0.1. Definition. Consider the following example (see **?**, §2.5). From a group of individuals, we must choose a number of committees, each of identical size, such that the appearances of each of the individuals among the various committees are equinumerous, as are the appearances of each of the t-subsets of the individuals. To be concrete, given eight individuals, can we arrange them into some number of committees of size four such that each individual is replicated the same number of times, and such that each triple of individuals is replicated precisely once.

This problem is solved by the following configuration where $\{a, b, c, d, e, f, g, h\}$ is our collection of individuals. The groups, or committees, are given by the following.

$$\begin{array}{llll}
\{a,b,e,f\}, & \{c,d,g,h\}, & \{a,c,e,g\}, & \{b,d,f,h\}, \\
\{a,d,e,h\}, & \{b,c,f,g\}, & \{a,b,c,d\}, & \{e,f,g,h\}, \\
\{a,b,g,h\}, & \{c,d,e,f\}, & \{a,c,f,h\}, & \{b,d,e,g\}, \\
\{a,d,f,g\}, & \{b,c,e,h\}. & &
\end{array}$$

This configuration is an example of a so-called t-design. The case that $t = 2$ is the case with which we will concern ourselves. In what follows, we use $\binom{X}{k}$ to denote the k-subsets of a set X.

0.1. Definition. Let X be a set of order v, called the set of varieties; and let $\mathscr{B} \subset \binom{X}{k}$, called the set of blocks, have order b. The ordered pair $\mathbf{D} = (X, \mathscr{B})$ is a *balanced incomplete block design* (henceforth BIBD) if there is a positive integer λ such that each 2-subset of varieties appears in λ blocks of $\mathscr{B}$.

* * *

0.2. Necessary Parametric Conditions. The conditions placed on a finite set and a collection of its subsets in order to form a BIBD are quite strong, and we have at once the following result using simple double counting arguments.

0.2. Proposition. Let $\mathbf{D} = (X, \mathscr{B})$ be a BIBD with $|X| = v$, and $\mathscr{B} \subset \binom{X}{k}$ for which $|\mathscr{B}| = b$. Then:

(0.2.a) Every point of X occurs in $r = \frac{\lambda(v-1)}{k-1}$ blocks, and

(0.2.b) there are $b = \frac{vr}{k} = \frac{\lambda v(v-1)}{k(k-1)}$ blocks in $\mathscr{B}$.

0.3. Corollary. For the parameters v, k, λ of a BIBD, it must hold that

(0.3.a) $\lambda(v-1) \equiv 0 \bmod (k-1)$, and

(0.3.b) $\lambda v(v-1) \equiv 0 \bmod k(k-1)$.

If $\mathbf{D} = (X, \mathscr{B})$ is a BIBD with the parameters shown above, then we denote this property as $\text{BIBD}(v, b, r, k, \lambda)$. As we have seen, however, the parameters b and r are expressible in terms of v, k, and λ; hence, we will usually shorten the denotation to $\text{BIBD}(v, k, \lambda)$ whenever no confusion will arise.

Corollary 0.3 imposes some necessary conditions on the parameters of a BIBD. Our next result, due to **?**, is a strong necessary condition relating the number of points to the number of blocks of a BIBD, and it has far reaching consequences in the applications of designs to fields like statistics.

This most important result admits several interesting derivations employing techniques ranging from determinants (see **?**) to variance counting (see **?**).

0.4. Fisher's Inequality. Let $\mathbf{D} = (X, \mathscr{B})$ be a $\text{BIBD}(v, b, r, k, \lambda)$. It follows that

(0.4.a) $b \geq v$.

The extremal case of Fisher's inequality is naturally very interesting and important. We single this case out thus.

0.5. Definition. Let $\mathbf{D} = (X, \mathscr{B})$ be some $\text{BIBD}(v, b, r, k, \lambda)$. If $v = b$ (equiv. $k = r$), then we say that $\mathbf{D}$ is a *symmetric* balanced incomplete block design, or simply symmetric.

* * *

0.3. Related Configurations.