VIBRATION OF POINT MASSES

Maarten van Walstijn

Aims

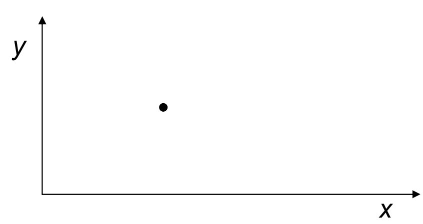
- To explore the motion of a point mass with forces exerting on it.
- To construct and formulate, from first principles the mathematical formulas that govern one- and two-mass oscillators.
- To analyse and interpret the oscillatory solutions of these systems.
- To derive useful concepts for analysis and representation of these systems.

Content

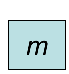
- Point Masses
- Forces and Acceleration
- One-mass Oscillator
- Resonance
- Damped Oscillator
- Driven Oscillator
- Time- and Frequency Response
- Two-Mass Systems
- Normal Modes
- Multi-Mass & Continuous Systems

Point Mass

A geometric (0-dimensional) point that may be assigned a finite mass (m). Since a point has zero volume, the density of a point mass having a finite mass is infinite, so point masses do not exist in reality!

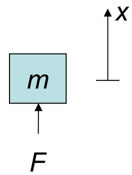


However, it is often a useful simplification in real problems. We represent the point mass here as follows:



Force and Acceleration

Force	F
Mass	m
Position	x
Acceleration	$\ddot{x} = \dfrac{\partial^2 x}{\partial t^2}$

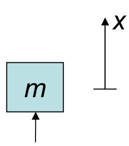


Newton's Second Law:

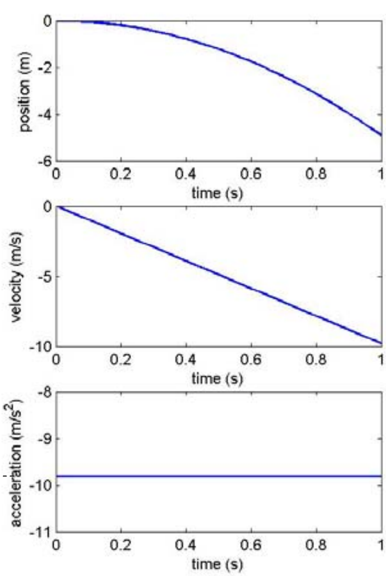
$$F = m\,\ddot{x}$$

total net force equals mass times acceleration

A Falling Mass



F (negative) gravitational force by earth



gravitational acceleration
$g = -9.81\ \mathrm{m/s^2}$

One-Mass Oscillator

Spring-mass system

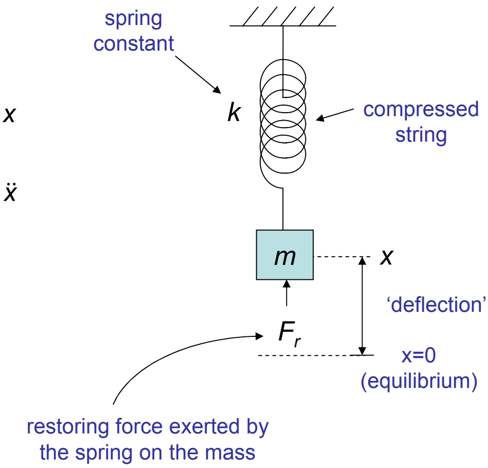
Hooke's law : $F_r = -k\,x$

Newton's 2nd law : $F_r = m\,\ddot{x}$

equation of motion:

$$\Rightarrow \quad m\,\ddot{x} + k\,x = 0$$

Ordinary differential equation (ODE)



One-Mass Oscillator: Solutions

Look for solution to the equation of motion (differential equation)

Try solution of the form:

$$x = e^{st}$$

thus:

$$\dot{x} = s e^{st} = sx$$

$$\ddot{x} = s^2 e^{st} = s^2 x$$

Substitute into:

$$m\,\ddot{x} + k\,x = 0$$

gives

$$ms^2 x + kx = 0$$

$$\Leftrightarrow \left(ms^2 + k\right)x = 0$$

for all x, thus

$$ms^2 + k = 0$$

$$s^2 + \left(\frac{k}{m}\right) = 0$$

$$s = \pm\sqrt{-\frac{K}{M}} = \pm\sqrt{j^2\frac{K}{M}} = \pm j\sqrt{\frac{K}{M}}$$

$$\left(j = \sqrt{-1}\right)$$

One-Mass Oscillator: Solutions (cont.)

<u>Dimension of sqrt(k/m)?</u>

$$K \rightarrow \frac{Force}{displacement} \rightarrow \frac{N}{m}$$

$$M \rightarrow \frac{Force}{Accelaration} \rightarrow \frac{N}{(m/\sec^2)} = \frac{N}{m}\sec^2$$

$$\sqrt{\frac{K}{M}} \rightarrow \sqrt{\frac{Nm}{Nm\sec^2}} = \frac{1}{\sec}$$ ⇨ dimension of frequency!

Turns out it is the (angular) natural frequency of the system:

$$\omega_0 = 2\pi f_0 = \sqrt{\frac{K}{M}} \quad \text{(rad/sec)}$$

resonance frequency in Hz

One-Mass Oscillator: Solutions (cont.)

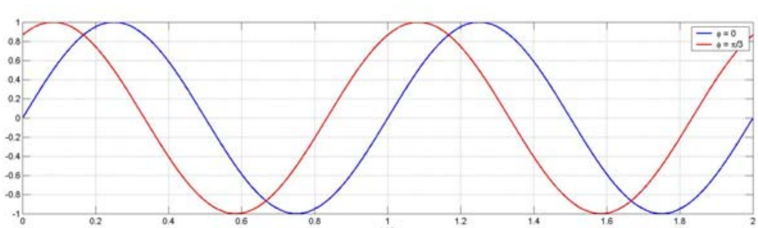
<u>General solution of the differential equation?</u>

By trying, found the solutions: $x = e^{j\omega_0 t}$ and $x = e^{-j\omega_0 t}$

Turns out that general (complex-valued) solution is: $x = C_1 e^{+j\omega_0 t} + C_2 e^{-j\omega_0 t}$

General **real-valued** solution can be written: $x = A\sin(\omega_0 t + \Phi)$

amplitude phase



Damped Oscillator

<u>Adding a damping term</u>

Damping force:

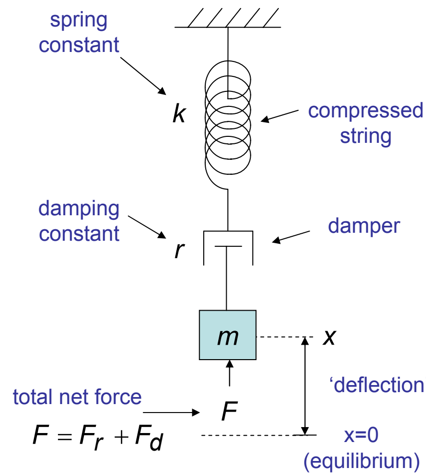
$$F_d = -r\,\dot{x}$$

Apply Newton's 2nd Law:

$$m\,\ddot{x} = F = F_r + F_d$$

Equation of motion:

$$m\,\ddot{x} + r\,\dot{x} + k\,x = 0$$



Damped Oscillator: Solutions

<u>Solution to the equation of motion:</u>

$$Ms^2x + Rsx + Kx = 0$$

$$\left(Ms^2 + Rs + K\right)x = 0$$

for all x, thus

$$Ms^2 + Rs + K = 0$$

$$s^2 + \left(\frac{R}{M}\right)s + \left(\frac{K}{M}\right) = 0$$

$$\omega_0 = \sqrt{\frac{K}{M}}$$ "natural frequency"

$$\alpha = \frac{R}{2M}$$ "decay rate"

$$\omega_r = \sqrt{\omega_0^2 - \alpha^2}$$ "resonance frequency"

$$s = \frac{-\left(\frac{R}{M}\right) \pm \sqrt{\left(\frac{R}{M}\right)^2 - 4\left(\frac{K}{M}\right)}}{2} = -\alpha \pm j\omega_r$$

Damped Oscillator: Solutions (cont.)

$$s = \frac{-\left(\frac{R}{M}\right) \pm \sqrt{\left(\frac{R}{M}\right)^2 - 4\left(\frac{K}{M}\right)}}{2} = -\alpha \pm j\omega_r$$

As long as damping is sufficiently small:

$$\left(\frac{R}{M}\right)^2 < 4\left(\frac{K}{M}\right)$$

If the damping (R) is so high that the square root operation is performed on a real number, *s* becomes real, which means no oscillation! The system is then called **overdamped**.

Damped Oscillator: Solutions (cont.)

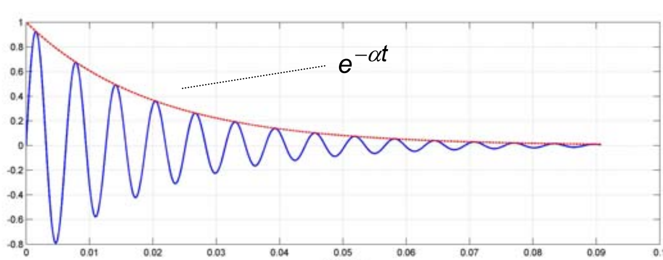
General solution of the differential equation

$$x = C_1 e^{-\alpha t} e^{+j\omega_r t} + C_2 e^{-\alpha t} e^{-j\omega_r t}$$

$$= \underset{\text{exponential decay}}{e^{-\alpha t}} \underbrace{\left(C_1 e^{+j\omega_r t} + C_2 e^{-j\omega_r t}\right)}_{\text{un-damped solution with } \omega_r}$$

Real-valued solution:

$$x = Ae^{-\alpha t}\sin(\omega_r t + \Phi)$$



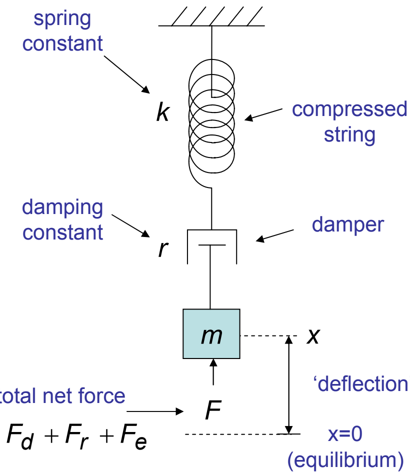
Driven Oscillator

Add external force:

$$m\,\ddot{x} = F_d + F_r + F_e$$

Equation of motion:

$$m\,\ddot{x} + r\,\dot{x} + k\,x = F_e$$



$$F = F_d + F_r + F_e$$

Frequency Response

Sinusoidal driving force:

$$m\,\ddot{x} + r\,\dot{x} + k\,x = Ae^{j\omega t}$$

Try

$$x = Be^{j\omega t}$$

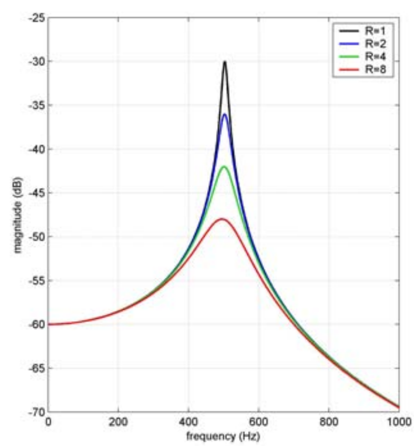
$$\left(-m\,\omega^2 + j\omega\,r + k\right)Be^{j\omega t} = Ae^{j\omega t}$$

Frequency response:

$$H(\omega) = \frac{x}{F} = \frac{Be^{j\omega t}}{Ae^{j\omega t}}$$

$$= \frac{1}{k + j\omega\,r - m\,\omega^2}$$

Frequency response plot (magnitude)



Impulse Response (Time-Domain)

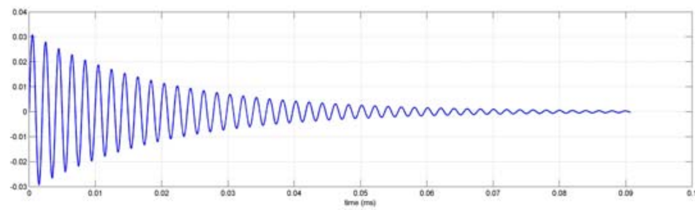
Frequency response = Fourier transform (FT) of impulse response function

$$H(\omega) = \frac{X(\omega)}{F(\omega)} = \frac{1}{k + j\omega r - m\omega^2}$$

$$h(t) = FT^{-1}\{H(\omega)\} \quad \text{(Inverse Fourier Transform)}$$

Impulse response = output response (x) to a single input impulse at t=0.

$$h(t) = \left(\frac{e^{-\alpha t}}{m\omega_r}\right)\sin(\omega_r t)$$



Two-Mass Systems

Total net force on first mass:

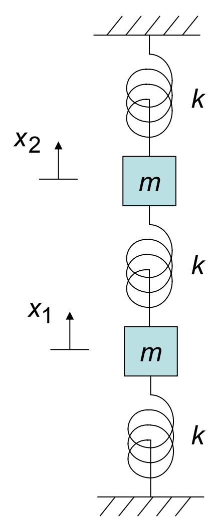
$$F_1 = -k\,x_1 + k(x_2 - x_1)$$

Total net force on second mass:

$$F_2 = -k\,x_2 + k(x_1 - x_2)$$

Thus the equations of motion are:

$$m\,\ddot{x}_1 + k\,x_1 + k(x_1 - x_2) = 0$$
$$m\,\ddot{x}_2 + k\,x_2 + k(x_2 - x_1) = 0$$



Two-Mass Systems (cont.)

Assume that both masses oscillate with the same frequency ω :

$$x_1 = A_1 e^{st} \qquad (s = j\omega)$$
$$x_2 = A_2 e^{st}$$

Substituting into equations of motion, we get, after some algebra:

$$\left(s^2 + 2\omega_0^2\right)A_1 - A_2\omega_0^2 = 0 \qquad \left(\omega_0^2 = k/m\right)$$
$$\left(s^2 + 2\omega_0^2\right)A_2 - A_1\omega_0^2 = 0$$

By eliminating A_1 and A_2, we find:

$$s^4 + 4\omega_0^2 s^2 + 3\omega_0^4 = 0$$

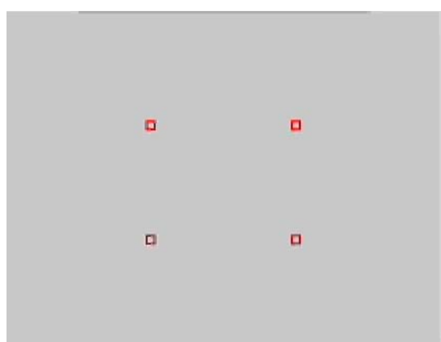
Two-Mass Systems (cont.)

This can be written:

$$\left(\omega^2\right)^2 - 4\omega_0^2\left(\omega^2\right) + 3\omega_0^4 = 0$$

and has the solutions:

$$\omega^2 = 2\omega_0^2 \pm \omega_0^2$$



There are thus two possible modes of vibration:

masses moving in:

$$\omega = \omega_0 : \left(\omega_0^2 - 2\omega_0^2\right)A_1 + \omega_0^2 A_2 = 0 \Rightarrow A_1 = A_2 \longrightarrow$$ same direction

$$\omega = \sqrt{3}\omega_0 : \left(3\omega_0^2 - 2\omega_0^2\right)A_1 + \omega_0^2 A_2 = 0 \Rightarrow A_1 = -A_2 \longrightarrow$$ opposite direction

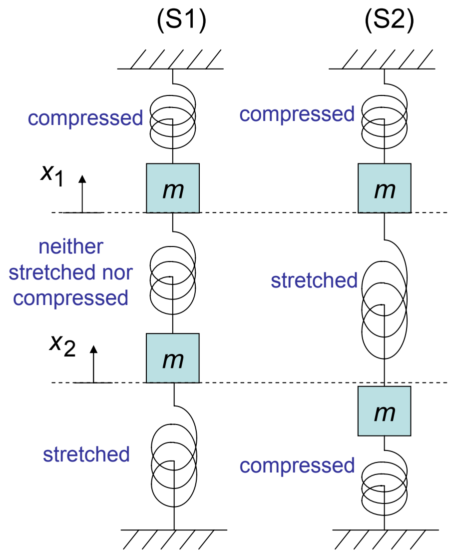
Two-Mass Systems (cont.)

General real-valued solution is a superposition of the two normal modes of vibration:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} A & B \\ A & -B \end{bmatrix} \cdot \begin{bmatrix} \sin(\omega_0 t) \\ \sin(\sqrt{3}\omega_0 t) \end{bmatrix}$$

Where the modal weights *A* and *B* depend on initial conditions (in case of free vibration), or on excitation position (in case of forced vibration)

Example: in initial scenario (S1) we have $B = 0$, and in initial scenario (S2) we have $A = 0$. In all other scenarios (except $A=B=0$), the motion of the each mass is a combination of two modal vibrations.



More Complex Systems

Two-mass systems can be more complex; different spring constants, different masses, transversal as well as longitudinal motion.

Multi-mass systems possible => *N* masses means *N* modes of vibration. Solution strategy is the same as for two-mass systems.

Continuous systems can be described as a system with infinite number of mass elements. They have an infinite number of modes.

Modes can be tested experimentally: modal analysis.

Modes of complex geometrical systems can also be determined via numerical simulation: finite element or finite difference method.

Summary

- Learned how basic forces act on a point mass.
- Understood how an equation of motion for a point mass (with spring and/or damper) is constructed from Newton's 2^{nd} law.
- Realised how an essential physical meaning of a solution-parameter of a physical system (one-mass oscillator) can be derived from dimensional analysis.
- Saw that a one-mass-oscillator exhibits 'free vibration' at its resonance frequency, which equals sqrt(K/M).
- Noticed and understood that damping forces impose an exponential decay onto the solution.
- Learned the formulas for the frequency response and impulse response.
- Learned about the formulation and solutions of a two-mass system.
- Learned that *N*-mass systems have *N* resonances.