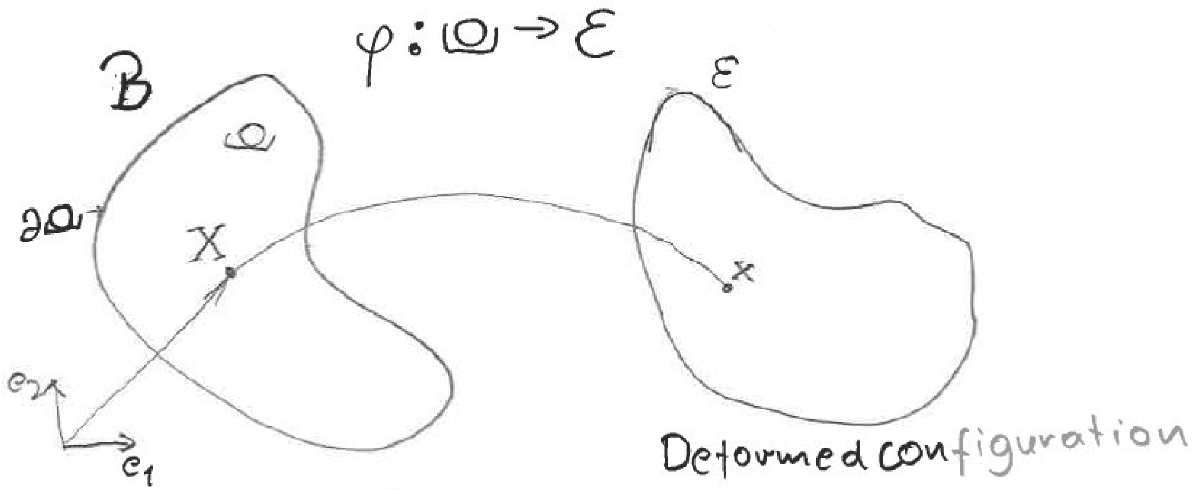
Finite Strains and Hyper-elasticity Brief Introduction

July 10th 2019

Literature

1. Non-Linear Elastic Deformations, R.W. Ogden
2. Computational Methods for Plasticity, de Souza Neto, D. Perić, DRJ Owen
3. Introduction to the Mechanics of a Continuous Medium, L.E. Malvern

Material and Spatial Configurations
Deformation Gradient



Reference configuration

$\varphi \to$ is a smooth one to one function that maps each material point X into a point $x = \varphi(X)$

Displacement of X is defined: $u(X) = \varphi(X) - X$

Special cases:

$\varphi(X) = X + u$ Rigid translation

$\varphi(X) = q + R(X - q)$ Rigid rotation about q;

R is a proper orthogonal tensor

Deformation Gradient: What Is It?

Deformation gradient is the second order tensor F defined by:

$$\underline{F(X) = \nabla_X \varphi(X) = \frac{\partial x}{\partial X}}$$

if you choose to remember just one equation from this class, rember this one. BB

Gradient Definition:

gradient of scalar ϕ:

$$\frac{d\phi}{ds} = \nabla\phi \cdot \vec{u}$$

where $\vec{u} = \frac{d\vec{P}}{ds}$ (unit vector)

Note: 1. In the definition above, we calculate gradient of a vector field

2. Gradient of a vector field is a tensor

From the definition:

$$\underline{dx = F\,dX}$$

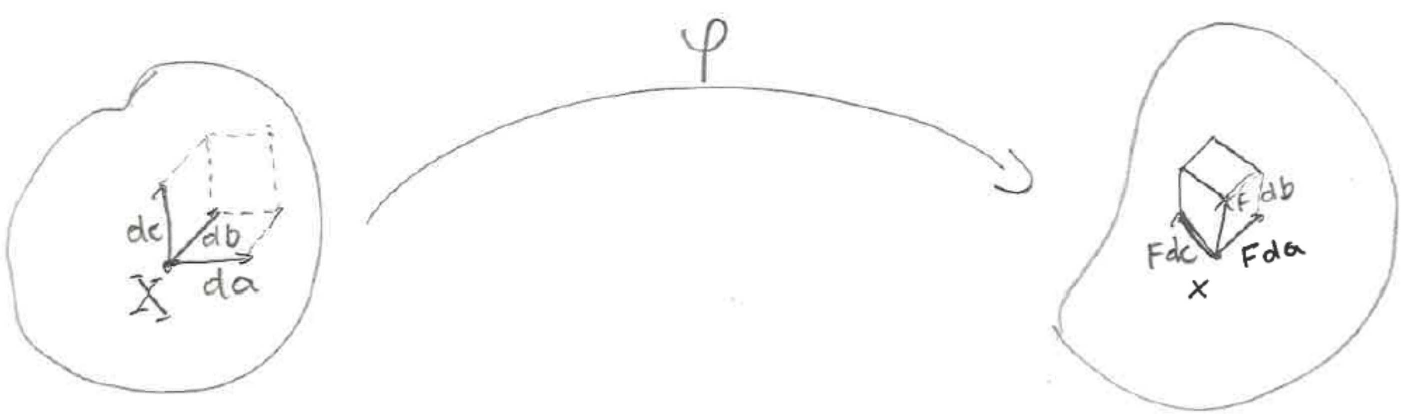
deformation gradient is the linear operator that relates dX (vector) into dx (vector). This is a definition of second order tensor!

Volume Change

Let us start with:

$$\underline{dx = F\, dX}$$

this equation defines how "fiber" element is deformed (remember it can include rotations and translations)



$$dV_0 = (d\vec{a} \times d\vec{b}) \cdot d\vec{c}$$

$$dV = (F d\vec{a} \times F d\vec{b}) \cdot F d\vec{c}$$

$$J = \det F = \frac{dV}{dV_0}$$ (see next page)

$$\underline{J\, dV_0 = dV}$$

Special cases:

$J = 1$ - isochoric deformation

$F = \alpha I$ - volumetric def.

Volume Change
Exercise 1

Show that for <u>any</u> tensor T and set $\vec{u}, \vec{v}, \vec{w}$ of <u>linearly independent</u> vectors:

$$\det T = \frac{(T\vec{u} \times T\vec{v}) \cdot T\vec{w}}{(\vec{u} \times \vec{v}) \cdot \vec{w}} \qquad (1)$$

Write (1) in index notation.

Polar Decomposition

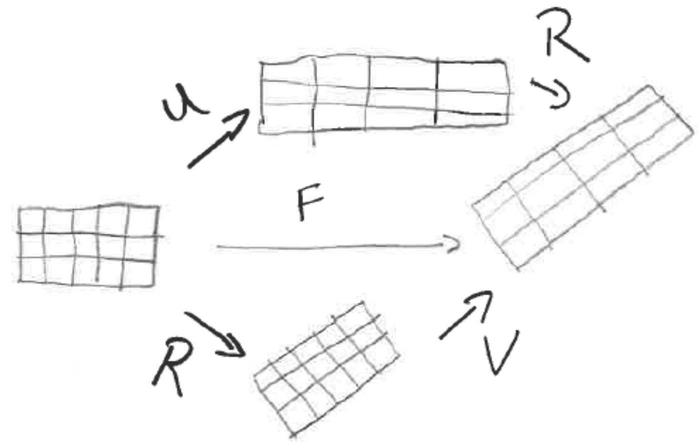
Polar Decomposition of Def. Gradient F:

$$F = RU = VR$$

R - <u>Local</u> rotation tensor

U - right stretch tensor

V - Left stretch tensor



$$U = \sqrt{C} \qquad V = \sqrt{B}$$

$$C = F^T F \qquad B = F F^T$$

Right Cauchy-Green strain tensor   Left Cauchy-Green strain tensor

Polar Decomposition: Principal Stretches
Exercise 2

Assume the following deformation:

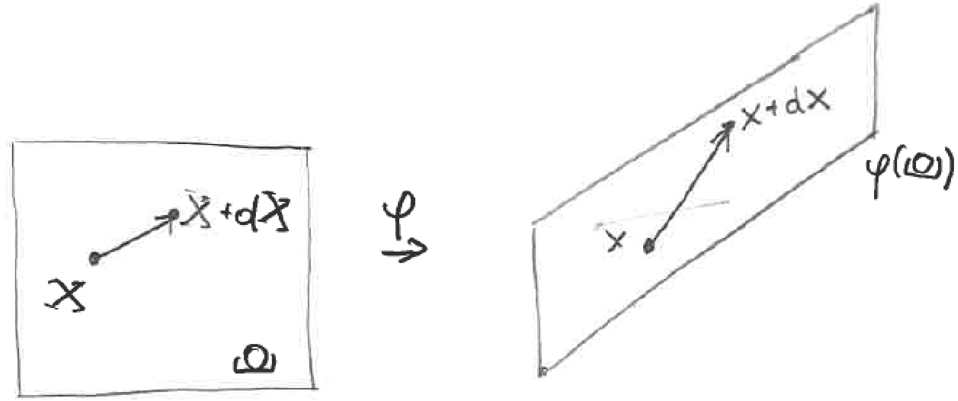
$$\varphi: \begin{cases} x_1 = X_1 \lambda_1 \cos\alpha - X_2 \lambda_2 \sin\alpha \\ x_2 = X_1 \lambda_1 \sin\alpha + X_2 \lambda_2 \cos\alpha \end{cases}$$

Calculate F, R, U, V.

Strain Measures

$\|x\|^2 = F\,dX \cdot F\,dX$

Deformed Length of the material "fibre"



$\|x\|^2 = C\,dX\,dX$

$= (I + 2E^{(2)})\,dX\,dX$

$E^{(2)} = \frac{1}{2}(C - I)$ ← Green Lagrange strain

In general, Lagrangian strain tensors are defined:

$$E^{(m)} = \begin{cases} \frac{1}{m}(U^m - I) & m \neq 0 \\ \ln(U) & m = 0 \end{cases}$$

Similar method can be used to define Eulerian strain tensors with respect to V.

$m=1$ ← Biot strain

$m=0$ ← Logarithmic strain

Stress Measures

Cauchy stress (true stress) : $\vec{t}(\vec{x}, \vec{n}) = \sigma(\vec{x})\,\vec{n}$ ← Cauchy theorem (momentum balance)

$\vec{t}$ - surface force

$\vec{n}$ - area element normal

σ - second order tensor (Cauchy stress)

Note that $\vec{t}$ is the force exerted across a material surface per unit deformed area.

$\vec{\underline{t}}$ → force that acts across any surface whose normal is $\vec{n}$ in the deformed configuration per unit reference area

$$\vec{\underline{t}} = \frac{da}{da_0}\vec{t}$$

$$P = J\sigma F^{-T}$$ ← Piola-Kirchhoff stress (nominal stress)

$J = \det F$

Green Elasticity (Hyperelasticity)

In hyperelasticity, there exists a strain-energy function defined on the space of deformation gradients:

$$P = \frac{\partial W}{\partial F} \quad \leftarrow \text{nominal stress}$$

$$\sigma = J^{-1} \frac{\partial W}{\partial F} F^T \quad \leftarrow \text{Cauchy stress}$$

$W(F) \leftarrow$ strain energy function (stored energy function)

Strain Energy Function
Invariant Formulation

Isotropic tensor-valued function of a tensor in 3D can be represented if the function is istropic:

$$Y(X) = \alpha_0 I + \alpha_1 X + \alpha_2 X^2$$

where α_i are scalar valued functions of the principal invariants of X.

Using an assumption that the material is isotropic $F = FQ$ it can be shown:

$$W(F) = \tilde{W}(B)$$

$$\bar{W} = \tilde{W}(I_0(B), I_2(B), I_3(B))$$

representation of energy function by principal invariants I_1, I_2, I_3

Strain Energy Function
Principal Stretches Formulation

Similarily, stress energy density function can be represented using principal stretches:

$$W(F) = \hat{W}(\lambda_1, \lambda_2, \lambda_3)$$

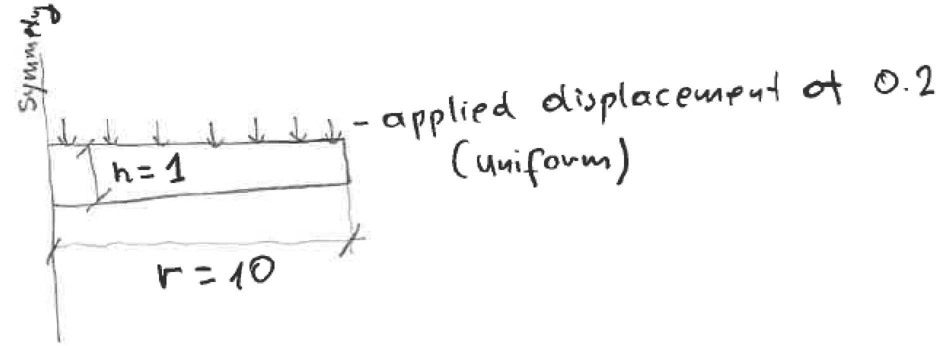
λ_i – principal stretches

Strain Energy Function
Exercise 3

Calculate Cauchy stress if W is represented using invariant and principal stretches.

Homework 1

Use Abaqus FE package, to model unidirectional stress compression of a thin disk ($r = 10$, $h = 1$). Apply uniform displacement to the top surface $\Delta h = 0.2$.



Use hyperelastic Mooney-Rivlin material with:

a) $C10 = 10$, $C01 = 1$, assume incompressibility

b) $C10 = 10$, $C01 = 1$, $D = 0.01$

Calculate principal stretches in a disk after deformation and lateral displacement.