Model order reduction for complex ocular simulations inside the human eyeball

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Motivations

Eye2brain project: develop a reliable and efficient mathematical and computational framework to simulate and predict the functioning and the connection between the eye and the brain

Model complexity
- Multiphysics / multiscale problem
- Numerous parameters and scarce real data
- Influence of multiple risk factors or a combination of them

In silico model

Uncertainty quantification (UQ)

Prohibitive cost in 3D

Model reduction

Model Order Reduction with Feel++ [2]

Model dependant on parameter $\mu \in D_\mu$

Input files: cfg and JSON

Problem-solve

$A(\mu)u(\mu) = F(\mu)$

$s_k(\mu) = L_k(\mu)u(\mu)$

Affine decomposition (Empirical Interpolation Method)

Model Order Reduction

- $A(\mu) \approx \sum_q \beta_A^q(\mu) A^q$
- $F(\mu) \approx \sum_p \beta_F^p(\mu) F^p$
- $L_k(\mu) \approx \sum_p \beta_{L_k}^p(\mu) L_k^p$

Offline Stage

(requires N finite element resoluion of high fidelity)

Offline stage

Reduced Basis construction

- $Z^N = [\xi_1, \cdots, \xi_N]$ reduced basis
- $S = \{\mu_1, \cdots, \mu_N\}$ sample

Computed only once, then stored

Online stage

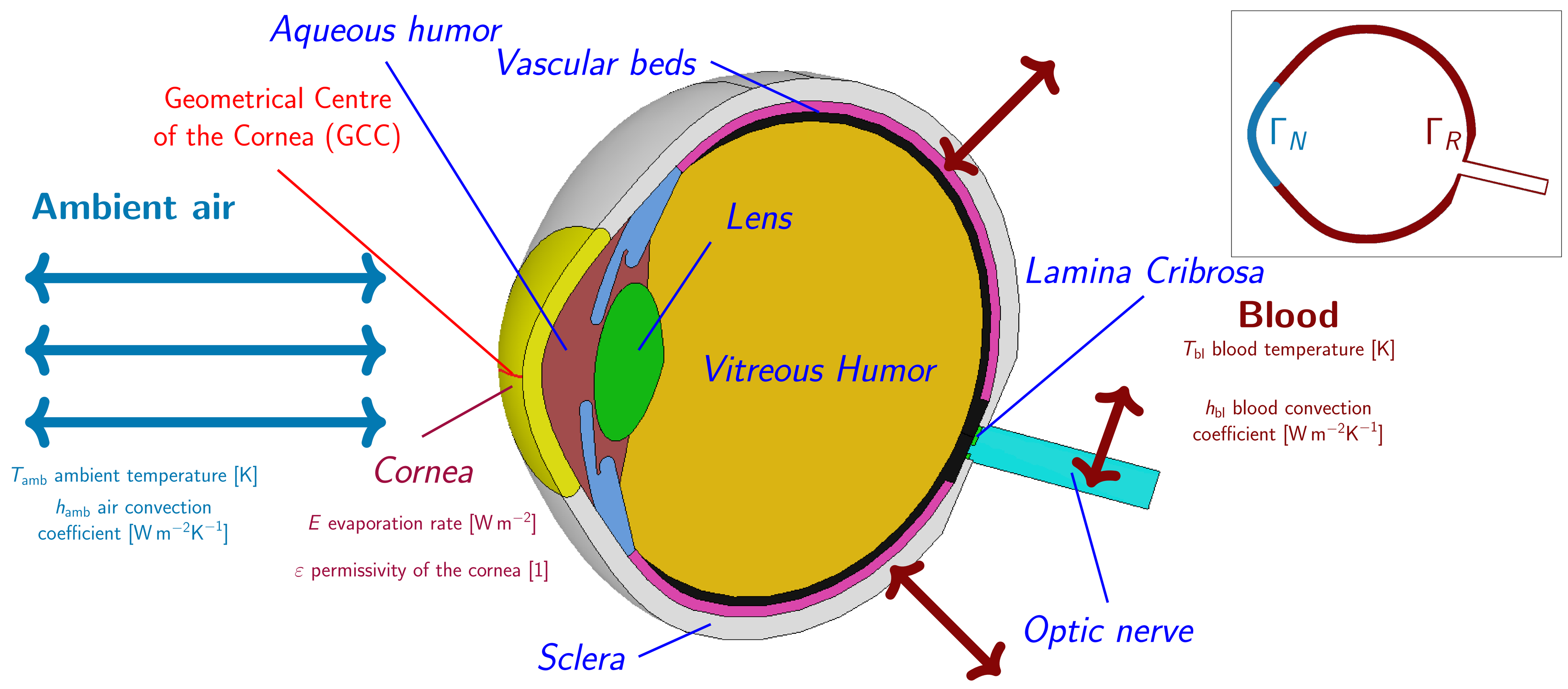
Given $\nu \in D_\mu$, compute efficiently $s_{k,N}(\nu)$

Online Stage

(efficient)

does **not** depend on finite element space

3D parametrized model of the human eye [1]



Heat transfer equation: $\nabla \cdot (k_i \nabla T_i) = 0$ on $\Omega = \cup_i \Omega_i$

- $i \in \{1, \dots, 10\}$ is the volume index (Cornea, VitreousHumor, Lens, Lamina, OpticNerve...),
- T_i [K] is the temperature in the volume i,
- k_i [W m$^{-1}K^{-1}$] is the thermal conductivity

Boundary conditions:

- Non linear Neumann condition on Γ_N:

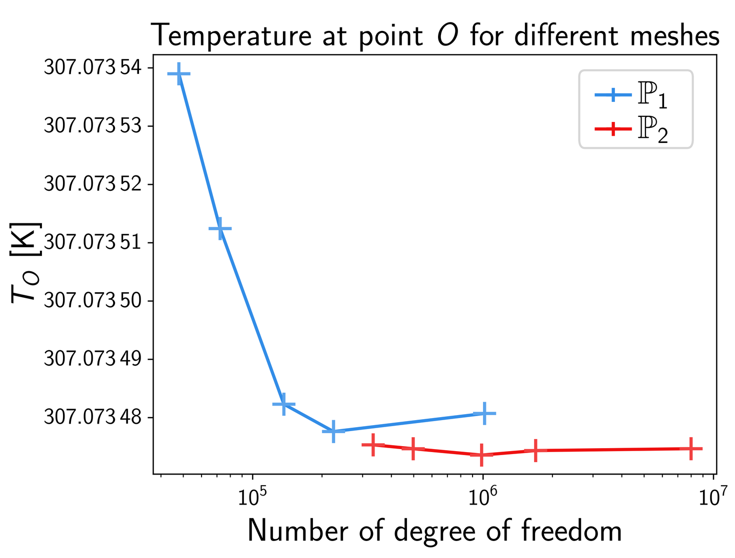
$$-k_{\text{cornea}} \frac{\partial T}{\partial \mathbf{n}} = h_{\text{amb}}(T - T_{\text{amb}}) + \sigma\varepsilon(T^4 - T_{\text{amb}}{}^4) + E$$

Stefan–Boltzmann constant: $\sigma = 5.670\,\mathrm{W\,m^{-2}K^{-4}}$

- Robin condition on Γ_R: $-k_i \dfrac{\partial T}{\partial \mathbf{n}} = h_{\text{bl}}(T - T_{\text{bl}})$

Verification and validation of the model

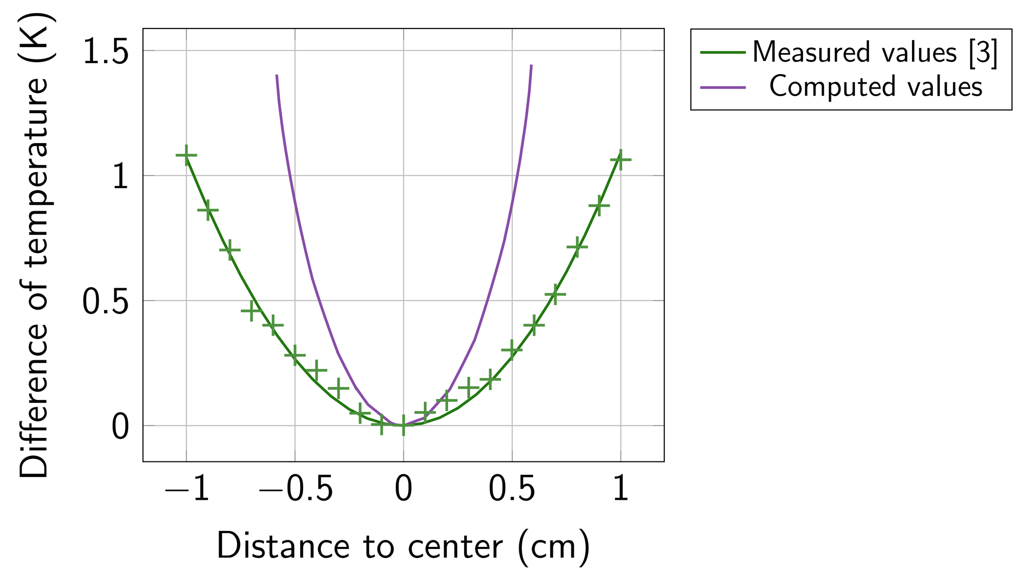
Mesh convergence study:



Order of convergence for a toy problem with analytical solution:

Norm	$\mathbb{P}_1$		$\mathbb{P}_2$	
	Theorical	Model output	Theorical	Model output
L^2	2	2.382	3	4.0304
H^1	1	1.176	2	2.613

Comparaison with measured data over GCC:



Ocular surface temperature:

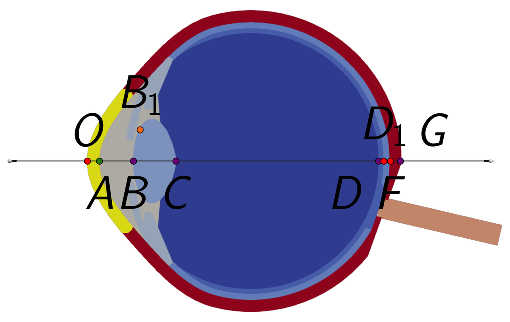
Model prediction	307.34 K
Experimental literature [4]	307.80 K

Uncertainty quantification

Deterministic analysis:

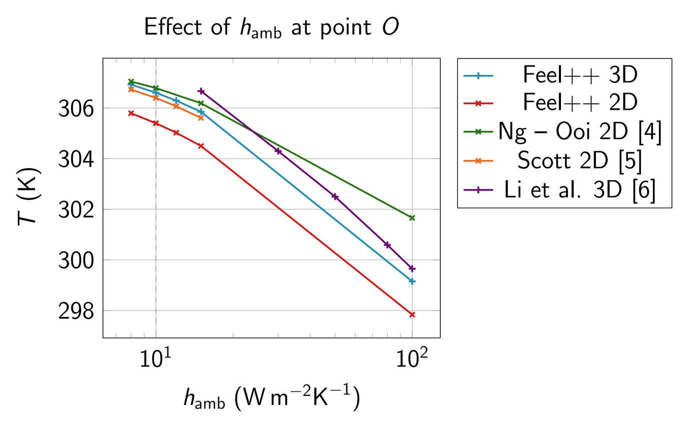
One parameter can vary, while all the other are fixed to a baseline value.

We focus on specific locations and the mean value of the temperature in the cornea.

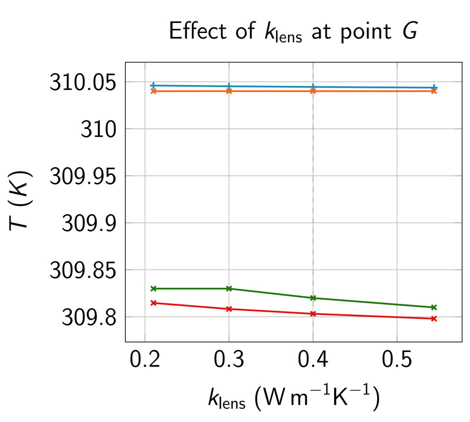


Featured geometrical locations for the output of interest (temperature)

Example of parameter with an impact on the output, and comparaison with litterature



Parameter with no impact on the output



Sensitivity analysis:

Uncertainty on parameters

- $\mu = (\mu_1, \dots, \mu_n)$ with $\mu_i \sim X_i$
- X_i are independant random variables

Output of interest

$Y = s_{k,N}(\mu) = f(X_1, \dots, X_n)$

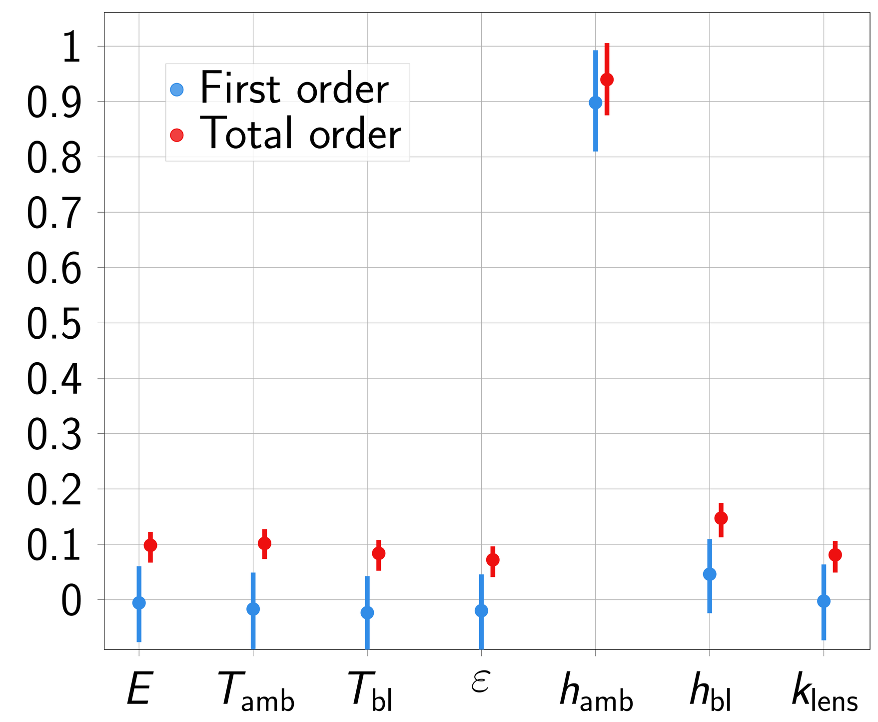
Sobol indices

- First order $S_i = \dfrac{\mathrm{Var}\left[\mathbb{E}\left[Y|X_i\right]\right]}{\mathrm{Var}[Y]}$
- Total order $S_i^{\text{tot}} = \dfrac{\mathrm{Var}\left[\mathbb{E}\left[Y|X_{-i}\right]\right]}{\mathrm{Var}[Y]}$

where $X_{-j} = (X_1, \dots, X_n) \setminus X_j$

Computed using Openturns

Sobol indices on the output T_O:



Parameters with significant impact: h_{amb} then h_{bl}.

Parameters with moderate or minimal impact: ε, T_{bl}...

Total order indices show that interaction of higher order are present.

Conclusion and next steps

- Set up of a complex framework to assess *via* model reduction and sensitivity analysis the influence of parameters on heat transfer in the human eye.
- More complex models: coupling with aqueous humor fluid dynamics, include multiscale aspects (IOP dynamics described by a non linear ODE), assess influence of geometric parameters (such as cornea thickness).
- Potential clinical application: local drug delivery in the eye, influence of aging.

References

1. E.-H. Ooi et al. (2008). "Simulation of aqueous humor hydrodynamics in human eye heat transfer". In: *Computers in Biology and Medicine* 38.2, pp. 252–262
2. C. Prud'homme et al. (2021). *feelpp/feelpp: Feel++ V109*. Version v0.109.0
3. N. Efron et al. (1989). "Ocular surface temperature." In: *Current eye research* 8 9, pp. 901–6
4. Ng, E.Y.K. and Ooi, E.H. (2006). "FEM simulation of the eye structure with bioheat analysis". In: *Computer Methods and Programs in Biomedicine* 82.3, pp. 268–276
5. J. Scott (1988). "A finite element model of heat transport in the human eye". In: *Physics in Medicine and Biology* 33.2, pp. 227–242
6. E. Li et al. (2010). "Modeling and simulation of bioheat transfer in the human eye using the 3D alpha finite element method (αFEM)". in: *International Journal for Numerical Methods in Biomedical Engineering* 26.8, pp. 955–976