EFFICIENT STRING INSTRUMENT SYNTHESIS

Maarten van Walstijn

Aims

- To introduce Digital Waveguide Modelling (DWG) approach to simulation of strings.

- To understand relationship with Finite Difference Schemes.

- To understand the functioning of all filter blocks involved in the DWG model.

- To understand how the SDL is constructed as an equivalent model.

- To understand the individual filter blocks and how they depend on the physical and numerical parameters.
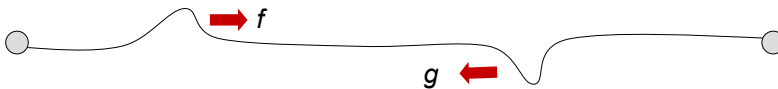
Travelling Waves

Wave equation:

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$
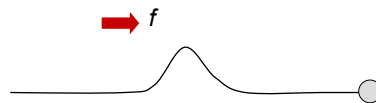
Travelling wave solution (d'Alembert, 1747):

$$y(x,t) = f(t - x/c) + g(t + x/c)$$
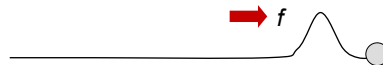$$= y^+(x,t) + y^-(x,t)$$

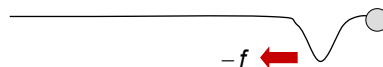## Ideal String: Reflection at Ends
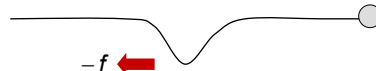
forward-travelling wave

wave travels towards end

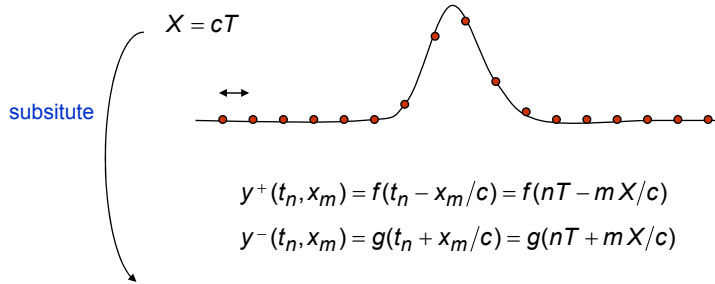wave is reflected with inverted amplitude (reflection $R = -1$)

wave travels backwards towards other end

In general, reflections at boundaries depend on boundary conditions (clamped, supported, free etc). R can be frequency-dependent.

Sampling the Travelling Wave

In one time step T, the wave travels one step X in space:

$$X = cT$$

subsitute

$$y^+(t_n, x_m) = f(t_n - x_m/c) = f(nT - mX/c)$$
$$y^-(t_n, x_m) = g(t_n + x_m/c) = g(nT + mX/c)$$

switch to different notation

$$y^+(t_n, x_m) = f([n-m]T) \rightarrow f(n-m)$$
$$y^-(t_n, x_m) = g([n+m]T) \rightarrow g(n+m)$$

$$y_m^n = \left(y^+\right)_m^n + \left(y^-\right)_m^n = f(n-m) + g(n+m)$$

Digital Waveguide: Bi-Directional Delay-Line

$$y_m^n = \left(y^+\right)_m^n + \left(y^-\right)_m^n = f(n-m) + g(n+m)$$

T $\equiv$ one sample period delay

delayed by m samples

$f(n)$ $f(n-m)$

end reflection

-1

$+ \rightarrow y_m^n$

-1

end reflection

$g(n)$ $g(n+m)$

delayed by m samples
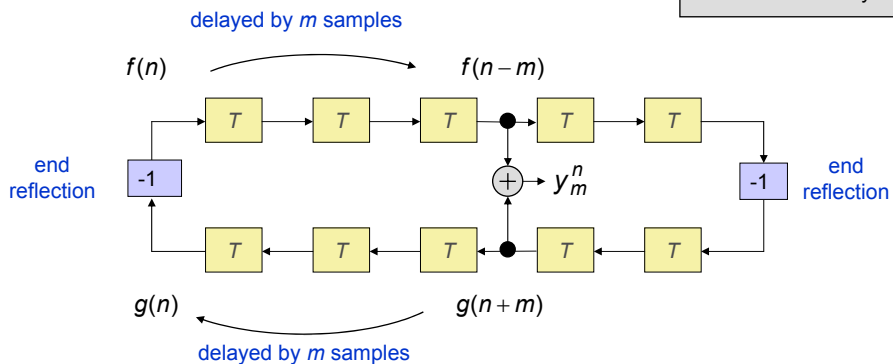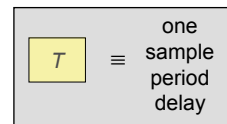
Relationship to Finite Difference Schemes

The difference equation for ideal string, using centered-difference scheme:

$$y_m^{n+1} = \underbrace{a_{10}}_{2-2\lambda^2} y_m^n + \underbrace{a_{11}}_{\lambda^2}\left(y_{m+1}^n + y_{m-1}^n\right) + \underbrace{a_{20}}_{-1} y_m^{n-1}$$

If we choose $\lambda^2 = (cT/X)^2 = 1$ then we have a simpler equation!

$$y_m^{n+1} = y_{m+1}^n + y_{m-1}^n - y_m^{n-1}$$

If we use decomposition into travelling waves, this turns out to be exactly equivalent to:

$$y_m^{n+1} = \left(y^+\right)_m^{n+1} + \left(y^-\right)_m^{n+1} = f((n+1)-m) + b((n+1)+m)$$

In other words, the digital waveguide model is equivalent to a finite difference scheme on the stability bound!

Alternative Wave Variables

Displacement waves (y^+, y^-)

Velocity waves $(\dot{y}^+, \dot{y}^-)$

Accelaration waves $(\ddot{y}^+, \ddot{y}^-)$

Slope waves (y_x^+, y_x^-)

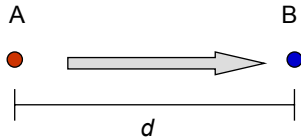| Any **temporal derivative** |
| --- |
| (nth derivative indicated with n dot) |
| or **spatial derivative** |
| (nth derivative indicated with n x subscripts) |
| is possible as wave variable. |

Force waves (F^+, F^-)

$$F = F^+ + F^- = R\dot{y}^+ - R\dot{y}^- = R(\dot{y}_+ - \dot{y}_-) \qquad (R = \sqrt{T_e\,\sigma})$$

Strings:

Velocity waves is a good choice for strings because they are directly related to force through the string wave impedance (R).
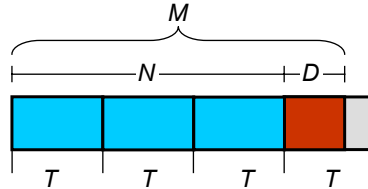
Fractional Delays

A   B

travel a distance d from (A) to (B):

propagation time (delay) is: $\tau_d = \dfrac{d}{c}$

d

The delay can be digitally realised with a delay-line of length $M = \dfrac{\tau_d}{T}$

PROBLEM: Number of delays M to implement the delay is normally not an integer! Solution: split up delay-time into an **integer** and a **fractional** part:

M

$\longleftarrow N \longrightarrow \; \vdash D \dashv$

$T \qquad T \qquad T \qquad T$

$M = t_d\, f_s = N + D$

$N = floor(M)$ $\longleftarrow$ integer number of delays $\longleftarrow$ **delay-line**

$D = M - N$ $\longleftarrow$ fractional number ($0<D<1$) $\longleftarrow$ **filter**

Fractional Delay Filters

Fractional Delay (FD) Filters approximate a non-integer delay. One option is a **first-order Thiran all-pass filter:**

$$H_{fd}(z) = \frac{a + z^{-1}}{1 + az^{-1}}, \quad a = \frac{1-D}{1+D}$$

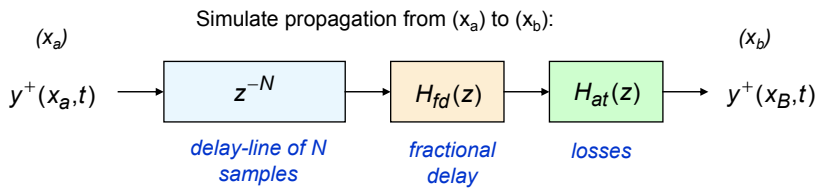**phase delay** expresses how much different frequency components are delayed by the filter

(x_a)   Simulate propagation from (x_a) to (x_b):   *(x_b)*

$y^+(x_a,t) \longrightarrow$ z^{-N} $\longrightarrow$ $H_{fd}(z)$ $\longrightarrow$ $y^+(x_B,t)$

delay-line of N samples   *fractional delay*
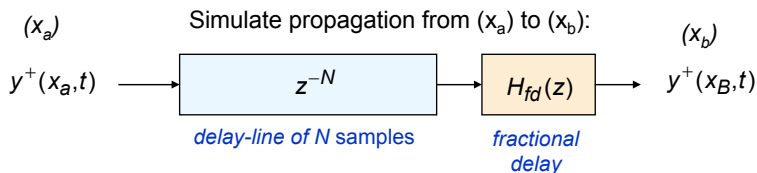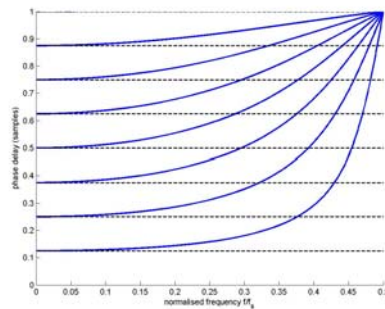
Attenuation Filters

b_1 air damping

b_2 internal friction damping

$\alpha(\omega_n) = b_1 + b_2 k_n^2$

$\alpha(\omega) \approx b_1 + b_2 (\omega/c)^2$

$H_{at}(\omega) = e^{-\alpha(\omega)d}$

Approximate with first-order filter:
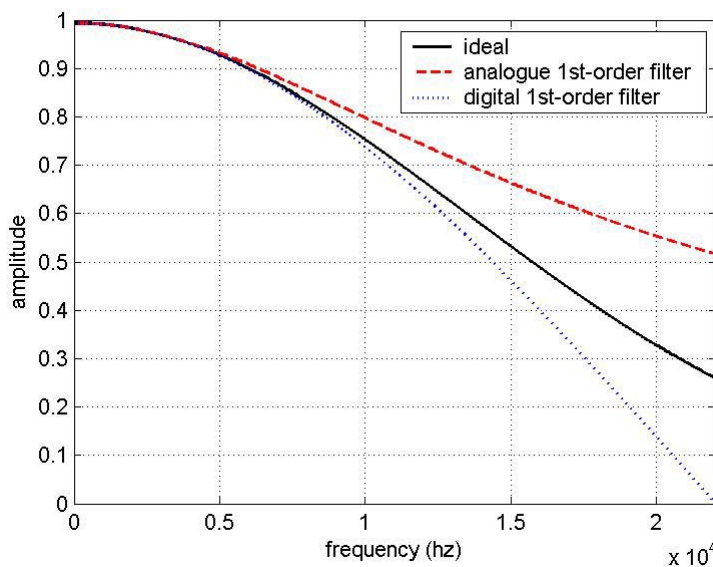
$$H_{at}(s) = \frac{g\beta}{s + \beta}$$

Bilinear transform

$$H_{at}(z) = \frac{b_0 + b_1 z^{-1}}{1 + a_1 z^{-1}}$$

Filter 'found' by matching analogue filter to ideal response at $\omega = 0$ and $\omega = \omega_1$

Simulate propagation from (x_a) to (x_b):

(x_a)

(x_b)

$y^+(x_a,t) \longrightarrow$ | z^{-N} | $\rightarrow$ | $H_{fd}(z)$ | $\rightarrow$ | $H_{at}(z)$ | $\rightarrow y^+(x_B,t)$

delay-line of N samples   *fractional delay*   *losses*

Attenuation Filters (cont.)

Summary of Modelling Wave Propagation

simulates propagation delay t_p simulates propagation losses

$$y^+(x_a,t) \longrightarrow \boxed{z^{-N}} \longrightarrow \boxed{H_{fd}(z)} \longrightarrow \boxed{H_{at}(z)} \longrightarrow y^+(x_B,t)$$

delay of N samples   delay of D samples
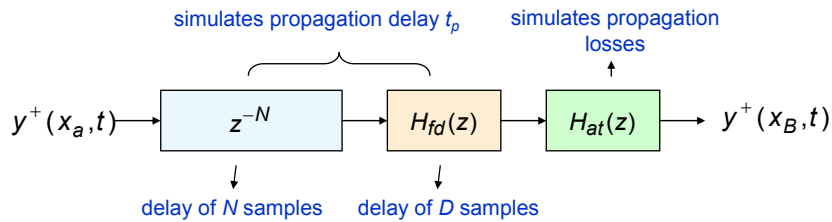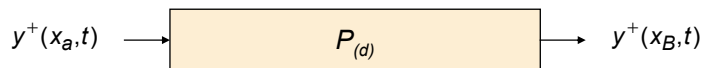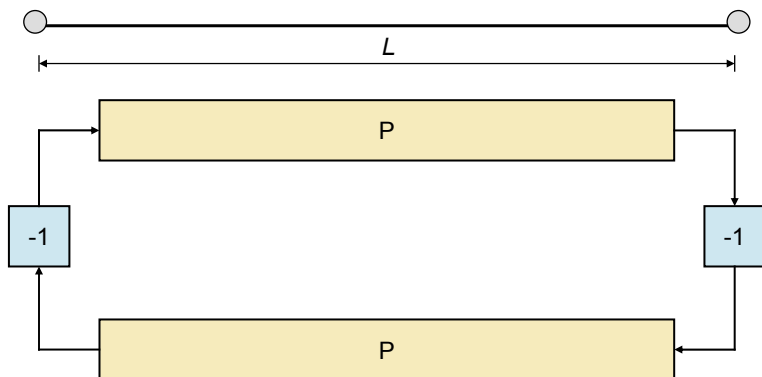
Implements propagation over distance: $d = MX = McT = c\underbrace{(NT + DT)}_{\tau_d}$

Let's depict one 'propagation block' with:

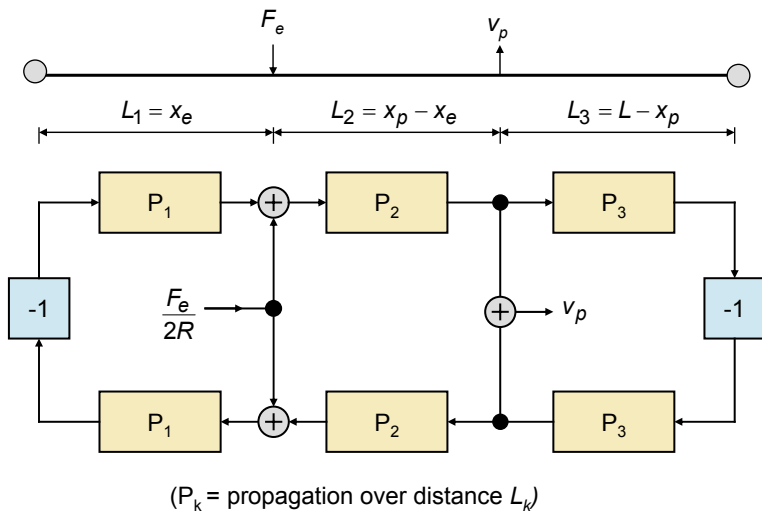$$y^+(x_a,t) \longrightarrow \boxed{P_{(d)}} \longrightarrow y^+(x_B,t)$$

Digital Waveguide Model of an Ideal String Formulated with Velocity Waves (no input, no output)

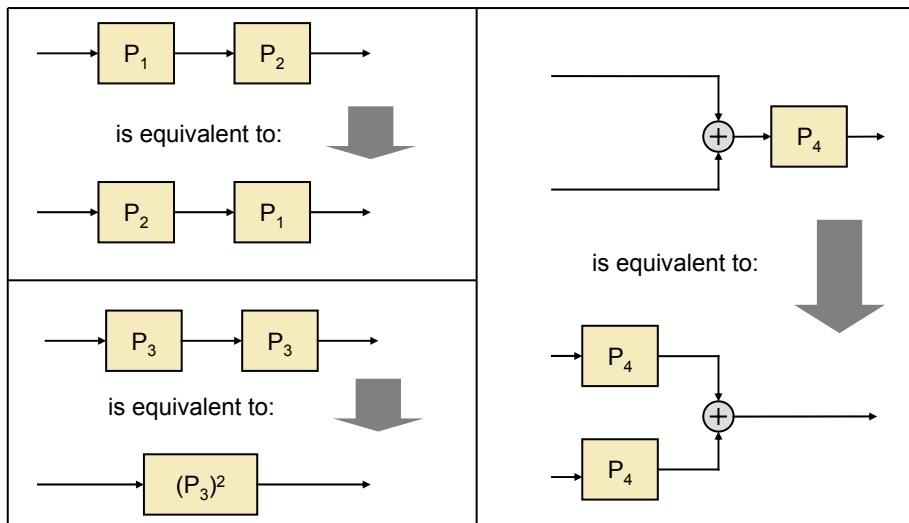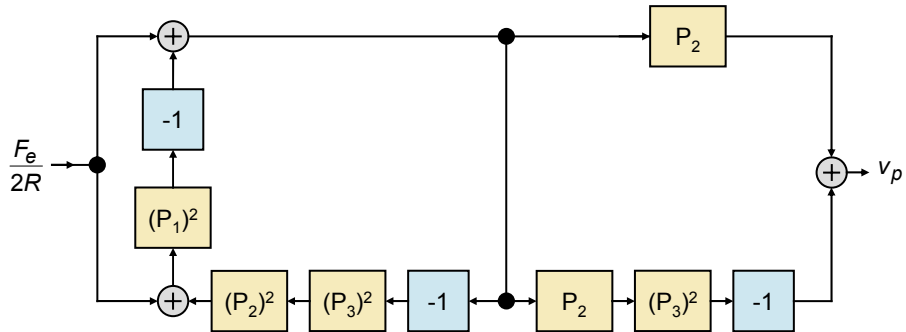

(P = propagation over distance L)

Digital Waveguide Model of an Ideal String Struck at $x=x_e$ and "Picked Up" at $x=x_p$
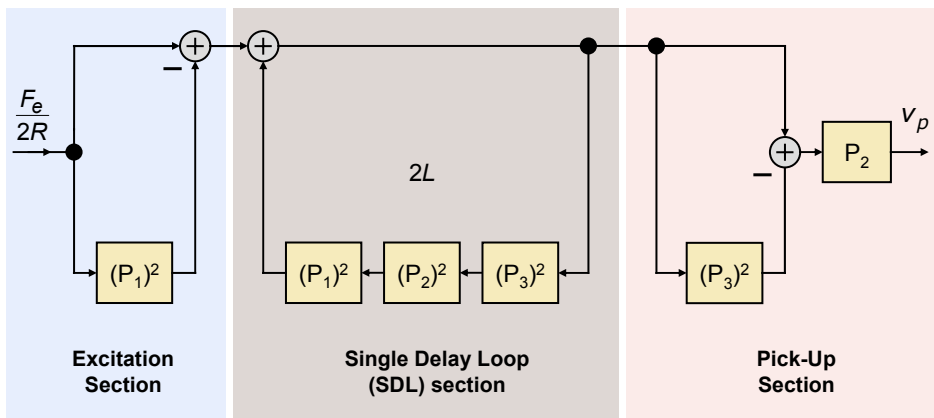

F_e v_p

$L_1 = x_e$ $L_2 = x_p - x_e$ $L_3 = L - x_p$

P_1 + P_2 P_3

-1 $\dfrac{F_e}{2R}$ + $\rightarrow v_p$ -1

P_1 + P_2 P_3

(P_k = propagation over distance L_k)

Equivalent Linear Block Structures

$P_1 \rightarrow P_2$

is equivalent to:

$P_2 \rightarrow P_1$

$P_3 \rightarrow P_3$

is equivalent to:

$(P_3)^2$

+ $\rightarrow P_4$

is equivalent to:

P_4

+

P_4

DW Equivalent Structure (1)

DW Equivalent Structure (2)
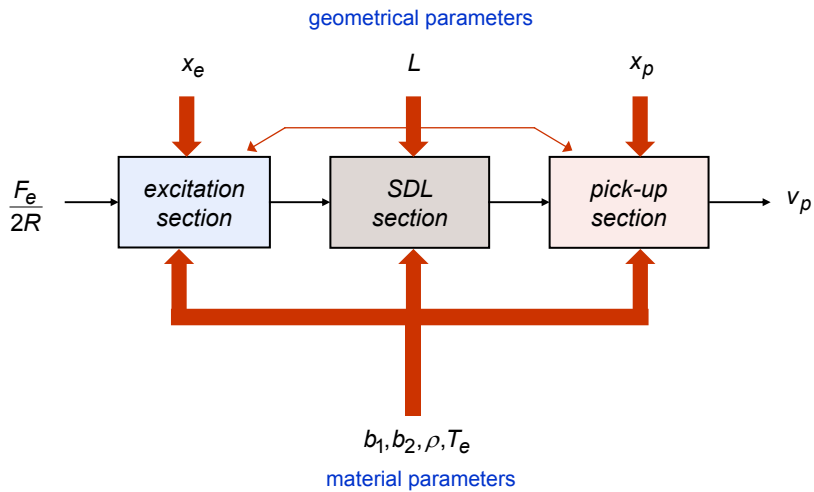
Excitation Section   Single Delay Loop (SDL) section   Pick-Up Section

Single Delay Loop (SDL) Model

geometrical parameters

$$x_e \qquad L \qquad x_p$$

$$\frac{F_e}{2R} \rightarrow$$ excitation section | SDL section | pick-up section $\rightarrow v_p$

$$b_1, b_2, \rho, T_e$$

material parameters
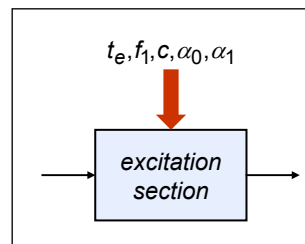

Excitation Section

$$P = (P_1)^2$$

P

P models the propagation of a wave over a string length $2x_e$:

travelling time $\qquad t_p = \dfrac{2x_e}{c}$

losses over distance $\qquad 2x_e$

$$\alpha_0 = b_1$$
$$\alpha_1 = b_1 + b_2(\omega_1/c)^2$$

$$t_e, f_1, c, \alpha_0, \alpha_1$$

excitation section

Pick-Up Section: Feed-Forward Section

$P = (P_3)^2$

Remove simple delay that does not effect the sound much
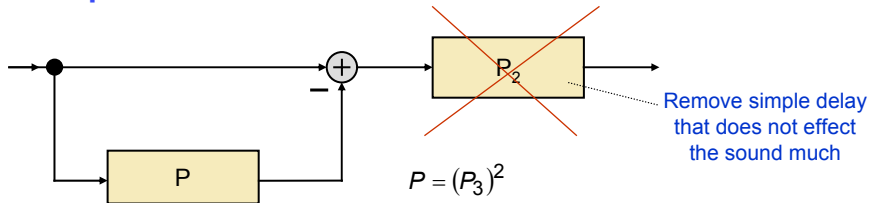
P models the propagation of a wave over a string length $2x_e$:
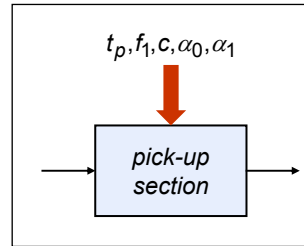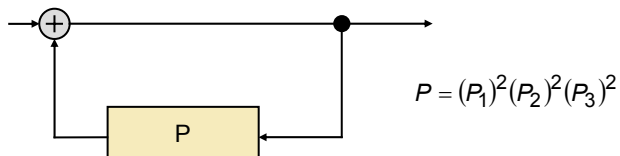
travelling time $\quad t_p = \dfrac{2(L - x_p)}{c}$

losses over distance $\quad 2(L - x_p)$

$\alpha_0 = b_1$

$\alpha_1 = b_1 + b_2(\omega_1/c)^2$

$t_p, f_1, c, \alpha_0, \alpha_1$

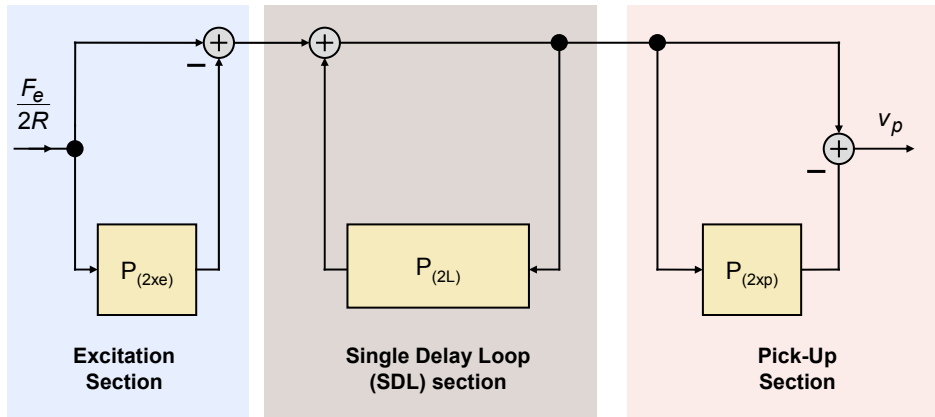pick-up section

---

Single Delay Loop (SDL) Section

$P = (P_1)^2(P_2)^2(P_3)^2$

P models the propagation of a wave over a string length $2L$:

travelling time $\quad t_{sdl} = \dfrac{2L}{c}$

losses over distance $\quad 2L$

$\alpha_0 = b_1$

$\alpha_1 = b_1 + b_2(\omega_1/c)^2$

$t_{sdl}, f_1, c, \alpha_0, \alpha_1$

SDL section

Final SDL Model

Excitation Section — $P_{(2xe)}$

Single Delay Loop (SDL) section — $P_{(2L)}$

Pick-Up Section — $P_{(2xp)}$

$\frac{F_e}{2R}$

v_p
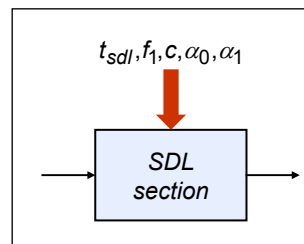
SDL & Finite Difference Scheme Sound Examples

FINITE DIFFERENCES

SDL

Created with sdl.m available
on the course website

Conclusions/Remarks (1)

- Vibrations governed by the wave equation can be simulated very efficiently using a digital waveguide.

- A basic digital waveguide (DWG) consists of two delay-lines that implement lossless forward- and backward wave propagation.

- Filters need to be inserted in order to model fractional delays and to incorporate propagation losses.

- The DWG model with one excitation and one pick-up point can be re-arranged into an SDL model that consist of three "filter blocks".

- Added bonus of the DWG/SDL string model is that it is stable for all passive boundary conditions.