***0.1. Lemmata.*** In §2.2, we introduced the linear simplex code $\mathscr{S}_{q,n}$. There it was shown that the code had constant weight $q^{n-1}$; in particular, it follows that it is equidistant with constant Hamming distance $q^{n-1}$ since the code is linear.

In **?**, and later reproduced in **?** using the language of BGW matrices, generalized Hadamard matrices $\mathrm{GH}(q, q^{n-1})$ were used recursively in conjunction with the classical parameter $\mathrm{BGW}((q^n-1)/(q-1), q^{n-1}, q^{n-1}-q^{n-2}; \mathrm{GF}(q)^*)$s in order to construct certain designs. It turns out out that the $\mathrm{GH}(q, q^{n-1})$ used in the construction can be replaced by $\mathscr{S}_{q,n}$, and so simplify the construction.

In order to apply the linear code $\mathscr{S}_{q,n}$, we will require the following lemma.

**0.1. Lemma.** Let $\mathrm{GF}(q) = \{a_0 = 0, a_1, \ldots, a_{q-1}\}$, and let $n > 1$. Then there exist disjoint $(0,1)$-matrices $A_{a_1}, \ldots, A_{a_{q-1}}$ of dimensions $q^n \times (q^n-1)/(q-1)$ such that $\mathscr{S}_{q,n} = \sum\limits_{\alpha \in \mathrm{GF}(q)^*} \alpha A_\alpha$. If we define $A_0 = J - \sum\limits_{\alpha \in \mathrm{GF}(q)^*} A_\alpha$, then the following hold.

(0.1.a) $\sum\limits_{\alpha \in \mathrm{GF}(q)} A_\alpha A_\alpha^t = \frac{q^{n-1}-1}{q-1} J + q^{n-1} I$, and

(0.1.b) $\sum\limits_{\substack{\alpha,\beta \in \mathrm{GF}(q)\\ \alpha \neq \beta}} A_\alpha A_\beta^t = q^{n-1}(J - I)$.

**Proof.** Labeling the rows of $\mathscr{S}_{q,n}$ by $r_0, \ldots, r_{q^n-1}$, and taking $A = \mathscr{S}_{q,n}$, we then have that

$$\begin{aligned}
(\sum_{\alpha \in \mathrm{GF}(q)} A_\alpha A_\alpha^t)_{ij} &= \sum_{\alpha \in \mathrm{GF}(q)} (A_\alpha A_\alpha^t)_{ij} \\
&= \sum_{\alpha \in \mathrm{GF}(q)} \sum_{\ell=0}^{\frac{q(q^{n-1}-1)}{q-1}} (A_\alpha)_{i\ell} (A_\alpha)_{j\ell} \\
&= \sum_{\alpha \in \mathrm{GF}(q)} \#\{\ell \in \{0, \ldots, \frac{q(q^{n-1}-1)}{q-1}\} : A_{i\ell} = A_{j\ell} = \alpha\} \\
&= \#\{\ell \in \{0, \ldots, \frac{q(q^{n-1}-1)}{q-1}\} : A_{i\ell} = A_{j\ell}\} \\
&= \frac{q^n-1}{q-1} - \mathrm{dist}(r_i, r_j),
\end{aligned}$$

which shows (0.1.a).

Since $\sum\limits_{\alpha \in \mathrm{GF}(q)} A_\alpha = J$, it follows that $\sum\limits_{\alpha,\beta} A_\alpha A_\beta^t = (\sum\limits_\alpha A_\alpha)(\sum\limits_\beta A_\beta)^t = \frac{q^n-1}{q-1} J$, and (0.1.b) has been proven. ∎

If $W$ is a $\mathrm{BGW}(v, k, \lambda; C_n)$ over some cyclic group $C_n = \{1, g, \ldots, g^{n-1}\}$ of order $n$, then there are $n$ disjoint $(0,1)$-matrices $W_0, \ldots, W_{n-1}$ such that $W =$

$W_0 + gW_1 + \cdots + g^{n-1}W_{n-1}$. We call $W_0$ and $W_1$ the *decomposition matrices* of the weighing matrix. Because $W$ is a BGW matrix, we have the following lemma.

**0.2. Lemma.**

(0.2.a) $\displaystyle\sum_{i,j} g^{i-j} W_i W_j^t = \sum_{i,j} g^{i-j} W_j^t W_i = kI + \tfrac{\lambda}{n}(\sum_i g_i)(J-I)$,

(0.2.b) $\displaystyle\sum_i W_i W_i^t = \sum_i W_i^t W_i = kI + \tfrac{\lambda}{n}(J-I)$, and

(0.2.c) $\displaystyle\sum_i W_i W_{i+j}^t = \sum_i W_{i+j}^t W_i = \tfrac{\lambda}{n}(J-I)$, for $j \in \{1,\ldots,n-1\}$.

**Proof.** (0.2.a) is simply a restatment of the fact that both $W$ and $W^*$ are BGW$(v,k,\lambda;C_n)$s (see **?**). (0.2.b) follows by noting that there are $k$ nonzero entries in every row of $W$, and that 1 appears $\lambda/n$ times in the conjugate inner product between distinct rows. Similarly, (0.2.c) follows by noting that each nonidentity element of the group appears $\lambda/n$ times in the conjugate inner product between distinct rows of $W$, and that $i \neq i+j$, for each $i$ whenever $j \not\equiv 0 \pmod{n}$. ∎

Now, consider the balanced $\mathrm{W}(19,9)$ shown to exist by computational means in **?**.

$$W_{19} = \begin{pmatrix}
0&+&+&+&+&+&+&+&+&+&0&0&0&0&0&0&0&0&0\\
0&0&-&-&0&+&0&0&0&+&-&+&+&0&+&0&0&0&+\\
0&-&0&-&0&0&+&+&0&0&+&-&+&0&0&+&+&0&0\\
0&-&-&0&+&0&0&0&+&0&+&+&-&+&0&0&0&+&0\\
0&0&0&+&0&-&-&0&+&0&0&0&+&-&+&+&0&+&0\\
0&+&0&0&-&0&-&0&0&+&+&0&0&+&-&+&0&0&+\\
0&0&+&0&-&-&0&+&0&0&0&+&0&+&+&-&+&0&0\\
0&0&+&0&0&0&+&0&-&-&0&+&0&0&0&+&-&+&+\\
0&0&0&+&+&0&0&-&0&-&0&0&+&+&0&0&+&-&+\\
0&+&0&0&0&+&0&-&-&0&+&0&0&0&+&0&+&+&-\\
+&0&0&0&+&0&-&+&-&0&0&-&-&0&+&0&0&0&+\\
+&0&0&0&-&+&0&0&+&-&-&0&-&0&0&+&+&0&0\\
+&0&0&0&0&-&+&-&0&+&-&-&0&+&0&0&0&+&0\\
+&+&-&0&0&0&0&+&0&-&0&0&+&0&-&-&0&+&0\\
+&0&+&-&0&0&0&-&+&0&+&0&0&-&0&-&0&0&+\\
+&-&0&+&0&0&0&0&-&+&0&+&0&-&-&0&+&0&0\\
+&+&0&-&+&-&0&0&0&0&0&+&0&0&0&+&0&-&-\\
+&-&+&0&0&+&-&0&0&0&0&0&+&+&0&0&-&0&-\\
+&0&-&+&-&0&+&0&0&0&+&0&0&0&+&0&-&-&0
\end{pmatrix}.$$

Take $R_1$ and $D$ to be the residual and derived parts, respectively, of $W_{19}$. Define the matrix $|R_1|$ by $|R_1|_{ij} = |R_{1_{ij}}|$. Then $|R_1|$ is the incidence matrix of a residual $\mathrm{BIBD}(10,18,9,5,4)$, hence $|R_2| = J - |R_1|$ is a BIBD with the same parameters. Moreover, $|R_2|$ is residual since $|R_2|$ together with $|D|$ also forms a symmetric design. We therefore seek a signing of $|R_2|$ over $\{-1,1\}$. Such a signing is given by

$$R_2 = \begin{pmatrix} 0&0&0&0&0&0&0&0&0&+&+&+&+&+&+&+&+&+ \\ -&0&0&+&0&+&+&+&0&0&0&0&+&0&-&+&-&0 \\ 0&-&0&+&+&0&0&+&+&0&0&0&-&+&0&0&+&- \\ 0&0&-&0&+&+&+&0&+&0&0&0&0&-&+&-&0&+ \\ +&+&0&-&0&0&+&0&+&+&-&0&0&0&0&+&0&- \\ 0&+&+&0&-&0&+&+&0&0&+&-&0&0&0&-&+&0 \\ +&0&+&0&0&-&0&+&+&-&0&+&0&0&0&0&-&+ \\ +&0&+&+&+&0&-&0&0&+&0&-&+&-&0&0&0&0 \\ +&+&0&0&+&+&0&-&0&-&+&0&0&+&-&0&0&0 \\ 0&+&+&+&0&+&0&0&-&0&-&+&-&0&+&0&0&0 \end{pmatrix}.$$

Remarkably, $R_2$ together with $D$ also forms a balanced $\mathrm{W}(19, 9)$. The matrices $R_1, R_2$, and $D$ then satisfy several properties.

**0.3. Lemma.**

(0.3.a) $R_1 R_1^t = R_2 R_2^t = I$, $R_1 R_2^t = R_2 R_1^t$;

(0.3.b) $DD^t = 9I - J$;

(0.3.c) $R_1 D^t = R_2 D^t = O$;

(0.3.d) $|R_1||R_1|^t = |R_2||R_2|^t = 5I + 4J$, $|R_1||R_2|^t = |R_2||R_1|^t = 5(J - I)$; and

(0.3.e) $|D||D|^t = 5I + 3J$, $|R_1||D|^t = |R_2||D^t| = 4J$.

**Proof.** Restatement of the facts that $R_1, R_2$, and $D$ form balanced $\mathrm{W}(19, 4)$ weighing matrices. ∎

* * *

***0.2. Construction.*** Having the lemmata of the previous subsection at our disposal, we are ready to present the construction of a new family of balanced weighing matrices. We desire to apply BGWs in the construction of these matrices, so we need an admissible group of symmetries. Take $\mathcal{M} = \{R_1, R_2\}$; then it isn't difficult to see that $-R_2 \mapsto -R_1 \mapsto R_2 \mapsto R_1 \mapsto -R_2$ is an admissible cyclic group of symmetries of order 4 for $\mathcal{M}$—though, this will be derived explicitly below.

Let $n > 1$, and take $\Xi$ to be a $\mathrm{BGW}((9^n - 1)/8, 9^{n-1}, 9^{n-1} - 9^{n-2}; C_4)$. We claim that $\Xi \otimes R_1$ is the residual part of a balanced $\mathrm{W}([9(9^n - 1)/4] + 1, 9^n)$. Note there are disjoint $(0, 1)$-matrices $\Xi_0, \Xi_1, \Xi_2, \Xi_3$ such that $\Xi = \Xi_0 + g\Xi_1 + g^2\Xi_2 + g^3\Xi_3$ if $C_4 = \{e, g, g^2, g^3\}$. Then $\Xi \otimes R_1 = \Xi_0 \otimes R_1 - \Xi_1 \otimes R_2 - \Xi_2 \otimes R_1 + \Xi_3 \otimes R_2$.

Next, let $\mathscr{S}_{9,n} = \sum_{\alpha \in \mathrm{GF}(9)^*} \alpha A_\alpha$, and define $A_0 = J - \sum_{\alpha \in \mathrm{GF}(9)^*} A_\alpha$. Finally, take $\Theta = \sum_{\alpha \in \mathrm{GF}(9)} A_\alpha \otimes D$. It will be shown that $\Theta$ is the derived part of a balanced $W([9(9^n - 1)/4] + 1, 9^n)$.

We require the following lemma.

**0.4. Lemma.**

(0.4.a) $(\Xi \otimes R_1)(\Xi \otimes R_1)^t = 9^n I$;

(0.4.b) $\Theta\Theta^* = 9^n I - J$;

(0.4.c) $(\Xi \otimes R_1)\Theta^t = \Theta(\Xi \otimes R_1)^t = O$;

(0.4.d) $|\Xi \otimes R_1||\Xi \otimes R_1|^t = 5 \cdot 9^n I + 4 \cdot 9^n J$;

(0.4.e) $|\Theta||\Theta|^t = 5 \cdot 9^n I + (4 \cdot 9^n - 1)J$; and

(0.4.f) $|\Xi \otimes R_1||\Theta|^t = |\Theta||\Xi \otimes R_1|^t = 4 \cdot 9^n J$.

**Proof.** By Lemma 0.2 and (0.3.a),

$$\begin{aligned}
(\Xi \otimes R_1)(\Xi \otimes R_1) &= 9\Xi_0\Xi_0^t \otimes I - \Xi_0\Xi_1^t \otimes R_1R_2^t - 9\Xi_0\Xi_2^t \otimes I + \Xi_0\Xi_3^t \otimes R_1R_2^t \\
&\quad - \Xi_1\Xi_0^t \otimes R_2R_1^t + 9\Xi_1\Xi_1^t \otimes I + \Xi_1\Xi_2^t \otimes R_2R_1^t - 9\Xi_1\Xi_3^t \otimes I \\
&\quad - 9\Xi_2\Xi_0^t \otimes I + \Xi_2\Xi_1^t \otimes R_1R_2^t + 9\Xi_2\Xi_2^t \otimes I - \Xi_2\Xi_3^t \otimes R_1R_2^t \\
&\quad + \Xi_3\Xi_0^t \otimes R_2R_1^t - 9\Xi_3\Xi_1^t \otimes I - \Xi_3\Xi_2^t \otimes R_2R_1^t + 9\Xi_3\Xi_3^t \otimes I \\
&= 9\sum_i (\Xi_i\Xi_i^t - \Xi_i\Xi_{i+2}^t) \otimes I - \sum_i (\Xi_i\Xi_{i+1} - \Xi_i\Xi_{i+3}) \otimes R_1R_2^t \\
&= 9^n I,
\end{aligned}$$

and (0.4.a) is shown.

Next, by Lemma 0.1 and (0.3.b), and upon indexing the rows of $D$ by elements of GF$(9)$,

$$\begin{aligned}
\Theta\Theta^t &= \sum_{\alpha,\beta\in \mathrm{GF}(9)} A_\alpha A_\beta^t \otimes r_\alpha r_\beta^t \\
&= \sum_{\alpha\in \mathrm{GF}(9)} A_\alpha A_\alpha^t \otimes r_\alpha r_\alpha^t + \sum_{\alpha\neq\beta} A_\alpha A_\beta^t \otimes r_\alpha r_\beta^t \\
&= 8 \sum_{\alpha\in \mathrm{GF}(9)} A_\alpha A_\alpha^t - \sum_{\alpha\neq\beta} A_\alpha A_\beta^t \\
&= (9^{n-1} - 1)J + 8 \cdot 9^{n-1} I - 9^{n-1}(J - I) \\
&= 9^n I - J,
\end{aligned}$$

which shows (0.4.b).

By Lemma (0.3.c),

$$\begin{aligned}
(\Xi \otimes R_1)\Theta^t &= \sum_{\alpha\in \mathrm{GF}(9)} (\Xi_0 A_\alpha^t \otimes R_1 r_\alpha^t - \Xi_1 A_\alpha^t \otimes R_2 r_\alpha^t - \Xi_2 A_\alpha^t \otimes R_2 r_\alpha^t + \Xi_3 A_\alpha^t \otimes R_2 r_\alpha^t) \\
&= O,
\end{aligned}$$

and (0.4.c) has been shown.

Since $|\Xi \otimes R_1| = (\Xi_0 + \Xi_2) \otimes |R_1| + (\Xi_1 + \Xi_3) \otimes |R_2|$, (0.4.d) is shown similarly to (0.4.a).

(0.4.e) is shown just as (0.4.b) after noting that $|\Theta| = \sum_{\alpha \in \mathrm{GF}(9)} A_\alpha \otimes |r_\alpha|$.

Finally, (0.4.f) is shown precisely as in (0.4.c). ∎

We are now ready to present the main construction.

**0.5. Theorem.** Given $\Xi \otimes R_1$ and $\Theta$ defined above,

$$\text{(0.5.a)} \quad \begin{pmatrix} \mathbf{0} & \Xi \otimes R_1 \\ \mathbf{1} & \Theta \end{pmatrix}$$

is a balanced W$([9(9^n - 1)/4] + 1, 9^n)$.

**Proof.** By the lemma, $(\Xi \otimes R_1)(\Xi \otimes R_1)^t = 9^n I$, $\Theta\Theta^t = 9^n I - J$, and $(\Xi \otimes R_1)\Theta^t = O$; thus, (0.5.a) is a weighing matrix with the appropriate parameters. It remains to show it is balanced. But the lemma again gives $|\Xi \otimes R_1||\Xi \otimes R_1|^t = 5 \cdot 9^n I + 4 \cdot 9^n J$, $|\Theta||\Theta|^t = 5 \cdot 9^n I + (4 \cdot 9^n - 1)J$, and $|\Xi \otimes R_1|\Theta^t = 4 \cdot 9^n J$. We have then shown that (0.5.a) is balanced, and the proof is complete. ∎