Balanced Weighing Matrices

Generalizations and Related Configurations

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Oct 24, 2023

Summary

1. Preliminaries
2. Novel Construction of Weighing Matrices
3. A New Class of Balanced Weighing Matrices

Preliminaries

Section Summary

1. Preliminaries
   - Weighing Matrices
   - Balanced Incomplete Block Designs
   - Balanced Weighing Matrices and Classical Constructions

Definition. Weighing Matrix

A $v \times v$ $(-1,0,1)$-matrix W such that

$$WW^t = kI_v.$$

Write $W(v,k)$.

- $W(v,v)$ is a *Hadamard matrix*
- $W(v,v-1)$ is a *conference matrix*

- A $W(19,9)$:

$$W = \begin{pmatrix}
0 & + & + & + & + & + & + & + & + & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & - & - & 0 & + & 0 & 0 & 0 & + & - & + & + & 0 & + & 0 & 0 & 0 & + \\
0 & - & 0 & - & 0 & 0 & + & + & 0 & 0 & + & - & + & 0 & 0 & + & + & 0 & 0 \\
0 & - & - & 0 & + & 0 & 0 & 0 & + & 0 & + & + & - & + & 0 & 0 & 0 & + & 0 \\
0 & 0 & 0 & + & 0 & - & - & 0 & + & 0 & 0 & 0 & + & - & + & + & 0 & + & 0 \\
0 & + & 0 & 0 & - & 0 & - & 0 & 0 & + & + & 0 & 0 & + & - & + & 0 & 0 & + \\
0 & 0 & + & 0 & - & - & 0 & + & 0 & 0 & 0 & + & 0 & + & + & - & + & 0 & 0 \\
0 & 0 & + & 0 & 0 & 0 & + & 0 & - & - & 0 & + & 0 & 0 & 0 & + & - & + & + \\
0 & 0 & 0 & + & + & 0 & 0 & - & 0 & - & 0 & 0 & + & + & 0 & 0 & + & - & + \\
0 & + & 0 & 0 & 0 & + & 0 & - & - & 0 & + & 0 & 0 & 0 & + & 0 & + & + & - \\
+ & 0 & 0 & 0 & + & 0 & - & + & - & 0 & 0 & - & - & 0 & + & 0 & 0 & 0 & + \\
+ & 0 & 0 & 0 & - & + & 0 & 0 & + & - & - & 0 & - & 0 & 0 & + & + & 0 & 0 \\
+ & 0 & 0 & 0 & 0 & - & + & - & 0 & + & - & - & 0 & + & 0 & 0 & 0 & + & 0 \\
+ & + & - & 0 & 0 & 0 & 0 & + & 0 & - & 0 & 0 & + & 0 & - & - & 0 & + & 0 \\
+ & 0 & + & - & 0 & 0 & 0 & - & + & 0 & + & 0 & 0 & - & 0 & - & 0 & 0 & + \\
+ & - & 0 & + & 0 & 0 & 0 & 0 & - & + & 0 & + & 0 & - & - & 0 & + & 0 & 0 \\
+ & + & 0 & - & + & - & 0 & 0 & 0 & 0 & 0 & + & 0 & 0 & 0 & + & 0 & - & - \\
+ & - & + & 0 & 0 & + & - & 0 & 0 & 0 & 0 & 0 & + & + & 0 & 0 & - & 0 & - \\
+ & 0 & - & + & - & 0 & + & 0 & 0 & 0 & + & 0 & 0 & 0 & + & 0 & - & - & 0
\end{pmatrix}$$

- Verify W is a weighing matrix:

$$WW^t = \begin{pmatrix}
\mathbf{9} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \mathbf{9} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mathbf{9} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \mathbf{9} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \mathbf{9} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \mathbf{9} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \mathbf{9} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{9} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{9} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{9} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{9} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{9} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{9} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{9} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{9} & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{9} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{9} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{9}
\end{pmatrix}$$

- A Hadamard matrix $W(16,16)$:

$$\begin{pmatrix}
+&+&+&+&+&+&+&+&+&+&+&+&+&+&+&+\\
+&-&+&-&+&-&+&-&+&-&+&-&+&-&+&-\\
+&+&-&-&+&+&-&-&+&+&-&-&+&+&-&-\\
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\end{pmatrix}$$

- A $W(2^n,2^n)$ as the character table for elementary abelian 2-group.

Theorem. Necessary conditions for existence

If a $W(v,k)$ exists, then

- v odd implies k a perfect square and $(v-k)^2-(v-k)\geq v-1$,
- $v\equiv 2 \pmod 4$ implies k a sum of two squares, and
- $v=k$ implies $v=1$, 2, or $v\equiv 0 \pmod 4$.

Theorem. Necessary conditions not necessarily sufficient

There does not exist a $W(2v+1,v)$ for any $v>2$.

Conjecture. Existence of Hadamard matrices

A $W(4v,4v)$ exists for every $v>1$.

Definition: Balanced Incomplete Block Design

- A binary $v \times b$ $(0,1)$-matrix A such that:
  1. $AA^t = rI_v + \lambda(J_v - I_v)$, and
  2. $J_v A = kJ_v$.

  Write 2-(v, k, λ)-*design*.
- The design is symmetric if $v = b$ (equiv. $k = r$).

- A symmetric 2-$(7,4,2)$-*design*:

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

- Verify A is a symmetric design:

$$AA^t = \begin{pmatrix} 4 & 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 4 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 4 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 4 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 4 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 4 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 4 \end{pmatrix}$$

Theorem. Necessary conditions for existence

A symmetric 2-(v, k, λ)-*design* exists only if

- $k - \lambda$ a perfect square whenever v is even, and
- the equation
$$x^2 = (k - \lambda)y^2 + (-1)^{(v-1)/2}\lambda z^2$$
has nontrivial integer solutions whenever v is odd.

Example. Necessary conditions not sufficient

A projective plane of order 10 (particular parameter family of symmetric BIBDs) is not ruled out by the Theorem (take $(x, y, z) = (3, 1, 1)$). It is known not to exist, however.

- The matrix is circulant:

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

- Cyclic group of automorphisms acting regularly on points and blocks of corresponding incidence structure.

- Index rows/columns by $H = \langle a : a^7 = 1 \rangle$:

$$A = \begin{array}{c} \\ 1 \\ a \\ a^2 \\ a^3 \\ a^4 \\ a^5 \\ a^6 \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & a & a^2 & a^3 & a^4 & a^5 & a^6 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix} \end{array}$$

- Take $A_{a^i, a^j} = A_{a^{i-j}, 1}$.
- 1st column is a characteristic vector for $D = \{1, a, a^2, a^4\} \subset H$.
- ith column is a characteristic vector of $D \cdot a^i$.

Definition. Difference Sets

A difference set (v, k, λ)-DS is a k-subset $D \subseteq G$ of finite group G of order v such that every nonidentity element of G appears λ times in the multiset $\{dd^{-1} : d \in D\}$ of differences (quotients) of elements of D.

Theorem. Difference sets and symmetric BIBDs

A symmetric 2-(v, k, λ)-*design* admits a regular group G of automorphisms if and only if the blocks of the design can be identified with the development (translates) of a (v, k, λ)-DS difference set in G.

- Consider the quotients amongst $D = \{1, a, a^2, a^4\} \subset H$:

	1	a	a^2	a^4
1	1	a^6	a^5	a^3
a	a	1	a^6	a^4
a^2	a^2	a	1	a^5
a^4	a^4	a^3	a^2	1

- Consider the quotients amongst $D = \{1, a, a^2, a^4\} \subset H$:

	1	a	a^2	a^4
1	1	a^6	a^5	a^3
a	a	1	a^6	a^4
a^2	a^2	a	1	a^5
a^4	a^4	a^3	a^2	1

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	1	a	a^2	a^4
1	1	a^6	a^5	a^3
a	a	1	a^6	a^4
a^2	a^2	a	1	a^5
a^4	a^4	a^3	a^2	1

Definition. Balanced Weighing Matrices

- If W is a $W(v,k)$, then W is balanced if $|W|$ is the incidence matrix of a symmetric 2-(v,k,λ)-*design*, $\lambda = k(k-1)/(v-1)$.
- Write $BW(v,k)$.

- Our example $W(19,9)$ is a $BW(19,9)$:

$$W = \begin{pmatrix}
0 & + & + & + & + & + & + & + & + & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & - & - & 0 & + & 0 & 0 & 0 & + & - & + & + & 0 & + & 0 & 0 & 0 & + \\
0 & - & 0 & - & 0 & 0 & + & + & 0 & 0 & + & - & + & 0 & 0 & + & + & 0 & 0 \\
0 & - & - & 0 & + & 0 & 0 & 0 & + & 0 & + & + & - & + & 0 & 0 & 0 & + & 0 \\
0 & 0 & 0 & + & 0 & - & - & 0 & + & 0 & 0 & 0 & + & - & + & + & 0 & + & 0 \\
0 & + & 0 & 0 & - & 0 & - & 0 & 0 & + & + & 0 & 0 & + & - & + & 0 & 0 & + \\
0 & 0 & + & 0 & - & - & 0 & + & 0 & 0 & 0 & + & 0 & + & + & - & + & 0 & 0 \\
0 & 0 & + & 0 & 0 & 0 & + & 0 & - & - & 0 & + & 0 & 0 & 0 & + & - & + & + \\
0 & 0 & 0 & + & + & 0 & 0 & - & 0 & - & 0 & 0 & + & + & 0 & 0 & + & - & + \\
0 & + & 0 & 0 & 0 & + & 0 & - & - & 0 & + & 0 & 0 & 0 & + & 0 & + & + & - \\
+ & 0 & 0 & 0 & + & 0 & - & + & - & 0 & 0 & - & - & 0 & + & 0 & 0 & 0 & + \\
+ & 0 & 0 & 0 & - & + & 0 & 0 & + & - & - & 0 & - & 0 & 0 & + & + & 0 & 0 \\
+ & 0 & 0 & 0 & 0 & - & + & - & 0 & + & - & - & 0 & + & 0 & 0 & 0 & + & 0 \\
+ & + & - & 0 & 0 & 0 & 0 & + & 0 & - & 0 & 0 & + & 0 & - & - & 0 & + & 0 \\
+ & 0 & + & - & 0 & 0 & 0 & - & + & 0 & + & 0 & 0 & - & 0 & - & 0 & 0 & + \\
+ & - & 0 & + & 0 & 0 & 0 & 0 & - & + & 0 & + & 0 & - & - & 0 & + & 0 & 0 \\
+ & + & 0 & - & + & - & 0 & 0 & 0 & 0 & 0 & + & 0 & 0 & 0 & + & 0 & - & - \\
+ & - & + & 0 & 0 & + & - & 0 & 0 & 0 & 0 & 0 & + & + & 0 & 0 & - & 0 & - \\
+ & 0 & - & + & - & 0 & + & 0 & 0 & 0 & + & 0 & 0 & 0 & + & 0 & - & - & 0
\end{pmatrix}$$

- A $BW(7,4)$:

$$B = \begin{pmatrix} - & 0 & 0 & + & 0 & + & + \\ + & - & 0 & 0 & + & 0 & + \\ + & + & - & 0 & 0 & + & 0 \\ 0 & + & + & - & 0 & 0 & + \\ + & 0 & + & + & - & 0 & 0 \\ 0 & + & 0 & + & + & - & 0 \\ 0 & 0 & + & 0 & + & + & - \end{pmatrix}$$

- The absolute value matrix is A.

- The $BW(7,4)$ B is constructed from a relative difference set.
- The $BW(19,9)$ W is not constructable from an RDS. Computationally found by Gibbons and Mathon (1987).

Definition. Relative Difference Sets

A relative difference set (m, n, k, λ)-*RDS* of a group G of order mn relative to a subgroup N of order n is a k-subset $R \subseteq G$ such that every nonidentity element of $G \backslash N$ appears λ times in the multiset $\{rr^{-1} : r \in R\}$ of differences (quotients) of elements of R.

- An RDS in a group G corresponds to a point regular automorphism group of a square divisible design isomorphic to G.

- Let $G = \langle a, b : a^7 = b^2 = 1 \rangle \cong C_{14}$.
- $R = \{b, a, a^2, a^4\}$ is an $(8, 2, 4, 1)$-RDS is G relative to $N = \{1, b\}$.

	b	a	a^2	a^4
b	1	ba^6	ba^5	ba^3
a	ab	1	a^6	a^4
a^2	a^2b	a	1	a^5
a^4	a^4b	a^3	a^2	1

- b does not appear!

- Let $G = \langle a, b : a^7 = b^2 = 1 \rangle \cong C_{14}$.
- $R = \{b, a, a^2, a^4\}$ is an $(8, 2, 4, 1)$-*RDS* is G relative to $N = \{1, b\}$.

	b	a	a^2	a^4
b	1	ba^6	ba^5	ba^3
a	ab	1	a^6	a^4
a^2	a^2b	a	1	a^5
a^4	a^4b	a^3	a^2	1

- b does not appear!

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- $R = \{b, a, a^2, a^4\}$ is an $(8, 2, 4, 1)$-RDS is G relative to $N = \{1, b\}$.

	b	a	a^2	a^4
b	1	ba^6	ba^5	ba^3
a	ab	1	a^6	a^4
a^2	a^2b	a	1	a^5
a^4	a^4b	a^3	a^2	1

- b does not appear!

- Form $a^i N$ and $a^i R$ for each $a^i \in \langle a \rangle$.
- Each $a^i N \cap a^j R = \emptyset$, $\{a^n\}$, or $\{a^m b\}$.
- Construct the 7×7 matrix B by

$$B_{a^i,a^j} = \begin{cases} 0 & \text{if } a^i N \cap a^j R = \emptyset, \\ 1 & \text{if } a^i N \cap a^j R = \{a^n\}, \text{ and} \\ b & \text{if } a^i N \cap a^j R = \{a^m b\}. \end{cases}$$

- General construction due to Jungnickel (1982).

- Using the methods above, the following is known.

Theorem. RDS construction of BWs

There is a BW with parameters

$$\left(\frac{q^{d+1}-1}{q-1}, q^d\right)$$

whenever (1) q odd and d arbitrary and (2) q and d even.

(1) Nonlinear hyperplanes of $GF(q^{d+1}) : GF(q)$ due to Bose (1942).
(2) Lifting of a "Waterloo decomposition" of classical difference sets due to Arasu, et al. (1995).

Novel Construction of Weighing Matrices

Section Summary

2. Novel Construction of Weighing Matrices
   - Ingredients
   - Recipie

- Equivalencies of weighing matrices (and BWs):
  - permutations of rows
  - permutations of columns
  - negation of rows
  - negation of columns
- Every weighing matrix is equivalent to one of the following form

$$\begin{pmatrix} \mathbf{0} & R \\ \mathbf{1} & D \end{pmatrix}.$$

- R is the residual-part.
- D is the derived-part.

Definition. Simplex Code

- q a prime power and $d > 0$.
- Form matrix G with columns given by reps. of 1-D subspaces of $GF(q^{d+1})$.
- The simplex code is $\mathcal{S}_{q,d} = row(G)$.

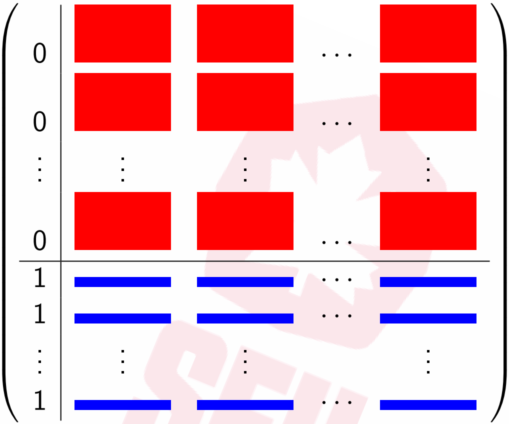
Proposition. Properties

For $\mathcal{S}_{q,d}$:

- $wt(x) = q^d$ for all $x \in \mathcal{S}_{q,d}/\{\mathbf{0}\}$, and
- $dist(x,y) = q^d$ for all $x, y \in \mathcal{S}_{q,d}$ and $x \neq y$

- Ingredients of unifying construction:
  - A normalized $W(v,q)$ (seed matrix) with residual-part R and derived-part D.
  - A $W((q^{d+1}-1)/(q-1), q^d)$, say W.
  - A simplex code $\mathcal{S}_{q,d}$.

- Recipie of unifying construction:
  - Form $A = W \otimes R$.
  - Form B by replacing elements of $\mathcal{S}_{q,d}$ by rows of D.
  - Then
$$\begin{pmatrix} \mathbf{0} & A \\ \mathbf{1} & B \end{pmatrix}$$
    is a $W((v-1)(q^{d+1}-1)/(q-1)+1, q^{d+1})$.


0
0
0
1
1
1

- A seed $W(8,5)$

$$\begin{pmatrix} \mathbf{0} & R \\ \mathbf{1} & D \end{pmatrix} = \left(\begin{array}{c|ccccccc} 0 & + & 0 & 0 & + & + & + & + \\ 0 & 0 & + & 0 & + & - & - & + \\ 0 & 0 & 0 & + & + & - & + & - \\ \hline + & 0 & 0 & 0 & + & + & - & - \\ + & + & + & + & - & 0 & 0 & 0 \\ + & + & - & - & 0 & - & 0 & 0 \\ + & - & + & - & 0 & 0 & + & 0 \\ + & - & - & + & 0 & 0 & 0 & + \end{array}\right)$$

- A classical parameter $W(6,5)$

$$W = \begin{pmatrix} - & + & - & 0 & + & + \\ - & - & + & - & 0 & + \\ - & - & - & + & - & 0 \\ 0 & - & - & - & + & - \\ + & 0 & - & - & - & + \\ - & + & 0 & - & - & - \end{pmatrix}.$$

- The simplex code $\mathcal{S}_{5,1}$ (transposed)

$$\begin{pmatrix} 1&2&3&4&0&1&2&3&4&0&1&2&3&4&0&1&2&3&4&0&1&2&3&4&0 \\ 1&1&1&1&1&2&2&2&2&2&3&3&3&3&3&4&4&4&4&4&0&0&0&0&0 \\ 2&4&0&3&1&4&3&1&0&2&0&1&4&2&3&3&0&2&1&4&1&2&3&4&0 \\ 4&0&3&2&1&3&1&0&4&2&1&4&2&0&3&0&2&1&3&4&2&3&4&1&0 \\ 0&3&2&4&1&1&0&4&3&2&4&2&0&1&3&2&1&3&0&4&3&4&1&2&0 \\ 3&2&4&0&1&0&4&3&1&2&2&0&1&4&3&1&3&0&2&4&4&1&2&3&0 \end{pmatrix}$$

- Take $A = W \otimes R$.
- Take B to be the matrix formed after replacing the entries of $S_{5,1}$ by the rows of D.
- Then

$$\begin{pmatrix} \mathbf{0} & A \\ \mathbf{1} & B \end{pmatrix}$$

is a $W(43, 25)$.

$$
\left(\begin{array}{c|*{42}{c}}
0 & - & 0 & 0 & - & - & - & - & + & 0 & 0 & + & + & + & + & - & 0 & 0 & - & - & - & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & 0 & 0 & + & + & + & + & + & 0 & 0 & + & + & + & + \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & - & 0 & 0 & - & - & - & - & - & 0 & 0 & - & - & - & - & - & 0 & 0 & - & - & - & - & + & 0 & 0 & + & + & + & + & - & 0 & 0 & - & - & - & - \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & - & 0 & - & + & + & - & 0 & - & 0 & - & + & + & - & 0 & - & 0 & - & + & + & - & 0 & + & 0 & + & - & - & + & 0 & - & 0 & - & + & + & - \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & - & - & + & - & + & 0 & 0 & - & - & + & - & + & 0 & 0 & - & - & + & - & + & 0 & 0 & + & + & - & + & - & 0 & 0 & - & - & + & - & + \\
0 & + & 0 & 0 & + & + & + & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & - & 0 & 0 & - & - & - & - & - & 0 & 0 & - & - & - & - & - & 0 & 0 & - & - & - & - & + & 0 & 0 & + & + & + & + \\
0 & 0 & + & 0 & + & - & - & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & - & 0 & - & + & + & - & 0 & - & 0 & - & + & + & - & 0 & - & 0 & - & + & + & - & 0 & + & 0 & + & - & - & + \\
0 & 0 & 0 & + & + & - & + & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & - & - & + & - & + & 0 & 0 & - & - & + & - & + & 0 & 0 & - & - & + & - & + & 0 & 0 & + & + & - & + & - \\
0 & - & 0 & 0 & - & - & - & - & + & 0 & 0 & + & + & + & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & - & 0 & 0 & - & - & - & - & - & 0 & 0 & - & - & - & - & - & 0 & 0 & - & - & - & - \\
0 & 0 & - & 0 & - & + & + & - & 0 & + & 0 & + & - & - & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & - & 0 & - & + & + & - & 0 & - & 0 & - & + & + & - & 0 & - & 0 & - & + & + & - \\
0 & 0 & 0 & - & - & + & - & + & 0 & 0 & + & + & - & + & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & - & - & + & - & + & 0 & 0 & - & - & + & - & + & 0 & 0 & - & - & + & - & + \\
\hline
+ & + & + & + & - & 0 & 0 & 0 & + & + & + & - & 0 & 0 & 0 & + & - & - & 0 & - & 0 & 0 & - & - & + & 0 & 0 & 0 & + & 0 & 0 & 0 & + & + & - & - & - & + & - & 0 & 0 & + & 0 \\
+ & + & - & - & 0 & - & 0 & 0 & + & + & + & - & 0 & 0 & 0 & - & - & + & 0 & 0 & 0 & + & 0 & 0 & 0 & + & + & - & - & - & + & - & 0 & 0 & + & 0 & + & - & - & 0 & - & 0 & 0 \\
+ & - & + & - & 0 & 0 & + & 0 & + & + & + & - & 0 & 0 & 0 & 0 & 0 & 0 & + & + & - & - & - & + & - & 0 & 0 & + & 0 & + & - & - & 0 & - & 0 & 0 & - & - & + & 0 & 0 & 0 & + \\
+ & - & - & + & 0 & 0 & 0 & + & + & + & + & - & 0 & 0 & 0 & - & + & - & 0 & 0 & + & 0 & + & - & - & 0 & - & 0 & 0 & - & - & + & 0 & 0 & 0 & + & 0 & 0 & 0 & + & + & - & - \\
+ & 0 & 0 & 0 & + & + & - & - & + & + & + & - & 0 & 0 & 0 & + & + & + & - & 0 & 0 & 0 & + & + & + & - & 0 & 0 & 0 & + & + & + & - & 0 & 0 & 0 & + & + & + & - & 0 & 0 & 0 \\
+ & + & + & + & - & 0 & 0 & 0 & + & - & - & 0 & - & 0 & 0 & - & - & + & 0 & 0 & 0 & + & - & + & - & 0 & 0 & + & 0 & + & + & + & - & 0 & 0 & 0 & 0 & 0 & 0 & + & + & - & - \\
+ & + & - & - & 0 & - & 0 & 0 & + & - & - & 0 & - & 0 & 0 & - & + & - & 0 & 0 & + & 0 & + & + & + & - & 0 & 0 & 0 & 0 & 0 & 0 & + & + & - & - & - & - & + & 0 & 0 & 0 & + \\
+ & - & + & - & 0 & 0 & + & 0 & + & - & - & 0 & - & 0 & 0 & + & + & + & - & 0 & 0 & 0 & 0 & 0 & 0 & + & + & - & - & - & - & + & 0 & 0 & 0 & + & - & + & - & 0 & 0 & + & 0 \\
+ & - & - & + & 0 & 0 & 0 & + & + & - & - & 0 & - & 0 & 0 & 0 & 0 & 0 & + & + & - & - & - & - & + & 0 & 0 & 0 & + & - & + & - & 0 & 0 & + & 0 & + & + & + & - & 0 & 0 & 0 \\
+ & 0 & 0 & 0 & + & + & - & - & + & - & - & 0 & - & 0 & 0 & + & - & - & 0 & - & 0 & 0 & + & - & - & 0 & - & 0 & 0 & + & - & - & 0 & - & 0 & 0 & + & - & - & 0 & - & 0 & 0 \\
+ & + & + & + & - & 0 & 0 & 0 & - & + & - & 0 & 0 & + & 0 & 0 & 0 & 0 & + & + & - & - & + & + & + & - & 0 & 0 & 0 & - & - & + & 0 & 0 & 0 & + & + & - & - & 0 & - & 0 & 0 \\
+ & + & - & - & 0 & - & 0 & 0 & - & + & - & 0 & 0 & + & 0 & + & + & + & - & 0 & 0 & 0 & - & - & + & 0 & 0 & 0 & + & + & - & - & 0 & - & 0 & 0 & 0 & 0 & 0 & + & + & - & - \\
+ & - & + & - & 0 & 0 & + & 0 & - & + & - & 0 & 0 & + & 0 & - & - & + & 0 & 0 & 0 & + & + & - & - & 0 & - & 0 & 0 & 0 & 0 & 0 & + & + & - & - & + & + & + & - & 0 & 0 & 0 \\
+ & - & - & + & 0 & 0 & 0 & + & - & + & - & 0 & 0 & + & 0 & + & - & - & 0 & - & 0 & 0 & 0 & 0 & 0 & + & + & - & - & + & + & + & - & 0 & 0 & 0 & - & - & + & 0 & 0 & 0 & + \\
+ & 0 & 0 & 0 & + & + & - & - & - & + & - & 0 & 0 & + & 0 & - & + & - & 0 & 0 & + & 0 & - & + & - & 0 & 0 & + & 0 & - & + & - & 0 & 0 & + & 0 & - & + & - & 0 & 0 & + & 0 \\
+ & + & + & + & - & 0 & 0 & 0 & - & - & + & 0 & 0 & 0 & + & - & + & - & 0 & 0 & + & 0 & 0 & 0 & 0 & + & + & - & - & + & - & - & 0 & - & 0 & 0 & + & + & + & - & 0 & 0 & 0 \\
+ & + & - & - & 0 & - & 0 & 0 & - & - & + & 0 & 0 & 0 & + & 0 & 0 & 0 & + & + & - & - & + & - & - & 0 & - & 0 & 0 & + & + & + & - & 0 & 0 & 0 & - & + & - & 0 & 0 & + & 0 \\
+ & - & + & - & 0 & 0 & + & 0 & - & - & + & 0 & 0 & 0 & + & + & - & - & 0 & - & 0 & 0 & + & + & + & - & 0 & 0 & 0 & - & + & - & 0 & 0 & + & 0 & 0 & 0 & 0 & + & + & - & - \\
+ & - & - & + & 0 & 0 & 0 & + & - & - & + & 0 & 0 & 0 & + & + & + & + & - & 0 & 0 & 0 & - & + & - & 0 & 0 & + & 0 & 0 & 0 & 0 & + & + & - & - & + & - & - & 0 & - & 0 & 0 \\
+ & 0 & 0 & 0 & + & + & - & - & - & - & + & 0 & 0 & 0 & + & - & - & + & 0 & 0 & 0 & + & - & - & + & 0 & 0 & 0 & + & - & - & + & 0 & 0 & 0 & + & - & - & + & 0 & 0 & 0 & + \\
+ & + & + & + & - & 0 & 0 & 0 & 0 & 0 & 0 & + & + & - & - & + & + & + & - & 0 & 0 & 0 & + & - & - & 0 & - & 0 & 0 & - & + & - & 0 & 0 & + & 0 & - & - & + & 0 & 0 & 0 & + \\
+ & + & - & - & 0 & - & 0 & 0 & 0 & 0 & 0 & + & + & - & - & + & - & - & 0 & - & 0 & 0 & - & + & - & 0 & 0 & + & 0 & - & - & + & 0 & 0 & 0 & + & + & + & + & - & 0 & 0 & 0 \\
+ & - & + & - & 0 & 0 & + & 0 & 0 & 0 & 0 & + & + & - & - & - & + & - & 0 & 0 & + & 0 & - & - & + & 0 & 0 & 0 & + & + & + & + & - & 0 & 0 & 0 & + & - & - & 0 & - & 0 & 0 \\
+ & - & - & + & 0 & 0 & 0 & + & 0 & 0 & 0 & + & + & - & - & - & - & + & 0 & 0 & 0 & + & + & + & + & - & 0 & 0 & 0 & + & - & - & 0 & - & 0 & 0 & - & + & - & 0 & 0 & + & 0 \\
+ & 0 & 0 & 0 & + & + & - & - & 0 & 0 & 0 & + & + & - & - & 0 & 0 & 0 & + & + & - & - & 0 & 0 & 0 & + & + & - & - & 0 & 0 & 0 & + & + & - & - & 0 & 0 & 0 & + & + & - & -
\end{array}\right)
$$

Theorem. (Kharaghani, et al., 2022b)

If there is a $W(v, q)$, then there is a weighing matrix with parameters

$$\left(\frac{(v-1)(q^{d+1}-1)}{q-1}+1, q^{d+1}\right)$$

whenever:

1. q is odd and every $d > 0$, and
2. q and d are both even.

Seed (v, k)	Succident (v', k')	Seed (v, k)	Succident (v', k')
$(6, 5)$:	$(31, 25)$, $(156, 125)$, $(781, 625)$	$(16, 3)$:	$(69, 9)$, $(196, 27)$, $(601, 81)$
$(8, 5)$:	$(43, 25)$, $(218, 125)$	$(16, 5)$:	$(91, 25)$, $(466, 125)$
$(8, 7)$:	$(57, 49)$, $(400, 343)$	$(16, 7)$:	$(121, 49)$, $(856, 343)$
$(10, 5)$:	$(55, 25)$, $(280, 125)$	$(16, 9)$:	$(151, 81)$
$(10, 9)$:	$(91, 81)$, $(820, 729)$	$(16, 11)$:	$(181, 121)$
$(12, 5)$:	$(67, 25)$, $(342, 125)$	$(16, 13)$:	$(211, 169)$
$(12, 7)$:	$(89, 49)$, $(628, 343)$	$(18, 13)$:	$(239, 169)$
$(12, 9)$:	$(111, 81)$	$(19, 9)$:	$(181, 81)$
$(13, 9)$:	$(121, 81)$	$(20, 7)$:	$(153, 49)$
$(14, 9)$:	$(131, 81)$	$(20, 13)$:	$(267, 169)$
$(14, 13)$:	$(183, 169)$		

A New Class of BWs

Section Summary

3. A New Class of Balanced Weighing Matrices
   - Seed Matrix
   - Construction

- Our example $BW(19,9)$:

$$
W = \left(\begin{array}{c|cccccccccccccccccc}
0 & + & + & + & + & + & + & + & + & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & - & - & 0 & + & 0 & 0 & 0 & + & - & + & + & 0 & + & 0 & 0 & 0 & + \\
0 & - & 0 & - & 0 & 0 & + & + & 0 & 0 & + & - & + & 0 & 0 & + & + & 0 & 0 \\
0 & - & - & 0 & + & 0 & 0 & 0 & + & 0 & + & + & - & + & 0 & 0 & 0 & + & 0 \\
0 & 0 & 0 & + & 0 & - & - & 0 & + & 0 & 0 & 0 & + & - & + & + & 0 & + & 0 \\
0 & + & 0 & 0 & - & 0 & - & 0 & 0 & + & + & 0 & 0 & + & - & + & 0 & 0 & + \\
0 & 0 & + & 0 & - & - & 0 & + & 0 & 0 & 0 & + & 0 & + & + & - & + & 0 & 0 \\
0 & 0 & + & 0 & 0 & 0 & + & 0 & - & - & 0 & + & 0 & 0 & 0 & + & - & + & + \\
0 & 0 & 0 & + & + & 0 & 0 & - & 0 & - & 0 & 0 & + & + & 0 & 0 & + & - & + \\
0 & + & 0 & 0 & 0 & + & 0 & - & - & 0 & + & 0 & 0 & 0 & + & 0 & + & + & - \\
\hline
+ & 0 & 0 & 0 & + & 0 & - & + & - & 0 & 0 & - & - & 0 & + & 0 & 0 & 0 & + \\
+ & 0 & 0 & 0 & - & + & 0 & 0 & + & - & - & 0 & - & 0 & 0 & + & + & 0 & 0 \\
+ & 0 & 0 & 0 & 0 & - & + & - & 0 & + & - & - & 0 & + & 0 & 0 & 0 & + & 0 \\
+ & + & - & 0 & 0 & 0 & 0 & + & 0 & - & 0 & 0 & + & 0 & - & - & 0 & + & 0 \\
+ & 0 & + & - & 0 & 0 & 0 & - & + & 0 & + & 0 & 0 & - & 0 & - & 0 & 0 & + \\
+ & - & 0 & + & 0 & 0 & 0 & 0 & - & + & 0 & + & 0 & - & - & 0 & + & 0 & 0 \\
+ & + & 0 & - & + & - & 0 & 0 & 0 & 0 & 0 & + & 0 & 0 & 0 & + & 0 & - & - \\
+ & - & + & 0 & 0 & + & - & 0 & 0 & 0 & 0 & 0 & + & + & 0 & 0 & - & 0 & - \\
+ & 0 & - & + & - & 0 & + & 0 & 0 & 0 & + & 0 & 0 & 0 & + & 0 & - & - & 0
\end{array}\right)
$$

$$R_1 = \begin{pmatrix}
+ & + & + & + & + & + & + & + & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & - & - & 0 & + & 0 & 0 & 0 & + & - & + & + & 0 & + & 0 & 0 & 0 & + \\
- & 0 & - & 0 & 0 & + & + & 0 & 0 & + & - & + & 0 & 0 & + & + & 0 & 0 \\
- & - & 0 & + & 0 & 0 & 0 & + & 0 & + & + & - & + & 0 & 0 & 0 & + & 0 \\
0 & 0 & + & 0 & - & - & 0 & + & 0 & 0 & 0 & + & - & + & + & 0 & + & 0 \\
+ & 0 & 0 & - & 0 & - & 0 & 0 & + & + & 0 & 0 & + & - & + & 0 & 0 & + \\
0 & + & 0 & - & - & 0 & + & 0 & 0 & 0 & + & 0 & + & + & - & + & 0 & 0 \\
0 & + & 0 & 0 & 0 & + & 0 & - & - & 0 & + & 0 & 0 & 0 & + & - & + & + \\
0 & 0 & + & + & 0 & 0 & - & 0 & - & 0 & 0 & + & + & 0 & 0 & + & - & + \\
+ & 0 & 0 & 0 & + & 0 & - & - & 0 & + & 0 & 0 & 0 & + & 0 & + & + & -
\end{pmatrix}$$

$$|R_1| = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1
\end{pmatrix}$$

$$|R_2| = J - |R_1| = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$$
R_2 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & + & + & + & + & + & + & + & + \\
- & 0 & 0 & + & 0 & + & + & + & 0 & 0 & 0 & 0 & + & 0 & - & + & - & 0 \\
0 & - & 0 & + & + & 0 & 0 & + & + & 0 & 0 & 0 & - & + & 0 & 0 & + & - \\
0 & 0 & - & 0 & + & + & + & 0 & + & 0 & 0 & 0 & 0 & - & + & - & 0 & + \\
+ & + & 0 & - & 0 & 0 & + & 0 & + & + & - & 0 & 0 & 0 & 0 & + & 0 & - \\
0 & + & + & 0 & - & 0 & + & + & 0 & 0 & + & - & 0 & 0 & 0 & - & + & 0 \\
+ & 0 & + & 0 & 0 & - & 0 & + & + & - & 0 & + & 0 & 0 & 0 & 0 & - & + \\
+ & 0 & + & + & + & 0 & - & 0 & 0 & + & 0 & - & + & - & 0 & 0 & 0 & 0 \\
+ & + & 0 & 0 & + & + & 0 & - & 0 & - & + & 0 & 0 & + & - & 0 & 0 & 0 \\
0 & + & + & + & 0 & + & 0 & 0 & - & 0 & - & + & - & 0 & + & 0 & 0 & 0
\end{pmatrix}
$$

$$
W' = \left(\begin{array}{c|cccccccccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & + & + & + & + & + & + & + & + \\
0 & - & 0 & 0 & + & 0 & + & + & + & 0 & 0 & 0 & 0 & + & 0 & - & + & - & 0 \\
0 & 0 & - & 0 & + & + & 0 & 0 & + & + & 0 & 0 & 0 & - & + & 0 & 0 & + & - \\
0 & 0 & 0 & - & 0 & + & + & + & 0 & + & 0 & 0 & 0 & 0 & - & + & - & 0 & + \\
0 & + & + & 0 & - & 0 & 0 & + & 0 & + & + & - & 0 & 0 & 0 & 0 & + & 0 & - \\
0 & 0 & + & + & 0 & - & 0 & + & + & 0 & 0 & + & - & 0 & 0 & 0 & - & + & 0 \\
0 & + & 0 & + & 0 & 0 & - & 0 & + & + & - & 0 & + & 0 & 0 & 0 & 0 & - & + \\
0 & + & 0 & + & + & + & 0 & - & 0 & 0 & + & 0 & - & + & - & 0 & 0 & 0 & 0 \\
0 & + & + & 0 & 0 & + & + & 0 & - & 0 & - & + & 0 & 0 & + & - & 0 & 0 & 0 \\
0 & 0 & + & + & + & 0 & + & 0 & 0 & - & 0 & - & + & - & 0 & + & 0 & 0 & 0 \\
\hline
+ & 0 & 0 & 0 & + & 0 & - & + & - & 0 & 0 & - & - & 0 & + & 0 & 0 & 0 & + \\
+ & 0 & 0 & 0 & - & + & 0 & 0 & + & - & - & 0 & - & 0 & 0 & + & + & 0 & 0 \\
+ & 0 & 0 & 0 & 0 & - & + & - & 0 & + & - & - & 0 & + & 0 & 0 & 0 & + & 0 \\
+ & + & - & 0 & 0 & 0 & 0 & + & 0 & - & 0 & 0 & + & 0 & - & - & 0 & + & 0 \\
+ & 0 & + & - & 0 & 0 & 0 & - & + & 0 & + & 0 & 0 & - & 0 & - & 0 & 0 & + \\
+ & - & 0 & + & 0 & 0 & 0 & 0 & - & + & 0 & + & 0 & - & - & 0 & + & 0 & 0 \\
+ & + & 0 & - & + & - & 0 & 0 & 0 & 0 & 0 & + & 0 & 0 & 0 & + & 0 & - & - \\
+ & - & + & 0 & 0 & + & - & 0 & 0 & 0 & 0 & 0 & + & + & 0 & 0 & - & 0 & - \\
+ & 0 & - & + & - & 0 & + & 0 & 0 & 0 & + & 0 & 0 & 0 & + & 0 & - & - & 0
\end{array}\right)
$$

- Let G be a finite group not containing the symbol 0.
- For $A \subseteq G$, identify $A = \sum_{g \in A} g$ in $\mathbb{Z}[G]$.
- For $A \in \mathbb{Z}[G]$, write $A^{(h)} = \sum_{g \in G} a_g g^h$.

- Let Θ be a $v \times v$ $(0, G)$-matrix.
- We interpret Θ as a matrix over $\mathbb{Z}[G]$.
- Define $\Theta^{(h)}$ by $\Theta_{ij}^{(h)}$.
- Write $\Theta^* = (\Theta^{(-1)})^t$.

Definition. Balanced generalized weighing matrices

- G a finite group of order n.
- A $v \times v$ $(0, G)$-matrix Θ is a $BGW(v, k, \lambda; G)$ if

$$\Theta\Theta^* = (k \cdot e)I + \frac{\lambda G}{n}(J - I).$$

Theorem. Classical BGWs

- Let q be a prime power and $d > 0$ an integer.
- For each q and d there is a BGW with parameters

$$\left(\frac{q^{d+1} - 1}{q - 1}, q^d, q^d - q^{d-1}\right)$$

over C_{q-1}.

- A $BGW(10,9,8;C_4)$:

$$\begin{pmatrix}
1 & a & 1 & a^3 & a & 0 & 1 & a & a & a \\
a^2 & 1 & a & 1 & a^3 & a & 0 & 1 & a & a \\
a^2 & a^2 & 1 & a & 1 & a^3 & a & 0 & 1 & a \\
a^2 & a^2 & a^2 & 1 & a & 1 & a^3 & a & 0 & 1 \\
a & a^2 & a^2 & a^2 & 1 & a & 1 & a^3 & a & 0 \\
0 & a & a^2 & a^2 & a^2 & 1 & a & 1 & a^3 & a \\
a^2 & 0 & a & a^2 & a^2 & a^2 & 1 & a & 1 & a^3 \\
1 & a^2 & 0 & a & a^2 & a^2 & a^2 & 1 & a & 1 \\
a & 1 & a^2 & 0 & a & a^2 & a^2 & a^2 & 1 & a \\
a^2 & a & 1 & a^2 & 0 & a & a^2 & a^2 & a^2 & 1
\end{pmatrix}$$

- Decomposition matrices:

$$\begin{pmatrix}
1 & a & 1 & a^3 & a & 0 & 1 & a & a & a \\
a^2 & 1 & a & 1 & a^3 & a & 0 & 1 & a & a \\
a^2 & a^2 & 1 & a & 1 & a^3 & a & 0 & 1 & a \\
a^2 & a^2 & a^2 & 1 & a & 1 & a^3 & a & 0 & 1 \\
a & a^2 & a^2 & a^2 & 1 & a & 1 & a^3 & a & 0 \\
0 & a & a^2 & a^2 & a^2 & 1 & a & 1 & a^3 & a \\
a^2 & 0 & a & a^2 & a^2 & a^2 & 1 & a & 1 & a^3 \\
1 & a^2 & 0 & a & a^2 & a^2 & a^2 & 1 & a & 1 \\
a & 1 & a^2 & 0 & a & a^2 & a^2 & a^2 & 1 & a \\
a^2 & a & 1 & a^2 & 0 & a & a^2 & a^2 & a^2 & 1
\end{pmatrix}$$

- Decomposition matrices:

$$\begin{pmatrix}
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0
\end{pmatrix}$$

- Let Θ be a BGW with parameters

$$\left(\frac{9^{d+1}-1}{8}, 9^d, 9^d - 9^{d-1}\right)$$

over the aroup $C_4 = \langle a : a^4 = 1 \rangle$.

- Decompose Θ as

$$\Theta = \Theta_1 + a\Theta_a + a^2\Theta_{a^2} + a^3\Theta_{a^3},$$

where the Θ_is are disjoint $(0, 1)$-matrices.

- Apply $R_1 \mapsto -R_2 \mapsto -R_1 \mapsto R_2 \mapsto R_1$.
- Form:

$$\begin{aligned}\Theta \otimes R_1 &= \Theta_1 \otimes R_1 + \Theta_a \otimes {R_1}^a + \Theta_{a^2} \otimes {R_1}^{a^2} + \Theta_{a^3} \otimes {R_1}^{a^3} \\ &= \Theta_1 \otimes R_1 - \Theta_a \otimes R_2 - \Theta_{a^2} \otimes R_1 + \Theta_{a^3} \otimes R_2\end{aligned}$$

- Form D by substituting for the elements of $S_{9,d}$ the rows of the derived part of W_{19}.

- The matrix

$$\begin{pmatrix} \mathbf{0} & \ominus \otimes R_1 \\ \mathbf{1} & D \end{pmatrix}$$

is a balanced weighing matrix.

Theorem. (Kharaghani, et al., 2022a)

For every $d > 0$, there is a balanced weighing matrix with parameters

$$\left(\frac{9^{d+2} - 9}{4} + 1, 9^{d+1} \right).$$

- These are signings of some of the Ionin-type symmetric designs (Ionin, 2001).

Why not more infinite families?

- A $BGW(15,7,3;C_3)$:

$$
\left(\begin{array}{c|cccccccccccccc}
0 & a & a & a & a & a & a & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & a^2 & 0 & a & 0 & 0 & 0 & 0 & 0 & a^2 & 0 & a^2 & a^2 & 1 \\
0 & 0 & 1 & a^2 & 0 & a & 0 & 0 & 1 & 0 & 0 & a^2 & 0 & a^2 & a^2 \\
0 & 0 & 0 & 1 & a^2 & 0 & a & 0 & a^2 & 1 & 0 & 0 & a^2 & 0 & a^2 \\
0 & 0 & 0 & 0 & 1 & a^2 & 0 & a & a^2 & a^2 & 1 & 0 & 0 & a^2 & 0 \\
0 & a & 0 & 0 & 0 & 1 & a^2 & 0 & 0 & a^2 & a^2 & 1 & 0 & 0 & a^2 \\
0 & 0 & a & 0 & 0 & 0 & 1 & a^2 & a^2 & 0 & a^2 & a^2 & 1 & 0 & 0 \\
0 & a^2 & 0 & a & 0 & 0 & 0 & 1 & 0 & a^2 & 0 & a^2 & a^2 & 1 & 0 \\
\hline
1 & 1 & a & 0 & a^2 & 0 & 0 & 0 & 1 & a^2 & 0 & a & 0 & 0 & 0 \\
1 & 0 & 1 & a & 0 & a^2 & 0 & 0 & 0 & 1 & a^2 & 0 & a & 0 & 0 \\
1 & 0 & 0 & 1 & a & 0 & a^2 & 0 & 0 & 0 & 1 & a^2 & 0 & a & 0 \\
1 & 0 & 0 & 0 & 1 & a & 0 & a^2 & 0 & 0 & 0 & 1 & a^2 & 0 & a \\
1 & a^2 & 0 & 0 & 0 & 1 & a & 0 & a & 0 & 0 & 0 & 1 & a^2 & 0 \\
1 & 0 & a^2 & 0 & 0 & 0 & 1 & a & 0 & a & 0 & 0 & 0 & 1 & a^2 \\
1 & a & 0 & a^2 & 0 & 0 & 0 & 1 & a^2 & 0 & a & 0 & 0 & 0 & 1
\end{array}\right)
$$

$$R = \left(\begin{array}{ccccccc|ccccccc}
a & a & a & a & a & a & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
1 & a^2 & 0 & a & 0 & 0 & 0 & 0 & 0 & a^2 & 0 & a^2 & a^2 & 1 \\
0 & 1 & a^2 & 0 & a & 0 & 0 & 1 & 0 & 0 & a^2 & 0 & a^2 & a^2 \\
0 & 0 & 1 & a^2 & 0 & a & 0 & a^2 & 1 & 0 & 0 & a^2 & 0 & a^2 \\
0 & 0 & 0 & 1 & a^2 & 0 & a & a^2 & a^2 & 1 & 0 & 0 & a^2 & 0 \\
a & 0 & 0 & 0 & 1 & a^2 & 0 & 0 & a^2 & a^2 & 1 & 0 & 0 & a^2 \\
0 & a & 0 & 0 & 0 & 1 & a^2 & a^2 & 0 & a^2 & a^2 & 1 & 0 & 0 \\
a^2 & 0 & a & 0 & 0 & 0 & 1 & 0 & a^2 & 0 & a^2 & a^2 & 1 & 0
\end{array}\right)$$

- This suggests the group $C_6 \cong \langle b : b^6 = 1 \rangle$ where

$$b = \left(\begin{array}{ccccccc|ccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} \\
\hline
\mathbf{a} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \mathbf{a} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mathbf{a} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \mathbf{a} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \mathbf{a} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \mathbf{a} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \mathbf{a} & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)$$

- The always exists a $BGW((7^{d+1}-1)/6, 7^d, 7^d - 7^{d-1}; C_6)$. Therefore ...

Theorem. New GBRD parameter family

For every $d > 0$, there is a simple, quasi-residual *GBRD* with parameters

$$\left(\frac{7^{d+2}-7}{3}, 4 \cdot 7^d, 3 \cdot 7^{d-1}\right)$$

over C_3.

Question.

Are these embeddable????

Done!