Sharp and Simple Bounds for the Raw Moments of the Binomial and Poisson Distributions

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Abstract

We prove the inequality $\mathrm{E}[(X/\mu)^k] \le (\frac{k/\mu}{\log(1+k/\mu)})^k \le \exp(k^2/(2\mu))$ for sub-Poissonian random variables X, such as Binomially or Poisson distributed variables, with mean μ. The asymptotic behaviour $\mathrm{E}[(X/\mu)^k] = 1 + O(k^2/\mu)$ matches a lower bound of $1+\Omega(k^2/\mu)$ for small k^2/μ. This improves over previous uniform raw moment bounds by a factor exponential in k.

1 Introduction

Suppose we sample an urn of n balls, each coloured *red* with probability p and otherwise *blue*. What is the probability that a sample of k balls, with replacement, *from this urn* consists of only red balls? Such questions are of interest to sample-efficient statistics and the derandomisation of algorithms.

If $R \sim \mathrm{Binomial}(n, p)$ denotes the number of red balls in the urn, the probability of drawing a single red ball from the urn is R/n. Thus, the probability that a sample of k balls from the urn is all red is given by $(R/n)^k$, or $P = \mathrm{E}[(R/n)^k]$ when the probability is taken over both sample phases. Whenever the urn is large (n is large), R/n concentrates around p, so sampling from the urn is equivalent to sampling from the original distribution and $P \approx p^k$. Indeed, from Jensen's inequality, we can see that p^k is always a lower bound: $P = \mathrm{E}[(R/n)^k] \ge \mathrm{E}[(R/n)]^k = p^k$. Previous authors have shown a nearly matching upper bound of $C^k p^k$ in the range $k/(np) = O(1)$ for some constant $C > 1$. (See eq. (1) below for details.) In this note, we improve the upper bound to $P \le p^k(1 + k/(2np))^k$, which shows that when $k = o(\sqrt{np})$, the factor C^k can be replaced by just $1 + o(1)$.

1.1 Related work

One direct approach to computing the Binomial moments expands them using the Stirling numbers of the second kind: $\mathrm{E}[X^k] = \sum_{i=0}^{k} {k \brace i} n^{\underline{i}} p^i$, where $n^{\underline{i}} = n(n-1)\cdots(n-i+1)$. This equality can be derived as a sum of the much easier to compute "factorial moments", $\mathrm{E}[X^{\underline{k}}] = n^{\underline{k}} p^k$. See Knoblauch (2008) for details. Taking the leading two terms of the sum, one finds that $\mathrm{E}[X^k] = (np)^k \left(1 + \binom{k}{2}\frac{1-p}{np} + O(1/n^2)\right)$ as $n \to \infty$. However, this approach does not work when k is not constant with respect to n. Similarly, for the Poisson distribution, the moments can be expressed as the so-called Bell (or Touchard) polynomials in μ: $\mathrm{E}[X^k] = \sum_{i=0}^k \left\{ {k \atop i} \right\} \mu^i$. This sum gives a simple lower bound $\mathrm{E}[X^k] \geq \left\{ {k \atop k} \right\} \mu^k + \left\{ {k \atop k-1} \right\} \mu^{k-1} = \mu^k (1 + \frac{k(k-1)}{2\mu})$, matching our upper bound asymptotically when $k = O(\sqrt{\mu})$. However, as in the Binomial case, the sum does not easily yield a uniform bound. We give the details of both lower bounds in Section 2.1.

A different approach uses the powerful results on moments of independent random variables by Latała (1997) and Pinelis (1995). In the case of Binomial and Poisson random variables, they yield:

$$\left(c \, \frac{k/\mu}{\log(1+k/\mu)} \right)^k \leq \mathrm{E}[(X/\mu)^k] \leq \left(C \, \frac{k/\mu}{\log(1+k/\mu)} \right)^k \tag{1}$$

for some universal constants $c < 1 < C$. The bound is tight up to the factor $(C/c)^k$, which is negligible when the overall growth is $O(k^k)$. However, when $k/\mu \to 0$, we expect the upper bound to be 1, and so the factor C^k in the upper bound can be overwhelmingly large.

A third option is to use a Rosenthal bound, such as the following by Berend and Tassa (2010), (see also Johnson et al., 1985):

$$\mathrm{E}[X^k] \leq B_k \max\{\mu, \mu^k\}. \tag{2}$$

Here, B_k is the kth Bell number, which Berend and Tassa show satisfies the uniform bound $B_k < \big(\frac{0.792k}{\log(k+1)}\big)^k$. For large k, a precise asymptotic bound, $B_k^{1/k} = \frac{k}{e \log k}(1 + o(1))$, is given by (e.g. de Bruijn, 1981; Ibragimov and Sharakhmetov, 1998). Unfortunately, the Rosenthal bound is incomparable to the other bounds in this paper when $\mu < 1$, as it grows with μ rather than μ^k. However, for $\mu \geq 1$ and integral, we show a matching asymptotic lower bound in the second half of Section 2.1. That indicates that the upper bound of this paper could be improved by a factor e^{-k} for large k.

Finally, Ostrovsky and Sirota (2017) give another asymptotically sharp bound in a recent preprint. Using a technique based on moment generating functions, similar to this paper, they bound the Bell polynomial, which as discussed above is equivalent to bounding the moments of a Poisson random variable. The bound holds when $k \geq 2\mu$:

$$\mathrm{E}[(X/\mu)^k]^{1/k} \leq \frac{k/\mu}{e \log(k/\mu)} \left(1 + C(\mu) \frac{\log\log(k/\mu)}{\log(k/\mu)} \right) \quad \text{if } k \geq 2\mu, \tag{3}$$

where $C(\mu) > 0$ is some "constant" depending only on μ. In the range $k < 2\mu$, Ostrovsky and Sirota only gives the bound $\mathrm{E}[(X/\mu)^k] \leq 8.9758^k$, so similarly to the other bounds presented, it loses an exponential factor in k compared to Theorem 1 below, for smaller k.

2 Bounds

Theorem 1. *Let X be a non-negative random variable with mean $\mu > 0$ and moment-generating function $\mathrm{E}[\exp(tX)]$ bounded by $\exp(\mu(e^t - 1))$ for all $t > 0$. Then for all*

$k > 0$:

$$\mathrm{E}[(X/\mu)^k] \le \left(\frac{k/\mu}{\log(1+k/\mu)}\right)^k.$$

A standard logarithmic bound, $\frac{x}{\log(1+x)} \le 1+x/2$ (see e.g. Topsøe, 2007, eq. 6), implies the corollary

$$\mathrm{E}[(X/\mu)^k] \le \big(1+k/(2\mu)\big)^k \le \exp\big(k^2/(2\mu)\big).$$

Random variables satisfying the requirement $\mathrm{E}[\exp(tX)] \le \exp(\mu(e^t-1))$ are known as sub-Poissonian and include many simple distributions, such as the Poisson or Binomial distribution. We give more examples in Section 3.

Technically our bound is shown using the moment-generating function and some new sharp inequalities involving the Lambert-W function, which is defined by $W(x)e^{W(x)} = x$.[1] We will use the following lemma:

Lemma 1 (Hoorfar and Hassani, 2008)**.** *For all $y > 1/e$ and $x > -1/e$,*

$$e^{W(x)} \le \frac{x+y}{1+\log y}. \tag{4}$$

We perform an elementary proof of this fact for completeness:

Proof. Starting from $1+t \le e^t$, substitute $\log(y)-t$ for t to get $1+\log y - t \le ye^{-t}$. Multiplying by e^t we get $e^t(1+\log y) \le te^t + y$. Let $t = W(x)$ s.t. $te^t = x$. Rearranging, we get eq. (4). □

Taking $y = e^{W(x)}$ in eq. (4) makes the two sides equal, so we can think of Lemma 1 as a way to turn a rough estimate into an upper bound. Hoorfar and Hassani make various substitutions, resulting in different bounds useful when $x \to \infty$. We will use the bound differently, focusing on having the right asymptotics as $x \to 0$.

We are now ready to prove the main theorem of the paper:

Proof of Theorem 1. Let $m(t) = \mathrm{E}[\exp(tX)]$ be the moment-generating function. We will bound the moments of X by

$$\mathrm{E}[X^k] \le m(t)(k/(et))^k, \tag{5}$$

which holds for all $k \ge 0$ and $t > 0$. This follows from the basic inequality $1+z \le e^z$, where we substitute $tz/k - 1$ for z to get $tz/k \le e^{tz/k-1} \implies z^k \le e^{tz}(k/(et))^k$. Letting $z = X$ and taking expectations, we get eq. (5).

We now define $x = k/\mu$ and take t such that $te^t = x$. In the notation of the Lambert-W function, this means $t = W(x)$. We note that $t > 0$ whenever $x > 0$. We proceed to bound the moments of X/μ using eq. (5):

$$\mathrm{E}[(X/\mu)^k] \le m(t)(k/(et))^k\mu^{-k}$$

[1] The Lambert-W function has multiple branches. We are interested in the main one (sometimes called the 0th), in which $W(x)$ and x are both positive.

$$
\begin{aligned}
&\le \exp\bigl(\mu(e^t-1)\bigr)\left(\frac{k}{e\mu t}\right)^k \\
&= \exp\bigl(\mu(x/t-1)\bigr)\left(\frac{e^t}{e}\right)^k && (6)\\
&= \exp\bigl((k/x)(x/t-1)+k(t-1)\bigr) \\
&= \exp(kf(x)), && (7)
\end{aligned}
$$

where we define $f(x) \coloneqq 1/t - 1/x + t - 1$. Here eq. (6) came from the simple rewriting of the definition of t, $1/t = e^t/x$

It remains to show $\exp(f(x)) \le \frac{x}{\log(1+x)}$. Taking logarithms, this means showing the bound

$$
f(x) = \frac{1}{W(x)} + W(x) - 1 - \frac{1}{x} \le \log\left(\frac{x}{\log(1+x)}\right)
$$

for all $x > 0$, where $W(x)$ is the Lambert-W function. The proof uses the identities $W(X) = \log x - \log(W(x))$ and $\frac{1}{W(x)} = \frac{1}{x}\exp(W(x))$ which are simple rewritings of the definition $W(x)e^{W(x)} = x$. The main idea is to introduce a new variable $z > 0$, to be determined later, which allows us to control the effect of applying the logarithmic inequality $\log x \ge 1 - 1/x$. We also use Lemma 1 which introduces another new variable $y > 1$ to be determined.

$$
\begin{aligned}
\frac{1}{W(x)} + W(x) - 1 - \frac{1}{x} &= \frac{1}{W(x)} - 1 - \frac{1}{x} + \log x - \log(W(x)) \\
&= \frac{1}{W(x)} - 1 - \frac{1}{x} + \log\left(\frac{x}{z}\right) - \log\left(\frac{W(x)}{z}\right) \\
&\le \frac{1}{W(x)} - 1 - \frac{1}{x} + \log\left(\frac{x}{z}\right) - \left(1 - \frac{z}{W(x)}\right) \\
&= \frac{1+z}{W(x)} - 2 - \frac{1}{x} + \log\left(\frac{x}{z}\right) \\
&= e^{W(x)}\frac{1+z}{x} - 2 - \frac{1}{x} + \log\left(\frac{x}{z}\right) \\
&\le \frac{x+y}{1+\log(y)}\frac{1+z}{x} - 2 - \frac{1}{x} + \log\left(\frac{x}{z}\right).
\end{aligned}
$$

Here the last inequality is eq. (4) in its general form. We finally take $z = \log(y)$ and $y = 1 + x$, which are both positive when $x > 0$. That simplifies the bound to

$$
f(x) \le \log\left(\frac{x}{\log(1+x)}\right).
$$

Backing up, we have shown $\mathrm{E}[(X/\mu)^k] \le \exp(kf(x)) \le (\frac{x}{\log(1+x)})^k$, which finishes the proof. $\square$

2.1 Lower bound

As mentioned in the introduction, the expansion for the Poisson moments $\mathrm{E}[X^k] = \sum_{i=0}^k \left\{ {k \atop i} \right\} \mu^i$ gives a simple lower bound by taking the two highest terms. We note that

$\left\{ {k \atop k} \right\} = 1$ and $\left\{ {k \atop k-1} \right\} = \binom{k}{2}$ to get

$$\mathrm{E}[X^k] \geq \mu^k \left(1 + \frac{k(k-1)}{2\mu}\right),$$

matching Theorem 1 asymptotically for $k = O(\sqrt{\mu})$.

The expansion for Binomial moments $\mathrm{E}[X^k] = \sum_{i=0}^{k} \left\{ {k \atop i} \right\} n^{\underline{i}} p^i$ yields a similar lower bound

$$\begin{aligned}
\mathrm{E}[X^k] &\geq n^{\underline{k}} p^k + \binom{k}{2} n^{\underline{k-1}} p^{k-1} \\
&= (np)^k \left(\frac{n^{\underline{k}}}{n^k}\right) \left(1 + \binom{k}{2} \frac{1}{(n-k+1)p}\right) \\
&= (np)^k \left(\prod_{i=0}^{k-1} 1 - \frac{i}{n}\right) \left(1 + \binom{k}{2} \frac{1}{(n-k+1)p}\right) \\
&\geq (np)^k \left(1 - \binom{k}{2} \frac{1}{n}\right) \left(1 + \binom{k}{2} \frac{1}{np}\right) \\
&= (np)^k \left(1 + \binom{k}{2} \frac{1-p}{np} \left(1 - \binom{k}{2} \frac{1}{n}\right)\right),
\end{aligned}$$

which matches Theorem 1 for $k = O(\sqrt{\mu})$ and p not too close to 1.

We will investigate some more precise lower bounds as k/μ gets large. As mentioned briefly in the introduction, there is a correspondence between the moments of a Poisson random variable and the Bell polynomials defined by $B(k, \mu) = \sum_i \left\{ {k \atop i} \right\} \mu^i$. In particular, $\mathrm{E}[X^k] = B(k, \mu)$, if μ is the mean of the Poissonian random variable. The Bell polynomials are so named because $B(k, 1)$ is the kth Bell number. By Dobiński's formula $B(k, 1) = \frac{1}{e} \sum_{i=0}^{\infty} \frac{i^k}{i!}$ the Bell numbers are generalised for real k. We write these as $B_x = B(x, 1)$.

We give a lower bound for $\mathrm{E}[(X/\mu)^k]$ by showing the following simple connection between the Bell polynomials and Bell numbers:

Theorem 2. *Let k be a positive real number and $\mu \geq 1$ be an integer. Then*

$$B(k, \mu)/\mu^k \geq B_{k/\mu}^{\mu}.$$

While the proof below assumes μ is an integer, we will conjecture Theorem 2 to be true for any $\mu \geq 1$. Now by de Bruijn's (1981) asymptotic expression for the Bell numbers:

$$\mathrm{E}[(X/\mu)^k] \geq B_{k/\mu}^{\mu} = \left(\frac{k/\mu}{e \log(k/\mu)} (1 + o(1))\right)^k \quad \text{as } k/\mu \to \infty.$$

matching the upper bound of Ostrovsky and Sirota, eq. (3), for large k, as well as Latała's uniform lower bound with a different constant.

Proof of Theorem 2. Let $X, X_1, \ldots, X_\mu$ be i.i.d. Poisson variables with mean 1, then $S = \sum_{i=1}^{\mu} X_i$ is Poisson with mean μ. We write $\|X\|_k = \mathrm{E}[X^k]^{1/k}$. Then by the AG inequality:

$$\|S/\mu\|_k = \left\|\frac{1}{\mu}\sum_{i=1}^{\mu} X_i\right\|_k \geq \left\|\left(\prod_{i=1}^{\mu} X_i\right)^{1/\mu}\right\|_k = \left\|\prod_{i=1}^{\mu} X_i\right\|_{k/\mu}^{1/\mu} = \left(\prod_{i=1}^{\mu} \|X_i\|_{k/\mu}\right)^{1/\mu} = \|X\|_{k/\mu}. \tag{8}$$

Since X has mean 1 we have $\|X\|_{k/\mu} = B_{k/\mu}^{\mu/k}$, and as S has mean μ we have $\|S/\mu\|_k = B(k,\mu)^{1/k}/\mu$. Thus, taking kth powers, eq. (8) is what we wanted to show. □

For small k/μ this bound is less interesting since $B_x \to 0$ as $x \to 0$, rather than 1 as our upper bound. However, it is pretty tight, as we conjecture by the following matching upper bound in terms of the Bell numbers:

Conjecture 1. *For all $k > 0$ and $\mu \geq 1$,*

$$B_{k/\mu}^{1/(k/\mu)} \leq \frac{B(k,\mu)^{1/k}}{\mu} \leq B_{k/\mu+1}^{1/(k/\mu+1)}.$$

Furthermore, for $0 < \mu \leq 1$, $\frac{B(k,\mu)^{1/k}}{\mu} \leq B_{k/\mu}^{1/(k/\mu)}$.

While the upper bound appears true numerically, it can't follow from our moment-generating function bound eq. (7), since it drops below that for k/μ bigger than 40. The conjectured upper bound is even incomparable with our Theorem 1, since it is slightly above $\frac{k/\mu}{\log(1+k/\mu)}$ for very small k/μ. In the region $k < 2$ and $\mu < 1$, the conjectured bound is weaker than eq. (2) by Berend and Tassa (2010), but for all other parameters, it is substantially tighter.

3 Sub-Poissonian Random Variables

We call a non-negative random variable X sub-Poissonian if $\mathrm{E}[X] = \mu$ and the moment-generating function, mgf., $\mathrm{E}[\exp(tX)] \leq \exp(\mu(e^t - 1))$ for all $t > 0$. We will briefly show that this notion includes all sums of bounded random variables, such as the Binomial distribution.

If $X_1, \ldots, X_n$ are sub-Poissonian with mgf. $m_1(t), \ldots, m_n(t)$ and mean $\mu_1, \ldots, \mu_n$ respectively, then $\sum_i X_i$ is sub-Poissonian as well, since

$$\mathrm{E}\big[\exp\big(t\sum_i X_i\big)\big] = \prod_i m_i(t) \leq \prod_i \exp\big(\mu_i(e^t-1)\big) = \exp\Big(\big(\sum_i \mu_i\big)\big(e^t - 1\big)\Big).$$

Next, a random variable bounded in $[0,1]$ with mean μ has mgf.

$$\mathrm{E}[\exp(tX)] = 1 + \sum_{k=1}^{\infty} \frac{t^k\,\mathrm{E}[X^k]}{k!} \leq 1 + \mu \sum_{k=1}^{\infty} \frac{t^k\,\mathrm{E}[1^{k-1}]}{k!} = 1 + \mu(e^t - 1) \leq \exp(\mu(e^t - 1)).$$

Hence if $X = X_1 + \cdots + X_n$ where each $X_i \in [0,1]$ we have $\mu = \mathrm{E}[X] = \sum_i \mathrm{E}[X_i]$ and by Theorem 1 that $\mathrm{E}[(X/\mu)^k] \leq \frac{k/\mu}{\log(k/\mu+1)}$. In particular this captures sum of Bernoulli variables with distinct probabilities.

An example of a non-sub-Poissonian distribution is the geometric distribution with mean μ. This has moment generating function $m(t) = \frac{1}{1-\mu(e^t-1)}$, which is larger than $\exp(\mu(e^t-1))$ for all $t > 0$. However, likely, similar methods to those in the proof of Theorem 1 will still apply to bound its moments.

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