2&4&0&5&1&3\\
0&0&0&3&3&3&0&2&4&0&4&2&0&0&0&0&4&2&0&2&4&3&5&1&3&1&5&0&0&0&0&4&2&0&2&4\\
0&0&0&3&3&3&4&0&2&2&0&4&0&0&0&2&0&4&4&0&2&1&3&5&5&3&1&0&0&0&2&0&4&4&0&2\\
0&0&0&3&3&3&2&4&0&4&2&0&0&0&0&4&2&0&2&4&0&5&1&3&1&5&3&0&0&0&4&2&0&2&4&0\\
0&4&2&3&1&5&0&2&4&0&2&4&0&4&2&0&0&0&0&4&2&0&2&4&3&5&1&3&1&5&0&0&0&0&4&2\\
0&4&2&3&1&5&4&0&2&4&0&2&2&0&4&0&0&0&2&0&4&4&0&2&1&3&5&5&3&1&0&0&0&2&0&4\\
0&4&2&3&1&5&2&4&0&2&4&0&4&2&0&0&0&0&4&2&0&2&4&0&5&1&3&1&5&3&0&0&0&4&2&0\\
0&2&4&3&5&1&0&4&2&0&2&4&0&2&4&0&4&2&0&0&0&0&4&2&0&2&4&3&5&1&3&1&5&0&0&0\\
0&2&4&3&5&1&2&0&4&4&0&2&4&0&2&2&0&4&0&0&0&2&0&4&4&0&2&1&3&5&5&3&1&0&0&0\\
0&2&4&3&5&1&4&2&0&2&4&0&2&4&0&4&2&0&0&0&0&4&2&0&2&4&0&5&1&3&1&5&3&0&0&0\\
\hline
3&1&5&3&1&5&0&0&0&0&4&2&0&2&4&3&5&1&3&1&5&0&0&0&0&4&2&0&2&4&0&2&4&0&4&2\\
3&1&5&3&1&5&0&0&0&2&0&4&4&0&2&1&3&5&5&3&1&0&0&0&2&0&4&4&0&2&4&0&2&2&0&4\\
3&1&5&3&1&5&0&0&0&4&2&0&2&4&0&5&1&3&1&5&3&0&0&0&4&2&0&2&4&0&2&4&0&4&2&0\\
3&5&1&3&5&1&3&1&5&0&0&0&0&4&2&0&2&4&3&5&1&0&4&2&0&0&0&0&4&2&0&2&4&0&2&4\\
3&5&1&3&5&1&5&3&1&0&0&0&2&0&4&4&0&2&1&3&5&2&0&4&0&0&0&2&0&4&4&0&2&4&0&2\\
3&5&1&3&5&1&1&5&3&0&0&0&4&2&0&2&4&0&5&1&3&4&2&0&0&0&0&4&2&0&2&4&0&2&4&0\\
3&3&3&0&0&0&3&5&1&3&1&5&0&0&0&0&4&2&0&2&4&0&2&4&0&4&2&0&0&0&0&4&2&0&2&4\\
3&3&3&0&0&0&1&3&5&5&3&1&0&0&0&2&0&4&4&0&2&4&0&2&2&0&4&0&0&0&2&0&4&4&0&2\\
3&3&3&0&0&0&5&1&3&1&5&3&0&0&0&4&2&0&2&4&0&2&4&0&4&2&0&0&0&0&4&2&0&2&4&0\\
3&1&5&0&4&2&0&2&4&3&5&1&3&1&5&0&0&0&0&4&2&0&2&4&0&2&4&0&4&2&0&0&0&0&4&2\\
3&1&5&0&4&2&4&0&2&1&3&5&5&3&1&0&0&0&2&0&4&4&0&2&4&0&2&2&0&4&0&0&0&2&0&4\\
3&1&5&0&4&2&2&4&0&5&1&3&1&5&3&0&0&0&4&2&0&2&4&0&2&4&0&4&2&0&0&0&0&4&2&0\\
3&5&1&0&2&4&0&4&2&0&2&4&3&5&1&3&1&5&0&0&0&0&4&2&0&2&4&0&2&4&0&4&2&0&0&0\\
3&5&1&0&2&4&2&0&4&4&0&2&1&3&5&5&3&1&0&0&0&2&0&4&4&0&2&4&0&2&2&0&4&0&0&0\\
3&5&1&0&2&4&4&2&0&2&4&0&5&1&3&1&5&3&0&0&0&4&2&0&2&4&0&2&4&0&4&2&0&0&0&0
\end{array}\right).
$$

which is a balancedly splittable $\mathrm{BW}(36, 36; 6)$.

A3. A Balancedly Splittable OD$(64; 16, 16, 16, 16)$

As before, we start with (6.16.a). The corresponding Hadamard matrix is (5.7.a), so we form the the auxiliary matrices given by

$$
c_0 = \begin{pmatrix} a & b & c & d \\ a & b & c & d \\ a & b & c & d \\ a & b & c & d \end{pmatrix}, c_1 = \begin{pmatrix} b & \overline{a} & d & \overline{c} \\ \overline{b} & a & \overline{d} & c \\ b & \overline{a} & d & \overline{c} \\ \overline{b} & a & \overline{d} & c \end{pmatrix}, c_2 = \begin{pmatrix} c & \overline{d} & \overline{a} & b \\ \overline{c} & d & a & \overline{b} \\ \overline{c} & d & a & \overline{b} \\ c & \overline{d} & \overline{a} & b \end{pmatrix}, c_3 = \begin{pmatrix} d & c & \overline{b} & \overline{a} \\ d & c & \overline{b} & \overline{a} \\ \overline{d} & \overline{c} & b & a \\ \overline{d} & \overline{c} & b & a \end{pmatrix}.
$$

The block-circulant matrix A has first row $(c_0, c_1, c_2, c_3, c_3, c_2, c_1)$ and is given by

$$A = \left(\begin{array}{cccc|cccc|cccc|cccc|cccc|cccc|cccc}
a & b & c & d & b & \bar{a} & d & \bar{c} & c & \bar{d} & \bar{a} & b & d & c & \bar{b} & \bar{a} & d & c & \bar{b} & \bar{a} & c & \bar{d} & \bar{a} & b & b & \bar{a} & d & \bar{c} \\
a & b & c & d & \bar{b} & a & \bar{d} & c & \bar{c} & d & a & \bar{b} & d & c & \bar{b} & \bar{a} & d & c & \bar{b} & \bar{a} & \bar{c} & d & a & \bar{b} & \bar{b} & a & \bar{d} & c \\
a & b & c & d & b & \bar{a} & d & \bar{c} & \bar{c} & d & a & \bar{b} & \bar{d} & \bar{c} & b & a & \bar{d} & \bar{c} & b & a & \bar{c} & d & a & \bar{b} & b & \bar{a} & d & \bar{c} \\
a & b & c & d & \bar{b} & a & \bar{d} & c & c & \bar{d} & \bar{a} & b & \bar{d} & \bar{c} & b & a & \bar{d} & \bar{c} & b & a & c & \bar{d} & \bar{a} & b & \bar{b} & a & \bar{d} & c \\
\hline
b & \bar{a} & d & \bar{c} & a & b & c & d & b & \bar{a} & d & \bar{c} & c & \bar{d} & \bar{a} & b & d & c & \bar{b} & \bar{a} & d & c & \bar{b} & \bar{a} & c & \bar{d} & \bar{a} & b \\
\bar{b} & a & \bar{d} & c & a & b & c & d & \bar{b} & a & \bar{d} & c & \bar{c} & d & a & \bar{b} & d & c & \bar{b} & \bar{a} & d & c & \bar{b} & \bar{a} & \bar{c} & d & a & \bar{b} \\
b & \bar{a} & d & \bar{c} & a & b & c & d & b & \bar{a} & d & \bar{c} & \bar{c} & d & a & \bar{b} & \bar{d} & \bar{c} & b & a & \bar{d} & \bar{c} & b & a & \bar{c} & d & a & \bar{b} \\
\bar{b} & a & \bar{d} & c & a & b & c & d & \bar{b} & a & \bar{d} & c & c & \bar{d} & \bar{a} & b & \bar{d} & \bar{c} & b & a & \bar{d} & \bar{c} & b & a & c & \bar{d} & \bar{a} & b \\
\hline
c & \bar{d} & \bar{a} & b & b & \bar{a} & d & \bar{c} & a & b & c & d & b & \bar{a} & d & \bar{c} & c & \bar{d} & \bar{a} & b & d & c & \bar{b} & \bar{a} & d & c & \bar{b} & \bar{a} \\
\bar{c} & d & a & \bar{b} & \bar{b} & a & \bar{d} & c & a & b & c & d & \bar{b} & a & \bar{d} & c & \bar{c} & d & a & \bar{b} & d & c & \bar{b} & \bar{a} & d & c & \bar{b} & \bar{a} \\
\bar{c} & d & a & \bar{b} & b & \bar{a} & d & \bar{c} & a & b & c & d & b & \bar{a} & d & \bar{c} & \bar{c} & d & a & \bar{b} & \bar{d} & \bar{c} & b & a & \bar{d} & \bar{c} & b & a \\
c & \bar{d} & \bar{a} & b & \bar{b} & a & \bar{d} & c & a & b & c & d & \bar{b} & a & \bar{d} & c & c & \bar{d} & \bar{a} & b & \bar{d} & \bar{c} & b & a & \bar{d} & \bar{c} & b & a \\
\hline
d & c & \bar{b} & \bar{a} & c & \bar{d} & \bar{a} & b & b & \bar{a} & d & \bar{c} & a & b & c & d & b & \bar{a} & d & \bar{c} & c & \bar{d} & \bar{a} & b & d & c & \bar{b} & \bar{a} \\
d & c & \bar{b} & \bar{a} & \bar{c} & d & a & \bar{b} & \bar{b} & a & \bar{d} & c & a & b & c & d & \bar{b} & a & \bar{d} & c & \bar{c} & d & a & \bar{b} & d & c & \bar{b} & \bar{a} \\
\bar{d} & \bar{c} & b & a & \bar{c} & d & a & \bar{b} & b & \bar{a} & d & \bar{c} & a & b & c & d & b & \bar{a} & d & \bar{c} & \bar{c} & d & a & \bar{b} & \bar{d} & \bar{c} & b & a \\
\bar{d} & \bar{c} & b & a & c & \bar{d} & \bar{a} & b & \bar{b} & a & \bar{d} & c & a & b & c & d & \bar{b} & a & \bar{d} & c & c & \bar{d} & \bar{a} & b & \bar{d} & \bar{c} & b & a \\
\hline
d & c & \bar{b} & \bar{a} & d & c & \bar{b} & \bar{a} & c & \bar{d} & \bar{a} & b & b & \bar{a} & d & \bar{c} & a & b & c & d & b & \bar{a} & d & \bar{c} & c & \bar{d} & \bar{a} & b \\
d & c & \bar{b} & \bar{a} & d & c & \bar{b} & \bar{a} & \bar{c} & d & a & \bar{b} & \bar{b} & a & \bar{d} & c & a & b & c & d & \bar{b} & a & \bar{d} & c & \bar{c} & d & a & \bar{b} \\
\bar{d} & \bar{c} & b & a & \bar{d} & \bar{c} & b & a & \bar{c} & d & a & \bar{b} & b & \bar{a} & d & \bar{c} & a & b & c & d & b & \bar{a} & d & \bar{c} & \bar{c} & d & a & \bar{b} \\
\bar{d} & \bar{c} & b & a & \bar{d} & \bar{c} & b & a & c & \bar{d} & \bar{a} & b & \bar{b} & a & \bar{d} & c & a & b & c & d & \bar{b} & a & \bar{d} & c & c & \bar{d} & \bar{a} & b \\
\hline
c & \bar{d} & \bar{a} & b & d & c & \bar{b} & \bar{a} & d & c & \bar{b} & \bar{a} & c & \bar{d} & \bar{a} & b & b & \bar{a} & d & \bar{c} & a & b & c & d & b & \bar{a} & d & \bar{c} \\
\bar{c} & d & a & \bar{b} & d & c & \bar{b} & \bar{a} & d & c & \bar{b} & \bar{a} & \bar{c} & d & a & \bar{b} & \bar{b} & a & \bar{d} & c & a & b & c & d & \bar{b} & a & \bar{d} & c \\
\bar{c} & d & a & \bar{b} & \bar{d} & \bar{c} & b & a & \bar{d} & \bar{c} & b & a & \bar{c} & d & a & \bar{b} & b & \bar{a} & d & \bar{c} & a & b & c & d & b & \bar{a} & d & \bar{c} \\
c & \bar{d} & \bar{a} & b & \bar{d} & \bar{c} & b & a & \bar{d} & \bar{c} & b & a & c & \bar{d} & \bar{a} & b & \bar{b} & a & \bar{d} & c & a & b & c & d & \bar{b} & a & \bar{d} & c \\
\hline
b & \bar{a} & d & \bar{c} & c & \bar{d} & \bar{a} & b & d & c & \bar{b} & \bar{a} & d & c & \bar{b} & \bar{a} & c & \bar{d} & \bar{a} & b & b & \bar{a} & d & \bar{c} & a & b & c & d \\
\bar{b} & a & \bar{d} & c & \bar{c} & d & a & \bar{b} & d & c & \bar{b} & \bar{a} & d & c & \bar{b} & \bar{a} & \bar{c} & d & a & \bar{b} & \bar{b} & a & \bar{d} & c & a & b & c & d \\
b & \bar{a} & d & \bar{c} & \bar{c} & d & a & \bar{b} & \bar{d} & \bar{c} & b & a & \bar{d} & \bar{c} & b & a & \bar{c} & d & a & \bar{b} & b & \bar{a} & d & \bar{c} & a & b & c & d \\
\bar{b} & a & \bar{d} & c & c & \bar{d} & \bar{a} & b & \bar{d} & \bar{c} & b & a & \bar{d} & \bar{c} & b & a & c & \bar{d} & \bar{a} & b & \bar{b} & a & \bar{d} & c & a & b & c & d
\end{array}\right).$$

The block-circulant matrix B has first row $(c_0, c_1, c_2, c_3, -c_3, -c_2, -c_1)$ and is given by

$$B = \left(\begin{array}{cccc|cccc|cccc|cccc|cccc|cccc|cccc}
a & b & c & d & b & \bar{a} & d & \bar{c} & c & \bar{d} & \bar{a} & b & d & c & \bar{b} & \bar{a} & \bar{d} & \bar{c} & b & a & \bar{c} & d & a & \bar{b} & \bar{b} & a & \bar{d} & c \\
a & b & c & d & \bar{b} & a & \bar{d} & c & \bar{c} & d & a & \bar{b} & d & c & \bar{b} & \bar{a} & \bar{d} & \bar{c} & b & a & c & \bar{d} & \bar{a} & b & b & \bar{a} & d & \bar{c} \\
a & b & c & d & b & \bar{a} & d & \bar{c} & \bar{c} & d & a & \bar{b} & \bar{d} & \bar{c} & b & a & d & c & \bar{b} & \bar{a} & c & \bar{d} & \bar{a} & b & \bar{b} & a & \bar{d} & c \\
a & b & c & d & \bar{b} & a & \bar{d} & c & c & \bar{d} & \bar{a} & b & \bar{d} & \bar{c} & b & a & d & c & \bar{b} & \bar{a} & \bar{c} & d & a & \bar{b} & b & \bar{a} & d & \bar{c} \\
\hline
\bar{b} & a & \bar{d} & c & a & b & c & d & b & \bar{a} & d & \bar{c} & c & \bar{d} & \bar{a} & b & d & c & \bar{b} & \bar{a} & \bar{d} & \bar{c} & b & a & \bar{c} & d & a & \bar{b} \\
b & \bar{a} & d & \bar{c} & a & b & c & d & \bar{b} & a & \bar{d} & c & \bar{c} & d & a & \bar{b} & d & c & \bar{b} & \bar{a} & \bar{d} & \bar{c} & b & a & c & \bar{d} & \bar{a} & b \\
\bar{b} & a & \bar{d} & c & a & b & c & d & b & \bar{a} & d & \bar{c} & \bar{c} & d & a & \bar{b} & \bar{d} & \bar{c} & b & a & d & c & \bar{b} & \bar{a} & c & \bar{d} & \bar{a} & b \\
b & \bar{a} & d & \bar{c} & a & b & c & d & \bar{b} & a & \bar{d} & c & c & \bar{d} & \bar{a} & b & \bar{d} & \bar{c} & b & a & d & c & \bar{b} & \bar{a} & \bar{c} & d & a & \bar{b} \\
\hline
\bar{c} & d & a & \bar{b} & \bar{b} & a & \bar{d} & c & a & b & c & d & b & \bar{a} & d & \bar{c} & c & \bar{d} & \bar{a} & b & d & c & \bar{b} & \bar{a} & \bar{d} & \bar{c} & b & a \\
c & \bar{d} & \bar{a} & b & b & \bar{a} & d & \bar{c} & a & b & c & d & \bar{b} & a & \bar{d} & c & \bar{c} & d & a & \bar{b} & d & c & \bar{b} & \bar{a} & \bar{d} & \bar{c} & b & a \\
c & \bar{d} & \bar{a} & b & \bar{b} & a & \bar{d} & c & a & b & c & d & b & \bar{a} & d & \bar{c} & \bar{c} & d & a & \bar{b} & \bar{d} & \bar{c} & b & a & d & c & \bar{b} & \bar{a} \\
\bar{c} & d & a & \bar{b} & b & \bar{a} & d & \bar{c} & a & b & c & d & \bar{b} & a & \bar{d} & c & c & \bar{d} & \bar{a} & b & \bar{d} & \bar{c} & b & a & d & c & \bar{b} & \bar{a} \\
\hline
\bar{d} & \bar{c} & b & a & \bar{c} & d & a & \bar{b} & \bar{b} & a & \bar{d} & c & a & b & c & d & b & \bar{a} & d & \bar{c} & c & \bar{d} & \bar{a} & b & d & c & \bar{b} & \bar{a} \\
\bar{d} & \bar{c} & b & a & c & \bar{d} & \bar{a} & b & b & \bar{a} & d & \bar{c} & a & b & c & d & \bar{b} & a & \bar{d} & c & \bar{c} & d & a & \bar{b} & d & c & \bar{b} & \bar{a} \\
d & c & \bar{b} & \bar{a} & c & \bar{d} & \bar{a} & b & \bar{b} & a & \bar{d} & c & a & b & c & d & b & \bar{a} & d & \bar{c} & \bar{c} & d & a & \bar{b} & \bar{d} & \bar{c} & b & a \\
d & c & \bar{b} & \bar{a} & \bar{c} & d & a & \bar{b} & b & \bar{a} & d & \bar{c} & a & b & c & d & \bar{b} & a & \bar{d} & c & c & \bar{d} & \bar{a} & b & \bar{d} & \bar{c} & b & a \\
\hline
d & c & \bar{b} & \bar{a} & \bar{d} & \bar{c} & b & a & \bar{c} & d & a & \bar{b} & \bar{b} & a & \bar{d} & c & a & b & c & d & b & \bar{a} & d & \bar{c} & c & \bar{d} & \bar{a} & b \\
d & c & \bar{b} & \bar{a} & \bar{d} & \bar{c} & b & a & c & \bar{d} & \bar{a} & b & b & \bar{a} & d & \bar{c} & a & b & c & d & \bar{b} & a & \bar{d} & c & \bar{c} & d & a & \bar{b} \\
\bar{d} & \bar{c} & b & a & d & c & \bar{b} & \bar{a} & c & \bar{d} & \bar{a} & b & \bar{b} & a & \bar{d} & c & a & b & c & d & b & \bar{a} & d & \bar{c} & \bar{c} & d & a & \bar{b} \\
\bar{d} & \bar{c} & b & a & d & c & \bar{b} & \bar{a} & \bar{c} & d & a & \bar{b} & b & \bar{a} & d & \bar{c} & a & b & c & d & \bar{b} & a & \bar{d} & c & c & \bar{d} & \bar{a} & b \\
\hline
c & \bar{d} & \bar{a} & b & d & c & \bar{b} & \bar{a} & \bar{d} & \bar{c} & b & a & \bar{c} & d & a & \bar{b} & \bar{b} & a & \bar{d} & c & a & b & c & d & b & \bar{a} & d & \bar{c} \\
\bar{c} & d & a & \bar{b} & d & c & \bar{b} & \bar{a} & \bar{d} & \bar{c} & b & a & c & \bar{d} & \bar{a} & b & b & \bar{a} & d & \bar{c} & a & b & c & d & \bar{b} & a & \bar{d} & c \\
\bar{c} & d & a & \bar{b} & \bar{d} & \bar{c} & b & a & d & c & \bar{b} & \bar{a} & c & \bar{d} & \bar{a} & b & \bar{b} & a & \bar{d} & c & a & b & c & d & b & \bar{a} & d & \bar{c} \\
c & \bar{d} & \bar{a} & b & \bar{d} & \bar{c} & b & a & d & c & \bar{b} & \bar{a} & \bar{c} & d & a & \bar{b} & b & \bar{a} & d & \bar{c} & a & b & c & d & \bar{b} & a & \bar{d} & c \\
\hline
b & \bar{a} & d & \bar{c} & c & \bar{d} & \bar{a} & b & d & c & \bar{b} & \bar{a} & \bar{d} & \bar{c} & b & a & \bar{c} & d & a & \bar{b} & \bar{b} & a & \bar{d} & c & a & b & c & d \\
\bar{b} & a & \bar{d} & c & \bar{c} & d & a & \bar{b} & d & c & \bar{b} & \bar{a} & \bar{d} & \bar{c} & b & a & c & \bar{d} & \bar{a} & b & b & \bar{a} & d & \bar{c} & a & b & c & d \\
b & \bar{a} & d & \bar{c} & \bar{c} & d & a & \bar{b} & \bar{d} & \bar{c} & b & a & d & c & \bar{b} & \bar{a} & c & \bar{d} & \bar{a} & b & \bar{b} & a & \bar{d} & c & a & b & c & d \\
\bar{b} & a & \bar{d} & c & c & \bar{d} & \bar{a} & b & \bar{d} & \bar{c} & b & a & d & c & \bar{b} & \bar{a} & \bar{c} & d & a & \bar{b} & b & \bar{a} & d & \bar{c} & a & b & c & d
\end{array}\right)$$

Taking Y to be the Kronecker product of $\left(\begin{smallmatrix} + & + \\ + & - \end{smallmatrix}\right)$ with X and K the corresponding Hadamard matrix, we form the matrices $F_i = k_i^t x_0$, for $i \in \{1, 2, 3, 4, 5, 6, 7\}$.

$$F_1 = \begin{pmatrix} \bar{a} & \bar{b} & \bar{c} & \bar{d} \\ a & b & c & d \\ \bar{a} & \bar{b} & \bar{c} & \bar{d} \\ a & b & c & d \\ \bar{a} & \bar{b} & \bar{c} & \bar{d} \\ a & b & c & d \\ \bar{a} & \bar{b} & \bar{c} & \bar{d} \\ a & b & c & d \end{pmatrix}, \quad F_2 = \begin{pmatrix} \bar{a} & \bar{b} & \bar{c} & \bar{d} \\ a & b & c & d \\ a & b & c & d \\ \bar{a} & \bar{b} & \bar{c} & \bar{d} \\ \bar{a} & \bar{b} & \bar{c} & \bar{d} \\ a & b & c & d \\ a & b & c & d \\ \bar{a} & \bar{b} & \bar{c} & \bar{d} \end{pmatrix}, \quad F_3 = \begin{pmatrix} \bar{a} & \bar{b} & \bar{c} & \bar{d} \\ \bar{a} & \bar{b} & \bar{c} & \bar{d} \\ a & b & c & d \\ a & b & c & d \\ \bar{a} & \bar{b} & \bar{c} & \bar{d} \\ \bar{a} & \bar{b} & \bar{c} & \bar{d} \\ a & b & c & d \\ a & b & c & d \end{pmatrix}, \quad F_4 = \begin{pmatrix} a & b & c & d \\ a & b & c & d \\ a & b & c & d \\ a & b & c & d \\ \bar{a} & \bar{b} & \bar{c} & \bar{d} \\ \bar{a} & \bar{b} & \bar{c} & \bar{d} \\ \bar{a} & \bar{b} & \bar{c} & \bar{d} \\ \bar{a} & \bar{b} & \bar{c} & \bar{d} \end{pmatrix},$$

$$F_5 = \begin{pmatrix} \bar{a} & \bar{b} & \bar{c} & \bar{d} \\ a & b & c & d \\ \bar{a} & \bar{b} & \bar{c} & \bar{d} \\ a & b & c & d \\ a & b & c & d \\ \bar{a} & \bar{b} & \bar{c} & \bar{d} \\ a & b & c & d \\ \bar{a} & \bar{b} & \bar{c} & \bar{d} \end{pmatrix}, \quad F_6 = \begin{pmatrix} \bar{a} & \bar{b} & \bar{c} & \bar{d} \\ a & b & c & d \\ a & b & c & d \\ \bar{a} & \bar{b} & \bar{c} & \bar{d} \\ a & b & c & d \\ \bar{a} & \bar{b} & \bar{c} & \bar{d} \\ \bar{a} & \bar{b} & \bar{c} & \bar{d} \\ a & b & c & d \end{pmatrix}, \quad F_7 = \begin{pmatrix} \bar{a} & \bar{b} & \bar{c} & \bar{d} \\ \bar{a} & \bar{b} & \bar{c} & \bar{d} \\ a & b & c & d \\ a & b & c & d \\ a & b & c & d \\ a & b & c & d \\ \bar{a} & \bar{b} & \bar{c} & \bar{d} \\ \bar{a} & \bar{b} & \bar{c} & \bar{d} \end{pmatrix}.$$

The matrix F is then given by

$$F = \left(\begin{array}{cccc|cccc|cccc|cccc|cccc|cccc|cccc}
\overline{a} & \overline{b} & \overline{c} & \overline{d} & \overline{a} & \overline{b} & \overline{c} & \overline{d} & \overline{a} & \overline{b} & \overline{c} & \overline{d} & a & b & c & d & \overline{a} & \overline{b} & \overline{c} & \overline{d} & \overline{a} & \overline{b} & \overline{c} & \overline{d} & \overline{a} & \overline{b} & \overline{c} & \overline{d} \\
a & b & c & d & a & b & c & d & \overline{a} & \overline{b} & \overline{c} & \overline{d} & a & b & c & d & a & b & c & d & a & b & c & d & \overline{a} & \overline{b} & \overline{c} & \overline{d} \\
\overline{a} & \overline{b} & \overline{c} & \overline{d} & a & b & c & d & a & b & c & d & a & b & c & d & \overline{a} & \overline{b} & \overline{c} & \overline{d} & a & b & c & d & a & b & c & d \\
a & b & c & d & \overline{a} & \overline{b} & \overline{c} & \overline{d} & a & b & c & d & a & b & c & d & a & b & c & d & \overline{a} & \overline{b} & \overline{c} & \overline{d} & a & b & c & d \\
\overline{a} & \overline{b} & \overline{c} & \overline{d} & \overline{a} & \overline{b} & \overline{c} & \overline{d} & \overline{a} & \overline{b} & \overline{c} & \overline{d} & \overline{a} & \overline{b} & \overline{c} & \overline{d} & a & b & c & d & a & b & c & d & a & b & c & d \\
a & b & c & d & a & b & c & d & \overline{a} & \overline{b} & \overline{c} & \overline{d} & \overline{a} & \overline{b} & \overline{c} & \overline{d} & \overline{a} & \overline{b} & \overline{c} & \overline{d} & \overline{a} & \overline{b} & \overline{c} & \overline{d} & a & b & c & d \\
\overline{a} & \overline{b} & \overline{c} & \overline{d} & a & b & c & d & a & b & c & d & \overline{a} & \overline{b} & \overline{c} & \overline{d} & a & b & c & d & \overline{a} & \overline{b} & \overline{c} & \overline{d} & \overline{a} & \overline{b} & \overline{c} & \overline{d} \\
a & b & c & d & \overline{a} & \overline{b} & \overline{c} & \overline{d} & a & b & c & d & \overline{a} & \overline{b} & \overline{c} & \overline{d} & \overline{a} & \overline{b} & \overline{c} & \overline{d} & a & b & c & d & \overline{a} & \overline{b} & \overline{c} & \overline{d}
\end{array}\right)$$

We next form the matrices $E_i = h_0^t y_i$, for $i \in \{1, 2, 3, 4, 5, 6, 7\}$.

$$E_1 = \begin{pmatrix}
\overline{b} & a & \overline{d} & c & \overline{b} & a & \overline{d} & c \\
\overline{b} & a & \overline{d} & c & \overline{b} & a & \overline{d} & c \\
\overline{b} & a & \overline{d} & c & \overline{b} & a & \overline{d} & c \\
\overline{b} & a & \overline{d} & c & \overline{b} & a & \overline{d} & c
\end{pmatrix}, \quad E_2 = \begin{pmatrix}
\overline{c} & d & a & \overline{b} & \overline{c} & d & a & \overline{b} \\
\overline{c} & d & a & \overline{b} & \overline{c} & d & a & \overline{b} \\
\overline{c} & d & a & \overline{b} & \overline{c} & d & a & \overline{b} \\
\overline{c} & d & a & \overline{b} & \overline{c} & d & a & \overline{b}
\end{pmatrix}, \quad E_3 = \begin{pmatrix}
\overline{d} & \overline{c} & b & a & \overline{d} & \overline{c} & b & a \\
\overline{d} & \overline{c} & b & a & \overline{d} & \overline{c} & b & a \\
\overline{d} & \overline{c} & b & a & \overline{d} & \overline{c} & b & a \\
\overline{d} & \overline{c} & b & a & \overline{d} & \overline{c} & b & a
\end{pmatrix},$$

$$E_4 = \begin{pmatrix}
a & b & c & d & \overline{a} & \overline{b} & \overline{c} & \overline{d} \\
a & b & c & d & \overline{a} & \overline{b} & \overline{c} & \overline{d} \\
a & b & c & d & \overline{a} & \overline{b} & \overline{c} & \overline{d} \\
a & b & c & d & \overline{a} & \overline{b} & \overline{c} & \overline{d}
\end{pmatrix}, \quad E_5 = \begin{pmatrix}
\overline{b} & a & \overline{d} & c & b & \overline{a} & d & \overline{c} \\
\overline{b} & a & \overline{d} & c & b & \overline{a} & d & \overline{c} \\
\overline{b} & a & \overline{d} & c & b & \overline{a} & d & \overline{c} \\
\overline{b} & a & \overline{d} & c & b & \overline{a} & d & \overline{c}
\end{pmatrix}, \quad E_6 = \begin{pmatrix}
\overline{c} & d & a & \overline{b} & c & \overline{d} & \overline{a} & b \\
\overline{c} & d & a & \overline{b} & c & \overline{d} & \overline{a} & b \\
\overline{c} & d & a & \overline{b} & c & \overline{d} & \overline{a} & b \\
\overline{c} & d & a & \overline{b} & c & \overline{d} & \overline{a} & b
\end{pmatrix},$$

$$E_7 = \begin{pmatrix}
\overline{d} & \overline{c} & b & a & d & c & \overline{b} & \overline{a} \\
\overline{d} & \overline{c} & b & a & d & c & \overline{b} & \overline{a} \\
\overline{d} & \overline{c} & b & a & d & c & \overline{b} & \overline{a} \\
\overline{d} & \overline{c} & b & a & d & c & \overline{b} & \overline{a}
\end{pmatrix}.$$

The matrix E^t is then given by

$$E^t = \left(\begin{array}{cccc|cccc|cccc|cccc|cccc|cccc|cccc}
\overline{b} & \overline{b} & \overline{b} & \overline{b} & \overline{c} & \overline{c} & \overline{c} & \overline{c} & \overline{d} & \overline{d} & \overline{d} & \overline{d} & a & a & a & a & \overline{b} & \overline{b} & \overline{b} & \overline{b} & \overline{c} & \overline{c} & \overline{c} & \overline{c} & \overline{d} & \overline{d} & \overline{d} & \overline{d} \\
a & a & a & a & d & d & d & d & \overline{c} & \overline{c} & \overline{c} & \overline{c} & b & b & b & b & a & a & a & a & d & d & d & d & \overline{c} & \overline{c} & \overline{c} & \overline{c} \\
\overline{d} & \overline{d} & \overline{d} & \overline{d} & a & a & a & a & b & b & b & b & c & c & c & c & \overline{d} & \overline{d} & \overline{d} & \overline{d} & a & a & a & a & b & b & b & b \\
c & c & c & c & \overline{b} & \overline{b} & \overline{b} & \overline{b} & a & a & a & a & d & d & d & d & c & c & c & c & \overline{b} & \overline{b} & \overline{b} & \overline{b} & a & a & a & a \\
\overline{b} & \overline{b} & \overline{b} & \overline{b} & \overline{c} & \overline{c} & \overline{c} & \overline{c} & \overline{d} & \overline{d} & \overline{d} & \overline{d} & \overline{a} & \overline{a} & \overline{a} & \overline{a} & b & b & b & b & c & c & c & c & d & d & d & d \\
a & a & a & a & d & d & d & d & \overline{c} & \overline{c} & \overline{c} & \overline{c} & \overline{b} & \overline{b} & \overline{b} & \overline{b} & \overline{a} & \overline{a} & \overline{a} & \overline{a} & \overline{d} & \overline{d} & \overline{d} & \overline{d} & c & c & c & c \\
\overline{d} & \overline{d} & \overline{d} & \overline{d} & a & a & a & a & b & b & b & b & \overline{c} & \overline{c} & \overline{c} & \overline{c} & d & d & d & d & \overline{a} & \overline{a} & \overline{a} & \overline{a} & \overline{b} & \overline{b} & \overline{b} & \overline{b} \\
c & c & c & c & \overline{b} & \overline{b} & \overline{b} & \overline{b} & a & a & a & a & \overline{d} & \overline{d} & \overline{d} & \overline{d} & \overline{c} & \overline{c} & \overline{c} & \overline{c} & b & b & b & b & \overline{a} & \overline{a} & \overline{a} & \overline{a}
\end{array}\right).$$

Finally, we form

$$G = k_0^t y_0 = \begin{pmatrix}
a & b & c & d & a & b & c & d \\
a & b & c & d & a & b & c & d \\
a & b & c & d & a & b & c & d \\
a & b & c & d & a & b & c & d \\
a & b & c & d & a & b & c & d \\
a & b & c & d & a & b & c & d \\
a & b & c & d & a & b & c & d \\
a & b & c & d & a & b & c & d
\end{pmatrix},$$

whereupon the matrix $\begin{pmatrix} G & F & -F \\ E & A & B \\ -E & B & A \end{pmatrix}$ is the required balancedly splittable OD$(64; 16, 16, 16, 16)$.

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