Orthogoval Notes

Intersection Enumeration

Let $\mathbf{D}_i = (V, B_i)$, $j = 1, 2, \dots, \alpha$, be Desarguesian affine planes of order 2^n on the same point set where the blocks of $\mathbf{D}_i$ are ovals in $\mathbf{D}_j$ for $i \neq j$; and let $\mathbf{D} = \mathbf{D}_1 \cup \mathbf{D}_2 \cup \cdots \cup \mathbf{D}_\alpha$. Then $\mathbf{D}$ is a resolvable 2–design with parameters

$$v = 2^n, \qquad b = \alpha 2^n(2^n + 1), \qquad r = \alpha(2^n + 1), \qquad k = 2^n, \qquad \lambda = \alpha.$$

Label the parallel classes of $\mathbf{D}$ as $\mathscr{C}_1, \mathscr{C}_2, \dots, \mathscr{C}_r$. Fix a block B. Then

$$\left|B \cap B'\right| = \begin{cases} 0 & \text{if } B,\ B' \in \mathscr{C}_i\text{; and} \\ 0,\ 1,\text{ or } 2 & B \in \mathscr{C}_i,\ B' \in \mathscr{C}_j,\ i \neq j. \end{cases}$$

Let n_0, n_1, n_2 be the number of blocks distinct from B which intersect it in 0, 1, or 2 points. The usual variance counting yields the nonsingular triangular system

$$\begin{aligned} n_0 + n_1 + n_2 &= \alpha 2^n(2^n + 1) - 1, \\ n_1 + 2n_2 &= 2^n\Big(\alpha(2^n + 1) - 1\Big), \\ 2n_2 &= 2^n(2^n - 1)(\alpha - 1). \end{aligned}$$

The unique solution is given by

$$\begin{aligned} n_0 &= 2^{n-1}\Big((2^n - 1)(\alpha - 1) + 2\Big) - 1, \\ n_1 &= 2^{n+1}(2^{n-1} + \alpha - 1), \\ n_2 &= 2^{n-1}(2^n - 1)(\alpha - 1). \end{aligned}$$

The number of blocks not in the $\mathscr{C}_i$ containing B that are disjoint to B is then

$$n_0 - 2^n + 1 = n_2.$$

The number of blocks that intersect B in a single point outside of the $\mathbf{D}_j$ containing B is

$$n_1 - 2^{2n} = 2^{n+1}(\alpha - 1).$$