CS 103: Mathematical Foundations of Computing
Problem Set #4

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Due Friday, October 28 at 2:30 pm Pacific

Sets can be written as $\{ 1, 2, 3 \}$ or as $\{ n \in \mathbb{N} \mid n \geq 137 \}$.

Multicharacter superscripts and subscripts need braces: $x^{137x+42}$ or x_{137}.

Problem One: Independent and Dominating Sets

i.

{b, g, d}

{a, c, g}

ii.

Theorem: Let $G = (V, E)$ be an arbitrary graph with the following property: every node in G is adjacent to at least one other node in G. Prove that if I is an independent set in G, then $V - I$ is a dominating set in G.

Proof: Consider an arbitrary graph $G = (V, E)$ with the following property: every node in G is adjacent to at least one other node in G. Pick an arbitrary node $x \in G$ such that $x \notin (V - I)$. There must be some node y where $\{x, y\} \in E$. We want to show that $y \in (V - I)$. Because $x \notin (V - I)$ and $x \in V$, we know that $x \in I$. Because $x, y\} \in E$, we know that y cannot be in I. Therefore, the y must be in $V - I$, which is what we wanted to prove.

iii.

$I = \{b, g, d\}$

$J = \{d, h, f, a\}$

iv.

Theorem: Prove that if I is a maximal independent set in $G = (V, E)$, then I is a dominating set of G.

Proof: Consider a graph, $G = (V, E)$ and a maximal independent set in G, I. For the sake of contradiction assume that I is not a dominating set of G. Pick an arbitrary node $x \in I$. Because I is not a dominating set of G, we know there exists a node $y \notin I$, such that $\{x, y\} \notin E$. Now consider the set, $I' = I \cup \{y\}$. Because y does not form any edges with any arbitrary x in I, I' is independent. Moreover, as $I \subseteq I'$ and $I \neq I'$, we know that I is a strict subset of I'. However, this is impossible, as we stated earlier that I is a maximal independent set. Therefore, our previous assumption must have been false.

Problem Two: Friends, Strangers, Enemies, and Hats

i.

Theorem: In a party with 36 attendees, each person wears a hat of seven possible colors: aureolin, bole, chartreuse, drab, ecru, fulvous, and gamboge. Prove that you can always find three mutual friends all wearing the same color hat or three mutual strangers all wearing the same color hat.

Proof: Given that there are 36 attendees and 7 possible color choices, by the pigeonhole principle, we have that there must be at least $\lceil \frac{36}{7} \rceil$ of people wearing the same colored hat. Let's consider three of the pairs. Because, by the Theorem on Friends and Strangers, we know that the pairs can either friends or strangers, we use the pigeonhole principle to claim that any person will have at least $\lceil \frac{6}{2} \rceil$ friends or strangers. Without loss of generality, assume that a person, x has three friends: a, b, and c. If a and b to be friends, b and c to be friends, or a and c to be friends, then one of those edges plus the two edges connecting back to node x would form three mutual friends. Otherwise, a and b are strangers, b and c are strangers, and a and c are strangers, so a, b, and c would be mutual strangers. Overall, this gives that you can always find three mutual friends all wearing the same color hat or three mutual strangers all wearing the same color hat, as required.

ii.

Theorem: There's a party with 17 attendees. For each pair of people at the party, either those people are strangers, those people are friends, or those people are enemies. Prove that you can always find three mutual friends, or three mutual strangers, or three mutual enemies.

Proof: By the pigeonhole principle, we know for any single person, x, there are at least $\lceil \frac{16}{3} \rceil$ friends, strangers, or enemies. Without loss of generality, assume that x has $\lceil \frac{16}{3} \rceil$, or 6 friends. Now pick any person, y, such that that x is friends with y. We proceed with two possible cases:

Case 1: If y is friends with anyone aside from x, three mutual friends are formed, as required.

Case 2: If y is not friends with anyone other than x, by the pigeonhole principle, we know that y has at least $\lceil \frac{5}{2} \rceil$ strangers, or enemies. Without loss of generality, assume that y is strangers with $\lceil \frac{5}{2} \rceil$, or 3, people. Consider a person z such that x is friends with z and y is strangers with z. We proceed with three cases, and for simplicity, only regarding the three people that x is friend with and y is strangers with:

Case 2a: If z is friends with anyone besides x, three mutual friends are formed, as required.

Case 2b: If z is strangers with anyone that y is strangers with, three mutual strangers are formed, as required.

Case 2c: Otherwise, z is enemies with the other two people. We know there are two people a and b, such that a and b are enemies with z, strangers with y, and friends with x. No matter what relationship a has with b, three mutual friends, strangers, or enemies are formed, as required.

Problem Three: Iterated Injections

i.

Theorem: Prove that the sequence must contain at least one duplicate value

Proof: Assume for the sake of contradiction that the sequence $f^0(a), f^1(a), f^2(a), ..., f^k(a)$, did not have at least one duplciate value. Because f is an injection, and f is applied k times, there must have been $k+1$ unique values $x \in A$ in the sequence, as $f^0(a) = a$. However, this is impossible because $|A| = k$. Therefore, we have reached a contradiction, so our original assumption must have been wrong. The sequence must contain at least one duplicate value.

ii.

Theorem: Using your result from part (i), prove that $f^{n+1}(a) = a$.

Proof: Suppose for the sake of contradiction that $f^{n+1}(a) \neq a$. Since f is an injection, there must exist some value x in the sequence such that $f^{n+1}(a) = x$. Since the sequence thus far consists of all unique values, we know there exists some k in range $0 < k \leq n$ such that $f^k(a) = x$. However, this is impossible because f is an injection. We have reached a contradiction so our original assumption must have been wrong. Thus, $f^{n+1}(a) = a$.

iii.

Theorem: Prove that f is bijective

Proof: Assume $f : A \to A$ is a well-defined injection from a finite set to itself. We want to show that f is also a bijection. To do so, we will show that f is surjective.
Pick any arbitrary $a \in A$. From our above proofs, we know that there exists some $n \in N$ where $f^{n+1}(a) = a$. Thus, for any arbitrary $a \in A$, there exists an $x \in A$ (namely $f^n(a)$) such that $f(x) = a$. As a was selected arbitrarily, we've shown that f is surjective. Therefore, because f is injective and surjective, we've shown that it is also bijection.

Optional Fun Problem: Forced Triangles

Write your answer to the Optional Fun Problem here.