Balancedly Splittable Orthogonal Designs

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Summary

- balancedly splittable Hadamard matrices
- balancedly splittable orthogonal designs
- construction
- related configurations

Hadamard Matrices

Definition: Hadmard Matrix

- A square matrix with entries from $\{-1, 1\}$ with pairwise orthogonal rows.
- Equiv. $HH^t = nI_n$, with H an $n \times n$ $(-1, 1)$-matrix of order n.

Hadamard Matrices

$$
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & - \\
1 & 1 & 1 & 1 & - & - & 1 & 1 & - & - & 1 & 1 & - & - & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & - & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & - & - & - & - & 1 & 1 & 1 & 1 & 1 & 1 & - & - \\
1 & - & 1 & - & 1 & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & - & 1 \\
1 & - & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - \\
1 & 1 & - & - & 1 & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - \\
1 & 1 & - & - & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & 1 \\
1 & - & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 \\
1 & - & - & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 \\
- & 1 & - & 1 & 1 & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - \\
- & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 \\
- & - & 1 & 1 & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - \\
- & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 \\
- & 1 & 1 & - & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & 1 \\
- & 1 & 1 & - & - & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & 1
\end{pmatrix}
$$

- a Hadamard matrix of order 16

Hadamard Matrices

Definition: Hadmard Matrix

- A square matrix with entries from $\{-1, 1\}$ with pairwise orthogonal rows.
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Hadamard Matrices

Definition: Hadmard Matrix

- A square matrix with entries from $\{-1, 1\}$ with pairwise orthogonal rows.
- Equiv. $HH^t = nI_n$, with H an $n \times n$ $(-1, 1)$-matrix of order n.

Definition: Balancedly Splittable (Kharaghani and Suda, 2019)

- A Hadamard matrix H of order n with $n \times \ell$ submatrix H_1 .
- H is (n, ℓ, a, b) balancedly splittable, $a < b$, if

$$H_1 H_1^t = \ell I_n + aA + b(J_n - I_n - A),$$

for symmetric $(0, 1)$-matrix A with zero diagonal.

Hadamard Matrices

$$\begin{pmatrix}
1&1&1&1&1&1&1&1&1&1&-&-&-&-&-&-\\
1&1&1&1&-&-&1&1&-&-&1&1&-&-&1&1\\
1&1&1&1&1&1&-&-&-&-&-&-&1&1&1&1\\
1&1&1&1&-&-&-&-&1&1&1&1&1&1&-&-\\
1&-&1&-&1&1&1&-&1&-&1&1&1&-&-&1\\
1&-&1&-&1&1&-&1&-&1&1&1&-&1&1&-\\
1&1&-&-&1&-&1&1&1&-&-&1&1&1&1&-\\
1&1&-&-&-&1&1&1&-&1&1&-&1&1&-&1\\
1&-&-&1&1&-&1&-&1&1&1&-&-&1&1&1\\
1&-&-&1&-&1&-&1&1&1&-&1&1&-&1&1\\
-&1&-&1&1&1&1&-&-&1&1&1&1&-&1&-\\
-&1&-&1&1&1&-&1&1&-&1&1&-&1&-&1\\
-&-&1&1&-&1&1&1&1&-&1&-&1&1&1&-\\
-&-&1&1&1&-&1&1&-&1&-&1&1&1&-&1\\
-&1&1&-&1&-&-&1&1&1&1&-&1&-&1&1\\
-&1&1&-&-&1&1&-&1&1&-&1&-&1&1&1
\end{pmatrix}$$

- a Hadamard matrix of order 16

Hadmard Matrices

$$
\left(\begin{array}{cccc|cccccc|cccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & - \\
1 & 1 & 1 & 1 & - & - & 1 & 1 & - & - & 1 & 1 & - & - & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & - & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & - & - & - & - & 1 & 1 & 1 & 1 & 1 & 1 & - & - \\
1 & - & 1 & - & 1 & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & - & 1 \\
1 & - & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - \\
1 & 1 & - & - & 1 & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - \\
1 & 1 & - & - & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & 1 \\
1 & - & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 \\
1 & - & - & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 \\
- & 1 & - & 1 & 1 & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - \\
- & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 \\
- & - & 1 & 1 & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - \\
- & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 \\
- & 1 & 1 & - & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & 1 \\
- & 1 & 1 & - & - & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & 1
\end{array}\right)
$$

- a balancedly splittable Hadamard matrix of order 16

Hadamard Matrices

$$
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
- & - & 1 & 1 & - & - \\
1 & 1 & - & - & - & - \\
- & - & - & - & 1 & 1 \\
1 & 1 & 1 & - & 1 & - \\
1 & 1 & - & 1 & - & 1 \\
1 & - & 1 & 1 & 1 & - \\
- & 1 & 1 & 1 & - & 1 \\
1 & - & 1 & - & 1 & 1 \\
- & 1 & - & 1 & 1 & 1 \\
1 & 1 & 1 & - & - & 1 \\
1 & 1 & - & 1 & 1 & - \\
- & 1 & 1 & 1 & 1 & - \\
1 & - & 1 & 1 & - & 1 \\
1 & - & - & 1 & 1 & 1 \\
- & 1 & 1 & - & 1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & - & 1 & - & 1 & 1 & 1 & - & 1 & - & 1 & 1 & - & 1 & 1 & - \\
1 & - & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & 1 & - & - & 1 \\
1 & 1 & - & - & 1 & - & 1 & 1 & 1 & - & 1 & - & 1 & 1 & - & 1 \\
1 & 1 & - & - & - & 1 & 1 & 1 & - & 1 & - & 1 & 1 & 1 & 1 & - \\
1 & - & - & 1 & 1 & - & 1 & - & 1 & 1 & - & 1 & 1 & - & 1 & 1 \\
1 & - & - & 1 & - & 1 & - & 1 & 1 & 1 & 1 & - & - & 1 & 1 & 1
\end{pmatrix}
$$

Hadamard Matrices

$$\begin{pmatrix}
6 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
\bar{2} & 6 & \bar{2} & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & \bar{2} \\
\bar{2} & \bar{2} & 6 & \bar{2} & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & \bar{2} \\
\bar{2} & \bar{2} & \bar{2} & 6 & \bar{2} & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & \bar{2} & 2 & 2 \\
2 & \bar{2} & 2 & \bar{2} & 6 & \bar{2} & 2 & \bar{2} & 2 & \bar{2} & 2 & 2 & 2 & \bar{2} & \bar{2} & 2 \\
2 & \bar{2} & 2 & \bar{2} & \bar{2} & 6 & \bar{2} & 2 & \bar{2} & 2 & 2 & 2 & \bar{2} & 2 & 2 & \bar{2} \\
2 & 2 & \bar{2} & \bar{2} & 2 & \bar{2} & 6 & \bar{2} & 2 & \bar{2} & \bar{2} & 2 & 2 & 2 & 2 & \bar{2} \\
2 & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & \bar{2} & 6 & \bar{2} & 2 & 2 & \bar{2} & 2 & 2 & \bar{2} & 2 \\
2 & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & 2 & \bar{2} & 6 & \bar{2} & 2 & \bar{2} & \bar{2} & 2 & 2 & 2 \\
2 & \bar{2} & \bar{2} & 2 & \bar{2} & 2 & \bar{2} & 2 & \bar{2} & 6 & \bar{2} & 2 & 2 & \bar{2} & 2 & 2 \\
2 & \bar{2} & 2 & \bar{2} & 2 & 2 & \bar{2} & 2 & 2 & \bar{2} & 6 & \bar{2} & \bar{2} & 2 & \bar{2} & 2 \\
2 & \bar{2} & 2 & \bar{2} & 2 & 2 & 2 & \bar{2} & \bar{2} & 2 & \bar{2} & 6 & 2 & \bar{2} & 2 & \bar{2} \\
2 & 2 & \bar{2} & \bar{2} & 2 & \bar{2} & 2 & 2 & \bar{2} & 2 & \bar{2} & 2 & 6 & \bar{2} & \bar{2} & 2 \\
2 & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & 2 & 2 & \bar{2} & 2 & \bar{2} & \bar{2} & 6 & 2 & \bar{2} \\
2 & \bar{2} & \bar{2} & 2 & \bar{2} & 2 & 2 & \bar{2} & 2 & 2 & \bar{2} & 2 & \bar{2} & 2 & 6 & \bar{2} \\
2 & \bar{2} & \bar{2} & 2 & 2 & \bar{2} & \bar{2} & 2 & 2 & 2 & 2 & \bar{2} & 2 & \bar{2} & \bar{2} & 6
\end{pmatrix}$$

- The rows of the split are equiangular.

Hadamard Matrices

$$
\left(\begin{array}{cccc|cccccc|cccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & - \\
1 & 1 & 1 & 1 & - & - & 1 & 1 & - & - & 1 & 1 & - & - & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & - & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & - & - & - & - & 1 & 1 & 1 & 1 & 1 & 1 & - & - \\
\hline
1 & - & 1 & - & 1 & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & - & 1 \\
1 & - & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - \\
1 & 1 & - & - & 1 & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - \\
1 & 1 & - & - & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & 1 \\
1 & - & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 \\
1 & - & - & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 \\
\hline
- & 1 & - & 1 & 1 & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - \\
- & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 \\
- & - & 1 & 1 & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - \\
- & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 \\
- & 1 & 1 & - & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & 1 \\
- & 1 & 1 & - & - & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & 1
\end{array}\right)
$$

- There are four splits.

Orthogonal Designs

Definition: Orthogonal Designs

- A square $\{\pm z_1, \ldots, \pm z_u\}$-matriz X of order n such that

$$XX^t = \sigma I_n, \qquad s_i \in \mathbb{N},$$

  where $\sigma = \sum_i s_i z_i^2$.
- Write $OD(n; s_1, \ldots, s_u)$.

Orthogonal Designs

$$
\begin{pmatrix}
a & b & a & b & a & b & a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} \\
a & b & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & \bar{b} & a & b \\
a & b & a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} & a & b & a & b \\
a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & a & b & a & b & a & b & \bar{a} & \bar{b} \\
b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a \\
b & \bar{a} & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} \\
a & b & \bar{a} & \bar{b} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} \\
a & b & \bar{a} & \bar{b} & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a \\
b & \bar{a} & \bar{b} & a & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b \\
b & \bar{a} & \bar{b} & a & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b \\
\bar{b} & a & \bar{b} & a & a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} \\
\bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a \\
\bar{a} & \bar{b} & a & b & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} \\
\bar{a} & \bar{b} & a & b & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a \\
\bar{b} & a & b & \bar{a} & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b \\
\bar{b} & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b
\end{pmatrix}
$$

- An $OD(16;8,8)$

Orthogonal Designs

$$
\left(\begin{array}{cccc|cccccc|cccccc}
a & b & a & b & a & b & a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} \\
a & b & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & \bar{b} & a & b \\
a & b & a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} & a & b & a & b \\
a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & a & b & a & b & a & b & \bar{a} & \bar{b} \\
b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a \\
b & \bar{a} & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} \\
a & b & \bar{a} & \bar{b} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} \\
a & b & \bar{a} & \bar{b} & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a \\
b & \bar{a} & \bar{b} & a & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b \\
b & \bar{a} & \bar{b} & a & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b \\
\bar{b} & a & \bar{b} & a & a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} \\
\bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a \\
\bar{a} & \bar{b} & a & b & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} \\
\bar{a} & \bar{b} & a & b & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a \\
\bar{b} & a & b & \bar{a} & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b \\
\bar{b} & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b
\end{array}\right)
$$

$$
(a^2+b^2)\begin{pmatrix}
3 & {-} & {-} & {-} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
{-} & 3 & {-} & {-} & {-} & {-} & 1 & 1 & {-} & {-} & {-} & {-} & 1 & 1 & {-} & {-} \\
{-} & {-} & 3 & {-} & 1 & 1 & {-} & {-} & {-} & {-} & 1 & 1 & {-} & {-} & {-} & {-} \\
{-} & {-} & {-} & 3 & {-} & {-} & {-} & {-} & 1 & 1 & {-} & {-} & {-} & {-} & 1 & 1 \\
1 & {-} & 1 & {-} & 3 & {-} & 1 & {-} & 1 & {-} & 1 & 1 & 1 & {-} & {-} & 1 \\
1 & {-} & 1 & {-} & {-} & 3 & {-} & 1 & {-} & 1 & 1 & 1 & {-} & 1 & 1 & {-} \\
1 & 1 & {-} & {-} & 1 & {-} & 3 & {-} & 1 & {-} & {-} & 1 & 1 & 1 & 1 & {-} \\
1 & 1 & {-} & {-} & {-} & 1 & {-} & 3 & {-} & 1 & 1 & {-} & 1 & 1 & {-} & 1 \\
1 & {-} & {-} & 1 & 1 & {-} & 1 & {-} & 3 & {-} & 1 & {-} & {-} & 1 & 1 & 1 \\
1 & {-} & {-} & 1 & {-} & 1 & {-} & 1 & {-} & 3 & {-} & 1 & 1 & {-} & 1 & 1 \\
1 & {-} & 1 & {-} & 1 & 1 & {-} & 1 & 1 & {-} & 3 & {-} & {-} & 1 & {-} & 1 \\
1 & {-} & 1 & {-} & 1 & 1 & 1 & {-} & {-} & 1 & {-} & 3 & 1 & {-} & 1 & {-} \\
1 & 1 & {-} & {-} & 1 & {-} & 1 & 1 & {-} & 1 & {-} & 1 & 3 & {-} & {-} & 1 \\
1 & 1 & {-} & {-} & {-} & 1 & 1 & 1 & 1 & {-} & 1 & {-} & {-} & 3 & 1 & {-} \\
1 & {-} & {-} & 1 & {-} & 1 & 1 & {-} & 1 & 1 & {-} & 1 & {-} & 1 & 3 & {-} \\
1 & {-} & {-} & 1 & 1 & {-} & {-} & 1 & 1 & 1 & 1 & {-} & 1 & {-} & {-} & 3
\end{pmatrix}
$$

Orthogonal Designs

$$
\left(\begin{array}{cccccccccccccccc}
a & b & a & b & a & b & a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} \\
a & b & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & \bar{b} & a & b \\
a & b & a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} & a & b & a & b \\
a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & a & b & a & b & a & b & \bar{a} & \bar{b} \\
\hline
b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a \\
b & \bar{a} & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} \\
a & b & \bar{a} & \bar{b} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} \\
a & b & \bar{a} & \bar{b} & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a \\
b & \bar{a} & \bar{b} & a & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b \\
b & \bar{a} & \bar{b} & a & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b \\
\hline
\bar{b} & a & \bar{b} & a & a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} \\
\bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a \\
\bar{a} & \bar{b} & a & b & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} \\
\bar{a} & \bar{b} & a & b & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a \\
\bar{b} & a & b & \bar{a} & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b \\
\bar{b} & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b
\end{array}\right)
$$

Orthogonal Designs

Definition: Orthogonal Designs

- A square $\{\pm z_1, \ldots, \pm z_u\}$-matrix X of order n such that $XX^t = \sigma I_n, s_i \in \mathbb{N}$, where $\sigma = \sum_i s_i z_i^2$.

Orthogonal Designs

Definition: Orthogonal Designs

- A square $\{\pm z_1, \ldots, \pm z_u\}$-matrix X of order n such that $XX^t = \sigma I_n, s_i \in \mathbb{N}$, where $\sigma = \sum_i s_i z_i^2$.

Definition: Balancedly Splittable (Kharaghani et al., 2021)

- An OD X with $n \times \ell$ submatrix X_1 admits a stable (n, ℓ, a, b) split if

$$X_1 X_1^t = \sigma(cI_n + aA + b(J_n - I_n - A)),$$

  where $c \mid \ell$.
- The split is unstable if it is not stable and

$$X_1 X_1^t = \ell I_n + aA + b(J_n - I_n - A)$$

  after replacing each variable with 1.

Auxiliary Matrices

Definition: Auxiliary Matrices

- X a full $OD(n; s_1, \ldots, s_u)$ and H the Hadamard matrix obtained by replacing each variable with 1.
- Label rows of X and H by $x_0, \ldots, x_{n-1}$, and $h_0, \ldots, h_{n-1}$.
- The auxiliary matrices of X are given by

$$c_i = h_i^t x_i, \qquad i = 0, \ldots, n-1.$$

Construction

- An $OD(2;1,1)$:

$$\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$$

- Auxiliary matrices:

$$c_0 = \begin{pmatrix} a & b \\ a & b \end{pmatrix}, \qquad c_1 = \begin{pmatrix} b & -a \\ -b & a \end{pmatrix}$$

Construction

- Form the sequences

$$\alpha = (c_0, c_1, c_1), \qquad \beta = (c_0, c_1, -c_1).$$

- The sum of the periodic correlations of α and β satisfy

$$\sum_i \alpha_i \alpha_{i+j}^t + \sum_i \beta_i \beta_{i+j}^t = O, \qquad j = 1, 2.$$

- Sums of cross-correlations

$$\sum_i \alpha_i \beta_{i+j}^t + \sum_i \beta_i \alpha_{i+j}^t = O, \qquad j = 1, 2.$$

Construction

- Form the matrices

$$A = circ(\alpha), \qquad B = circ(\beta),$$

- Then the matrix

$$\Theta = \begin{pmatrix} A & B \\ B & A \end{pmatrix}.$$

Construction

$$\Theta = \left(\begin{array}{cccccc|cccccc}
a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a \\
a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} \\
b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} \\
\bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a \\
b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b \\
\bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b \\
\hline
a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} \\
a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a \\
\bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} \\
b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a \\
b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b \\
\bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b
\end{array}\right)$$

Construction

$$\Theta\Theta^t = \begin{pmatrix} AA^t + BB^t & AB^t + BA^t \\ AB^t + BA^t & AA^t + BB^t \end{pmatrix}.$$

Construction

$$\Theta\Theta^t = (a^2+b^2)\left(\begin{array}{cccccc|cccccc}
6 & \bar{2} & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 \\
\bar{2} & 6 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 6 & \bar{2} & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 \\
0 & 0 & \bar{2} & 6 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 6 & \bar{2} & 0 & 0 & 0 & 0 & 2 & 2 \\
0 & 0 & 0 & 0 & \bar{2} & 6 & 0 & 0 & 0 & 0 & 2 & 2 \\
\hline
2 & 2 & 0 & 0 & 0 & 0 & 6 & \bar{2} & 0 & 0 & 0 & 0 \\
2 & 2 & 0 & 0 & 0 & 0 & \bar{2} & 6 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 6 & \bar{2} & 0 & 0 \\
0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & \bar{2} & 6 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 6 & \bar{2} \\
0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & \bar{2} & 6
\end{array}\right)$$

Construction

$$
\left(\begin{array}{cccc|cccccc|cccccc}
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
\hline
* & * & * & * & a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a \\
* & * & * & * & a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} \\
* & * & * & * & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} \\
* & * & * & * & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a \\
* & * & * & * & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b \\
* & * & * & * & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b \\
\hline
* & * & * & * & a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} \\
* & * & * & * & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a \\
* & * & * & * & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} \\
* & * & * & * & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a \\
* & * & * & * & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b \\
* & * & * & * & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b
\end{array}\right)
$$

Construction

- Define:

$$Y = \left(\begin{smallmatrix} 1 & 1 \\ 1 & - \end{smallmatrix}\right) \otimes X,$$
$$K = \left(\begin{smallmatrix} 1 & 1 \\ 1 & - \end{smallmatrix}\right) \otimes H.$$

- Index rows as $y_0, \ldots, y_{2n-1}$, and $k_0, \ldots, k_{2n-1}$.

Construction

$$Y = \left(\begin{smallmatrix} 1 & 1 \\ 1 & - \end{smallmatrix}\right) \otimes X \qquad K = \left(\begin{smallmatrix} 1 & 1 \\ 1 & - \end{smallmatrix}\right) \otimes H$$

Rows: $y_0, \ldots, y_{2n-1}$ Rows: $k_0, \ldots, k_{2n-1}$

- Form

$$F = \begin{pmatrix} k_1^t x_0 & k_2^t x_0 & k_3^t x_0 \end{pmatrix},$$

$$E = \begin{pmatrix} h_0^t y_1 \\ h_0^t y_2 \\ h_0^t y_3 \end{pmatrix},$$

$$G = k_0^t y_0.$$

Construction

$$
\left(\begin{array}{c|cccccc|cccccc}
G & \multicolumn{6}{c|}{F} & \multicolumn{6}{c}{-F} \\
\hline
 & a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a \\
 & a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} \\
 & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} \\
E & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a \\
 & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b \\
 & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b \\
\hline
 & a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} \\
 & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a \\
 & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} \\
-E & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a \\
 & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b \\
 & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b
\end{array}\right)
$$

Construction

$$
\left(\begin{array}{cccc|cccccc|cccccc}
a & b & a & b & a & b & a & b & a & b & \bar a & \bar b & \bar a & \bar b & \bar a & \bar b \\
a & b & a & b & \bar a & \bar b & a & b & \bar a & \bar b & a & b & \bar a & \bar b & a & b \\
a & b & a & b & a & b & \bar a & \bar b & \bar a & \bar b & \bar a & \bar b & a & b & a & b \\
a & b & a & b & \bar a & \bar b & \bar a & \bar b & a & b & a & b & a & b & \bar a & \bar b \\
\hline
b & \bar a & b & \bar a & a & b & b & \bar a & b & \bar a & a & b & b & \bar a & \bar b & a \\
b & \bar a & b & \bar a & a & b & \bar b & a & \bar b & a & a & b & \bar b & a & b & \bar a \\
a & b & \bar a & \bar b & b & \bar a & a & b & b & \bar a & \bar b & a & a & b & b & \bar a \\
a & b & \bar a & \bar b & \bar b & a & a & b & \bar b & a & b & \bar a & a & b & \bar b & a \\
b & \bar a & \bar b & a & b & \bar a & b & \bar a & a & b & b & \bar a & \bar b & a & a & b \\
b & \bar a & \bar b & a & \bar b & a & \bar b & a & a & b & \bar b & a & b & \bar a & a & b \\
\hline
\bar b & a & \bar b & a & a & b & b & \bar a & \bar b & a & a & b & b & \bar a & b & \bar a \\
\bar b & a & \bar b & a & a & b & \bar b & a & b & \bar a & a & b & \bar b & a & \bar b & a \\
\bar a & \bar b & a & b & \bar b & a & a & b & b & \bar a & b & \bar a & a & b & b & \bar a \\
\bar a & \bar b & a & b & b & \bar a & a & b & \bar b & a & \bar b & a & a & b & \bar b & a \\
\bar b & a & b & \bar a & b & \bar a & \bar b & a & a & b & b & \bar a & b & \bar a & a & b \\
\bar b & a & b & \bar a & \bar b & a & b & \bar a & a & b & \bar b & a & \bar b & a & a & b
\end{array}\right)
$$

Construction

$$
\begin{pmatrix}
a & b & a & b & a & b & a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} \\
a & b & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & \bar{b} & a & b \\
a & b & a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} & a & b & a & b \\
a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & a & b & a & b & a & b & \bar{a} & \bar{b} \\
b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a \\
b & \bar{a} & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} \\
a & b & \bar{a} & \bar{b} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} \\
a & b & \bar{a} & \bar{b} & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a \\
b & \bar{a} & \bar{b} & a & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b \\
b & \bar{a} & \bar{b} & a & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b \\
\bar{b} & a & \bar{b} & a & a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} \\
\bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a \\
\bar{a} & \bar{b} & a & b & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} \\
\bar{a} & \bar{b} & a & b & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a \\
\bar{b} & a & b & \bar{a} & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b \\
\bar{b} & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b
\end{pmatrix}
$$

Complex and Quaternion Elements

- The following is a $OD(2;1,1)$ of quaternions.

$$\begin{pmatrix} \bar{a} & bi \\ \bar{b}j & ak \end{pmatrix},$$

- where a and b are real variables.

Complex and Quaternion Elements

$$
\left(\begin{array}{cccc|cccccc|cccccc}
\bar{a} & bi & \bar{a} & bi & \bar{a} & bi & \bar{a} & bi & \bar{a} & bi & a & \bar{b}i & a & \bar{b}i & a & \bar{b}i \\
\bar{a} & bi & \bar{a} & bi & a & \bar{b}i & \bar{a} & bi & a & \bar{b}i & \bar{a} & bi & a & \bar{b}i & \bar{a} & bi \\
\bar{a} & bi & \bar{a} & bi & \bar{a} & bi & a & \bar{b}i & a & \bar{b}i & a & \bar{b}i & \bar{a} & bi & \bar{a} & bi \\
\bar{a} & bi & \bar{a} & bi & a & \bar{b}i & a & \bar{b}i & \bar{a} & bi & \bar{a} & bi & \bar{a} & bi & a & \bar{b}i \\
\hline
\bar{b}j & ak & \bar{b}j & ak & \bar{a} & bi & \bar{b}j & ak & \bar{b}j & ak & \bar{a} & bi & \bar{b}j & ak & bj & \bar{a}k \\
\bar{b}j & ak & \bar{b}j & ak & \bar{a} & bi & bj & \bar{a}k & bj & \bar{a}k & \bar{a} & bi & bj & \bar{a}k & \bar{b}j & ak \\
\bar{a} & bi & a & \bar{b}i & \bar{b}j & ak & \bar{a} & bi & \bar{b}j & ak & bj & \bar{a}k & \bar{a} & bi & \bar{b}j & ak \\
\bar{a} & bi & a & \bar{b}i & bj & \bar{a}k & \bar{a} & bi & bj & \bar{a}k & \bar{b}j & ak & \bar{a} & bi & bj & \bar{a}k \\
\bar{b}j & ak & bj & \bar{a}k & \bar{b}j & ak & \bar{b}j & ak & \bar{a} & bi & \bar{b}j & ak & bj & \bar{a}k & \bar{a} & bi \\
\bar{b}j & ak & bj & \bar{a}k & bj & \bar{a}k & bj & \bar{a}k & \bar{a} & bi & bj & \bar{a}k & \bar{b}j & ak & \bar{a} & bi \\
\hline
bj & \bar{a}k & bj & \bar{a}k & \bar{a} & bi & \bar{b}j & ak & bj & \bar{a}k & \bar{a} & bi & \bar{b}j & ak & \bar{b}j & ak \\
bj & \bar{a}k & bj & \bar{a}k & \bar{a} & bi & bj & \bar{a}k & \bar{b}j & ak & \bar{a} & bi & bj & \bar{a}k & bj & \bar{a}k \\
a & \bar{b}i & \bar{a} & bi & bj & \bar{a}k & \bar{a} & bi & \bar{b}j & ak & \bar{b}j & ak & \bar{a} & bi & \bar{b}j & ak \\
a & \bar{b}i & \bar{a} & bi & \bar{b}j & ak & \bar{a} & bi & bj & \bar{a}k & bj & \bar{a}k & \bar{a} & bi & bj & \bar{a}k \\
bj & \bar{a}k & \bar{b}j & ak & \bar{b}j & ak & bj & \bar{a}k & \bar{a} & bi & \bar{b}j & ak & \bar{b}j & ak & \bar{a} & bi \\
bj & \bar{a}k & \bar{b}j & ak & bj & \bar{a}k & \bar{b}j & ak & \bar{a} & bi & bj & \bar{a}k & bj & \bar{a}k & \bar{a} & bi
\end{array}\right)
$$

Equiangular Lines and Unbiased Hadamard Matrices

Definition: Equiangular Lines

A collection $\mathcal{L} \subset \mathbb{R}^\ell$ of lines (vectors) are equiangular if there is a constant a such that $|\langle u, v\rangle| = a$ for every $u, v \in L$.

Definition: Unbiased Hadamard Matrices

Hadamard matrices H and K are unbiased if

$$HK^t = \sqrt{n}L$$

for some Hadamard matrix L.

Equiangular Lines and Unbiased Hadamard Matrices

$$H = \begin{pmatrix}
1&1&1&1&1&1&1&1&1&1&-&-&-&-&-&-\\
1&1&1&1&-&-&1&1&-&-&1&1&-&-&1&1\\
1&1&1&1&1&1&-&-&-&-&-&-&1&1&1&1\\
1&1&1&1&-&-&-&-&1&1&1&1&1&1&-&-\\
1&-&1&-&1&1&1&-&1&-&1&1&1&-&-&1\\
1&-&1&-&1&1&-&1&-&1&1&1&-&1&1&-\\
1&1&-&-&1&-&1&1&1&-&-&1&1&1&1&-\\
1&1&-&-&-&1&1&1&-&1&1&-&1&1&-&1\\
1&-&-&1&1&-&1&-&1&1&1&-&-&1&1&1\\
1&-&-&1&-&1&-&1&1&1&-&1&1&-&1&1\\
-&1&-&1&1&1&1&-&-&1&1&1&1&-&1&-\\
-&1&-&1&1&1&-&1&1&-&1&1&-&1&-&1\\
-&-&1&1&-&1&1&1&1&-&1&-&1&1&1&-\\
-&-&1&1&1&-&1&1&-&1&-&1&1&1&-&1\\
-&1&1&-&1&-&-&1&1&1&1&-&1&-&1&1\\
-&1&1&-&-&1&1&-&1&1&-&1&-&1&1&1
\end{pmatrix}$$

- a Hadamard matrix of order 16

Equiangular Lines and Unbiased Hadamard Matrices

$$
H = \left(\begin{array}{cccc|cccccc|cccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & - \\
1 & 1 & 1 & 1 & - & - & 1 & 1 & - & - & 1 & 1 & - & - & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & - & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & - & - & - & - & 1 & 1 & 1 & 1 & 1 & 1 & - & - \\
1 & - & 1 & - & 1 & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & - & 1 \\
1 & - & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - \\
1 & 1 & - & - & 1 & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - \\
1 & 1 & - & - & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & 1 \\
1 & - & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 \\
1 & - & - & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 \\
- & 1 & - & 1 & 1 & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - \\
- & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 \\
- & - & 1 & 1 & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - \\
- & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & 1 \\
- & 1 & 1 & - & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & 1 \\
- & 1 & 1 & - & - & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & 1
\end{array}\right)
$$

- a balanced split

Equiangular Lines and Unbiased Hadamard Matrices

$$\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
- & - & 1 & 1 & - & - \\
1 & 1 & - & - & - & - \\
- & - & - & - & 1 & 1 \\
1 & 1 & 1 & - & 1 & - \\
1 & 1 & - & 1 & - & 1 \\
1 & - & 1 & 1 & 1 & - \\
- & 1 & 1 & 1 & - & 1 \\
1 & - & 1 & - & 1 & 1 \\
- & 1 & - & 1 & 1 & 1 \\
1 & 1 & 1 & - & - & 1 \\
1 & 1 & - & 1 & 1 & - \\
- & 1 & 1 & 1 & 1 & - \\
1 & - & 1 & 1 & - & 1 \\
1 & - & - & 1 & 1 & 1 \\
- & 1 & 1 & - & 1 & 1
\end{pmatrix}$$

- a collection of equiangular lines
- products are the same in absolute value

Equiangular Lines and Unbiased Hadamard Matrices

$$K = \left(\begin{array}{cccc|cccccc|cccccc}
1 & 1 & 1 & 1 & - & - & - & - & - & - & - & - & - & - & - & - \\
1 & 1 & 1 & 1 & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & - & 1 & 1 \\
1 & 1 & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & - & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & - \\
1 & - & 1 & - & - & - & - & 1 & - & 1 & 1 & 1 & 1 & - & - & 1 \\
1 & - & 1 & - & - & - & 1 & - & 1 & - & 1 & 1 & - & 1 & 1 & - \\
1 & 1 & - & - & - & 1 & - & - & - & 1 & - & 1 & 1 & 1 & 1 & - \\
1 & 1 & - & - & 1 & - & - & - & 1 & - & 1 & - & 1 & 1 & - & 1 \\
1 & - & - & 1 & - & 1 & - & 1 & - & - & 1 & - & - & 1 & 1 & 1 \\
1 & - & - & 1 & 1 & - & 1 & - & - & - & - & 1 & 1 & - & 1 & 1 \\
- & 1 & - & 1 & - & - & - & 1 & 1 & - & 1 & 1 & 1 & - & 1 & - \\
- & 1 & - & 1 & - & - & 1 & - & - & 1 & 1 & 1 & - & 1 & - & 1 \\
- & - & 1 & 1 & 1 & - & - & - & - & 1 & 1 & - & 1 & 1 & 1 & - \\
- & - & 1 & 1 & - & 1 & - & - & 1 & - & - & 1 & 1 & 1 & - & 1 \\
- & 1 & 1 & - & - & 1 & 1 & - & - & - & 1 & - & 1 & - & 1 & 1 \\
- & 1 & 1 & - & 1 & - & - & 1 & - & - & - & 1 & - & 1 & 1 & 1
\end{array}\right)$$

- H and K are unbiased

Equiangular Lines and Unbiased Hadamard Matrices

Proposition (Kharaghani and Suda, 2019)

$H = \begin{pmatrix} H_1 & H_2 \end{pmatrix}$ a Hadamard matrix of order n with

$$H_1 H_1^t = \ell I_n + aS$$

where S is a $(-1, 0, 1)$-matrix. The following are equivalent:

1. $K = \begin{pmatrix} -H_1 & H_2 \end{pmatrix}$ is unbiased with H, and
2. $(\ell, a) = ((n \pm \sqrt{n})/2, \sqrt{n}/2)$.

Equiangular Lines and Unbiased Hadamard Matrices

$$\begin{pmatrix} G & F & -F \\ E & A & B \\ -E & B & A \end{pmatrix}$$

Equiangular Lines and Unbiased Hadamard Matrices

$$\begin{pmatrix} G & \color{blue}{F} & -F \\ E & \color{blue}{A} & B \\ -E & \color{blue}{B} & A \end{pmatrix}$$

Equiangular Lines and Unbiased Hadamard Matrices

$$\begin{pmatrix} G & F & -F \\ \color{blue}{E} & \color{blue}{A} & \color{blue}{B} \\ -E & B & A \end{pmatrix}$$

Equiangular Lines and Unbiased Hadamard Matrices

$$\begin{pmatrix} G & F & -F \\ E & A & B \\ \color{blue}{-E} & \color{blue}{B} & \color{blue}{A} \end{pmatrix}$$

Equiangular Lines and Hadamard Matrices

- each split gives equiangular lines
- parametric conditions of the proposition are satisfied
- we can always construct unbiased Hadamard matrices

Unbiased ODs

Definition: Unbiased ODs (Kharaghani and Suda, 2018)

- X_1 and X_2 be two $OD(n, s_1, \ldots, s_u)$ in $\{\pm z_1, \ldots, \pm z_u\}$.
- X_1 and X_2 are unbiased if there is a Hadamard matrix H such that

$$X_1 X_2^t = \left(\alpha^{-\frac{1}{2}} \sum_i s_i z_i^2 \right) H.$$

for some $\alpha \in \mathbb{R}^+$.

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- $$\left.\begin{array}{l} \text{stable split} \\ \text{equiangular lines} \\ \text{param. cond.} \end{array}\right\} \Rightarrow \text{unbiased ODs}$$

Unbiased ODs

- Recall:
$$X = \begin{pmatrix} G & F & -F \\ E & A & B \\ -E & B & A \end{pmatrix}.$$
- Take
$$Y = \begin{pmatrix} G & F & F \\ E & A & -B \\ -E & B & -A \end{pmatrix}.$$
- Then $XY^t = 2\left(\sum_i s_i z_i^2\right) H$, where H is Hadamard.

Unbiased ODs

$$\begin{pmatrix}
a & b & a & b & a & b & a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} \\
a & b & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & \bar{b} & a & b \\
a & b & a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & \bar{a} & \bar{b} & a & b & a & b \\
a & b & a & b & \bar{a} & \bar{b} & \bar{a} & \bar{b} & a & b & a & b & a & b & \bar{a} & \bar{b} \\
b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a \\
b & \bar{a} & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} \\
a & b & \bar{a} & \bar{b} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} \\
a & b & \bar{a} & \bar{b} & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a \\
b & \bar{a} & \bar{b} & a & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} & \bar{b} & a & a & b \\
b & \bar{a} & \bar{b} & a & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b \\
\bar{b} & a & \bar{b} & a & a & b & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} \\
\bar{b} & a & \bar{b} & a & a & b & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a \\
\bar{a} & \bar{b} & a & b & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b & b & \bar{a} \\
\bar{a} & \bar{b} & a & b & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b & \bar{b} & a \\
\bar{b} & a & b & \bar{a} & b & \bar{a} & \bar{b} & a & a & b & b & \bar{a} & b & \bar{a} & a & b \\
\bar{b} & a & b & \bar{a} & \bar{b} & a & b & \bar{a} & a & b & \bar{b} & a & \bar{b} & a & a & b
\end{pmatrix} \cdot \begin{pmatrix}
a & a & a & a & b & b & a & a & b & b & \bar{b} & \bar{b} & \bar{a} & \bar{a} & \bar{b} & \bar{b} \\
b & b & b & b & \bar{a} & \bar{a} & b & b & \bar{a} & \bar{a} & a & a & \bar{b} & \bar{b} & a & a \\
a & a & a & a & b & b & \bar{a} & \bar{a} & \bar{b} & \bar{b} & \bar{b} & \bar{b} & a & a & b & b \\
b & b & b & b & \bar{a} & \bar{a} & \bar{b} & \bar{b} & a & a & a & a & b & b & \bar{a} & \bar{a} \\
a & \bar{a} & a & \bar{a} & a & a & b & \bar{b} & b & \bar{b} & a & a & \bar{b} & b & b & \bar{b} \\
b & \bar{b} & b & \bar{b} & b & b & \bar{a} & a & \bar{a} & a & b & b & a & \bar{a} & \bar{a} & a \\
a & a & \bar{a} & \bar{a} & b & \bar{b} & a & a & b & \bar{b} & b & \bar{b} & a & a & \bar{b} & b \\
b & b & \bar{b} & \bar{b} & \bar{a} & a & b & b & \bar{a} & a & \bar{a} & a & b & b & a & \bar{a} \\
a & \bar{a} & \bar{a} & a & b & \bar{b} & b & \bar{b} & a & a & \bar{b} & b & b & \bar{b} & a & a \\
b & \bar{b} & \bar{b} & b & \bar{a} & a & \bar{a} & a & b & b & a & \bar{a} & \bar{a} & a & b & b \\
a & \bar{a} & a & \bar{a} & \bar{a} & \bar{a} & b & \bar{b} & \bar{b} & b & \bar{a} & \bar{a} & \bar{b} & b & \bar{b} & b \\
b & \bar{b} & b & \bar{b} & \bar{b} & \bar{b} & \bar{a} & a & a & \bar{a} & \bar{b} & \bar{b} & a & \bar{a} & a & \bar{a} \\
a & a & \bar{a} & \bar{a} & \bar{b} & b & \bar{a} & \bar{a} & b & \bar{b} & \bar{b} & b & \bar{a} & \bar{a} & \bar{b} & b \\
b & b & \bar{b} & \bar{b} & a & \bar{a} & \bar{b} & \bar{b} & \bar{a} & a & a & \bar{a} & \bar{b} & \bar{b} & a & \bar{a} \\
a & \bar{a} & \bar{a} & a & b & \bar{b} & \bar{b} & b & \bar{a} & \bar{a} & \bar{b} & b & \bar{b} & b & \bar{a} & \bar{a} \\
b & \bar{b} & \bar{b} & b & \bar{a} & a & a & \bar{a} & \bar{b} & \bar{b} & a & \bar{a} & a & \bar{a} & \bar{b} & \bar{b}
\end{pmatrix}$$

Unbiased ODs

$$2(a^2+b^2)\begin{pmatrix}
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&-&-&1&1&-&-&-&-&1&1&-&-\\
1&1&1&1&1&1&-&-&-&-&1&1&-&-&-&-\\
1&1&1&1&-&-&-&-&1&1&-&-&-&-&1&1\\
1&-&1&-&1&1&1&-&1&-&-&-&1&-&-&1\\
1&-&1&-&1&1&-&1&-&1&-&-&-&1&1&-\\
1&1&-&-&1&-&1&1&1&-&-&1&-&-&1&-\\
1&1&-&-&-&1&1&1&-&1&1&-&-&-&-&1\\
1&-&-&1&1&-&1&-&1&1&1&-&-&1&-&-\\
1&-&-&1&-&1&-&1&1&1&-&1&1&-&-&-\\
1&-&1&-&-&-&-&1&1&-&1&1&-&1&-&1\\
1&-&1&-&-&-&1&-&-&1&1&1&1&-&1&-\\
1&1&-&-&1&-&-&-&-&1&-&1&1&1&-&1\\
1&1&-&-&-&1&-&-&1&-&1&-&1&1&1&-\\
1&-&-&1&-&1&1&-&-&-&-&1&-&1&1&1\\
1&-&-&1&1&-&-&1&-&-&1&-&1&-&1&1
\end{pmatrix}$$

The End!!

Thank You!

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