THE STRESS DISTRIBUTION AT THE NECK OF A TENSION SPECIMEN

By P. W. Bridgman

Abstract

By approximating to the contour of a tension specimen at the neck by a circle and by using a circle to approximate the lines of principal stress in the neighborhood of the neck, the distribution of stress across the neck has been found which rigorously satisfies the conditions of plasticity in the conventional form of von Mises. The same solution also applies with an error of only a few per cent when strain hardening occurs of the amount found under actual conditions. The solution differs qualitatively from the stress distribution in an elastically strained specimen. In the plastic specimen, the tension is greatest on the axis and least on the periphery; the stress system consists of an axial tension, uniform all the way across the neck, plus a hydrostatic tension, which is zero on the periphery and increases to its maximum value on the axis.

The effect of the variation of tension across the section is to make the mean tension higher than the true tension of flow. Numerical values and curves are given for converting one tension to the other. Under extreme conditions recently attained experimentally, the correction may amount to 40 per cent. The correction depends on a single parameter, a/R, the ratio of the radius of the neck to the radius of curvature of the contour of the neck. Under experimental conditions a/R is determined to a certain degree of approximation by the reduction of area only, so that the correction may be roughly applied given the reduction of area only.

There appears to be a close connection between the "cup and cone" fracture often observed and the stress distribution at the neck. The brittle fracture on the axis is associated with the hydrostatic tension prevailing there, while the shearing fracture nearer the outer surface is connected with the shearing stresses which become important near the outer surface.

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THE stress-strain curve which is deduced from the ordinary tension test is properly a curve of average stress against average strain at the neck. It is almost certain, however, that the stress is not constant across the neck, and the question arises as to whether the variation is extensive enough to alter materially any deductions that we make from the behavior of the averages, in particular our deductions about strain hardening. A rigorous solution of the problem in plastic flow involved here does not seem possible at the present time. Apparently the only approximate solution is that of Siebel,[1] and this involves several doubtful simplifications, which will be referred to again later.

One might perhaps hope to get some hint as to a plausible order of magnitude for the variation of stress across the neck from the solution of the corresponding elastic problem. The elastic distortion in the immediate neighborhood of the neck must be roughly like the solution worked out rigorously by Neuber[2] for a hyperboloid of revolution. One may draw the conclusion from this known solution that probably if the reduction of area is only 50 or 60 per cent, which is of the order of magnitude of the reduction at fracture of ordinary ductile steels, the effect of lack of uniformity of stress distribution is not important. Recently, however, I have found[3] that by conducting the tensile test in a fluid at pressures of the order of 400,000 lb/in^2 reductions in area up to nearly 100 per cent may be realized without fracture, and strain hardenings, calculated on the basis of the average stress, may be realized up to a factor of 3 or more. Under these conditions the contour of the specimen at the neck may have such a small radius of curvature that the elastic solution indicates variations of stress across the section of several hundred per cent, and it obviously becomes important to make an approximate estimate of the departures from uniformity of the stresses in actual cases of plastic flow.

A complete solution should be capable of giving, among other things, the shape of the external contour of the tension sample. In fact, one might anticipate that one of the uses that could be made of a complete solution would be to provide quick information about the strain hardening curve from the shape of the contour. This, however, seems impossible at present. The best that I have been

[1] E. Siebel, Berichte der Fachauschüsse des Vereins deutscher Eisenhüttenleute. Werkstoffausschuss. Bericht Nr. 71, Nov. 5, 1925.

[2] H. Neuber, Beiträge für den achssymmetrischen Spannungszustand; Thesis, München, 1932.

[3] P. W. Bridgman, American Scientist, 31, 1, 1943.

able to do in the solution given in the appendix is to assume the external contour given, and to deduce a set of stresses that satisfies the conditions of plasticity in the immediate neighborhood of the neck.

The factor $\left(1 + 2\frac{R}{a}\right) \log \left(1 + \frac{1}{2}\frac{a}{R}\right)$ and certain other related data are given in Table I. The inverse of the factor, which may be called simply the "correction factor," since it is the factor by which the "uncorrected true stress" is to be multiplied to obtain the "corrected true stress" is shown as a curve against a/R in Fig. 1 The "corrected true stresses" given by the present analysis are progressively larger than the "corrected true stresses" that would be given by Siebel's formula.

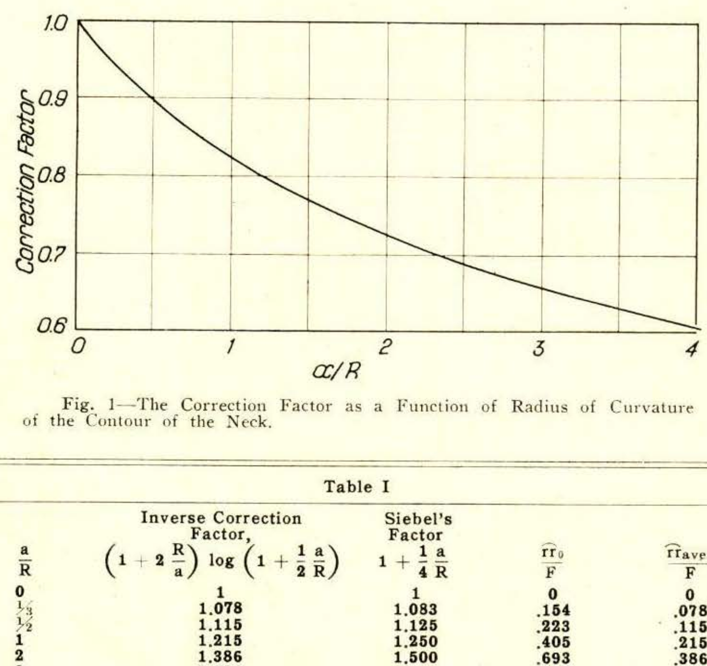

Fig. 1—The Correction Factor as a Function of Radius of Curvature of the Contour of the Neck.

Table I

$\frac{a}{R}$	Inverse Correction Factor, $\left(1 + 2\frac{R}{a}\right) \log \left(1 + \frac{1}{2}\frac{a}{R}\right)$	Siebel's Factor $1 + \frac{1}{4}\frac{a}{R}$	$\frac{\widehat{rr}_0}{F}$	$\frac{\widehat{rr}_{aver}}{F}$
0	1	1	0	0
1/3	1.078	1.083	.154	.078
1/2	1.115	1.125	.223	.115
1	1.215	1.250	.405	.215
2	1.386	1.500	.693	.386
3	1.524	1.750	.916	.524
4	1.649	2.000	1.099	.649

The corrections are quite appreciable even under ordinary conditions of testing. Thus for a steel of approximately 0.45 carbon content a representative reduction of area at fracture is 60 per cent, and at this reduction a representative value of a/R is 0.8. (a is the radius of the cross section and R is the radius of curvature of the longitudinal surface fibers, at the cross section of least diameter.) Under these conditions F (the "flow stress," or stress required for flow in simple tension) is 15 per cent lower than zz_{aver} (the average longitudinal tension), and therefore the strain hardening, calculated without making the correction, 17.6 per cent too high. Under greater strains, such as may be produced by pulling under hydrostatic pressure, the correction may change even the sign of certain terms and thus change the qualitative aspect of the picture. The change will be greatest with respect to fracture. It appears that the fracture is essentially determined by the hydrostatic tension on the axis, and this tension is large enough numerically, even in ordinary cases, to demand that it be given as well as the longitudinal tension in specifying the conditions of fracture in an ordinary tensile test. This matter is elaborated further in one of the symposia of this meeting. One important result remains, however, even after the correction is applied. After the first initial stages of a tension test, it is coming to be recognized that the stress-strain curve is linear in the natural strain, that is, the "uncorrected true stress" gives a linear plot against $\log_e A_0/A$, A_0 and A being the initial and final neck areas. Experiments indicate that over a range of conditions "corrected true stress" is also linear. The subject requires further study, but at least I have found no cases in which the corrected curve was not also linear within experimental error.

In practise, in order to apply the correction, the radius of curvature at the neck must be measured in addition to the other data which it is now conventional to collect. This may be done by various devices, but it would be more convenient if it should prove that a/R is a function only of the reduction of area. There are present indications that any departures of a/R from being a function of reduction of area only may be by amounts unimportant for purposes of the correction. The matter demands extensive study over a wide range of conditions, but in the meantime a curve is shown in Fig. 2 representing the average results of some 50 experiments on a variety of steels. Most of these were on a 1045 steel, heat treatments varying from annealed to quenched and drawn at 800 degrees Fahr., but also included a dead soft 1020 steel, a silicon-manganese alloy steel treated to a strength of around 300,000 lb/in^2 and a tough nickel steel. At a value of $\log_e A_0/A$ of 3.0 the extreme variation found in a/R was from 1.3 to 2.0. There seems no correlation between the composition of the steel and departure of a/R from the median curve. Even brass and a bearing bronze were found to fall in the same limits as the steels. In the extreme case the correction factor computed from the directly measured a/R does not differ by more than 4 per cent from the factor from the median curve. There are factors yet to be investigated. For instance, does the ratio of

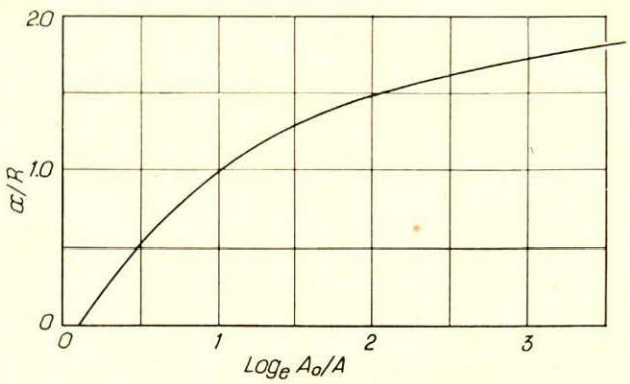

Fig. 2—Average Contour Factor $\left(\frac{a}{R}\right)$ from a Number of Experiments as a Function of the "Natural Strain" at the Neck.

length to diameter of the original specimen affect the values of a/R? All the measurements referred to above were made with a value for this ratio in the neighborhood of 3. However, in the absence of direct measurements of a/R the corrected value of "true stress" obtained by assuming the a/R of Fig. 2 is doubtless better than the "uncorrected true stress." To assist in making the reduction, Figs. 1 and 2 have been combined to give Fig. 3, in which the correction factor is plotted against $\log_e A_0/A$. This curve is to be used only subject to the precautions suggested by the discussion.

The solution has assumed that the hardening is constant all the way across the neck, that is, it has assumed that the hardening is not affected by the hydrostatic tension that acts in addition to the stress $\widehat{zz}_a$ at the boundary. The two last columns of the Table show the ratio of the maximum (on the axis) and the average hydrostatic tension to the $\widehat{zz}$ component of stress at the edge. Consider a typical example in which F = 355,000 pounds per square inch for a natural strain of 2, at which a/R is approximately 1.5. $\widehat{rr}_0/F$ is approximately 0.55, which means a hydrostatic tension on the axis of approximately 196,000 pounds per square inch, and an average hydrostatic tension over the section of 106,000 pounds per square inch. Now I have found by experiment that the flow stress when necking begins is increased from 10 to 15 per cent by a hydrostatic *pressure* of 355,000 pounds per square inch. This means that under the condition in our example the flow stress would be decreased by

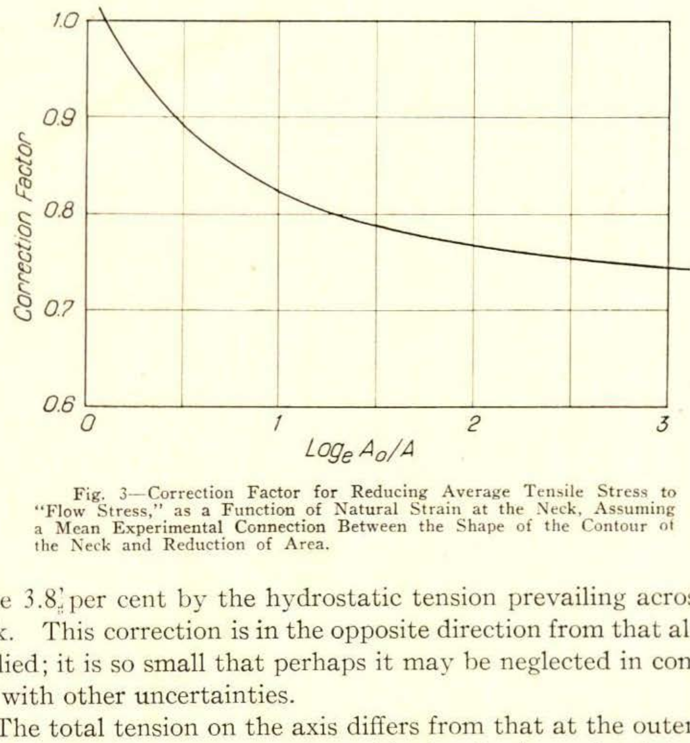

Fig. 3—Correction Factor for Reducing Average Tensile Stress to "Flow Stress," as a Function of Natural Strain at the Neck, Assuming a Mean Experimental Connection Between the Shape of the Contour of the Neck and Reduction of Area.

some 3.8 per cent by the hydrostatic tension prevailing across the neck. This correction is in the opposite direction from that already applied; it is so small that perhaps it may be neglected in comparison with other uncertainties.

The total tension on the axis differs from that at the outer edge by $\widehat{rr}_0$ (the radial tension on the axis), that is, the ratio of the axial to the peripheral tension is $1 + \widehat{rr}_0/F$. In Fig. 4 this is plotted against $\log_e A_0/A$, assuming the same empirical relation between a/R and $\log_e A_0/A$ that was used in constructing Fig. 3. The practical use of the figure is therefore subject to the same precautions as use of Fig. 3. This curve will be of service in computing the fracture stress, since as will appear in the discussion presently, fracture is doubtless initiated on the axis.

Finally, we consider certain applications that may be made of our solution to the problem of the nature of the fracture. The maximum shearing stress at the neck, assuming the solution given, is constant across the neck, has the value $\widehat{zz}_a/2$, and is equal on the $r - z$ and $r - \theta$ planes. That is, at every point there is a cone of shear. The shearing stress in the $r - \theta$ plane vanishes. Because of the geometrical limitations any actual slip of the material must probably be on the $r - z$ planes. The greatest shearing stress,

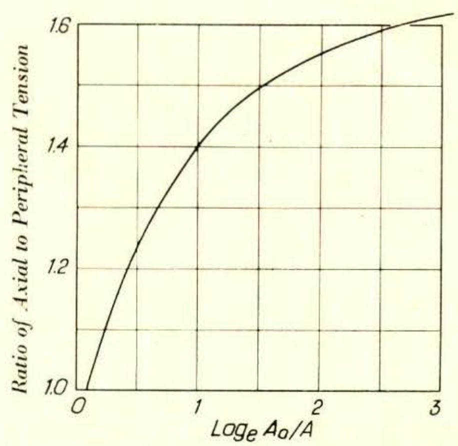

Fig. 4—The Ratio of the Tensile Stress $(\widehat{zz})$ on the Axis to its Value at the Periphery, as a Function of the Natural Strain at the Neck, Assuming the Experimental Connection Between Contour Shape and Strain Shown in Fig. 2.

however, is not at the neck. Referring to Fig. 5 of the appendix, and recalling the boundary conditions, at the point P there is a shearing stress along the plane bisecting AB and CD of $\widehat{zz}/2 \cos^2 \alpha$, and along the plane cutting the plane of the paper in CD at an angle of 45 degrees of $\frac{1}{2} [\widehat{zz}/\cos^2 \alpha - \widehat{\theta\theta}]$. The sign of $\widehat{\theta\theta}$ at points on the boundary not on the neck would seem to be in some doubt, so that it is not at once obvious which of these two shearing stresses is the larger. Geometrical limitations, however, would seem to demand that actual slip of the material take place in the first plane, bisecting AB and CD. This agrees with observation; slip on the other plane takes place less commonly, and even then usually only when slip can spread right across the neck to a symmetrical point on the other side.

The shearing stress $\widehat{zz}/2 \cos^2 \alpha$ is composed of two factors, the first of which, $\widehat{zz}$, decreases on receding from the neck, and the second of which, $1/\cos^2 \alpha$, at first increases. The variation of $\widehat{zz}$ is not fixed by the analysis above, only the average $\widehat{zz}$. If the neck is abrupt, $1/\cos^2 \alpha$ increases so rapidly in the neighborhood of the neck that it swamps the decrease of zz_{aver}, and also any plausible decrease of the actual $\widehat{zz}$. Observation of many tensile fractures in necked samples shows that at the outside failure is frequently along a shear plane, and this shear never issues at the neck itself, but a little removed from it. To give a numerical example, the specimen for which a/R was 2 at a reduction in diameter to one quarter had a maximum $zz/\cos^2 \alpha$, assuming $\widehat{zz}$ uniform across every section, 14 per cent greater than at the neck. The point of maximum was

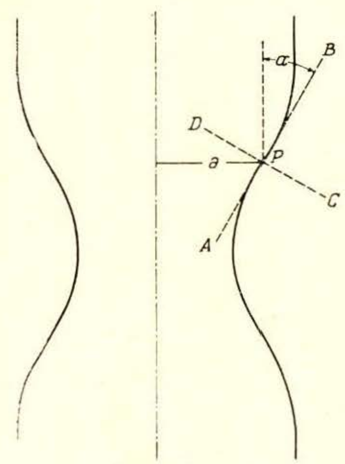

Fig. 5—Certain Geometrical Relations Used in the Analysis (p. 562).

displaced along the axis from the neck by about 0.4 of the neck diameter. With regard to the assumption of uniformity of $\widehat{zz}$ it is probable that as one proceeds away from the neck toward the elastic region $\widehat{zz}$ will tend to assume the property of the elastic solution, that is, to have the greatest value at the outside surface instead of on the axis as in the full plastic region at the neck. The effect of this direction of variation in the lack of uniformity of $\widehat{zz}$ will accentuate the tendency of the shearing stress to have its maximum value away from the neck.

The conclusion that at the neck the tension is greatest on the axis and least at the outside, instead of inversely as in the elastic case, has interesting reactions on our picture of the mechanism of fracture. We have seen that the stress on the axis may be regarded as differing from that at the edge, where it is a simple tension, merely by the addition of a hydrostatic tension. This superposed hydrostatic tension will have two effects. In the first place there will be a reduction in the tension component in excess of the hydrostatic stress required to produce fracture. It is true that this has not been established by direct experiment, but it has been established for the grades of steel that permit necking that the excess tension component at fracture is increased by hydrostatic pressure, and since these effects are usually linear, we may expect a decrease with hydrostatic tension. In the second place, it is known that hydrostatic pressure increases ductility, so that a hydrostatic tension may be expected to decrease it. It may be that these two effects are not completely independent of each other, but in any event they are both in the same direction, and conspire to produce the brittle tensile fracture at the axis which is usually shown by the grades of steel that exhibit necking. Although there has in the past been difference of opinion, it seems to be becoming accepted now that the fracture of a necked specimen starts on the axis. This is shown convincingly, for example, by a photograph on page 55 of Gensamer's Strength of Metals under Combined Stresses.

Our solution demands that the tendency to brittle fracture diminishes as one recedes from the axis. On the other hand, the tendency to shearing slip increases at distances from the axis. On the axis the geometrical conditions are unfavorable because slip will have to occur on curved conical surfaces, whereas toward the outside the slip occurs more nearly on planes. Another important feature in the geometrical situation is that shearing fracture cannot get initiated unless there is freedom of slip all the way across the slip planes; slip cannot start if one end of the slip planes is anchored on the axis, whereas a tensile break spreading from the axis provides a possible point of initiation for shearing slip. It is the balance between the two tendencies to tensile and shearing fracture that gives the ordinary "cup and cone" fracture.

This view would demand that the fracture of a necked test piece begin on the axis as a tensile break, where the tendency to tensile fracture is the greatest, and also that the tensile break be situated at the narrowest part of the neck. This latter agrees with observation; longitudinal sections made through a number of "cup and cone" breaks shows that the flat bottom of the cup is always located at the narrowest part of the neck. The tensile fracture travels toward the outside until the geometry is so modified that the shearing fracture can take over. Precisely what the critical condition is that determines when shear is ready to take over is a subject for future investigation.

If the pulling is done in a medium under hydrostatic pressure the general ductility increases, the tendency to brittle tensile fracture diminishes, and the shearing type of fracture will become more prominent. This agrees with recent observations, as yet unpublished, that above a certain pressure the fracture becomes almost, if not entirely, shearing in character.

Appendix

Summary of Principal Symbols Used.

The stress system is referred to conventional cylindrical co-ordinates, r, θ, and z, and the notation for the stress components is that adopted, for example, in Love's Mathematical Theory of Elasticity. In particular, the stress components are: $\widehat{rr}$, $\widehat{\theta\theta}$, $\widehat{zz}$, $\widehat{\theta z}$, $\widehat{zr}$, and $\widehat{r\theta}$. Other symbols are:

a, outside radius of the cross section at the neck.

R, radius of curvature at the neck, of the section through the neck containing the axis

F, "flow stress," tension at which plastic flow occurs, the two other principal stress components being zero.

A_0, initial cross section of the tension sample.

A, final section at the neck of the tension sample.

Derivation.

Certain necessary conditions are imposed on any solution by the stress equations of equilibrium. The problem may be assumed to have rotational symmetry about the axis. Using ordinary cylindrical co-ordinates, this means that all derivatives with respect to θ vanish, and the $\widehat{r\theta}$ and $\widehat{\theta z}$ stress components vanish identically. The usual three stress equations of equilibrium reduce to two:

$$\left.\begin{aligned} \frac{\partial \widehat{rr}}{\partial r} + \frac{\partial \widehat{rz}}{\partial z} + \frac{\widehat{rr} - \widehat{\theta\theta}}{r} &= 0 \\ \frac{\partial \widehat{rz}}{\partial r} + \frac{\partial \widehat{zz}}{\partial z} + \frac{\widehat{rz}}{r} \quad &= 0 \end{aligned}\right\} \qquad (1)$$

The external curved surface is free from stress, and the boundary conditions become:

$$\left.\begin{aligned} \widehat{rr} \cos \alpha - \widehat{rz} \sin \alpha &= 0 \\ \widehat{rz} \cos \alpha - \widehat{zz} \sin \alpha &= 0 \end{aligned}\right\} \text{ at external surface.} \qquad (2)$$

Here α is the angle between the generating line of the external surface and the axis, as indicated in Fig. 5. The conditions at the surface may be rewritten as:

$$\left.\begin{aligned}\widehat{rr} &= \widehat{zz}\tan^2\alpha\\ \widehat{rz} &= \widehat{zz}\tan\alpha\end{aligned}\right\}\ \text{at external surface.} \tag{3}$$

At the external surface the planes of principal stress are: (1) the plane of the axis (plane of the paper in Fig. 5) across which the normal stress is $\widehat{\theta\theta}$, (2) the tangent plane (perpendicular to the plane of the paper and with trace AB in Fig. 5) across which the normal stress is zero, and (3) the plane perpendicular to (1) and (2) (trace CD in Fig. 5) across which the normal stress is $\widehat{zz}/\cos^2\alpha$.

At any section perpendicular to the axis,

$$2\pi\int_0^a r\,\widehat{zz}\,dr = \text{Load} = \text{constant} \tag{4}$$

The conditions simplify at the neck. It is assumed that the specimen is symmetrical on both sides of the neck. Hence at the neck:

$$\text{also}\qquad \left.\begin{aligned}\frac{\partial\widehat{zz}}{\partial z} &= 0 \qquad \text{for all r}\\ \widehat{rz} &= 0 \qquad \text{for all r}\\ \widehat{rr}_a &= 0\\ \widehat{rr}_0 &= \widehat{\theta\theta}_0\end{aligned}\right\}\ \text{at the neck.} \tag{5}$$

A stress symbol with subscript "a" denotes the value at the external boundary, subscript 0 at the axis. The first two conditions of (5) arise from symmetry. The third expresses the freedom of the external surface from stress. The last of the four conditions, which also holds at all points on the axis, is necessary in order to avoid infinities in the first of equations (1).

Let us now inquire what is a plausible solution at the neck for $\widehat{rr}$; $\widehat{\theta\theta}$, and $\widehat{zz}$, $\widehat{rz}$ not being considered. If $\widehat{rz}$ is eliminated between the two stress equations, a single equation results for the three stresses:

$$\frac{\partial}{\partial r}(r\cdot\widehat{rr}) = \widehat{\theta\theta} + \int_0^r r\,\frac{\partial^2\widehat{zz}}{\partial z^2}\,dr \tag{6}$$

If we put

$$\widehat{zz} = f_1(r) + \tfrac{1}{2}z^2 f_2(r), \tag{7}$$

where $f_2(r) \equiv \dfrac{\partial^2\widehat{zz}}{\partial z^2}$, f_1 and f_2 are subject only to the restrictions:

$$2\pi\int_0^a f_1(r)\,r\,dr = \text{Load} \tag{8}$$

and

$$\frac{a}{R}\,\widehat{zz}_a = -\int_0^a f_2(r)\,r\,dr, \tag{9}$$

where R is the radius of curvature of the contour at the neck and a is now the radius of the neck. Equation (9) comes from applying condition (4) to a plane section just above the neck.

Subject to these conditions, a great deal of latitude is possible in the solution; in fact, having assumed f_1 and f_2 subject to restrictions (8) and (9), one may assume any $\widehat{rr}$ subject only to the condition $\widehat{rr}_a = 0$, and then solve (6) for $\widehat{\theta\theta}$. The problem is to pick out from this infinity of solutions those which satisfy the plasticity conditions.

We first proceed on the assumption of no strain hardening. The plasticity conditions are now of two sorts. First there is the condition that the material be in the plastic condition all the way across the neck. We assume the form of von Mises:

$$(\widehat{rr} - \widehat{\theta\theta})^2 + (\widehat{\theta\theta} - \widehat{zz})^2 + (\widehat{zz} - \widehat{rr})^2 = \text{constant, independent of r at the neck.} \tag{10}$$

Second, there are the flow conditions, written in the accepted form in terms of the strain velocities:

$$\left.\begin{aligned} \dot{e}_r &= \beta[\widehat{rr} - \tfrac{1}{2}(\widehat{\theta\theta} + \widehat{zz})] \\ \dot{e}_\theta &= \beta[\widehat{\theta\theta} - \tfrac{1}{2}(\widehat{zz} + \widehat{rr})] \\ \dot{e}_z &= \beta[\widehat{zz} - \tfrac{1}{2}(\widehat{rr} + \widehat{\theta\theta})] \end{aligned}\right\} \tag{11}$$

In the general case the coefficient β of these equations is neither a material constant nor a function of the physical parameters such as the stresses. It is rather a freely disposable function of the co-ordinates and varies from problem to problem with the geometry. The equations merely express the isotropy of flow at any single point.

If it is not possible to find a set of stresses satisfying all the conditions one may draw the conclusion that the state of affairs at the neck is of necessity non-isotropic. Conversely, if a solution is possible, then an isotropic state of affairs at the neck is at least not ruled out.

Since the plasticity equations hold across the neck, we may particularize by writing them for an external point and for a point on the axis. Remembering that $\widehat{rr}_a = 0$, and $\widehat{rr}_0 = \widehat{\theta\theta}_0$, these become:

$$2\widehat{\theta\theta}_a{}^2 + 2\widehat{zz}_a{}^2 - 2\widehat{\theta\theta}_a\widehat{zz}_a = 2\widehat{\theta\theta}_0{}^2 + 2\widehat{zz}_0{}^2 - 4\widehat{\theta\theta}_0\widehat{zz}_0 \tag{12}$$

$$\left.\begin{aligned} \dot{e}_{r_a} &= \beta[-\tfrac{1}{2}(\widehat{\theta\theta}_a + \widehat{zz}_a)]; & \dot{e}_{r_0} &= \beta[-\tfrac{1}{2}\widehat{zz}_0 + \tfrac{1}{2}\widehat{\theta\theta}_0] \\ \dot{e}_{\theta_a} &= \beta[\widehat{\theta\theta}_a - \tfrac{1}{2}\widehat{zz}_a]; & \dot{e}_{\theta_0} &= \beta[-\tfrac{1}{2}\widehat{zz}_0 + \tfrac{1}{2}\widehat{\theta\theta}_0] \\ \dot{e}_{z_a} &= \beta[\widehat{zz}_a - \tfrac{1}{2}\widehat{\theta\theta}_a]; & \dot{e}_{z_0} &= \beta[\widehat{zz}_0 - \widehat{\theta\theta}_0] \end{aligned}\right\} \tag{13}$$

The condition of isotropy of flow is automatically satisfied on the axis, but elsewhere special adjustment will be necessary to meet both the flow conditions and the plasticity condition of von Mises.

Before proceeding further it will pay to look at the numerical values, to see whether we are concerned only with departures from linearity which may be treated as small, or whether they are of a larger order of magnitude. If the tensile specimen is pulled to a reduction of area of about 94 per cent, it has been observed in the case of one experiment that a/R at the neck is approximately 2. Reductions up to nearly 100 per cent have been observed, so that values of a/R in excess of 2 are to be expected. If we assume $\partial^2\widehat{zz}/\partial z^2$ = const, then the stress equation (6) becomes

$$\frac{\partial}{\partial r}(r \cdot \widehat{rr}) = \widehat{\theta\theta} - \frac{a}{R}\widehat{zz}_a\frac{r^2}{a^2} \tag{14}$$

For the sake of example put a/R = 2, and write the equation for the outer edge:

$$\frac{\partial}{\partial r}(r \cdot \widehat{rr})\bigg|_{r=a} = \widehat{\theta\theta}_a - 2\widehat{zz}_a \tag{15}$$

It would appear then that in general $\widehat{\theta\theta}$ and $\widehat{rr}$ must be of the same order of magnitude as $\widehat{zz}$ and that we cannot hope to get an approximate solution of the flow and plasticity conditions by treating $\widehat{\theta\theta}$ and $\widehat{rr}$ as small compared with $\widehat{zz}$.

In searching for a qualitatively satisfactory solution it is natural to start with the solution for the elastically strained body worked out by Neuber for the case of a solid of revolution whose outer surface is a hyperbola. We may take this as approximating our problem in the neighborhood of the neck. In the elastic solution $\widehat{zz}$ is not uniform at the neck but has its smallest value at the axis and then rises continually, at first slowly, and then more abruptly toward the outer surface, where it reaches its maximum value. The total extent of the rise and the abruptness of the upturn both increase as a/R increases. For values of a/R between 0 and 2 the maximum value of $\widehat{zz}$ differs from the average $\widehat{zz}$ by approximately the fraction $\frac{1}{4}$a/R. On the axis, $\widehat{\theta\theta}$ and $\widehat{rr}$ are both positive, that is, of the same sign as $\widehat{zz}$. At first $\widehat{rr}$ and $\widehat{\theta\theta}$ both increase on leaving the axis; $\widehat{rr}$ presently passes through a maximum and plunges to the value zero at the outer surface with an abruptness increasing with a/R. $\widehat{\theta\theta}$ rises continually toward the outer edge with an abruptness the greater the greater a/R. In the range from 0 to 2 for a/R, $\widehat{\theta\theta}$ at the outer edge is roughly $\frac{1}{3}(a/R)\widehat{zz}_{aver}$.

If one tries to find a solution of the stress equations of equilibrium (1), which of course must always be rigorously satisfied, which has the qualitative aspects of the elastic solution, one will find that the plasticity and flow conditions cannot be satisfied. Some experimenting with simple algebraic forms that have the requisite rough qualitative behavior, combined with a study of the demands expressed in equations (12) and (13), will, I think, convince one of this. Furthermore, since the $\widehat{rr}$'s and $\widehat{\theta\theta}$'s are of the same order of magnitude as $\widehat{zz}$, the failure of the elastic solution to meet the plasticity conditions is by a large amount. Large failures of isotropy are necessary if the stresses are to have the same qualitative configuration as in the elastic case. But the isotropic conditions can be maintained by a change in the qualitative picture. The isotropy of flow at the outside demands, according to (13) that $\widehat{\theta\theta}_a = 0$. But already $\widehat{rr}_a = 0$, and in all cases $\widehat{rr}_0 = \widehat{\theta\theta}_0$. It suggests itself therefore to try setting $\widehat{rr} = \widehat{\theta\theta}$ for all values of r. This automatically ensures the isotropy of flow at all points of the neck. If we put $\widehat{\theta\theta} = \widehat{rr}$ in the equations of equilibrium in the form (6) and integrate, we get:

$$\widehat{rr} = -\int_r^a \left\{\frac{1}{r}\int_0^r r\frac{\partial^2 \widehat{zz}}{\partial z^2}\,dr\right\} dr \qquad (16)$$

We are free to assume any form for $\partial^2\widehat{zz}/\partial z^2$ subject only to the restriction (9). Equation (16) shows that for any ordinary smooth variation of $\partial^2 zz/\partial z^2$ with r, $\widehat{rr}_0$ is going to be positive, and there is a two dimensional hydrostatic *tension* on the axis, as in the elastic case.

Inspection now shows that the von Mises plasticity condition is satisfied for all values of r if we set

$$\widehat{zz} = \widehat{zz}_a + \widehat{rr} \qquad (17)$$

We are free to do this because $\widehat{zz}_a$ is not yet fixed, and $\widehat{zz}$ is subject to the single condition $2\pi\int_0^a (\widehat{zz}_a + \widehat{rr})rdr = \text{Load}$.

It is to be remarked that under these special conditions the von Mises plasticity condition becomes identical with the maximum shearing stress condition.

The stress system which we have just found consists of a uniform tension $\widehat{zz}_a$ in the z direction all the way across the neck, plus a superposed hydrostatic tension (three equal tensions along three mutually perpendicular directions) increasing from zero at the outside surface to $\widehat{rr}_0$ on the axis.

A complete solution requires that the coefficient β of the flow equations be determined. This is to be so determined that, given the stresses, the components of flow satisfy the conditions imposed by the geometry. In general, when there is radial symmetry and the elongation along the axis, e_z, is constant, the radial displacement ρ must satisfy the condition:

$$\rho = -\frac{e_z}{2}r + \frac{c}{r} \qquad (18)$$

In this particular situation, because the material occupies the axis and infinities must therefore be avoided on the axis, $c = 0$, $\rho = -\frac{e_z}{2} r$, and $e_r = e_\theta = -\frac{1}{2}e_z$. That is, the strains are uniform all the way across the neck, and this holds independent of time. The flow equations must therefore also demand a uniform flow, and the coefficient β of equations (11) is in this particular case a constant all the way across. At any instant of time the equations for the strains e_r, e_θ, and e_z have the same appearance as the flow equations. That is, the dots may be removed from the left hand side of equations (11). The coefficient β may still be a function of time, but its precise value is of no particular interest to us.

How must this solution be modified when strain hardening is considered? The question is not academic, because we have seen that it is possible to increase the average flow stress by a factor of at least two or three under extreme but realizable conditions of necking. Furthermore, the phenomenon of necking is itself a strain hardening phenomenon and only occurs when the slope of the strain hardening curve decreases below a critical value.

In general the whole mathematical set-up must be modified when there is strain hardening. The general case may be very complicated; we simplify by assuming the strain hardening to be isotropic. Under these conditions equations of the same formal appearance as (11) remain, but these are now equations on the strains and not on the strain velocities, and represent the strains reached asymptotically in time under constant stress. The coefficient β now becomes determinate in terms of material constants and such physical parameters as the stresses or strains. The coefficient β must be consistent with experimental stress-strain curves for all particular and possible stress systems. How, or whether, β can be uniquely determined from a single type of stress-strain curve, as, for example, simple tension, need not be discussed here. It is obvious, however, since β has now become deprived of its role as a disposable function, that it is no longer possible to satisfy all the conditions of equilibrium etc. if there is an independent plasticity condition analogous to (10) still to be satisfied. Any equation of the type of (10) which persists in the strain hardening case must play the role of a mathematical identity expressing conditions already inherent in the three equations for the three principal strains. Under such conditions the right hand side of (10) now becomes a variable, a function perhaps of the strains, determined so as to be consistent with experiment.

In the special case where the strains are uniform the whole picture very much simplifies. The solution we have just found, assuming no strain hardening, continues to apply to the strain hardening case if we are at liberty to assume that at every stage of the necking process the β of equations (11) is constant all the way across the neck and that the right hand side of equation (10) is similarly constant all the way across. In the solution we have found the strains are constant all the way across, and the stress system is a tension constant all the way across with a variable superposed hydrostatic tension. If the superposed hydrostatic tension has no effect on the strain hardening, the physical conditions would be met and we could take over our simple solution. Now it is known that to a first approximation such is indeed the case; numerical values have been given in the body of the text indicating how good an approximation may be expected. We shall assume in the following that our solution applies also to the strain hardening case, with an appropriate change in the meaning of the right hand side of (10).

It is possible, therefore, to meet all the conditions, assuming isotropy of flow across the neck, with the qualitative difference compared with the elastic case that $\widehat{zz}$ increases toward the axis instead of decreasing. Such an increase would demand only a very small excess yielding of the material near the surface and furthermore meets certain qualitative demands imposed by the nature of the fracture, as will appear later. It therefore is not unplausible. The steps by which this stress system is set up are doubtless somewhat as follows. Yield starts at the outside surface, where the tensile stress in the elastic condition is a maximum. A very slight amount of plastic flow is sufficient to redistribute the stress approximately uniformly. The plastic condition works its way toward the axis; as the material in the neighborhood of the axis assumes the plastic condition the two dimensional hydrostatic tension ($\widehat{rr}_0 = \widehat{\theta\theta}_0 =$ positive quantity) remaining from the previous elastic condition produces extensional creep along the axis and so an additional component of tension along the axis.

We still have to find an explicit solution; with the formulation above this will depend on finding the proper expression for $\partial^2\widehat{zz}/\partial z^2$, which is still at our disposal. No direct method for obtaining the proper expression presents itself; another method will therefore be adopted for obtaining a differential equation for the stresses at the neck. Fig. 6 represents the state of affairs in a neighborhood close enough to the neck to permit the external contour to be represented by a circle. The radius of the neck, a, and the radius of curvature, R, of the contour at the neck, are given. At the external surface we have seen that one of the lines of principal stress is normal to the

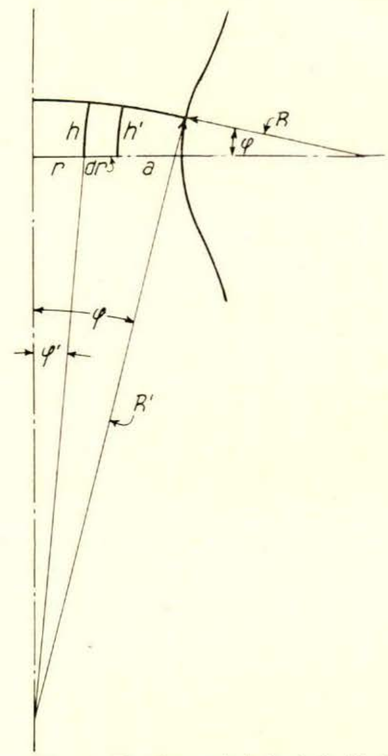

Fig. 6—The Geometrical Analysis Used in the Approximate Solution in the Neighborhood of the Neck.

surface, and we know that on the axis the lines are normal to the axis. We assume that in the immediate neighborhood of the neck the complete line of principal stress is a circle, with center on the axis. Consider the particular stress circle that cuts the contour at such a point as to subtend the small angle φ at the center of curvature of the contour. The radius of curvature, R′, of the stress circle, is a/φ. Consider now a point at the neck distant r from the axis. Let the length r subtend an angle φ' at the center of the circle R′.

Then $\varphi' = \frac{r}{a}\varphi$. Through the point r, perpendicular to the radius, there passes one member of the family of lines of principal stress of which the external contour is another member. This line is perpendicular to the radius r and the circle of radius R′. We may assume that this line is also a circle, and we might compute its radius. It happens, however, that to the order of small quantities in which we are interested this refinement is not necessary, and it will be sufficient to treat this line of principal stress as a straight line perpendicular to the radius at r. Consider now the element of volume bounded by two axial planes including between them the small angle θ, by the cylinders with radii r and r + dr, by the plane perpendicular to the axis at the neck, and by the spherical surface of radius R′. By construction the forces across the faces of this element are entirely normal and are given by the principal stress components. The condition that the net component of force in the r direction on the six faces of this element vanishes gives the equation:

$$\left(\widehat{zz} + h\frac{\partial\widehat{zz}}{\partial z}\right)\sin\varphi'\left(r + \frac{dr}{2}\right)\theta dr$$
$$= \widehat{rr}\cdot h\cdot r\cdot\theta - \left(\widehat{rr} + dr\frac{\partial\widehat{rr}}{\partial r}\right)(r + dr)h'\cdot\theta + \widehat{\theta\theta}\sin\theta\cdot h\cdot dr.$$

We have further:

$$h = R\varphi + R'[\cos\varphi' - \cos\varphi]$$
$$h' = R\varphi + R'[\cos(\varphi' + d\varphi') - \cos\varphi]$$

Expansion of this equation, keeping only terms of the lowest order, gives:

$$\widehat{zz}\frac{r^2}{a} = \widehat{rr}\left[\frac{3}{2}\frac{r^2}{a} - \frac{a}{2} - R\right] - r\frac{d\widehat{rr}}{dr}\left[R + \frac{1}{2}\frac{a^2 - r^2}{a}\right]$$
$$+ \widehat{\theta\theta}\left[R + \frac{1}{2}\frac{a^2 - r^2}{a}\right] \quad (19)$$

Into this equation involving the three stress components we may now substitute the particular relations given by our analysis above, namely, $\widehat{\theta\theta} = \widehat{rr}$, and $\widehat{zz} = \widehat{rr} + \widehat{zz}_a$. $\widehat{zz}_a$ is the flow stress at the external surface where the other two components of stress vanish, and is therefore the flow stress for an ordinary tension test at the particular elongation under homogeneous conditions. We replace the notation $\widehat{zz}_a$ by F, for flow stress. The equation (19) now becomes an equation for the single component $\widehat{rr}$:

$$\frac{d\widehat{rr}}{dr}\left[R + \frac{1}{2}\frac{a^2 - r^2}{a}\right] + \frac{r}{a}F = 0 \quad (20)$$

The variables are separated, and the equation may be at once integrated, giving, together with the boundary condition, the closed solution:

$$\left.\begin{aligned} \widehat{rr} &= F \log \frac{a^2 + 2aR - r^2}{2aR} \\ \widehat{zz} &= F\left[1 + \log \frac{a^2 + 2aR - r^2}{2aR}\right] \end{aligned}\right\} \tag{21}$$

The connection with the load is given by:

$$\text{Load} = \int_0^a 2\pi r \widehat{zz} dr = \pi F(a^2 + 2aR) \log \left(1 + \frac{1}{2}\frac{a}{R}\right)$$

Also

$$\widehat{zz}_{aver} = \text{Load}/\pi a^2 = F\left(1 + 2\frac{R}{a}\right) \log \left(1 + \frac{1}{2}\frac{a}{R}\right) \tag{22}$$

The factor $\left(1 + 2\frac{R}{a}\right) \log \left(1 + \frac{1}{2}\frac{a}{R}\right)$ should be divided into $\widehat{zz}_{aver}$ in order to obtain $\widehat{zz}_a$ or F. $\widehat{zz}_{aver}$ has been used hitherto in constructing strain hardening curves or stress-strain curves. It is F or $\widehat{zz}_a$ which should properly be used for such curves, for this is the flow stress at the given strain under conditions such that the other two principal components of stress vanish. $\widehat{zz}_{aver}$ is often referred to as the "true stress." In view of the magnitude of the correction factor, this now appears to be an unfortunate and confusing nomenclature. It is always confusing to propose a revision of accepted nomenclature, but it is sometimes necessary. I propose the following as perhaps leading to a minimum of confusion. Call $\widehat{zz}_{aver}$ the "uncorrected true stress" and F or $\widehat{zz}_a$ the "corrected true stress."

By substituting the explicit expression for $\widehat{rr}$ into equation (16) it is possible by successive differentiations to obtain an explicit expression for $\partial^2 zz/\partial z^2$, but there seems no particular interest in writing this out.

The procedure in the last step of the solution, from page 569 on, was suggested by the method used by Siebel. The nature of the approximations made by Siebel was such that it is difficult to estimate the degree of approximation to be expected of his final solution. He solved the problem for two dimensions, in rectangular co-ordinates x and z, the y co-ordinate running to infinity and not entering the solution. The Y_y component of stress was set equal to zero. Under these assumptions his solution was approximate because he effectively set the h and h′ of the analysis above equal to each other, this discarding a finite term, and he also effectively set $\widehat{zz}$ = const for a first integration. The resultant solution was finally applied to the cylindrical case merely by substituting r for x in the formula and integrating around the circle, and without any consideration of the effect of the stress component $\widehat{\theta\theta}$ which must of necessity appear when passing to cylindrical co-ordinates. Siebel's final formula was:

$$\widehat{zz}_{aver} = F\left(1 + \frac{1}{4}\frac{a}{R}\right)$$

Siebel's formula for the hydrostatic tension on the axis is a less good approximation than his formula for $\widehat{zz}_{aver}$.

DISCUSSION

Written Discussion: By R. W. Mebs, associate physicist in metallurgy, National Bureau of Standards, Washington, D. C.

Experimental evidence is far from convincing that the von Mises condition holds true for states of triaxial tension. It would require only a small deviation from von Mises assumption in order to cause an effective decrease in the flow stress at the center of the specimen.

The assumption that hardening is constant all the way across the neck, independent of hydrostatic tension, appears to be untenable. Micro-examination of polished specimens of fractured tensile bars, short distances removed from the fractures, reveals a decrease in strain hardening as one approaches the axis. The longitudinal stress-gradient across the specimen thus is largely replaced by a deformation gradient. The initiation of a "brittle", or better, "separation" fracture at the center of the specimen could be attributed, in part, to the smaller strain hardening at that point, as well as to greater hydrostatic tension.[4]

Oral Discussion

M. Gensamer:[5] This paper entered prominently into the cohesive strength symposium discussion Tuesday afternoon and perhaps it might be well to point out why it came into that discussion. There is a possibility that it could be used to help us in our understanding of what happens in notched bars. The elastic solution for notched bars tells us that the stresses are greatest at the root of the notch, that is, at the surface of the specimen. The plastic solution that Dr. Bridgman has given us tells us that the situation is just the reverse of this. The stresses are a maximum at the axis. When we test a notched bar, if it breaks without any elastic deformation, the stress is greatest on the outside. If it flows enough so that the deformation is uniform across the cross section as Dr. Bridgman has assumed, then we have the maximum stresses on the center line.

When the specimen deforms relatively little before it breaks so that the elastic stress distribution is not changed too much, we should have a combination of the two states of affairs; and it may be, as Dr. McAdam has contended,

[4]D. J. McAdam, Jr. and R. W. Mebs, "An Investigation of the Technical Cohesive Strength of Metals," American Institute of Mining and Metallurgical Engineers, Tech. Pub. No. 1615, *Metals Technology*, Aug. 1943.

[5]Professor of metallurgical engineering, Carnegie Institute of Technology, Pittsburgh.

that the stresses are not as nonuniformly distributed across the section of the specimen as we might think; certainly not as nonuniformly distributed as we would guess from the elastic solution. There is a fair chance that Dr. McAdam's conclusions and arguments based upon a relatively uniform stress distribution are not so bad.

I think Dr. Bridgman's contribution will not only enable us to correct stress-strain curves but it will help greatly in our understanding of the behavior of notched bars, which is a practical problem.

Author's Reply

The discussion of Mr. Mebs opens the question of the validity of various approximations made in the solution. It must of course be recognized that various approximations were made; the question is how good were these approximations, and whether it is better to apply the correction for nonuniformity of stress deduced on the basis of the approximations or to neglect the correction.

With regard to the failure of the von Mises condition under triaxial stress it is to be remarked that fortunately this question need not be considered because the actual solution shows a biaxial stress (two of the principal components equal) all the way across the section, and the von Mises condition is as applicable here as it is in the situations from which it is derived.

With regard to the assumption that hydrostatic tension does not affect hardening, numerical data were given in the body of the paper that indicate that the effect must be small. It is difficult to know what quantitative weight to attach to Mr. Mebs' statement that micro-examination of fractured tensile bars shows a decrease in strain hardening toward the axis, since no numerical values were given nor was the method of measurement indicated. If the method was, as may be suspected, a micro-hardness method then it is to be considered that the parallelism is by no means close between hardness and "strain hardening", as some unpublished experiments of my own show.

The most important deviations from exactness in our solution doubtless arise from strain hardening. But strain hardening does not proceed at a rapid enough rate to be a very important factor in my opinion. This may be illustrated by a numerical example. Consider a representative steel which fractures at a natural strain of unity, at which the ratio a/R also has approximately the value unity. For the tensile stress at fracture on the periphery of the neck we take a representative value 225,000 pounds per square inch. The analysis of the paper shows that under these conditions the tensile stress on the axis, derived on the assumption of uniform strain, is 1.4 times greater, or 315,000 p.s.i. Let us now suppose that the stress is uniform all the way across, the difference of stress between axis and periphery which would naturally arise from the curvature of the contour being compensated by an enhanced stress at the periphery due to greater strain there and the accompanying greater strain hardening. The strain hardening at the periphery must be sufficient to raise the stress from 225,000 to 315,000. Representative experimental values for the strain hardening of this steel indicate an increase of flow stress of 90,000 p.s.i. for an increase of strain of unity. This means that the natural strain at the periphery would have to be greater by unity than that at the center, or the elongation at the periphery 2.7 times greater than that at the center. If one were to make a rough free-hand graphical construction of the lines of flow which might give rise to this large difference of strain, I think it will be plain that any such difference of strain between axis and periphery cannot be entertained. Although the mathematical problem has not been exactly solved I believe that on the basis of geometrical experiments like this one can convince oneself that the approximation is not a bad one and that it is much better to apply the correction than to neglect it.

With regard to notched bar experiments brought up by Dr. Gensamer the assumption of uniform strain is doubtless less applicable, but I believe that nevertheless the deviation of the stress from uniformity across the section at the notch must be qualitatively like that found above for necked bars and quantitatively probably exaggerated in comparison with values ordinarily attained in necked specimens because of the very large value of a/R. This it seems to me may have an important bearing on the interpretation of notched bar experiments.