Equivalent Reduced Model Technique Development for Nonlinear System Dynamic Response

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ABSTRACT

The dynamic response of structural systems commonly involves nonlinear effects. Often times, structural systems are made up of several components, whose individual behavior is essentially linear compared to the total assembled system. However, the assembly of linear components using highly nonlinear connection elements or contact regions causes the entire system to become nonlinear. Conventional transient nonlinear integration of the equations of motion can be extremely computationally intensive, especially when the finite element models describing the components are very large and detailed.

In this work, the Equivalent Reduced Model Technique (ERMT) is developed to address complicated nonlinear contact problems. ERMT utilizes a highly accurate model reduction scheme, the System Equivalent Reduction Expansion Process (SEREP). Extremely reduced order models that provide dynamic characteristics of linear components, which are interconnected with highly nonlinear connection elements, are formulated with SEREP for the dynamic response evaluation using direct integration techniques. The full-space solution will be compared to the response obtained using drastically reduced models to make evident the usefulness of the technique for a variety of analytical cases.

INTRODUCTION

Nonlinear response analysis typically involves significant computation, especially if the system matrices for the full analytical model are used to obtain the forced nonlinear response solution. Due to the significant computational resources required for these types of nonlinear problems, the analyst may often be unable to investigate specific nonlinear scenarios in depth, particularly if the nonlinear elements are characterized with a set of performance characteristics related to temperature, preload, deflection, etc. Thus, there is significant motivation to develop several reduced order models that can accurately predict nonlinear system response with substantially reduced computation time.

A particular area of interest is the dynamic response of systems with nonlinear connections. These systems are typically made up of several components, whose individual behavior is essentially linear compared to the total assembled system. Local regions where component interconnections exist cause the entire system to become nonlinear. The components that make up the system may be linear but the response of the system is nonlinear due to the nature of the nonlinear component interconnection. The technique employed in this paper, the Equivalent Reduced Model Technique (ERMT), was developed to address this class of nonlinear problem.

ERMT [1] is implemented in this work using the System Equivalent Reduction Expansion Process (SEREP) [2], which allows for the formulation of a dramatically reduced model that accurately preserves the full analytical model dynamics with very few degrees of freedom (DOFs). Discrete nonlinear connection elements are then assembled into the reduced model in the local regions where component interconnections occur. Using the reduced models that are developed with ERMT in conjunction with direct integration techniques allows computationally efficient forced response solutions to be obtained. These techniques can also be easily extended to experimental components if the system matrices are updated using any of the direct model updating techniques such as those identified in [3].

This approach was first presented by Avitabile and O'Callahan [4] where a detailed overview of the applicable theory was provided, along with a simple analytical example. Friswell *et al.* [5] looked at reducing models with local nonlinearities using several different reduction schemes for a periodic solution. Lamarque and Janin [6] looked at modal superposition for simple single-DOF and two-DOF systems with impact and concluded that modal superposition had limitations due to difficulties in developing the general formulas with the nonlinear impacts. Ozguven and Kuran [7] converted nonlinear Ordinary Differential Equations (ODE's) into a set of nonlinear algebraic equations, which could be reduced by using linear modes. This technique was found to provide the best reduction in computation time when the structure was excited at a forcing frequency that corresponded to a resonance of the structure. An alternative approach that has been studied uses Nonlinear Normal Modes (NNMs) which are formulated by Ritz vectors [8, 9]. This approach seeks to extend the concept of linear orthogonal modes to nonlinear systems.

This paper presents the analytical time response results of a cantilevered beam system subjected to an input force pulse. Four cases are studied: single beam with no contact, single beam with single contact, two beams with single contact, and two beams with two contacts. For each of these cases, two types of contact stiffness are considered, a soft contact representing a rubber/isolation material, and a hard contact representing a metal on metal contact. For all cases, the time response results of the full-space model will be used as the reference solution.

THEORY

Equivalent Reduced Model Technique (ERMT)

The Equivalent Reduced Model Technique (ERMT) is based on concepts related to model reduction, which are summarized herein.

General Reduction Techniques

Model reduction is typically performed to reduce the size of a large analytical model to develop a more efficient model for further analytical studies. Most reduction or condensation techniques affect the dynamic characteristics of the resulting reduced model. Model reduction is performed for a number of reasons, but the technique is used primarily as a mapping technique for expansion. In general, a relationship between the full set of finite element DOFs and the reduced set of DOFs needs to be formed as

$$\{X_n\} = \begin{Bmatrix} X_a \\ X_d \end{Bmatrix} = [T]\{X_a\} \tag{1}$$

The 'n' subscript denotes the full set of finite element DOFs, the 'a' subscript denotes the active set of DOFs (sometimes referred to as master DOFs), and the subscript 'd' denotes the deleted DOFs (sometimes referred to as omitted DOFs); the [T] transformation matrix relates the mapping between these two sets of DOFs.

The reduced mass and stiffness matrices are related to the full-space mass and stiffness matrices using congruent matrix operations as

$$[M_a] = [T]^T [M_n][T] \quad \text{and} \quad [K_a] = [T]^T [K_n][T] \tag{2}$$

What is most important in model reduction is that the eigenvalues and eigenvectors of the original system are preserved as accurately as possible in the reduction process. If this is not maintained then the matrices are of questionable value. The eigensolution is then given by

$$\big[[K_a] - \lambda [M_a]\big]\{X_a\} = \{0\} \tag{3}$$

Because reduction schemes such as Guyan Condensation [10] and Improved Reduced System Technique [11] are based primarily on the stiffness of the system, the eigenvalues and eigenvectors will not be exactly reproduced in the reduced model. However, the System Equivalent Reduction Expansion Process (SEREP) [2] exactly preserves the eigenvalues and eigenvectors in the reduced model.

System Equivalent Reduction Expansion Process (SEREP)

The SEREP modal transformation relies on the partitioning of the modal equations representing the system DOFs relative to the modal DOFs using

$$\{X_n\} = \begin{Bmatrix} X_a \\ X_d \end{Bmatrix} = \begin{bmatrix} U_a \\ U_d \end{bmatrix} \{p\} \tag{4}$$

Using a generalized inverse, this can be manipulated to give

$$\{p\} = \left([U_a]^T [U_a] \right)^{-1} [U_a]^T \{X_a\} = [U_a]^g \{X_a\} \tag{5}$$

which is then used to relate the ‘n’ DOFs to the ‘a’ DOFs as

$$\{X_n\} = [U_n][U_a]^g \{X_a\} = [T_U]\{X_a\} \tag{6}$$

with

$$[T_U] = [U_n][U_a]^g \tag{7}$$

Equation (7) represents the SEREP transformation matrix that is used in the reduction of the finite element mass and stiffness matrices as described in Equation (2).

SEREP relies heavily on a “well developed” finite element dynamic model from which an ‘n’ dimensional eigensolution is obtained. In addition, the quality of the SEREP reduced model depends on the selection of active ‘a’ DOFs used in the formulation of the generalized matrix inverse of the ‘a’ DOFs modal vector partition. Both conditions affect the rank and matrix conditioning required to define a good SEREP transformation matrix needed to develop well-behaved reduced system matrices. The reduced models developed in this work are for the equivalent condition, where the number of modes used is equal to the number of DOFs retained in the reduction process.

Since the transformation matrix is formed from the eigenvectors of the full-space finite element model, the reduced matrices preserve the eigenvalues and eigenvectors of the full-space model. This implies that any collection of desired eigenvectors can be retained in an exact sense for the reduced model. This fact is significant in terms of the development of efficient models from large finite element models used for forced response studies, especially those that contain discrete nonlinear effects that are typical in joints and connections for many structural systems.

An additional model reduction approach [12] can be used where a highly accurate reduced model is formulated by using Guyan reduction in conjunction with analytical model improvement. The advantages of this technique are that the desired mode shapes are preserved in an exact sense while the fully ranked and well-conditioned system matrices obtained from Guyan reduction are maintained in the process.

Mode Contribution Identification

To ensure that the reduced models are minimally affected by modal truncation when assembling system models using multiple reduced component models, the contribution of component modes to the assembled system is computed using

$$\text{Mode Contribution} = \begin{bmatrix} [U_n^A] & \\ & [U_n^B] \end{bmatrix}^T [M_n^{AB}][U_n^{AB}] \tag{8}$$

where $[U^A]$ and $[U^B]$ are the unmodified component modal matrices that are organized into a partitioned matrix, $[M^{AB}]$ is the modified system mass matrix, and $[U^{AB}]$ is the modified system modal matrix. The mode contribution matrix is computed for all possible system configurations using full-space component models.

The resulting mode contribution matrix is the key to identifying the necessary set of modal vectors to accurately obtain the final modified system modes. This is similar to using the $[U_{12}]$ matrix that is computed in Structural Dynamic Modification (SDM) [13] to identify the contributions of component modes in the assembled system modes. The $[U_{12}]$ matrix contains the scaling coefficients needed to form the final modified set of modal vectors $[U_2]$ from the initial unmodified set of modal vectors $[U_1]$. Figure 1 illustrates how the $[U_{12}]$ matrix is used in forming the final modified set of modes $[U_2]$, where 'm' modes of the $[U_{12}]$ matrix are used, and 'n-m' modes are excluded.

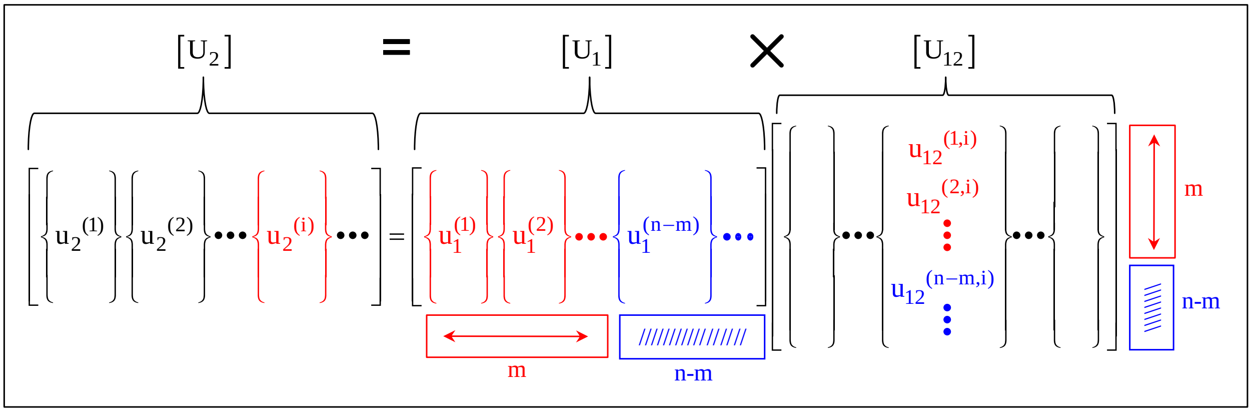


Figure 1. Mode Contribution Identification Using $[U_{12}]$ Matrix from SDM

Response Analysis Technique

The reduced component mass and stiffness matrices described in Equation (2) are used in a normal system assembly to connect the linear components with highly nonlinear connection elements. Nonlinear direct integration of the equations of motion is then performed to obtain the system response. The ERMT process is shown schematically in Figure 2.

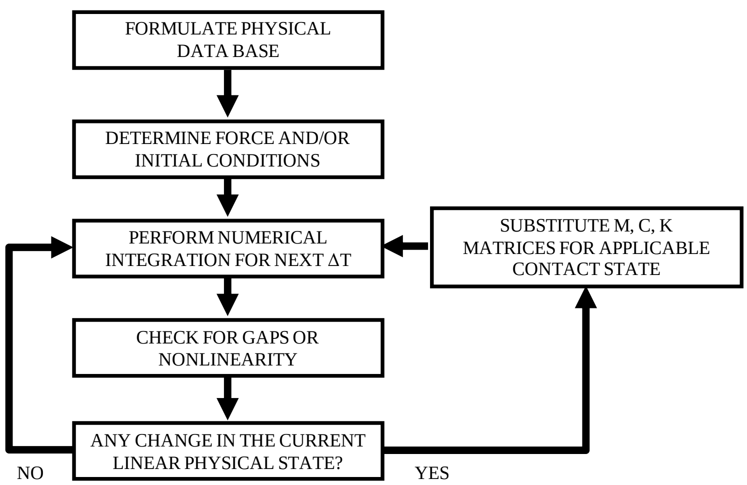


Figure 2. Schematic for ERMT Process

Newmark Direct Integration Technique

In this work, the Newmark Method [14] is used to perform the direct integration of the equations of motion for the ERMT solution process. From the known initial conditions for displacement and velocity, the initial acceleration vector is computed using the equation of motion and the applied forces as

$$\ddot{\vec{x}}_0 = [M]^{-1}\left(\vec{F}_0 - [C]\dot{\vec{x}}_0 - [K]\vec{x}_0\right) \tag{9}$$

where $\ddot{\vec{x}}_0$ = Initial Acceleration Vector

$\dot{\vec{x}}_0$ = Initial Velocity Vector

x_0 = Initial Displacement Vector

$\vec{F}_0$ = Initial Force Vector

Choosing an appropriate Δt, α, and β, the displacement vector is

$$\begin{aligned}\vec{x}_{i+1} = \left[\frac{1}{\alpha(\Delta t)^2}[M] + \frac{\beta}{\alpha\Delta t}[C] + [K]\right]^{-1} \Bigg\{ & \vec{F}_{i+1} + [M]\left(\frac{1}{\alpha(\Delta t)^2}\vec{x}_i + \frac{1}{\alpha\Delta t}\dot{\vec{x}}_i + \left(\frac{1}{2\alpha} - 1\right)\ddot{\vec{x}}_i\right) \\ & + [C]\left(\frac{\beta}{\alpha\Delta t}\vec{x}_i + \left(\frac{\beta}{\alpha} - 1\right)\dot{\vec{x}}_i + \left(\frac{\beta}{\alpha} - 2\right)\frac{\Delta t}{2}\ddot{\vec{x}}_i\right)\Bigg\}\end{aligned} \tag{10}$$

The values chosen for α and β were ¼ and ½, respectively. This assumes constant acceleration and the integration process is unconditionally stable, where a reasonable solution will always be reached regardless of the time step used. However, the time step should be chosen such that the highest frequency involved in the system response can be characterized properly to avoid numerical damping in the solution. The time step should be chosen to be at least 10 times smaller than the period of the highest frequency involved in the system response. The time step used for the analytical cases studied in this paper was 0.0001 seconds.

Following the displacement vector calculation, the acceleration and velocity vectors are computed for the next time step using

$$\dot{\vec{x}}_{i+1} = \dot{\vec{x}}_i + (1-\beta)\Delta t\ddot{\vec{x}}_i + \beta\Delta t\ddot{\vec{x}}_{i+1} \tag{11}$$

$$\ddot{\vec{x}}_{i+1} = \frac{1}{\alpha(\Delta t)^2}\left(\vec{x}_{i+1} - \vec{x}_i\right) - \frac{1}{\alpha\Delta t}\dot{\vec{x}}_i - \left(\frac{1}{2\alpha} - 1\right)\ddot{\vec{x}}_i \tag{12}$$

This process is repeated at each time step for the duration of the time response solution desired.

Time Response Correlation Tools

In order to quantitatively compare two different time solutions, two correlation tools were employed: The Modal Assurance Criterion (MAC) and the Time Response Assurance Criterion (TRAC).

Modal Assurance Criterion (MAC)

The Modal Assurance Criterion (MAC) [15] is widely used as a vector correlation tool. In this work, the MAC was used to correlate all DOF at a single instance in time. The MAC is written as

$$MAC_{ij} = \frac{\left[\{X1_i\}^T\{X2_j\}\right]^2}{\left[\{X1_i\}^T\{X1_i\}\right]\left[\{X2_j\}^T\{X2_j\}\right]} \tag{13}$$

where X1 and X2 are displacement vectors. MAC values close to 1.0 indicate strong similarity between vectors, where values close to 0.0 indicate minimal or no similarity.

Time Response Assurance Criterion (TRAC)

The Time Response Assurance Criterion (TRAC) [3] quantifies the similarity between a single DOF across all instances in time. The TRAC is written as

$$\mathrm{TRAC}_{ji} = \frac{\left[\{X1_j\}^T\{X2_i\}\right]^2}{\left[\{X1_j\}^T\{X1_j\}\right]\left[\{X2_i\}^T\{X2_i\}\right]} \tag{14}$$

where X1 and X2 are displacement vectors. TRAC values close to 1.0 indicate strong similarity between vectors, where values close to 0.0 indicate minimal or no similarity.

MODEL DESCRIPTION AND CASES STUDIED

This section presents the analytical models developed as well as the cases studied. The full-space time solution is used as the reference solution for all cases.

Linear Component Models: Beam A and Beam B

Two planar element beam models were generated using MAT_SAP [16], which is a finite element modeling (FEM) program developed for MATLAB [17], and were used for all of the cases studied. Figure 3 shows the two beams assembled into the linear system, where the red points are the active DOFs in the reduced order models, and the black arrow denotes the force pulse input location (DOF 105). Note that 3 inches of each beam are clamped for the cantilevered boundary condition that was applied.

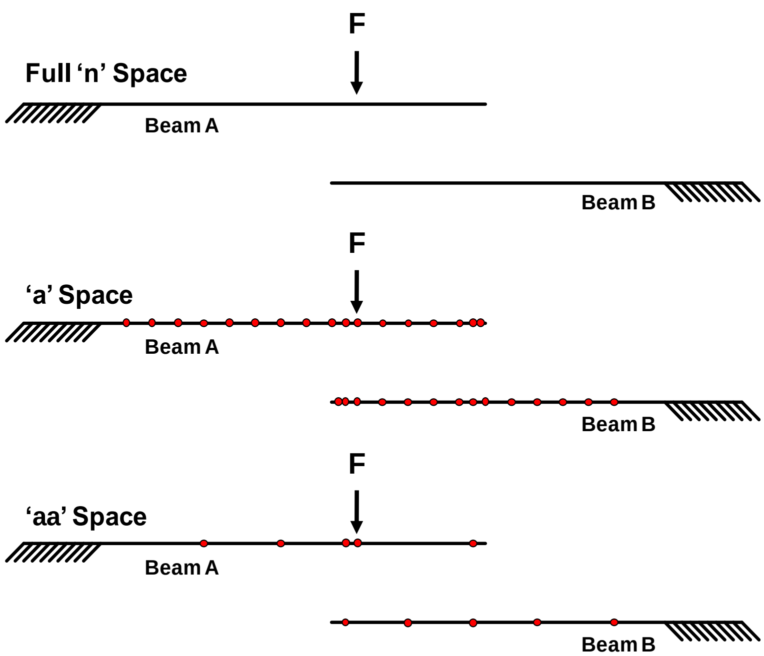


Figure 3. Schematic of Linear Beam Models with Force Pulse Input Location

Table 1 lists the characteristics of the beam models and Table 2 lists the natural frequencies for the first 10 modes of each beam component model. The mode shapes for the unmodified beam components are provided in Appendix A. Damping was assumed 1% of critical damping for all unmodified component modes as well as for all modified system modes in all of the cases studied.

Table 1. Beam Model Characteristics

Property	Beam A	Beam B
Length (in)	18	16
Width (in)	2	4
Thickness (in)	0.123	0.123
# of Elements	72	64
# of Nodes	73	65
# of DOF	146	130
Node Spacing (in)	0.25	0.25
Material	Aluminum	Aluminum
Mass Density (lb/in^3)	2.54E^{-4}	2.54E^{-4}
Young's Modulus (Msi)	10	10

Table 2. Natural Frequencies of Unmodified Beam Component Models

Mode #	Beam A	Beam B
1	12.91	22.62
2	84.12	141.56
3	252.34	396.60
4	519.59	776.92
5	806.16	1284.71
6	1256.55	1918.28
7	1682.96	2678.33
8	2201.36	3563.89
9	2755.52	4572.70
10	3510.01	5707.04

The force pulse input to the system is an analytic force pulse designed to be frequency band-limited, exciting modes up to 1000 Hz while minimally exciting higher order modes. Using this force pulse, the number of modes involved in the response can be determined easily, as modes above 1000 Hz can be considered to have negligible participation in the response. Modes 1 to 5 will be primarily excited in Beam A, while modes 1 to 4 will be primarily excited in Beam B. Figure 4 shows the analytical force pulse in the time and frequency domain.

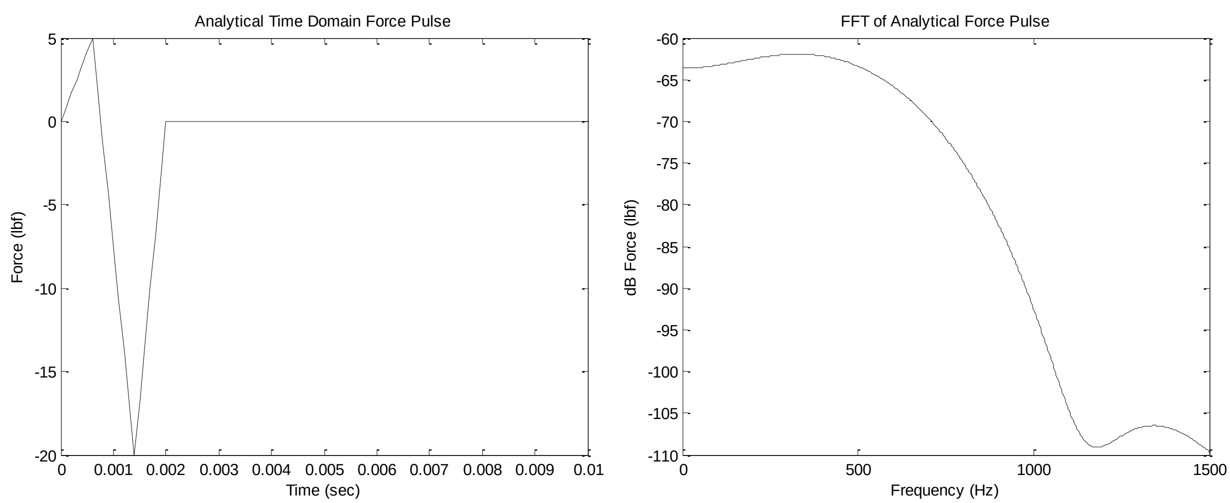


Figure 4. Analytical Force Pulse in the Time Domain (left) and Frequency Domain (right)

The full-space linear component models were reduced down to ‘a’ and ‘aa’ space using SEREP. The active DOFs and modes retained in the reduced component models are listed in Table 3. Note that only translational DOF were used in the reduced component models.

Table 3. Active DOFs and Modes Retained in the Reduced Component Models

Model	# of DOF	Retained Modes	Active DOF
Beam A - 'a' Space	17	1-17	33,41,49,57,65,73,81,89,97,101,105,113,121,129,137,141,143
Beam A - 'aa' Space	5	1-5	57,81,101,105,141
Beam B - 'a' Space	14	1-14	3,5,9,17,25,33,41,45,49,57,65,73,81,89
Beam B - 'aa' Space	5	1-5	5,25,45,65,89

CASE 1: SINGLE BEAM WITH NO CONTACT

For the first case, the system is a single beam (Beam A) with no contact. This system is linear and there is no change in the state of the system, where the number of modes needed is only a function of the input force spectrum. Based on the input force spectrum seen previously in Figure 4 and the natural frequencies for Beam A in Table 2, five modes are needed to accurately compute the system time response. Therefore, the smallest reduced model that can be used to prevent the effects of mode truncation is 5 DOF to maintain the SEREP condition, as discussed previously. To determine if this assumption is accurate, the system response was plotted in the time and frequency domain for ‘aa’ space with comparison to the full ‘n’ space solution, as shown in Figure 5. The response for the ‘a’ space model was not shown, due to the accurate results observed from the ‘aa’ space model response.

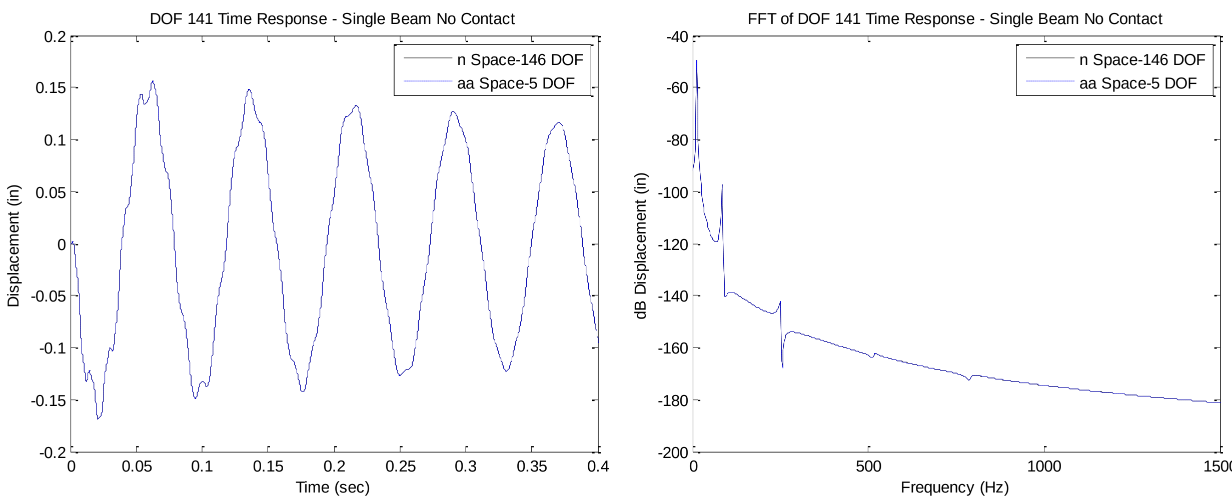


Figure 5. Comparison of ‘n’ and ‘aa’ Space Models for Single Beam with No Contact for DOF 141 – Time Response (left) and FFT of Time Response (right)

The system response in Figure 5 shows that the reduced model is able to accurately capture the response of the system due to the inclusion of the modes primarily excited by the input force pulse. To quantify the similarity of the reduced model results with the full-space solution, the MAC and TRAC were computed between the full model and the reduced model time responses, which were then averaged, as listed in Table 4. The solution time for each model is also listed to show the decrease in computation time when reduced models are used.

Table 4. Average MAC and TRAC for Reduced Models and Solution Times for Single Beam with No Contact

Model	# of DOF	Average MAC	Average TRAC	Solution Time (sec)
'n' Space	146	1	1	34.6
'a' Space	17	0.99999687	0.99999999	1.9
'aa' Space	5	0.99993281	0.99999992	0.7

Table 4 provides further confirmation that the ‘aa’ space reduced model is sufficient for accurately computing the time response for this particular case. In addition, the solution time for the full-space model is over 30 seconds in contrast to the reduced ‘aa’ space model, which is less than 1 second. ERMT was shown in this case to provide significant reduction in computation time for a linear system.

This first case demonstrates that an accurate time response solution can be obtained using a drastically reduced model with limited number of modes, if the primary modes excited by the structure are retained in the reduced model. However, if the retained modes are selected incorrectly, an accurate time response will not be obtained, regardless of how many additional modes and active DOFs are retained in the reduced model.

The following cases will show the application of ERMT for contact situations, which causes the system to become nonlinear. In addition, both soft and hard contacts will be studied to show the effect that different contact stiffness has on the accuracy of the reduced model results. The soft stiffness case will be studied first, as this contact stiffness is unlikely to excite a higher frequency range than the input force pulse applied to the models.

CASE 2: SOFT CONTACT

Case 2-A: Single Beam with Single Soft Contact

This case consists of the tip of Beam A coming into contact with a fixed object once the beam has displaced a known gap distance of 0.05 inches, as shown in Figure 6 for the full-space and reduced space models.

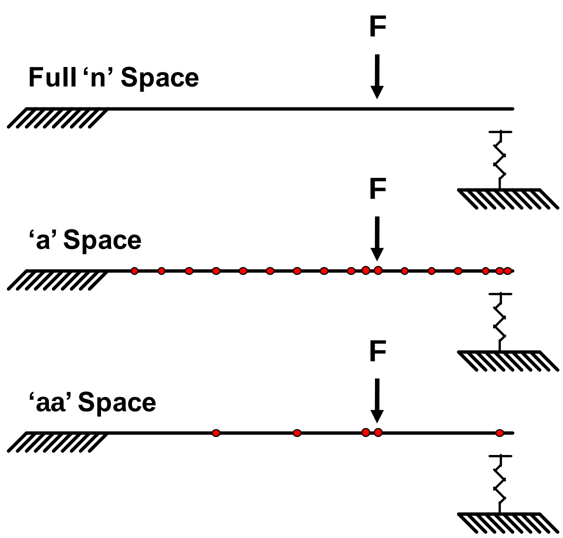
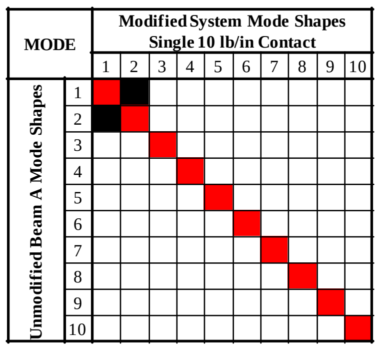


Figure 6. Diagram of Single Beam with Single Contact for Full-Space and Reduced Space Models

The contact is represented as an additional spring stiffness that is added when the contact location on Beam A closes the specified gap distance. This is performed by exchanging the physical system matrices from the initial no contact state to the new contact state until the contact is separated. The contact stiffness of 10 lb/in will be used to represent a soft contact, which is typically seen in a damper or isolation mount. For the purposes of ERMT, the spring stiffness was applied to the full-space physical model, prior to model reduction. In addition, the contact is represented as a spring stiffness in only the translational DOF, and not in the rotational DOF as well. The contact location is at DOF 141 of Beam A. Table 5 lists the first 10 natural frequencies of the system with the soft contact stiffness applied. Figure 7 shows the mode contribution matrix, which indicates the unmodified component modes that participate in the modified system modes. The mode contributions are computed using full-space models so that modal truncation is not a concern. The various box colors indicate the amount that each of the unmodified component modes contributes to a modified system mode; the actual contribution ranges for each color are shown. The mode shapes for the modified system are provided in Appendix A.

Table 5. Natural Frequencies for Single Beam with Single Soft Contact

Mode #	Unmodified	Single Soft Contact
1	12.91	26.09
2	84.12	86.02
3	252.34	252.54
4	519.59	519.65
5	806.16	806.18
6	1256.55	1256.55
7	1682.96	1682.96
8	2201.36	2201.36
9	2755.52	2755.53
10	3510.01	3510.02



Bar Color	Min Value	Max Value
Black	0.005	0.1
Blue	0.1	0.2
Green	0.2	0.3
Cyan	0.3	0.5
Magenta	0.5	0.7
Yellow	0.7	0.9
Red	0.9	1.0

Figure 7. Mode Contribution Matrix for Single Beam with Single Soft Contact

Table 5 and Figure 7 show that the addition of the soft spring has a pronounced effect on the first two lower order modes, with modes 3 and higher remaining relatively unaffected. Figure 7 indicates that both modes 1 and 2 of the unmodified component are needed to obtain either mode 1 or 2 of the modified system. The frequencies for modes 3 and higher of the unmodified component are minimally affected for the modified system. Therefore, additional unmodified component mode shapes are not needed to form the modified system modes above mode 3 (as indicated by the red main diagonal). For a soft contact, the input force pulse primarily dominates the frequency spectrum excited. Therefore, the modes required to accurately predict the system response are modes 1 to 5 for the unmodified component as well as for the modified system. To confirm this, the system response was plotted in the time and frequency domain for ‘aa’ space with comparison to the full ‘n’ space solution, as shown in Figure 8. The response for the ‘a’ space model was not shown, due to the accurate results observed from the ‘aa’ space model response.

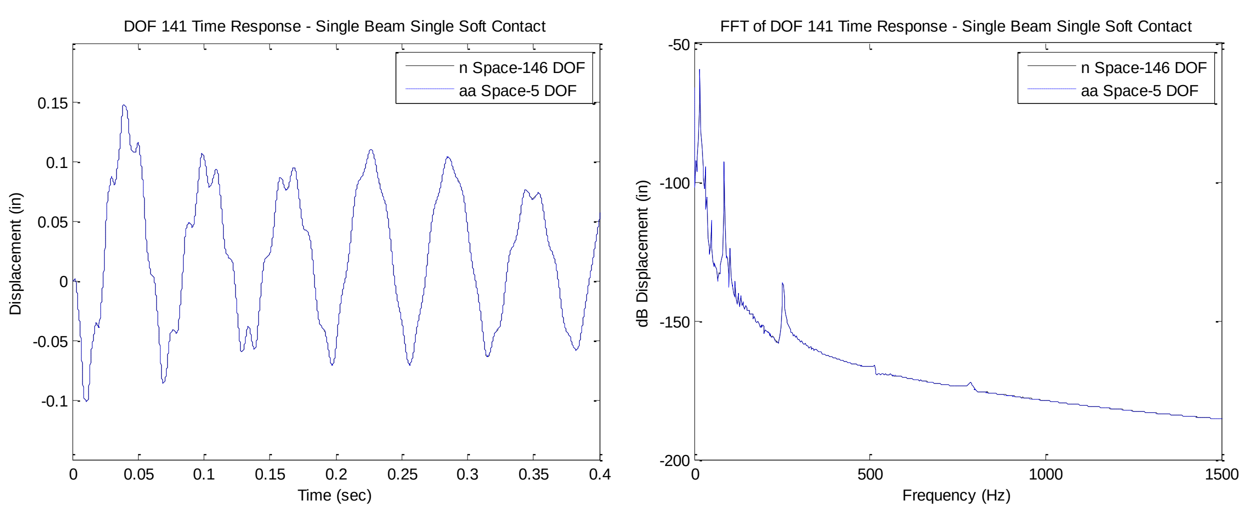


Figure 8. Comparison of ‘n’ and ‘aa’ Space Models for Single Beam with Single Soft Contact for DOF 141 – Time Response (left) and FFT of Time Response (right)

The system response in Figure 8 shows that the reduced model accurately captures the response of the system due to the inclusion of the modes primarily excited by the input force pulse. To quantify the similarity of the reduced model results with the full-space solution, the MAC and TRAC were computed between the full model and the reduced model time responses, which were then averaged, as listed in Table 6. The solution time for each model is also listed to show the decrease in computation time when reduced models are used.

Table 6. Average MAC and TRAC for Reduced Models and Solution Times for Single Beam with Single Soft Contact

Model	# of DOF	Average MAC	Average TRAC	Solution Time (sec)
'n' Space	146	1	1	34.8
'a' Space	17	0.99999687	0.99999999	1.9
'aa' Space	5	0.99993256	0.99999979	0.7

Table 6 provides further confirmation that the ‘aa’ space reduced model is sufficient for accurately computing the time response for this particular case. In addition, the solution time for the full-space model is over 30 seconds in contrast to the reduced ‘aa’ space model, which is less than 1 second. ERMT was shown in this case to provide significant reduction in computation time for a nonlinear system that consists of a single component with a single soft contact.

This second case demonstrated that the mode contribution matrix should be examined to determine the number of unmodified component modes that are needed to obtain the modified system modes of interest. The input force pulse may excite only a few modified system modes, but several unmodified component modes may be required to produce the modified system modes. Failure to include the contributing modes will result in an incorrect time response, regardless of how many additional modes and active DOFs are retained in the reduced model. When the mode contribution matrix is examined and the correct unmodified component modes are used in the reduced model, an accurate nonlinear time response solution can be obtained using a drastically reduced model.

Case 2-B: Two Beams with Single Soft Contact

This case consists of the tip of Beam A coming into contact with Beam B once the specified gap distance of 0.05 inches between Beams A and B is closed, as shown in Figure 9 for the full-space and reduced space models.

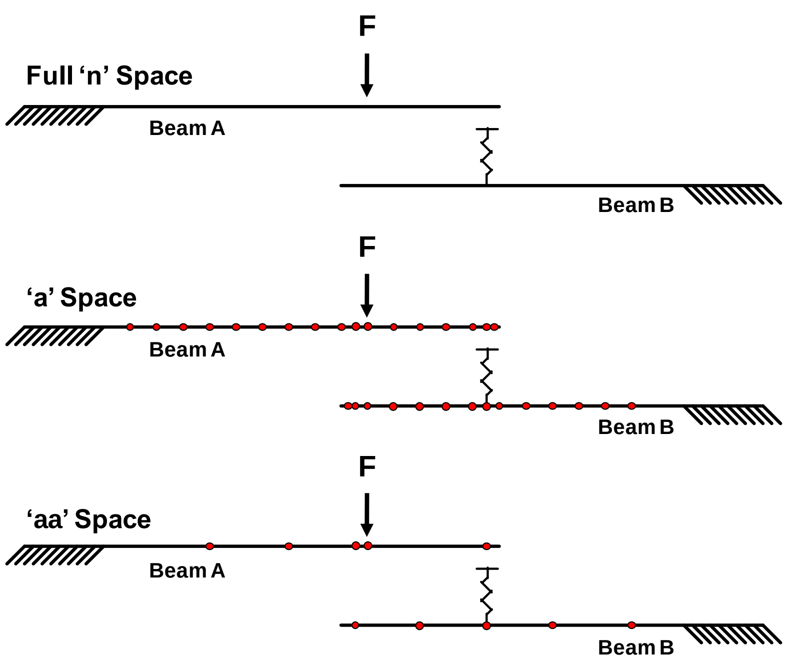


Figure 9. Diagram of Two Beams with Single Contact for Full-Space and Reduced Space Models for Configuration 1

The same soft contact stiffness of 10 lb/in was used to represent the contact of the beams as explained in Case 2-A. The contact occurs at DOF 141 of Beam A and DOF 45 of Beam B. Table 7 lists the natural frequencies for the modified system as well as for the unmodified components. The mode shapes for the modified system are provided in Appendix A.

Figure 10 shows the mode contribution matrix used for identifying the unmodified component modes that contribute in the modified system modes.

Table 7. Natural Frequencies for Two Beams with Single Soft Contact for Configuration 1

Mode #	**2 Beams, Single Soft Contact**	**Unmodified Components**	
		Beam A	**Beam B**
1	20.35	12.91	22.62
2	29.58	84.12	141.56
3	86.01	252.34	396.60
4	142.43	519.59	776.92
5	252.54	806.16	1284.71
6	396.71	1256.55	1918.28
7	519.65	1682.96	2678.33
8	777.00	2201.36	3563.89
9	806.18	2755.52	4572.70
10	1256.55	3510.01	5707.04
11	1284.80	-	-
12	1682.96	-	-
13	1918.28	-	-
14	2201.36	-	-
15	2678.39	-	-
16	2755.53	-	-
17	3510.02	-	-
18	3563.89	-	-
19	3948.55	-	-
20	4572.72	-	-

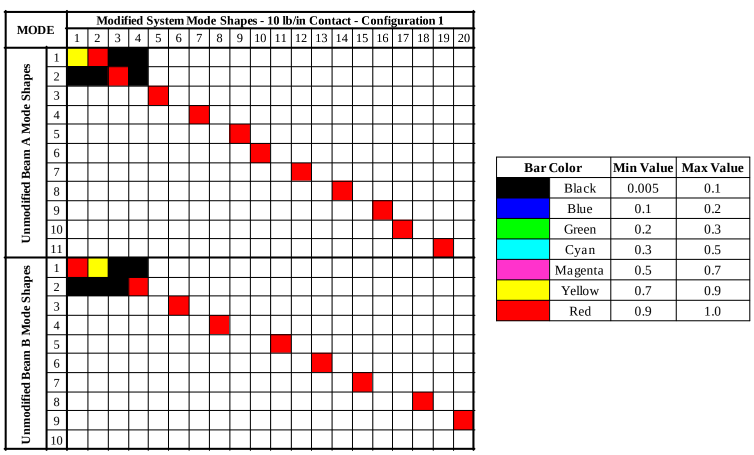


Figure 10. Mode Contribution Matrix for Two Beams with Single Soft Contact for Configuration 1

Due to the contact between the two beams, the first two modes from both unmodified components are needed to obtain the first mode of the modified system, which increases the number of modes needed in the reduced models. Based on the input force pulse, which excites up to 1000 Hz, the first 10 modes of the modified system are expected to be excited, which can be seen in Table 7. Examining Figure 10 shows that for the first 10 modes of the modified system, the first 6 modes of Beam A and the first 4 modes of Beam B are needed to obtain the first 10 modes of the modified system. For a soft contact, the input force pulse primarily dominates the frequency spectrum excited. Therefore, the modes required to accurately predict the system response are those observed in the mode contribution matrix, as stated previously. To show the effects of mode truncation, the system response was plotted in the time and frequency domain for ‘a’ space and ‘aa’ space with comparison to the full ‘n’ space solution in Figure 11, where the ‘aa’ space model is missing a key contributing mode in the modified system response – mode 6 of Beam A.

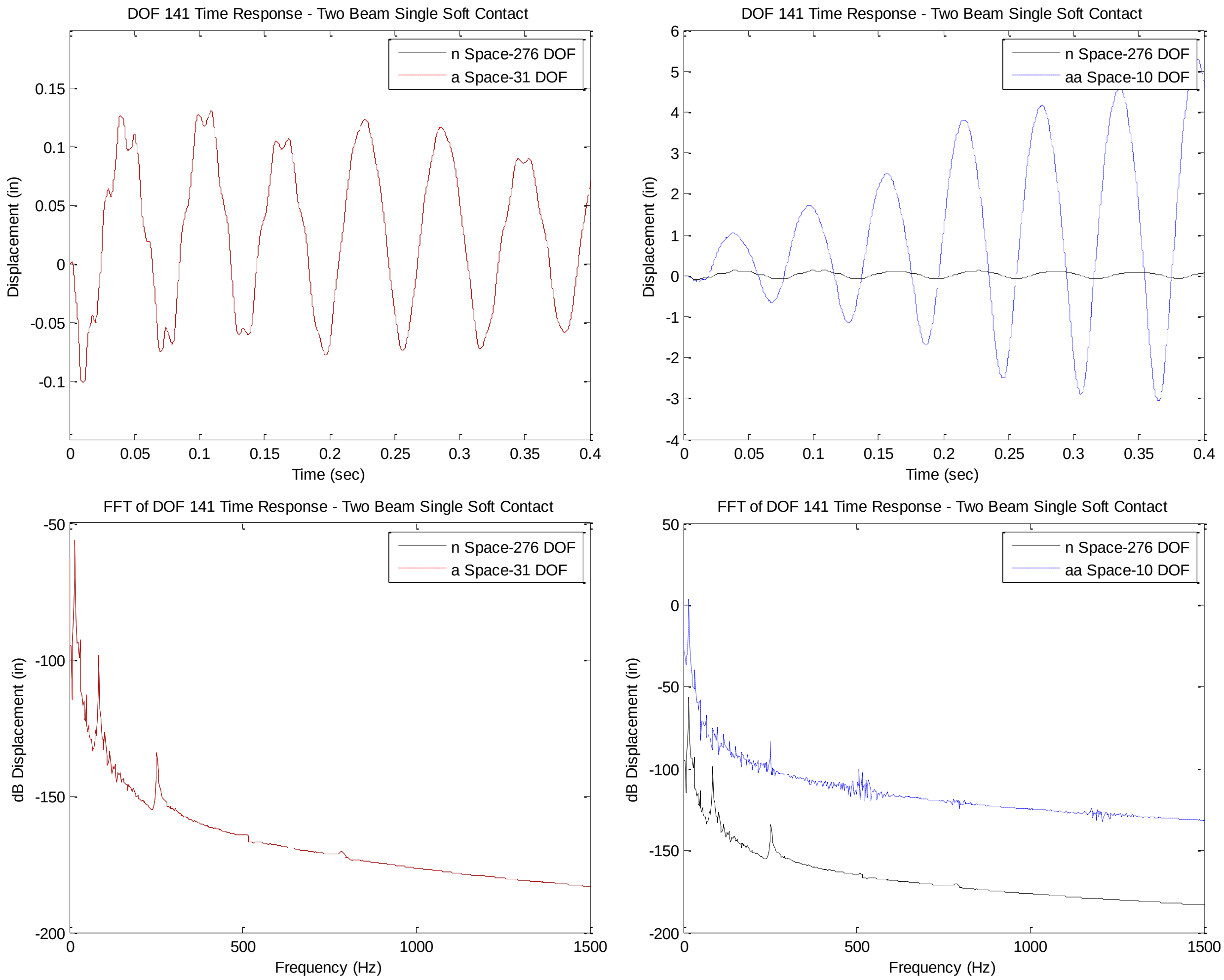


Figure 11. Comparison of ‘n’, ‘a’, and ‘aa’ Space Models for Two Beams with Single Soft Contact for DOF 141 – Time Response (top) and FFT of Time Response (bottom)

The ‘a’ space model response in Figure 11 can be observed to have very good correlation with the full-space solution; however, the ‘aa’ space model response indicates that not including mode 6 of Beam A degrades the results obtained. To quantify the similarity of the reduced model results with the full-space solution, the MAC and TRAC were computed between the full model and the reduced model time responses, which were then averaged, as listed in Table 8.a. The solution time for each model is also listed to show the decrease in computation time when reduced models are used.

To improve the ‘aa’ space model results, the addition of DOF 143 and mode 6 from Beam A was included in the reduced model. From the mode contribution matrix in Figure 10, mode 6 of Beam A is the primary contributing mode that is needed to obtain mode 10 of the modified system. The reduced system response that uses 11 DOF and 11 modes is plotted in the time and frequency domain with comparison to the full ‘n’ space solution in Figure 12.

The ‘aa’ space model time response in Figure 12 shows significant improvement in the correlation with the full-space model when an additional DOF and key contributing mode of Beam A is included in the reduced model. In Table 8.b, the MAC is improved from 0.678 to 0.999 and the TRAC is improved from 0.104 to 0.999 with essentially no change in the solution time when DOF 143 and mode 6 from Beam A are included in the reduced model. This shows that significant errors can result when key contributing component modes are not included in the assembled system for reduced models. The improved MAC and TRAC results further confirm that the 11 DOF reduced model is sufficient for accurately computing the time response for this particular case. In addition, the solution time for the full-space model is approximately 90 seconds in contrast to the ‘aa’ space reduced model, which is only 1 second. ERMT was shown in this case to provide significant reduction in computation time for a nonlinear multiple component system with a single soft contact while maintaining a highly accurate time response solution.

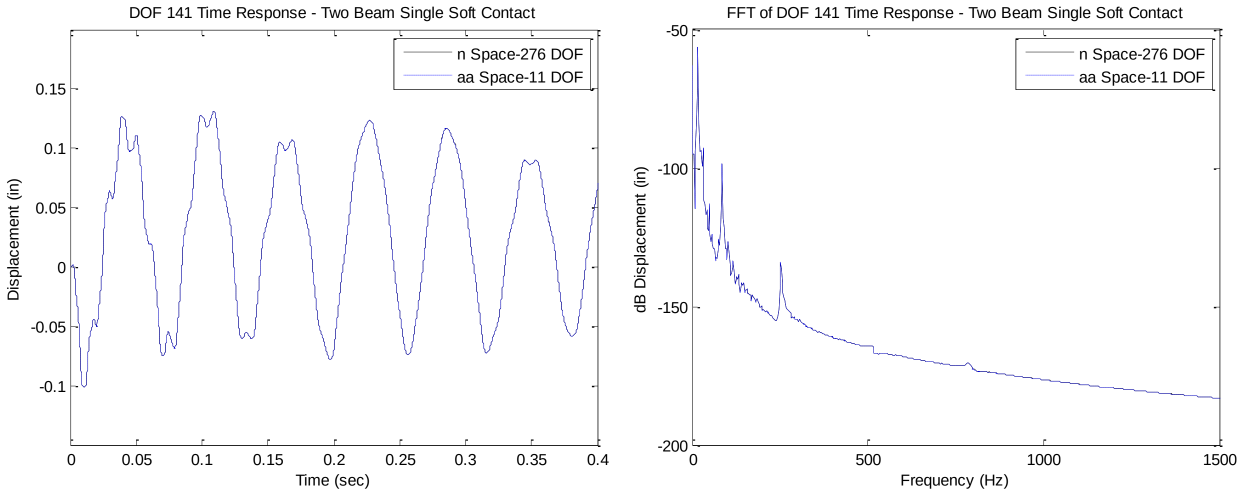


Figure 12. Comparison of 'n' and 'aa' Space Models for Two Beams with Single Soft Contact for DOF 141 – Time Response (left) and FFT of Time Response (right)

Table 8. Average MAC and TRAC for Reduced Models and Solution Times for Two Beams with Single Soft Contact

Table 8.a. Incorrect Number of Modes used in 'aa' Space Model

Model	# of DOF	Average MAC	Average TRAC	Solution Time (sec)
'n' Space	276	1	1	87.9
'a' Space	31	0.99999686	0.99999952	4.8
'aa' Space	10	0.67818808	0.10422630	1.1

Table 8.b. Suitable Number of Modes used in 'aa' Space Model

Model	# of DOF	Average MAC	Average TRAC	Solution Time (sec)
'n' Space	276	1	1	87.9
'a' Space	31	0.99999686	0.99999952	4.8
'aa' Space	11	0.99996327	0.99999980	1.1

Case 2-C: Two Beams with Multiple Soft Contacts

This case consists of Beam A coming into contact with Beam B in three different configurations at two possible contact locations with a specified gap of 0.05 inches, as shown in Figure 13. Note that each system is a potential configuration of the two components depending on the relative displacements of the two beams; no contact is also a possible configuration.

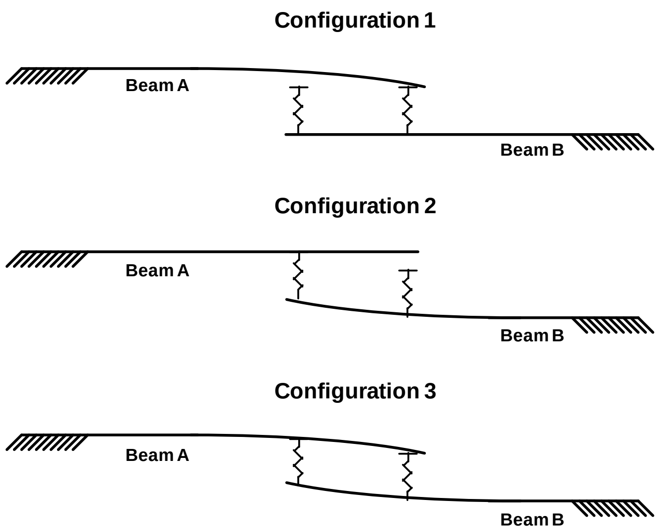


Figure 13. Diagram of Two Beams with Multiple Contacts for Configurations 1, 2, and 3

The same soft contact stiffness of 10 lb/in was used to represent the contact of the beams as explained in Case 2-A. The contact for configuration 1 occurs at DOF 141 of Beam A and DOF 45 of Beam B. The contact for configuration 2 occurs at DOF 101 of Beam A and DOF 5 of Beam B. The contact for configuration 3 occurs when both contacts for configurations 1 and 2 are closed simultaneously. Table 9 lists the natural frequencies for the modified system configurations as well as for the unmodified components. The mode shapes for the modified system configurations are provided in Appendix A. Figure 14 shows the mode contribution matrices used for identifying the unmodified component modes that participate in the modified system modes for the three possible system configurations.

Table 9. Frequencies for Two Beams with Multiple Soft Contact Configurations

Mode #	**Modified System - Soft Contacts**			**Unmodified Components**	
	Config. 1	**Config. 2**	**Config. 3**	**Beam A**	**Beam B**
1	20.35	14.78	21.08	12.91	22.62
2	29.58	33.04	39.23	84.12	141.56
3	86.01	85.71	87.24	252.34	396.60
4	142.43	142.98	143.82	519.59	776.92
5	252.54	252.61	252.82	806.16	1284.71
6	396.71	396.96	397.07	1256.55	1918.28
7	519.65	519.61	519.68	1682.96	2678.33
8	777.00	777.04	777.12	2201.36	3563.89
9	806.18	806.28	806.30	2755.52	4572.70
10	1256.55	1256.56	1256.56	3510.01	5707.04
11	1284.80	1284.76	1284.85	-	-
12	1682.96	1682.99	1682.99	-	-
13	1918.28	1918.29	1918.30	-	-
14	2201.36	2201.37	2201.37	-	-
15	2678.39	2678.34	2678.39	-	-
16	2755.53	2755.53	2755.53	-	-
17	3510.02	3510.02	3510.03	-	-
18	3563.89	3563.89	3563.89	-	-
19	3948.55	3948.54	3948.55	-	-
20	4572.72	4572.70	4572.72	-	-

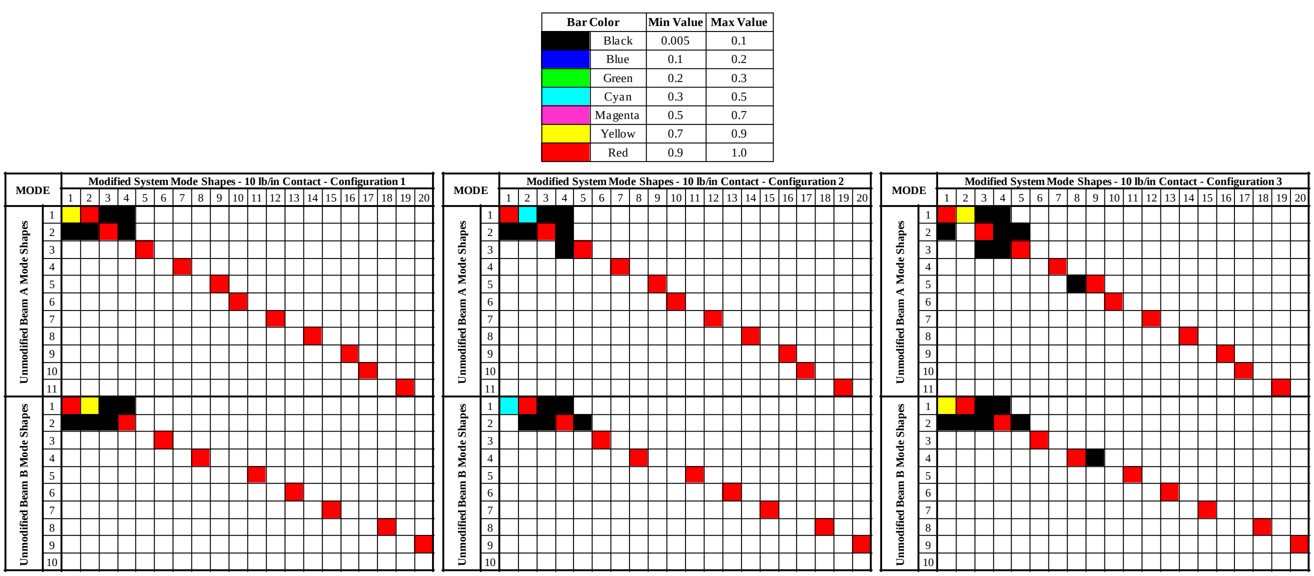


Figure 14. Mode Contribution Matrix for Two Beams with Soft Contacts for Configurations 1, 2, and 3

For this case, three potential modified system configurations exist, which results in three separate mode contribution matrices. Based on the input force pulse, which excites up to 1000 Hz, the first 10 modes of the modified system for all three configurations are expected to be excited, which can be seen in Table 9. Examining Figure 14 indicates that for the first 10 modes of the modified system for all three configurations, the first 6 modes of Beam A and the first 4 modes of Beam B are needed to obtain the first 10 modes of the modified system for all potential configurations. For a soft contact, the input force pulse primarily dominates the frequency spectrum excited. Therefore, the unmodified component modes required to accurately predict the system response are those observed in the mode contribution matrices, as stated previously.

Table 9 and Figure 14 also show that not only does the additional spring stiffness affect the number of unmodified component modes needed, but the location of the spring affects the combination of modes needed for each configuration as well. Depending on whether the spring is at the tip of Beam A or the tip of Beam B, the mode shapes and frequencies change noticeably. For example, the second mode of Beam B is needed to obtain the first mode of the modified system for configurations 1 and 3, but is not needed in configuration 2. To show the effects mode truncation has on the solution, the system response was plotted in the time and frequency domain for ‘a’ space and ‘aa’ space with comparison to the full ‘n’ space solution in Figure 15, where the ‘aa’ space model is missing a key contributing mode in the modified system response – mode 6 of Beam A.

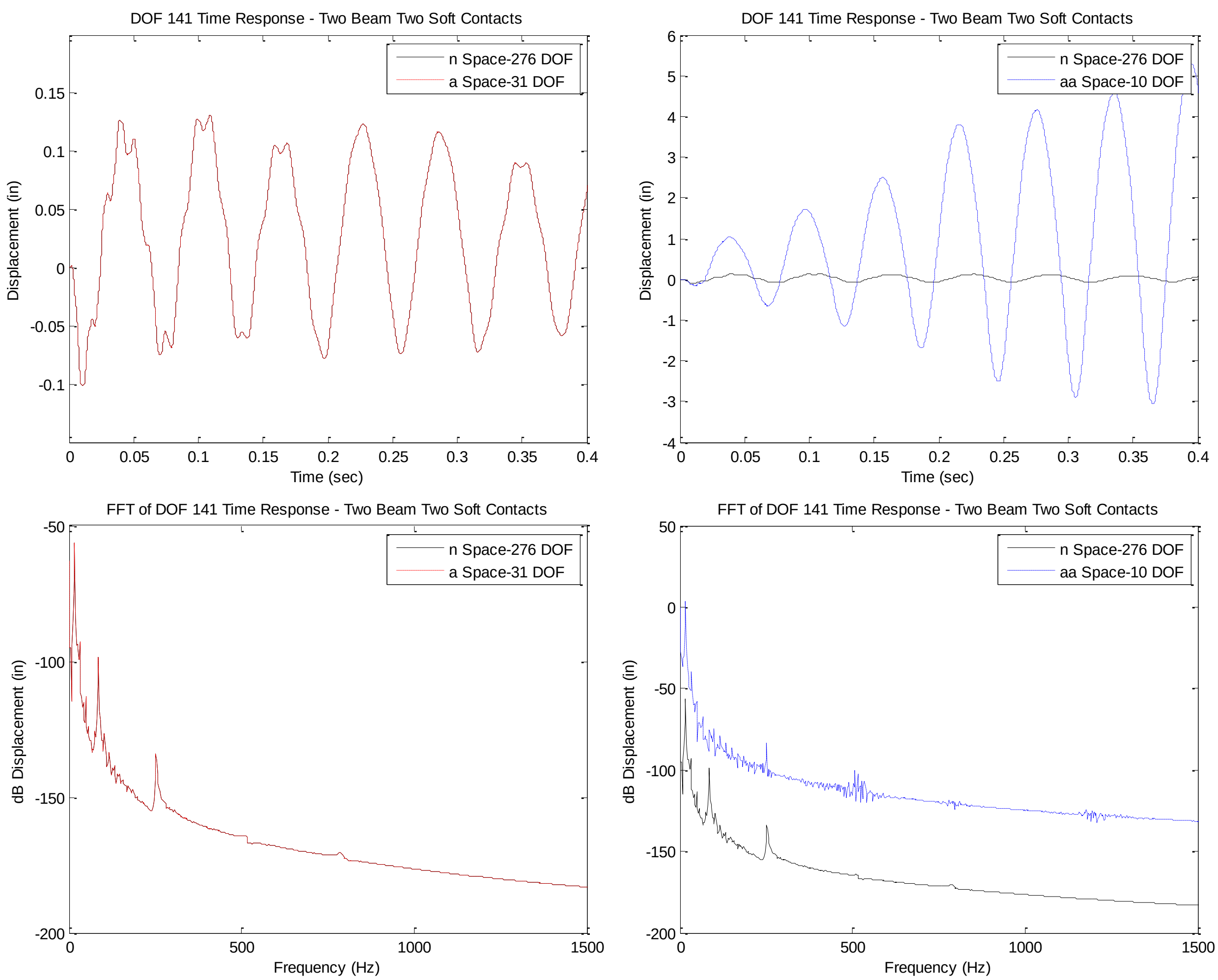
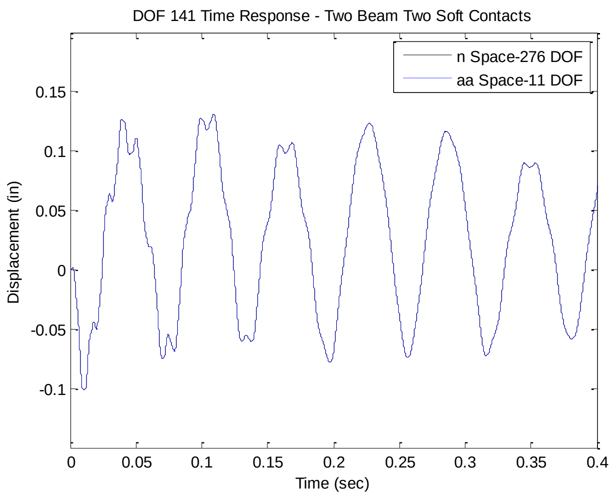


Figure 15. Comparison of ‘n’, ‘a’, and ‘aa’ Space Models for Two Beams with Multiple Soft Contacts for DOF 141 – Time Response (top) and FFT of Time Response (bottom)

The ‘a’ space model response in Figure 15 can be observed to have very good correlation with the full-space solution; however, the ‘aa’ space model response indicates that not including mode 6 of Beam A degrades the results obtained. To quantify the similarity of the reduced model results with the full-space solution, the MAC and TRAC were computed between the full model and the reduced model time responses, which were then averaged, as listed in Table 10.a. The solution time for each model is also listed to show the decrease in computation time when reduced models are used.

As seen in Case 2-B, the mode contribution matrix provides a good understanding of the effects mode truncation has on the solution obtained. Therefore, to improve the ‘aa’ space results, the addition of DOF 143 and mode 6 from Beam A was included in the reduced model. From the mode contribution matrices in Figure 14, mode 6 of Beam A is the primary contributing mode that is needed to obtain mode 10 of the modified system in all three configurations. The reduced system response that uses 11 DOF and 11 modes is plotted in the time and frequency domain with comparison to the full ‘n’ space solution in Figure 16.



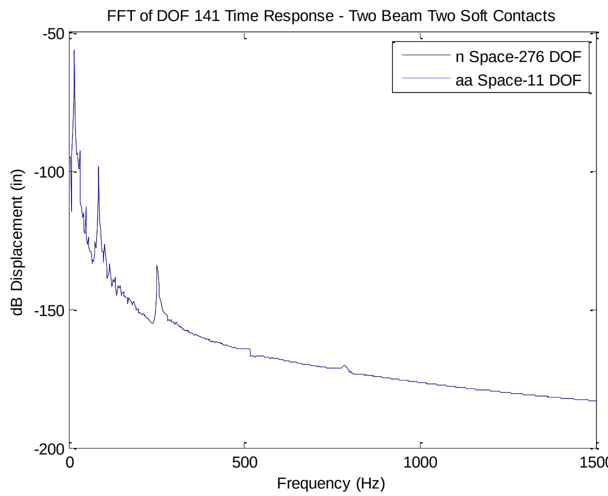


Figure 16. Comparison of ‘n’ and ‘aa’ Space Models for Two Beams with Multiple Soft Contacts for DOF 141 – Time Response (left) and FFT of Time Response (right)

The ‘aa’ space model time response in Figure 16 shows significant improvement in the correlation with the full-space model when an additional DOF and key contributing mode of Beam A is included in the reduced model. In Table 10.b, the MAC is improved from 0.678 to 0.999 and the TRAC is improved from 0.104 to 0.999 with essentially no change in the solution time when DOF 143 and mode 6 from Beam A are included in the reduced model. The improved MAC and TRAC results further confirm that the 11 DOF reduced model is sufficient for accurately computing the time response for this particular case. In addition, the solution time for the full-space model is approximately 90 seconds in contrast to the ‘aa’ space reduced model, which is only 1 second. ERMT was shown in this case to provide significant reduction in computation time for a nonlinear multiple component system with multiple soft contacts while maintaining a highly accurate time response solution.

Table 10. Average MAC and TRAC for Reduced Models and Solution Times for Two Beams with Multiple Soft Contacts

Table 10.a. Incorrect Number of Modes used in ‘aa’ Space Model

Model	# of DOF	Average MAC	Average TRAC	Solution Time (sec)
'n' Space	276	1	1	89.7
'a' Space	31	0.99999686	0.99999952	4.9
'aa' Space	10	0.67818808	0.10422630	1.1

Table 10.b. Suitable Number of Modes used in ‘aa’ Space Model

Model	# of DOF	Average MAC	Average TRAC	Solution Time (sec)
'n' Space	276	1	1	89.7
'a' Space	31	0.99999686	0.99999952	4.9
'aa' Space	11	0.99996327	0.99999980	1.1

The system response for this case is the same as in Case 2-B, which only has a single potential contact location, rather than two as in this case. In the cases to follow where hard contact stiffness is studied, the second contact location does come into contact, however for the soft contact stiffness, contact did not occur at all of the possible locations. This case was included to maintain continuity with the hard contact stiffness cases studied in following section. The soft contact models analyzed in Case 2 demonstrate how the mode participation matrix can be used to identify the component modes that are needed to generate accurate time response solutions using reduced models. Case 2 also shows that ERMT can provide highly accurate results with significantly reduced computation time for nonlinear systems with soft contacts.

CASE 3: HARD CONTACT

For the soft contact stiffness models discussed in Case 2, the contact stiffness was soft enough that the contact did not excite a frequency bandwidth beyond the input force spectrum. Thus, the number of modes needed in the time response was controlled by the forcing function, and the contributing unmodified component modes needed to obtain the modified system modes were based on the mode contribution matrix. As long as the frequency bandwidth excited by the contact stiffness is below the highest frequency excited by the input force pulse, the procedure for identifying the number of modes needed works well, as in Case 2. However, for a harder contact stiffness situation, there is a possibility that the contact can excite a frequency bandwidth beyond the input spectrum. Under this scenario, the number of unmodified component modes needed in the modified system response would not be a function of the input spectrum, but of the contact stiffness. In order to examine this possible scenario in detail, the same cases were studied as in Case 2, but with 1000 lb/in contact stiffness.

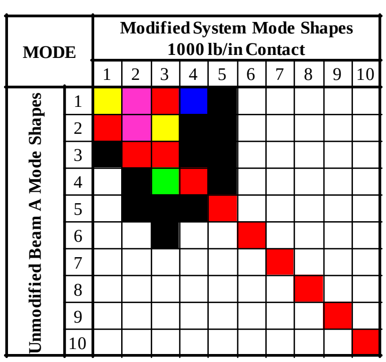
Case 3-A: Single Beam with Single Hard Contact

This case consists of the tip of Beam A coming into contact with a fixed object once the beam has displaced a known gap distance of 0.05 inches, as explained previously in Case 2-A. Table 11 lists the first 10 natural frequencies of the modified system with the hard contact stiffness applied, while Figure 17 shows the mode contribution matrix. The mode shapes for the modified system are provided in Appendix A.

Table 11 and Figure 17 indicate that the first 6 modes of the unmodified component are needed to obtain the first 5 modes of the modified system. This is expected, as the mode shapes with the harder spring attached look less like the original model and therefore requires more mode mixing to form the modified system modes. The frequencies of the unmodified component for mode 6 and higher are minimally affected for the modified system, which do not require additional unmodified component mode shapes to form the modified system modes (as indicated by the red main diagonal). Therefore, six active DOF and the first 6 modes of the unmodified component are needed to accurately obtain the modified system response. In order to confirm this, the system response was plotted in the time and frequency domain for ‘a’ space and ‘aa’ space with comparison to the full ‘n’ space solution in Figure 18. The response for the ‘aa’ space model is shown to illustrate when a key contributing mode (mode 6 of Beam A) is not included in the reduced model for the modified system.

Table 11. Natural Frequencies for Single Beam with Single Hard Contact

Mode #	Unmodified	Single Hard Contact
1	12.91	66.68
2	84.12	223.55
3	252.34	326.04
4	519.59	529.68
5	806.16	808.16
6	1256.55	1256.73
7	1682.96	1682.97
8	2201.36	2201.52
9	2755.52	2755.76
10	3510.01	3510.46



Bar Color	Min Value	Max Value
Black	0.005	0.1
Blue	0.1	0.2
Green	0.2	0.3
Cyan	0.3	0.5
Magenta	0.5	0.7
Yellow	0.7	0.9
Red	0.9	1.0

Figure 17. Mode Contribution Matrix for Single Beam with Single Hard Contact

The ‘a’ space model response in Figure 18 can be observed to have very good correlation with the full-space solution; however, in contrast to Case 2-A the ‘aa’ space model response indicates that not including mode 6 of Beam A degrades the results obtained. The first 6 modes of the unmodified component are needed to obtain the first 5 modes of the modified system as indicated in Figure 17, which were not able to be adequately represented using only modes 1 to 5 for the ‘aa’ space model. Even though the input force spectrum was limited primarily to the first 5 modes of the unmodified component, the first 6 modes are needed for the modified system. The hard contact does not appear to contribute significant energy beyond 1000 Hz when contact occurs, which shows that the number of modes needed to accurately compute the time response for the reduced models is governed by the forcing function and not the contact stiffness for this case. To quantify the similarity of the reduced model results with the full-space solution, the MAC and TRAC were computed between the full model and the reduced model time responses, which were then averaged, as listed in Table 12. The solution time for each model is also listed to show the decrease in computation time when reduced models are used.

Table 12. Average MAC and TRAC for Reduced Models and Solution Times for Single Beam with Single Hard Contact

Model	**# of DOF**	**Average MAC**	**Average TRAC**	**Solution Time (sec)**
'n' Space	146	1	1	35.1
'a' Space	17	0.99999686	0.99999980	1.9
'aa' Space	5	0.99854646	0.98781940	0.7

The system response for the hard contact is observably more nonlinear compared to the soft contact case, where the modes of the system are more difficult to identify in the FFT of the response. The FFT of the time response for both the soft and hard contact is shown in Figure 19 for comparison.

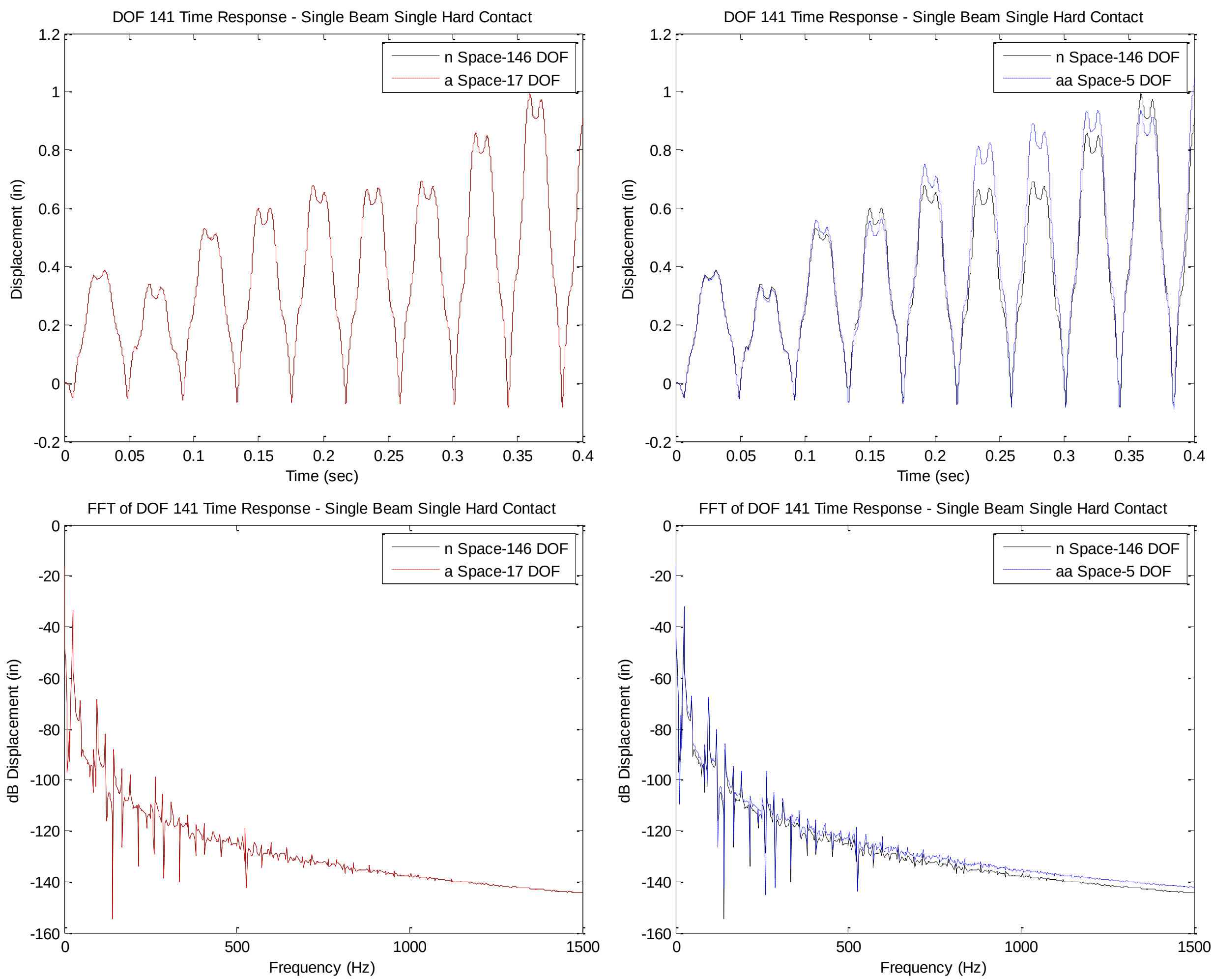


Figure 18. Comparison of ‘n’, ‘a’, and ‘aa’ Space Models for Single Beam with Single Hard Contact for DOF 141 – Time Response (top) and FFT of Time Response (bottom)

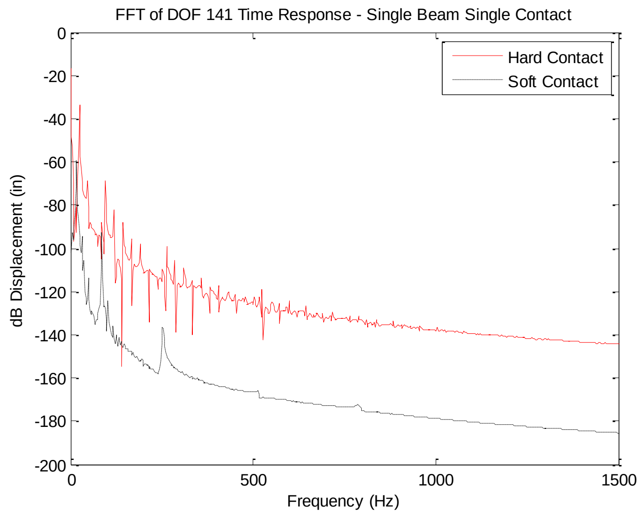


Figure 19. Comparison of the FFT for Single Beam with Single Soft and Hard Contact

The favorable MAC and TRAC results in Table 12 further confirm that the 17 DOF 'a' space model is sufficient for accurately computing the time response for this particular case. In addition, the solution time for the full-space model is over 30 seconds in contrast to the reduced 'a' space model, which is less than 2 seconds. This case showed that an accurate time response solution can be obtained using ERMT with drastically reduced models for a single component with a single hard contact.

Case 3-B: Two Beams with Single Hard Contact

This case consists of the tip of Beam A coming into contact with Beam B once the specified gap distance of 0.05 inches between Beams A and B is closed, as discussed previously in Case 2-B. The same hard contact stiffness of 1000 lb/in was used to represent the contact of the beams as explained in Case 3-A. Table 13 lists the natural frequencies for the modified system as well as for the unmodified components. Figure 20 shows the mode contribution matrix for identifying the unmodified component modes that contribute in the modified system modes. The mode shapes for the modified system are provided in Appendix A.

Based on the input force pulse, which excites up to 1000 Hz, the first 10 modes of the modified system are expected to be excited, which can be seen in Table 13. Examining Figure 20 shows that for the first 10 modes of the modified system, the first 6 modes of Beam A and the first 7 modes of Beam B are needed to obtain the first 10 modes of the modified system. For a soft contact, the input force pulse primarily dominates the frequency spectrum excited, however this may not always be true for a hard contact scenario. The modes required to accurately predict the system response are provided in the mode contribution matrix; however, there may be additional higher order modes excited by the hard contact that cannot be identified based solely on the input force spectrum. For hard contact situations, care must be taken when generating reduced models such that higher order modes that participate in the system response are included. To show the effects of severe mode truncation, the system response was plotted in the time and frequency domain for 'a' space and 'aa' space with comparison to the full 'n' space solution in Figure 21, where the 'aa' space model is missing several key contributing modes in the modified system response.

Note that in contrast to the soft contact, the hard contact excited modes well above the 1000 Hz bandwidth of the input force pulse (up to approximately 3000 Hz). Accordingly, the number of unmodified component modes identified using the mode contribution matrix to accurately obtain the modified system time response is no longer governed by the input force spectrum, but by the frequency bandwidth excited by the contact stiffness impact. To quantify the similarity of the reduced model results with the full-space solution, the MAC and TRAC were computed between the full model and the reduced model time responses, which were then averaged, as listed in Table 14.a. The solution time for each model is also listed to show the decrease in computation time when reduced models are used.

Table 13. Natural Frequencies for Two Beams with Single Hard Contact for Configuration 1

Mode #	2 Beams, Single Hard Contact	Unmodified Components	
		Beam A	Beam B
1	21.25	12.91	22.62
2	67.74	84.12	141.56
3	127.04	252.34	396.60
4	236.02	519.59	776.92
5	338.31	806.16	1284.71
6	424.26	1256.55	1918.28
7	532.04	1682.96	2678.33
8	786.59	2201.36	3563.89
9	809.25	2755.52	4572.70
10	1256.69	3510.01	5707.04
11	1294.66	-	-
12	1682.97	-	-
13	1918.72	-	-
14	2201.53	-	-
15	2683.90	-	-
16	2755.78	-	-
17	3510.46	-	-
18	3564.14	-	-
19	3948.79	-	-
20	4575.19	-	-

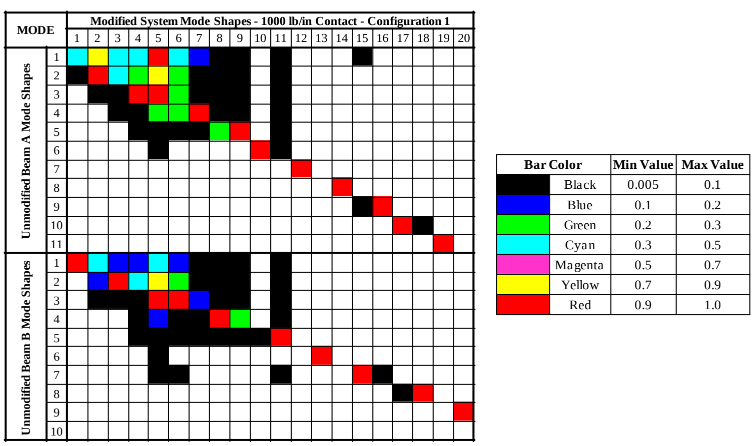


Figure 20. Mode Contribution Matrix for Two Beams with Single Hard Contact for Configuration 1

The transient portion of the ‘a’ space model response in Figure 21 can be observed to have very good correlation with the full-space solution. However, in contrast to Case 3-A the hard contact appears to contribute noticeable energy beyond the 1000 Hz input force spectrum, which was not anticipated when identifying the unmodified component modes that contribute in the modified system modes. In Figure 21, the ‘aa’ space model produced less desirable results due to severe mode truncation. The ‘a’ space model provides very good results for the initial transient portion of the system response, however degraded correlation can be observed after the initial transient. This is due to higher order modes that are missing in the reduced model not being excited until a high force impact occurs at the hard stiffness contact location. This behavior will be different depending on the location and level of the input excitation, due to the nonlinear characteristics of the system. The number of modes included in the ‘a’ space model is sufficient when only the forces pulse is considered. However, due to the hard contact, more modes are needed to accurately compute the system response beyond the initial transient portion. To observe the high frequency content excited by the hard contact, the FFT of the system response is shown in Figure 22 over a 5000 Hz band with comparison to the soft contact case.

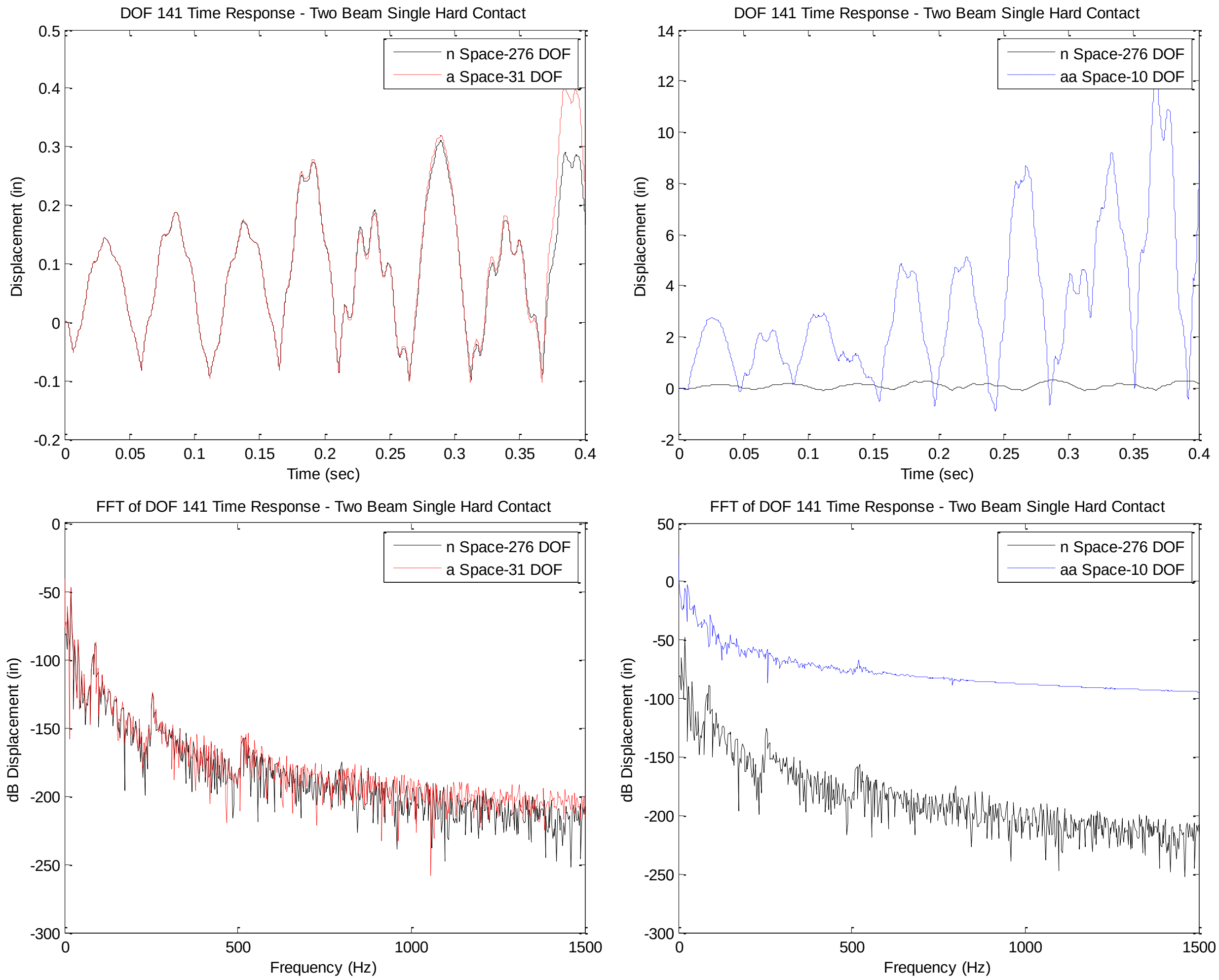


Figure 21. Comparison of ‘n’, ‘a’, and ‘aa’ Space Models for Two Beams with Single Hard Contact for DOF 141 – Time Response (top) and FFT of Time Response (bottom)

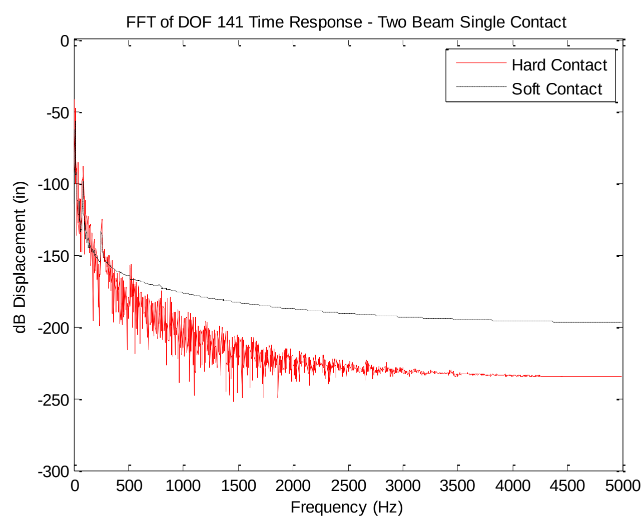


Figure 22. Comparison of the FFT for Two Beams with Single Soft and Hard Contact

Often times for nonlinear systems, the initial transient portion of the time response is of interest, not the entire solution. When the ‘a’ space solution is evaluated for only the transient portion of the response (the first 0.2 seconds), the MAC improves from 0.997 to 0.999 and the TRAC improves from 0.976 to 0.999, as shown in Table 14.b.

Table 14. Average MAC and TRAC for Reduced Models and Solution Times for Two Beams with Single Hard Contact

Table 14.a. MAC and TRAC for Full Time Response (0 to 0.4 seconds)

Model	# of DOF	Average MAC	Average TRAC	Solution Time (sec)
'n' Space	276	1	1	87.9
'a' Space	31	0.99684042	0.97628620	4.9
'aa' Space	10	0.40375102	0.12895379	1.1

Table 14.b. MAC and TRAC for Transient Portion of Time Response (0 to 0.2 seconds)

Model	# of DOF	Average MAC	Average TRAC	Solution Time (sec)
'n' Space	276	1	1	87.9
'a' Space	31	0.99995692	0.99958379	4.9
'aa' Space	10	0.45859557	0.11002554	1.1

The ‘a’ space model for the two beam system with a single hard contact produced very good results for the initial transient portion of the response, which is generally the portion of the system response of interest for nonlinear systems. However, less desirable results were obtained using the ‘aa’ space model, due to not enough modes being included to obtain the modified system response. As seen in Figure 20, the first 10 modes of the modified system requires the first 6 modes of Beam A and the first 7 modes of Beam B, which were not able to be adequately represented using the ‘aa’ space model. The number of modes included in the ‘a’ space model is sufficient when only the force pulse is considered. However, due to the hard contact stiffness, more modes are needed to accurately compute the system response beyond the initial transient portion. This shows that throughout the nonlinear system time response, hard contacts can potentially excite higher frequencies than the input excitation. Therefore, a reduced model with greater than 31 DOF and 31 modes is needed to obtain an accurate time response solution beyond the initial transient for this case. The favorable MAC and TRAC results in Table 14.b show that the 31 DOF ‘a’ space model is sufficient for accurately computing the initial transient portion of the time response for this particular case. In addition, the solution time for the full-space model is almost 90 seconds in contrast to the reduced ‘a’ space model, which is approximately 5 seconds.

ERMT was shown in this case to provide significant reduction in computation time for a nonlinear multiple component system with a single hard contact while maintaining a highly accurate initial transient time response solution. However, care must be taken when generating reduced models for nonlinear systems with hard contact stiffness, due to the higher frequencies that can potentially be excited in the system.

Case 3-C: Two Beams with Multiple Hard Contacts

This case consists of Beam A coming into contact with Beam B in three different configurations with hard contact stiffness at two possible contact locations with a specified gap distance of 0.05 inches, as discussed previously in Case 2-C. Table 15 lists the natural frequencies for the modified system as well as for the unmodified components. Figure 23 shows the mode contribution matrices used for identifying the unmodified component modes that participate in the modified system modes for the three possible system configurations. The mode shapes for the modified system configurations are provided in Appendix A.

Table 15 and Figure 23 show that for the input force pulse applied, ten system modes are excited and modes 1 to 8 of Beam A and modes 1 to 7 of Beam B are needed. However, Case 3-B showed that additional modes beyond the input excitation spectrum are activated by the hard contact stiffness. Therefore, larger reduced models are needed for the inclusion of additional modes that participate in the modified system response to obtain an accurate time response solution. The system response was plotted in the time and frequency domain for ‘a’ space and ‘aa’ space with comparison to the full ‘n’ space solution in Figure 24, where the ‘aa’ space model is missing several key contributing modes in the modified system response.

The transient portion of the ‘a’ space model response in Figure 24 can be observed to have very good correlation with the full-space solution. As seen in Case 3-B, the hard contact contributes significant energy beyond the 1000 Hz input force spectrum. In Figure 24, the ‘aa’ space model produced less desirable results, due to severe mode truncation. The ‘a’ space model provides very good results for the initial transient portion of the response, however degraded correlation is observed after the initial transient. The number of modes included in the ‘a’ space model is sufficient when only the forces pulse is considered. However, due to the hard contact, more modes are needed to accurately compute the system response beyond the initial transient portion, which was discussed in Case 3-B. To observe the high frequency content excited by the hard contact, the FFT of the system response is shown in Figure 25 over a 5000 Hz band with comparison to the soft contact case.

Table 15. Natural Frequencies for Two Beams with Multiple Hard Contact Configurations

Mode #	Modified System - Hard Contacts			Unmodified Components	
	Config. 1	Config. 2	Config. 3	Beam A	Beam B
1	21.25	15.50	36.22	12.91	22.62
2	67.74	69.52	113.69	84.12	141.56
3	127.04	119.79	228.57	252.34	396.60
4	236.02	234.14	312.68	519.59	776.92
5	338.31	325.85	339.40	806.16	1284.71
6	424.26	467.09	504.56	1256.55	1918.28
7	532.04	527.49	532.41	1682.96	2678.33
8	786.59	786.04	790.33	2201.36	3563.89
9	809.25	828.78	833.74	2755.52	4572.70
10	1256.69	1257.17	1257.22	3510.01	5707.04
11	1294.66	1290.03	1300.10	-	-
12	1682.97	1686.00	1686.01	-	-
13	1918.72	1920.05	1920.48	-	-
14	2201.53	2202.67	2202.84	-	-
15	2683.90	2678.85	2684.38	-	-
16	2755.78	2756.01	2756.28	-	-
17	3510.46	3511.34	3511.79	-	-
18	3564.14	3563.98	3564.23	-	-
19	3948.79	3948.54	3948.79	-	-
20	4575.19	4572.70	4575.19	-	-

Bar Color		Min Value	Max Value
	Black	0.005	0.1
	Blue	0.1	0.2
	Green	0.2	0.3
	Cyan	0.3	0.5
	Magenta	0.5	0.7
	Yellow	0.7	0.9
	Red	0.9	1.0

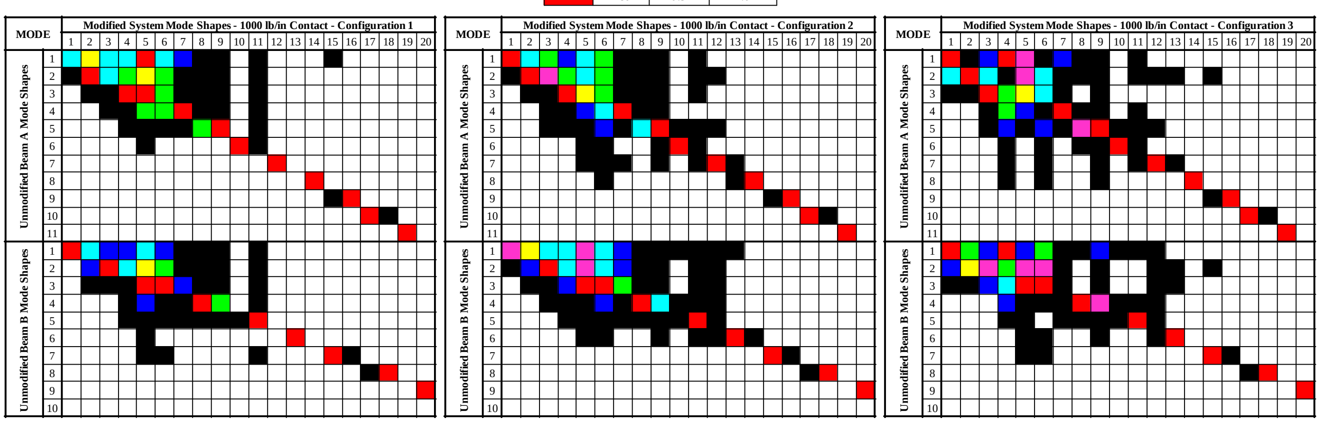


Figure 23. Mode Contribution Matrix for Two Beams with Hard Contacts for Configurations 1, 2, and 3

As observed in Case 3-B, the frequency excited by the hard contact stiffness was well above the bandwidth of the input force pulse. The mode contribution matrix provided a good indication of the unmodified component modes that participate in the modified system modes for the soft contact stiffness scenario. However, higher order modes beyond the input force spectrum are excited by the hard contact stiffness and the modes identified in the mode contribution matrix are governed by the frequency bandwidth excited by the hard contact impact. To quantify the similarity of the reduced model results with the full-space solution, the MAC and TRAC were computed between the full model and the reduced model time responses, which were then averaged, as listed in Table 16.a. The solution time for each model is also listed to show the decrease in computation time when reduced models are used.

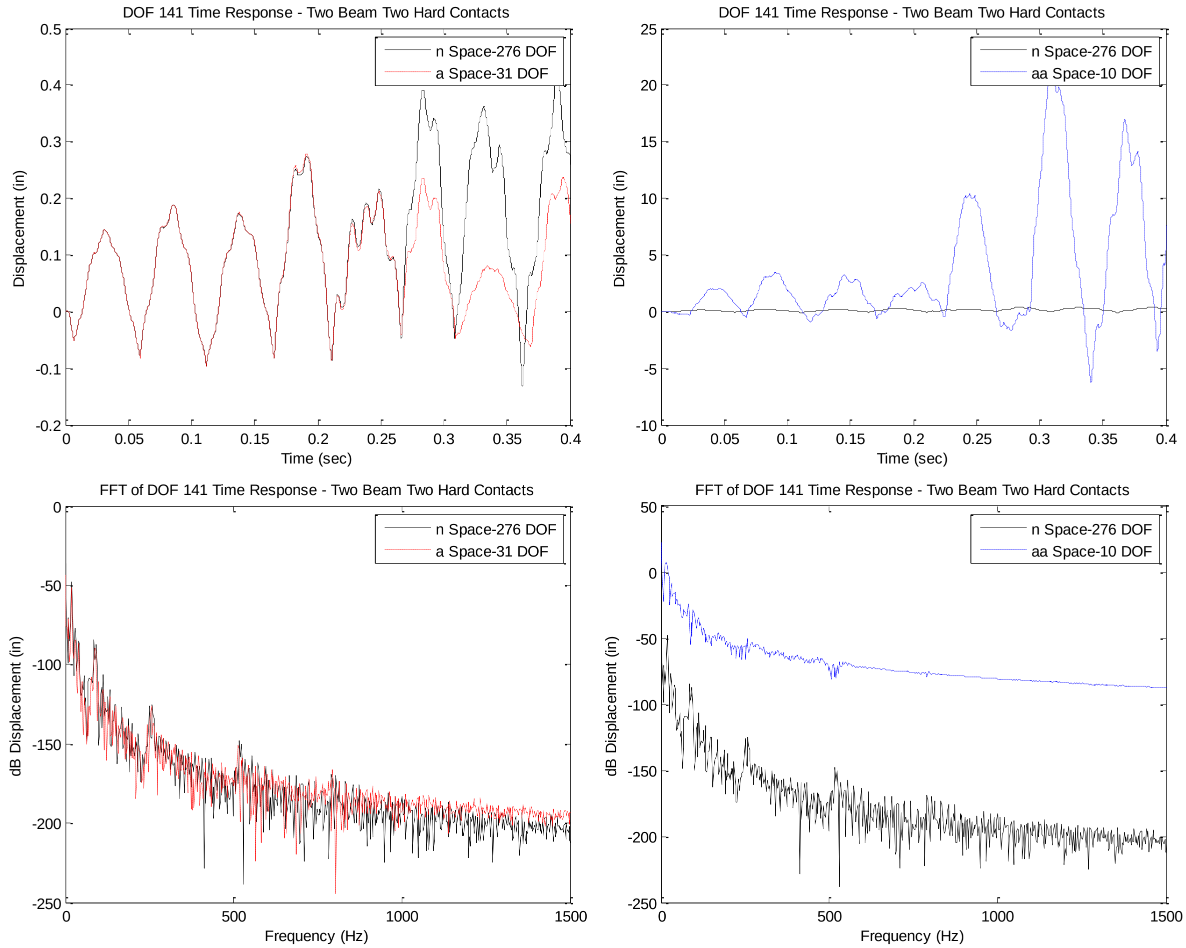


Figure 24. Comparison of ‘n’, ‘a’, and ‘aa’ Space Models for Two Beams with Multiple Hard Contacts for DOF 141 – Time Response (top) and FFT of Time Response (bottom)

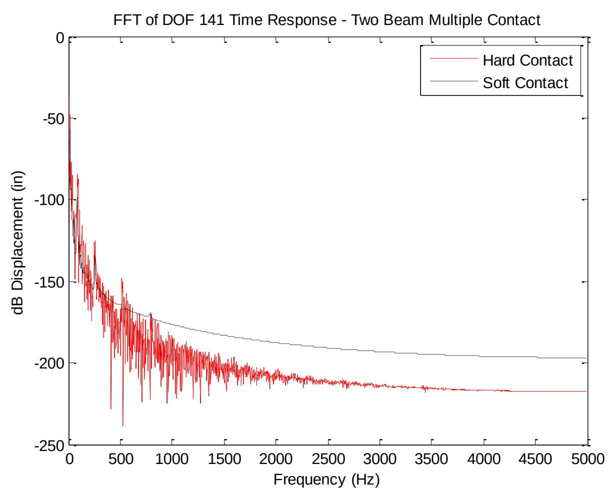


Figure 25. Comparison of the FFT for Two Beams with Multiple Soft and Hard Contacts

Often times for nonlinear systems, the initial transient portion of the time response is of interest, not the entire solution. When the ‘a’ space solution is evaluated for only the transient portion of the response (the first 0.2 seconds), the MAC improves from 0.890 to 0.999 and the TRAC improves from 0.574 to 0.999, as shown in Table 16.b.

Table 16. Average MAC and TRAC for Reduced Models and Solution Times for Two Beams with Hard Contacts

Table 16.a. MAC and TRAC for Full Time Response (0 to 0.4 seconds)

Model	# of DOF	Average MAC	Average TRAC	Solution Time (sec)
'n' Space	276	1	1	88.5
'a' Space	31	0.88955510	0.57445998	4.9
'aa' Space	10	0.45974452	0.07460348	1.1

Table 16.b. MAC and TRAC for Transient Portion of Time Response (0 to 0.2 seconds)

Model	# of DOF	Average MAC	Average TRAC	Solution Time (sec)
'n' Space	276	1	1	88.5
'a' Space	31	0.99995692	0.99958379	4.9
'aa' Space	10	0.53251049	0.24627142	1.1

The ‘a’ space model for the two beam system with multiple hard contacts produced very good results for the initial transient portion of the response, which is generally the portion of the system response of interest for nonlinear systems. However, less desirable results were obtained using the ‘aa’ space model, due to not enough modes being included to obtain the modified system response. As seen in Figure 23, the first 10 modes of the modified system requires the first 8 modes of Beam A and the first 7 modes of Beam B, which were not able to be adequately represented using the ‘aa’ space model. The number of modes included in the ‘a’ space model is sufficient when only the forces pulse is considered. However, due to the hard contact, more modes are needed to accurately compute the system response beyond the initial transient portion, as discussed in Case 3-B. Therefore, a reduced model with greater than 31 DOF and 31 modes is needed to obtain an accurate time response solution beyond the initial transient for this case. The favorable MAC and TRAC results in Table 16.b show that the 31 DOF ‘a’ space model is sufficient for accurately computing the initial transient portion of the time response for this particular case. In addition, the solution time for the full-space model is almost 90 seconds in contrast to the reduced ‘a’ space model, which is approximately 5 seconds.

ERMT was shown in this case to provide significant reduction in computation time for a nonlinear multiple component system with multiple hard contacts where significant component interaction occurs. In addition, a highly accurate initial transient time response solution was obtained using drastically reduced models.

Comparing the soft and hard contact cases shows that the number of modes needed in the reduced models can be predicted accurately as long as the input spectrum bandwidth defines the frequency range that is excited, as seen in the soft contact cases. However, once this condition is no longer true, determining the number of modes needed requires additional knowledge of the frequency bandwidth excited by the contacts ahead of time. Care must be taken when generating reduced models for nonlinear systems with hard contact stiffness, due to the higher frequencies that can potentially be excited in the system. For both soft and hard contact cases, ERMT was shown to provide a substantial reduction in computation time while maintaining highly accurate time response solutions, which demonstrates the usefulness of this technique.

TIME STEP SELECTION EFFECT ON SOLUTION

For the time step of 0.0001 seconds used in the cases studied, Raleigh Criteria and Shannon’s Sampling Theorem state that the maximum frequency range that can be observed is 5 kHz, which is well above the 1000 Hz frequency bandwidth excited by the input force pulse. Although this time resolution may be fine enough to accurately capture the time response for the linear system, this resolution may not be adequate when the response becomes nonlinear. In addition, the contact stiffness of the nonlinear system also affects the frequency bandwidth excited, as discussed previously in Case 3. Since the time step chosen affects the instance in time that contact occurs, the system may respond differently depending on whether the time duration of the impact is short (soft contact) or long (hard contact). For a soft contact, the system remains in contact for a longer time duration, and the time response should be relatively consistent regardless of the time step chosen. For the hard contact, however, the system may experience high frequency impact chatter, which may not be captured when a coarse time step is used. The system response was computed for the single beam with soft and hard contact using a time step of 0.00005seconds and the previously used time step of 0.0001seconds, which are compared in Figure 26 for the first 0.1 seconds of the time response solutions.

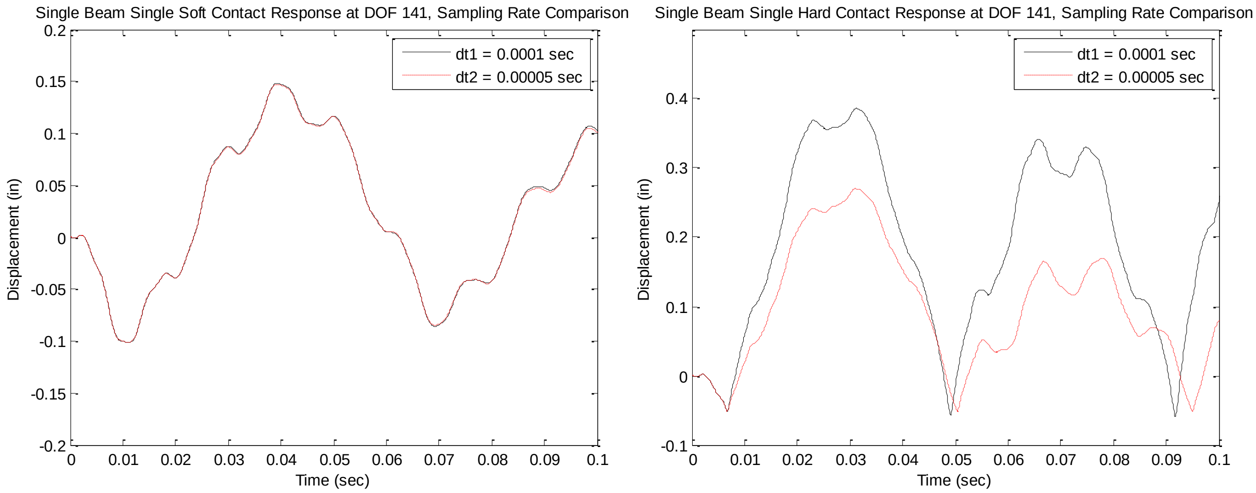


Figure 26. Comparison of System Response Using Different Time Steps for Soft Contact (left) and Hard Contact (right)

The left plot in Figure 26 shows that for the soft contact, the smaller time step minimally effected the time response solution. The results compare very well for the soft contact, due to the system remaining in contact with the soft spring for a long time duration. For the hard contact in the right plot of Figure 26, the system comes into contact and then immediately bounces off. Since the time step directly affects the acceleration, there is a divergence in the solution immediately after the first contact occurs. Although this study shows that the time step selected was not sufficiently small enough for the hard contact cases, the time step of 0.0001 seconds was used for all of the previous cases in order to demonstrate the main principles and computational advantages of ERMT. Further study is needed to determine the time step required when the contact stiffness dictates the spectral energy content that participates in the nonlinear system response.

COMPARISON TO LARGE FINITE ELEMENT MODELS

ERMT was shown to provide substantial reduction in computation time for the simple beam models studied in this paper. The computational advantages of the technique would be further emphasized when used for a detailed finite element model, which are typically found in industry. Figure 27 shows a very detailed model of a helicopter/missile/wing configuration.

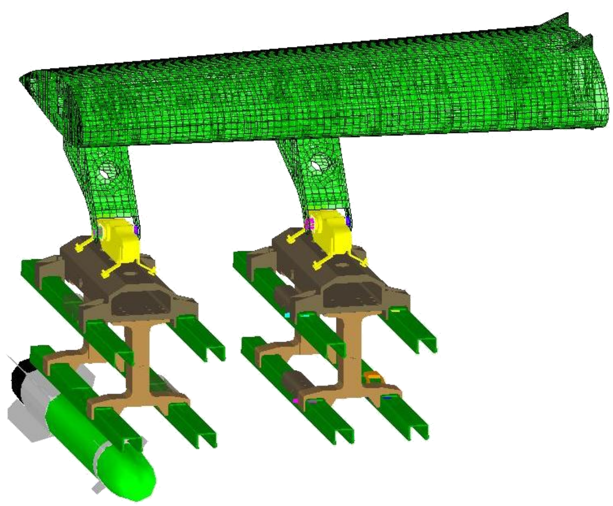


Figure 27. Detailed FEM of Helicopter/Missile/Wing Configuration

These detailed models are used in dynamic simulations to compute the transient dynamics that occur during a missile firing under a variety of conditions. Due to the system being very complex with many detailed components, the time response solutions require significant computation (on the order of hours to days). However, utilizing reduced order models of the various components allows for substantial reduction in computation time. In addition, multiple system configurations can be efficiently studied in detail to provide the analyst with the wealth of information needed to improve the system design.

CONCLUSION

A computationally efficient technique – Equivalent Reduced Model Technique (ERMT) – that utilizes reduced linear component models assembled with discrete nonlinear connection elements to perform nonlinear forced response analysis is presented. Four cases of increasing complexity are studied. The mode contribution matrix is used to identify the modes required to form accurate reduced order models. The technique was shown to yield accurate results when compared to the reference solution and provided significant improvement in computational efficiency for the analytical nonlinear forced response cases studied.

FUTURE WORK

Future work will be performed using ERMT to demonstrate the usefulness of the technique with experimental data validation. For comparison to experimental data, several additional considerations are needed to yield accurate results. First, the underlying linear system will need to be a highly accurate model in order to predict the system time responses correctly. Care will need to be taken in modeling and updating the component models to test data to ensure that both the model and measurements are reflective of the physical structure. Second, the damping will have to be measured experimentally for the linear system for as many modes as possible in order to have high correlation in the time domain. In addition, efforts will be needed to determine the correct damping for the physical system models, as the damping may change once the component(s) are in the state of contact. Third, the stiffness of the contact will have to be determined to accurately model the system and compute the time response. In addition, the impact force from the beam contact may need to be accounted for in the analytical models, where this additional force input to the structure may potentially affect the time response. Finally, variation in the time step used was found to affect the results obtained and therefore, additional work is needed to remedy this item of concern.

NOMENCLATURE

Symbols:

$\{X_n\}$	Full Set Displacement Vector
$\{X_a\}$	Reduced Set Displacement Vector
$\{X_d\}$	Deleted Set Displacement Vector
$[M_a]$	Reduced Mass Matrix
$[M_n]$	Expanded Mass Matrix
$[K_a]$	Reduced Stiffness Matrix
$[K_n]$	Expanded Stiffness Matrix
$[U_a]$	Reduced Set Shape Matrix
$[U_n]$	Full Set Shape Matrix
$[U_a]^g$	Generalized Inverse
$[T]$	Transformation Matrix
$[T_U]$	SEREP Transformation Matrix
$\{p\}$	Modal Displacement Vector
$[M]$	Physical Mass Matrix
$[C]$	Physical Damping Matrix
$[K]$	Physical Stiffness Matrix
$\{F\}$	Physical Force Vector
$\{\ddot{x}\}$	Physical Acceleration Vector
$\{\dot{x}\}$	Physical Velocity Vector
$\{x\}$	Physical Displacement Vector
α	Parameter for Newmark Integration
β	Parameter for Newmark Integration
Δt	Time Step
$[U_{12}]$	Mode Contribution Matrix

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APPENDIX A: COMPONENT AND SYSTEM MODE SHAPES

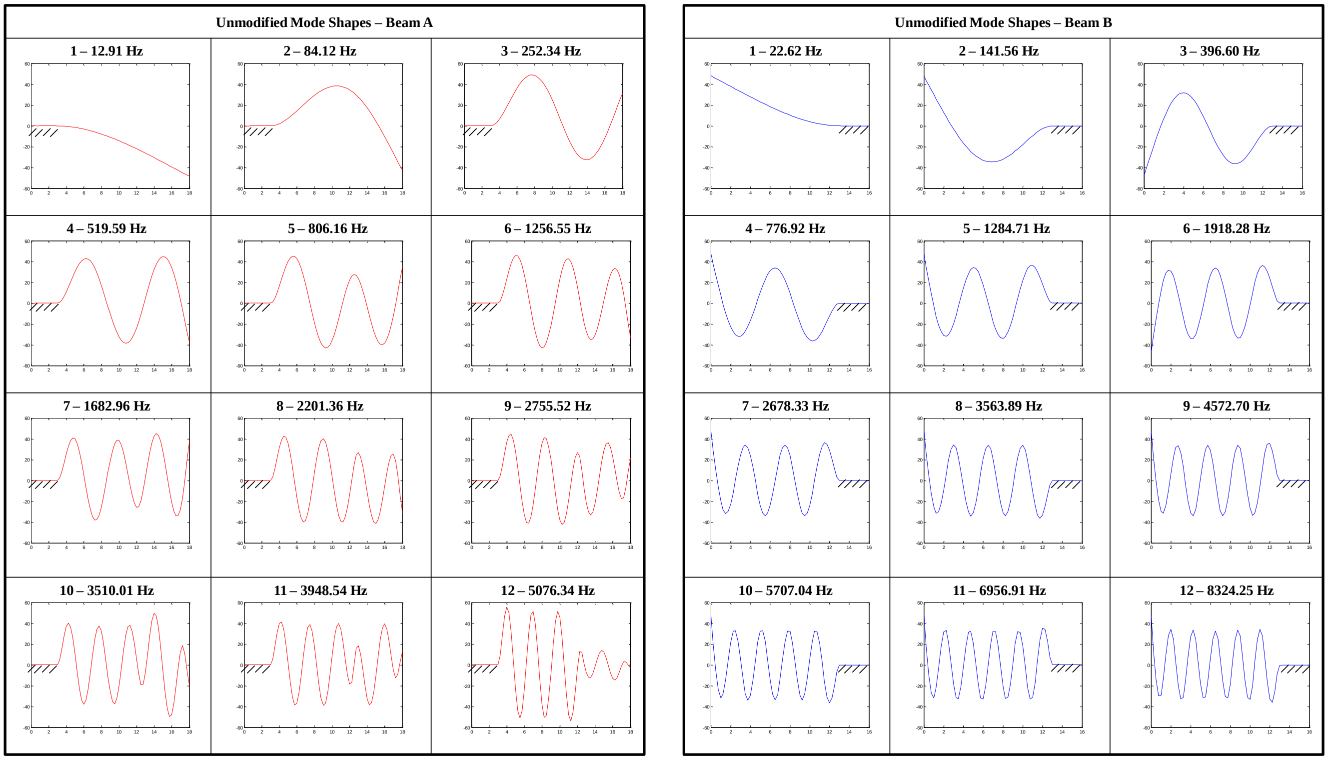


Figure A.1. Mode Shapes for Unmodified Components A and B

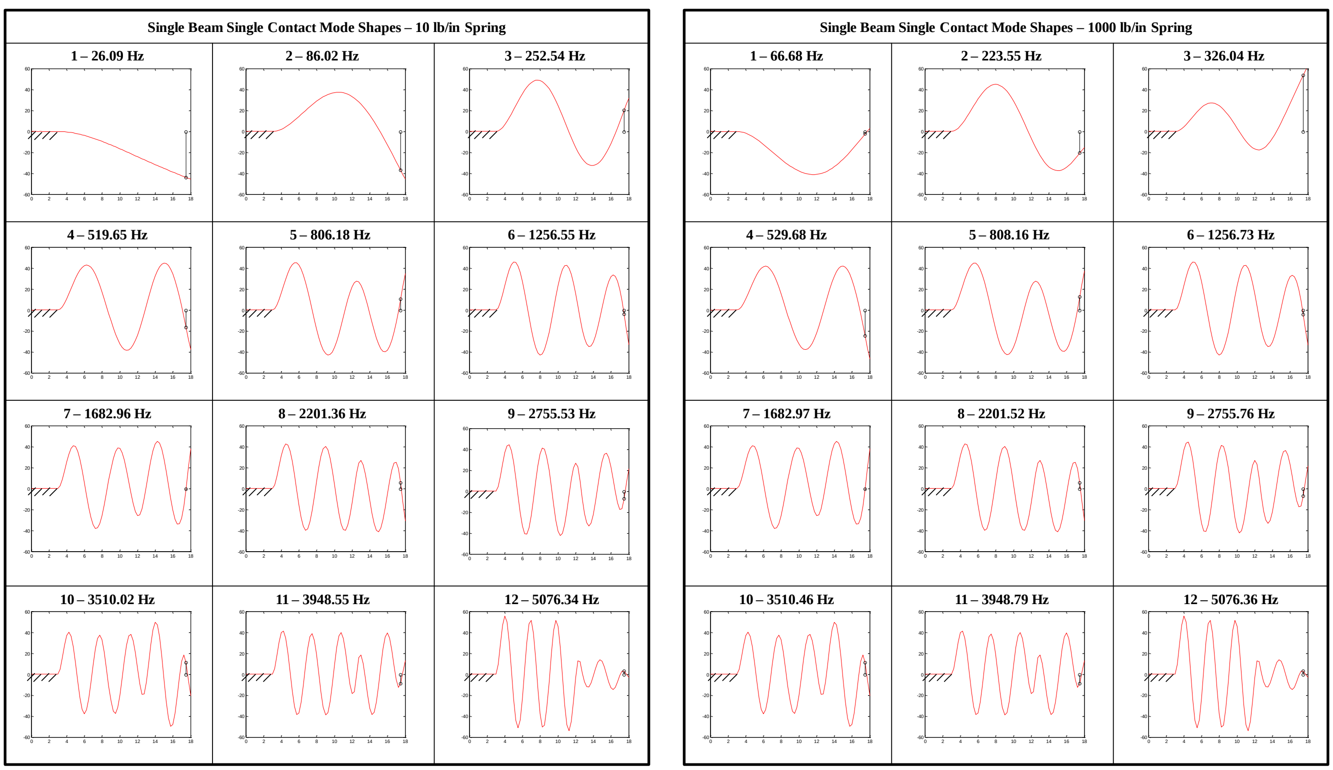
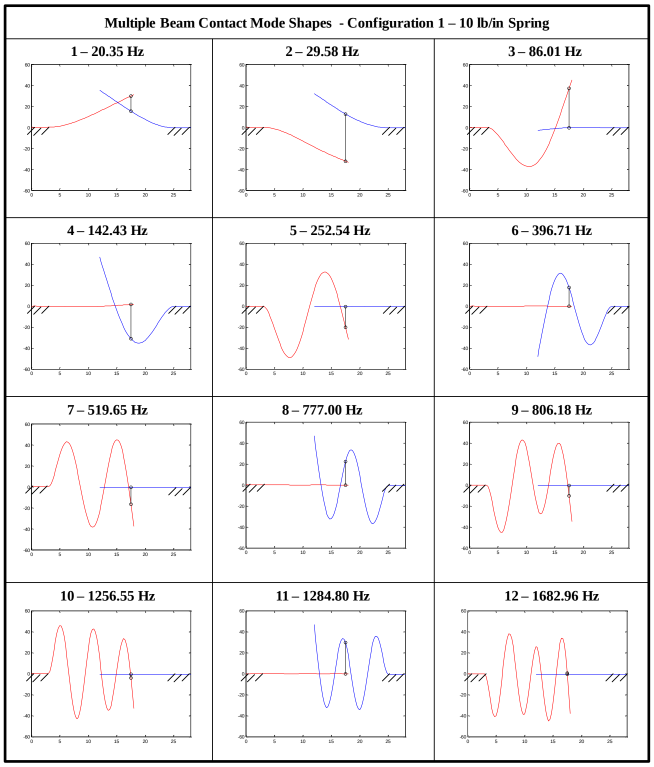
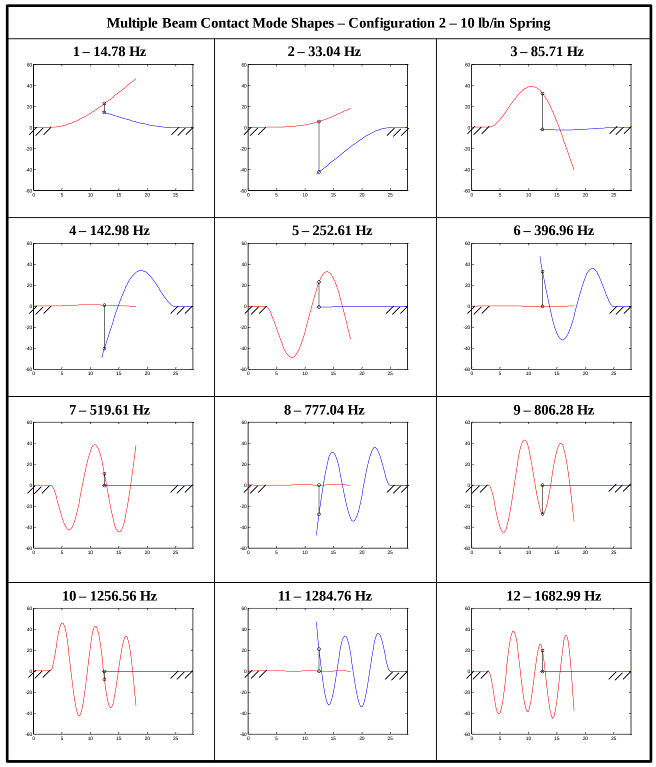


Figure A.2. Mode Shapes for Single Beam with Single Soft (left) and Single Hard (right) Contact





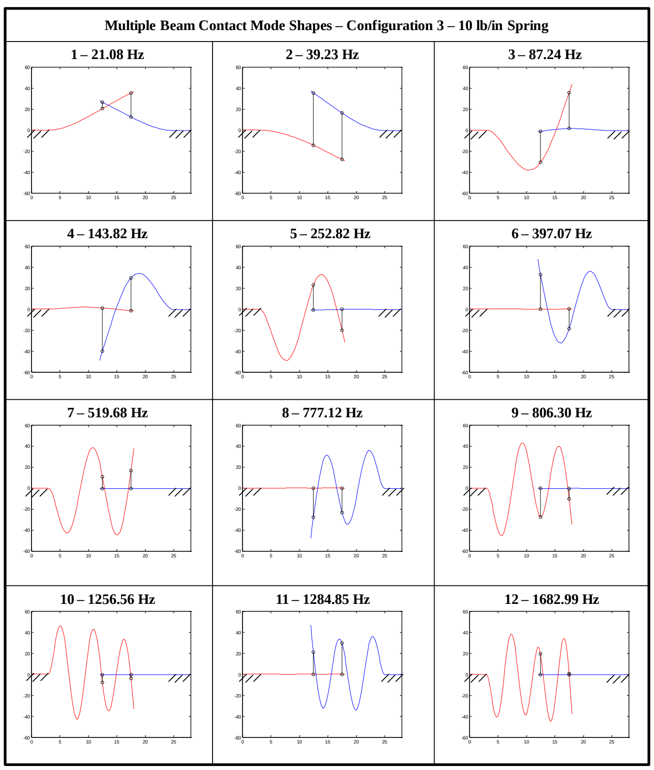


Figure A.3. Mode Shapes for Two Beam System for Multiple Soft Contact Configurations

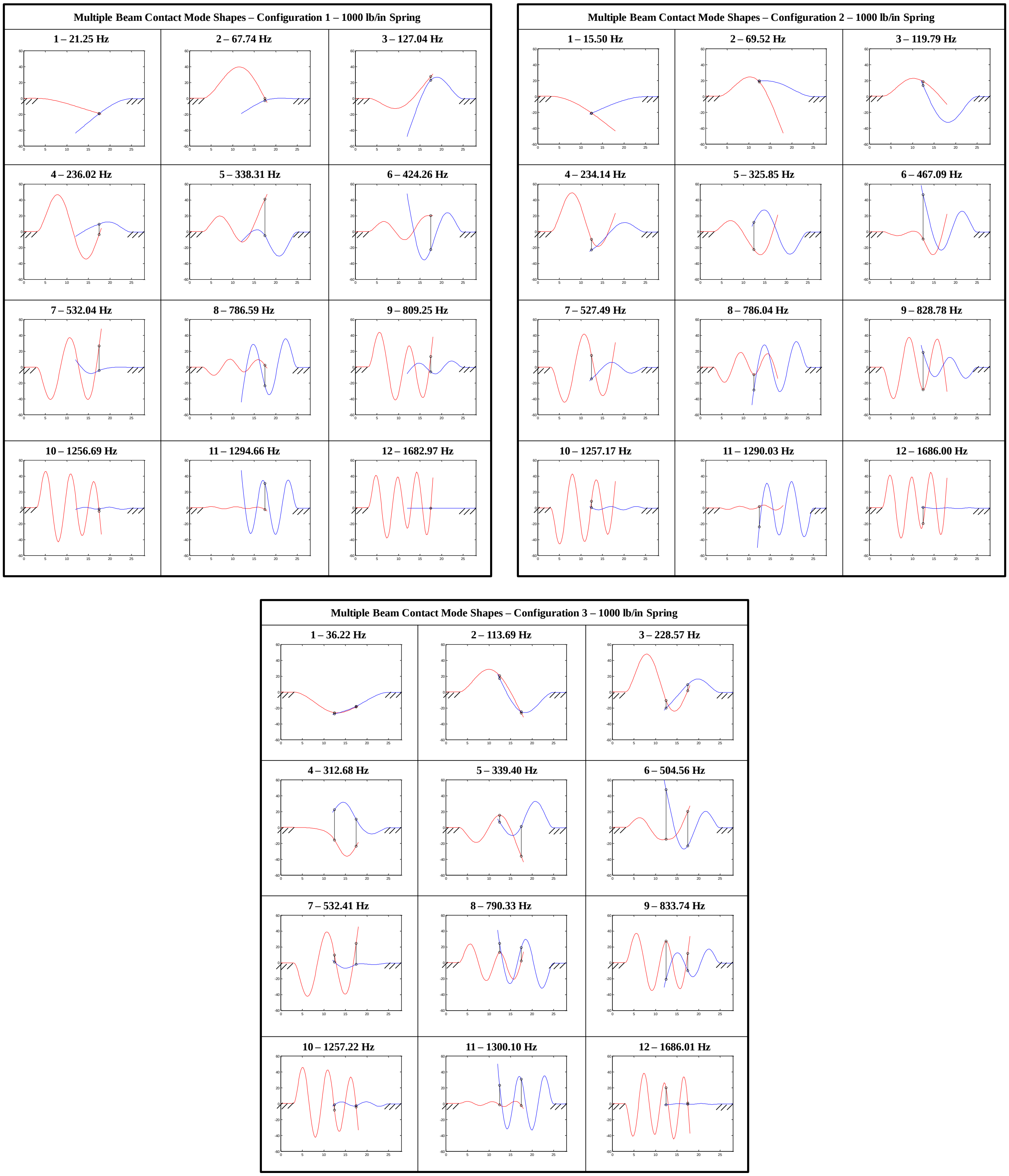


Figure A.4. Mode Shapes for Two Beam System for Multiple Hard Contact Configurations