合肥大學
HEFEI UNIVERSITY

Frequency Fitness Assignment
频率适应度分配

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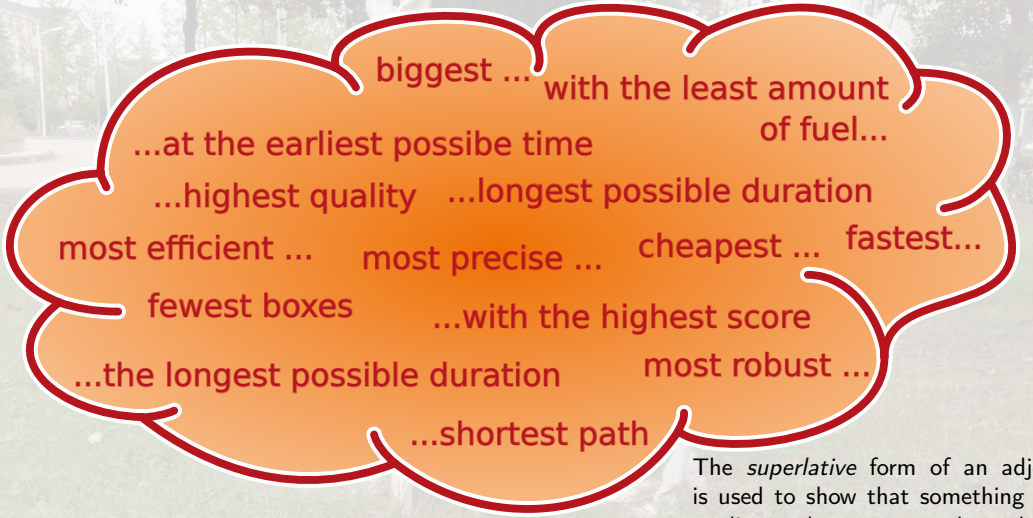
Outline

Introduction

Introduction to Optimization

- Optimization means finding superlatives.



The *superlative* form of an adjective is used to show that something has a quality to the greatest or least degree.

Introduction to Optimization

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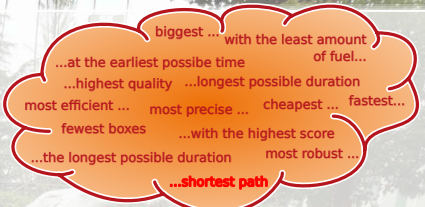
- Find the shortest path from start to goal[26,55].



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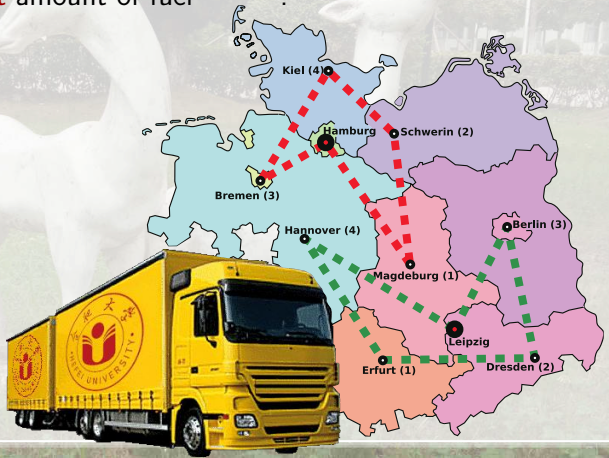

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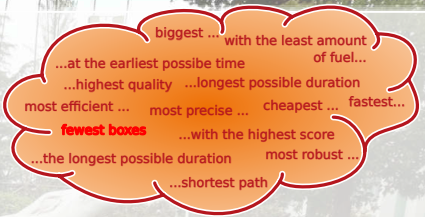

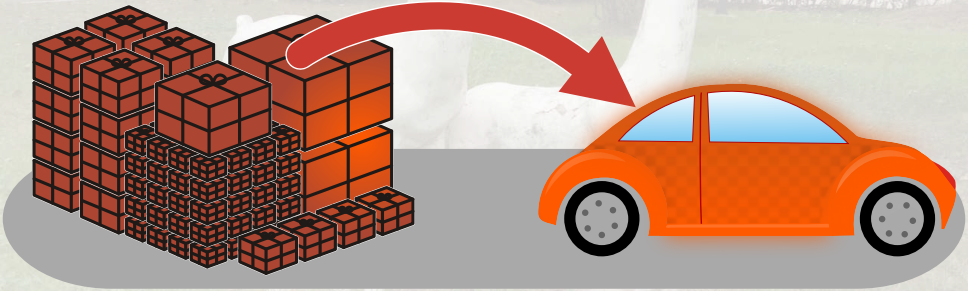

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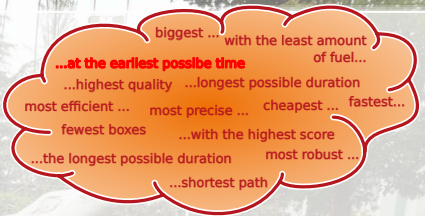

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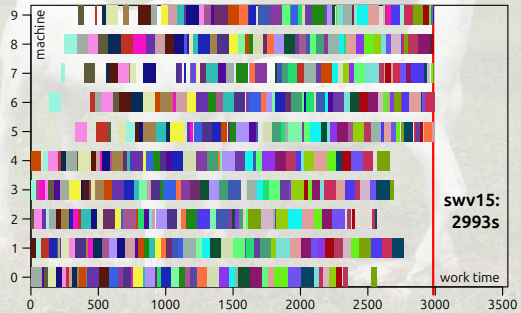

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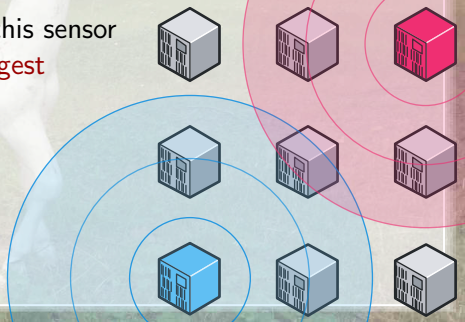

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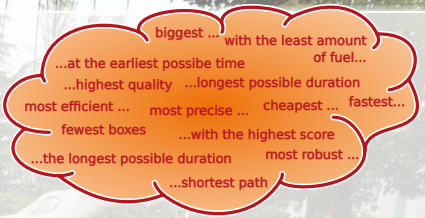

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- And so on.

Views on Optimization

- There are two ways to look at optimization.

Views on Optimization

- The economic view.

Optimization

An optimization problem is a situation which requires deciding for one choice from a set of possible alternatives in order to reach a predefined or required benefit at minimal costs.

Views on Optimization

- The mathematical view.

Optimization

An optimization problem is a situation which requires deciding for one choice from a set of possible alternatives in order to reach a predefined or required benefit at minimal costs.

Solving an optimization problem requires finding an input element $x^\star$ within a set $\mathbb{X}$ of allowed elements for which a mathematical function $f\colon \mathbb{X} \mapsto \mathbb{R}$ takes on the smallest possible value.

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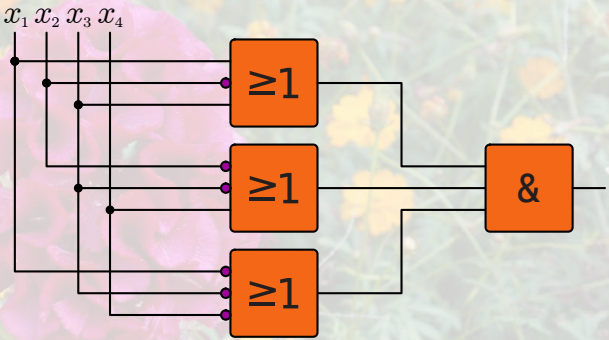
- The optimal solution $x^\star \in \mathbb{X}$ is the shortest possible tour.

Example: Maximum Satisfiability Problem

- The goal of the Maximum Satisfiability (MaxSAT)[13,28] problem is to find a setting of n variables that makes a Boolean formula F become True. The variables appear directly or negated in m OR-clauses, whose results flow into one AND-clause.



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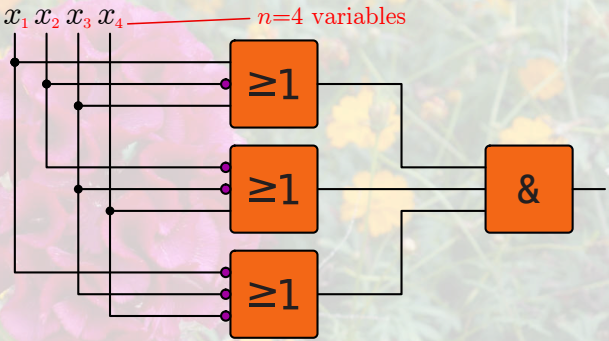

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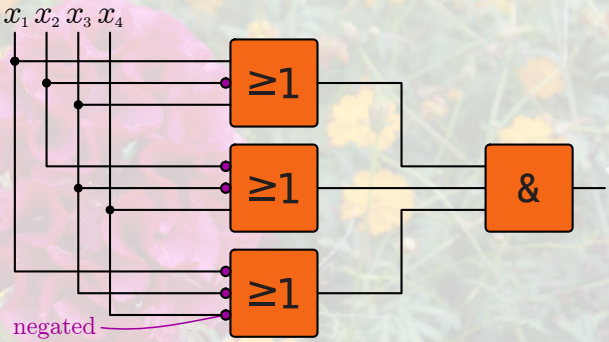

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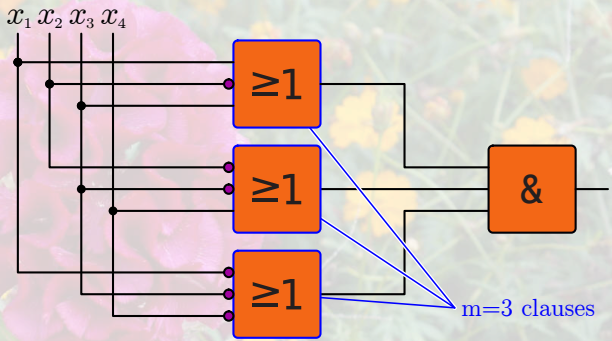

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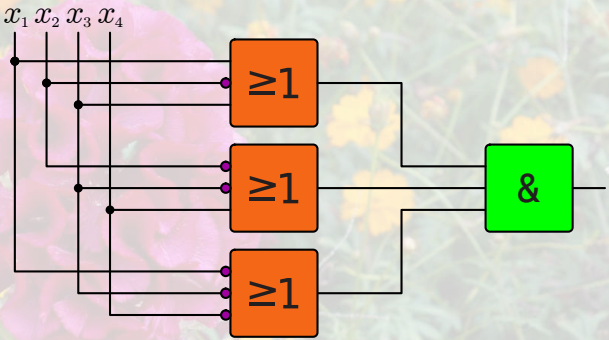

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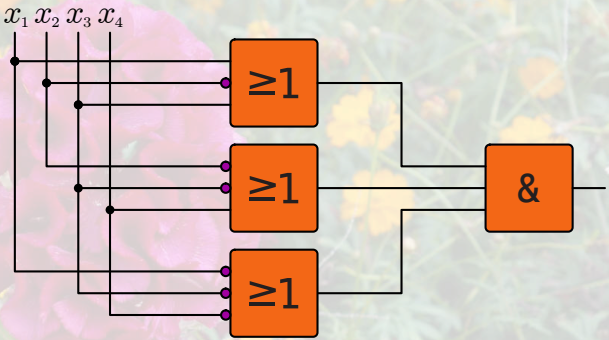

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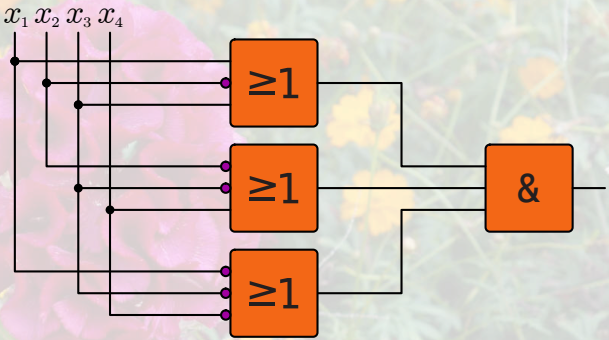

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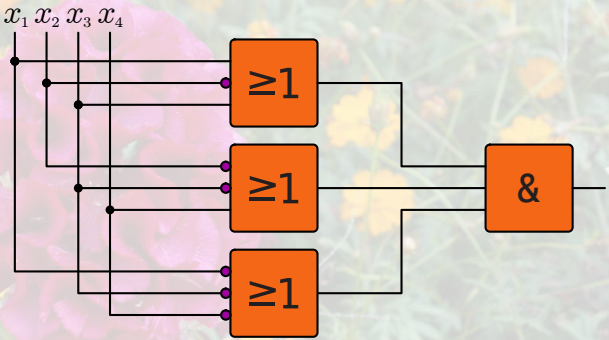

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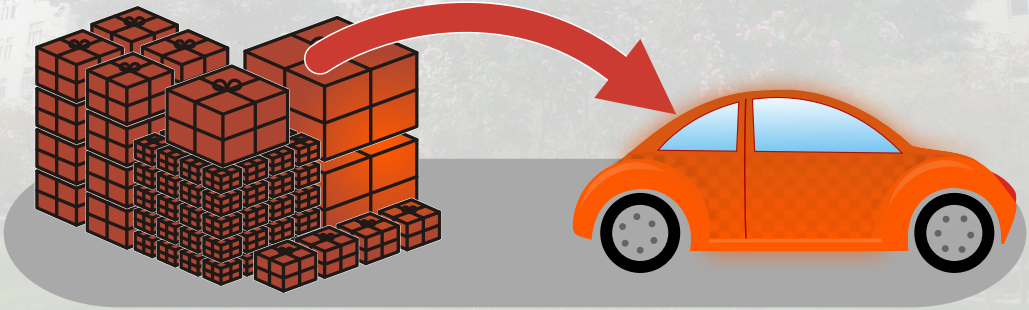

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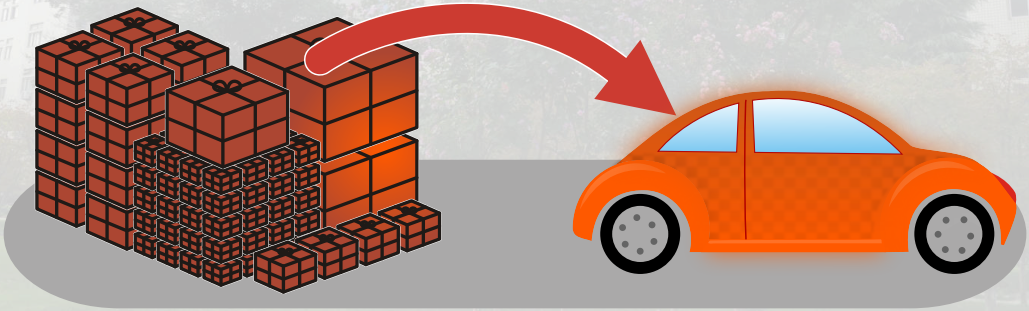

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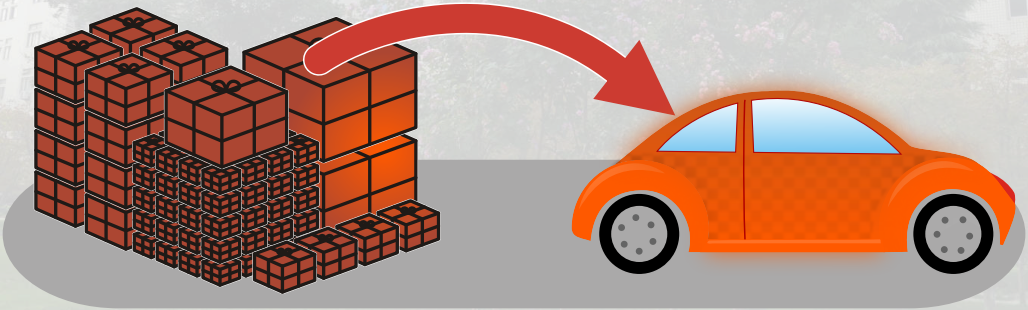

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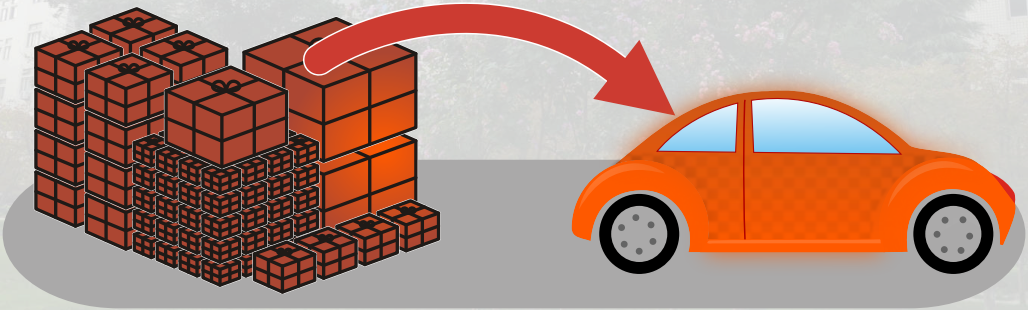

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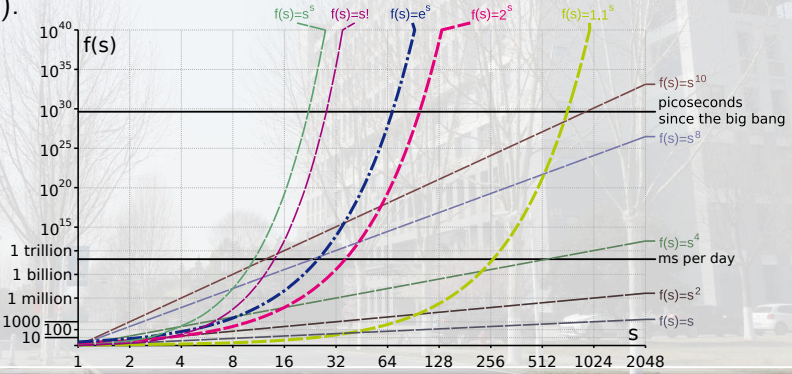

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- In other words, if we want to guarantee to find the best possible solution $x^\star$ for all possible instances of a problem, we often cannot really be much faster than testing all possible candidate solutions $x \in \mathbb{X}$ in the worst case.

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Begin with a set $S_0 \subset \mathbb{X}$ of one or multiple randomly sampled solutions

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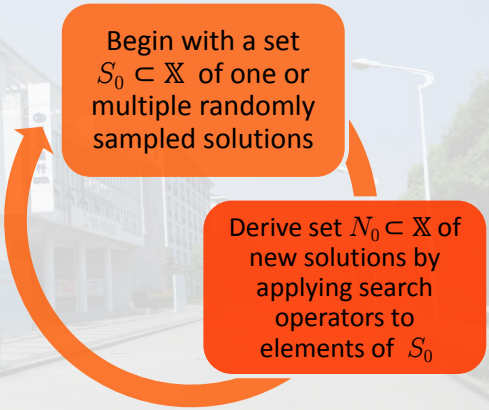

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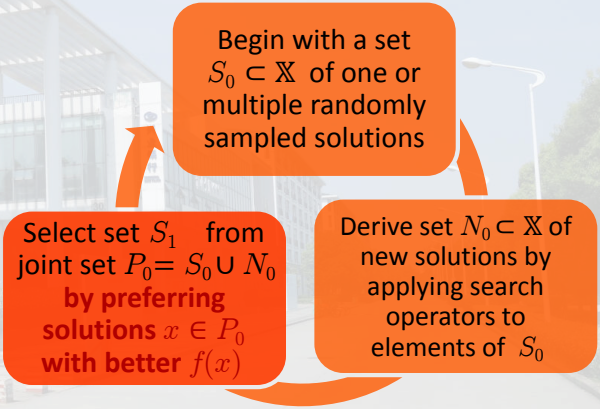

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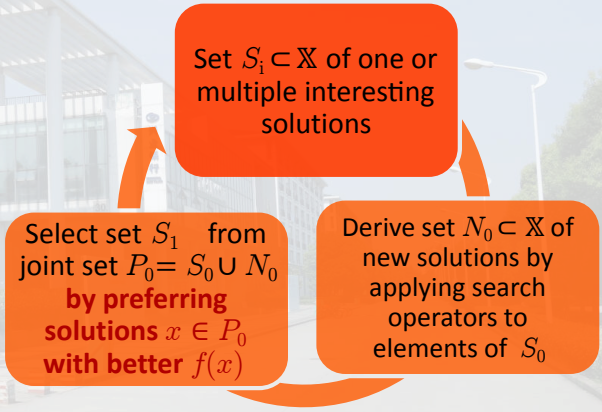

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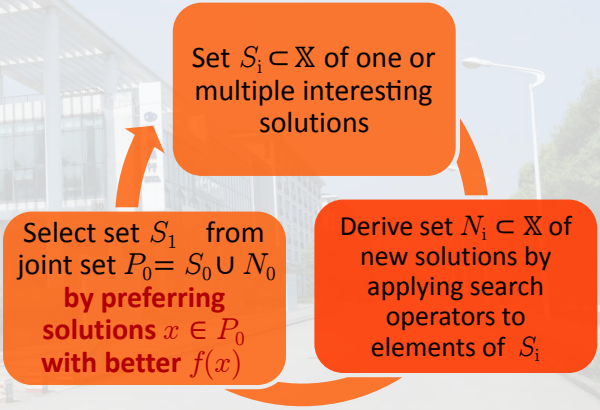

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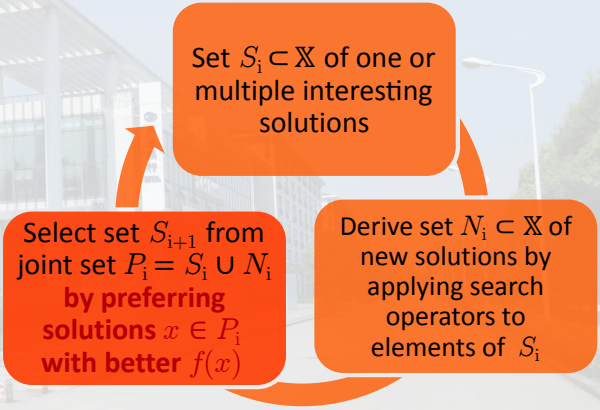

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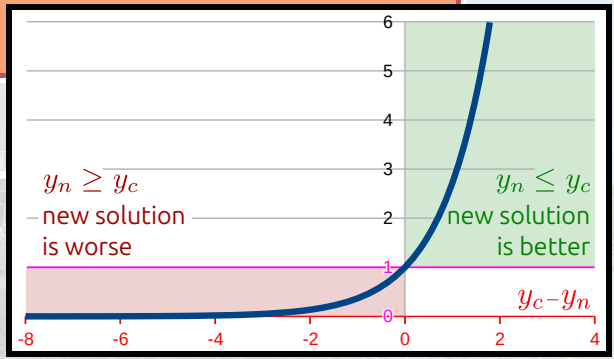

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- The Standard Genetic Algorithm (SGA) with Fitness Proportionate Selection (Roulette Wheel) is for maximization[5,15,22,48,49,69].
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procedure SGA($f : \mathbb{X} \mapsto \mathbb{R}^+$, ps, cr) ▷ **for maximization!**
  $x_B \leftarrow \emptyset$; $y_B \leftarrow -\infty$; ▷ best-so-far solution
  for $j \in 1 \ldots ps$ **do** ▷ random initial population
    randomly sample $S_0[j].x$ from $\mathbb{X}$; $S_0[j].y \leftarrow f(S_0[j].x)$;
    if $S_0[j].y > y_B$ **then** $x_B \leftarrow S_0[j].x$; $y_B \leftarrow S_0[j].y$;
  for $i \in 0 \ldots \infty$ **do** ▷ iterate "generations"
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- Let's keep these things in mind.

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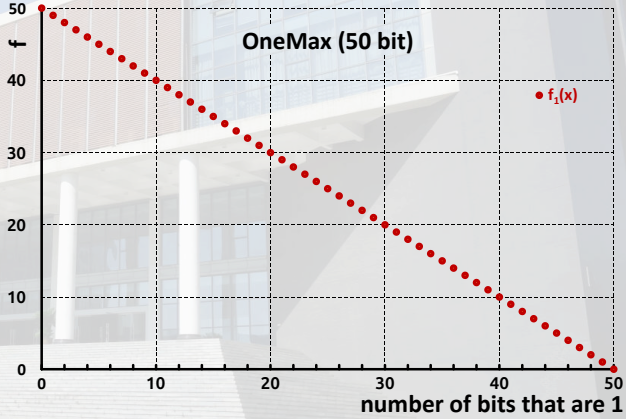
- OneMax is the simplest benchmark problem in discrete optimization[50].



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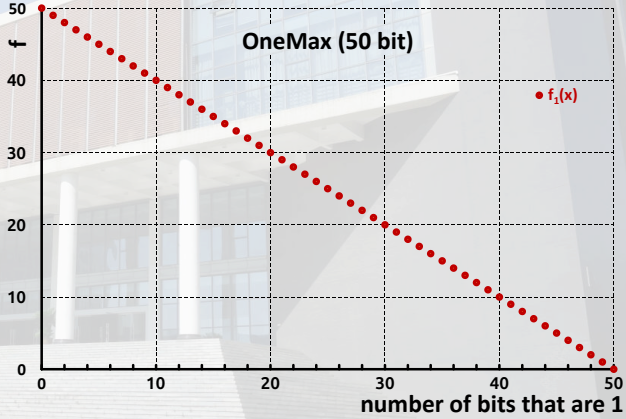

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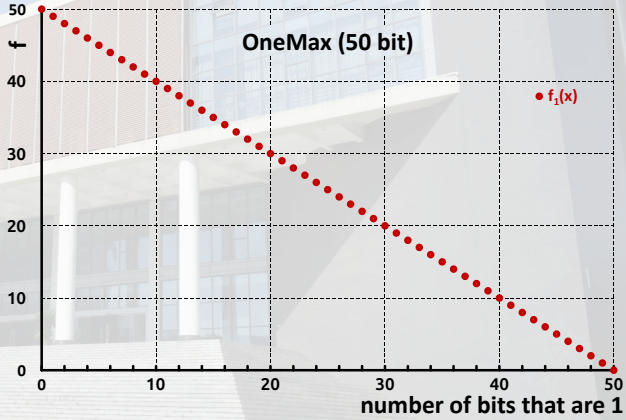

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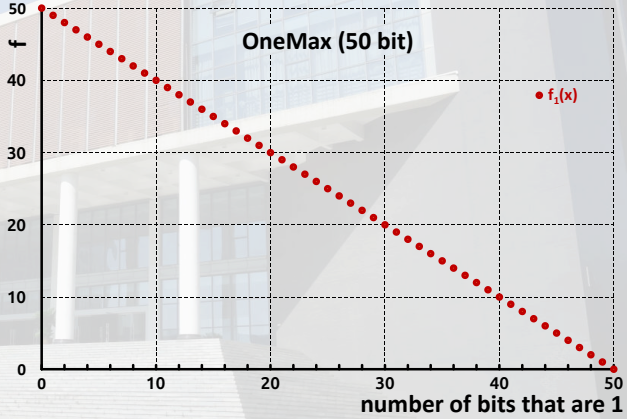

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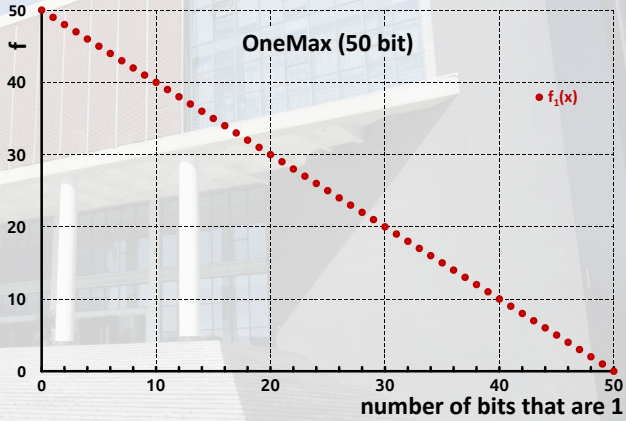

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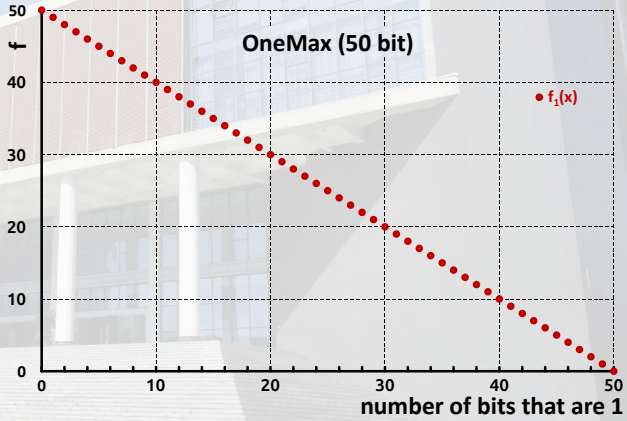

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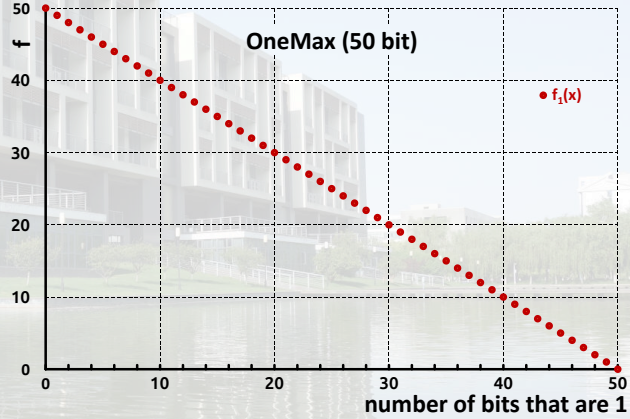

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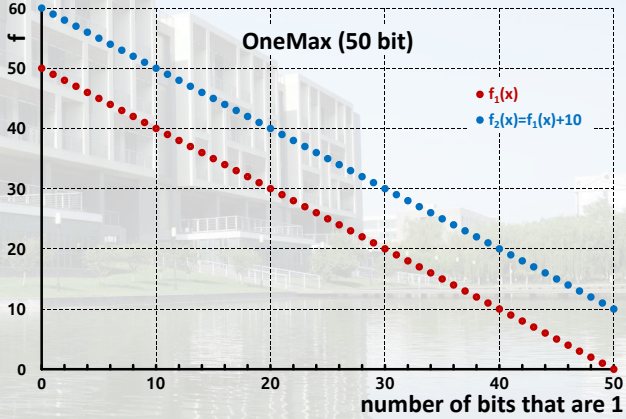
- $f_2(x) = f_1(x) + 10$.



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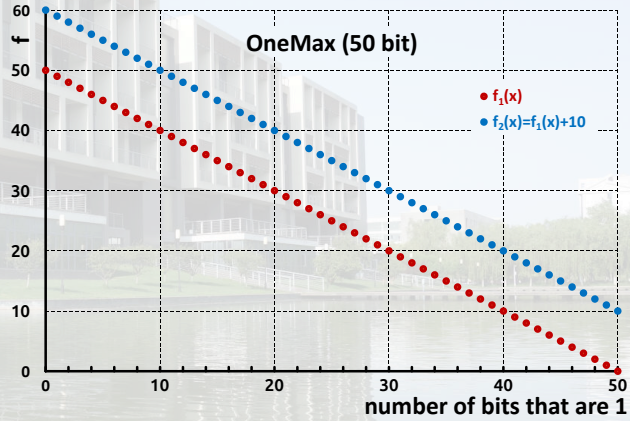

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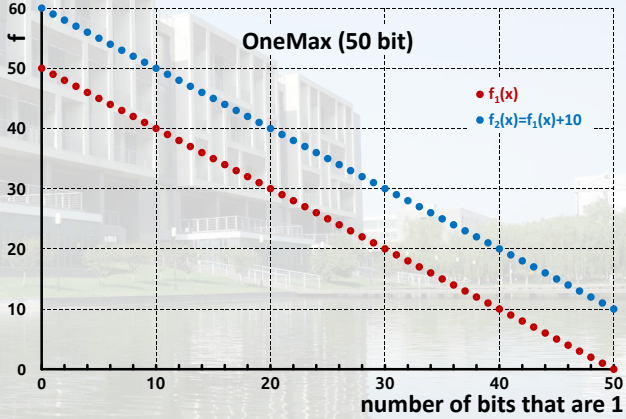

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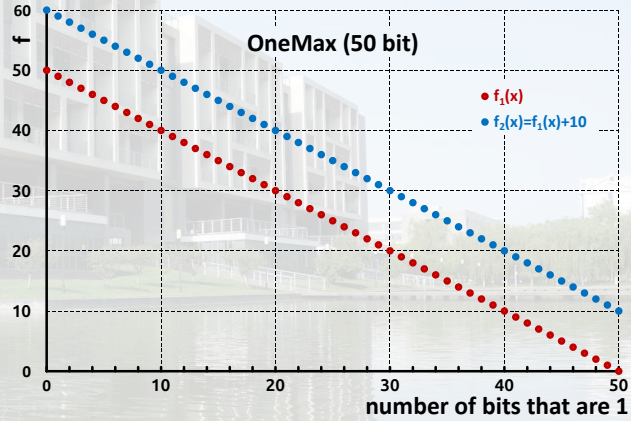

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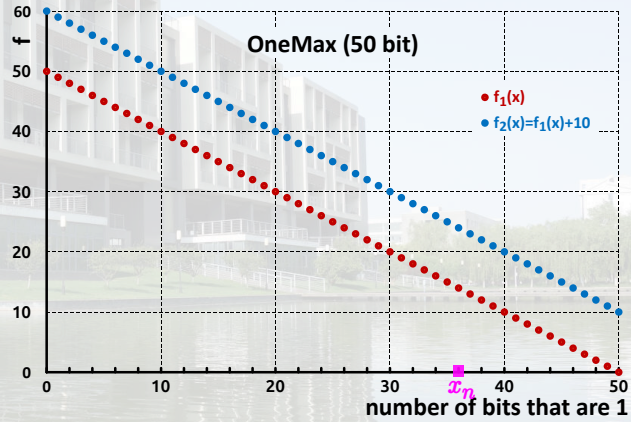

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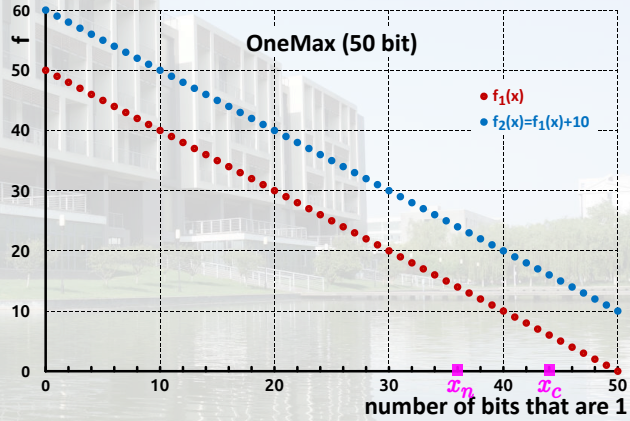

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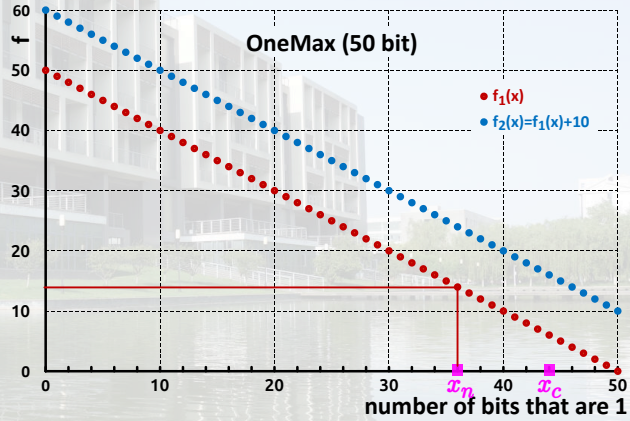

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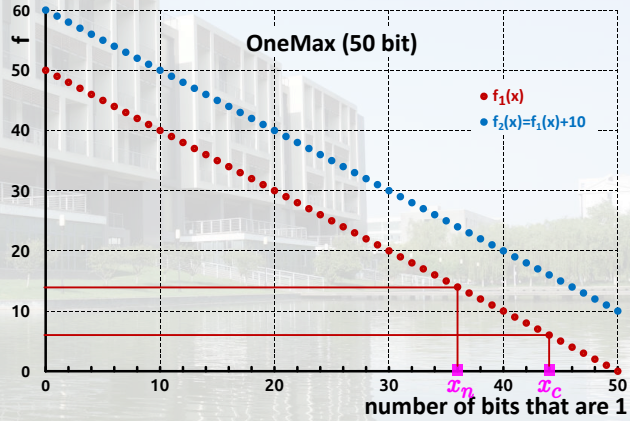

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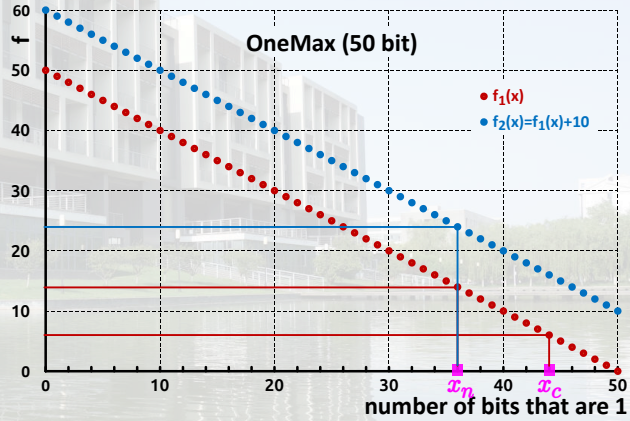

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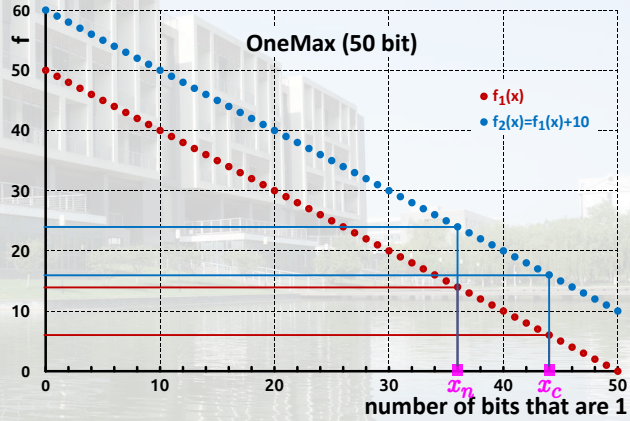

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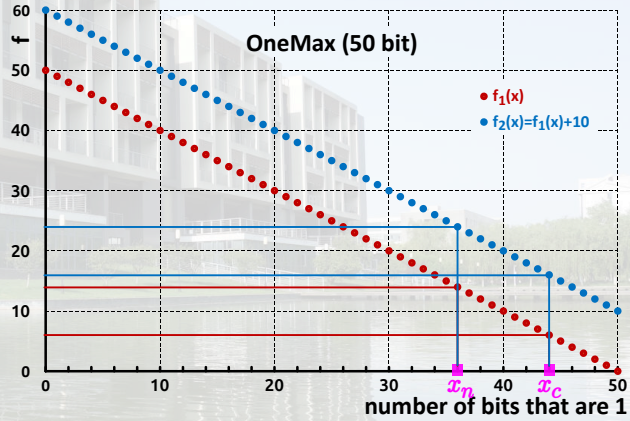
- $(1+1)$ EA: $14 \leq 6 = 24 \leq 16$ ✓



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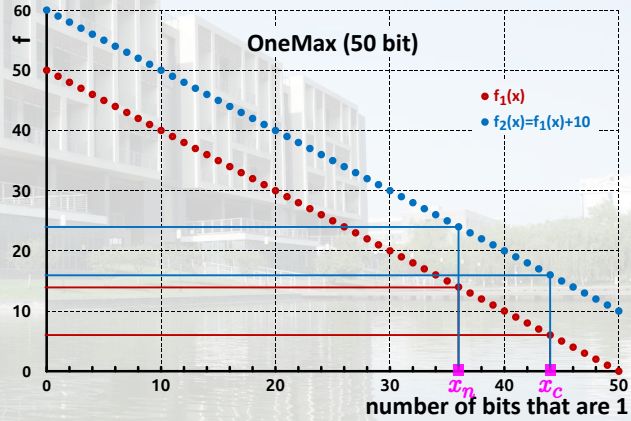
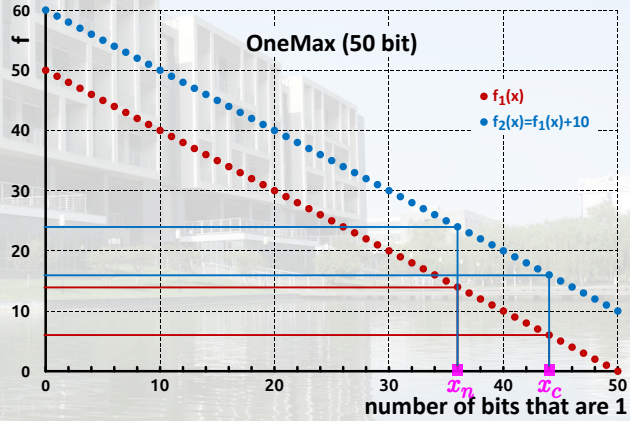
- SA: $6 - 14 = 16 - 24$ ✓



Invariance Properties: OneMax Example – Translation

- Now I create a slightly modified variant of this problem.
- $f_2(x) = f_1(x) + 10$.
- Expectation: Any reasonable algorithm should perform the same on f_1 and f_2, i.e., make the exact same decisions given the same random seeds.
- $f_1(x_n) = 14,\ f_1(x_c) = 6,$ $f_2(x_n) = 24,\ f_2(x_c) = 16$
- $(1+1)$ EA: $14 \leq 6 = 24 \leq 16$ ✓
- SA: $6 - 14 = 16 - 24$ ✓
- SGA: $1-14/(14+6) = 0.3 \neq 1-24/(24+16) = 0.4$ ✗



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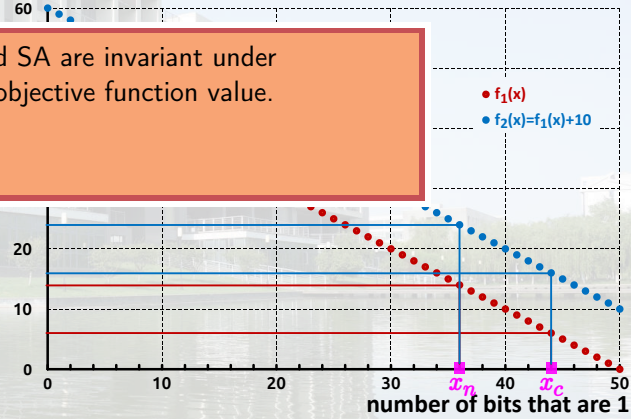
- The $(1+1)$ EA and SA are invariant under translations of the objective function value.



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- The $(1+1)$ EA and SA are invariant under translations of the objective function value.
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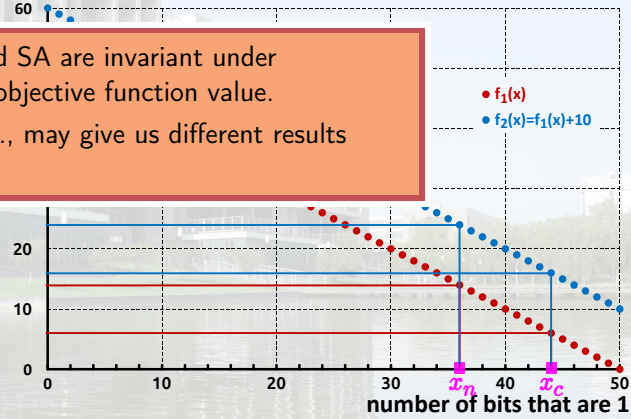

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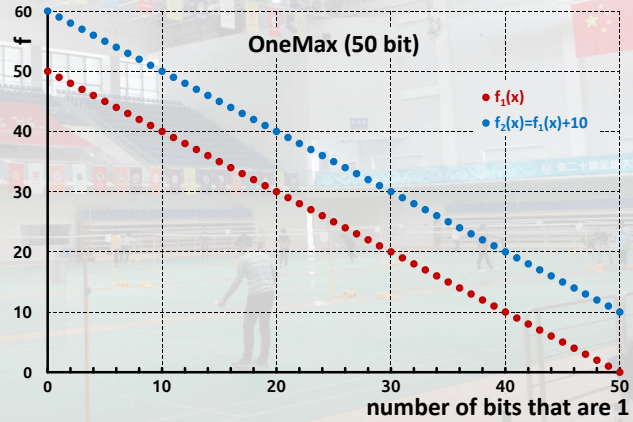

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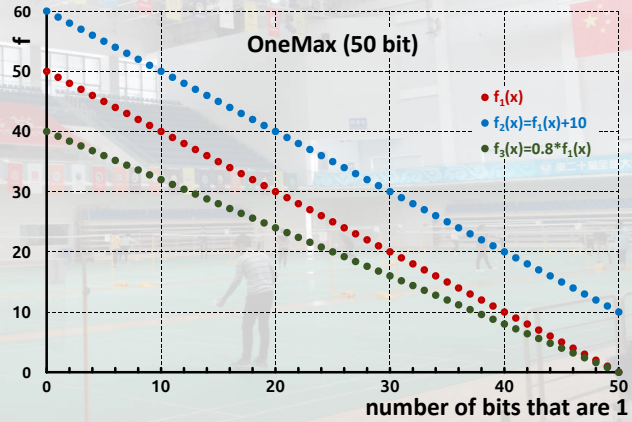
- $f_3(x) = 0.8 f_1(x)$.



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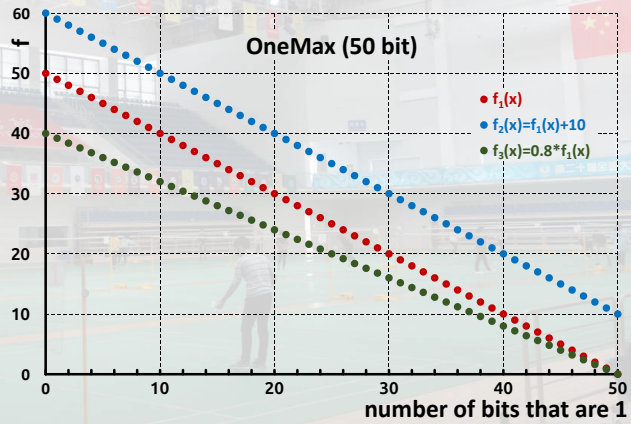

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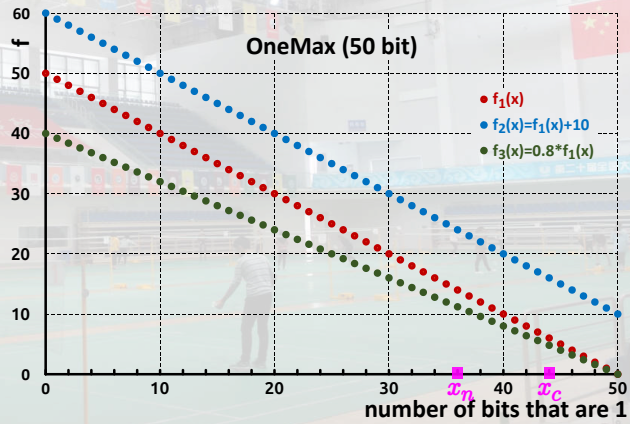

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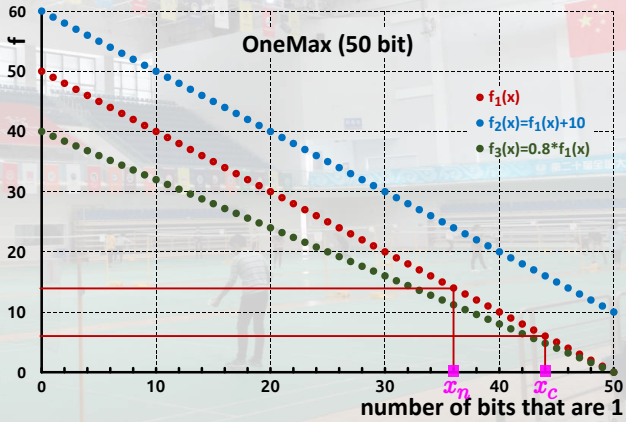

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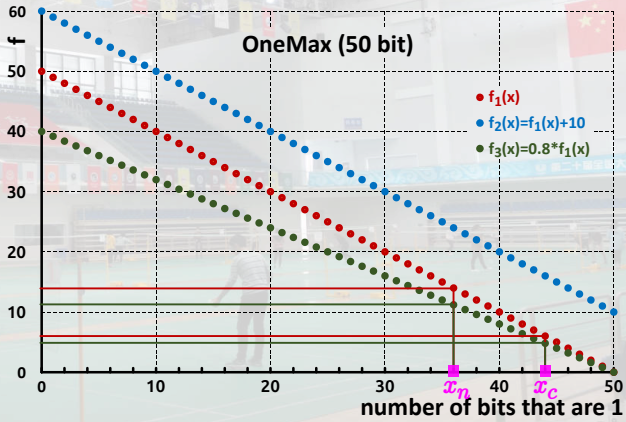
- $f_1(x_n) = 14$, $f_1(x_c) = 6$, $f_3(x_n) = 11.2$, $f_3(x_c) = 4.8$



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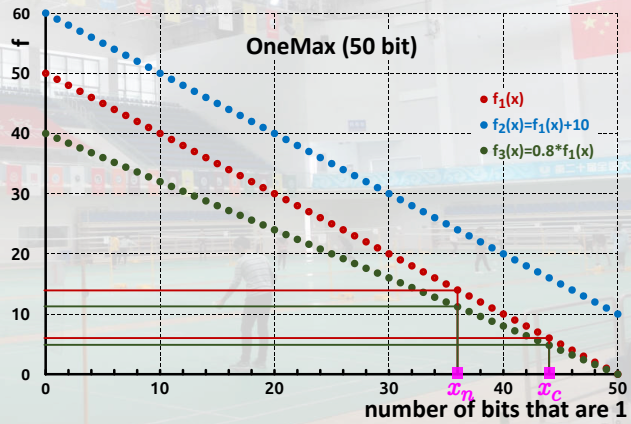

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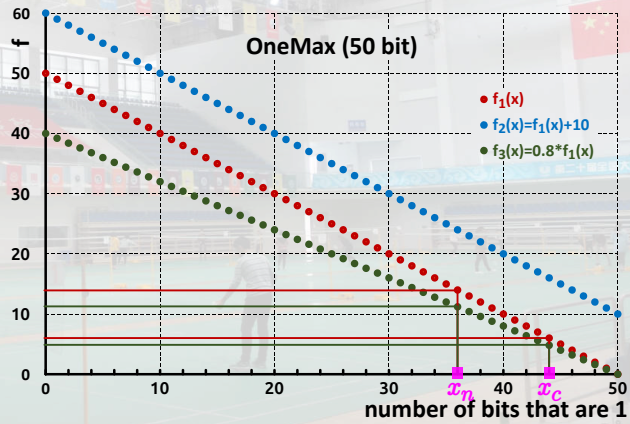

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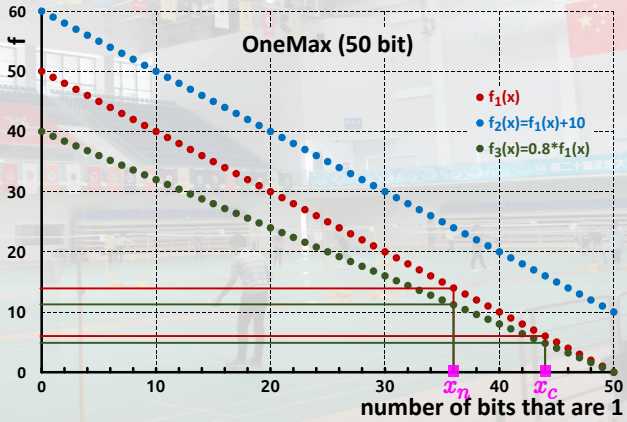

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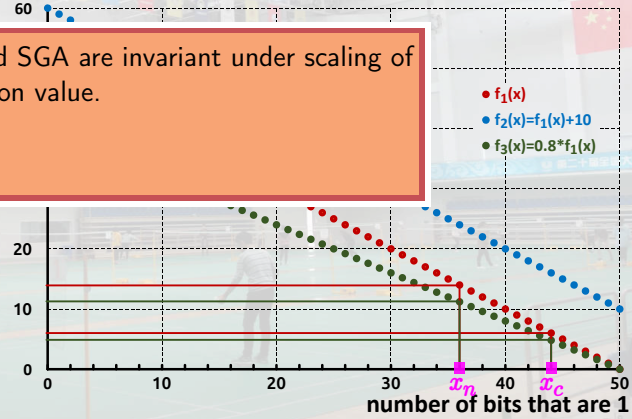

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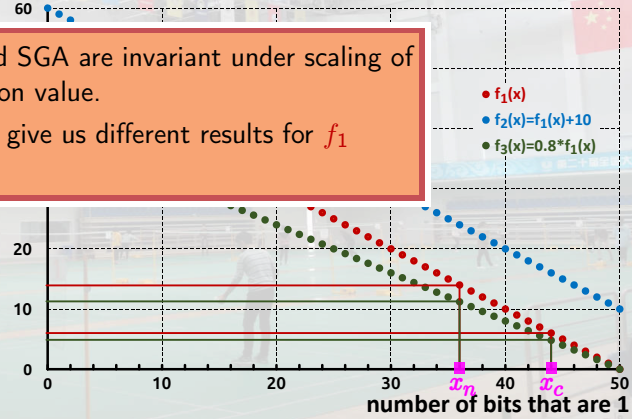

Invariance Properties: OneMax Example – Squaring

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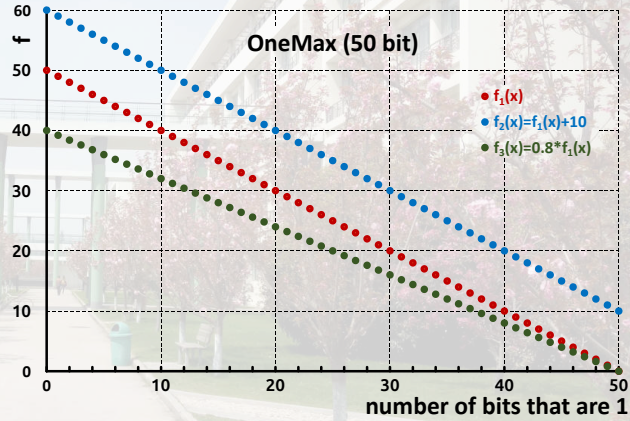

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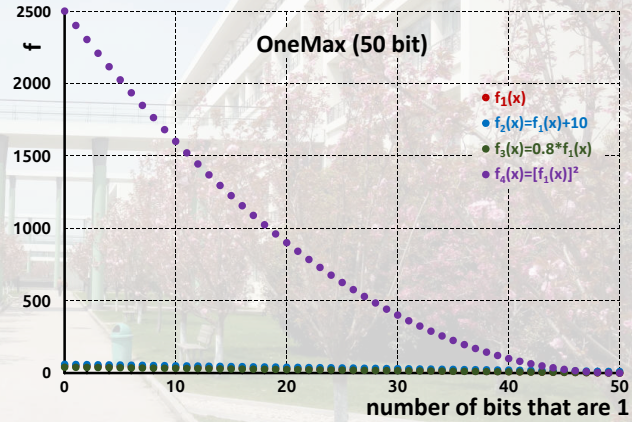
- $f_4(x) = [f_1(x)]^2$.



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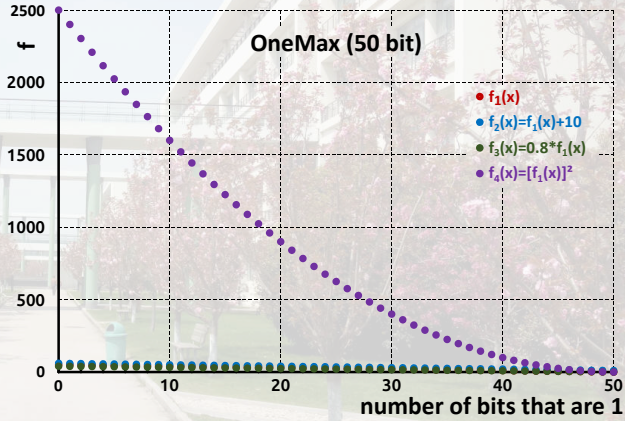

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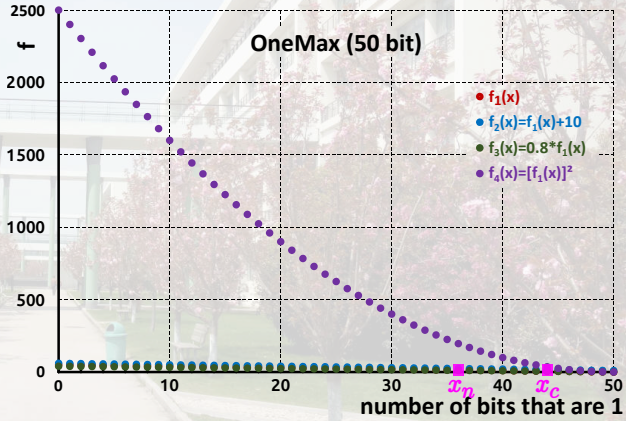

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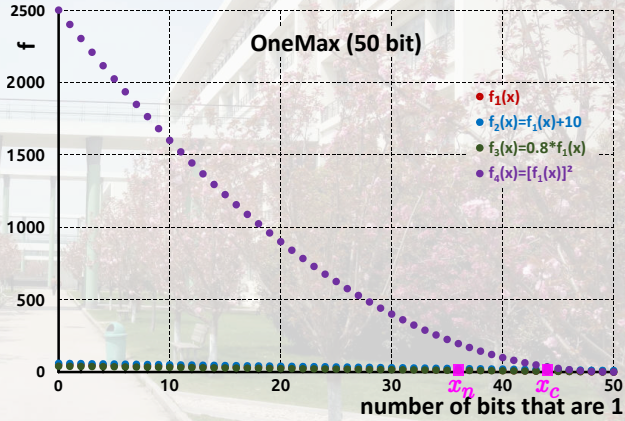
- $f_1(x_n) = 14$, $f_1(x_c) = 6$, $f_4(x_n) = 196$, $f_4(x_c) = 36$



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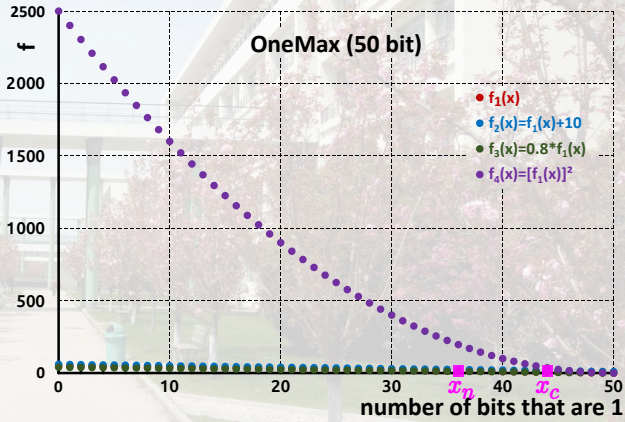
- $(1+1)$ EA: $14 \leq 6 = 196 \leq 36$ ✓



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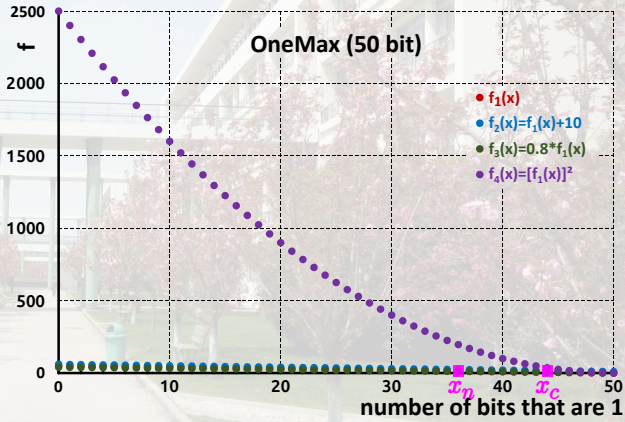

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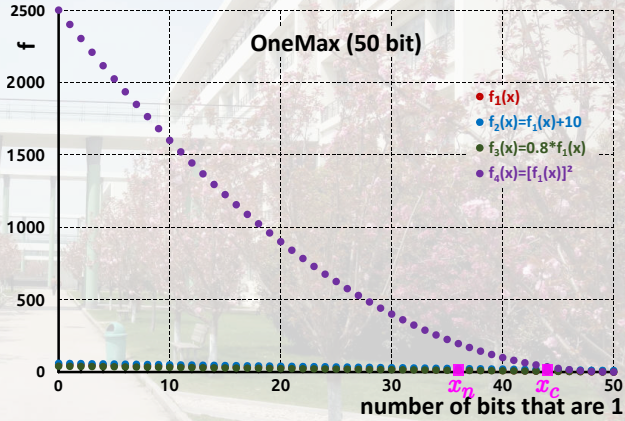

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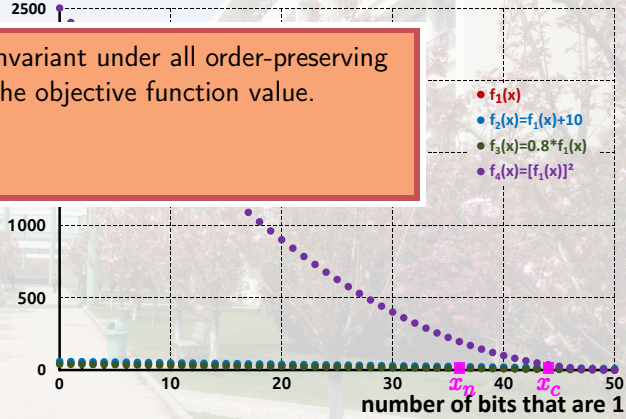
- The $(1+1)$ EA is invariant under all order-preserving transformations of the objective function value.



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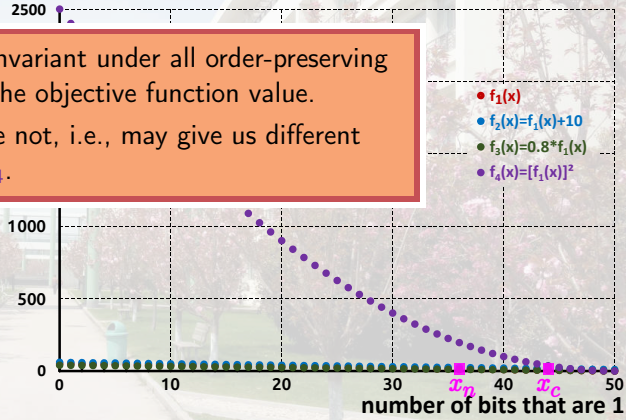
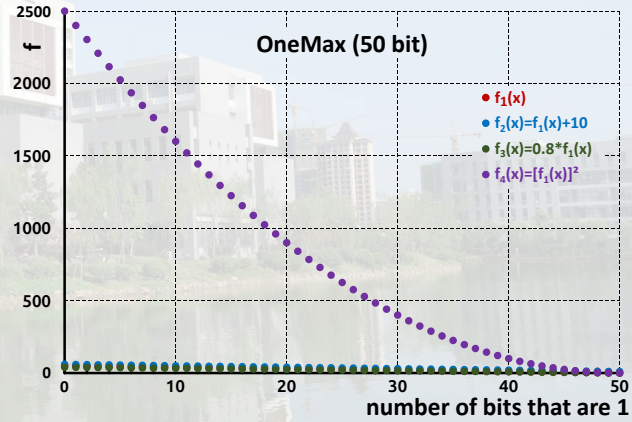



Now let's enter eerie territory...

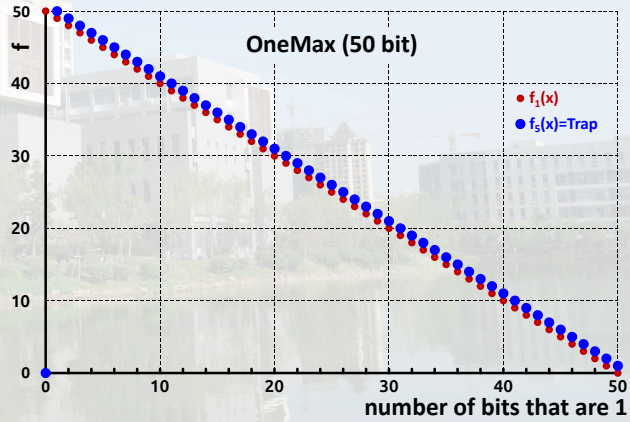
Invariance Properties: OneMax Example – Trap

- Now I create another modified variant of this problem: a trap[17,51].



Invariance Properties: OneMax Example – Trap

- Now I create another modified variant of this problem: a trap[17,51].
- $f_5(x) = \begin{cases} 0 & \text{if } f_1(x) = 50 \\ 1 + f_1(x) & \text{otherwise} \end{cases}$



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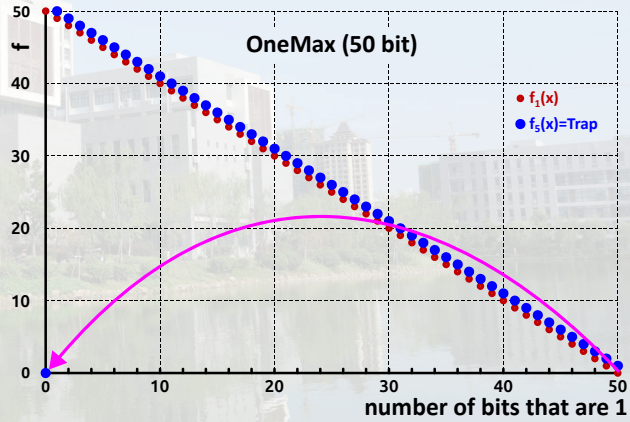

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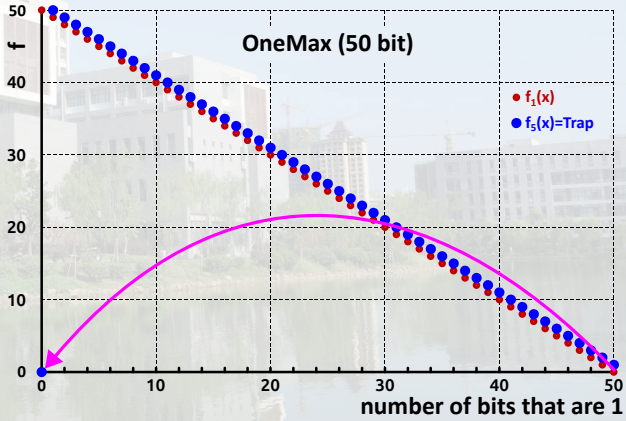
- $f_5(x) = \begin{cases} 0 & \text{if } f_1(x) = 50 \\ 1 + f_1(x) & \text{otherwise} \end{cases}$
- Expectation: Algorithm performance on f_1 does not carry over to f_5 .



Invariance Properties: OneMax Example – Trap

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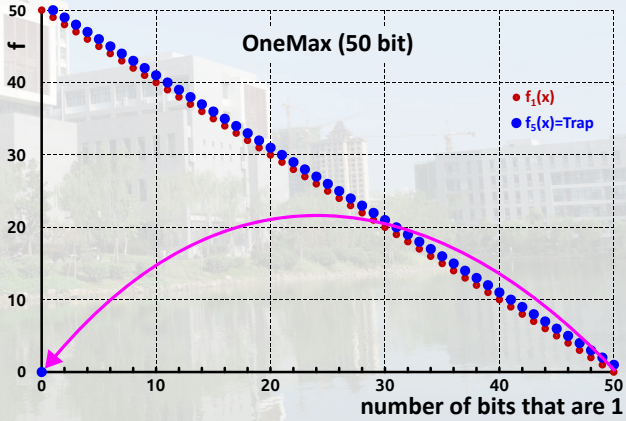

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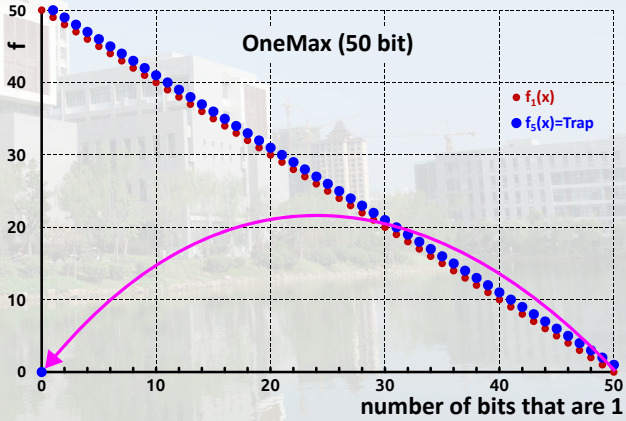

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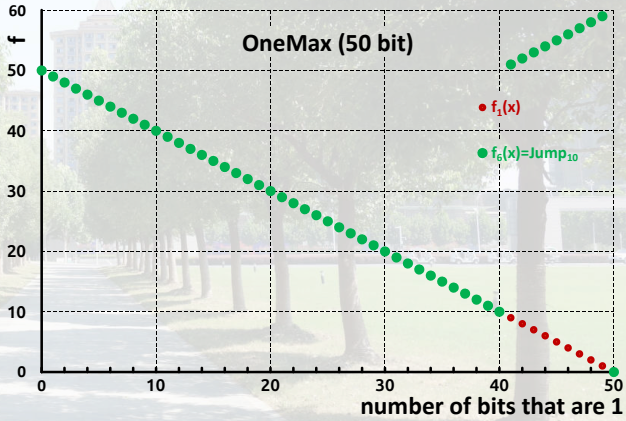

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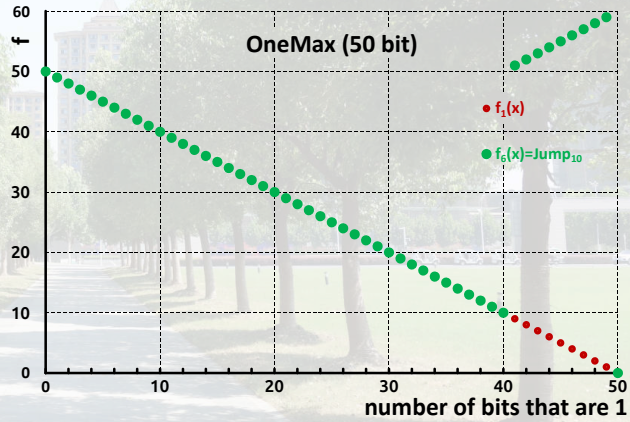

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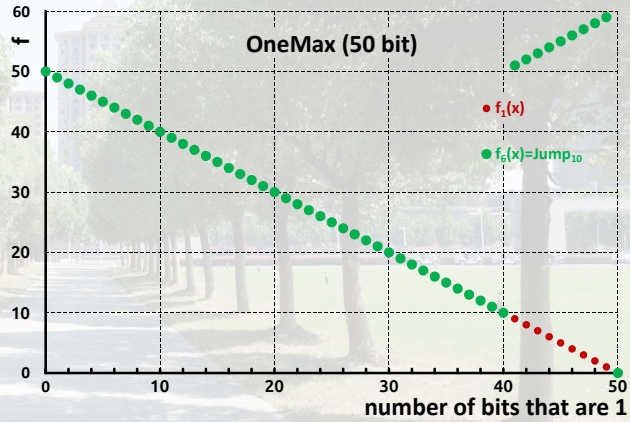

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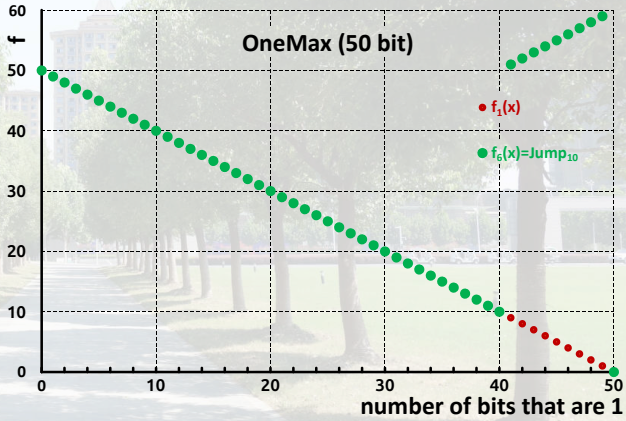

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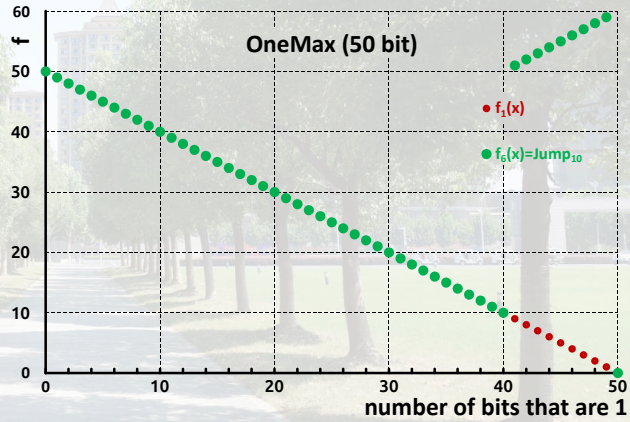
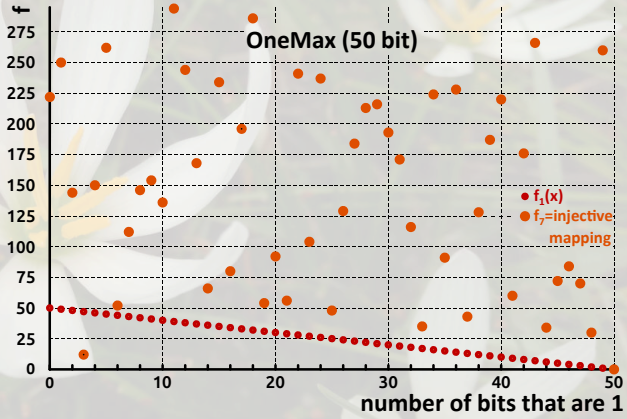


Invariance Properties: OneMax Example – Injection

- How about I apply an arbitrary injection g that preserves the location of optimum to f_1 and get $f_7(x) = g(f_1(x))$?.



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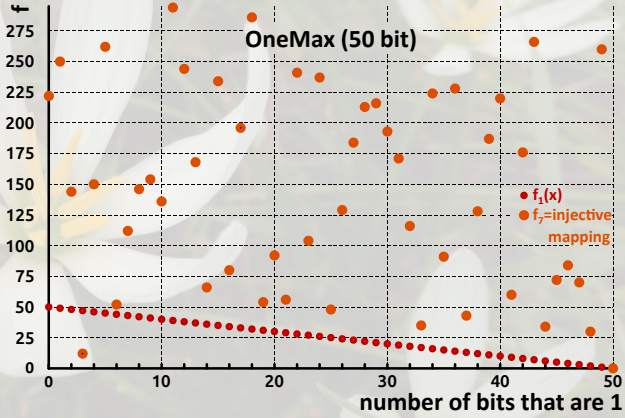

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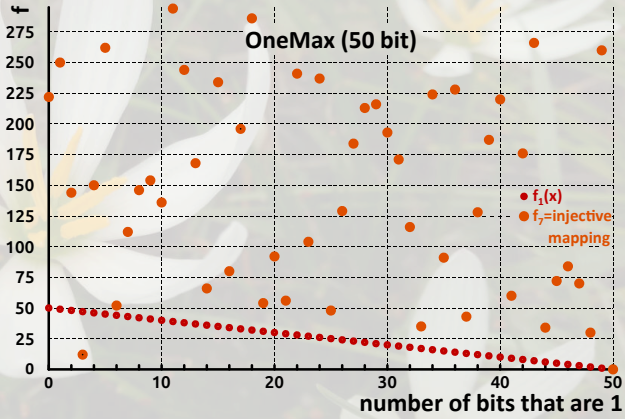

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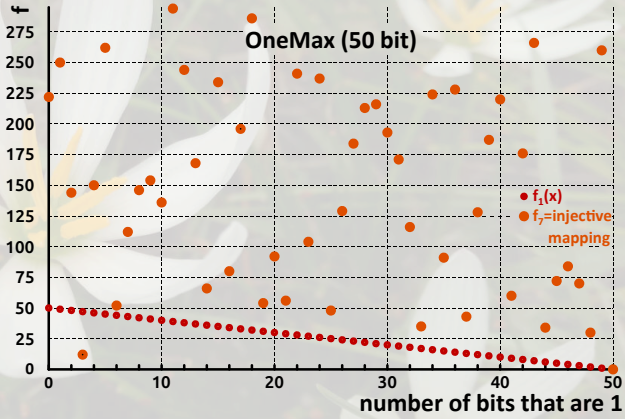

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IEEE TRANSACTIONS ON EVOLUTIONARY COMPUTATION, VOL. 25, NO. 2, APRIL 2021 307

Frequency Fitness Assignment: Making Optimization Algorithms Invariant Under Bijective Transformations of the Objective Function Value

Thomas Weise, *Member, IEEE*, Zhize Wu, Xinlu Li, and Yan Chen

***Abstract*—Under frequency fitness assignment (FFA), the fitness corresponding to an objective value is its encounter frequency in fitness assignment steps and is subject to minimization. FFA renders optimization processes invariant under bijective transformations of the objective function value. On TwoMax, Jump, and Trap functions of dimension s, the classical $(1+1)$-EA with standard mutation at rate 1/s can have expected runtimes exponential in s. In our experiments, a $(1+1)$-FEA, the same algorithm but using FFA, exhibits mean runtimes that seem to scale as $s^2 \ln s$. Since Jump and Trap are bijective transformations of OneMax, it behaves identical on all three. On OneMax, LeadingOnes, and Plateau problems, it seems to be slower than the $(1+1)$-EA by a factor linear in s. The $(1+1)$-FEA performs much better than the $(1+1)$-EA on W-Model and MaxSat instances. We further verify the bijection invariance by applying the Md5 checksum computation as transformation to some of the above problems and yield the same behaviors. Finally, we show that FFA can improve the performance of a memetic algorithm for job shop scheduling.**

Index Terms—(1+1)-EA, ev... tionary algorithm (EA), ...ency fitness ...gnment ...A), j... shop sc... duling ...n (JS...), ... Ju... ...lems, ...meti...

which are then the basis for selection. Frequency fitness assignment (FFA) [1], [2] was developed to enable algorithms to escape from local optima. In FFA, the fitness corresponding to an objective value is its encounter frequency so far in fitness assignment steps and is subject to minimization. As we discuss in detail in Section II, FFA turns a static optimization problem into a dynamic one where objective values that are often encountered will receive worse and worse fitness.

In this article, we uncover a so-far unexplored property of FFA: it is invariant under any bijective transformation of the objective function values. This is the strongest invariance known to us and encompasses all order-preserving mappings. Other examples for bijective t...ansformations include the negation, permutation, or even en...ryption of the objec... tive valu...s. According to [3], invarianc... extends performan... observed ...n a single function to an en...e associa...d inv... ance clas... ...at is, it generalizes from ...ngle p... class ... p... ...en...s. Th..., it hop...fully ... ...es ... ...es... ...e pre... ...atio...

T Weise, Z Wu, X Li, and Y Chen. "Frequency Fitness Assignment: Making Optimization Algorithms Invariant under Bijective Transformations of the Objective Function Value". IEEE Transactions on Evolutionary Computation 25(2):307–319. 2021. [81] [copyright IEEE]

Frequency Fitness Assignment

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- The table H is initially filled with zeros.
- Before the selection step of the algorithm, $H[f(P_i[j])]$ for all $j \in 1..|P_i|$ is incremented by 1.
- Then, the frequencies $H[f(P_i[j])]$ replace the objective values $f(P_i[j])$ in the actual selection decisions.

FFA: (1+1) EA and (1+1) FEA

- Let's plug FFA into the $(1+1)$ EA and obtain the $(1+1)$ EA with FFA ($(1+1)$ FEA).

procedure $(1+1)$ EA$(f : \mathbb{X} \mapsto \mathbb{R})$
  randomly sample x_c from $\mathbb{X}$; $y_c \leftarrow f(x_c)$;
  while $\neg$ terminate **do**
    $x_n \leftarrow \texttt{move}(x_c)$; $y_n \leftarrow f(x_n)$;
    if $y_n \leq y_c$ **then** $x_c \leftarrow x_n$; $y_c \leftarrow y_n$;
  return x_c, y_c

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- We start with the $(1+1)$ EA.

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procedure $(1+1)$ EA$(f : \mathbb{X} \mapsto \mathbb{N})$

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FFA: (1+1) EA and (1+1) FEA

- We begin by initializing the frequency table H with all zeros.

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procedure $(1+1)$ FEA$(f : \mathbb{X} \mapsto \mathbb{N})$
  $H \leftarrow (0, 0, \cdots, 0)$;
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  while $\neg$ terminate **do**
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FFA: (1+1) EA and (1+1) FEA

- *Before* the selection decision, we increment the frequency values of the objective values of all current solutions.

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    $H[y_c] \leftarrow H[y_c] + 1$; $H[y_n] \leftarrow H[y_n] + 1$;
    if $y_n \leq y_c$ **then**
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FFA: (1+1) EA and (1+1) FEA

- Now the frequency values replace the objective values in the selection decisions.

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- ... which are then the return values of the $(1+1)$ FEA.

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- Algorithms using FFA are invariant under all injective transformations of the objective function value.

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- Solutions get less attractive the more often their corresponding objective values have been seen. This also holds for local optima...
- Solutions with better objective values are no longer preferred over such with worse objective value.
- Instead, solutions with less-frequent objective values are preferred.
- Algorithms using FFA are invariant under all injective transformations of the objective function value.
- They perform identical on ALL of the OneMax-based functions from before!

Discrete Optimization Theory Benchmarks

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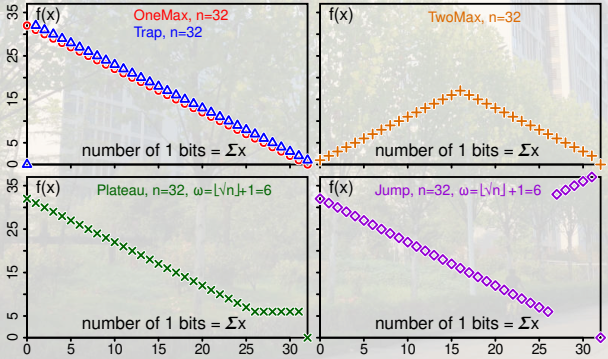
- So we tested it on discrete optimization benchmarks.



(1+1) FEA on OneMax

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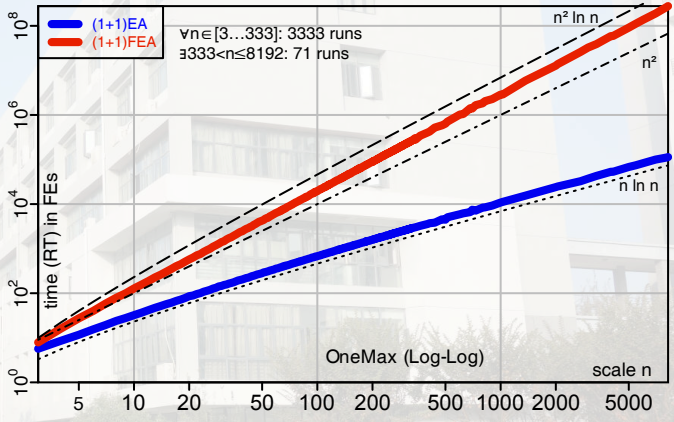

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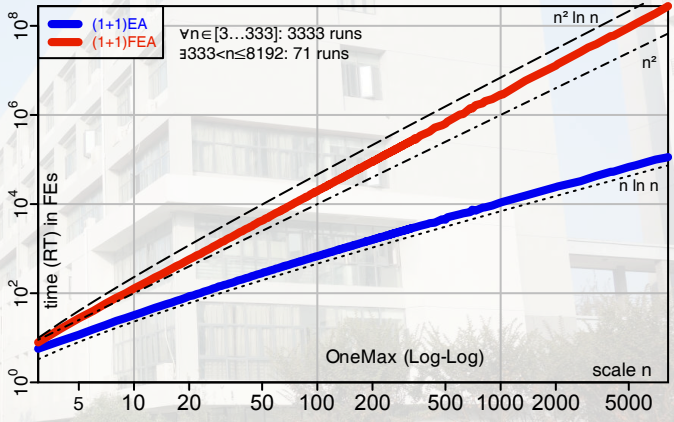

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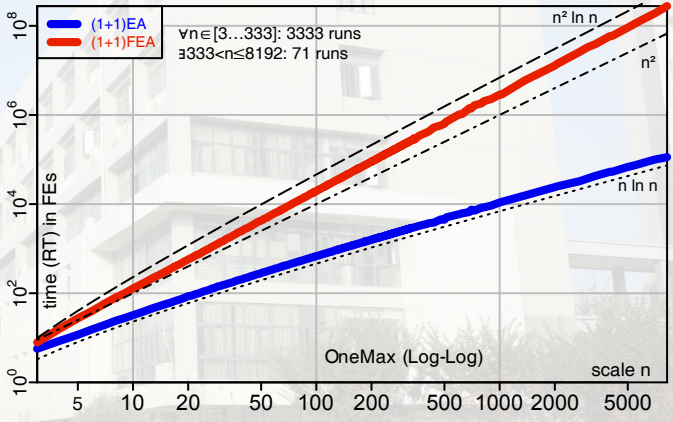

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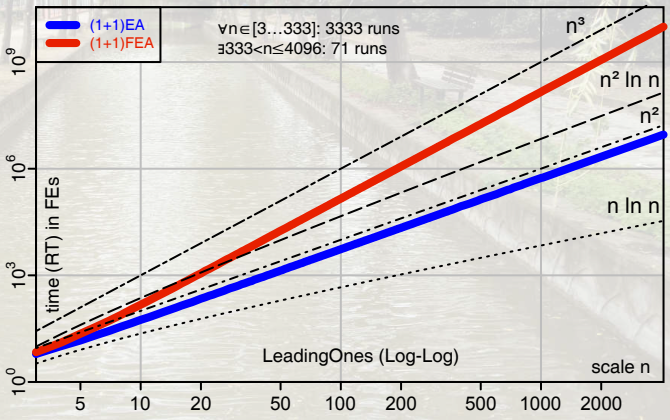

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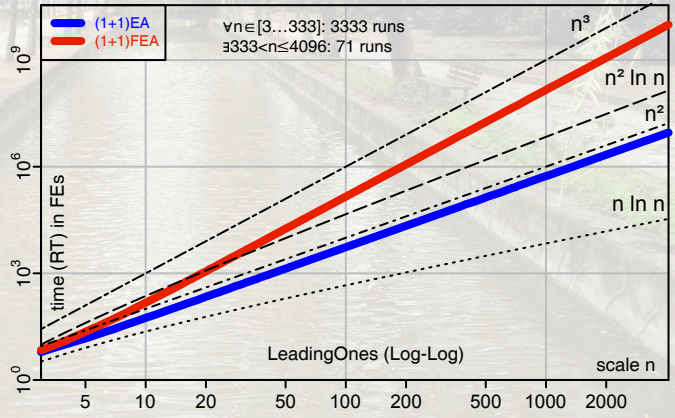

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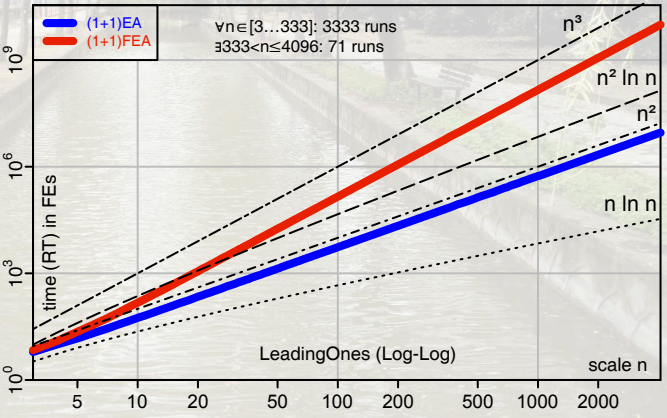

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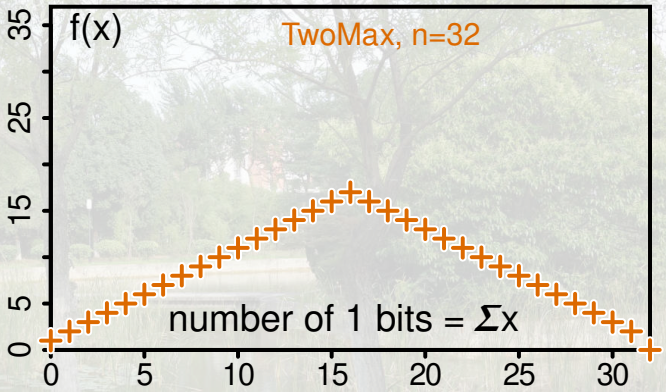

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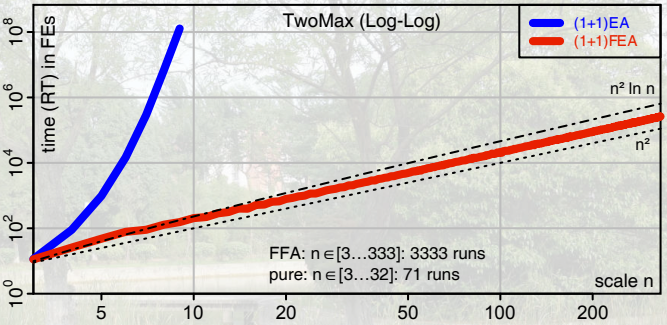

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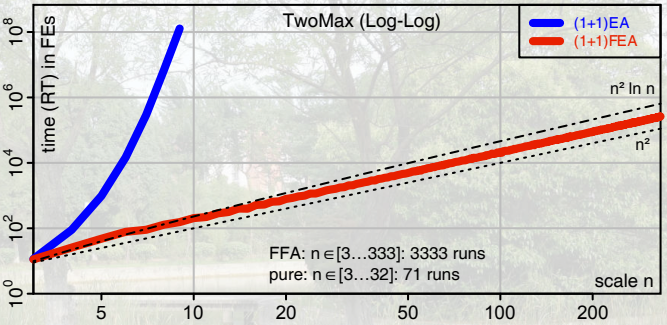

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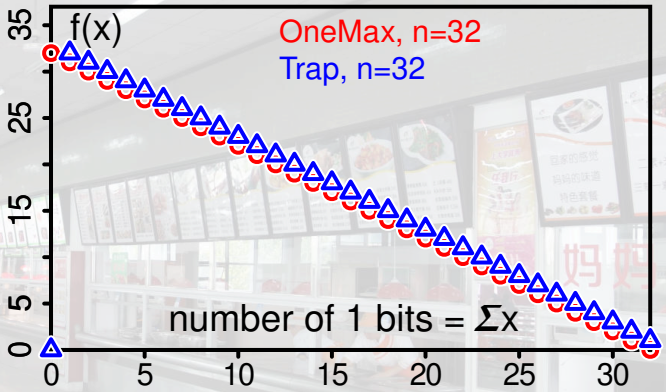

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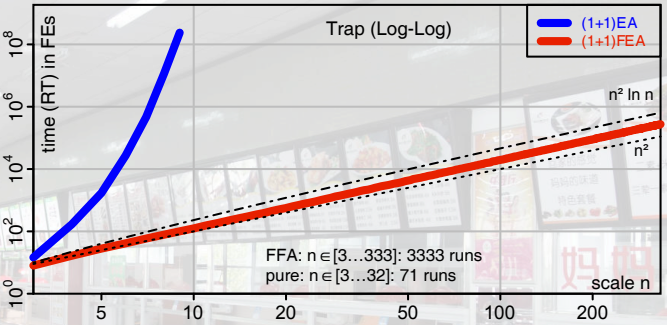

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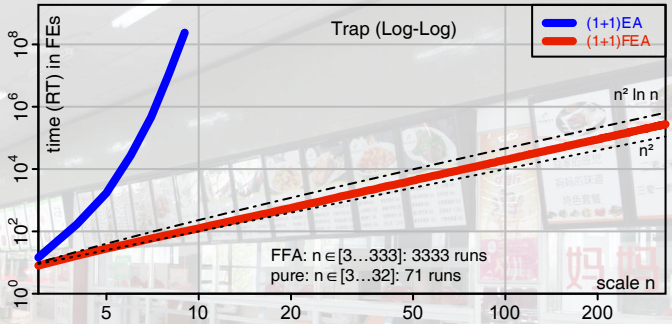

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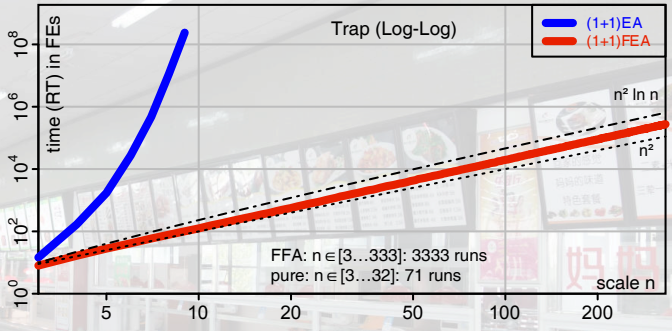

- The expected runtime is the same as on OneMax, because Trap is an injective transformation of OneMax, symmetric, and FFA "converges" into both directions.

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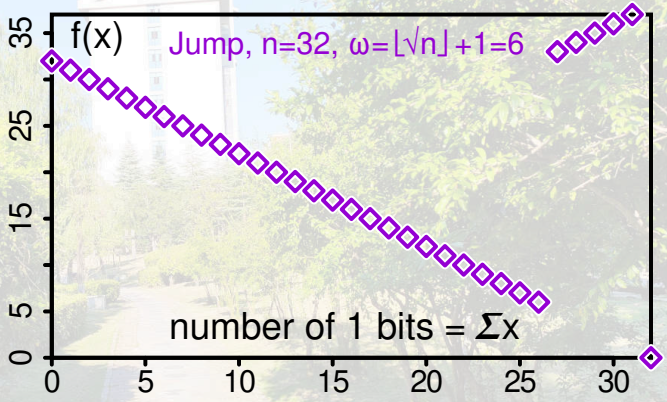

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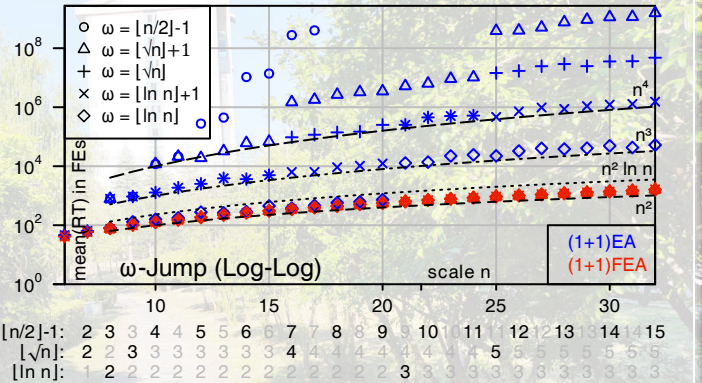

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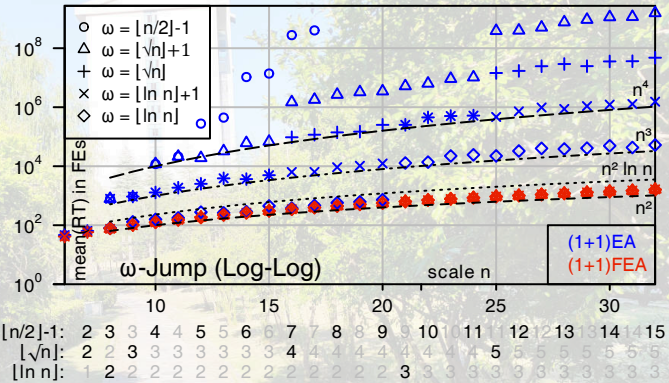

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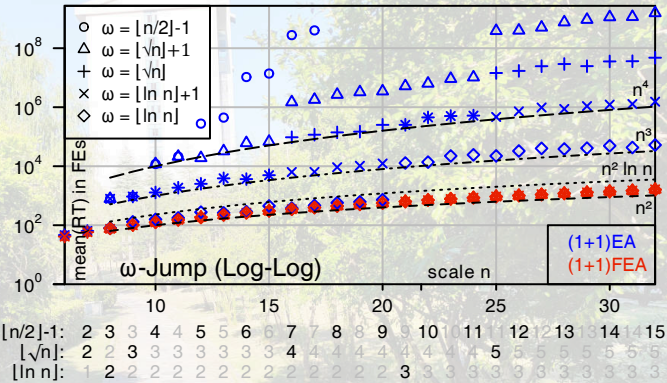

- It is identical to its performance on OneMax regardless of ω, because all Jumps are an injective transformation of OneMax!

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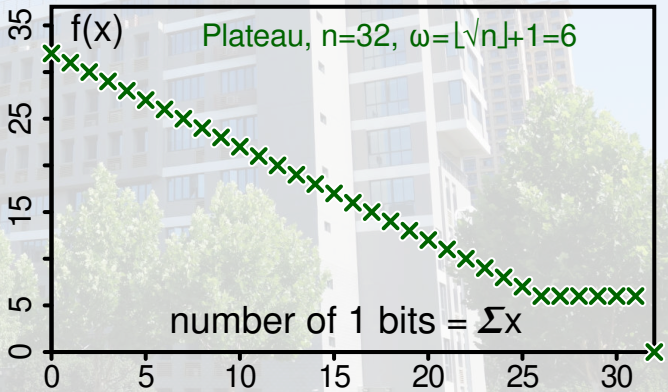

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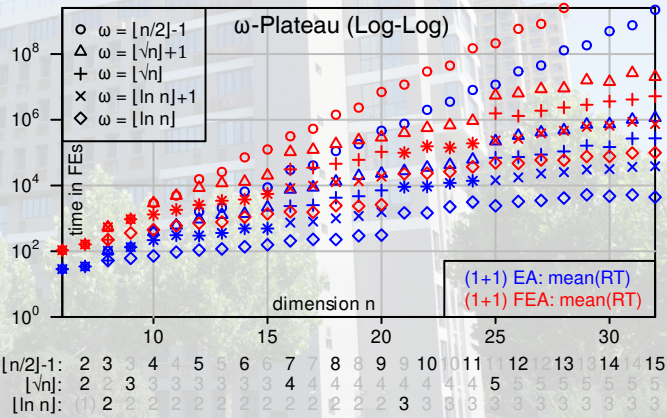

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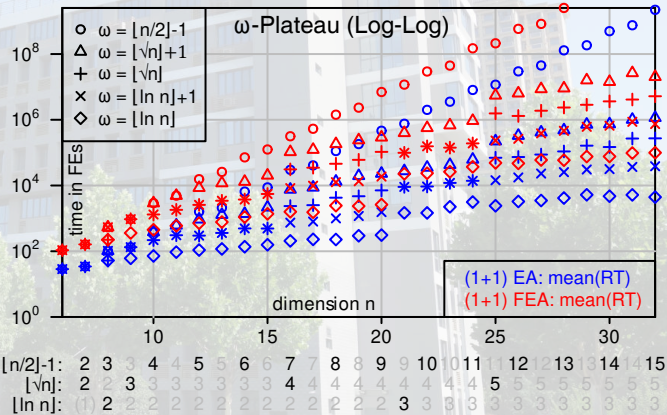

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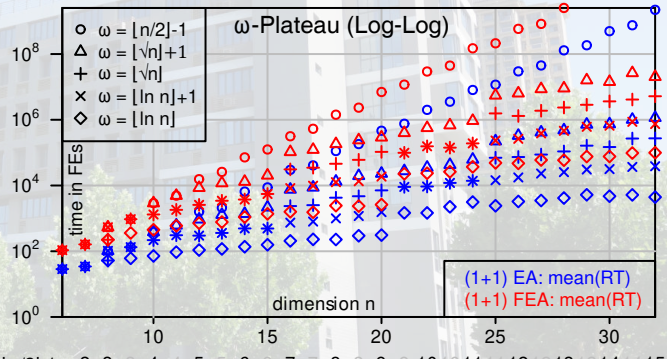
- Plateaus remain plateaus under the identity-based fitness assignment FFA.



(1+1) FEA on MaxSAT

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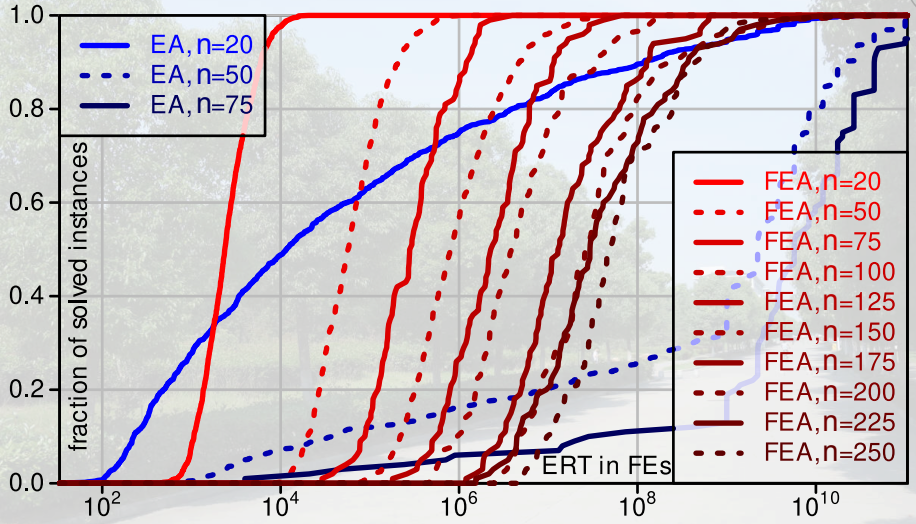
- Our computational budget was always 10^{10} objective function evaluations (FEs).

(1+1) FEA on MaxSAT

- The $(1+1)$ FEA is better on problems with 250 variables than the $(1+1)$ EA on problems with 50.

instance type	(1+1) EA			(1+1) FEA		
	success rate	ERT	mean y_c	success rate	ERT	mean y_B
uf20_*	0.985	$1.91 * 10^8$	0.015	1	3′091	0
uf50_*	0.748	$3.56 * 10^9$	0.299	1	93′459	0
uf75_*	0.583	$7.41 * 10^9$	0.528	1	490′166	0
uf100_*				1	$2.14 * 10^6$	0
uf125_*				1	$5.27 * 10^6$	0
uf150_*				1	$1.40 * 10^7$	0
uf175_*				1	$5.78 * 10^7$	0
uf200_*				0.991	$2.44 * 10^8$	0.00945
uf225_*				0.994	$2.43 * 10^8$	0.00555
uf250_*				0.992	$2.43 * 10^8$	0.00782

(1+1) FEA on MaxSAT: ERT-ECDF



"Frequency Fitness Assignment: Making Optimization Algorithms Invariant under Bijective Transformations of the Objective Function Value" [81]

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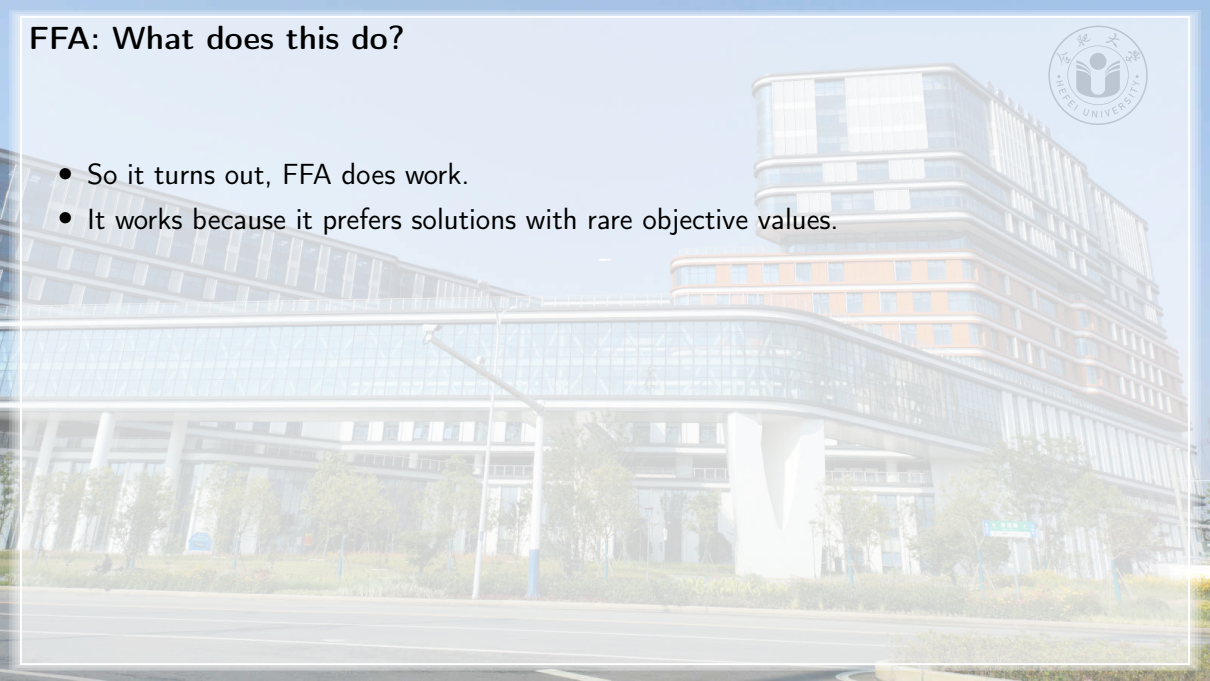

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- $g(f(x))$ will be totally random. No reasonable optimization algorithm could work with that.
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- How incredibly strange is that?

Where did we try FFA?

Applications of FFA

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- We picked some of the most popular $\mathcal{NP}$-hard problems as testbeds.
- We could not (yet) win against the state-of-the-art that matured over decades.
- But we did get some interesting results on real, hard, and well-known problems.

Maximum Satisfiability Problem

- FFA on the Maximum Satisfiability (MaxSAT) Problem: [79, 81, 82].

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> From Table I, we can see that **the highest number of failed runs at scale $s = 250$ of *any* algorithm using FFA is lower than the lowest number of failed runs of *any* pure algorithm at $s = 50$.** From Table II, we find that no FFA-based algorithm has a higher ERT at scale $s = 250$ than its pure variant on $s = 50$. On the scales $s \leq 75$, the FFA-based algorithms have a mean runtime which is between three and four orders of magnitude smaller that the ERT of the pure algorithms.

- Snippet of page 10 of [82] (copyright IEEE).
- Several different EAs with and without FFA
- Budget: 10^{10} FEs

Traveling Salesperson Problem

- FFA on the Traveling Salesperson Problem (TSP): [42–44].

Traveling Salesperson Problem

- FFA on the Traveling Salesperson Problem (TSP): [42–44].
- Snippet of page 12 of [42] (copyright Springer).
- $(1+1)$ EA, $(1+1)$ FEA, SA w/o FFA, hybrids
- Budget: 10^{10} FEs

In this work, we explored both the EA and SA on 56 symmetric instances from the benchmark set TSPLib (Reinelt 1991, 1995). The EA is unsuitable for this problem, but SA can find the optimum on many small and mid-sized instances. We then plug FFA into both algorithms and obtain the FEA and the FSA, respectively, which both exhibit very similar performance. **The FEA solves 27 of the instances in all of its runs, which SA can only achieve for 19 instances.** Plugging FFA into either the EA or SA thus substantially improved the number of runs in which the algorithms can find the optimum, however, both FFA-based variants suffer when the number of unique objective values is high.

• • •

Both types of **hybridization** methods significantly improve the average result quality compared to both the objective-guided and FFA-based FEA variants. **The SAFEA$_A$ discovers the optimal solutions in more runs than any other algorithm setup in our study.** It also finds the optimum most often in most instances and delivers the best approximation quality on most instances, compared to the other algorithms.

Quadratic Assignment Problem

- FFA on the Quadratic Assignment Problem (QAP): [12, 65].

Quadratic Assignment Problem

- FFA on the Quadratic Assignment Problem (QAP): [12, 65].
- Snippet of page 7 of [12] (copyright SciTePress).
- RLS w/o FFA
- Budget: 10^8 FEs

We find that the **FFA-based** randomized local search **FRLS does not just find better solutions than the objective-guided RLS algorithm on the vast majority of the QAPLIB instances**, it also keeps improving its current best solution for the complete computational budget of 10^8 FEs that we assigned to the runs. With this budget, **it can discover the optimal solutions of over 58% of the QAPLIB instances**. Had we assigned a larger budget – (Liang et al., 2022; Liang et al., 2024; Weise et al., 2021b; Weise et al., 2023) use 10^{10} FEs – we would likely have seen even more instances solved.

We furthermore confirm the remarkable ability of FFA to discover very diverse solutions (at least from the perspective of the objective function). It is known that on the QAP, many solutions tend to have the same objective values (Tayarani-N. and Prügel-Bennett, 2015). Yet, on some of the instances, more than half of the objective values discovered by FRLS were unique.

Job Shop Scheduling Problem

- FFA on the Job Shop Scheduling Problem (JSSP) Problem: [14, 72, 81].

Job Shop Scheduling Problem

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> The **end result quality delivered by (1+1)-FEA is better in average on the abz*, ft*, la*, orb*, and yn4* instance sets, both in terms of *best* and *mean*.** On swv*, the average for *mean* is better for (1+1)-FEA, while (1+1)-EA has a slight lead in *best*. The (1+1)-EA performs better on the dmu* and ta* instances. Since these two sets are larger (holding 160 out of the 242 instances), the (1+1)-EA comes out ahead in the overall averages, but with no more than a 1.5% advantage.

- Snippet of page 10 of [72] (copyright ACM).
- Budget: $2^{30} \approx 10^9$ FEs

Algorithm Synthesis and Genetic Programming

- FFA for algorithm synthesis and Genetic Programming (GP): [78, 79].

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> Fourth, we were able to confirm with great significance that **FFA** has a tremendous positive impact on the performance of GP, as it **can increase the success rate by 40%.**

- Snippet of page 7 of [78] (copyright IEEE).
- Budget: $\approx 100\,000$ FEs

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- This includes, surprisingly, also some computationally hard problems:
  - the Traveling Tournament Problem (TTP), where RLS can find surprisingly good results: [85].
  - the two-dimensional bin packing problem, where RLS can actually outperform several complex state-of-the-art metaheuristics [88, 89].

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- It can speed them up on hard problems or lead them to solution qualities they otherwise cannot reach.
- It is limited to objective functions that cannot take on too many different objective values.

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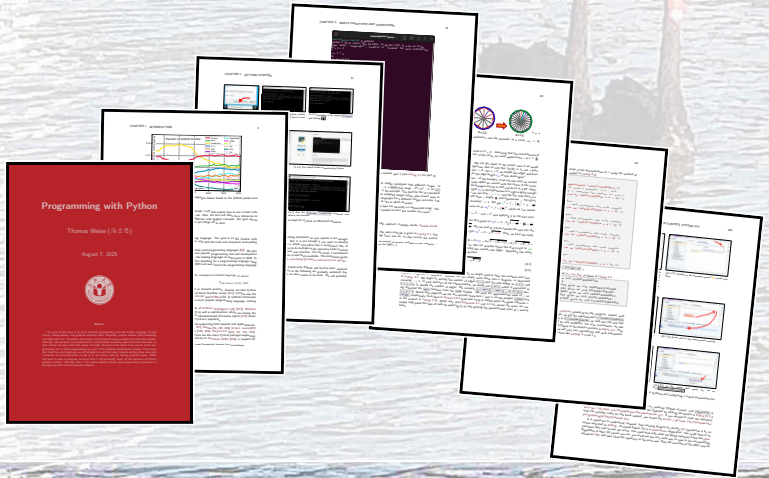

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We have a freely available course book on *Databases* at `https://thomasweise.github.io/databases`, with actual practical examples using a real database management system (DBMS)[68].

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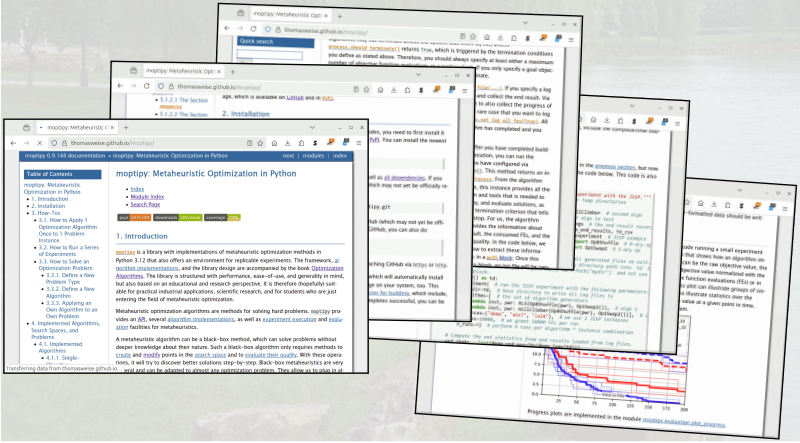
We offer moptipy[80] a mature open source Python package for metaheuristic optimization, which implements several algorithms, can run self-documenting experiments in parallel and in a distributed fashion, and offers statistical evaluation tools.



谢谢您们！

Thank you!

Vielen Dank!

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Glossary I

$(1+1)$ EA The $(1+1)$ EA is a local search algorithm that retains the best solution x_c discovered so far during the search[9,17]. In each step, it applies a unary search operator to this best-so-far solution x_c and derives a new solution x_n. If the new solution x_n is *better or equally good* when compared with x_c, i.e., not worse, then it replaces it, i.e., is stored as the new x_c. If the search space are bit strings of length n, then the $(1+1)$ EA uses a unary search operator that flips each bit independently with probability m/n, where usually $m = 1$. This operator is the main difference to randomized local search (RLS). The $(1+1)$ EA is a special case of the $(\mu+\lambda)$ evolutionary algorithm ($(\mu+\lambda)$ EA) where $\mu = \lambda = 1$.

$(1+1)$ FEA The $(1+1)$ EA with FFA plugged in.

EA An *evolutionary algorithm* is a metaheuristic optimization method that maintains a population of candidate solutions, which undergo selection (where better solutions are chosen with higher probability) and reproduction (where mutation and recombination create a new candidate solution from one or two existing ones, repectively)[5,69].

$(\mu+\lambda)$ EA The $(\mu+\lambda)$ EA is an evolutionary algorithm (EA) where, in each generation, λ offspring solutions are generated from the current population of μ parent solutions. The offspring and parent populations are merged, yielding $\mu+\lambda$ solutions, from which then the best μ solutions are ratained to form the parent population of the next generation. If the search space is the bit strings of length n, then this algorithm usually applies a mutation operator flipping each bit independently with probability $1/n$.

AI Artificial Intelligence, see, e.g.,[62]

DB A *database* is an organized collection of structured information or data, typically stored electronically in a computer system. Databases are discussed in our book *Databases*[68].

DBMS A *database management system* is the software layer located between the user or application and the database (DB). The DBMS allows the user/application to create, read, write, update, delete, and otherwise manipulate the data in the DB[86].

FE *Objective function evaluations* are an implementation-independent measure of runtime for optimization algorithms[6,71,73,80]. 1 FE equals to one evaluated candidate solution during the optimization process.

Glossary II

FFA *Frequency Fitness Assignment* is a algorithm plugin for optimization methods applied to discrete or combinatorial problems with not-too-many different possible objective values. It replaces the objective values in all comparisons with their absolute encounter frequency so far during the search. FFA has successfully been applied to the QAP[12].

GP Genetic Programming[37,58,69,77] is the application of metaheuristic optimization, usually in form of an EAs, to a search space comprised of tree datastructures. These tree datastructures often represent programs or mathematical expressions.

JSSP The *Job Shop Scheduling Problem*[7,39] is one of the most prominent and well-studied scheduling tasks. In a JSSP instance, there are k machines and m jobs. Each job must be processed once by each machine in a job-specific sequence and has a job-specific processing time on each machine. The goal is to find an assignment of jobs to machines that results in an overall shortest makespan, i.e., the schedule which can complete all the jobs in the shortest time. The JSSP is $\mathcal{NP}$-complete[11,39].

MaxSAT The goal of satisfiability problems is to find an assignment for n Boolean variables that make a given Boolean formula $F : \{0,1\}^n \mapsto \{0,1\}$ become true. In the *Maximum Satisfiability (MaxSAT)* problem[28], F is given in conjunctive normal form, i.e., the variables appear either directly or negated in m "or" clauses, which are all combined into one "and." The objective function $f(x)$, subject to minimization, computes the number of clauses which are false under the variable setting x. If $f(x) = 0$, then all clauses of F are true, which solves the problem. The MaxSAT problem is $\mathcal{NP}$-complete[13].

ML Machine Learning, see, e.g.,[64]

moptipy is the *Metaheuristic Optimization in Python* library[80]. It has been used in several different research works, including[12,42–44,65,85,88,89]. Learn more at https://thomasweise.github.io/moptipy and https://thomasweise.github.io/moptipyapps.

Python The Python programming language[29,41,46,70], i.e., what you will learn about in our book[70]. Learn more at https://python.org.

Glossary III

QAP The *Quadratic Assignment Problem* is an optimization problem where the goal is to assign a set of n facilities to a set of n locations[8,12,36,45]. Such an assignment can be represented as a permutation x of the first n natural numbers, where x_i specifies the location where facility i should be placed. For each QAP, a distance matrix D is given, where D_{pq} specifies the distance from location p to location q, as well as a flow matrix F, where F_{ij} is the amount of material flowing from facility i to facility j. The objective function f then rates a permutation x as $f(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} D_{x_i x_j} F_{ij}$. The QAP is $\mathcal{NP}$-complete[63].

RLS Randomized local search retains the best solution x_c discovered so far during the search and, in each step, it applies a unary search operator to this best-so-far solution x_c and derives a new solution x_n. If the new solution x_n is *better or equally good* when compared with x_c, i.e., not worse, then it replaces it, i.e., is stored as the new x_c. If the search space are bit strings of length n, then RLS uses a unary search operator that flips exactly one bit. This operator is the main difference to $(1 + 1)$ EA.

RSA The RSA algorithm[60,87], named after its inventors Rivest, Shamir, and Adleman, is a widely used asymmetric cryptographic technique for digital signatures, key exchange, and encryption. In the latter case, it uses a public key for encryption and a private key for decryption. It is based on the mathematical complexity of factorizing large integers.

SA Simulated Annealing is a local search that sometimes accepts a worse solution[10,31,33,56]. The probability to do so decreases over time and with the difference in objective values, i.e.,is the lower the worse the new solution is.

SATLIB is a library of benchmark instances for the Maximum Satisfiability (MaxSAT) available at https://www.cs.ubc.ca/~hoos/SATLIB/index-ubc.html[27].

SGA The Standard Genetic Algorithm[5,15,22,48,49,69] was the first population EA. It maintains a population of solutions and applies mutation and crossover to generate offspring solutions. It uses fitness proportionate selection to choose which solutions should "survive" into the next generation, which today is considered a very bad design choice, see, e.g., [84].

Glossary IV

TSP In an instance of the *Traveling Salesperson Problem*, also known as *Traveling Salesman Problem*, a set of n cities or locations as well as the distances between them are defined[3,24,40,42,71,83]. The goal is to find the shortest round-trip tour that starts at one city, visits all the other cities one time each, and returns to the origin. The TSP is one of the most well-known $\mathcal{NP}$-hard combinatorial optimization problems[24].

TTP The *Traveling Tournament Problem* (TTP) is the combinatorial optimization problem of both efficiently and fairly organizing a tournament of n teams that play against each other in a pairwise fashion[19,85]. The efficient part boils down to arranging the games such that the total travel length is short, which is somewhat similar to the classical TSP. Initially, each team is at its home location. On each day, a team needs to travel if its scheduled game is not at its present location. On the last day, each team may need to travel back home unless their last game is a home game. The total travel length sums up the lengths of all travels over all teams. The fair part is represented in several constraints, such as doubleRoundRobin, compactness, maxStreak, and noRepeat. The TTP is $\mathcal{NP}$-hard[67].

$i..j$ with $i, j \in \mathbb{Z}$ and $i \leq j$ is the set that contains all integer numbers in the inclusive range from i to j. For example, 5..9 is equivalent to $\{5, 6, 7, 8, 9\}$

$\mathbb{N}_0$ the set of the natural numbers *including* 0, i.e., 0, 1, 2, 3, and so on. It holds that $\mathbb{N}_0 \subset \mathbb{Z}$.

$\mathcal{NP}$ is the class of computational problems that can be solved in polynomial time by a non-deterministic machine and can be verified in polynomial time by a deterministic machine (such as a normal computer)[23].

$\mathcal{NP}$-complete A decision problem is $\mathcal{NP}$-complete if it is in $\mathcal{NP}$ and all problems in $\mathcal{NP}$ are reducible to it in polynomial time[23,59]. A problem is $\mathcal{NP}$-complete if it is $\mathcal{NP}$-hard and if it is in $\mathcal{NP}$.

$\mathcal{NP}$-hard Algorithms that guarantee to find the correct solutions of $\mathcal{NP}$-hard problems[11,13,39] need a runtime that is exponential in the problem scale in the worst case. A problem is $\mathcal{NP}$-hard if all problems in $\mathcal{NP}$ are reducible to it in polynomial time[23].

$\Omega(g(x))$ If $f(x) = \Omega(g(x))$, then there exist positive numbers $x_0 \in \mathbb{R}^+$ and $c \in \mathbb{R}^+$ such that $f(x) \geq c * g(x) \geq 0 \forall x \geq x_0$[34,35]. In other words, $\Omega(g(x))$ describes a lower bound for function growth.

Glossary V

$\mathcal{O}(g(x))$ If $f(x) = \mathcal{O}(g(x))$, then there exist positive numbers $x_0 \in \mathbb{R}^+$ and $c \in \mathbb{R}^+$ such that $0 \leq f(x) \leq c * g(x) \forall x \geq x_0$[4,34,35,38]. In other words, $\mathcal{O}(g(x))$ describes an upper bound for function growth.

$\Theta(g(x))$ If $f(x) = \Theta(g(x))$, then $f(x) = \mathcal{O}(g(x))$ and $f(x) = \Omega(g(x))$[34,35]. In other words, $\Theta(g(x))$ describes an exact order of function growth.

$\mathbb{R}$ the set of the real numbers.

$\mathbb{R}^+$ the set of the positive real numbers, i.e., $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$.

$\mathbb{Z}$ the set of the integers numbers including positive and negative numbers and 0, i.e., . . . , -3, -2, -1, 0, 1, 2, 3, . . . , and so on. It holds that $\mathbb{Z} \subset \mathbb{R}$.