0.1. Definition. Generalizing balanced splittability to orthogonal designs presents several difficulties. The various cases are encapsulated in the next definition.

0.1. Definition. Let X be a full QOD$(n; s_1, \ldots, s_u)$. X is *balancedly splittable* if there is an $\ell \times n$ submatrix X_1 where one of the following conditions holds. In what follows $\alpha, \beta \in \{a + ib + jc + kd : a, b, c, d \in \mathbf{R}\}$.

(0.1.a) The off-diagonal entries of $X_1^* X_1$ are in the set

$$\{\pm\varepsilon c x_1^{m_1} \cdots x_u^{m_u} x_1^{*m_1'} \cdots x_u^{*m_u'} : m_i, m_i' \in \mathbf{N}, \varepsilon \in \{1, i, j, k\}, c \in \{\alpha, \alpha^*, \beta, \beta^*\}\}.$$

(0.1.b) The off-diagonal entries of $X_1^* X_1$ are in the set

$$S = \left\{ \sum_{i=1}^{u} t_i |x_i|^2 : t_i \in \mathbf{N}, \sum_{i=0}^{u} t_i = m \right\}$$

or in the set

$$\{\pm\varepsilon c x_1^{m_1} \cdots x_u^{m_u} x_1^{*m_1'} \cdots x_u^{*m_u'} : m_i, m_i' \in \mathbf{N}, \varepsilon \in \{1, i, j, k\}, c \in \{\alpha, \alpha^*, \beta, \beta^*\}\}.$$

(0.1.c) The off-diagonal entries of $X_1^* X_1$ are in the set

$$\{\pm\varepsilon c \sigma : \varepsilon \in \{1, i, j, k\}, c \in \{\alpha, \alpha^*, \beta, \beta^*\}\},$$

where $\sigma = \sum_i s_i |x_i|^2$ (cf. **??**).

In the first case, the split is *unstable*; in the second, the split is *unfaithfully unstable*; and in the third, the split is *stable*. The term *faithful* is used to describe the first and third cases.

From the definition, we see that if α and β are the same in absolute value, the split corresponds to a set of equiangular lines. Interestingly, we will see that both conditions in (0.1.b) can hold simultaneously.

The next two subsections will present constructions for both the unfaithful, and the faithful case.

* * *

0.2. Unfaithful Constructions. The constructions of this section are similar to those presented in **?** and **?**, and are applicable to real and complex orthogonal designs.

To begin, if W is s skew-symmetric W$(q + 1, q)$, then we take Q to be its core, i.e. the submatrix obtaind by deleting the first row and column. Further, we can assume that $W = \left(\begin{smallmatrix} 0 & \mathbf{1}^t \\ -\mathbf{1} & Q \end{smallmatrix}\right)$, hence $JQ = QJ = O$ and $Q^2 = J - qI$. We recursively define the following family of matrices.

$$\mathcal{J}_m = \begin{cases} aJ_1 & \text{if } m = 0, \text{ and} \\ J_q \otimes \mathcal{A}_{m-1} & \text{if } m > 0; \end{cases}$$

$$\mathcal{A}_m = \begin{cases} bJ_1 & \text{if } m = 0, \text{ and} \\ I_q \otimes \mathcal{J}_{m-1} + Q \otimes \mathcal{A}_{m-1} & \text{if } m > 0; \end{cases}$$

where a and b are commuting indeterminants.

We require the following lemma.

0.2. Lemma. (0.2.a) $\mathcal{J}_m\mathcal{A}_m^t = \mathcal{A}_m\mathcal{J}_m^t$;

(0.2.b) $\mathcal{J}_m\mathcal{J}_m^t + q\mathcal{A}_m\mathcal{A}_m^t = (q^m a^2 + q^{m+1}b^2)I$; and

(0.2.c) $\mathcal{J}_1^t\mathcal{J}_1 = qa^2 J, \mathcal{A}_1^t\mathcal{A}_1 = a^2 I + b^2(qI - J)$, and $\mathcal{A}_1^t\mathcal{J}_1 = \mathcal{J}_1^t\mathcal{A}_1 = abJ$.

Proof. We have $\mathcal{J}_0\mathcal{A}_0^t = ab = ba = \mathcal{A}_0\mathcal{J}_0^t$. Assume $\mathcal{J}_{m-1}\mathcal{A}_{m-1}^t = \mathcal{A}_{m-1}\mathcal{J}_{m-1}^t$. Then

$$\begin{aligned} \mathcal{J}_m\mathcal{A}_m^t &= (J \otimes \mathcal{A}_{m-1})(I \otimes \mathcal{J}_{m-1} + Q \otimes \mathcal{A}_{m-1})^t \\ &= J \otimes \mathcal{A}_{m-1}\mathcal{J}_{m-1}^t + JQ^t \otimes \mathcal{A}_{m-1}\mathcal{A}_{m-1}^t \\ &= J \otimes \mathcal{J}_{m-1}\mathcal{A}_{m-1}^t + QJ \otimes \mathcal{A}_{m-1}\mathcal{A}_{m-1}^t \\ &= (I \otimes \mathcal{J}_{m-1} + Q \otimes \mathcal{A}_{m-1})(J \otimes \mathcal{A}_{m-1})^t, \end{aligned}$$

and (0.2.a) has been shown.

Clearly, $\mathcal{J}_0\mathcal{J}_0^t + q\mathcal{A}_0\mathcal{A}_0^t = a^2 + qb^2$; so, assume $\mathcal{J}_{m-1}\mathcal{J}_{m-1}^t + q\mathcal{A}_{m-1}\mathcal{A}_{m-1}^t = (a^2 + qb^2)I$. Then

$$\begin{aligned} \mathcal{J}_m\mathcal{J}_m^t + q\mathcal{A}_m\mathcal{A}_m^t &= qJ \otimes \mathcal{A}_{m-1}\mathcal{A}_{m-1}^t + q(I \otimes \mathcal{J}_{m-1}\mathcal{J}_{m-1}^t - Q^2 \otimes \mathcal{A}_{m-1}\mathcal{A}_{m-1}^t) \\ &= qI \otimes (\mathcal{J}_{m-1}\mathcal{J}_{m-1}^t + q\mathcal{A}_{m-1}\mathcal{A}_{m-1}^t) \\ &= qI \otimes (q^{m-1}q^2 + q^m b^2)I \\ &= (q^m a^2 + q^{m+1}b^2)I, \end{aligned}$$

and (0.2.b) is proven.

Finally, (0.2.c) is simply a restatement of the definitions of $\mathcal{J}_m$ and $\mathcal{A}_m$. ■

We can now present the first construction of the novel balancedly splittable ODs.

0.3. Theorem. Let $W, \mathcal{A}_m$, and $\mathcal{J}_m$ be as above. Define $X_m = I \otimes \mathcal{J}_m + W \otimes \mathcal{A}_m$. Then:

(0.3.a) X_m is an OD$(q^m(q+1); q^m, q^{m+1})$, and

(0.3.b) The matrix X_1 is an unfaithful balancedly splittable OD$(q(q+1); q, q^2)$.

Proof. X_m has entries from $\{\pm a, \pm b\}$. Observe:

$$\begin{aligned}
X_m X_m^t &= (I \otimes \mathcal{J}_m + W \otimes \mathcal{A}_m)(I \otimes \mathcal{J}_m + W \otimes \mathcal{A}_m)^t \\
&= I \otimes \mathcal{J}_m \mathcal{J}_m^t + W W^t \otimes \mathcal{A}_m \mathcal{A}_m^t \\
&= I \otimes \mathcal{J}_m \mathcal{J}_m^t + q I \otimes \mathcal{A}_m \mathcal{A}_m^t \\
&= I \otimes (\mathcal{J}_m \mathcal{J}_m^t + q \mathcal{A}_m \mathcal{A}_m^t) \\
&= I \otimes (q^m a^2 + q^{m+1} b^2) I \\
&= (q^m a^2 + q^{m+1} b^2) I,
\end{aligned}$$

which shows that X_m is an OD$(q^m(q+1); q^m; q^{m+1})$. It remains to prove the balanced splittability of the base case.

Take $Y = \left(\mathcal{J}_1 \ \mathcal{A}_1 \ \dots \ \mathcal{A}_1 \right)$, the first block row of X_1. Then

$$\begin{aligned}
Y^t Y &= \begin{pmatrix} \mathcal{J}_1^t \mathcal{J}_1 & \mathbf{1}^t \otimes \mathcal{J}_1^t \mathcal{A}_1 \\ \mathbf{1} \otimes \mathcal{A}_1^t \mathcal{J}_1 & J \otimes \mathcal{A}_1^t \mathcal{A}_1 \end{pmatrix} \\
&= \begin{pmatrix} q a^2 J & ab \mathbf{1}^t \otimes J \\ ab \mathbf{1} \otimes J & J \otimes [(a^2 - b^2) J + q b^2 I] \end{pmatrix}.
\end{aligned}$$

Hence, X_1 admits an unfaithfully balanced split. ■

0.4. Corollary. For every prime power $q \equiv -1 \pmod 4$, and for every integer $m > 0$, there is an OD$(q^m(q+1); q^m, q^{m+1})$

Proof. By Propositions **??** and **??**, there is a skew-symmetric W$(q+1, q)$. Apply the theorem to this matrix. ■

0.5. Corollary. For every prime power $q \equiv -1 \pmod 4$, there is an unfaithful balancedly splittable OD$(q(q+1); q, q^2)$.

0.6. Example. Using the skew-symmetric Paley weighing matrixpaley-note $W(4,3)$ given by

$$\text{(0.6.a)} \quad \begin{pmatrix} 0 & + & + & + \\ - & 0 & + & - \\ - & - & 0 & + \\ - & + & - & 0 \end{pmatrix},$$

we construct the smallest case of an OD$(12; 3, 9)$ given by the theorem

$$
\text{(0.6.b)} \qquad \begin{pmatrix}
\mathbf{b} & \mathbf{b} & \mathbf{b} & \mathbf{a} & \mathbf{b} & \overline{\mathbf{b}} & \mathbf{a} & \mathbf{b} & \overline{\mathbf{b}} & \mathbf{a} & \mathbf{b} & \overline{\mathbf{b}} \\
\mathbf{b} & \mathbf{b} & \mathbf{b} & \overline{\mathbf{b}} & \mathbf{a} & \mathbf{b} & \overline{\mathbf{b}} & \mathbf{a} & \mathbf{b} & \overline{\mathbf{b}} & \mathbf{a} & \mathbf{b} \\
\mathbf{b} & \mathbf{b} & \mathbf{b} & \mathbf{b} & \overline{\mathbf{b}} & \mathbf{a} & \mathbf{b} & \overline{\mathbf{b}} & \mathbf{a} & \mathbf{b} & \overline{\mathbf{b}} & \mathbf{a} \\
\overline{\mathbf{a}} & \overline{\mathbf{b}} & \mathbf{b} & \mathbf{b} & \mathbf{b} & \mathbf{b} & \mathbf{a} & \mathbf{b} & \overline{\mathbf{b}} & \overline{\mathbf{a}} & \overline{\mathbf{b}} & \mathbf{b} \\
b & \overline{a} & \overline{b} & b & b & b & \overline{b} & a & b & b & \overline{a} & \overline{b} \\
\overline{b} & b & \overline{a} & b & b & b & b & \overline{b} & a & \overline{b} & b & \overline{a} \\
\overline{a} & \overline{b} & b & \overline{a} & \overline{b} & b & b & b & b & a & b & \overline{b} \\
b & \overline{a} & \overline{b} & b & \overline{a} & \overline{b} & b & b & b & \overline{b} & a & b \\
\overline{b} & b & \overline{a} & \overline{b} & b & \overline{a} & b & b & b & b & \overline{b} & a \\
\overline{a} & \overline{b} & b & a & b & \overline{b} & \overline{a} & \overline{b} & b & b & b & b \\
b & \overline{a} & \overline{b} & \overline{b} & a & b & b & \overline{a} & \overline{b} & b & b & b \\
\overline{b} & b & \overline{a} & b & \overline{b} & a & \overline{b} & b & \overline{a} & b & b & b
\end{pmatrix},
$$

where the unfaithful split is shown in bold.

Our first construction yields real ODs and is applicable in the case that we have a prime power $q \equiv -1 \pmod 4$. Of course, since $(q-1)/2$ is odd, we can apply the results of §4 to construct a weighing matrix that is skew-symmetric, a property essential to the construction. If $q \equiv 1 \pmod 4$, then $(q-1)/2$ is even and the ensuing weighing matrix is symmetric. In this event, we need to appeal to complex ODs in order apply the construction.

To apply the complex units, we make the following recursive definitions where $W = \left(\begin{smallmatrix} 0 & \mathbf{1}^t \\ \mathbf{1} & Q \end{smallmatrix}\right)$ is a $\mathrm{W}(q+1, q)$ with $Q^t = Q$.

$$
\mathcal{C}_m = \begin{cases} aJ_1 & \text{if } m = 0, \text{ and} \\ J_q \otimes \mathcal{D}_{m-1} & \text{if } m > 0; \end{cases}
$$

$$
\mathcal{D}_m = \begin{cases} bJ_1 & \text{if } m = 0, \text{ and} \\ I_q \otimes \mathcal{C}_{m-1} + iQ \otimes \mathcal{D}_{m-1} & \text{if } m > 0; \end{cases}
$$

where again a and b are real commuting indeterminants. As above, we have the following lemma that is shown in precisely the same way as before, save one replaces transposition with conjugate transposition.

0.7. Lemma. (0.7.a) $\mathcal{C}_m \mathcal{D}_m^* = \mathcal{D}_m \mathcal{C}_m^*$;

(0.7.b) $\mathcal{C}_m \mathcal{C}_m^* + q\mathcal{D}_m \mathcal{D}_m^* = (q^m a^2 + q^{m+1} b^2)I$; and

(0.7.c) $\mathcal{C}_1^* \mathcal{C}_1 = qa^2 J, \mathcal{D}_1^* \mathcal{D}_1 = a^2 I + b^2(qI - J)$, and $\mathcal{D}_1^* \mathcal{C}_1 = \mathcal{C}_1^* \mathcal{D}_1 = abJ$.

0.8. Theorem. Let W, $\mathcal{C}_m$, and $\mathcal{D}_m$ be as above, and define $Y_m = iI \otimes \mathcal{C}_m + W \otimes \mathcal{D}_m$. Then:

(0.8.a) The matrix Y_m is a $\mathrm{COD}(q^m(q+1); q^m, q^{m+1})$, and

(0.8.b) Y_1 admits an unfaithfully balanced split.

0.9. Corollary. For every prime power $q \equiv 1 \pmod 4$, and for every integer $m > 0$, there is a $\mathrm{COD}(q^m(q+1); q^m, q^{m+1})$.

0.10. Corollary. For every prime power $q \equiv 1 \pmod 4$, there is an unfaithful balancedly splittable COD$(q(q+1); q, q^2)$.

We again explore the smallest case.

0.11. Example. Take $q = 5$ and consider the symmetric Paley weighing matrix W$(6,5)$ given by

$$\text{(0.11.a)} \quad \begin{pmatrix} 0 & + & + & + & + & + \\ + & 0 & + & - & - & + \\ + & + & 0 & + & - & - \\ + & - & + & 0 & + & - \\ + & - & - & + & 0 & + \\ + & + & - & - & + & 0 \end{pmatrix}.$$

Applying the construction, we obtain

$$\text{(0.11.b)} \quad \left(\begin{smallmatrix}
\mathbf{ib} & \mathbf{ib} & \mathbf{ib} & \mathbf{ib} & \mathbf{ib} & \mathbf{a} & \mathbf{ib} & \mathbf{i\overline{b}} & \mathbf{i\overline{b}} & \mathbf{ib} & \mathbf{a} & \mathbf{ib} & \mathbf{i\overline{b}} & \mathbf{i\overline{b}} & \mathbf{ib} & \mathbf{a} & \mathbf{ib} & \mathbf{i\overline{b}} & \mathbf{i\overline{b}} & \mathbf{ib} & \mathbf{a} & \mathbf{ib} & \mathbf{i\overline{b}} & \mathbf{i\overline{b}} & \mathbf{ib} & \mathbf{a} & \mathbf{ib} & \mathbf{i\overline{b}} & \mathbf{i\overline{b}} & \mathbf{ib} \\
\mathbf{ib} & \mathbf{ib} & \mathbf{ib} & \mathbf{ib} & \mathbf{ib} & \mathbf{ib} & \mathbf{a} & \mathbf{ib} & \mathbf{i\overline{b}} & \mathbf{i\overline{b}} & \mathbf{ib} & \mathbf{a} & \mathbf{ib} & \mathbf{i\overline{b}} & \mathbf{i\overline{b}} & \mathbf{ib} & \mathbf{a} & \mathbf{ib} & \mathbf{i\overline{b}} & \mathbf{i\overline{b}} & \mathbf{ib} & \mathbf{a} & \mathbf{ib} & \mathbf{i\overline{b}} & \mathbf{i\overline{b}} & \mathbf{ib} & \mathbf{a} & \mathbf{ib} & \mathbf{i\overline{b}} & \mathbf{i\overline{b}} \\
\mathbf{ib} & \mathbf{ib} & \mathbf{ib} & \mathbf{ib} & \mathbf{ib} & \mathbf{i\overline{b}} & \mathbf{ib} & \mathbf{a} & \mathbf{ib} & \mathbf{i\overline{b}} & \mathbf{i\overline{b}} & \mathbf{ib} & \mathbf{a} & \mathbf{ib} & \mathbf{i\overline{b}} & \mathbf{i\overline{b}} & \mathbf{ib} & \mathbf{a} & \mathbf{ib} & \mathbf{i\overline{b}} & \mathbf{i\overline{b}} & \mathbf{ib} & \mathbf{a} & \mathbf{ib} & \mathbf{i\overline{b}} & \mathbf{i\overline{b}} & \mathbf{ib} & \mathbf{a} & \mathbf{ib} & \mathbf{i\overline{b}} \\
\mathbf{ib} & \mathbf{ib} & \mathbf{ib} & \mathbf{ib} & \mathbf{ib} & \mathbf{i\overline{b}} & \mathbf{i\overline{b}} & \mathbf{ib} & \mathbf{a} & \mathbf{ib} & \mathbf{i\overline{b}} & \mathbf{i\overline{b}} & \mathbf{ib} & \mathbf{a} & \mathbf{ib} & \mathbf{i\overline{b}} & \mathbf{i\overline{b}} & \mathbf{ib} & \mathbf{a} & \mathbf{ib} & \mathbf{i\overline{b}} & \mathbf{i\overline{b}} & \mathbf{ib} & \mathbf{a} & \mathbf{ib} & \mathbf{i\overline{b}} & \mathbf{i\overline{b}} & \mathbf{ib} & \mathbf{a} & \mathbf{ib} \\
\mathbf{ib} & \mathbf{ib} & \mathbf{ib} & \mathbf{ib} & \mathbf{ib} & \mathbf{ib} & \mathbf{i\overline{b}} & \mathbf{i\overline{b}} & \mathbf{ib} & \mathbf{a} & \mathbf{ib} & \mathbf{i\overline{b}} & \mathbf{i\overline{b}} & \mathbf{ib} & \mathbf{a} & \mathbf{ib} & \mathbf{i\overline{b}} & \mathbf{i\overline{b}} & \mathbf{ib} & \mathbf{a} & \mathbf{ib} & \mathbf{i\overline{b}} & \mathbf{i\overline{b}} & \mathbf{ib} & \mathbf{a} & \mathbf{ib} & \mathbf{i\overline{b}} & \mathbf{i\overline{b}} & \mathbf{ib} & \mathbf{a} \\
a & ib & i\overline{b} & i\overline{b} & ib & ib & ib & ib & ib & ib & a & ib & i\overline{b} & i\overline{b} & ib & \overline{a} & i\overline{b} & ib & ib & i\overline{b} & \overline{a} & i\overline{b} & ib & ib & i\overline{b} & a & ib & i\overline{b} & i\overline{b} & ib \\
ib & a & ib & i\overline{b} & i\overline{b} & ib & ib & ib & ib & ib & ib & a & ib & i\overline{b} & i\overline{b} & i\overline{b} & \overline{a} & i\overline{b} & ib & ib & i\overline{b} & \overline{a} & i\overline{b} & ib & ib & ib & a & ib & i\overline{b} & i\overline{b} \\
i\overline{b} & ib & a & ib & i\overline{b} & ib & ib & ib & ib & ib & i\overline{b} & ib & a & ib & i\overline{b} & ib & i\overline{b} & \overline{a} & i\overline{b} & ib & ib & i\overline{b} & \overline{a} & i\overline{b} & ib & i\overline{b} & ib & a & ib & i\overline{b} \\
i\overline{b} & i\overline{b} & ib & a & ib & ib & ib & ib & ib & ib & i\overline{b} & i\overline{b} & ib & a & ib & ib & ib & i\overline{b} & \overline{a} & i\overline{b} & ib & ib & i\overline{b} & \overline{a} & i\overline{b} & i\overline{b} & i\overline{b} & ib & a & ib \\
ib & i\overline{b} & i\overline{b} & ib & a & ib & ib & ib & ib & ib & ib & i\overline{b} & i\overline{b} & ib & a & i\overline{b} & ib & ib & i\overline{b} & \overline{a} & i\overline{b} & ib & ib & i\overline{b} & \overline{a} & ib & i\overline{b} & i\overline{b} & ib & a \\
a & ib & i\overline{b} & i\overline{b} & ib & a & ib & i\overline{b} & i\overline{b} & ib & ib & ib & ib & ib & ib & a & ib & i\overline{b} & i\overline{b} & ib & \overline{a} & i\overline{b} & ib & ib & i\overline{b} & \overline{a} & i\overline{b} & ib & ib & i\overline{b} \\
ib & a & ib & i\overline{b} & i\overline{b} & ib & a & ib & i\overline{b} & i\overline{b} & ib & ib & ib & ib & ib & ib & a & ib & i\overline{b} & i\overline{b} & i\overline{b} & \overline{a} & i\overline{b} & ib & ib & i\overline{b} & \overline{a} & i\overline{b} & ib & ib \\
i\overline{b} & ib & a & ib & i\overline{b} & i\overline{b} & ib & a & ib & i\overline{b} & ib & ib & ib & ib & ib & i\overline{b} & ib & a & ib & i\overline{b} & ib & i\overline{b} & \overline{a} & i\overline{b} & ib & ib & i\overline{b} & \overline{a} & i\overline{b} & ib \\
i\overline{b} & i\overline{b} & ib & a & ib & i\overline{b} & i\overline{b} & ib & a & ib & ib & ib & ib & ib & ib & i\overline{b} & i\overline{b} & ib & a & ib & ib & ib & i\overline{b} & \overline{a} & i\overline{b} & ib & ib & i\overline{b} & \overline{a} & i\overline{b} \\
ib & i\overline{b} & i\overline{b} & ib & a & ib & i\overline{b} & i\overline{b} & ib & a & ib & ib & ib & ib & ib & ib & i\overline{b} & i\overline{b} & ib & a & i\overline{b} & ib & ib & i\overline{b} & \overline{a} & i\overline{b} & ib & ib & i\overline{b} & \overline{a} \\
a & ib & i\overline{b} & i\overline{b} & ib & \overline{a} & i\overline{b} & ib & ib & i\overline{b} & a & ib & i\overline{b} & i\overline{b} & ib & ib & ib & ib & ib & ib & a & ib & i\overline{b} & i\overline{b} & ib & \overline{a} & i\overline{b} & ib & ib & i\overline{b} \\
ib & a & ib & i\overline{b} & i\overline{b} & i\overline{b} & \overline{a} & i\overline{b} & ib & ib & ib & a & ib & i\overline{b} & i\overline{b} & ib & ib & ib & ib & ib & ib & a & ib & i\overline{b} & i\overline{b} & i\overline{b} & \overline{a} & i\overline{b} & ib & ib \\
i\overline{b} & ib & a & ib & i\overline{b} & ib & i\overline{b} & \overline{a} & i\overline{b} & ib & i\overline{b} & ib & a & ib & i\overline{b} & ib & ib & ib & ib & ib & i\overline{b} & ib & a & ib & i\overline{b} & ib & i\overline{b} & \overline{a} & i\overline{b} & ib \\
i\overline{b} & i\overline{b} & ib & a & ib & ib & ib & i\overline{b} & \overline{a} & i\overline{b} & i\overline{b} & i\overline{b} & ib & a & ib & ib & ib & ib & ib & ib & i\overline{b} & i\overline{b} & ib & a & ib & ib & ib & i\overline{b} & \overline{a} & i\overline{b} \\
ib & i\overline{b} & i\overline{b} & ib & a & i\overline{b} & ib & ib & i\overline{b} & \overline{a} & ib & i\overline{b} & i\overline{b} & ib & a & ib & ib & ib & ib & ib & ib & i\overline{b} & i\overline{b} & ib & a & i\overline{b} & ib & ib & i\overline{b} & \overline{a} \\
a & ib & i\overline{b} & i\overline{b} & ib & \overline{a} & i\overline{b} & ib & ib & i\overline{b} & \overline{a} & i\overline{b} & ib & ib & i\overline{b} & a & ib & i\overline{b} & i\overline{b} & ib & ib & ib & ib & ib & ib & a & ib & i\overline{b} & i\overline{b} & ib \\
ib & a & ib & i\overline{b} & i\overline{b} & i\overline{b} & \overline{a} & i\overline{b} & ib & ib & i\overline{b} & \overline{a} & i\overline{b} & ib & ib & ib & a & ib & i\overline{b} & i\overline{b} & ib & ib & ib & ib & ib & ib & a & ib & i\overline{b} & i\overline{b} \\
i\overline{b} & ib & a & ib & i\overline{b} & ib & i\overline{b} & \overline{a} & i\overline{b} & ib & ib & i\overline{b} & \overline{a} & i\overline{b} & ib & i\overline{b} & ib & a & ib & i\overline{b} & ib & ib & ib & ib & ib & i\overline{b} & ib & a & ib & i\overline{b} \\
i\overline{b} & i\overline{b} & ib & a & ib & ib & ib & i\overline{b} & \overline{a} & i\overline{b} & ib & ib & i\overline{b} & \overline{a} & i\overline{b} & i\overline{b} & i\overline{b} & ib & a & ib & ib & ib & ib & ib & ib & i\overline{b} & i\overline{b} & ib & a & ib \\
ib & i\overline{b} & i\overline{b} & ib & a & i\overline{b} & ib & ib & i\overline{b} & \overline{a} & i\overline{b} & ib & ib & i\overline{b} & \overline{a} & ib & i\overline{b} & i\overline{b} & ib & a & ib & ib & ib & ib & ib & ib & i\overline{b} & i\overline{b} & ib & a \\
a & ib & i\overline{b} & i\overline{b} & ib & a & ib & i\overline{b} & i\overline{b} & ib & \overline{a} & i\overline{b} & ib & ib & i\overline{b} & \overline{a} & i\overline{b} & ib & ib & i\overline{b} & a & ib & i\overline{b} & i\overline{b} & ib & ib & ib & ib & ib & ib \\
ib & a & ib & i\overline{b} & i\overline{b} & ib & a & ib & i\overline{b} & i\overline{b} & i\overline{b} & \overline{a} & i\overline{b} & ib & ib & i\overline{b} & \overline{a} & i\overline{b} & ib & ib & ib & a & ib & i\overline{b} & i\overline{b} & ib & ib & ib & ib & ib \\
i\overline{b} & ib & a & ib & i\overline{b} & i\overline{b} & ib & a & ib & i\overline{b} & ib & i\overline{b} & \overline{a} & i\overline{b} & ib & ib & i\overline{b} & \overline{a} & i\overline{b} & ib & i\overline{b} & ib & a & ib & i\overline{b} & ib & ib & ib & ib & ib \\
i\overline{b} & i\overline{b} & ib & a & ib & i\overline{b} & i\overline{b} & ib & a & ib & ib & ib & i\overline{b} & \overline{a} & i\overline{b} & ib & ib & i\overline{b} & \overline{a} & i\overline{b} & i\overline{b} & i\overline{b} & ib & a & ib & ib & ib & ib & ib & ib \\
ib & i\overline{b} & i\overline{b} & ib & a & ib & i\overline{b} & i\overline{b} & ib & a & i\overline{b} & ib & ib & i\overline{b} & \overline{a} & i\overline{b} & ib & ib & i\overline{b} & \overline{a} & ib & i\overline{b} & i\overline{b} & ib & a & ib & ib & ib & ib & ib
\end{smallmatrix}\right),$$

a COD$(30; 5, 25)$, where the unfaithful split is shown in bold.

* * *

0.3. Faithful Construction. Here we will introduce a most useful construction of orthogonal designs admitting a faithful split. We will see later how we can use these constructed matrices in constructing other objects.

To begin, we assume the existence of a full OD$(n; s_1, \ldots, s_u)$, say X, and label the rows of X as $x_0, \ldots, x_{n-1}$. We need to extend the idea of an auxiliary matrix given in Example **??**. To do this, we will follow **?** in defining the auxiliary matrix of an OD thus: Let H be the Hadamard matrix obtained by setting each indeterminant of X to $+1$, and label the rows of H as $h_0, \ldots, h_{n-1}$.aux-note Then the auxiliary matrices of X are given by $c_i = h_i^t x_i$. We have the following result.

0.12. Lemma. Let $c_i = h_i^t x_i$, for $i \in \{0, \dots, n-1\}$, be the auxiliary matrices of an OD$(n; s_1, \dots, s_u)$ X where $XX^t = \sigma I$. Then:

(0.12.a) $\sum_i c_i = n\sigma I_n$,

(0.12.b) $c_i c_i^t = \sigma h_i^t h_i$, and

(0.12.c) $c_i c_j^t = O$ whenever $i \neq j$.

We need the simple fact that if (a, b) denotes the concatination of sequences a and b, then (a, b) and $(a, -b)$ is a complementary sequence (see §5). Continuing to let $c_0, \dots, c_{n-1}$ be the auxiliary matrices of the OD$(n; s_1, \dots, s_u)$ X. Then $a = (c_0, c_1, \dots, c_{n-1}, c_{n-1}, \dots, c_1)$ and $b = (c_0, c_1, \dots, c_{n-1}, -c_{n-1}, \dots, -c_1)$ form a complementary pair. Let A and B be the block-circulant matrices with first rows a and b, respectively.

Now, take $\tilde{X} = \left(\begin{smallmatrix} + & + \\ + & - \end{smallmatrix}\right) \otimes X$ and $\tilde{H} = \left(\begin{smallmatrix} + & + \\ + & - \end{smallmatrix}\right) \otimes H$, and label the block rows $\tilde{x}_0, \dots, \tilde{x}_{2n-1}$ and $\tilde{h}_0, \dots, \tilde{h}_{2n-1}$. Define $G = \tilde{h}_0^t \tilde{x}_0$, and define the block matrices $E^t = \left(\begin{smallmatrix} E_1^t & \cdots & E_{2n-1}^t \end{smallmatrix}\right)^t$ and $F = \left(\begin{smallmatrix} F_1 & \cdots & F_{2n-1} \end{smallmatrix}\right)$ by $E_i = h_0^t \tilde{x}_i^t$ and $F = \tilde{h}_i^t x_0$.

As before, we then take $Z = \left(\begin{smallmatrix} G & F & -F \\ E & A & B \\ -E & B & A \end{smallmatrix}\right)$. The next result then follows.

0.13. Theorem. If there is an OD$(n; s_1, \dots, s_u)$, then the block matrix Z is an OD$(4n^2; 4ns_1, \dots, 4ns_u)$.

Proof. The proof amounts to checking the block entries of ZZ^t.

To begin, $GE_i^t = (\tilde{h}_0^t \tilde{x}_0)(h_0^t \tilde{x}_i)^t = \tilde{h}_0^t (\tilde{x}_0 \tilde{x}_i^t) h_0 = O$, hence $GE^t = EG^t = O$. Then $F_i c_j^t = (\tilde{h}_i^t x_0)(h_j^t x_j)^t = \tilde{h}_i^t (x_0 x_j^t) h_j = \delta_{0j} \sigma \tilde{h}_i^t h_0$ so that

$$FA^t = FB^t = \sigma \begin{pmatrix} \tilde{h}_1 \\ \vdots \\ \tilde{h}_{2n-1} \end{pmatrix}^t (\mathbf{1}_{2n-1} \otimes h_0).$$

We have, therefore, that $FA^t - FB^t = O$. Then the inner product between the first and second, and the first and third, block rows of Z are orthogonal.

Next, $E_i E_j^t = (h_0^t \tilde{x}_i)(h_0^t \tilde{x}_j)^t = h_0^t (\tilde{x}_i \tilde{x}_j^t) h_0 = \delta_{ij} \sigma J_n$. It follows that $EE^t = (E_i E_j^t) = (\delta_{ij} \sigma J_n) = \sigma(I_{2n-1} \otimes J_n)$.

We need to examine the product AB^t and in order to do that, we need to examine the cross-product correlations (see §5). To begin, the product between the first block row and column of A and B^t is given by $c_0 c_0^t + \sum_{i=1}^{n-1} c_i c_i^t - \sum_{i=1}^{n-1} c_i c_i^t = \sigma J_n$.

Next, let a and b be two sequences of length $2n-1$ defined by

$$a_i = \begin{cases} c_i & \text{if } 0 \le i < n \text{, and} \\ c_{2n-i-1} & \text{if } n \le i < 2n-1, \end{cases}$$

$$b_i = \begin{cases} c_0 & \text{if } i = 0, \\ -c_i & \text{if } 0 < i < n \text{, and} \\ c_{2n-i-1} & \text{if } n \le i < 2n-1. \end{cases}$$

For $j \in \{1, \ldots 2n-2\}$, and using the fact that $c_i c_j^t = \delta_{ij} \sigma h_i^t h_i$, we have that

$$\begin{aligned} C_j(a,b) &= \sum_{0 \le i < 2n-1} a_i b_{i+j}^t \\ &= \sum_{0 \le i < n} a_i b_{i+j}^t + \sum_{n \le i < 2n-1} a_i b_{i+j}^t \\ &= \sum_{n-j \le i < n} a_i b_{i+j}^t + \sum_{2n-j-1 \le i < 2n-1} a_i b_{i+j}^t \\ &= \sum_{0 \le i < j} c_i c_{n+i}^t + \sum \end{aligned}$$

where precisely one of the sums is nonzero for any given $j \in \{1, \ldots, 2n-2\}$. ■