CS 103: Mathematical Foundations of Computing
Problem Set #3

[Caleb Liu, Justin Shen]

February 4, 2023

Due Friday, October 21 at 2:30 pm Pacific

Problem One is autograded. You won't include your answers to that problem here.

Symbols Reference

Here are some symbols that may be useful for this problem set.

- f is a function from A to B: $f : A \to B$.
- Function composition: $f \circ g$.
- Power sets: $\wp(() S)$.
- Unions and intersections: $S \cup T$, $S \cap T$.
- Cardinality: $|S|$.
- Sets: $\{ 1, 2, 3 \}$ or $\{ n \in \mathbb{N} \mid n \geq 137 \}$
- Exponents (use curly braces if exponent is more than 1 character): x^2, 2^{3x}
- Subscripts (use curly braces if subscript is more than 1 character): x_0, x_{10}

Problem Two: $|\mathbb{N}| = |\mathbb{Z}|$

i. Fill in the blanks for Problem Two, part i. below.

- $f(0) = 0$.
- $f(1) = -1$.
- $f(2) = 1$.
- $f(3) = -2$.
- $f(4) = 2$.
- $f(5) = -3$.

ii.

Theorem: Prove that f is a bijection.

Proof: To prove that f is a bijection, we need to prove that f is injective and surjective.

Proof of injectivity: Pick an arbitrary $a \in N$ and $b \in N$, such that $a = b$. We need to show that $f(a) = f(b)$. We can split the proof into two cases:

Case 1: a and b are even. Then, $\exists k_1, k_2 \in N$ such that $a = 2k_1$ and $b = 2k_2$. Since $a = b$, we see

$$\begin{aligned} a &= b \\ (2k_1) &= (2k_2) \\ k_1 &= k_2 \end{aligned} \tag{1}$$

and thus, $f(a) = k_1 = f(b) = k_2$.

Case 2: a and b are odd. Then, $\exists k_1, k_2 \in N$ such that $a = 2k_1 + 1$ and $b = 2k_2 + 1$. Since $a = b$, we see

$$\begin{aligned} a &= b \\ (2k_1 + 1) &= (2k_2 + 1) \\ 2k_1 &= 2k_2 \\ k_1 &= k_2 \end{aligned} \tag{2}$$

and thus $f(a) = -(k_1 + 1) = f(b) = -(k_2 + 1)$.

which is what we wanted to show.

Proof of Surjectivity: Pick an arbitrary $y \in \mathbb{Z}$. We need to show there exists an $x \in N$ such that $f(x) = y$. We can split the proof into two cases:

Case 1: $y = k$ for some $k \in N$. Then, $x = 2k$ is a solution.

Case 2: $y = -(k + 1)$ for some $k \in N$. Then, $x = 2k + 1$ is a solution.

Therefore, f is surjective. Since f is both injective and surjective, it is a bijection.

Problem Three: Strictly Increasing Functions

i.

$f(x) = x^2$ is not strictly increasing, but $f(x) = x^3$ is strictly increasing.

ii.

Theorem: Let $f : \mathbb{Z} \to \mathbb{Z}$ and $g : \mathbb{Z} \to \mathbb{Z}$ be arbitrary strictly increasing functions. Prove that $g \circ f$ is strictly increasing.

Proof: Consider any $x, y \in \mathbb{Z}$ such that $x < y$. We need to show that $g(f(x)) < g(f(y))$. Because function f is strictly increasing and $x < y$, we know that $f(x) < f(y)$. Furthermore, because function g is strictly increasing and $f(x) < f(y)$, we know that $g(f(x)) < g(f(y))$, which is what we need to show.

iii.

Theorem: Let $f : \mathbb{Z} \to \mathbb{Z}$ be an arbitrary strictly increasing function. Prove that f is injective.

Proof: We will prove the contrapositive, that if $x \neq y$, $f(x) \neq f(y)$. To do so, consider any $x, y \in \mathbb{Z}$ such that $x \neq y$. We want to show that $f(x) \neq f(y)$. As $x \neq y$, either $x < y$ or $y < x$. Assume, without loss of generality, that $x < y$. Because $x < y$ and f is strictly increasing, we know that $f(x) < f(y)$. Because $f(x) < f(y)$, $f(x) \neq f(y)$, which is what we needed to show.

iv.

Theorem: Let $f : \mathbb{Z} \to \mathbb{Z}$ be a strictly increasing function and consider any integers x and y. Prove that if $f(x) = y$, and $f(y) = x$, then $x = y$.

Proof: We will take the contrapositive of the above theorem, namely if $x \neq y$, then $f(x) \neq y$ or $f(y) \neq x$. For the sake of contradiction, consider any $x, y \in \mathbb{Z}$ such that $x \neq y$ and $f(x) = y$ and $f(y) = x$.

As $x \neq y$, either $x < y$ or $y < x$. Assume, without loss of generality, that $x < y$. Because $x < y$ and f is strictly increasing, we know that $f(x) < f(y)$. We then see

$$\begin{gathered} f(x) < f(y) \\ y < x \end{gathered} \tag{3}$$

This is impossible because x < y. We have reached a contradiction, so our assumption must have been wrong, so if $x \neq y$, then $f(x) \neq y$ or $f(y) \neq x$, which is what we wanted to show.

Problem Four: Eventual Bijections

i. Fill in the blanks for Problem Four, part i. below.

- $f^3(2) = 9$.
- $f^{137}(1) = 1$.
- $f^0(137) = 137$.

ii.

Theorem: Let $f : A \to A$ be a function. Prove that if f^3 is surjective, then f is surjective.

Proof: Pick an arbitrary $y \in A$. We need to show that there exists an $x \in A$, such that $f(x) = y$. Because $f^3(x)$ is surjective, we know that there exists an $a \in A$, such that $f(f(f(a))) = y$. Now, let $x = f(f(a))$. We see that

$$\begin{aligned} f(f(f(a))) &= y \\ f(x) &= y \end{aligned} \tag{4}$$

which is what we wanted to prove.

iii.

Theorem: Let $f : A \to A$ be a function. Prove that if f^3 is injective, then f is injective.

Proof: We will prove the the contrapositive: if f is not injective, f^3 is not injective. Pick an $a, b \in A$, such that $a \neq b$. We want to show that $f^3(a) \neq f^3(b)$. Because f is injecitve and $a \neq b$, we know that $f(a) \neq f(b)$. We then see

$$\begin{aligned} f(a) &\neq f(b) \\ f(f(a)) &\neq f(f(b)) \\ f(f(f(a))) &\neq f(f(f(b))) \end{aligned} \tag{5}$$

which is what we needed to show.

Problem Five: Understanding Diagonalization

i.

$\mathbb{N}$

ii.

Any element that is not the empty set and that does not contain all the elements in the domain (i.e. $\{5\}$)

iii.

This set contains all natural numbers that are $< n$.

iv.

All natural numbers that are $< n - 1$

v.

$f(n) = \{n \in \mathbb{N} | 2n - 1\}$. This function produces all natural numbers that are odd. Therefore, the set D will contain any natural number that is no odd, and not contain any number that is odd (even natural numbers).

Problem Six: Simplifying Cantor's Theorem?

The incorrect statement is the following: "Since f is not surjective, it is not a bijection. Thus $|S| \neq |\wp(S)|$." f not being surjective does not prove that $|S| \neq |\wp(S)|$ because there could be some other function that is bijective. In order to prove that $|S| \neq |\wp(S)|$, we must pick an ***arbitrary*** function $f : S \to \wp(S)$ and prove that it is not bijective.

Problem Seven: Proofs on Sets

iii.

Theorem: For all sets A and B, if $\wp(A) = \wp(B)$, then $A = B$.

Proof: For the sake of contradiction, assume that $\wp(A) = \wp(B)$ and $A \neq B$. Therefore, there exists an $x \in A$, where $x \notin B$. If $x \in A$, $\{x\} \in \wp(A)$. Because $\wp(A) = \wp(B)$, $\{x\} \in \wp(B)$. However, this is impossible because $x \notin B$. We have reached a contradiction, so our previous assumption must be false.

Problem Eight: The Universal Set

i.

Theorem: Prove that if A and B are arbitrary subsets where $A \subseteq B$, then $|A| \leq |B|$.

Proof: Pick two arbitrary subsets A and B where $A \subseteq B$. Secondly, consider the function $f : A \to B$ where $f(x) = x$. We want to show that f is an injection.

First, we'll show that f is a well-defined function. Because $A \subseteq B$, we know that $\forall x\ (x \in A \to x \in B)$. Therefore, for any $x \in A$, we have $f(x) = x \in B$

Second, we'll show that f is injective. Pick any $x_1, x_2, \in A$ where $f(x_1) = f(x_2)$. We need to show that $x_1 = x_2$. Since $f(x_1) = f(x_2)$, we see by definition of f that $x_1 = x_2$, as required.

Because f is an injection, it follows that $|A| \leq |B|$.

ii.

Theorem: Let U be the universal set discussed in problem 8's description. Using your result from (i), prove that if U exists, then $|\wp(U)| \leq |U|$.

Proof: Consider the function $f : \wp(U) \to U$ where $f(x) = x$. Because we proved earlier that if $A \subseteq B$, then $|A| \leq |B|$, we want to prove that $\wp(U) \subseteq U$.

Assuming that U exists, we know that $\forall x(x \in U)$. It follows that for any $x \in \wp(U)$, $x \in U$. Therefore, $\wp(U) \subseteq U$. Because $\wp(U) \subseteq U$, we know from the earlier proof that $|\wp(U)| \leq |U|$.

iii.

Write your answer to Problem Eight, part iii. here.

iv.

Theorem: Let U be the universal set discussed in problem 8's description. Prove that U does not exist.

Proof: Above, we proved that if U exists, then $|\wp(U)| \leq |U|$. By contrapositive, if $|\wp(U)| > |U|$, then U does not exist. Thus, we need to show that $|\wp(U)| > |U|$.

By Cantor's Theorem, we know that because U is a set, $|U| < |\wp(U)|$, which is what we needed to show.

Optional Fun Problem: Infinity Minus Two

Write your answer to the Optional Fun Problem here.