Weighing Matrices

Generalizations and Related Configurations

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Summary

Preliminaries

Section Summary

- $J_{n\times m}$ the $n \times m$ matrix of all 1s.
- I_n the identity matrix of order n.

Definition: Balanced Incomplete Block Design

- A binary $v \times b$ $(0,1)$-matrix A such that:
  1. $AA^t = rI_v + \lambda(J_v - I_v)$, and
  2. $J_vA = kJ_v$.

  Write $BIBD(v, b, r, k, \lambda)$.
- The design is symmetric if $v = b$ (equiv. $k = r$).

- A $BIBD(13, 9, 6)$

$$A = \begin{pmatrix} 0&0&1&0&1&1&1&1&1&0&1&1&1 \\ 1&0&0&1&0&1&1&1&1&1&0&1&1 \\ 1&1&0&0&1&0&1&1&1&1&1&0&1 \\ 1&1&1&0&0&1&0&1&1&1&1&1&0 \\ 0&1&1&1&0&0&1&0&1&1&1&1&1 \\ 1&0&1&1&1&0&0&1&0&1&1&1&1 \\ 1&1&0&1&1&1&0&0&1&0&1&1&1 \\ 1&1&1&0&1&1&1&0&0&1&0&1&1 \\ 1&1&1&1&0&1&1&1&0&0&1&0&1 \\ 1&1&1&1&1&0&1&1&1&0&0&1&0 \\ 0&1&1&1&1&1&0&1&1&1&0&0&1 \\ 1&0&1&1&1&1&1&0&1&1&1&0&0 \\ 0&1&0&1&1&1&1&1&0&1&1&1&0 \end{pmatrix}.$$

$$AA^t = \begin{pmatrix} 9&6&6&6&6&6&6&6&6&6&6&6&6 \\ 6&9&6&6&6&6&6&6&6&6&6&6&6 \\ 6&6&9&6&6&6&6&6&6&6&6&6&6 \\ 6&6&6&9&6&6&6&6&6&6&6&6&6 \\ 6&6&6&6&9&6&6&6&6&6&6&6&6 \\ 6&6&6&6&6&9&6&6&6&6&6&6&6 \\ 6&6&6&6&6&6&9&6&6&6&6&6&6 \\ 6&6&6&6&6&6&6&9&6&6&6&6&6 \\ 6&6&6&6&6&6&6&6&9&6&6&6&6 \\ 6&6&6&6&6&6&6&6&6&9&6&6&6 \\ 6&6&6&6&6&6&6&6&6&6&9&6&6 \\ 6&6&6&6&6&6&6&6&6&6&6&9&6 \\ 6&6&6&6&6&6&6&6&6&6&6&6&9 \end{pmatrix}$$

Definition: Weighing Matrix

A $v \times v$ $(-1,0,1)$-matrix W such that

$$WW^t = kI_v.$$

Write $W(v,k)$.

- $W(v,v)$ is a *Hadamard matrix*
- $W(v,v-1)$ is a *conference matrix*

- A $W(13,9)$

$$W = \begin{pmatrix}
0 & 0 & - & 0 & - & 1 & - & - & 1 & 0 & - & - & - \\
1 & 0 & 0 & - & 0 & - & 1 & - & - & 1 & 0 & - & - \\
1 & 1 & 0 & 0 & - & 0 & - & 1 & - & - & 1 & 0 & - \\
1 & 1 & 1 & 0 & 0 & - & 0 & - & 1 & - & - & 1 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & - & 0 & - & 1 & - & - & 1 \\
- & 0 & 1 & 1 & 1 & 0 & 0 & - & 0 & - & 1 & - & - \\
1 & - & 0 & 1 & 1 & 1 & 0 & 0 & - & 0 & - & 1 & - \\
1 & 1 & - & 0 & 1 & 1 & 1 & 0 & 0 & - & 0 & - & 1 \\
- & 1 & 1 & - & 0 & 1 & 1 & 1 & 0 & 0 & - & 0 & - \\
1 & - & 1 & 1 & - & 0 & 1 & 1 & 1 & 0 & 0 & - & 0 \\
0 & 1 & - & 1 & 1 & - & 0 & 1 & 1 & 1 & 0 & 0 & - \\
1 & 0 & 1 & - & 1 & 1 & - & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & - & 1 & 1 & - & 0 & 1 & 1 & 1 & 0
\end{pmatrix}$$

$$WW^t = \begin{pmatrix} 9&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&9&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&9&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&9&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&9&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&9&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&9&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&9&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&9&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&9&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&9&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&9&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&9 \end{pmatrix}$$

Definition: Generalized Weighing Matrix

- Abelian group G not containg 0.
- A $v \times v$ $(0, G)$-matrix W such that

$$WW^* = (k \cdot 1_G)I_v \pmod{\mathbb{Z}G}.$$

Definition: Balanced Generalized Weighing Matrix

- Abelian group G not containing 0.
- A $v \times v$ $(0, G)$-matrix W such that

$$WW^* = (k \cdot 1_G)I_v + \frac{\lambda}{|G|}G(J_v - I_v).$$

- Write $BGW(v, k, \lambda; G)$.

- A $BGW(v, k, \lambda; C_2)$ is a balanced weighing matrix.
- $\lambda = k(k-1)/(v-1)$

- A $BGW(13, 9, 6; C_2)$

$$W = \begin{pmatrix}
0 & 0 & - & 0 & - & 1 & - & - & 1 & 0 & - & - & - \\
1 & 0 & 0 & - & 0 & - & 1 & - & - & 1 & 0 & - & - \\
1 & 1 & 0 & 0 & - & 0 & - & 1 & - & - & 1 & 0 & - \\
1 & 1 & 1 & 0 & 0 & - & 0 & - & 1 & - & - & 1 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & - & 0 & - & 1 & - & - & 1 \\
- & 0 & 1 & 1 & 1 & 0 & 0 & - & 0 & - & 1 & - & - \\
1 & - & 0 & 1 & 1 & 1 & 0 & 0 & - & 0 & - & 1 & - \\
1 & 1 & - & 0 & 1 & 1 & 1 & 0 & 0 & - & 0 & - & 1 \\
- & 1 & 1 & - & 0 & 1 & 1 & 1 & 0 & 0 & - & 0 & - \\
1 & - & 1 & 1 & - & 0 & 1 & 1 & 1 & 0 & 0 & - & 0 \\
0 & 1 & - & 1 & 1 & - & 0 & 1 & 1 & 1 & 0 & 0 & - \\
1 & 0 & 1 & - & 1 & 1 & - & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & - & 1 & 1 & - & 0 & 1 & 1 & 1 & 0
\end{pmatrix}$$

- Necessary conditions for BGWs:
  - weighing matrix
  - symmetric BIBD
- Sufficient in the case that $|G| = \text{prime}$ (Lam and Leung, 2000)

Definition: Orthogonal Design

- Real commuting variables $\{z_1, \ldots, z_u\}$.
- A $v \times v$ $(0, \pm z_1, \ldots, \pm z_u)$-matrix X such that

$$XX^t = \sigma I_v,$$

where $\sigma = \sum_i s_i z_i^2$.
- Write $OD(v; s_1, \ldots, s_u)$.

- An $OD(4; 1, 1, 1, 1)$

$$X = \begin{pmatrix} a & b & c & d \\ \bar{b} & a & \bar{d} & c \\ \bar{c} & d & a & \bar{b} \\ \bar{d} & \bar{c} & b & a \end{pmatrix}.$$

$$XX^t = (a^2 + b^2 + c^2 + d^2)\begin{pmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{pmatrix}$$

Balancedly Splittable ODs

Section Summary

2. Balancedly Splittable Orthogonal Designs
    - Balancedly Splittable Hadamard Matrices
    - Balancedly Splittable Orthogonal Designs
    - Faithful Construction
    - Related Configurations
        - Immediate Generalizations
        - Configurations

- Recall a Hadamard matrix is a $W(v, v)$.

Definition: Balancedly Splittable Hadamard Matrix (Kharaghani and Suda, 2019)

- A $W(v, v)$, say H.
- An $v \times \ell$ submatrix H_1 such that

$$H_1 H_1^t = \ell I_v + aA + b(J_v - I_v - A),$$

where A is a symmetric $(0, 1)$-matrix with zero diagonal.

- H is (v, ℓ, a, b) balancedly splittable.

- A Hadamard matrix of order 16

$$
\begin{pmatrix}
1&1&1&1&1&1&1&1&1&1&-&-&-&-&-&-\\
1&1&1&1&-&-&1&1&-&-&1&1&-&-&1&1\\
1&1&1&1&1&1&-&-&-&-&-&-&1&1&1&1\\
1&1&1&1&-&-&-&-&1&1&1&1&1&1&-&-\\
1&-&1&-&1&1&1&-&1&-&1&1&1&-&-&1\\
1&-&1&-&1&1&-&1&-&1&1&1&-&1&1&-\\
1&1&-&-&1&-&1&1&1&-&-&1&1&1&1&-\\
1&1&-&-&-&1&1&1&-&1&1&-&1&1&-&1\\
1&-&-&1&1&-&1&-&1&1&1&-&-&1&1&1\\
1&-&-&1&-&1&-&1&1&1&-&1&1&-&1&1\\
-&1&-&1&1&1&1&-&-&1&1&1&1&-&1&-\\
-&1&-&1&1&1&-&1&1&-&1&1&-&1&-&1\\
-&-&1&1&-&1&1&1&1&-&1&-&1&1&1&-\\
-&-&1&1&1&-&1&1&-&1&-&1&1&1&-&1\\
-&1&1&-&1&-&-&1&1&1&1&-&1&-&1&1\\
-&1&1&-&-&1&1&-&1&1&-&1&-&1&1&1
\end{pmatrix}.
$$

- A Hadamard matrix of order 16

$$\left(\begin{array}{cccc|cccccc|cccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & - \\
1 & 1 & 1 & 1 & - & - & 1 & 1 & - & - & 1 & 1 & - & - & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & - & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & - & - & - & - & 1 & 1 & 1 & 1 & 1 & 1 & - & - \\
1 & - & 1 & - & 1 & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & - & 1 \\
1 & - & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - \\
1 & 1 & - & - & 1 & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - \\
1 & 1 & - & - & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & 1 \\
1 & - & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 \\
1 & - & - & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 \\
- & 1 & - & 1 & 1 & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - \\
- & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 \\
- & - & 1 & 1 & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - \\
- & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 \\
- & 1 & 1 & - & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & 1 \\
- & 1 & 1 & - & - & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & 1
\end{array}\right).$$

- A balanced split.

$$\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1\\
- & - & 1 & 1 & - & -\\
1 & 1 & - & - & - & -\\
- & - & - & - & 1 & 1\\
1 & 1 & 1 & - & 1 & -\\
1 & 1 & - & 1 & - & 1\\
1 & - & 1 & 1 & 1 & -\\
- & 1 & 1 & 1 & - & 1\\
1 & - & 1 & - & 1 & 1\\
- & 1 & - & 1 & 1 & 1\\
1 & 1 & 1 & - & - & 1\\
1 & 1 & - & 1 & 1 & -\\
- & 1 & 1 & 1 & 1 & -\\
1 & - & 1 & 1 & - & 1\\
1 & - & - & 1 & 1 & 1\\
- & 1 & 1 & - & 1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & - & 1 & - & 1 & 1 & 1 & - & 1 & - & 1 & 1 & - & 1 & 1 & -\\
1 & - & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & 1 & - & - & 1\\
1 & 1 & - & - & 1 & - & 1 & 1 & 1 & - & 1 & - & 1 & 1 & - & 1\\
1 & 1 & - & - & - & 1 & 1 & 1 & - & 1 & - & 1 & 1 & 1 & 1 & -\\
1 & - & - & 1 & 1 & - & 1 & - & 1 & 1 & - & 1 & 1 & - & 1 & 1\\
1 & - & - & 1 & - & 1 & - & 1 & 1 & 1 & 1 & - & - & 1 & 1 & 1
\end{pmatrix}$$

$$
\begin{pmatrix}
6 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
\bar{2} & 6 & \bar{2} & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & \bar{2} \\
\bar{2} & \bar{2} & 6 & \bar{2} & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & \bar{2} \\
\bar{2} & \bar{2} & \bar{2} & 6 & \bar{2} & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & \bar{2} & 2 & 2 \\
2 & \bar{2} & 2 & \bar{2} & 6 & \bar{2} & 2 & \bar{2} & 2 & \bar{2} & 2 & 2 & 2 & \bar{2} & \bar{2} & 2 \\
2 & \bar{2} & 2 & \bar{2} & \bar{2} & 6 & \bar{2} & 2 & \bar{2} & 2 & 2 & 2 & \bar{2} & 2 & 2 & \bar{2} \\
2 & 2 & \bar{2} & \bar{2} & 2 & \bar{2} & 6 & \bar{2} & 2 & \bar{2} & \bar{2} & 2 & 2 & 2 & 2 & \bar{2} \\
2 & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & \bar{2} & 6 & \bar{2} & 2 & 2 & \bar{2} & 2 & 2 & \bar{2} & 2 \\
2 & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & 2 & \bar{2} & 6 & \bar{2} & 2 & \bar{2} & \bar{2} & 2 & 2 & 2 \\
2 & \bar{2} & \bar{2} & 2 & \bar{2} & 2 & \bar{2} & 2 & \bar{2} & 6 & \bar{2} & 2 & 2 & \bar{2} & 2 & 2 \\
2 & \bar{2} & 2 & \bar{2} & 2 & 2 & \bar{2} & 2 & 2 & \bar{2} & 6 & \bar{2} & \bar{2} & 2 & \bar{2} & 2 \\
2 & \bar{2} & 2 & \bar{2} & 2 & 2 & 2 & \bar{2} & \bar{2} & 2 & \bar{2} & 6 & 2 & \bar{2} & 2 & \bar{2} \\
2 & 2 & \bar{2} & \bar{2} & 2 & \bar{2} & 2 & 2 & \bar{2} & 2 & \bar{2} & 2 & 6 & \bar{2} & \bar{2} & 2 \\
2 & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & 2 & 2 & \bar{2} & 2 & \bar{2} & \bar{2} & 6 & 2 & \bar{2} \\
2 & \bar{2} & \bar{2} & 2 & \bar{2} & 2 & 2 & \bar{2} & 2 & 2 & \bar{2} & 2 & \bar{2} & 2 & 6 & \bar{2} \\
2 & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & \bar{2} & 2 & 2 & 2 & 2 & \bar{2} & 2 & \bar{2} & \bar{2} & 6
\end{pmatrix}
$$

$$
\left(\begin{array}{cccc|cccccc|cccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & - \\
1 & 1 & 1 & 1 & - & - & 1 & 1 & - & - & 1 & 1 & - & - & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & - & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & - & - & - & - & 1 & 1 & 1 & 1 & 1 & 1 & - & - \\
1 & - & 1 & - & 1 & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & - & 1 \\
1 & - & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - \\
1 & 1 & - & - & 1 & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - \\
1 & 1 & - & - & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & 1 \\
1 & - & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 \\
1 & - & - & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 \\
- & 1 & - & 1 & 1 & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - \\
- & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 \\
- & - & 1 & 1 & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - \\
- & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 \\
- & 1 & 1 & - & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & 1 \\
- & 1 & 1 & - & - & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & 1
\end{array}\right)
$$

$$\left(\begin{array}{cccccccccc|cccccc}
1&1&1&1&1&1&1&1&1&1&-&-&-&-&-&-\\
1&1&1&1&-&-&1&1&-&-&1&1&-&-&1&1\\
1&1&1&1&1&1&-&-&-&-&-&-&1&1&1&1\\
1&1&1&1&-&-&-&-&1&1&1&1&1&1&-&-\\
1&-&1&-&1&1&1&-&1&-&1&1&1&-&-&1\\
1&-&1&-&1&1&-&1&-&1&1&1&-&1&1&-\\
1&1&-&-&1&-&1&1&1&-&-&1&1&1&1&-\\
1&1&-&-&-&1&1&1&-&1&1&-&1&1&-&1\\
1&-&-&1&1&-&1&-&1&1&1&-&-&1&1&1\\
1&-&-&1&-&1&-&1&1&1&-&1&1&-&1&1\\
-&1&-&1&1&1&1&-&-&1&1&1&1&-&1&-\\
-&1&-&1&1&1&-&1&1&-&1&1&-&1&-&1\\
-&-&1&1&-&1&1&1&1&-&1&-&1&1&1&-\\
-&-&1&1&1&-&1&1&-&1&-&1&1&1&-&1\\
-&1&1&-&1&-&-&1&1&1&1&-&1&-&1&1\\
-&1&1&-&-&1&1&-&1&1&-&1&-&1&1&1
\end{array}\right)$$

$$\left(\begin{array}{cccccccccccccccc}
1&1&1&1&1&1&1&1&1&1&-&-&-&-&-&-\\
1&1&1&1&-&-&1&1&-&-&1&1&-&-&1&1\\
1&1&1&1&1&1&-&-&-&-&-&-&1&1&1&1\\
1&1&1&1&-&-&-&-&1&1&1&1&1&1&-&-\\
\hline
1&-&1&-&1&1&1&-&1&-&1&1&1&-&-&1\\
1&-&1&-&1&1&-&1&-&1&1&1&-&1&1&-\\
1&1&-&-&1&-&1&1&1&-&-&1&1&1&1&-\\
1&1&-&-&-&1&1&1&-&1&1&-&1&1&-&1\\
1&-&-&1&1&-&1&-&1&1&1&-&-&1&1&1\\
1&-&-&1&-&1&-&1&1&1&-&1&1&-&1&1\\
\hline
-&1&-&1&1&1&1&-&-&1&1&1&1&-&1&-\\
-&1&-&1&1&1&-&1&1&-&1&1&-&1&-&1\\
-&-&1&1&-&1&1&1&1&-&1&-&1&1&1&-\\
-&-&1&1&1&-&1&1&-&1&-&1&1&1&-&1\\
-&1&1&-&1&-&-&1&1&1&1&-&1&-&1&1\\
-&1&1&-&-&1&1&-&1&1&-&1&-&1&1&1
\end{array}\right)$$

$$
\left(\begin{array}{cccccccccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & - \\
1 & 1 & 1 & 1 & - & - & 1 & 1 & - & - & 1 & 1 & - & - & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & - & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & - & - & - & - & 1 & 1 & 1 & 1 & 1 & 1 & - & - \\
1 & - & 1 & - & 1 & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & - & 1 \\
1 & - & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - \\
1 & 1 & - & - & 1 & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - \\
1 & 1 & - & - & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & 1 \\
1 & - & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 \\
1 & - & - & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 \\
\hline
- & 1 & - & 1 & 1 & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - \\
- & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 \\
- & - & 1 & 1 & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - \\
- & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 \\
- & 1 & 1 & - & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & 1 \\
- & 1 & 1 & - & - & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & 1
\end{array}\right)
$$

- An $OD(16; 8, 8)$

$$
\begin{pmatrix}
a & b & a & b & a & b & a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} \\
a & b & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & \bar{b} & a & b \\
a & b & a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} & a & b & a & b \\
a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & a & b & a & b & a & b & \bar{a} & \bar{b} \\
b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a \\
b & \bar{a} & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} \\
a & b & \bar{a} & \bar{b} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} \\
a & b & \bar{a} & \bar{b} & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a \\
b & \bar{a} & \bar{b} & a & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b \\
b & \bar{a} & \bar{b} & a & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b \\
\bar{b} & a & \bar{b} & a & a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} \\
\bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a \\
\bar{a} & \bar{b} & a & b & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} \\
\bar{a} & \bar{b} & a & b & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a \\
\bar{b} & a & b & \bar{a} & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b \\
\bar{b} & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b
\end{pmatrix}
$$

$$\left(\begin{array}{cccc|cccccc|cccccc}
a & b & a & b & a & b & a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} \\
a & b & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & \bar{b} & a & b \\
a & b & a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} & a & b & a & b \\
a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & a & b & a & b & a & b & \bar{a} & \bar{b} \\
b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a \\
b & \bar{a} & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} \\
a & b & \bar{a} & \bar{b} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} \\
a & b & \bar{a} & \bar{b} & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a \\
b & \bar{a} & \bar{b} & a & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b \\
b & \bar{a} & \bar{b} & a & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b \\
\bar{b} & a & \bar{b} & a & a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} \\
\bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a \\
\bar{a} & \bar{b} & a & b & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} \\
\bar{a} & \bar{b} & a & b & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a \\
\bar{b} & a & b & \bar{a} & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b \\
\bar{b} & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b
\end{array}\right)$$

$$
(a^2+b^2)\begin{pmatrix}
3 & - & - & - & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
- & 3 & - & - & - & - & 1 & 1 & - & - & - & - & 1 & 1 & - & - \\
- & - & 3 & - & 1 & 1 & - & - & - & - & 1 & 1 & - & - & - & - \\
- & - & - & 3 & - & - & - & - & 1 & 1 & - & - & - & - & 1 & 1 \\
1 & - & 1 & - & 3 & - & 1 & - & 1 & - & 1 & 1 & 1 & - & - & 1 \\
1 & - & 1 & - & - & 3 & - & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - \\
1 & 1 & - & - & 1 & - & 3 & - & 1 & - & - & 1 & 1 & 1 & 1 & - \\
1 & 1 & - & - & - & 1 & - & 3 & - & 1 & 1 & - & 1 & 1 & - & 1 \\
1 & - & - & 1 & 1 & - & 1 & - & 3 & - & 1 & - & - & 1 & 1 & 1 \\
1 & - & - & 1 & - & 1 & - & 1 & - & 3 & - & 1 & 1 & - & 1 & 1 \\
1 & - & 1 & - & 1 & 1 & - & 1 & 1 & - & 3 & - & - & 1 & - & 1 \\
1 & - & 1 & - & 1 & 1 & 1 & - & - & 1 & - & 3 & 1 & - & 1 & - \\
1 & 1 & - & - & 1 & - & 1 & 1 & - & 1 & - & 1 & 3 & - & - & 1 \\
1 & 1 & - & - & - & 1 & 1 & 1 & 1 & - & 1 & - & - & 3 & 1 & - \\
1 & - & - & 1 & - & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 & 3 & - \\
1 & - & - & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - & - & 3
\end{pmatrix}
$$

$$
\left(\begin{array}{cccccccccccccccc}
a & b & a & b & a & b & a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} \\
a & b & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & \bar{b} & a & b \\
a & b & a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} & a & b & a & b \\
a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & a & b & a & b & a & b & \bar{a} & \bar{b} \\
\hline
b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a \\
b & \bar{a} & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} \\
a & b & \bar{a} & \bar{b} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} \\
a & b & \bar{a} & \bar{b} & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a \\
b & \bar{a} & \bar{b} & a & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b \\
b & \bar{a} & \bar{b} & a & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b \\
\hline
\bar{b} & a & \bar{b} & a & a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} \\
\bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a \\
\bar{a} & \bar{b} & a & b & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} \\
\bar{a} & \bar{b} & a & b & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a \\
\bar{b} & a & b & \bar{a} & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b \\
\bar{b} & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b
\end{array}\right)
$$

- Products in the set $\{\pm 2a^2, \pm 2b^2, \pm 2ab\}$.

Definition: Balancedly Splittable (Kharaghani et al., 2021)

- An OD X with $v \times \ell$ submatrix X_1 admits a stable (v, ℓ, a, b) split if

$$X_1 X_1^t = \sigma(cI_v + aA + b(J_v - I_v - A)).$$

- The split is unstable if it is not stable and

$$X_1 X_1^t = \ell I_v + aA + b(J_v - I_v - A)$$

after replacing each variable with 1.

Definition: Auxiliary Matrices

- X a full $OD(v; s_1, \ldots, s_u)$ and H the Hadamard matrix obtained by replacing each variable with 1.
- Label rows of X and H by $x_0, \ldots, x_{v-1}$, and $h_0, \ldots, h_{v-1}$.
- The auxiliary matrices of X are given by

$$c_i = h_i^t x_i, \qquad i = 0, \ldots, v-1.$$

- An $OD(2;1,1)$:

$$\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$$

- Auxiliary matrices:

$$c_0 = \begin{pmatrix} a & b \\ a & b \end{pmatrix}, \qquad c_1 = \begin{pmatrix} b & -a \\ -b & a \end{pmatrix}$$

- Form the sequences

$$\alpha = (c_0, c_1, c_1), \qquad \beta = (c_0, c_1, -c_1).$$

- The sum of the periodic autocorrelations of α and β satisfy

$$\sum_i \alpha_i \alpha_{i+j}^t + \sum_i \beta_i \beta_{i+j}^t = O, \qquad j = 1, 2.$$

- Sums of cross-correlations

$$\sum_i \alpha_i \beta_{i+j}^t + \sum_i \beta_i \alpha_{i+j}^t = O, \qquad j = 1, 2.$$

- Form the matrices

$$A = circ(\alpha), \qquad B = circ(\beta),$$

- Then the matrix

$$\Theta = \begin{pmatrix} A & B \\ B & A \end{pmatrix}.$$

$$\Theta = \left(\begin{array}{cccccc|cccccc}
a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a \\
a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} \\
b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} \\
\bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a \\
b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b \\
\bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b \\
\hline
a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} \\
a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a \\
\bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} \\
b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a \\
b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b \\
\bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b
\end{array}\right)$$

$$\Theta\Theta^t = \begin{pmatrix} AA^t + BB^t & AB^t + BA^t \\ AB^t + BA^t & AA^t + BB^t \end{pmatrix}.$$

$$\Theta\Theta^t = (a^2+b^2)\left(\begin{array}{cccccc|cccccc}
6 & \bar{2} & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 \\
\bar{2} & 6 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 6 & \bar{2} & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 \\
0 & 0 & \bar{2} & 6 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 6 & \bar{2} & 0 & 0 & 0 & 0 & 2 & 2 \\
0 & 0 & 0 & 0 & \bar{2} & 6 & 0 & 0 & 0 & 0 & 2 & 2 \\
\hline
2 & 2 & 0 & 0 & 0 & 0 & 6 & \bar{2} & 0 & 0 & 0 & 0 \\
2 & 2 & 0 & 0 & 0 & 0 & \bar{2} & 6 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 6 & \bar{2} & 0 & 0 \\
0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & \bar{2} & 6 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 6 & \bar{2} \\
0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & \bar{2} & 6
\end{array}\right)$$

$$
\left(\begin{array}{cccc|cccccc|cccccc}
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
\hline
* & * & * & * & a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a \\
* & * & * & * & a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} \\
* & * & * & * & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} \\
* & * & * & * & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a \\
* & * & * & * & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b \\
* & * & * & * & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b \\
\hline
* & * & * & * & a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} \\
* & * & * & * & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a \\
* & * & * & * & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} \\
* & * & * & * & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a \\
* & * & * & * & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b \\
* & * & * & * & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b
\end{array}\right)
$$

- Define:

$$\begin{aligned} Y &= \left(\begin{smallmatrix} 1 & 1 \\ 1 & - \end{smallmatrix}\right) \otimes X, \\ K &= \left(\begin{smallmatrix} 1 & 1 \\ 1 & - \end{smallmatrix}\right) \otimes H. \end{aligned}$$

- Index rows as $y_0, \ldots, y_{2n-1}$, and $k_0, \ldots, k_{2n-1}$.

$$Y = \begin{pmatrix} 1 & 1 \\ 1 & - \end{pmatrix} \otimes X \qquad K = \begin{pmatrix} 1 & 1 \\ 1 & - \end{pmatrix} \otimes H$$
$$\text{Rows: } y_0, \ldots, y_{2n-1} \quad \text{Rows: } k_0, \ldots, k_{2n-1}$$

- Form

$$F = \begin{pmatrix} k_1^t x_0 & k_2^t x_0 & k_3^t x_0 \end{pmatrix},$$
$$E = \begin{pmatrix} h_0^t y_1 \\ h_0^t y_2 \\ h_0^t y_3 \end{pmatrix},$$
$$G = k_0^t y_0.$$

$$
\left(\begin{array}{c|cccccc|cccccc}
G & & & F & & & & & & -F & & & \\
\hline
 & a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a \\
 & a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} \\
 & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} \\
E & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a \\
 & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b \\
 & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b \\
\hline
 & a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} \\
 & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a \\
 & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} \\
-E & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a \\
 & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b \\
 & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b
\end{array}\right)
$$

$$\left(\begin{array}{cccc|cccccc|cccccc}
a & b & a & b & a & b & a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} \\
a & b & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & \bar{b} & a & b \\
a & b & a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} & a & b & a & b \\
a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & a & b & a & b & a & b & \bar{a} & \bar{b} \\
\hline
b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a \\
b & \bar{a} & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} \\
a & b & \bar{a} & \bar{b} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} \\
a & b & \bar{a} & \bar{b} & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a \\
b & \bar{a} & \bar{b} & a & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b \\
b & \bar{a} & \bar{b} & a & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b \\
\hline
\bar{b} & a & \bar{b} & a & a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} \\
\bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a \\
\bar{a} & \bar{b} & a & b & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} \\
\bar{a} & \bar{b} & a & b & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a \\
\bar{b} & a & b & \bar{a} & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b \\
\bar{b} & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b
\end{array}\right)$$

Theorem (Kharaghani et al., 2021)

If there is a full $OD(v, s_1, \ldots, s_u)$, then there is a full $OD(4v^2; 4vs_1, \ldots, 4vs_u)$ for which:

- there are two verticle splits, and
- there are two unstable horizontal splits.

Proof

- In the general case:

$$\alpha = (c_0, c_1, \ldots, c_{n-1}, c_{n-1}, \ldots, c_1)$$
$$\beta = (c_0, c_1, \ldots, c_{n-1}, -c_{n-1}, \ldots, -c_1).$$

- Tedious checking of products between block rows.

- The following is a $OD(2;1,1)$ of quaternions.

$$\begin{pmatrix} \bar{a} & bi \\ \bar{b}j & ak \end{pmatrix},$$

- where a and b are real variables.

$$
\left(\begin{array}{cccc|cccccc|cccccc}
\bar{a} & bi & \bar{a} & bi & \bar{a} & bi & \bar{a} & bi & \bar{a} & bi & a & \bar{b}i & a & \bar{b}i & a & \bar{b}i \\
\bar{a} & bi & \bar{a} & bi & a & \bar{b}i & \bar{a} & bi & a & \bar{b}i & \bar{a} & bi & a & \bar{b}i & \bar{a} & bi \\
\bar{a} & bi & \bar{a} & bi & \bar{a} & bi & a & \bar{b}i & a & \bar{b}i & a & \bar{b}i & \bar{a} & bi & \bar{a} & bi \\
\bar{a} & bi & \bar{a} & bi & a & \bar{b}i & a & \bar{b}i & \bar{a} & bi & \bar{a} & bi & \bar{a} & bi & a & \bar{b}i \\
\hline
\bar{b}j & ak & \bar{b}j & ak & \bar{a} & bi & \bar{b}j & ak & \bar{b}j & ak & \bar{a} & bi & \bar{b}j & ak & bj & \bar{a}k \\
\bar{b}j & ak & \bar{b}j & ak & \bar{a} & bi & bj & \bar{a}k & bj & \bar{a}k & \bar{a} & bi & bj & \bar{a}k & \bar{b}j & ak \\
\bar{a} & bi & a & \bar{b}i & \bar{b}j & ak & \bar{a} & bi & \bar{b}j & ak & bj & \bar{a}k & \bar{a} & bi & \bar{b}j & ak \\
\bar{a} & bi & a & \bar{b}i & bj & \bar{a}k & \bar{a} & bi & bj & \bar{a}k & \bar{b}j & ak & \bar{a} & bi & bj & \bar{a}k \\
\bar{b}j & ak & bj & \bar{a}k & \bar{b}j & ak & \bar{b}j & ak & \bar{a} & bi & \bar{b}j & ak & bj & \bar{a}k & \bar{a} & bi \\
\bar{b}j & ak & bj & \bar{a}k & bj & \bar{a}k & bj & \bar{a}k & \bar{a} & bi & bj & \bar{a}k & \bar{b}j & ak & \bar{a} & bi \\
\hline
bj & \bar{a}k & bj & \bar{a}k & \bar{a} & bi & \bar{b}j & ak & bj & \bar{a}k & \bar{a} & bi & \bar{b}j & ak & \bar{b}j & ak \\
bj & \bar{a}k & bj & \bar{a}k & \bar{a} & bi & bj & \bar{a}k & \bar{b}j & ak & \bar{a} & bi & bj & \bar{a}k & bj & \bar{a}k \\
a & \bar{b}i & \bar{a} & bi & bj & \bar{a}k & \bar{a} & bi & \bar{b}j & ak & \bar{b}j & ak & \bar{a} & bi & \bar{b}j & ak \\
a & \bar{b}i & \bar{a} & bi & \bar{b}j & ak & \bar{a} & bi & bj & \bar{a}k & bj & \bar{a}k & \bar{a} & bi & bj & \bar{a}k \\
bj & \bar{a}k & \bar{b}j & ak & \bar{b}j & ak & bj & \bar{a}k & \bar{a} & bi & \bar{b}j & ak & \bar{b}j & ak & \bar{a} & bi \\
bj & \bar{a}k & \bar{b}j & ak & bj & \bar{a}k & \bar{b}j & ak & \bar{a} & bi & bj & \bar{a}k & bj & \bar{a}k & \bar{a} & bi
\end{array}\right)
$$

- Related configurations:
  - equiangular lines
  - Unbiased orthogonal designs
  - quasi-symmetric designs
  - quaternion frames

Unifying Construction of Weighing Matrices

Section Summary

3. Unifying Construction of Weighing Matrices
   - Classical Parameter BGWs
   - Normalization
   - Generalized Simplex Codes
   - Construction

Proposition: Classical Paramaeter BGWs

- Let q be a prime power and $d > 0$.
- BGWs are known to exist with the following parameters:
  - $$\left(v = \frac{q^{d+1}-1}{q-1}, k = q^d, \lambda = q^d - q^{d-1}; G = C_{q-1}\right)$$
    for every q and $d > 0$, and
  - $$\left(v = \frac{q^{d+1}-1}{q-1}, k = q^d, \lambda = q^d - q^{d-1}; G = C_{2q-2}\right)$$
    for even q and d.

- Applying an epimorphism to the nonzero entries of a BGW gives a BGW with the same parameters.
- Equivalence operations for generalized weighing matrices:
  - permutation of rows
  - permutation of columns
  - post-multiplying column elements by a group element
  - pre-multiplying row elements by a group element
- Every weighing matrix is equivalent to one of the following form

$$\begin{pmatrix} \mathbf{0} & R \\ \mathbf{1} & D \end{pmatrix}.$$

- R is the residual-part
- D is the derived-part

Definition: Simplex Code

- q a prime power and $d > 0$
- Form matrix G with columns given by reps. of 1-D subspaces of $GF(q^{d+1})$
- simplex code $\mathcal{S}_{q,d} = row(G)$

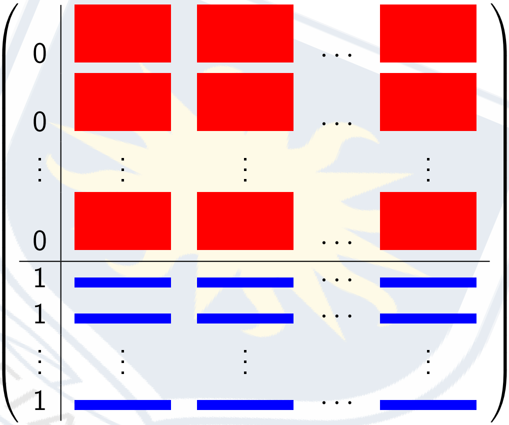
Proposition

For $\mathcal{S}_{q,d}$:

- $wt(x) = q^d$ for all $x \in \mathcal{S}_{q,d}$
- $dist(x, y) = q^d$ for all $x, y \in \mathcal{S}_{q,d}$ and $x \neq y$

- Ingredients of unifying construction:
  - a normalized $W(v, q)$ (seed matrix) with residual-part R and derived-part D
  - a $W((q^{d+1}-1)/(q-1), q^d)$, say W
  - $\mathcal{S}_{q,d}$

- Recipie of unifying construction:
  - Form $A = W \otimes R$.
  - Form B by replacing elements of $\mathcal{S}_{q,d}$ by rows of D.
  - Then
$$\begin{pmatrix} \mathbf{0} & A \\ \mathbf{1} & B \end{pmatrix}$$
    is a $W((v-1)(q^{d+1}-1)/(q-1)+1, q^{d+1})$.


0
0
0
1
1
1

- A seed $W(8,5)$

$$\begin{pmatrix} \mathbf{0} & R \\ \mathbf{1} & D \end{pmatrix} = \left(\begin{array}{c|ccccccc} 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & - & - & 1 \\ 0 & 0 & 0 & 1 & 1 & - & 1 & - \\ \hline 1 & 0 & 0 & 0 & 1 & 1 & - & - \\ 1 & 1 & 1 & 1 & - & 0 & 0 & 0 \\ 1 & 1 & - & - & 0 & - & 0 & 0 \\ 1 & - & 1 & - & 0 & 0 & 1 & 0 \\ 1 & - & - & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

- A classical parameter $W(6, 5)$

$$W = \begin{pmatrix} - & 1 & - & 0 & 1 & 1 \\ - & - & 1 & - & 0 & 1 \\ - & - & - & 1 & - & 0 \\ 0 & - & - & - & 1 & - \\ 1 & 0 & - & - & - & 1 \\ - & 1 & 0 & - & - & - \end{pmatrix}.$$

- The simplex code $\mathcal{S}_{5,1}$ (transposed)

$$\begin{pmatrix} 1\,2\,3\,4\,0\,1\,2\,3\,4\,0\,1\,2\,3\,4\,0\,1\,2\,3\,4\,0\,1\,2\,3\,4\,0 \\ 1\,1\,1\,1\,1\,2\,2\,2\,2\,2\,3\,3\,3\,3\,3\,4\,4\,4\,4\,4\,0\,0\,0\,0\,0 \\ 2\,4\,0\,3\,1\,4\,3\,1\,0\,2\,0\,1\,4\,2\,3\,3\,0\,2\,1\,4\,1\,2\,3\,4\,0 \\ 4\,0\,3\,2\,1\,3\,1\,0\,4\,2\,1\,4\,2\,0\,3\,0\,2\,1\,3\,4\,2\,3\,4\,1\,0 \\ 0\,3\,2\,4\,1\,1\,0\,4\,3\,2\,4\,2\,0\,1\,3\,2\,1\,3\,0\,4\,3\,4\,1\,2\,0 \\ 3\,2\,4\,0\,1\,0\,4\,3\,1\,2\,2\,0\,1\,4\,3\,1\,3\,0\,2\,4\,4\,1\,2\,3\,0 \end{pmatrix}$$

- Take $A = W \otimes R$.
- Take B to be the matrix formed after replacing the entries of $S_{5,1}$ by the rows of D.
- Then

$$\begin{pmatrix} \mathbf{0} & A \\ \mathbf{1} & B \end{pmatrix}$$

is a $W(43, 25)$.

$$
\left(\begin{array}{c|cccccccccccccccccccccccccccccccccccccccccc}
0 & - & 0 & 0 & - & - & - & - & 1 & 0 & 0 & 1 & 1 & 1 & 1 & - & 0 & 0 & - & - & - & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & - & 0 & - & 1 & 1 & - & 0 & 1 & 0 & 1 & - & - & 1 & 0 & - & 0 & - & 1 & 1 & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & - & - & 1 & 0 & 1 & 0 & 1 & - & - & 1 \\
0 & 0 & 0 & - & - & 1 & - & 1 & 0 & 0 & 1 & 1 & - & 1 & - & 0 & 0 & - & - & 1 & - & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & - & 1 & - & 0 & 0 & 1 & 1 & - & 1 & - \\
0 & - & 0 & 0 & - & - & - & - & - & 0 & 0 & - & - & - & - & 1 & 0 & 0 & 1 & 1 & 1 & 1 & - & 0 & 0 & - & - & - & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & - & 0 & - & 1 & 1 & - & 0 & - & 0 & - & 1 & 1 & - & 0 & 1 & 0 & 1 & - & - & 1 & 0 & - & 0 & - & 1 & 1 & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & - & - & 1 \\
0 & 0 & 0 & - & - & 1 & - & 1 & 0 & 0 & - & - & 1 & - & 1 & 0 & 0 & 1 & 1 & - & 1 & - & 0 & 0 & - & - & 1 & - & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & - & 1 & - \\
0 & - & 0 & 0 & - & - & - & - & - & 0 & 0 & - & - & - & - & - & 0 & 0 & - & - & - & - & 1 & 0 & 0 & 1 & 1 & 1 & 1 & - & 0 & 0 & - & - & - & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & - & 0 & - & 1 & 1 & - & 0 & - & 0 & - & 1 & 1 & - & 0 & - & 0 & - & 1 & 1 & - & 0 & 1 & 0 & 1 & - & - & 1 & 0 & - & 0 & - & 1 & 1 & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & - & - & 1 & - & 1 & 0 & 0 & - & - & 1 & - & 1 & 0 & 0 & - & - & 1 & - & 1 & 0 & 0 & 1 & 1 & - & 1 & - & 0 & 0 & - & - & 1 & - & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & - & 0 & 0 & - & - & - & - & - & 0 & 0 & - & - & - & - & - & 0 & 0 & - & - & - & - & 1 & 0 & 0 & 1 & 1 & 1 & 1 & - & 0 & 0 & - & - & - & - \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & - & 0 & - & 1 & 1 & - & 0 & - & 0 & - & 1 & 1 & - & 0 & - & 0 & - & 1 & 1 & - & 0 & 1 & 0 & 1 & - & - & 1 & 0 & - & 0 & - & 1 & 1 & - \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & - & - & 1 & - & 1 & 0 & 0 & - & - & 1 & - & 1 & 0 & 0 & - & - & 1 & - & 1 & 0 & 0 & 1 & 1 & - & 1 & - & 0 & 0 & - & - & 1 & - & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & - & 0 & 0 & - & - & - & - & - & 0 & 0 & - & - & - & - & - & 0 & 0 & - & - & - & - & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
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0 & 0 & - & 0 & - & 1 & 1 & - & 0 & 1 & 0 & 1 & - & - & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & - & 0 & - & 1 & 1 & - & 0 & - & 0 & - & 1 & 1 & - & 0 & - & 0 & - & 1 & 1 & - \\
0 & 0 & 0 & - & - & 1 & - & 1 & 0 & 0 & 1 & 1 & - & 1 & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & - & - & 1 & - & 1 & 0 & 0 & - & - & 1 & - & 1 & 0 & 0 & - & - & 1 & - & 1 \\
\hline
1 & 1 & 1 & 1 & - & 0 & 0 & 0 & 1 & 1 & 1 & - & 0 & 0 & 0 & 1 & - & - & 0 & - & 0 & 0 & - & - & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & - & - & - & 1 & - & 0 & 0 & 1 & 0 \\
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1 & 1 & - & - & 0 & - & 0 & 0 & 1 & - & - & 0 & - & 0 & 0 & - & 1 & - & 0 & 0 & 1 & 0 & 1 & 1 & 1 & - & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & - & - & - & - & 1 & 0 & 0 & 0 & 1 \\
1 & - & 1 & - & 0 & 0 & 1 & 0 & 1 & - & - & 0 & - & 0 & 0 & 1 & 1 & 1 & - & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & - & - & - & - & 1 & 0 & 0 & 0 & 1 & - & 1 & - & 0 & 0 & 1 & 0 \\
1 & - & - & 1 & 0 & 0 & 0 & 1 & 1 & - & - & 0 & - & 0 & 0 & 0 & 0 & 0 & 1 & 1 & - & - & - & - & 1 & 0 & 0 & 0 & 1 & - & 1 & - & 0 & 0 & 1 & 0 & 1 & 1 & 1 & - & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 1 & - & - & 1 & - & - & 0 & - & 0 & 0 & 1 & - & - & 0 & - & 0 & 0 & 1 & - & - & 0 & - & 0 & 0 & 1 & - & - & 0 & - & 0 & 0 & 1 & - & - & 0 & - & 0 & 0 \\
1 & 1 & 1 & 1 & - & 0 & 0 & 0 & - & 1 & - & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & - & - & 1 & 1 & 1 & - & 0 & 0 & 0 & - & - & 1 & 0 & 0 & 0 & 1 & 1 & - & - & 0 & - & 0 & 0 \\
1 & 1 & - & - & 0 & - & 0 & 0 & - & 1 & - & 0 & 0 & 1 & 0 & 1 & 1 & 1 & - & 0 & 0 & 0 & - & - & 1 & 0 & 0 & 0 & 1 & 1 & - & - & 0 & - & 0 & 0 & 0 & 0 & 0 & 1 & 1 & - & - \\
1 & - & 1 & - & 0 & 0 & 1 & 0 & - & 1 & - & 0 & 0 & 1 & 0 & - & - & 1 & 0 & 0 & 0 & 1 & 1 & - & - & 0 & - & 0 & 0 & 0 & 0 & 0 & 1 & 1 & - & - & 1 & 1 & 1 & - & 0 & 0 & 0 \\
1 & - & - & 1 & 0 & 0 & 0 & 1 & - & 1 & - & 0 & 0 & 1 & 0 & 1 & - & - & 0 & - & 0 & 0 & 0 & 0 & 0 & 1 & 1 & - & - & 1 & 1 & 1 & - & 0 & 0 & 0 & - & - & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 1 & - & - & - & 1 & - & 0 & 0 & 1 & 0 & - & 1 & - & 0 & 0 & 1 & 0 & - & 1 & - & 0 & 0 & 1 & 0 & - & 1 & - & 0 & 0 & 1 & 0 & - & 1 & - & 0 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & - & 0 & 0 & 0 & - & - & 1 & 0 & 0 & 0 & 1 & - & 1 & - & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & - & - & 1 & - & - & 0 & - & 0 & 0 & 1 & 1 & 1 & - & 0 & 0 & 0 \\
1 & 1 & - & - & 0 & - & 0 & 0 & - & - & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & - & - & 1 & - & - & 0 & - & 0 & 0 & 1 & 1 & 1 & - & 0 & 0 & 0 & - & 1 & - & 0 & 0 & 1 & 0 \\
1 & - & 1 & - & 0 & 0 & 1 & 0 & - & - & 1 & 0 & 0 & 0 & 1 & 1 & - & - & 0 & - & 0 & 0 & 1 & 1 & 1 & - & 0 & 0 & 0 & - & 1 & - & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & - & - \\
1 & - & - & 1 & 0 & 0 & 0 & 1 & - & - & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & - & 0 & 0 & 0 & - & 1 & - & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & - & - & 1 & - & - & 0 & - & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 1 & - & - & - & - & 1 & 0 & 0 & 0 & 1 & - & - & 1 & 0 & 0 & 0 & 1 & - & - & 1 & 0 & 0 & 0 & 1 & - & - & 1 & 0 & 0 & 0 & 1 & - & - & 1 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & - & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & - & - & 1 & 1 & 1 & - & 0 & 0 & 0 & 1 & - & - & 0 & - & 0 & 0 & - & 1 & - & 0 & 0 & 1 & 0 & - & - & 1 & 0 & 0 & 0 & 1 \\
1 & 1 & - & - & 0 & - & 0 & 0 & 0 & 0 & 0 & 1 & 1 & - & - & 1 & - & - & 0 & - & 0 & 0 & - & 1 & - & 0 & 0 & 1 & 0 & - & - & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & - & 0 & 0 & 0 \\
1 & - & 1 & - & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & - & - & - & 1 & - & 0 & 0 & 1 & 0 & - & - & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & - & 0 & 0 & 0 & 1 & - & - & 0 & - & 0 & 0 \\
1 & - & - & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & - & - & - & - & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & - & 0 & 0 & 0 & 1 & - & - & 0 & - & 0 & 0 & - & 1 & - & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 & 1 & - & - & 0 & 0 & 0 & 1 & 1 & - & - & 0 & 0 & 0 & 1 & 1 & - & - & 0 & 0 & 0 & 1 & 1 & - & - & 0 & 0 & 0 & 1 & 1 & - & - & 0 & 0 & 0 & 1 & 1 & - & -
\end{array}\right)
$$

Theorem (Kharaghani et al., 2022a)

If there is a $W(v,q)$, then there is a weighing matrix with parameters

$$\left(\frac{(v-1)(q^{d+1}-1)}{q-1}+1, q^{d+1}\right)$$

whenever:

1. q is odd and every $d>0$, and
2. q and d are both even.

Seed (v, k)	Succident (v', k')	Seed (v, k)	Succident (v', k')
$(6, 5)$:	$(31, 25)$, $(156, 125)$, $(781, 625)$	$(16, 3)$:	$(69, 9)$, $(196, 27)$, $(601, 81)$
$(8, 5)$:	$(43, 25)$, $(218, 125)$	$(16, 5)$:	$(91, 25)$, $(466, 125)$
$(8, 7)$:	$(57, 49)$, $(400, 343)$	$(16, 7)$:	$(121, 49)$, $(856, 343)$
$(10, 5)$:	$(55, 25)$, $(280, 125)$	$(16, 9)$:	$(151, 81)$
$(10, 9)$:	$(91, 81)$, $(820, 729)$	$(16, 11)$:	$(181, 121)$
$(12, 5)$:	$(67, 25)$, $(342, 125)$	$(16, 13)$:	$(211, 169)$
$(12, 7)$:	$(89, 49)$, $(628, 343)$	$(18, 13)$:	$(239, 169)$
$(12, 9)$:	$(111, 81)$	$(19, 9)$:	$(181, 81)$
$(13, 9)$:	$(121, 81)$	$(20, 7)$:	$(153, 49)$
$(14, 9)$:	$(131, 81)$	$(20, 13)$:	$(267, 169)$
$(14, 13)$:	$(183, 169)$		

A New Class of BWs

Section Summary

- Consider the following $BGW(19, 9, 4; C_2)$:

$$W_{19} = \left(\begin{array}{c|cccccccccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & - & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & - & 1 & - & 0 \\
0 & 0 & - & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & - & 1 & 0 & 0 & 1 & - \\
0 & 0 & 0 & - & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & - & 1 & - & 0 & 1 \\
0 & 1 & 1 & 0 & - & 0 & 0 & 1 & 0 & 1 & 1 & - & 0 & 0 & 0 & 0 & 1 & 0 & - \\
0 & 0 & 1 & 1 & 0 & - & 0 & 1 & 1 & 0 & 0 & 1 & - & 0 & 0 & 0 & - & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & - & 0 & 1 & 1 & - & 0 & 1 & 0 & 0 & 0 & 0 & - & 1 \\
0 & 1 & 0 & 1 & 1 & 1 & 0 & - & 0 & 0 & 1 & 0 & - & 1 & - & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & - & 0 & - & 1 & 0 & 0 & 1 & - & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & - & 0 & - & 1 & - & 0 & 1 & 0 & 0 & 0 \\
\hline
1 & 0 & 0 & 0 & 1 & 0 & - & 1 & - & 0 & 0 & - & - & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & - & 1 & 0 & 0 & 1 & - & - & 0 & - & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & - & 1 & - & 0 & 1 & - & - & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & - & 0 & 0 & 0 & 0 & 1 & 0 & - & 0 & 0 & 1 & 0 & - & - & 0 & 1 & 0 \\
1 & 0 & 1 & - & 0 & 0 & 0 & - & 1 & 0 & 1 & 0 & 0 & - & 0 & - & 0 & 0 & 1 \\
1 & - & 0 & 1 & 0 & 0 & 0 & 0 & - & 1 & 0 & 1 & 0 & - & - & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & - & 1 & - & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & - & - \\
1 & - & 1 & 0 & 0 & 1 & - & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & - & 0 & - \\
1 & 0 & - & 1 & - & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & - & - & 0
\end{array}\right)$$

- Computationally found in de Launey and Sarvate (1984).

$$R_1 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
- & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & - & 1 & - & 0 \\
0 & - & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & - & 1 & 0 & 0 & 1 & - \\
0 & 0 & - & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & - & 1 & - & 0 & 1 \\
1 & 1 & 0 & - & 0 & 0 & 1 & 0 & 1 & 1 & - & 0 & 0 & 0 & 0 & 1 & 0 & - \\
0 & 1 & 1 & 0 & - & 0 & 1 & 1 & 0 & 0 & 1 & - & 0 & 0 & 0 & - & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & - & 0 & 1 & 1 & - & 0 & 1 & 0 & 0 & 0 & 0 & - & 1 \\
1 & 0 & 1 & 1 & 1 & 0 & - & 0 & 0 & 1 & 0 & - & 1 & - & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 & - & 0 & - & 1 & 0 & 0 & 1 & - & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & - & 0 & - & 1 & - & 0 & 1 & 0 & 0 & 0
\end{pmatrix}$$

$$|R_1| = \begin{pmatrix}
0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1\\
1&0&0&1&0&1&1&1&0&0&0&0&1&0&1&1&1&0\\
0&1&0&1&1&0&0&1&1&0&0&0&1&1&0&0&1&1\\
0&0&1&0&1&1&1&0&1&0&0&0&0&1&1&1&0&1\\
1&1&0&1&0&0&1&0&1&1&1&0&0&0&0&1&0&1\\
0&1&1&0&1&0&1&1&0&0&1&1&0&0&0&1&1&0\\
1&0&1&0&0&1&0&1&1&1&0&1&0&0&0&0&1&1\\
1&0&1&1&1&0&1&0&0&1&0&1&1&1&0&0&0&0\\
1&1&0&0&1&1&0&1&0&1&1&0&0&1&1&0&0&0\\
0&1&1&1&0&1&0&0&1&0&1&1&1&0&1&0&0&0
\end{pmatrix}$$

$$
|R_2| = J - |R_1| = \begin{pmatrix}
1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0\\
0&1&1&0&1&0&0&0&1&1&1&1&0&1&0&0&0&1\\
1&0&1&0&0&1&1&0&0&1&1&1&0&0&1&1&0&0\\
1&1&0&1&0&0&0&1&0&1&1&1&1&0&0&0&1&0\\
0&0&1&0&1&1&0&1&0&0&0&1&1&1&1&0&1&0\\
1&0&0&1&0&1&0&0&1&1&0&0&1&1&1&0&0&1\\
0&1&0&1&1&0&1&0&0&0&1&0&1&1&1&1&0&0\\
0&1&0&0&0&1&0&1&1&0&1&0&0&0&1&1&1&1\\
0&0&1&1&0&0&1&0&1&0&0&1&1&0&0&1&1&1\\
1&0&0&0&1&0&1&1&0&1&0&0&0&1&0&1&1&1
\end{pmatrix}
$$

$$R_2 = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & - & - & 0 & 1 & 0 & 0 & 0 & 1 & - & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
- & 0 & - & 0 & 0 & 1 & 1 & 0 & 0 & 1 & - & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
- & - & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & - & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & - & - & 0 & 1 & 0 & 0 & 0 & 1 & - & 1 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & - & 0 & - & 0 & 0 & 1 & 1 & 0 & 0 & 1 & - & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & - & - & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & - & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & - & - & 0 & 1 & 0 & 0 & 0 & 1 & - & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & - & 0 & - & 0 & 0 & 1 & 1 & 0 & 0 & 1 & - & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & - & - & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & -
\end{pmatrix}$$

- Computationally found viz. Maple.

$$W'_{19} = \left(\begin{array}{c|cccccccccccccccccc}
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & - & - & 0 & 1 & 0 & 0 & 0 & 1 & - & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & - & 0 & - & 0 & 0 & 1 & 1 & 0 & 0 & 1 & - & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & - & - & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & - & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & - & - & 0 & 1 & 0 & 0 & 0 & 1 & - & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & - & 0 & - & 0 & 0 & 1 & 1 & 0 & 0 & 1 & - & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & - & - & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & - & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & - & - & 0 & 1 & 0 & 0 & 0 & 1 & - & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & - & 0 & - & 0 & 0 & 1 & 1 & 0 & 0 & 1 & - & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & - & - & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & - \\
\hline
1 & 0 & 0 & 0 & 1 & 0 & - & 1 & - & 0 & 0 & - & - & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & - & 1 & 0 & 0 & 1 & - & - & 0 & - & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & - & 1 & - & 0 & 1 & - & - & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & - & 0 & 0 & 0 & 0 & 1 & 0 & - & 0 & 0 & 1 & 0 & - & - & 0 & 1 & 0 \\
1 & 0 & 1 & - & 0 & 0 & 0 & - & 1 & 0 & 1 & 0 & 0 & - & 0 & - & 0 & 0 & 1 \\
1 & - & 0 & 1 & 0 & 0 & 0 & 0 & - & 1 & 0 & 1 & 0 & - & - & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & - & 1 & - & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & - & - \\
1 & - & 1 & 0 & 0 & 1 & - & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & - & 0 & - \\
1 & 0 & - & 1 & - & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & - & - & 0
\end{array}\right)$$

- Let Ξ be a BGW with parameters

$$\left(v = \frac{9^{d+1}-1}{8}, k = 9^d, \lambda = 9^d - 9^{d-1}\right)$$

  over the group $C_4 = \{1, g, g^2, g^3\}$.
- Decompose Ξ as

$$\Xi = \Xi_1 + g\Xi_g + g^2\Xi_{g^2} + g^3\Xi_{g^3},$$

  where the Ξ_is are disjoint $(0,1)$-matrices.

$$\begin{pmatrix}
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\\
1 & 0 & g & g^2 & g^3 & 1 & g & g^2 & g^3 & 1\\
1 & g & 0 & 1 & g^2 & g^2 & g & g^3 & 1 & g^3\\
1 & g^2 & 1 & 0 & g & g^3 & g^3 & g^2 & 1 & g\\
1 & g^3 & g^2 & g & 0 & g^2 & 1 & 1 & g^3 & g\\
1 & 1 & g^2 & g^3 & g^2 & 0 & g^3 & g & g & 1\\
1 & g & g & g^3 & 1 & g^3 & 0 & 1 & g^2 & g^2\\
1 & g^2 & g^3 & g^2 & 1 & g & 1 & 0 & g & g^3\\
1 & g^3 & 1 & 1 & g^3 & g & g^2 & g & 0 & g^2\\
1 & 1 & g^3 & g & g & 1 & g^2 & g^3 & g^2 & 0
\end{pmatrix}$$

$$\begin{pmatrix}
0&1&1&1&1&1&1&1&1&1\\
1&0&0&0&0&1&0&0&0&1\\
1&0&0&1&0&0&0&0&1&0\\
1&0&1&0&0&0&0&0&1&0\\
1&0&0&0&0&0&1&1&0&0\\
1&1&0&0&0&0&0&0&0&1\\
1&0&0&0&1&0&0&1&0&0\\
1&0&0&0&1&0&1&0&0&0\\
1&0&1&1&0&0&0&0&0&0\\
1&1&0&0&0&1&0&0&0&0
\end{pmatrix}
+ g \begin{pmatrix}
0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&1&0&0&0\\
0&1&0&0&0&0&1&0&0&0\\
0&0&0&0&1&0&0&0&0&1\\
0&0&0&1&0&0&0&0&0&1\\
0&0&0&0&0&0&0&1&1&0\\
0&1&1&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&1&0\\
0&0&0&0&0&1&0&1&0&0\\
0&0&0&1&1&0&0&0&0&0
\end{pmatrix}
+ g^2 \begin{pmatrix}
0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&1&0&0\\
0&0&0&0&1&1&0&0&0&0\\
0&1&0&0&0&0&0&1&0&0\\
0&0&1&0&0&1&0&0&0&0\\
0&0&1&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&1\\
0&1&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&1\\
0&0&0&0&0&0&1&0&1&0
\end{pmatrix}
+ g^3 \begin{pmatrix}
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&1&0\\
0&0&0&0&0&0&0&1&0&1\\
0&0&0&0&0&1&1&0&0&0\\
0&1&0&0&0&0&0&0&1&0\\
0&0&0&1&0&0&1&0&0&0\\
0&0&0&1&0&1&0&0&0&0\\
0&0&1&0&0&0&0&0&0&1\\
0&1&0&0&1&0&0&0&0&0\\
0&0&1&0&0&0&0&1&0&0
\end{pmatrix}$$

- Form:

$$\Xi \otimes R_1 = \Xi_1 \otimes R_1 - \Xi_g \otimes R_2 + \Xi_{g^2} \otimes R_1 + \Xi_{g^3} \otimes R_2$$

- Applying $R_1 \mapsto -R_2 \mapsto -R_1 \mapsto R_2 \mapsto R_1$.
- Form D by substituting for the elements of $S_{9,d}$ the rows of the derived part of W_{19}.

- The matrix

$$\begin{pmatrix} \mathbf{0} & \Xi \otimes R_1 \\ 1 & D \end{pmatrix}$$

is a balanced weighing matrix.

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$$\begin{pmatrix} \mathbf{0} & \Xi \otimes R_1 \\ 1 & D \end{pmatrix}$$

is a balanced weighing matrix.

Theorem: (Kharaghani et al., 2022b)

For every $d > 0$, there is a balanced weighing matrix with parameters

$$\left(v = \frac{9(9^{d+1} - 1)}{4}, k = 9^{d+1} \right).$$

Equivalence to Association Schemes

Section Summary

Definition (Commutative) Association Scheme

A (commutative) association scheme is a pair $\mathfrak{X} = (X, \mathscr{A})$ where:

- X is a point set and $\mathscr{A} = \{A_0, \ldots, A_d\}$ a collection of $(0,1)$-matrices indexed by elements of X.
- $A_0 = I$;
- $\displaystyle\sum_{i=0}^{d} A_i = J$;
- $\mathscr{A}$ is closed under transposition;
- there exist non-negative integers p_{ij}^k such that $A_iA_j = \displaystyle\sum_{k=0}^{d} p_{ij}^k A_k$; and
- $A_iA_j = A_jA_i$.

*** $\mathfrak{X}$ is symmetric if $A_i^t = A_i$, for every i.

- $\mathfrak{U} = \langle \mathscr{A} \rangle$ is the adjacency algebra of $\mathfrak{X}$.
- $\mathfrak{U}$ has a dual basis of idempotents $\{E_0 = |X|^{-1}J, \ldots, E_d\}$.
- There are change of basis matrices P and Q such that

$$A_j = \sum_{i=0}^{d} P_{ij} E_i, \qquad E_j = |X|^{-1} \sum_{i=0}^{d} Q_{ij} A_i$$

- Let W be a $BGW(v, k, \lambda; C_2)$.
- Decompose...

$$W = W_1 - W_2,$$

where W_1 and W_2 are disjoint $(0, 1)$-matrices.

- Properties...

- Properties...
  - $(W_1 - W_2)(W_1 - W_2)^t = (W_1 - W_2)^t(W_1 - W_2) = kI$

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  - $(W_1 - W_2)(W_1 - W_2)^t = (W_1 - W_2)^t(W_1 - W_2) = kI$
  - $(W_1 + W_2)(W_1 + W_2)^t = (W_1 + W_2)^t(W_1 + W_2) = kI + \lambda(J - I)$

- Properties...
  - $(W_1 - W_2)(W_1 - W_2)^t = (W_1 - W_2)^t(W_1 - W_2) = kI$
  - $(W_1 + W_2)(W_1 + W_2)^t = (W_1 + W_2)^t(W_1 + W_2) = kI + \lambda(J - I)$
  - $W_1 W_1^t + W_2 W_2^t = W_1^t W_1 + W_2^t W_2 = kI + \frac{\lambda}{2}(J - I)$

- Properties...
  - $(W_1 - W_2)(W_1 - W_2)^t = (W_1 - W_2)^t(W_1 - W_2) = kI$
  - $(W_1 + W_2)(W_1 + W_2)^t = (W_1 + W_2)^t(W_1 + W_2) = kI + \lambda(J - I)$
  - $W_1 W_1^t + W_2 W_2^t = W_1^t W_1 + W_2^t W_2 = kI + \frac{\lambda}{2}(J - I)$
  - $W_1 W_2^t + W_2 W_1^t = W_1^t W_2 + W_2^t W_1 = \frac{\lambda}{2}(J - I)$

- With $P = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, form the $(0,1)$-matrices...

$$A_0 = I_{4v}$$
$$A_1 = \begin{pmatrix} P \otimes I_v & O \\ O & P \otimes I_v \end{pmatrix}$$
$$A_2 = \begin{pmatrix} J_2 \otimes (J_v - I_v) & O \\ O & J_2 \otimes (J_v - I_v) \end{pmatrix}$$
$$A_3 = \begin{pmatrix} O & I_2 \otimes W_1 + P \otimes W_2 \\ I_2 \otimes W_1^t + P \otimes W_2^t & O \end{pmatrix}$$
$$A_4 = \begin{pmatrix} O & I_2 \otimes W_2 + P \otimes W_1 \\ I_2 \otimes W_2^t + P \otimes W_1^t & O \end{pmatrix}$$
$$A_5 = \begin{pmatrix} O & J_2 \otimes (J_v - W_1 - W_2) \\ J_2 \otimes (J_v - W_1^t - W_2^t) & O \end{pmatrix}$$

- Character tables...

$$P = \begin{pmatrix} 1 & 1 & 2(v-1) & k & k & 2(v-k) \\ 1 & -1 & 0 & \sqrt{k} & -\sqrt{k} & 0 \\ 1 & -1 & 0 & -\sqrt{k} & \sqrt{k} & 0 \\ 1 & 1 & 2(v-1) & -k & -k & 2(k-v) \\ 1 & 1 & -2 & -\sqrt{\frac{k(v-k)}{v-1}} & -\sqrt{\frac{k(v-k)}{v-1}} & 2\sqrt{\frac{k(v-k)}{v-1}} \\ 1 & 1 & -2 & \sqrt{\frac{k(v-k)}{v-1}} & \sqrt{\frac{k(v-k)}{v-1}} & -2\sqrt{\frac{k(v-k)}{v-1}} \end{pmatrix}$$

$$Q = \begin{pmatrix} 1 & v & v & 1 & v-1 & v-1 \\ 1 & -v & -v & 1 & v-1 & v-1 \\ 1 & 0 & 0 & 1 & -1 & -1 \\ 1 & \frac{v}{\sqrt{k}} & -\frac{v}{\sqrt{k}} & -1 & -\sqrt{\frac{(v-1)(v-k)}{k}} & \sqrt{\frac{(v-1)(v-k)}{k}} \\ 1 & -\frac{v}{\sqrt{k}} & \frac{v}{\sqrt{k}} & -1 & -\sqrt{\frac{(v-1)(v-k)}{k}} & \sqrt{\frac{(v-1)(v-k)}{k}} \\ 1 & 0 & 0 & -1 & \sqrt{\frac{k(v-1)}{v-k}} & -\sqrt{\frac{k(v-1)}{v-k}} \end{pmatrix}$$

Theorem: Weighing Schemes (Kharaghani et al., 2022b)

There is a commutative scheme with character tables P and Q given above if and only if there is a balanced $W(v, k)$.

Future Work

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