Robert M. Hackett

Hyperelasticity Primer

Second Edition

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To Tricia, my always encouraging dear wife
and best friend, and all our offspring

Foreword: Nonlinear Elasticity

A fundamental property of elastic materials is that the stress depends only on the current level of the strain. This implies that the loading and unloading stress–strain curves are identical and that the original shape is recovered upon unloading. In this case, the strains are said to be *reversible*. If the stress state in a material is independent of the history of the material point, the behavior of the material is said to be *path independent*. In a purely mechanical theory, reversibility and path independence also imply the absence of energy *dissipation* in the process of deformation. In other words, in an elastic material, deformation is not accompanied by any dissipation of energy and all energy expended in deformation is stored in the body and can be recovered upon unloading. The existence of a potential implies reversibility, path independence, and absence of dissipation in the deformation process. Thus, the implied close relationship of path independence, reversibility, and non-dissipative behavior leads to the degree of path independence being viewed as a measure of the elasticity of a material model. It should also be noted that triaxial *nonlinear elasticity* is fundamentally a shear deformation phenomenon, since volumetric strain is linear in nature.

In finite strain elasticity, many different constitutive relationships have been developed for multiaxial elasticity. Also, the same constitutive relationship can be written in several different ways. The generalization of finite strain elasticity to multiaxial large strains is a formidable mathematical problem that has been addressed by some of the keenest minds in the twentieth century and still encompasses open questions.

The widely accepted finite strain elasticity formulations can be classified into three formats—*algebraic*, *integral*, and *differential*. The most notable formulation of the algebraic format is *Cauchy elasticity*, characterized by $\boldsymbol{\sigma} = \mathbf{f}(\boldsymbol{\varepsilon})$ and limited to rate- and history-independent material behavior. In Cauchy elasticity, the stress is path independent but the energy is not. Cauchy elasticity has a non-conservative structure, i.e., the stress is not derivable from a scalar potential function.

Within the integral format, *Green elasticity*, or *hyperelasticity* (**hyper:** over, above, beyond, super), is of primary consideration. It dates back to the original work of George Green (1793–1841) and is characterized by $\boldsymbol{\sigma} = \partial W/\partial \boldsymbol{\varepsilon}$. For hyperelasticity, the response is fully path independent and reversible and the stress is derived from a strain (or stored) energy potential.

The differential format nests the *hypoelastic* (**hypo**: below, beneath, under) material models, designated as such by Clifford Truesdell (1919–2000). Hypoelastic material laws relate the rate of stress to the rate of deformation, as characterized by $d\boldsymbol{\sigma} = E_t : d\boldsymbol{\varepsilon}$ or by $\dot{\boldsymbol{\sigma}} = \mathbf{g}(\boldsymbol{\sigma}, \mathbf{d})$ (**d** being the rate-of-deformation tensor). Hypoelasticity is used to model materials that exhibit nonlinear, but reversible, stress–strain behavior even at small strains. Hypoelastic models are, in general, strictly path independent only in the one-dimensional case. If the elastic strains are small, the behavior is close enough to path independence to model elastic behavior. Because of the simplicity of hypoelastic laws, a multiaxial generalization is often used in finite element software to model elastic response of materials in large-strain elastic-plastic problems. Hypoelastic models are the most weakly path independent, followed by Cauchy elasticity.

Therefore, generally speaking, hyperelastic material models enjoy the widest range of applicability in modeling polymeric or rubbery elastic response.

Preface to the First Edition

One of the definitions of primer (**prim′·er**) is that it is a textbook that gives the first principles of any subject (Webster, 1962). In light of the nature of the material that follows, it is thus appropriate to refer to this as a primer. The reason for writing this primer is to provide a vehicle for engineers who may need to apply this technology but are not well-versed in the underlying theory, nor aware of some of the limitations associated with the application.

The material covered herein was explored during the period July 1, 2005 through December 31, 2012, during which time I was engaged as a consultant by ERC, Incorporated in Huntsville, Alabama to perform services as directed by the Weapons Development and Integration (WDI) Directorate of the Aviation and Missile Research, Development and Engineering Center (AMRDEC), U.S. Army Research, Development and Engineering Command at Redstone Arsenal, Alabama. A large portion of the services I performed related directly to the formulation of models to simulate the response and behavior of solid propellant material systems.

The subjects presented cover a sufficiently broad range of topics that define hyperelasticity. The material is, by and large, introductory in nature, but probes the subjects in some depth in most cases. It is developed to be understood and useable by engineers who understand and are able to employ the principles of mechanics. An important aspect of the presented material is that a number of illustrative numerical examples are included. This usually serves well to soften the bluntness of new, to the reader, and challenging theoretical material. The coverage focuses on the topics of *finite elasticity*, *strain-energy functions*, *polar decomposition*, *strain measures*, *stress measures*, *tangent moduli*, *conjugate pairs*, *incrementation*, *objectivity*, *finite viscoelasticity*, and *finite element implementation*. In addition to these topics, some emphasis is placed on obtaining *model parameters from test data*. Most emphasis is placed on, and coverage given to, the topic of incrementation, since this is how we all solve mechanics problems today.

Jeremy R. Rice, an employee and a Team Leader in the WDI Directorate and a Ph.D. candidate at the University of Alabama in Huntsville (UAH) at the time that I worked there, made innumerable invaluable suggestions relative to the

development of this monograph. Robert R. Little, Chief of Missile Sustainment of the WDI Directorate, made available the funding that supported my work, for which I am especially grateful. Numerous discussions with Q.H. Ken Zuo, Associate Professor in the Department of Mechanical and Aerospace Engineering (MAE) at UAH, were also extremely beneficial.

Special recognition and appreciation is extended to Simone Lanza Calvert, ERC, Incorporated Program Manager, for her valuable guidance and support during the time that I was involved in that effort/activity. I am also grateful to Melanie D. Williams for expediting the compensation and expense support I received while employed by ERC, Incorporated.

Deep appreciation is expressed to Michael Luby, Senior Editor, Physical Sciences & Engineering, who has provided exceptional direction, support and encouragement from the very beginning of the undertaking to publish this monograph, and to Brinda Megasyamalan, Project Co-ordinator, Production Editor (Books), who has continuously coordinated the project activities, providing most helpful information and feedback on the process at every step.

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Preface to the Second Edition

The primary reason for writing the second edition of *Hyperelasticity Primer* was to include extensive coverage on the modeling of *soft biological tissue*, which was only alluded to in the first edition—all topics covered in the first edition are still covered in the second edition. Since the author's background is heavily weighted toward the study and research of polymeric materials, the emphasis in the first edition was in that direction. However, there is no doubt that the relatively recent emergence and awareness of the important role played by the mechanical modeling of soft biological tissue in this life science in the medical field, and the fact that hyperelastic models can play a highly important role in that activity, call attention to the fact that this topic should be included, and emphasized, in a monograph of this nature. Hopefully, this might result in some diagnostic benefit at some level within the medical field for those many who strive toward the ultimate eradication of cancer. Because of this topical coverage, considerable information related to obtaining *model parameters from test data* is added in Chap. 15.

Other reasons are to increase the coverage of *nearly incompressible* materials—two related example problems are also included; to develop a much more extensive Index than was included in the first edition—it has been pointed out that the most useful textbooks are the ones with broad and inclusive indexes, facilitating the tracking of any and all key terms; and to incorporate corrections, mostly typographical, made to the first edition.

Special appreciation is expressed to Michael Luby, Senior Publishing Editor, Physical Sciences and Engineering, and to Nicole Lowary, Assistant Editor, Physical Sciences and Engineering, for their exceptional direction, assistance, and encouragement throughout this project/effort. Sincere appreciation is also expressed to Ms. Gayathri Sharma, Production Editor (Books) for Springer Nature, and Ms. Hemachandrane Sarumathi, Sr. Project Manager, Content Solutions, at SPi Global for their extremely valuable services in bringing this manuscript to fruition.

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Notation

$\underline{\underline{A}}$ (or $\mathbf{A}$) is a second-order tensor.

$\underline{\underline{A}}^{-1}$ $\left(\text{or } \mathbf{A}^{-1}\right)$ is the inverse of $\underline{\underline{A}}$ (or $\mathbf{A}$).

$\underline{\underline{A}}^{T}$ (or $\mathbf{A}^{T}$) is the transpose of $\underline{\underline{A}}$ (or $\mathbf{A}$).

$\underline{\underline{A}}^{-T} = \left(\underline{\underline{A}}^{T}\right)^{-1} = \left(\underline{\underline{A}}^{-1}\right)^{T}$ and $\mathbf{A}^{-T} = \left(\mathbf{A}^{T}\right)^{-1} = \left(\mathbf{A}^{-1}\right)^{T}$

tr $\underline{\underline{A}}$ is the trace of $\underline{\underline{A}}$.

det $\underline{\underline{A}}$ is the determinant of $\underline{\underline{A}}$.

$\dot{\underline{\underline{A}}}$ is the derivative of $\underline{\underline{A}}$ with respect to time.

$\underline{\underline{A}}\,\underline{\underline{B}}^{T} = A_{ik}B_{jk}$

$\underline{\underline{\underline{\underline{C}}}}$ is a fourth-order tensor.

D_{iJ} is a two-point tensor.
D_{Ji} is the inverse of D_{iJ}.
$\underline{\underline{A}} = \underline{\underline{\underline{\underline{C}}}} : \underline{\underline{B}} \equiv A_{ij} = C_{ijkl}B_{kl}$

$\underline{\underline{\underline{\underline{\underline{\underline{A}}}}}}$ is a sixth-order tensor.

$\underline{\underline{I}}$ (or $\mathbf{1}$) is the second-order identity tensor.

$\underline{\underline{\underline{\underline{I}}}}$ (or $\mathbf{I}$) is the fourth-order identity tensor.

δ_{ij} is the Kronecker delta.
$\hat{\varepsilon}_{ijk}$ is the permutation symbol or alternator.
$\underline{\nabla}$ is the del operator.
$\nabla \cdot \mathbf{v}$ is the divergence of $\mathbf{v}$.
$\nabla \times \mathbf{v}$ is the curl of $\mathbf{v}$.
$\mathbf{u} \otimes \mathbf{v}$ is the tensor product of $\mathbf{u}$ and $\mathbf{v}$.

$$\mathbf{u} \cdot \mathbf{v} (\text{or } \mathbf{u}\mathbf{v}) = \sum_{i=1}^{3} u_i v_i$$

$$(\mathbf{u} \otimes \mathbf{v})\mathbf{w} \equiv (\mathbf{v} \cdot \mathbf{w})\mathbf{u}$$

$$\mathrm{tr}(\mathbf{u} \otimes \mathbf{v}) = (\mathbf{u} \cdot \mathbf{v})$$

Introduction

The purpose of this monograph is to provide an introduction to the subject of finite (hyper)elasticity for engineers, physicists, and mathematicians, and even some physicians, who are interested in studying the behavior of materials that may undergo large strains. An elastic material is defined by a constitutive relation giving the stress as a function of current deformation. Finite elasticity is a theory of elastic materials capable of undergoing large deformations. In the case of these materials, linear elastic theory coupled with that of infinitesimal deformations is inadequate to characterize mechanical or thermomechanical response. This theory is inherently nonlinear and is, in reality, far more difficult than most theories of mathematical physics.

Hyperelasticity, although not a new realm of study, has been saturated, and enhanced, with theory and concepts, some of which were developed as recently as 25 years ago. The phenomenological theory treats the problem from the viewpoint of continuum mechanics. This approach constructs a mathematical framework to characterize rubbery behavior so that stress analysis and strain analysis problems may be solved without reference to microscopic structure or molecular concepts. The subject material is formidable and not readily grasped by most engineers and scientists who have no formal educational background in the subject of continuum mechanics.

The spectrum of materials, the behavior of which can be classified as being hyperelastic, includes vulcanized rubber, solid propellant, polymeric foams, and biological soft materials. The design of rocket motors has for the most part in the past been based upon the employment of the principles of linear elasticity and ad hoc modeling, which attempts to recognize, and deal with, the inherent nonlinearity of highly complex composite solid propellant material. It has long been recognized that principles of nonlinear mechanics should be applied to the analysis of this material. It is also widely recognized, in more recent years, that it is vitally important that the same analytical principles and technology be applied in the study and characterization of biological soft materials and systems. This recognition has fostered in recent years an effort to establish modeling approaches based upon the theory of finite deformation and of hyperelastic behavior. Support and extension of this effort requires that those involved be well grounded in the fundamental aspects of nonlinear mechanics as applied to finite deformation and hyperelasticity.

This monograph is by no means exhaustive but attempts to introduce the subject matter in such a way as to facilitate the learning of it, while minimizing the inherent difficulties of mastering it. This is aided and abetted by the inclusion of numerous examples to illustrate the theory presented. Obviously, this provides only a beginning but, hopefully, directs the inquisitive reader to seek to build onto the basic information provided. All of the coverage is limited to the consideration of isotropic material systems, except where orthotropic material systems are considered in Chap. 13, which deals with finite viscoelasticity, and in Chap. 15 where biological soft materials are focused upon to some degree. Additionally, important aspects of solid propellant nonlinear behavior include viscoelasticity and internal damage; these must be included in any meaningful modeling of this material response, but because this monograph is introductory in nature and focuses on hyperelasticity per se, internal damage is not included in the presented material. Also, much information on applying the same principles to the modeling of plasticity is found in the pertinent literature.

It is also obvious that some specialized background is necessary for one to attempt to follow the theory that is unfolded. One should have significant knowledge of the concepts of linear elasticity coupled with small strain theory. One should also have an understanding of matrix operations and, most importantly, of mathematical tensors and the rules governing operations with them. The reader should have a degree of mastery of tensors and tensor equations. In short, the background requirements are, in a sense, somewhat strict.

With regard to tensor notation, some explanation is probably in order. Tensor operations are carried out within this text usually using indicial notation, for example, noting first- and second-order tensors as e_i and A_{ij}, respectively. These tensors might also be expressed as $\mathbf{e}$ and $\mathbf{A}$, respectively, depending upon the application, appropriateness, etc. It is important to note that herein these same type tensors may be written as $\underline{e}$ and $\underline{\underline{A}}$, respectively. Consequently, a fourth-order tensor would be written as $\underline{\underline{\underline{\underline{D}}}}$. We also write the second-order identity tensor as $\underline{\underline{I}}$, or as $\mathbf{1}$, and the fourth-order identity tensor as $\underline{\underline{\underline{\underline{I}}}}$, or as $\mathbf{I}$. Of course, in indicial notation the second-order identity symbol is the *Kronecker* delta, δ_{ij}.

The author first asked a class of graduate students to distinguish between *hyperelasticity* and *hypoelasticity* in a course at the University of Alabama in Huntsville (UAH) in 1984. We must, to begin with, make sure that we understand what the term *elasticity* actually means. A fundamental property of elastic materials is that the stress depends only on the current level of the strain. This implies that the loading and unloading stress–strain curves are identical and that the original shape is recovered upon unloading. In this case the strains are said to be *reversible*. If the stress state in a material is independent of the history of the material point, the behavior of the material is said to be *path independent*. In a purely mechanical theory, reversibility and path independence also imply the absence of energy *dissipation* in the process of deformation. In other words, in an elastic material, deformation is not accompanied by any dissipation of energy and all energy expended in deformation is stored in the body and can be recovered upon unloading. The existence of a potential implies reversibility, path independence, and absence of dissipation in the deformation process. Thus, the implied close relationship of path independence, reversibility, and nondissipative behavior leads to the degree of path independence being viewed as a measure of the elasticity of a material model. It should also be noted that triaxial *nonlinear elasticity* is fundamentally a shear deformation phenomenon, since volumetric strain is linear in nature.

In finite strain elasticity, many different constitutive relationships have been developed for multiaxial elasticity. Also, the same constitutive relationship can be written in several different ways. The generalization of finite strain elasticity to multiaxial large strains is a formidable mathematical problem that has been addressed by some of the keenest minds in the twentieth century and still encompasses open questions.

The widely accepted finite strain elasticity formulations can be classified into three formats—*algebraic*, *integral*, and *differential*. The most notable formulation of the algebraic format is *Cauchy elasticity*, characterized by $\boldsymbol{\sigma} = \mathbf{f}(\boldsymbol{\varepsilon})$ and limited to rate- and history-independent material behavior. In Cauchy elasticity, the stress is path independent but the energy is not. Cauchy elasticity has a nonconservative structure, i.e., the stress is not derivable from a scalar potential function.

Within the integral format, *Green elasticity*, or *hyperelasticity* (**hyper-:** over, above, beyond, super), is of primary consideration. It dates back to the original work of George Green (1793–1841) and is characterized by $\boldsymbol{\sigma} = \partial W/\partial \boldsymbol{\varepsilon}$. For hyperelasticity, the response is fully path independent and reversible and the stress is derived from a strain (or stored) energy potential.

The differential format nests the *hypoelastic* (**hypo-:** below, beneath, under) material models, designated as such by Clifford Truesdell (1919–2000). Hypoelastic material laws relate the rate of stress to the rate of deformation, as characterized by $d\boldsymbol{\sigma} = E_t : d\boldsymbol{\varepsilon}$, or by $\dot{\boldsymbol{\sigma}} = \mathbf{g}(\boldsymbol{\sigma}, \mathbf{d})$ ($\mathbf{d}$ being the rate-of-deformation tensor). Hypoelasticity is used to model materials that exhibit nonlinear, but reversible, stress–strain behavior even at small strains. Hypoelasticity is defined by equations that are based upon objective time rate tensors, which, unlike those of hyperelasticity, require incrementally objective solution formulations.

Hypoelastic models are, in general, strictly path independent only in the one-dimensional case. If the elastic strains are small, the behavior is close enough to path independence to model elastic behavior. Because of the simplicity of hypoelastic laws, a multiaxial generalization is often used in finite element software to model elastic response of materials in large-strain elastic-plastic problems. Hypoelastic models are the most weakly path independent, followed by Cauchy elasticity.

Therefore, generally speaking, hyperelastic material models enjoy the widest range of applicability in modeling problems that fall within the regime of finite strain elasticity.

Chapter 1
Finite Elasticity

Abstract Definitions abound as the stage is set for the introduction and engagement of the terminology that underlies the foundation of hyperelasticity. The introduction of a strain-energy (or stored-energy) function into elasticity is due to George Green (1793–1841) and elastic solids for which such a function is assumed to exist are said to be Green elastic or hyperelastic. Elasticity without an underlying strain-energy function is called Cauchy elasticity. A formal definition is: hyperelasticity is the finite strain encompassing constitutive theory which describes the mechanical behavior of elastic solids with the use of one material function. The formulation of finite strain elasticity is considered with uncoupled, volumetric/deviatoric response and is based on the multiplicative decomposition of the deformation gradient. Additive decomposition, although formally valid, loses its physical content in the nonlinear theory. The volume-preserving, or isochoric, part of the deformation gradient is referred to as the distortion gradient, while the Jacobian determinant defines the volume change. Because the deformation gradient provides a complete description of homogeneous local deformations, it is considered to be the primitive measure of deformation.

The introduction of the strain-energy (or stored-energy) function into elasticity is due to George Green (1793–1841) and elastic solids for which such a function is assumed to exist are said to be *Green elastic* or *hyperelastic*. A formal definition is given by Drozdov (1996): the constitutive theory which describes the mechanical behavior of elastic solids with the use of (only) one material function is called *hyperelasticity*. Hyperelastic materials are truly elastic in the sense that if a load is applied to such a material and then removed, the material returns to its original shape without any dissipation of energy in the process. In other words, a hyperelastic material stores energy during loading and releases exactly the same amount of energy during unloading. There is no path dependence. Elasticity without an underlying strain-energy function is called *Cauchy elasticity*.

Through vector addition, we can directly write the relationship between the position vectors in the initial and deformed configurations of a deformable body:

$$x_i = X_I + u_i \qquad i = 1, 2, 3, \quad I = 1, 2, 3 \tag{1.1}$$

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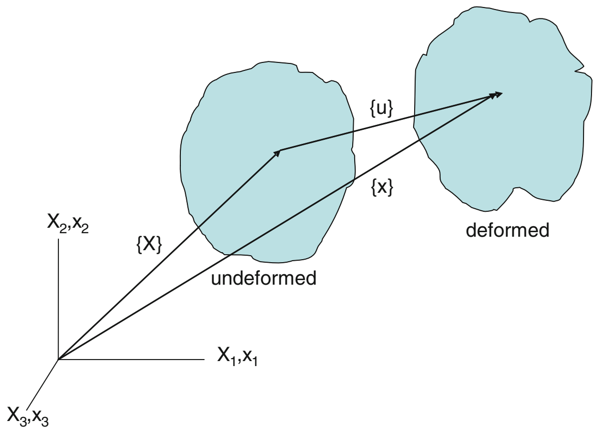

Fig. 1.1 Three-dimensional vectorial representation of Lagrangian and Eulerian systems

Then, we consider vectors that describe material orientation in each configuration, as shown in Fig. 1.1. These vectors essentially describe the orientation of an infinitesimal piece of material in the body. We denote these vectors as $\mathbf{X}$, in the reference configuration, and $\mathbf{x}$, in the deformed configuration. Employing the chain rule, we can directly write a mapping between the material orientation vector in the initial and in the deformed configuration as

$$dx_i = \frac{\partial x_i}{\partial X_J} dX_J \tag{1.2}$$

This equation gives a relationship between a material vector in the undeformed and the deformed configurations. We define the mapping as

$$F_{iJ} \equiv \nabla_0 \mathbf{x} \equiv x_{i,J} \equiv \frac{\partial x_i}{\partial X_J} \tag{1.3}$$

where F_{iJ} is the *deformation gradient*. The components of F_{iJ} are such that the first index refers to an Eulerian (current) and the second to a Lagrangian (reference) basis. If we take the derivative of Equation (1.1), which relates the initial and deformed configurations, we obtain

$$\frac{\partial x_i}{\partial X_J} = \frac{\partial X_I}{\partial X_J} + \frac{\partial u_i}{\partial X_J} \tag{1.4}$$

Thus, we define the deformation gradient tensor as the Kronecker delta plus a displacement gradient:

$$F_{iJ} = \delta_{IJ} + \frac{\partial u_i}{\partial X_J} \tag{1.5}$$

We can consider the formulation of finite strain elasticity with uncoupled, volumetric/deviatoric response. Additive decomposition, although formally valid, loses its physical content in the *nonlinear theory*. The correct split of the volumetric/deviatoric response takes the following form. The volume-preserving, or *isochoric*, part of F_{iJ} is referred to as the *distortion gradient* $\overline{F}_{iJ}$ and is given by Simo and Hughes (1998),

$$\overline{F}_{iJ} = J^{-1/3} F_{iJ} \quad \text{or} \quad \overline{\mathbf{F}} = J^{-1/3}\mathbf{F} \tag{1.6}$$

where the Jacobian determinant

$$J = \det\ F_{iJ} \tag{1.7}$$

defines the volume change. Accordingly, $\det \overline{F}_{iJ} \equiv 1$. Because F_{iJ} provides a complete description of homogeneous local deformations, it is natural to consider F_{iJ} as the primitive measure of deformation. We also note that the volume change is, by and large, linear in nature.

It is an interesting side note that this decomposition is possible due to a more general property of tensors and their determinants. Any tensor $\mathbf{A}$ of rank n has the following property:

$$\overline{\mathbf{A}} = \Theta^{-\frac{1}{n}}\mathbf{A} \tag{1.8}$$

where

$$\Theta = \det\ \mathbf{A} \tag{1.9}$$

Then,

$$\det\ \overline{\mathbf{A}} \equiv 1 \tag{1.10}$$

Second-order tensors defined through contractions with a Lagrangian vector and an Eulerian vector, such as the deformation gradient F_{iJ}, belong to a class of *two-point tensors* called Eulerian-Lagrangian, since the contraction is with an Eulerian vector on the left and a Lagrangian vector on the right. The inverse of the deformation gradient, F_{Ji}, is correspondingly a Lagrangian-Eulerian two-point tensor.

Chapter 2
Strain Measures

Abstract Strain measures for hyperelastic materials must model the effect of finite deformations. They are single-based second-order tensors, either Eulerian or Lagrangian, and are defined in terms of the Cauchy-Green deformation tensors, which are derived from the deformation gradient. The Green-Lagrange strain tensor is Lagrangian based, while the Almansi strain tensor is Eulerian based. Both of these strain measures are described in detail. The Green-Lagrange strain tensor is in terms of the right Cauchy-Green deformation tensor, while the Almansi strain tensor is in terms of the left Cauchy-Green deformation tensor. The reduced invariants of the right and left Cauchy-Green deformation tensors, known as the invariants of the right and left Cauchy-Green distortion tensors, are introduced, and the derivation of the reduced invariants is presented and defined. Since the strain measures are derived from the deformation gradient, they are related, the relationship is easily demonstrated. An additional strain measure, one which is less commonly employed, is the Biot strain tensor. The different strain measures can be formally reduced to those of linear elastic systems, this being demonstrated.

The important *right Cauchy-Green deformation tensor*, or the *Green deformation tensor*, is defined by

$$C_{IJ} = \frac{\partial x_k}{\partial X_I}\frac{\partial x_k}{\partial X_J} = F_{kI}F_{kJ} \quad \text{or} \quad \mathbf{C} = \mathbf{F}^T\mathbf{F} \tag{2.1}$$

or, in terms of the *distortion gradient*,

$$C_{IJ} = J^{2/3}\overline{F}_{kI}\overline{F}_{kJ} \quad \text{or} \quad \mathbf{C} = J^{2/3}\overline{\mathbf{F}}^T\overline{\mathbf{F}} \tag{2.2}$$

Then, with

$$\overline{C}_{IJ} = \overline{F}_{kI}\overline{F}_{kJ} \quad \text{or} \quad \overline{\mathbf{C}} = \overline{\mathbf{F}}^T\overline{\mathbf{F}} \tag{2.3}$$

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we also have the relationship

$$\overline{C}_{IJ} = J^{-2/3} C_{IJ} \quad \text{or} \quad \overline{\mathbf{C}} = J^{-2/3} \mathbf{C} \tag{2.4}$$

The inverse of C_{IJ}, C_{IJ}^{-1}, is called the *Piola deformation tensor*.

The also important *left Cauchy-Green deformation tensor*, which is also sometimes called the *Finger tensor*, is defined by

$$b_{ij} = \frac{\partial x_i}{\partial X_K} \frac{\partial x_j}{\partial X_K} = F_{iK} F_{jK} \quad \text{or} \quad \mathbf{b} = \mathbf{F}\mathbf{F}^T \tag{2.5a}$$

where, also

$$\overline{b}_{ij} = \overline{F}_{iK} \overline{F}_{jK} \quad \text{or} \quad \overline{\mathbf{b}} = \overline{\mathbf{F}}\,\overline{\mathbf{F}}^T \tag{2.5b}$$

The inverse of b_{ij}, b_{ij}^{-1}, is called the *Cauchy deformation tensor*.

We can also note here that

$$\overline{I}_1 = \overline{b}_{ii} = \overline{C}_{II} \quad \text{or} \quad \overline{I}_1 = \operatorname{tr} \overline{\mathbf{b}} = \operatorname{tr} \overline{\mathbf{C}} \tag{2.6a}$$

$$\overline{I}_2 = \frac{1}{2}\left(\overline{I}_1^2 - \overline{I}_2'\right) \tag{2.6b}$$

where

$$\overline{I}_2' = \overline{b}_{ik} \overline{b}_{ki} = \overline{C}_{IK} \overline{C}_{KI} \quad \text{or} \quad \overline{I}_2' = \operatorname{tr}\left(\overline{\mathbf{b}}\,\overline{\mathbf{b}}\right) = \operatorname{tr}\left(\overline{\mathbf{C}}\,\overline{\mathbf{C}}\right) \tag{2.6c}$$

and

$$\overline{I}_3 = \det \overline{b}_{ij} = \det \overline{C}_{IJ} \quad \text{or} \quad \overline{I}_3 = \det \overline{\mathbf{b}} = \det \overline{\mathbf{C}} \tag{2.6d}$$

where $\bar{I}_1, \bar{I}_2, \bar{I}_3$ are the conventionally defined principal invariants of the right and left Cauchy-Green distortion, or reduced, tensors. Sometimes the alternative definition $\overline{I}_2 = \overline{I}_2'$ is used (Bonet and Wood 2008).

We can write the *Green*, or *Green-Lagrange*, *strain tensor* components as

$$E_{IJ} = \frac{1}{2}(F_{kI} u_{k,J} + F_{kJ} u_{k,I}) \tag{2.7a}$$

or

$$E_{IJ} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_J} + \frac{\partial u_j}{\partial X_I} + \frac{\partial u_k}{\partial X_I} \frac{\partial u_k}{\partial X_J}\right) \tag{2.7b}$$

or, considering Equation (2.1), as

$$E_{IJ} = \frac{1}{2}(C_{IJ} - \delta_{IJ}) \quad \text{or} \quad \mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{1}) \tag{2.8}$$

We can note that by dropping the third term on the right-hand side of Equation (2.7b) we have, significantly, the standard small strain relationship for linear elasticity which can also be written as

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left(\mathbf{F} + \mathbf{F}^T\right) - \mathbf{1} \tag{2.9}$$

The *Eulerian strain tensor*, or *Almansi strain tensor* (Almansi 1911), is defined in terms of the left Cauchy-Green deformation tensor, as

$$e_{ij} = \frac{1}{2}\left(\delta_{ij} - b_{ij}^{-1}\right) \tag{2.10a}$$

or, in terms of displacement as

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right) \tag{2.10b}$$

We note that the two defined strain states are related through the expressions

$$E_{IJ} = F_{kI} e_{kl} F_{lJ} \quad \text{or} \quad \mathbf{E} = \mathbf{F}^T \mathbf{e} \mathbf{F} \tag{2.11a}$$

and

$$e_{ij} = F_{Ki} E_{KL} F_{Lj} \quad \text{or} \quad \mathbf{e} = \mathbf{F}^{-T} \mathbf{E} \mathbf{F}^{-1} \tag{2.11b}$$

The Green-Lagrange and Almansi strains are the two classical strain measures.

Now, let us consider *deviatoric* strain. We can write for the deviatoric Green-Lagrange strain

$$\overline{E}_{IJ} = \frac{1}{2}\left(\overline{F}_{kI}\overline{F}_{kJ} - \delta_{IJ}\right) \tag{2.12}$$

and, for the deviatoric Almansi strain,

$$\overline{e}_{ij} = \frac{1}{2}\left(\delta_{ij} - \overline{F}_{Ki}\overline{F}_{Kj}\right) \tag{2.13}$$

and, defining

$$\overline{B}_{iJ} = \overline{F}_{Ji}^{T} \quad \text{or} \quad \overline{\mathbf{B}} = \overline{\mathbf{F}}^{-T} \tag{2.14}$$

where $\overline{F}_{Ji}^{T}$ is the transpose of the inverse of the distortion gradient tensor, we can also write for the deviatoric Almansi strain,

$$\overline{e}_{ij} = \frac{1}{2}\left(\delta_{ij} - \overline{B}_{iK}\overline{B}_{jK}\right) \tag{2.15}$$

Another strain measure, one which is not too commonly employed, is the *Biot strain tensor* (Biot 1939). It is defined as

$$\overline{U}_{IJ} = U_{IJ} - \delta_{IJ} \quad \text{or} \quad \overline{\mathbf{U}} = \mathbf{U} - \mathbf{1} \tag{2.16}$$

U_{IJ} being the *right stretch tensor* which is obtained from *polar decomposition*, as shown in the next chapter.

Chapter 3
Polar Decomposition

Abstract Within the framework of hyperelastic materials, the polar decomposition theorem of Trusdell and Noll* occupies a position of primary importance. The polar decomposition theorem states that any deformation gradient tensor can be multiplicatively decomposed into the product of an orthogonal tensor, known as the rotation tensor, and a symmetric tensor called the right stretch tensor, or into the product of a symmetric tensor called the left stretch tensor and the same rotation tensor. The rotation tensor is a two-point Eulerian-Lagrangian second-order tensor, the right stretch tensor is a Lagrangian single-based second-order tensor and the left stretch tensor is single-order and Eulerian based. The relationship between right and left stretch tensors is derived through eigenvalue extraction. The polar decomposition theorem is the principal tool for studying finite deformations and the corresponding strains. The detailed formulation of the polar decomposition theorem is presented in terms of the right Cauchy-Green deformation tensor. A detailed numerical example is presented to demonstrate the polar decomposition theorem.

Within the framework of *hyperelastic* materials, let us now consider the *polar decomposition theorem* (Truesdell and Noll 1965) wherein we can write

$$F_{iJ} = R_{iK} U_{KJ} \quad \text{or} \quad \mathbf{F} = \mathbf{RU} \tag{3.1}$$

The polar decomposition theorem states that any deformation gradient tensor F_{iJ} can be multiplicatively decomposed into the product of an orthogonal tensor R_{iJ}, known as the *rotation tensor*, and a symmetric tensor U_{IJ} called the *right stretch tensor*. This is graphically demonstrated in Fig. 3.1.

We can also define a different decomposition of the deformation gradient tensor,

$$F_{iJ} = v_{ik} R_{kJ} \quad \text{or} \quad \mathbf{F} = \mathbf{vR} \tag{3.2}$$

*The author, as a graduate student at Carnegie Mellon University in 1964, took the course *Tensor Analysis* taught by Professor Walter Noll.

R. M. Hackett, *Hyperelasticity Primer*,
https://doi.org/10.1007/978-3-319-73201-5_3

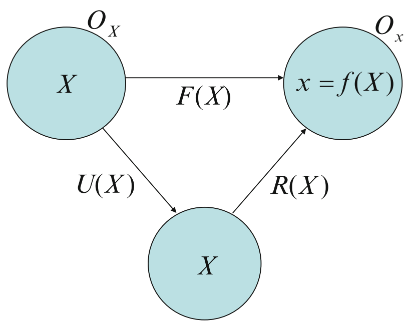

Fig. 3.1 A polar decomposition

where v_{ij} is called the *left stretch tensor*. This form of the polar decomposition is employed less frequently than the one defined in Equation (3.1). We can now examine some properties of the stretch and rotation tensors. First, we note that

$$\mathbf{R}^{-1} = \mathbf{R}^T \tag{3.3a}$$

and

$$\mathbf{U} = \mathbf{U}^T \tag{3.3b}$$

We are also able to determine that

$$U_{IJ}^2 = F_{kI}^T F_{kJ} = C_{IJ} \quad \text{or} \quad \mathbf{U}^2 = \mathbf{F}^T\mathbf{F} = \mathbf{C} \tag{3.4a}$$

$$v_{ij}^2 = F_{iK} F_{jK}^T = b_{ij} \quad \text{or} \quad \mathbf{v}^2 = \mathbf{F}\mathbf{F}^T = \mathbf{b} \tag{3.4b}$$

$$U_{IJ} = R_{kI}^T v_{kl} R_{lJ} \quad \text{or} \quad \mathbf{U} = \mathbf{R}^T\mathbf{v}\mathbf{R} \tag{3.4c}$$

and

$$v_{ij} = R_{iK} U_{KL} R_{jL}^T \quad \text{or} \quad \mathbf{v} = \mathbf{R}\mathbf{U}\mathbf{R}^T \tag{3.4d}$$

Equations (3.4c and 3.4d) show that the eigenvalues of **U** and **v** are identical. We can also relate the right and left Cauchy-Green deformation tensors through the expression

$$b_{ij} = R_{iJ} C_{JK} R_{jK}^T \tag{3.5}$$

There exists a proper orthogonal tensor (say **A**) which transforms **C** into a diagonal form (Chung 1988):

$$\mathbf{U}^2 = \mathbf{C} = \mathbf{A}\ \ diag\left[\lambda_1^2, \lambda_2^2, \lambda_3^2\right]\mathbf{A}^T \tag{3.6}$$

where λ_i^2 represents the eigenvalues of $\mathbf{U}^2$ and $\mathbf{A} = \left[\mathbf{n}^{(i)}\right]$, with each $\mathbf{n}^{(i)}$ being the eigenvector for one eigenvalue of $\mathbf{U}^2$. Thus

$$\mathbf{U} = \mathbf{A}\ \ diag[\lambda_1, \lambda_2, \lambda_3]\mathbf{A}^T \tag{3.7}$$

where $\lambda_1, \lambda_2, \lambda_3$ are the *principal stretches*.

Any significant study of deformation tensors must include a reference to *observer transformations*. Observer motion consists of translation and rotation, hence it can only modify those kinematical tensors which also relate to translation and rotation. The principle of *objectivity* is therefore explored in some detail in Chap. 12.

The polar decomposition theorem is the principal tool for studying finite deformations. By way of polar decomposition the Green-Lagrange strain tensor is defined by

$$\underline{\underline{E}} = \frac{1}{2}\left(\underline{\underline{U}}^2 - \underline{\underline{I}}\right) \tag{3.8}$$

Generalizing on the Green-Lagrange strain formulation, we can write, from Equation (3.8), the Seth-Hill strain measure (Seth 1964):

$$\underline{\underline{\varepsilon}} = \frac{1}{\kappa}\left(\underline{\underline{U}}^\kappa - \underline{\underline{I}}\right) \tag{3.9}$$

where some commonly employed strain measures correspond to different choices of the parameter κ. For example, for $\kappa = 1$, $\underline{\underline{\varepsilon}}$ is the *Biot strain tensor*, and for $\kappa \to 0$, $\underline{\underline{\varepsilon}} = \ln \underline{\underline{U}}$, the *logarithmic*, or *Hencky*, *strain tensor*. For the uniaxial case we would write Equation (3.9) as

$$\varepsilon = \frac{1}{\kappa}(\lambda^\kappa - 1) \tag{3.10}$$

where the stretch $\lambda = \frac{l}{L}$; for $\kappa = -1$, ε is the *true strain*; for $\kappa \to 0$, ε is the *logarithmic strain*; for $\kappa = 1$, ε is the *engineering strain*; and for $\kappa = 2$, ε is the Green-Lagrange strain.

We can quite easily derive the expression for uniaxial logarithmic strain. Given

$$d\varepsilon = \frac{dl}{l} \tag{3.11}$$

we can integrate and obtain

$$\int d\varepsilon = \int_L^l \frac{dl}{l} = \ln\ l\Big|_L^l = \ln\ l - \ln\ L = \ln\left(\frac{l}{L}\right) = \ln\ \lambda \tag{3.12}$$

which is the logarithmic strain. We can also write the expression for engineering strain,

$$\varepsilon_e = (\lambda - 1) = \frac{l - L}{L} \tag{3.13}$$

and note that

$$\ln(\varepsilon_e + 1) = \ln\left(\frac{l}{L}\right) \tag{3.14}$$

We can also note that the uniaxial true strain ε_{true} is given by the expression

$$\varepsilon_{true} = -\left(\lambda^{-1} - 1\right) = \left(1 - \frac{L}{l}\right) = \frac{l - L}{l} \tag{3.15}$$

Also, it is straightforward to extract the eigenvalues in Equation (3.10) if κ is an even integer, otherwise complex eigenvalue computation is required.

At this point, we consider a numerical example in order to demonstrate an application of the theory presented in the foregoing material.

3.1 Example 1: Polar Decomposition and Strain

Given a homogeneous, isotropic body in the shape of a cube, consider the deformation defined by

$$x_1 = X_1 + u_1; \tag{3.16a}$$

$$u_1 = \gamma X_3 \tag{3.16b}$$

$$x_2 = X_2 + u_2; \tag{3.16c}$$

$$u_2 = 0 \tag{3.16d}$$

$$x_3 = X_3 + u_3; \tag{3.16e}$$

$$u_3 = 0 \tag{3.16f}$$

and shown in Fig. 3.2.

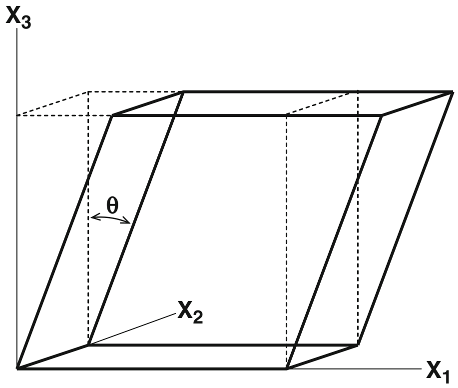

Fig. 3.2 Simple shear deformation

We can write the deformation gradient in matrix form, with $\gamma = 0.1$,

$$[F] = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix} = \begin{bmatrix} 1+\frac{\partial u_1}{\partial X_1} & \frac{\partial u_1}{\partial X_2} & \frac{\partial u_1}{\partial X_3} \\ \frac{\partial u_2}{\partial X_1} & 1+\frac{\partial u_2}{\partial X_2} & \frac{\partial u_2}{\partial X_3} \\ \frac{\partial u_3}{\partial X_1} & \frac{\partial u_3}{\partial X_2} & 1+\frac{\partial u_3}{\partial X_3} \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0.1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.17}$$

and

$$[U]^2 = [C] = [F]^T[F] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0.1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0.1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0.1 \\ 0 & 1 & 0 \\ 0.1 & 0 & 1.01 \end{bmatrix} \tag{3.18}$$

where $[U]$ is the *right stretch*. Now, from Equation (3.6) we can write

$$[U]^2 = [A]diag\left[\lambda_1^2, \lambda_2^2, \lambda_3^2\right][A]^T \tag{3.19}$$

which yields

$$[U] = [A]diag[\lambda_1, \lambda_2, \lambda_3][A]^T \tag{3.20}$$

or

$$[U] = \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} \tag{3.21}$$

where l_i, m_i, and n_i are the *direction cosines*. The *characteristic equation* of **C** is

$$\lambda^6 - I_1\lambda^4 + I_2\lambda^2 - I_3 = 0 \tag{3.22}$$

where I_1, I_2, I_3 are the first, second, and third invariants, respectively; then

$$I_1 = C_{11} + C_{22} + C_{33} = 3.01 \tag{3.23a}$$

$$I_2 = C_{11}C_{22} - C_{12}C_{21} + C_{22}C_{33} - C_{23}C_{32} + C_{33}C_{11} - C_{31}C_{13} = 3.01 \tag{3.23b}$$

$$\begin{aligned} I_3 &= C_{11}C_{22}C_{33} + C_{23}C_{31}C_{12} + C_{32}C_{13}C_{21} - C_{23}C_{32}C_{11} - C_{31}C_{13}C_{22} \\ &\quad - C_{12}C_{21}C_{33} \\ &= 1.00 \end{aligned} \tag{3.23c}$$

giving

$$\lambda^6 - 3.01\lambda^4 + 3.01\lambda^2 - 1.00 = 0 \tag{3.24}$$

The *eigenvalues*, or roots, of this equation are

$$\lambda_1^2 = 1.105125 \tag{3.25a}$$

$$\lambda_2^2 = 1.000000 \tag{3.25b}$$

$$\lambda_3^2 = 0.904875 \tag{3.25c}$$

The square roots of the eigenvalues are

$$\lambda_1 = 1.0512492 \tag{3.26a}$$

$$\lambda_2 = 1.000000 \tag{3.26b}$$

$$\lambda_3 = 0.9512492 \tag{3.26c}$$

which are the principal stretches. Hill (1970) points out that the principal stretches have the relationship

$$\lambda_1 = \lambda \tag{3.27a}$$

$$\lambda_2 = 1 \tag{3.27b}$$

$$\lambda_3 = \lambda^{-1} \tag{3.27c}$$

where $\lambda \geq 1$ and $J = 1$. The logarithmic strains are then

$$\ln(1.0512492) = 0.0499792 \tag{3.28a}$$

$$\ln(1) = 0 \tag{3.28b}$$

$$\ln(0.9512492) = -0.0499792 \tag{3.28c}$$

Now, solving for the *eigenvectors*,

$$\begin{bmatrix} 1 - 1.105125 & 0 & 0.1 \\ 0 & 1 - 1.105125 & 0 \\ 0.1 & 0 & 1.01 - 1.105125 \end{bmatrix} \begin{Bmatrix} l_1 \\ m_1 \\ n_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{3.29}$$

$$l_1 = 0.689225 \tag{3.30a}$$

$$m_1 = 0 \tag{3.30b}$$

$$n_1 = 0.724548 \tag{3.30c}$$

$$\begin{bmatrix} 0 & 0 & 0.1 \\ 0 & 0 & 0 \\ 0.1 & 0 & 0.01 \end{bmatrix} \begin{Bmatrix} l_2 \\ m_2 \\ n_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{3.31}$$

$$l_2 = 0 \tag{3.32a}$$

$$m_2 = 1 \tag{3.32b}$$

$$n_2 = 0 \tag{3.32c}$$

$$\begin{bmatrix} 1 - 0.904875 & 0 & 0.1 \\ 0 & 1 - 0.904875 & 0 \\ 0.1 & 0 & 1.01 - 0.904875 \end{bmatrix} \begin{Bmatrix} l_3 \\ m_3 \\ n_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{3.33}$$

$$l_3 = 0.724547 \tag{3.34a}$$

$$m_3 = 0 \tag{3.34b}$$

$$n_3 = -0.689225 \tag{3.34c}$$

Then,

$$[U] = \begin{bmatrix} 0.689225 & 0 & 0.724547 \\ 0 & 1 & 0 \\ 0.724548 & 0 & -0.689225 \end{bmatrix} \begin{bmatrix} 1.051249 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.951249 \end{bmatrix} \tag{3.35}$$

$$\begin{bmatrix} 0.689225 & 0 & 0.724548 \\ 0 & 1 & 0 \\ 0.724547 & 0 & -0.689225 \end{bmatrix}$$

$$[U] = \begin{bmatrix} 0.998752 & 0 & 0.049938 \\ 0 & 1 & 0 \\ 0.049938 & 0 & 1.003747 \end{bmatrix} \tag{3.36}$$

i.e., the right stretch. As a check,

$$[U]^2 = \begin{bmatrix} 0.998752 & 0 & 0.049938 \\ 0 & 1 & 0 \\ 0.049938 & 0 & 1.003747 \end{bmatrix} \begin{bmatrix} 0.998752 & 0 & 0.049938 \\ 0 & 1 & 0 \\ 0.049938 & 0 & 1.003747 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0.1 \\ 0 & 1 & 0 \\ 0.1 & 0 & 1.01 \end{bmatrix} = [C] \tag{3.37}$$

Obtaining the eigenvalues of $[U]$,

$$\begin{vmatrix} 0.998752 - \lambda_i & 0 & 0.049938 \\ 0 & 1 - \lambda_i & 0 \\ 0.049938 & 0 & 1.003747 - \lambda_i \end{vmatrix} = 0 \tag{3.38}$$

$$\lambda_i^3 - 3.0025\lambda_i^2 + 3.0025\lambda_i - 1 = 0 \tag{3.39}$$

$$(\lambda_i - 1)\left(\lambda_i^2 - 2.0025\lambda_i + 1\right) = 0 \tag{3.40}$$

$$\lambda_{1,3} = \frac{2.0025 \pm \sqrt{(-2.0025)^2 - 4(1)(1)}}{2(1)} = \frac{2.0025 \pm \sqrt{0.01}}{2}$$

$$= \frac{2.0025 \pm 0.1}{2} \tag{3.41}$$

$$\lambda_1 = 1.05125 \tag{3.42a}$$

$$\lambda_2 = 1 \tag{3.42b}$$

$$\lambda_3 = 0.95125 \tag{3.42c}$$

$$\ln(1.05125) = 0.0499799 \tag{3.43a}$$

$$\ln(0.95125) = -0.0499784 \tag{3.43b}$$

Now, we want to obtain the rotation matrix $[R]$ so we write

$$[U]^{-1} = \begin{bmatrix} 1.003747 & 0 & -0.049939 \\ 0 & 1 & 0 \\ -0.049938 & 0 & 0.998753 \end{bmatrix} \tag{3.44}$$

and

$$[R] = [F][U]^{-1} \tag{3.45}$$

Then,

$$\begin{aligned} [R] &= \begin{bmatrix} 1 & 0 & 0.1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1.003747 & 0 & -0.049939 \\ 0 & 1 & 0 \\ -0.049938 & 0 & 0.998753 \end{bmatrix} \\ &= \begin{bmatrix} 0.998753 & 0 & 0.049936 \\ 0 & 1 & 0 \\ -0.049938 & 0 & 0.998753 \end{bmatrix} \end{aligned} \tag{3.46}$$

As we know, $[R]$ is a proper orthogonal matrix such that

$$[R]^{-1} = [R]^T \tag{3.47}$$

Then,

$$[R]^T[R] = [I] \tag{3.48}$$

Thus, as a check,

$$\begin{bmatrix} 0.998753 & 0 & -0.049938 \\ 0 & 1 & 0 \\ 0.049936 & 0 & 0.998753 \end{bmatrix} \begin{bmatrix} 0.998753 & 0 & 0.049936 \\ 0 & 1 & 0 \\ -0.049938 & 0 & 0.998753 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.49}$$

Now, we want to solve for the strains. We can write

$$\underline{\underline{E}} = \frac{1}{2}\left(\underline{\underline{C}} - \underline{\underline{I}}\right) \tag{3.50}$$

where $\underline{\underline{E}}$ is the Green-Lagrange strain tensor, thus we obtain

$$[E] = \frac{1}{2}\left\{ \begin{bmatrix} 1 & 0 & 0.1 \\ 0 & 1 & 0 \\ 0.1 & 0 & 1.01 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\} = \begin{bmatrix} 0 & 0 & 0.05 \\ 0 & 0 & 0 \\ 0.05 & 0 & 0.005 \end{bmatrix} \tag{3.51}$$

We can write the left Cauchy-Green deformation tensor in matrix form,

$$[b] = \begin{bmatrix} 1 & 0 & 0.1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0.1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1.01 & 0 & 0.1 \\ 0 & 1 & 0 \\ 0.1 & 0 & 1 \end{bmatrix} \tag{3.52}$$

yielding

$$[b]^{-1} = \begin{bmatrix} 1 & 0 & -0.1 \\ 0 & 1 & 0 \\ -0.1 & 0 & 1.01 \end{bmatrix} \tag{3.53}$$

We can write

$$\underset{=}{e} = \frac{1}{2}\left(\underset{=}{I} - \underset{=}{b}^{-1}\right) \tag{3.54}$$

where $\underset{=}{e}$ is the Almansi strain tensor; thus we obtain

$$[e] = \frac{1}{2}\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & -0.1 \\ 0 & 1 & 0 \\ -0.1 & 0 & 1.01 \end{bmatrix} \right\} = \begin{bmatrix} 0 & 0 & 0.05 \\ 0 & 0 & 0 \\ 0.05 & 0 & -0.005 \end{bmatrix} \tag{3.55}$$

From Equation (3.9), we can write, for $\kappa = -1$ (the "true" strain condition),

$$\begin{aligned} [\varepsilon] &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1.003747 & 0 & -0.049939 \\ 0 & 1 & 0 \\ -0.049938 & 0 & 0.998753 \end{bmatrix} \\ &= \begin{bmatrix} -0.003747 & 0 & 0.049939 \\ 0 & 0 & 0 \\ 0.049938 & 0 & 0.001247 \end{bmatrix} \end{aligned} \tag{3.56}$$

and for $\kappa = 1$ (the engineering strain condition),

$$\begin{aligned} [\varepsilon] &= \begin{bmatrix} 0.998753 & 0 & 0.049939 \\ 0 & 1 & 0 \\ 0.049938 & 0 & 1.003747 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -0.001247 & 0 & 0.049939 \\ 0 & 0 & 0 \\ 0.049938 & 0 & 0.003747 \end{bmatrix} \end{aligned} \tag{3.57}$$

Chapter 4
Strain-Energy Functions

Abstract The isotropic elastic properties of a hyperelastic material model are described in terms of a strain-energy (stored-energy) function, typically as a function of the three invariants of each of the two Cauchy-Green deformation tensors, given in terms of the principal extension ratios, or stretches. A number of different strain-energy formulations exist, having properties and characteristics that make them appropriate for characterizing different hyperelastic material systems. The primary, and probably best known and most widely employed, strain-energy function formulation is the Mooney-Rivlin model, which reduces to the widely known neo-Hookean model. Other models which have been demonstrated to be quite appropriate and desirable for modeling rubberlike materials are the Ogden, Yeoh, Arruda-Boyce (statistically based), and Gent models. Flexible foams which exhibit finite elasticity characteristics can be modeled as hyperelastic material systems to a large extent. Strain-energy function models which are designated as foam models are the Blatz-Ko model and the Ogden-Storaker model. Soft tissues can also be modeled employing hyperelastic material models. Models designated as soft tissue models are the Fung model, the Holzapfel-Gasser-Ogden (HGO) model, and the Veronda-Westmann model.

The isotropic elastic properties of a hyperelastic material model may be described in terms of a *strain-energy function*

$$\breve{W} = f(I_1, I_2, I_3) \tag{4.1}$$

where $\breve{W}$ is the strain-energy density and I_1, I_2, and I_3 are the three invariants of each of the two *Cauchy-Green deformation tensors*, given in terms of the principal extension ratios λ_1, λ_2, and λ_3 by

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \tag{4.2a}$$

$$I_2 = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 \tag{4.2b}$$

R. M. Hackett, *Hyperelasticity Primer*,
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$$I_3 = \lambda_1^2 \lambda_2^2 \lambda_3^2 \tag{4.2c}$$

For reference, the derivatives of the principal invariants of a second-order tensor with respect to the tensor itself are often required in the formulation of constitutive equations. For example,

$$\frac{\partial I_1}{\partial b_{ij}} = \delta_{ij} \quad \text{or} \quad \frac{\partial I_1}{\partial \underset{=}{b}} = \underset{=}{I} \tag{4.3a}$$

$$\frac{\partial I_2}{\partial b_{ij}} = b_{kk}\delta_{ij} - b_{ji} \quad \text{or} \quad \frac{\partial I_2}{\partial \underset{=}{b}} = I_1 \underset{=}{I} - \underset{=}{b}^T \tag{4.3b}$$

$$\frac{\partial I_3}{\partial b_{ij}} = I_3 b_{ji}^{-1} \quad \text{or} \quad \frac{\partial I_3}{\partial \underset{=}{b}} = I_3 \underset{=}{b}^{-T} \tag{4.3c}$$

Rivlin (1956) specified the form of Equation (4.1) with the power series

$$\breve{W} = \sum_{i+j+k=1}^{\infty} C_{ijk}(I_1 - 3)^i (I_2 - 3)^j (I_3 - 1)^k \tag{4.4}$$

The 3s and the 1 are included in the formulation to force $\breve{W} = 0$ under conditions of zero deformation. For *incompressible* materials, $I_3 = 1$, and Equation (4.4) reduces to

$$\breve{W} = \sum_{i+j=1}^{\infty} C_{ij}(I_1 - 3)^i (I_2 - 3)^j \tag{4.5}$$

The power series in Equation (4.5) is usually truncated by taking only the leading terms. Taking the first two terms of Equation (4.5) yields

$$\breve{W} = C_{10}(I_1 - 3) + C_{01}(I_2 - 3) \tag{4.6}$$

which is the *Mooney-Rivlin equation*, (Mooney 1940; Rivlin 1948). Taking only the first term of this equation yields the *neo-Hookean* model

$$\breve{W} = C_{10}(I_1 - 3) \tag{4.7}$$

A neo-Hookean material description is thus a Mooney-Rivlin material description, Equation (4.6), with $C_{01} = 0$. The Mooney-Rivlin material description is typically used to characterize rubber-like materials undergoing large strains. The conventional Mooney-Rivlin material is defined by the strain-energy function

$$\breve{W} = C_{10}(I_1 - 3) + C_{01}(I_2 - 3); \; I_3 = 1 \tag{4.8}$$

where C_{10} and C_{01} are material constants and the invariants I_1, I_2, and I_3 are expressed in terms of the right Cauchy-Green deformation tensor C_{IJ}, i.e.,

$$I_1 = C_{KK} \tag{4.9a}$$

$$I_2 = \frac{1}{2}\left(I_1^2 - C_{IK}C_{KI}\right) \tag{4.9b}$$

$$I_3 = \det \; C_{IJ} \tag{4.9c}$$

We can note that, for small strains, $2(C_{10}+C_{01})$ represents the shear modulus and $6(C_{10}+C_{01})$ represents the Young's modulus. A Mooney-Rivlin material description having $C_{01}=0$ is a neo-Hookean material, with $C_{10} = G/2$, G being the shear modulus, which implies that the shear modulus is equal to one-third of the Young's modulus.

This material description is based on the assumption that the material is totally *incompressible*. A better assumption is that the bulk modulus K is several hundred times as large as the shear modulus, or that the material is almost incompressible. This can be accomplished by removing the restriction $I_3 = 1$ and adding a hydrostatic work term W_H to the expression for the strain-energy function, Equation (4.8),

$$\breve{W} = C_{10}(I_1 - 3) + C_{01}(I_2 - 3) + W_H(I_3) \tag{4.10}$$

However, we cannot directly use this description since we want uncoupled deviatoric response and volumetric response and all three terms contribute to the pressure. To circumvent this problem, we employ the reduced invariants presented in Chap. 2,

$$\bar{I}_1 = I_1 I_3^{-1/3} \tag{4.11a}$$

$$\bar{I}_2 = I_2 I_3^{-2/3} \tag{4.11b}$$

$\bar{I}_1$ and $\bar{I}_2$ being the first and second invariants of $\overline{C}_{IJ}$, respectively, We also invoke the relationship

$$J = \sqrt{I_3} \tag{4.12}$$

which leads to the expression for the hydrostatic work term

$$W_H = D_1(J-1)^2 \tag{4.13}$$

where $D_1 = K/2$. With these substitutions made in Equation (4.10), the expression for the strain-energy function becomes

$$\widehat{W} = C_{10}\left(\overline{I}_1 - 3\right) + C_{01}\left(\overline{I}_2 - 3\right) + D_1(J-1)^2 \tag{4.14}$$

or, compacted,

$$\widehat{W} = \overline{W}_D\left(\overline{C}_{IJ}\right) + W_H(J) \tag{4.15}$$

We refer to $\overline{W}_D\left(\overline{C}_{IJ}\right)$ and $W_H(J)$ as the *deviatoric* and *hydrostatic* parts of $\widehat{W}$, respectively.

Although we concentrate on the Mooney-Rivlin and neo-Hookean hyperelastic material models in this monograph, there are a number of other strain-energy function-based models that are utilized in the characterization and analysis of hyperelastic materials. We list here, and briefly describe, a selection of additional representative examples of forms of strain-energy function which characterize isotropic hyperelastic materials within the isothermal regime.

The *Ogden* model (Ogden 1972a),

$$\widetilde{W}(\lambda_1, \lambda_2, \lambda_3) = \sum_{p=1}^{N} \frac{\mu_p}{\alpha_p}\left(\lambda_1^{\alpha_p} + \lambda_2^{\alpha_p} + \lambda_3^{\alpha_p} - 3\right) \tag{4.16}$$

where $\lambda_1, \lambda_2, \lambda_3$ are the principal stretches and μ_p and α_p are material properties, is frequently employed to model biological materials and incompressible rubberlike materials such as solid polymers, the only materials that undergo finite strains relative to an equilibrium state (Holzapfel 2000). On comparison with the linear theory we obtain the (consistency) condition

$$2G = \sum_{p=1}^{N} \mu_p \alpha_p \quad \text{with} \quad \mu_p \alpha_p > 0 \tag{4.17}$$

The Mooney-Rivlin incompressible material model, Equation (4.6), results from Equation (4.16) if $N = 2, \alpha_1 = 2, \alpha_2 = -2$ and $\lambda_1\lambda_2\lambda_3 = 1$, with the constants $C_{10} = \mu_1/2$ and $C_{01} = -\mu_2/2$. Then, from Equation (4.17), the shear modulus G has the value $\mu_1 - \mu_2$. The corresponding neo-Hookean model results from Equation (4.16) if $N = 1$, $\alpha_1 = 2$, and $\lambda_1\lambda_2\lambda_3 = 1$, with the constant $C_{10} = \mu_1/2$. Then, from Equation (4.17), the shear modulus G is equal to μ_1. Strain-energy functions of the form given by Equation (4.16) have led to very successful correlations with experimental stress-deformation data for rubber-like materials (Ogden 1997).

Nearly all practical engineering elastomers contain reinforcing fillers such as *carbon black* (in natural rubber vulcanizate) or *silica* (in silicone rubber). These finely distributed fillers, which have typical dimensions on the order of $1.0 - 2.0 \times 10^{-12}$m, form physical and chemical bonds with the polymer chains. Carbon-black-filled rubbers have important applications in the manufacture of automotive tires and other engineered components. It turns out that the strain-energy functions described herein up to this point are not sufficient to approximate the observed physical behavior of carbon-black-filled rubbers. Apparently the Mooney-Rivlin (and its neo-Hookean specialization) model doesn't adequately characterize the behavior/response of the carbon-black-filled rubber vulcanizates (Holzapfel 2000).

The motivation for the development of the phenomenological material *Yeoh* model (Yeoh 1990) was the simulation of the mechanical behavior of carbon-black filler-stiffened rubber vulcanizates in the large-strain domain. Yeoh (1990) proposed the following three-term strain-energy function where only the first strain invariant I_1 appears. It has the specific form

$$\breve{W} = c_1(I_1 - 3) + c_2(I_1 - 3)^2 + c_3(I_1 - 3)^3 \tag{4.18}$$

where c_1, c_2, and c_3 are material constants which must satisfy certain restrictions. The appropriate restrictions can be determined from the discriminants of the respective quadratic and cubic equations in $(I_1 - 3)$ (Holzapfel 2000).

Another material model for characterizing the response of rubber, which has a structure similar to Equation (4.18), the *Arruda-Boyce* model (Arruda and Boyce 1993),

$$\breve{W} = G\left[\frac{1}{2}(\bar{I}_1 - 3) + \frac{1}{20n}\left(\bar{I}_1^2 - 9\right) + \frac{11}{1050n^2}\left(\bar{I}_1^3 - 27\right) + \cdots\right] + \frac{K}{2}(J - 1)^2 \tag{4.19}$$

is statistically-based, with the parameters physically linked to the chain orientations involved in the deformation of the three-dimensional network of the rubber. The molecular network structure is represented by an eight-chain model which replaces classical three- and four-chain models. The parameter n is the number of segments (each of the same length) in a chain, freely joined together at chemical cross-links. The series is the result of a Taylor expansion of the *inverse Langevin function*, an integral component for statistically-based network models that describe rubber-like materials (Bower 2010). Arruda and Boyce (1993) list additional terms in the model if needed.

We also point out the *Gent* model (Gent 1996), an empirical formulation suitable for use over a wide range of strains. For an incompressible material model, the strain-energy function form is

$$\breve{W} = -G\frac{J_m}{2}\ln\left(1 - \frac{I_1 - 3}{J_m}\right) \tag{4.20}$$

where J_m is the maximum value of J_1, where

$$J_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3 \tag{4.21}$$

The value of J_m is on the order of 10^2 for unfilled rubber vulcanizates (Gent 1996); it corresponds to a maximum extension ratio λ_m of approximately 10 and represents the limit state of the material. This model is extended to compressible materials by introducing the reduced invariant $\bar{I}_1$ (Dill 2007),

$$\widehat{W} = -G\frac{J_m}{2}\ln\left(1 - \frac{\bar{I}_1 - 3}{J_m}\right) + \frac{K}{2}\left(\frac{1}{2}\left(J^2 - 1\right) + \ln J\right) \tag{4.22}$$

We now consider foam models, which occupy an important place in the realm of hyperelastic material behavior. For foamed elastomers, which cannot be regarded as being incompressible, Blatz and Ko (1962) and Ogden (1972b) proposed a strain-energy function which combines theory and experimental data from solid polyurethane rubbers and foamed polyurethane elastomers. The *Blatz-Ko* model is thus obtained from

$$\begin{aligned}\breve{W}(I_1, I_2, I_3) = {} & \frac{Gf}{2}\left[(I_1 - 3) + \frac{1}{\beta}\left(I_3^{-\beta} - 1\right)\right] \\ & + \frac{G(1-f)}{2}\left[\left(\frac{I_2}{I_3} - 3\right) + \frac{1}{\beta}\left(I_3^{\beta} - 1\right)\right]\end{aligned} \tag{4.23}$$

where

$$\beta = \frac{v}{(1 - 2v)}, \quad 0.0 \leq \beta \leq 1.0 \tag{4.24}$$

v is Poisson's ratio, and $f \in [0, 1]$ is an interpolation parameter. The constant β defines the degree of compressibility. Interestingly, employing the incompressibility constraint $I_3 = 1$ results in Equation (4.23) reducing to the Mooney-Rivlin form, with the constants

$$C_{10} = \frac{Gf}{2} \tag{4.25a}$$

and

$$C_{01} = \frac{G(1-f)}{2} \tag{4.25b}$$

The *Ogden-Storakers* model strain-energy function proposed by Storakers (1986) for describing the mechanical behavior of highly compressible foams is

$$\widetilde{W}(\lambda_1, \lambda_2, \lambda_3) = \sum_{k=1}^{N} \frac{2\mu_k}{\alpha_k^2} \left[\lambda_1^{\alpha_k} + \lambda_2^{\alpha_k} + \lambda_3^{\alpha_k} - 3 + g(J) \right] \tag{4.26}$$

where μ_k and α_k are material properties and $g(J)$ is a volumetric function of the form

$$g(J) = \frac{1}{\beta_k} \left(J^{-\alpha_k \beta_k} - 1 \right) \tag{4.27}$$

where β_k defines the degree of compressibility, with the shear modulus G and bulk modulus K obtained from

$$G = \sum_{k=1}^{N} \mu_k \tag{4.28a}$$

and

$$K = \sum_{k=1}^{N} 2\mu_k \left(\beta_k + \frac{1}{3} \right) \tag{4.28b}$$

The material properties α_k, β_k, and μ_k are determined from experimental test data. Poisson's ratio is then obtained from

$$v_k = \frac{\beta_k}{1 + 2\beta_k} \tag{4.29}$$

The Ogden-Storakers model is a HYPERFOAM model in the Abaqus/Standard® commercial finite element code.

We now focus on a material of a different origin which can be effectively modeled employing hyperelasticity models, soft biological tissue.

4.1 Soft Biological Tissue

The foremost pioneer in the field encompassing the study and characterization of the mechanical behavior of *soft biological tissue* is Yuan-Cheng Fung (1919–) who is widely regarded as the founder of modern biomechanics. *Biomechanics* is the application of mechanics to the study of biological systems. Biological systems are, generally speaking, much more complex than man-made systems. The biomechanical model of soft tissue is a key requirement for developing a reality-based model for minimally invasive surgical simulation (Ahmadian and Nikooyan 2006). Research is normally carried out in an iterative process of hypothesis and verification, including several steps of modeling, computer simulation, and experimental measurement. Analytical and experimental techniques are used to reveal the biomechanical mechanisms of, for example, cardiovascular disease, pulmonary/respiratory disorders, ocular disease, and tissue regeneration.

Soft tissue is the term that is generally used to characterize biological material that comprises the organs, blood vessels, skin, tendons, and ligaments, among a host of other vital parts, of living beings. Soft tissue also provides the connectivity between bones, teeth, and other hard tissue which exhibit a much different stress-strain response—a response that is more like that of engineering materials. It is the soft tissue that requires special consideration when the material models that simulate its response are formulated.

Soft tissue displays viscoelastic behavior to various extents, i.e., stress is a function of strain rate, as is shown/demonstrated by Holzapfel and Gasser (2001) (see also Chap. 13), but its response can also be approximated by a hyperelastic material model, after preconditioning to a load pattern (Fung 1993). After some cycles of loading and unloading a specimen of the material, the mechanical response becomes independent of strain rate. Despite the independence of strain rate, the preconditioned soft tissue can still exhibit hysteresis; thus its mechanical response can be modeled as a hyperelastic material with different material constants for loading and unloading (Fung 1993). Fung refers to his formulated model as being *pseudoelastic*—distinguishing it from formal hyperelasticity material models. There are additional highly developed hyperelasticity material models that have been successfully adapted to the modeling of the mechanical response of soft tissue. They will be presented and discussed in some detail, along with the *Fung* model. A primary and unifying characteristic of soft tissue finite strain models is that they employ an exponential function. The exponential function allows modeling stiffening that is typical of soft biological tissues. Soft tissue is readily modeled as being nearly incompressible—the models presented in this section will reflect that.

The *Fung* model (Fung 1993) is a purely phenomenological model; that is, it is not derived from first principles, but is consistent with fundamental theory. Fung (1973) proposed the generalized anisotropic strain-energy function, expressed here in a nearly incompressible formulation (Fung 1973, 1993; Fung et al. 1979; Chuong and Fung 1986):

$$\begin{aligned}\overline{W} &= \frac{1}{2}\overline{E}_{IJ}\alpha_{IJKL}\overline{E}_{KL} + \left(\beta_0 + \overline{E}_{MN}\beta_{MNPQ}\overline{E}_{PQ}\right)\exp\left(\gamma_{IJ}\overline{E}_{IJ} + \overline{E}_{IJ}\delta_{IJKL}\overline{E}_{KL}\right) \\ &\quad + \frac{K}{2}\left(\frac{\left(J^{el}\right)^2 - 1}{2} - \ln J^{el}\right) \qquad (4.30)\end{aligned}$$

where $\overline{E}_{IJ}$ is the reduced Green-Lagrange strain tensor; α_{IJKL}, β_0, β_{IJKL}, γ_{IJ}, and δ_{IJKL} are material constants; and K is the bulk modulus. He later dropped the terms $\gamma_{IJ}\overline{E}_{IJ}$. If small strains are of no concern, the first term in Equation (4.30) can also be omitted. The third term facilitates the nearly incompressible modeling, with J^{el} being the elastic volume ratio, which can be replaced by J for an isothermal model.

A widely employed generalized version of the Fung strain-energy function has the simplified form:

$$\overline{W} = \frac{c}{2}(\exp Q - 1) + \frac{K}{2}\left(\frac{\left(J^{el}\right)^2 - 1}{2} - \ln J^{el}\right) \tag{4.31}$$

where c is a material property with the units of stress and where

$$Q = \overline{E}_{IJ}\delta_{IJKL}\overline{E}_{KL} \tag{4.32}$$

The number of independent components δ_{IJKL} that must be specified in the model depends upon the level of anisotropy of the material. The full 81 tensor-expansion components in Equation (4.32) are reduced to 21 independent components to model *anisotropic* conditions and to 9 to model *orthotropic* conditions. *Transverse isotropy*, i.e., isotropy in the planes perpendicular to the preferred direction, is the special case of orthotropy where there are five independent constants. Materials displaying transverse isotropy have one characteristic (preferred) direction. The material behavior of arterial walls is transversely isotropic, in general.

As an example of a calibrated Fung strain-energy model, the constitutive model of a rabbit carotid artery (Volokh 2016) is formulated in Chap. 5.

The following strain-energy function (Gasser et al. 2006), which models a fibrous soft tissue,

$$\begin{aligned} \overline{W} = {} & \frac{G}{2}\left(\bar{I}_1 - 3\right) + \frac{k_1}{2k_2}\sum_{i=1}^{N}\left[\exp\left(k_2\left\{\kappa\bar{I}_1 + (1-3\kappa)\bar{I}_{4(i)} - 1\right\}^2\right) - 1\right] \\ & + \frac{K}{2}\left(\frac{\left(J^{el}\right)^2 - 1}{2} - \ln J^{el}\right) \end{aligned} \tag{4.33}$$

provides a different constitutive framework for arterial wall mechanics. In this formulation, Equation (4.33), $\bar{I}_{4(i)}$ is a defined anisotropic pseudo-invariant having the clear physical meaning of the squared stretch in the characteristic direction, and is equal to $\boldsymbol{\alpha}_{0i} \otimes \boldsymbol{\alpha}_{0i} : \overline{\mathbf{C}}$, κ is a dispersion parameter that characterizes the distribution of fibers within the families of fibers ($0 \leq \kappa \leq 1/3$), k_1 is a material property with the units of stress that relates to the stiffness of the fibers, k_2 is a dimensionless material parameter that is related to fiber nonlinear behavior, and N is the number of fiber families ($N \leq 3$). The tensor product $\boldsymbol{\alpha}_0 \otimes \boldsymbol{\alpha}_0$ is often called the *structural* or *structure tensor* (Volokh 2016), where unit vector $\boldsymbol{\alpha}_0$ designates the preferred direction in the reference configuration. For $\kappa = 0$ (perfect alignment of the fibers, i.e., transverse isotropy), Equation (4.33) becomes

$$\overline{W} = \frac{G}{2}(\bar{I}_1 - 3) + \frac{k_1}{2k_2}\sum_{i=1}^{N}\left[\exp\left(k_2\left(\bar{I}_{4(i)} - 1\right)^2\right) - 1\right]$$
$$+ \frac{K}{2}\left(\frac{\left(J^{el}\right)^2 - 1}{2} - \ln J^{el}\right) \tag{4.34}$$

which is the analytical form of the *Holzapfel-Gasser-Ogden* (HGO) model (Holzapfel et al. 2000), a micromechanically based formulation, which models nearly incompressible anisotropic materials. For κ = 1/3 (random distribution of the fibers), Equation (4.33) becomes

$$\overline{W} = \frac{G}{2}(\bar{I}_1 - 3) + \frac{k_1}{2k_2}\left[\exp\left(\frac{k_2}{9}(\bar{I}_1 - 3)^2\right) - 1\right]$$
$$+ \frac{K}{2}\left(\frac{\left(J^{el}\right)^2 - 1}{2} - \ln J^{el}\right) \tag{4.35}$$

which corresponds to an isotropic distribution. Fiber dispersion implies multiple characteristic directions. Many soft biological tissues, e.g., arterial walls, which are structurally composed of a mostly isotropic matrix (elastin) and fibers (collagen), readily lend themselves to modeling with the Holzapfel-Gasser-Ogden (HGO) formulation.

Murphy (2013) demonstrates the modeling of transversely isotropic biological soft tissue using two anisotropic pseudo-invariants: the previously defined I_4 and I_5 which is equal to $\boldsymbol{\alpha}_0 \otimes \boldsymbol{\alpha}_0 : \mathbf{C}^2$.

The soft tissue formulations shown in Equations (4.31) and (4.33) are both ANISOTROPIC HYPERELASTIC designated models in the Abaqus/Explicit® commercial finite element code.

One of the more widely applied models for the analysis of soft biological tissues, especially in the case of model parameter restructuring for an *inverse problem* (Oberai et al. 2003, 2004; see Chap. 15), is the *Veronda-Westmann* model (Veronda and Westmann 1970) having the strain-energy function:

$$W = \mu_0\left(\frac{\exp[\beta(I_1 - 3)] - 1}{\beta} - \frac{I_2 - 3}{2}\right) \tag{4.36}$$

where μ_0 is the shear modulus of the material at zero strain and β is a parameter that denotes the degree of nonlinearity of the material. Experimental results of displacement versus external force on a variety of soft tissues indicate a nonlinear stiffness coefficient behavior for these materials (Ahmadian and Nikooyan 2006). An example of the employment of this model is found in Chap. 5.

Chapter 5
Stress Measures

Abstract Within the framework of hyperelasticity, there are as many different stress measures as there are strain measures. The second Piola-Kirchhoff stress tensor, a Lagrangian formulation, is the most significant of the stress measures. The formulation and steps for computing it are presented in terms of the Mooney-Rivlin strain-energy function model. The Cauchy stress tensor, a Eulerian formulation, is obtained directly from the second Piola-Kirchhoff stress tensor. The first Piola-Kirchhoff stress tensor, a Eulerian-Lagrangian two-point tensor, is also obtained directly from the second Piola-Kirchhoff stress tensor. The transpose of the first Piola-Kirchhoff stress tensor is the so-called nominal stress tensor. Both the first Piola-Kirchhoff stress tensor and the nominal stress tensor are widely used in the field of hyperelasticity. The Kirchhoff stress tensor (weighted Cauchy stress tensor) is related to the Cauchy stress tensor through a multiplication by the Jacobian (the determinant of the deformation gradient). The Biot stress, a Lagrangian-based stress tensor, is also an important stress measure. Only somewhat recently has it been recognized that the Biot stress tensor is helpful in the understanding of certain fundamental problems in elasticity theory. Four detailed numerical examples are presented, two of which employ a uniaxial elongation model in comparing/contrasting the solutions of incompressible and nearly incompressible material formulations.

There are as many different stress measures as there are strain measures. The *second Piola-Kirchhoff stress tensor*, also known as the *Kirchhoff-Trefftz stress tensor*, is the most significant stress measure within the framework of hyperelasticity. It can be obtained from

$$S_{IJ} = \frac{1}{2}\left(\frac{\partial W}{\partial E_{IJ}} + \frac{\partial W}{\partial E_{JI}}\right) \tag{5.1}$$

with the abbreviations

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$$()^*_{IJ} = \frac{1}{2}\left(\frac{\partial}{\partial E_{IJ}} + \frac{\partial}{\partial E_{JI}}\right) = \frac{\partial}{\partial C_{IJ}} + \frac{\partial}{\partial C_{JI}} \tag{5.2}$$

used in subsequent chain-rule differentiation operations (Sussman and Bathe 1987). Note that the operator $()^*_{IJ}$ is a linear operator and may be manipulated using the usual chain rules of differentiation. Performing the operations thus indicated in Equations (5.1) and (5.2), we get

$$S_{IJ} = C_{10}\left(\bar{I}_1\right)^*_{IJ} + C_{01}\left(\bar{I}_2\right)^*_{IJ} + 2D_1(J-1)(J)^*_{IJ} \tag{5.3}$$

where

$$\left(\bar{I}_1\right)^*_{IJ} = (I_3)^{-1/3}(I_1)^*_{IJ} - \frac{1}{3}\left(I_1 I_3^{-4/3}\right)(I_3)^*_{IJ} \tag{5.4a}$$

$$\left(\bar{I}_2\right)^*_{IJ} = (I_3)^{-2/3}(I_2)^*_{IJ} - \frac{2}{3}\left(I_2 I_3^{-5/3}\right)(I_3)^*_{IJ} \tag{5.4b}$$

$$(J)^*_{IJ} = \frac{1}{2}(I_3)^{-1/2}(I_3)^*_{IJ} \tag{5.4c}$$

where

$$(I_1)^*_{IJ} = 2\delta_{IJ} \tag{5.5a}$$

$$(I_2)^*_{IJ} = 2I_1\delta_{IJ} - (C_{IJ} + C_{JI}) \tag{5.5b}$$

$$(I_3)^*_{IJ} = \frac{1}{2}(\hat{\varepsilon}_{IBC}\hat{\varepsilon}_{JDF} + \hat{\varepsilon}_{JBC}\hat{\varepsilon}_{IDF})C_{BD}C_{CF} \tag{5.5c}$$

or

$$(I_3)^*_{IJ} = I_3\left(C^{-1}_{IJ} + C^{-1}_{JI}\right) \tag{5.5d}$$

where δ_{IJ} is the Kronecker delta and $\hat{\varepsilon}_{IJK}$ is the permutation symbol, or alternator. The alternator is defined (Gould 1983), such that

$$\hat{\varepsilon}_{ijk} = \frac{1}{2}(i-j)(j-k)(k-i) \tag{5.6}$$

Thus,

$\hat{\varepsilon} = 0$, if any two of i, j, k are equal
$\hat{\varepsilon} = 1$ for an even permutation (forward on the number line 1, 2, 3)
$\hat{\varepsilon} = -1$ for an odd permutation (backward on the number line)

We can write

$$t_{ij} = \frac{1}{J}F_{iK}S_{KL}F_{jL} \quad \text{or} \quad \mathbf{t} = \frac{1}{J}\mathbf{FSF}^T \tag{5.7}$$

where t_{ij} is the well-known *Cauchy stress tensor*, which defines the "true" stress. We can also write

$$P_{iJ} = F_{iK} S_{KJ} \quad \text{or} \quad \mathbf{P} = \mathbf{FS} \tag{5.8}$$

where P_{iJ} is known as the *first Piola-Kirchhoff stress tensor*, or more simply, the *Piola stress*; the Piola stress is the multi-axial generalization of the uniaxial *nominal*, or *engineering*, *stress*; the nomenclature used for nominal stress and Piola stress by different leading authors is somewhat contradictory, although not misleading, in that Truesdell and Noll (1965), Marsden and Hughes (1983), Ogden (1997), and Belytschko et al. (2000) define nominal stress as the transpose of the first Piola-Kirchhoff stress, whereas Malvern (1969) and Simo and Hughes (1998) refer to the first Piola-Kirchhoff stress as the nominal stress. The Piola stress tensor is usually nonsymmetric. The *nominal stress tensor* N_{Ji} is given by (only in the cases of the nominal stress tensor and a corresponding finite viscoelasticity stress tensor does this order of the Lagrange/Euler indices not indicate "inverse" in this monograph.)

$$N_{Ji} = S_{JK} F_{iK} \quad \text{or} \quad \mathbf{N} = \mathbf{SF}^T \tag{5.9a}$$

as well as by

$$N_{Ij} = J F_{Ik} t_{kj} \quad \text{or} \quad \mathbf{N} = J\mathbf{F}^{-1}\mathbf{t} \tag{5.9b}$$

The nominal stress tensor is a Lagrangian-Eulerian *two-point tensor* since it is the contraction of a Lagrangian vector on the left and an Eulerian vector on the right.

The *Kirchhoff* stress (weighted *Cauchy* stress) τ_{ij} is related to the *Cauchy* stress through

$$\tau_{ij} = J t_{ij} \quad \text{or} \quad \boldsymbol{\tau} = J\mathbf{t} \tag{5.10}$$

The Cauchy stress and the Kirchhoff stress are *symmetric* tensors defined on the *current configuration* of the body. For a linearized problem, we speak of *the* stress tensor, since there is only one.

The *Biot stress tensor* T_{IJ}, which is also referred to as the *Jaumann stress tensor*, is important because it is power conjugate to the rate of the right stretch tensor $\dot{U}_{IJ}$ (Biot 1965). Only fairly recently has it been recognized that T_{IJ} is helpful in the understanding of certain fundamental problems in elasticity theory (Ogden 1997); thus it has not been used extensively in the literature heretofore. It is defined by the following relationships:

$$T_{IJ} = \frac{1}{2}\left(R_{iI}^T P_{iJ} + P_{iI}^T R_{iJ}\right) \quad \text{or} \quad \mathbf{T} = \frac{1}{2}\left(\mathbf{R}^T\mathbf{P} + \mathbf{P}^T\mathbf{R}\right) \tag{5.11a}$$

and

$$T_{IJ} = \frac{1}{2}\left(N_{Ii}R_{iJ} + R_{iI}^T N_{Ji}^T\right) \quad \text{or} \quad \mathbf{T} = \frac{1}{2}\left(\mathbf{NR} + \mathbf{R}^T\mathbf{N}^T\right) \tag{5.11b}$$

as well as by

$$T_{IJ} = \frac{1}{2}(S_{IK}U_{KJ} + U_{IK}S_{KJ}) \quad \text{or} \quad \mathbf{T} = \frac{1}{2}(\mathbf{SU} + \mathbf{US}) \tag{5.11c}$$

5.1 Example 2: Stress

At this point, we again consider the previously formulated simple shear example, shown in Fig. 3.2, this time for the purpose of calculating corresponding stresses. We first employ Equations (5.3) through (5.5c) to calculate the second Piola-Kirchhoff stress tensor S_{IJ}. We can write, as earlier,

$$[C] = \begin{bmatrix} 1 & 0 & 0.1 \\ 0 & 1 & 0 \\ 0.1 & 0 & 1.01 \end{bmatrix} \tag{5.12}$$

We choose values of $C_{10} = 150$ psi and $C_{01} = 0$, and we know that $J = \det F_{iJ} = 1$. We then obtain, after some computational steps, S_{IJ}:

$$[S] = \begin{bmatrix} -4.01 & 0 & 30.1 \\ 0 & -1 & 0 \\ 30.1 & 0 & -1 \end{bmatrix} \text{psi} \tag{5.13}$$

Then, employing Equation (5.7), we can write t_{ij}:

$$\begin{aligned} [t] &= \begin{bmatrix} 1 & 0 & 0.1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -4.01 & 0 & 30.1 \\ 0 & -1 & 0 \\ 30.1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0.1 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 & 30 \\ 0 & -1 & 0 \\ 30 & 0 & -1 \end{bmatrix} \text{psi} \end{aligned} \tag{5.14}$$

and, from Equation (5.9a), we can also write N_{Ji} and P_{iJ}:

$$[N] = [P]^T = \begin{bmatrix} -4.01 & 0 & 30.1 \\ 0 & -1 & 0 \\ 30.1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0.1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 30.1 \\ 0 & -1 & 0 \\ 30 & 0 & -1 \end{bmatrix} \text{psi} \tag{5.15}$$

and, from Equation (5.10), τ_{ij}:

$$[\tau] = \begin{bmatrix} 2 & 0 & 30 \\ 0 & -1 & 0 \\ 30 & 0 & -1 \end{bmatrix} \text{psi} \tag{5.16}$$

We can note, from Equation (5.14), that

$$t_{11} \neq t_{33} \tag{5.17}$$

which represents the *Poynting effect* (Jaunzemis 1967; Gurtin 1981), i.e., the existence of unequal pressures in simple shear. We also see that

$$t_{11} - t_{33} = \gamma t_{13} \tag{5.18}$$

where γ is the shear strain. This relationship is independent of the material properties of the body; it is satisfied by every isotropic elastic body in simple shear. It is readily apparent then that the normal stresses cannot be equal. If the necessary normal stresses are not provided, a shearing stress will produce a dilatation or compression of a specimen, depending on the sign of the mean stresses. This result was conjectured by Kelvin, and is referred to as the *Kelvin effect*. We also note that the nominal stress tensor N_{Ji} is nonsymmetric—it was earlier implied that this is usually the case.

We now consider the development of the stress tensor τ_{ij} as a function of the left Cauchy-Green deformation tensor b_{ij} (Bower 2010). We are still modeling the nearly incompressible Mooney-Rivlin material defined in Equation (4.14) and so we have for the strain-energy density

$$W(F_{iJ}) = \hat{W}(I_1, I_2, I_3) = \overset{\frown}{W}(\bar{I}_1, \bar{I}_2, J) = \tilde{W}(\lambda_1, \lambda_2, \lambda_3) \tag{5.19}$$

where

$$\overset{\frown}{W} = C_{10}(\bar{I}_1 - 3) + C_{01}(\bar{I}_2 - 3) + D_1(J-1)^2 \tag{5.20}$$

$$I_1 = b_{kk} \tag{5.21a}$$

$$I_2 = \frac{1}{2}\left(I_1^2 - b_{ik}b_{ki}\right) \tag{5.21b}$$

$$I_3 = \det b_{ij} = J^2 \tag{5.21c}$$

and

$$\bar{I}_1 = \frac{I_1}{J^{2/3}} = \frac{b_{kk}}{J^{2/3}} \tag{5.22a}$$

$$\bar{I}_2 = \frac{I_2}{J^{4/3}} = \frac{1}{2}\left(\bar{I}_1^2 - \frac{b_{ik}b_{ki}}{J^{4/3}}\right) \tag{5.22b}$$

$$J = \sqrt{\det b_{ij}} \tag{5.22c}$$

$\bar{I}_1$ and $\bar{I}_2$ being, as indicated, the first and second invariants of $\overline{b}_{ij}$. Employing the chain rule we can write

$$\frac{\partial W}{\partial F_{iJ}} = \frac{\partial \widehat{W}}{\partial \bar{I}_1}\frac{\partial \bar{I}_1}{\partial F_{iJ}} + \frac{\partial \widehat{W}}{\partial \bar{I}_2}\frac{\partial \bar{I}_2}{\partial F_{iJ}} + \frac{\partial \widehat{W}}{\partial J}\frac{\partial J}{\partial F_{iJ}} \tag{5.23}$$

where

$$\begin{aligned}
\frac{\partial \bar{I}_1}{\partial F_{iJ}} &= \frac{1}{J^{2/3}}\frac{\partial I_1}{\partial F_{iJ}} - \frac{2I_1}{3J^{5/3}}\frac{\partial J}{\partial F_{iJ}} \\
&= \frac{2}{J^{2/3}}\left(F_{iJ} - \frac{I_1}{3}F_{Ji}\right) \\
&= \frac{2}{J^{2/3}}F_{iJ} - \frac{2}{3}\bar{I}_1 F_{Ji}
\end{aligned} \tag{5.24}$$

$$\begin{aligned}
\frac{\partial \bar{I}_2}{\partial F_{iJ}} &= \frac{1}{J^{4/3}}\frac{\partial I_2}{\partial F_{iJ}} - \frac{4I_2}{3J^{7/3}}\frac{\partial J}{\partial F_{iJ}} \\
&= \frac{2}{J^{4/3}}\left(I_1 F_{iJ} - b_{ik}F_{kJ} - \frac{2I_2}{3}F_{Ji}\right) \\
&= \frac{2}{J^{2/3}}\bar{I}_1 F_{iJ} - \frac{2}{J^{4/3}}b_{ik}F_{kJ} - \frac{4\bar{I}_2}{3}F_{Ji}
\end{aligned} \tag{5.25}$$

and

$$\frac{\partial J}{\partial F_{iJ}} = JF_{Ji}^T \tag{5.26}$$

This relationship is derived in Appendix C.

The Cauchy stress is obtained from the strain-energy function by

$$t_{ij} = \frac{1}{J} F_{iK} N_{Kj} = \frac{1}{J} F_{iK} \frac{\partial W}{\partial F_{jK}} \tag{5.27}$$

Then

$$t_{ij} = \frac{2}{J^{5/3}} \left(\frac{\partial \widehat{W}}{\partial \bar{I}_1} + \bar{I}_1 \frac{\partial \widehat{W}}{\partial \bar{I}_2} \right) b_{ij} - \frac{2}{3J} \left(\bar{I}_1 \frac{\partial \widehat{W}}{\partial \bar{I}_1} + 2\bar{I}_2 \frac{\partial \widehat{W}}{\partial \bar{I}_2} \right) \delta_{ij} - \frac{2}{J^{7/3}} \frac{\partial \widehat{W}}{\partial \bar{I}_2} b_{ik} b_{kj} + \frac{\partial \widehat{W}}{\partial J} \delta_{ij} \tag{5.28}$$

or

$$t_{ij} = \frac{2}{J} \left[\frac{1}{J^{2/3}} \left(\frac{\partial \widehat{W}}{\partial \bar{I}_1} + \bar{I}_1 \frac{\partial \widehat{W}}{\partial \bar{I}_2} \right) b_{ij} - \left(\bar{I}_1 \frac{\partial \widehat{W}}{\partial \bar{I}_1} + 2\bar{I}_2 \frac{\partial \widehat{W}}{\partial \bar{I}_2} \right) \frac{\delta_{ij}}{3} - \frac{1}{J^{4/3}} \frac{\partial \widehat{W}}{\partial \bar{I}_2} b_{ik} b_{kj} \right] + \frac{\partial \widehat{W}}{\partial J} \delta_{ij} \tag{5.29}$$

With

$$\frac{\partial \widehat{W}}{\partial \bar{I}_1} = C_{10}, \tag{5.30a}$$

$$\frac{\partial \widehat{W}}{\partial \bar{I}_2} = C_{01} \tag{5.30b}$$

and

$$\frac{\partial \widehat{W}}{\partial J} = 2D_1(J-1) \tag{5.30c}$$

we obtain

$$t_{ij} = \frac{2}{J} \left[J^{-2/3} \left(C_{10} + \bar{I}_1 C_{01} \right) b_{ij} - \left(\bar{I}_1 C_{10} + 2\bar{I}_2 C_{01} \right) \frac{\delta_{ij}}{3} - J^{-4/3} C_{01} b_{ik} b_{kj} \right] + 2D_1(J-1)\delta_{ij} \tag{5.31}$$

The Kirchhoff stress tensor is given by

$$\begin{aligned}\tau_{ij} &= Jt_{ij} \\ &= 2\left(C_{10} + \bar{I}_1 C_{01}\right)\bar{b}_{ij} - 2\left(\bar{I}_1 C_{10} + 2\bar{I}_2 C_{01}\right)\frac{\delta_{ij}}{3} - 2C_{01}\bar{b}_{ik}\bar{b}_{kj} \\ &\quad + 2D_1 J(J-1)\delta_{ij}\end{aligned} \tag{5.32}$$

Then, with

$$C_{10} = \frac{G}{2}, \tag{5.33a}$$

$$C_{01} = 0 \tag{5.33b}$$

and

$$D_1 = \frac{K}{2} \tag{5.33c}$$

we get

$$\tau_{ij} = G\left(\bar{b}_{ij} - \frac{\bar{b}_{kk}}{3}\,\delta_{ij}\right) + KJ(J-1)\delta_{ij} \tag{5.34}$$

or

$$\tau_{ij} = G\mathrm{dev}\left[\bar{b}_{ij}\right] + KJ(J-1)\delta_{ij} \tag{5.35}$$

where

$$\mathrm{dev}[\bullet] = (\bullet) - \frac{1}{3}[(\bullet) : \mathbf{1}]\mathbf{1} \tag{5.36}$$

This Eulerian formulation is the one that is employed in the hyperelastic material model in the Abaqus/Standard® finite element code. It should be noted that if we had not assigned a value of 0 to C_{01}, we would have for τ_{ij} (Bower 2010),

$$\begin{aligned}\tau_{ij} = 2C_{10}\left(\bar{b}_{ij} - \frac{\bar{b}_{kk}}{3}\delta_{ij}\right) + 2C_{01}\left(\bar{b}_{kk}\bar{b}_{ij} - \frac{\left[\bar{b}_{kk}\right]^2}{3}\delta_{ij} - \bar{b}_{ik}\bar{b}_{kj} + \frac{\bar{b}_{kn}\bar{b}_{nk}}{3}\,\delta_{ij}\right) \\ + KJ(J-1)\delta_{ij}\end{aligned} \tag{5.37}$$

Now, we once again revisit the simple shear model shown in Fig. 3.2 for an example.

5.2 Example 3: Stress

$$[F] = \begin{bmatrix} 1 & 0 & 0.1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5.38a}$$

$$[\overline{F}] = \begin{bmatrix} 1 & 0 & 0.1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5.38b}$$

$$[\overline{b}] = [\overline{F}]\,[\overline{F}]^T = \begin{bmatrix} 1 & 0 & 0.1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0.1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1.01 & 0 & 0.1 \\ 0 & 1 & 0 \\ 0.1 & 0 & 1 \end{bmatrix} \tag{5.39}$$

$$\bar{I}_1 = 3.01 \tag{5.40a}$$

$$J = 1 \tag{5.40b}$$

$$G = 300 \text{ psi} \tag{5.40c}$$

yielding, from Equation (5.34),

$$\tau_{11} = 300\big(1.01 - 3.01/3\big] - 0 = 2 \text{ psi} \tag{5.41a}$$

$$\tau_{22} = 300(1 - 3.01/3) - 0 = -1 \text{ psi} \tag{5.41b}$$

$$\tau_{33} = 300(1 - 3.01/3) - 0 = -1 \text{ psi} \tag{5.41c}$$

$$\tau_{23} = \tau_{32} = 300(0) = 0 \tag{5.41d}$$

$$\tau_{31} = \tau_{13} = 300(0.1) = 30 \text{ psi} \tag{5.41e}$$

$$\tau_{12} = \tau_{21} = 300(0) = 0 \tag{5.41f}$$

or

$$[\tau] = \begin{bmatrix} 2 & 0 & 30 \\ 0 & -1 & 0 \\ 30 & 0 & -1 \end{bmatrix} \text{ psi} \tag{5.42}$$

which matches our earlier determined value of the Kirchhoff stress for the simple shear example.

We consider now the uniaxial elongation, in the 1-direction, of an *incompressible* uniform rod. The deformation gradient, in terms of the principal stretches λ_1, λ_2, λ_3, is

$$[F] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \tag{5.43}$$

The corresponding Mooney-Rivlin model strain-energy function can be written as

$$\breve{W} = C_{10}(I_1 - 3) + C_{01}(I_2 - 3); \quad I_3 = 1 \tag{5.44}$$

or as

$$\begin{aligned} \widetilde{W} &= C_{10}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3\right) + C_{01}\left(\lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 - 3\right) \\ &\quad + D_1(J - 1)^2 \end{aligned} \tag{5.45}$$

where $J = 1$. Then we can write

$$\lambda_1 \frac{\partial \widetilde{W}}{\partial \lambda_1} = 2C_{10}\lambda_1^2 + 2C_{01}\lambda_1^2\left(\lambda_2^2 + \lambda_3^2\right) \tag{5.46a}$$

$$\lambda_2 \frac{\partial \widetilde{W}}{\partial \lambda_2} = 2C_{10}\lambda_2^2 + 2C_{01}\lambda_2^2\left(\lambda_3^2 + \lambda_1^2\right) \tag{5.46b}$$

and

$$\lambda_3 \frac{\partial \widetilde{W}}{\partial \lambda_3} = 2C_{10}\lambda_3^2 + 2C_{01}\lambda_3^2\left(\lambda_1^2 + \lambda_2^2\right) \tag{5.46c}$$

Further, considering that the material is incompressible, with

$$\lambda_1\lambda_2\lambda_3 = 1 \tag{5.47}$$

we can write

$$\lambda_1 \frac{\partial \widetilde{W}}{\partial \lambda_1} = 2C_{10}\lambda_1^2 + 2C_{01}\left(\frac{1}{\lambda_2^2} + \frac{1}{\lambda_3^2}\right) \tag{5.48a}$$

$$\lambda_2 \frac{\partial \widetilde{W}}{\partial \lambda_2} = 2C_{10}\lambda_2^2 + 2C_{01}\left(\frac{1}{\lambda_3^2} + \frac{1}{\lambda_1^2}\right) \tag{5.48b}$$

and

$$\lambda_3 \frac{\partial \widetilde{W}}{\partial \lambda_3} = 2C_{10}\lambda_3^2 + 2C_{01}\left(\frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2}\right) \tag{5.48c}$$

We can also write

$$\lambda_2 = \lambda_3 = \lambda_1^{-1/2} \tag{5.49}$$

and thus obtain an expression for the axial Cauchy stress:

$$\begin{aligned} t_{11} &= -p + \lambda_1 \frac{\partial \widetilde{W}}{\partial \lambda_1} = -p + 2C_{10}\lambda_1^2 + 2C_{01}\left(\frac{1}{\lambda_2^2} + \frac{1}{\lambda_2^2}\right) \\ &= -p + 2C_{10}\lambda_1^2 + 2C_{01}(\lambda_1 + \lambda_1) \\ &= -p + 2C_{10}\lambda_1^2 + 4C_{01}\lambda_1 \end{aligned} \tag{5.50}$$

The hydrostatic pressure p acts as a *Lagrange multiplier* (a negative stress) to enforce the incompressibility constraint—a pressure can be applied to an incompressible body without changing its shape. Thus we see that stress is not completely determined by the strain for the incompressible condition because a hydrostatic pressure can be added to t_{ij} without changing λ_i. And, we can write an expression for the lateral Cauchy stress, equating it to zero,

$$\begin{aligned} t_{22} &= -p + \lambda_2 \frac{\partial \widetilde{W}}{\partial \lambda_2} = -p + 2C_{10}\lambda_2^2 + 2C_{01}\left(\frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2}\right) = 0 \\ &= -p + 2C_{10}\lambda_1^{-1} + 2C_{01}\left(\frac{1}{\lambda_1^2} + \lambda_1\right) = 0 \end{aligned} \tag{5.51}$$

obtaining the expression

$$p = 2C_{10}\lambda_1^{-1} + 2C_{01}\left(\frac{1}{\lambda_1^2} + \lambda_1\right) \tag{5.52}$$

which, when substituted into Equation (5.50), gives

$$\begin{aligned} t_{11} &= -2C_{10}\lambda_1^{-1} - 2C_{01}\left(\frac{1}{\lambda_1^2} + \lambda_1\right) + 2C_{10}\lambda_1^2 + 4C_{01}\lambda_1 \\ &= 2C_{10}\left(\lambda_1^2 - \lambda_1^{-1}\right) + 2C_{01}\left(2\lambda_1 - \frac{1}{\lambda_1^2} - \lambda_1\right) \\ &= 2C_{10}\left(\lambda_1^2 - \lambda_1^{-1}\right) + 2C_{01}\left(\lambda_1 - \lambda_1^{-2}\right) \\ &= 2\left(C_{10} + \frac{C_{01}}{\lambda_1}\right)\left(\lambda_1^2 - \frac{1}{\lambda_1}\right) \end{aligned} \tag{5.53}$$

We can obtain the expression for the axial second Piola-Kirchhoff stress for this case from

$$S_{JI} = F_{Ji} t_{ij} F_{Ij}^T \tag{5.54}$$

finding that

$$S_{11} = \lambda_1^{-2} t_{11} = 2\left(C_{10} + \frac{C_{01}}{\lambda_1}\right)\left(1 - \frac{1}{\lambda_1^3}\right) \tag{5.55}$$

and an expression for the axial nominal stress for this case from

$$N_{Ij} = S_{IJ} F_{jJ}^T \tag{5.56}$$

finding that

$$N_{11} = \lambda_1 S_{11} = 2\left(C_{10} + \frac{C_{01}}{\lambda_1}\right)\left(\lambda_1 - \frac{1}{\lambda_1^2}\right) \tag{5.57}$$

We will now employ a numerical example to illustrate this formulation.

5.3 Example 4: Uniaxial Stress—Incompressible Material

We consider a uniform rod of an incompressible neo-Hookean material, loaded in the axial 1-direction. With $\frac{\partial u_1}{\partial X_1}$ set equal to 0.1 and the longitudinal stretch λ_1 thus equal to 1 + 0.1, with λ_2 and λ_3 defining the transverse stretch,

$$G = 300 \text{ psi} \tag{5.58}$$

and

$$C_{10} = \frac{G}{2} \tag{5.59a}$$

$$C_{01} = 0 \tag{5.59b}$$

we obtain from Equation (5.57)

$$N_{11} = 82.0661157\,\text{psi} \tag{5.60}$$

from Equation (5.53)

$$t_{11} = 90.2727273\,\text{psi} \tag{5.61}$$

and from Equation (5.55)

$$S_{11} = 74.6055597\,\text{psi} \tag{5.62}$$

We will now employ a numerical example to demonstrate a nearly incompressible material model.

5.4 Example 5: Uniaxial Stress—Nearly Incompressible Material

We again consider a uniform rod, but now of a *nearly incompressible* neo-Hookean material, loaded in the axial 1-direction. With $\frac{\partial u_1}{\partial X_1}$ set equal to 0.1 and the longitudinal stretch λ_1 thus equal to 1 + 0.1, with λ_2 and λ_3 defining the transverse stretch, we can write

$$\lambda_2 = \lambda_3 = \lambda_1^{-1/2} J^{1/2} \tag{5.63a}$$

$$G = 300\text{ psi} \tag{5.63b}$$

$$K = 200000\text{ psi} \tag{5.63c}$$

$$[F] = \begin{bmatrix} 1.1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix} \tag{5.63d}$$

$$[F]^{-T} = \begin{bmatrix} \frac{1}{1.1} & 0 & 0 \\ 0 & \frac{1}{\lambda_2} & 0 \\ 0 & 0 & \frac{1}{\lambda_2} \end{bmatrix} \tag{5.63e}$$

We can then also note that

$$\lambda_2 = \sqrt{\frac{J}{1.1}} \tag{5.64a}$$

$$[b] = \begin{bmatrix} 1.1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} 1.1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 1.21 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_2^2 \end{bmatrix} \tag{5.64b}$$

and

$$\bar{I}_1 = \frac{b_{ii}}{J^{2/3}} = (1.1)^{-2/3} \lambda_2^{-4/3} \left(1.21 + 2\lambda_2^2\right) \tag{5.64c}$$

We then input these properties and expressions into Equation (6.66) and, employing a simple convergence routine, obtain the value of J that results from P_{22} and P_{33} being set equal to zero, which is, by observation, a necessary boundary condition, and consequently the components of P_{iJ}, with the converged value of J being 1.000150348529, consistent with there being some volume change (J = 1$\rightarrow$ zero volume change), and of λ_2 being 0.953534262,

$$[N]^T = [P] = \begin{bmatrix} 82.0181235 & 0 & 0 \\ 0 & -0.0029034 & 0 \\ 0 & 0 & -0.0029034 \end{bmatrix} \text{psi} \tag{5.65}$$

With these values of J and λ_2, we can compute, using the same procedure, the Cauchy stress tensor t_{ij} from Equation (5.34):

$$[t] = \begin{bmatrix} 90.2063767 & 0 & 0 \\ 0 & -0.0027681 & 0 \\ 0 & 0 & -0.0027681 \end{bmatrix} \text{psi} \tag{5.66}$$

Knowing that

$$S_{JK} = P_{iJ}^T F_{Ki}^T \tag{5.67}$$

we can also compute the value of the second Piola-Kirchhoff stress tensor:

$$[S] = \begin{bmatrix} 82.0181235 & 0 & 0 \\ 0 & -0.0029034 & 0 \\ 0 & 0 & -0.0029034 \end{bmatrix} \begin{bmatrix} 0.909090909 & 0 & 0 \\ 0 & 1.048730014 & 0 \\ 0 & 0 & 1.048730014 \end{bmatrix}$$
$$= \begin{bmatrix} 74.5619304 & 0 & 0 \\ 0 & -0.0030449 & 0 \\ 0 & 0 & -0.0030449 \end{bmatrix} \text{psi} \tag{5.68}$$

In these stress expressions, the 22 and 33 locations have nonzero values only because of the non-exactness of the numerical solution algorithm.

When we examine and compare the values of N_{11}, t_{11}, and S_{11} obtained from Example **4** and Example **5**, we see that Example **5** truly represents a *nearly incompressible* solution. Also, we should be able to see that if we were operating in the small strain regime, the values of N_{11}, t_{11}, and S_{11} would converge to the same value. We should also make note of the fact that most commercial finite element (FE) codes solve hyperelasticity problems using a *nearly incompressible* model.

We shift our focus now from deliberation of the Mooney-Rivlin and neo-Hookean models to the consideration of other hyperelastic strain-energy models.

The Arruda-Boyce strain-energy function (Arruda and Boyce 1993) is described in some detail in Chap. 4, with the strain-energy function given by Equation (4.19). The expression for the Cauchy stress tensor for this model, derived from Equation (4.19) (Bower 2010), is

$$t_{ij} = \frac{G}{J^{5/3}}\left(1 + \frac{b_{kk}}{5J^{2/3}n} + \frac{33b_{kk}^2}{525n^2J^{4/3}} + \cdots\right)\left(b_{ij} - \frac{b_{kk}}{3}\delta_{ij}\right) + K(J-1)\delta_{ij} \tag{5.69}$$

Additionally, we consider another hyperelastic strain-energy model, in this case a *foam model*, the Ogden-Storakers hyperelastic foam model (Storakers 1986). This formulation is also described in some detail in Chap. 4, with the strain-energy function defined in Equation (4.26). The expression for the Cauchy stress tensor for this model, derived from Equation (4.26) (Schrodt et al. 2005), is

$$\mathbf{t} = \frac{2}{J}\sum_{i=1}^{3}\sum_{k=1}^{N}\left\{\frac{\mu_k}{\alpha_k}\left[\lambda_i^{\alpha_k} + \frac{1}{\alpha_k}J\frac{\partial g(J)}{\partial J}\right]\mathbf{n}^{(i)}\mathbf{n}^{(i)}\right\} \tag{5.70}$$

with λ_i being the eigenvalues of the right stretch tensor $\mathbf{U}$ and $\mathbf{n}^{(i)}$ being the eigenvectors of the left stretch tensor $\mathbf{v}$, referencing Chap 3. A form of the volumetric function $g(J)$ is given by Storakers (1986):

$$g(J) = \frac{1}{\beta_k}\left(J^{-\alpha_k\beta_k} - 1\right) \tag{5.71}$$

where μ_k, α_k, and β_k are material properties, β_k defining the degree of compressibility.

Now, we will focus attention on yet a different set of strain-energy functions, those which characterize *soft biological tissues*.

The Fung strain-energy function (Fung 1993), defined in Equation (4.30), is probably the foremost soft biological tissue model. It is widely used, and is the forerunner of a number of other widely applied soft tissue models. It is described in some detail in Chap. 4. It will be illustrated in the following numerical example.

5.5 Example 6: Soft Biological Tissue

We consider here an example related to the modeling of the carotid artery of a rabbit; we begin with the strain-energy expression for a Fung model of this material (Volokh 2016):

$$W = \frac{c}{2}\left\{\exp\left(c_1 E_{RR}^2 + c_2 E_{\phi\phi}^2 + c_3 E_{ZZ}^2 + 2c_4 E_{RR}E_{\phi\phi} + 2c_5 E_{\phi\phi}E_{ZZ} + 2c_6 E_{ZZ}E_{RR}\right) - 1\right\} \tag{5.72}$$

where the material constant $c = 3.909$ psi and where the c_i dimensionless constants are

$$c_1 = 0.0089 \quad c_2 = 0.9925 \quad c_3 = 0.4180 \quad c_4 = 0.0193 \quad c_5 = 0.0749$$
$$c_6 = 0.0295$$

This material is thus considered orthotropic, but with no shear strain-energy terms. The artery wall is not incompressible, i.e., its volume can change, and hence there is no indeterminacy in the mean stress (pressure).

The deformation gradient and right Cauchy-Green deformation tensor are given by

$$[F]^T = [F] = \begin{bmatrix} \lambda_R & 0 & 0 \\ 0 & \lambda_\Phi & 0 \\ 0 & 0 & \lambda_Z \end{bmatrix} \tag{5.73a}$$

and

$$[C] = [F]^T[F] = \begin{bmatrix} \lambda_R^2 & 0 & 0 \\ 0 & \lambda_\Phi^2 & 0 \\ 0 & 0 & \lambda_Z^2 \end{bmatrix} \tag{5.73b}$$

The Green-Lagrange strain tensor is

$$E_{IJ} = \frac{1}{2}(C_{IJ} - \delta_{IJ}) \tag{5.74}$$

or, in terms of the stretches,

$$E_{RR} = \frac{1}{2}\left(\lambda_R^2 - 1\right) \tag{5.75a}$$

$$E_{\Phi\Phi} = \frac{1}{2}\left(\lambda_\Phi^2 - 1\right) \tag{5.75b}$$

$$E_{\Phi\Phi} = \frac{1}{2}\left(\lambda_\Phi^2 - 1\right) \tag{5.75c}$$

The corresponding second Piola-Kirchhoff stress tensor is given by

$$[S] = \begin{bmatrix} S_{RR} & 0 & 0 \\ 0 & S_{\Phi\Phi} & 0 \\ 0 & 0 & S_{ZZ} \end{bmatrix} \tag{5.76}$$

where

$$S_{IJ} = \frac{\partial W}{\partial E_{IJ}} \tag{5.77}$$

Then, solving for the second Piola-Kirchhoff stresses, we get

$$\begin{aligned} S_{RR} = \frac{\partial W}{\partial E_{RR}} = c(c_1E_{RR} + c_4E_{\Phi\Phi} + c_6E_{ZZ})\exp\big(&c_1E_{RR}^2 + c_2E_{\Phi\Phi}^2 + c_3E_{ZZ}^2 \\ &+2c_4E_{RR}E_{\Phi\Phi} + 2c_5E_{\Phi\Phi}E_{ZZ} + 2c_6E_{ZZ}E_{RR}\big) \end{aligned} \tag{5.78a}$$

$$\begin{aligned} S_{\Phi\Phi} = \frac{\partial W}{\partial E_{\Phi\Phi}} = c(c_2E_{\Phi\Phi} + c_4E_{RR} + c_5E_{ZZ})\exp\big(&c_1E_{RR}^2 + c_2E_{\Phi\Phi}^2 + c_3E_{ZZ}^2 \\ &+2c_4E_{RR}E_{\Phi\Phi} + 2c_5E_{\Phi\Phi}E_{ZZ} + 2c_6E_{ZZ}E_{RR}\big) \end{aligned} \tag{5.78b}$$

$$\begin{aligned} S_{ZZ} = \frac{\partial W}{\partial E_{ZZ}} = c(c_3E_{ZZ} + c_5E_{\Phi\Phi} + c_6E_{RR})\exp\big(&c_1E_{RR}^2 + c_2E_{\Phi\Phi}^2 + c_3E_{ZZ}^2 \\ +2c_4E_{RR}E_{\Phi\Phi} + 2c_5E_{\Phi\Phi}E_{ZZ} + 2c_6E_{ZZ}E_{RR}\big)& \end{aligned} \tag{5.78c}$$

Then, inputting the numerical values of c and c_i, and numerical values of E_{RR}, $E_{\Phi\Phi}$, and E_{ZZ} obtained from Equations (5.75a), (5.75b), and (5.75c), respectively, into Equations (5.78a), (5.78b), and (5.78c), we can obtain the numerical values of the components of the second Piola-Kirchhoff stress tensor. Of course this requires our having measured values of λ_R, λ_Φ, and λ_Z to be inserted into Equations (5.75a), (5.75b), and (5.75c), respectively, *a priori*.

We can further determine the components of the *stiffness tensor* of the modeled tissue

$$\left[D^{SE}\right] = \begin{bmatrix} D_{RRRR} & D_{RR\Phi\Phi} & D_{RRZZ} \\ D_{\Phi\Phi RR} & D_{\Phi\Phi\Phi\Phi} & D_{\Phi\Phi ZZ} \\ D_{ZZRR} & D_{ZZ\Phi\Phi} & D_{ZZZZ} \end{bmatrix} \tag{5.79}$$

by evaluating the partial derivatives $D_{RRRR} = \dfrac{\partial S_{RR}}{\partial E_{RR}}$, $D_{RR\Phi\Phi} = \dfrac{\partial S_{RR}}{\partial E_{\Phi\Phi}}$, etc.

We consider now the uniaxial elongation, in the 1-direction, of an *incompressible* soft biological tissue specimen. The deformation gradient, in terms of the principal stretches λ_1, λ_2, λ_3, is

$$[F] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \tag{5.80}$$

The Veronda-Westmann model strain-energy function (Veronda and Westmann 1970), which is highly appropriate for modeling this type of material, is typically expressed as

$$\breve{W} = \mu_0\left(\frac{\exp[\beta(I_1 - 3)] - 1}{\beta} - \frac{I_2 - 3}{2}\right) \tag{5.81}$$

or it can be written as

$$\widetilde{W} = \frac{\mu_0}{\beta}\left[\exp\left(\beta\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3\right)\right) - 1\right] - \frac{\mu_0}{2}\left(\lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 - 3\right) \tag{5.82}$$

Then we can write

$$\lambda_1\frac{\partial\widetilde{W}}{\partial\lambda_1} = 2\mu_0\lambda_1^2\exp\left(\beta\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3\right)\right) - \mu_0\left(\lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2\right) \tag{5.83a}$$

and

$$\lambda_2\frac{\partial\widetilde{W}}{\partial\lambda_2} = 2\mu_0\lambda_2^2\exp\left(\beta\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3\right)\right) - \mu_0\left(\lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2\right) \tag{5.83b}$$

Further, considering that the material is incompressible, with

$$\lambda_1\lambda_2\lambda_3 = 1 \tag{5.84a}$$

$$\lambda_2 = \lambda_3 = \lambda_1^{-1/2} \tag{5.84b}$$

and with

$$t_{11} = -p + \lambda_1 \frac{\partial \widetilde{W}}{\partial \lambda_1} \tag{5.85a}$$

and

$$t_{22} = -p + \lambda_2 \frac{\partial \widetilde{W}}{\partial \lambda_2} = 0 \tag{5.85b}$$

we can write, combining Equations (5.83a) and (5.85a),

$$t_{11} = -p + 2\mu_0 \lambda_1^2 \exp\left(\beta\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3\right)\right) - \mu_0\left(\lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2\right) \tag{5.86}$$

and, combining Equations (5.83b) and (5.85b),

$$p = 2\mu_0 \lambda_2^2 \exp\left(\beta\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3\right)\right) - \mu_0\left(\lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2\right) \tag{5.87}$$

Then, substituting Equation (5.87) into Equation (5.86), we obtain the expression for the longitudinal Cauchy stress

$$t_{11} = 2\mu_0\left(\lambda_1^2 - \lambda_1^{-1}\right)\left[\exp\left(\beta\left(\lambda_1^2 + 2\lambda_1^{-1} - 3\right)\right) - \frac{1}{2\lambda_1}\right] \tag{5.88}$$

where μ_0 is the initial unstrained shear modulus and β is a parameter that denotes the degree of nonlinearity of the material. In order that the strain energy shall increase with the deformation from the natural state, $\beta \geq 0$ is necessary (Beatty 1987).

We can write

$$\begin{aligned} \mathrm{E} &= \frac{dt_{11}}{d\varepsilon_{11}} = \frac{dt_{11}}{d\lambda_1}\frac{d\lambda_1}{d\varepsilon_{11}} \\ &= 2\mu_0\left(\exp\left(\beta\left(\lambda_1^2 + 2\lambda_1^{-1} - 3\right)\right)\left[2\lambda_1 + \lambda_1^{-2} + 2\beta\left(\lambda_1^3 - 2 + \lambda_1^{-3}\right)\right] - \left(\frac{1}{2} + \lambda_1^{-3}\right)\right) \end{aligned} \tag{5.89}$$

where E is the Young's modulus. With $\lambda_1 = 1$ (zero strain), we get

$$\mathrm{E} = 3\mu_0 \tag{5.90}$$

which implies that Young's modulus at zero strain, the only condition wherein it is meaningful in the case of finite deformation, is completely determined by the initial shear modulus μ_0, and that any change in Young's modulus as a function of strain is solely determined by β (Oberai et al. 2009). The Young's modulus of a typical soft biological tissue, e.g., human skin (E $\simeq$ 1MPa = 145psi), is very low compared to that of engineering materials (Akhtar et al. 2011). The range of shear modulus values for adipose tissue (human body fat, breast tissue) is 0.5–25 kPa (Gefen and Dilmoney 2007).

Chapter 6
Tangent Moduli

Abstract Clearly, in the case of small strain linear elasticity, the tangent modulus is constant regardless of deformation, i.e., since the stress–strain curve is linear, the stiffness does not change as deformation changes. However, for a hyperelastic model, differentiating the strain-energy function with respect to either the finite strain tensor or one of the two Cauchy-Green deformation tensors yields elastic "constants," the magnitude of which depend upon the level of deformation. Mathematically, taking the second derivative of the strain-energy function is equivalent to taking the first derivative of the stress–strain curve, yielding the fourth-order tangent stiffness tensor. Hence, for any point on the stress–strain curve the tangent to the curve at that point, i.e., at that amount of deformation, is obtained. Thus, the elastic "constants" obtained by differentiating the strain-energy function twice are referred to as the tangent elastic properties. Depending upon the combination of stress and strain tensors employed, corresponding constitutive models are derived. The correct employment of these model-developing procedures is very critical to solving problems in large deformation nonlinear elasticity. A very important new approach to deriving the fourth-order "first elasticity tensor" is given. A numerical example is presented to augment the developed theory.

It is clear that for linear elasticity the tangent modulus D_{ijkl} will be constant regardless of deformation. If we differentiate any strain-energy function with respect to either the *finite strain tensor* or the *right Cauchy-Green deformation tensor* we obtain elastic "constants." However, for the case of a nonlinear material, these elastic "constants" will differ depending on the level of deformation. Mathematically, when we take the second derivative of the strain-energy function, we are taking the first derivative of the stress–strain curve. Hence, for any point on the stress–strain curve we are obtaining the tangent to the curve at that point, i.e., at that amount of deformation. Thus, we refer to the elastic "constants" obtained by differentiating the strain-energy function twice as the *tangent elastic properties*. This procedure is very critical to solving problems in large deformation nonlinear elasticity.

For small deformation linear elasticity, the tangent elastic constants are equivalent to the overall elastic properties since the stress–strain curve is linear and therefore stiffness does not change as deformation changes. However, for a nonlinear elastic material, the slope of the stress–strain curve changes with deformation, hence the instantaneous stiffness of the material changes with deformation.

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Depending upon the combination of stress and strain tensors employed, we derive corresponding constitutive models.

Operating on Equation (5.1) with (5.2), we get for the fourth-order tangent stiffness tensor,

$$D_{IJKL} = \frac{\partial S_{IJ}}{\partial C_{KL}} + \frac{\partial S_{IJ}}{\partial C_{LK}} \tag{6.1}$$

yielding

$$D^{SE}_{IJKL} = C_{10}\left(\bar{I}_1\right)^{**}_{IJKL} + C_{01}\left(\bar{I}_2\right)^{**}_{IJKL} + 2D_1\left[(J)^{*}_{IJ}(J)^{*}_{KL} + (J-1)(J)^{**}_{IJKL}\right] \tag{6.2}$$

where

$$\begin{aligned}\left(\bar{I}_1\right)^{**}_{IJKl} = {} & (I_3)^{-1/3}(I_1)^{**}_{IJKL} - \frac{1}{3}(I_3)^{-4/3}\left[(I_1)^{*}_{IJ}(I_3)^{*}_{KL} + (I_3)^{*}_{IJ}(I_1)^{*}_{KL}\right.\\ & \left. + (I_1)(I_3)^{**}_{IJKL}\right] + \frac{4}{9}\left(I_1 I_3^{-7/3}\right)(I_3)^{*}_{IJ}(I_3)^{*}_{KL}\end{aligned} \tag{6.3a}$$

$$\begin{aligned}\left(\bar{I}_2\right)^{**}_{IJKL} = {} & (I_3)^{-2/3}(I_2)^{**}_{IJKL} - \frac{2}{3}(I_3)^{-5/3}\left[(I_2)^{*}_{IJ}(I_3)^{*}_{KL} + (I_3)^{*}_{IJ}(I_2)^{*}_{KL} + (I_2)(I_3)^{**}_{IJKL}\right]\\ & + \frac{10}{9}\left(I_2 I_3^{-8/3}\right)(I_3)^{*}_{IJ}(I_3)^{*}_{KL}\end{aligned} \tag{6.3b}$$

$$(J)^{**}_{IJKL} = -\frac{1}{4}(I_3)^{-3/2}(I_1)^{*}_{IJ}(I_3)^{*}_{KL} + (I_3)^{*}_{IJ}(I_1)^{*}_{KL} + \frac{1}{2}(I_3)^{-1/2}(I_3)^{**}_{IJKL} \tag{6.3c}$$

where

$$(I_1)^{**}_{IJKL} = 0 \tag{6.4a}$$

$$(I_2)^{**}_{IJKL} = 4\delta_{IJ}\delta_{KL} - 2(\delta_{IK}\delta_{JL} + \delta_{IL}\delta_{JK}) \tag{6.4b}$$

$$(I_3)^{**}_{IJKL} = (\hat{\varepsilon}_{IKC}\hat{\varepsilon}_{JLF} + \hat{\varepsilon}_{ILC}\hat{\varepsilon}_{JKF} + \hat{\varepsilon}_{JKC}\hat{\varepsilon}_{ILF} + \hat{\varepsilon}_{JLC}\hat{\varepsilon}_{IKF})C_{CF} \tag{6.4c}$$

Since $C_{IJ} = C_{JI}$, we could also write, from Equation (6.1),

$$D^{SE}_{IJKL} = 2\frac{\partial S_{IJ}}{\partial C_{KL}} = 4\frac{\partial^2 W}{\partial C_{IJ}\partial C_{KL}} \tag{6.5}$$

The formulation comprising Equations (5.2) through (5.5) and (6.2) through (6.5) is from Sussman and Bathe (1987). A slightly different derivation of these equations is found in Appendix B, one that is somewhat more easily programmed.

An internal strain-energy function for a *compressible* neo-Hookean isotropic material (Belytschko et al. 2000), is expressed as

$$W\left(\underset{=}{C}\right) = \frac{1}{2}G\left(\mathrm{tr}\ \underset{=}{C} - 3\right) - G\ln J + \left(K - \frac{2G}{3}\right)(\ln J)^2 \tag{6.6}$$

The elasticity tensors corresponding to, and derived from, this function are

$$D^{SE}_{IJKL} = \lambda C^{-1}_{IJ}C^{-1}_{KL} + \mu\left(C^{-1}_{IK}C^{-1}_{JL} + C^{-1}_{IL}C^{-1}_{KJ}\right) \tag{6.7a}$$

$$D^{\tau}_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) \tag{6.7b}$$

where $\lambda \equiv K - 2G/3$ and $\mu = G(1 + 2\ln J/3) - K\ln J$. Nearly incompressible behavior is obtained for $K >> G$. It can be demonstrated numerically (computationally) that the tangent moduli obtained from Equation (6.7a), for $K >> G$, match those obtained from Equation (6.2). It can also be demonstrated that the push-forward transformation

$$D^{\tau}_{ijkl} = F_{iI}F_{jJ}F_{kK}F_{lL}D^{SE}_{IJKL} \tag{6.8}$$

applied to the tangent moduli obtained from Equation (6.2) will yield values that match those that would be obtained from Equation (6.7b), for $K >> G$.

We can also develop an expression for the Eulerian tangent stiffness by differentiating the expression for the Kirchhoff stress tensor given in Equation (5.34). We begin with the relationship

$$D^{\tau}_{ijkl} = \frac{\partial\tau_{ij}}{\partial F_{kM}}F_{lM} \tag{6.9}$$

where the Kirchhoff stress tensor is given by

$$\tau_{ij} = G\left(\overline{b}_{ij} - \frac{1}{3}\overline{b}_{kk}\delta_{ij}\right) + K(J-1)J\delta_{ij} \tag{6.10}$$

Evaluating the derivatives is a tedious, but straightforward, exercise in indicial notation, aided by the following identity,

$$\frac{\partial J}{\partial F_{kM}} = JF^{T}_{Mk} \tag{6.11}$$

which is derived in Appendix C. We know that

$$\overline{b}_{ij} = J^{-2/3}F_{iM}F_{jM} \tag{6.12}$$

Then, taking the derivative,

$$\frac{\partial \overline{b}_{ij}}{\partial F_{kM}} = \frac{\partial \left(J^{-2/3} b_{ij}\right)}{\partial F_{kM}} = \frac{1}{J^{2/3}} \frac{\partial b_{ij}}{\partial F_{kM}} + \frac{\partial J^{-2/3}}{\partial F_{kM}} b_{ij} = \frac{1}{J^{2/3}} \frac{\partial b_{ij}}{\partial F_{kM}} + \frac{\partial J^{-2/3}}{\partial J} \frac{\partial J}{\partial F_{kM}} b_{ij}$$
$$= \frac{1}{J^{2/3}} \frac{\partial b_{ij}}{\partial F_{kM}} - \frac{2}{3} J^{-5/3} J F^{T}_{Mk} b_{ij} = \frac{1}{J^{2/3}} \left(\frac{\partial b_{ij}}{\partial F_{kM}} - \frac{2}{3} F^{T}_{Mk} b_{ij} \right) \tag{6.13}$$

Evaluating the derivative,

$$\frac{\partial b_{ij}}{\partial F_{kM}} = \frac{\partial F_{iM}}{\partial F_{kM}} F_{jM} + F_{iM} \frac{\partial F_{jM}}{\partial F_{kM}} = \delta_{ik} F_{jM} + \delta_{jk} F_{iM} \tag{6.14}$$

leads to

$$\frac{\partial \overline{b}_{ij}}{\partial F_{kM}} = \frac{1}{J^{2/3}} \left(\delta_{ik} F_{jM} + \delta_{jk} F_{iM} \right) - \frac{2}{3} F^{T}_{Mk} \overline{b}_{ij} \tag{6.15}$$

and

$$\frac{\partial \overline{b}_{ij}}{\partial F_{kM}} F_{lM} = \delta_{ik} \overline{b}_{jl} + \delta_{jk} \overline{b}_{il} - \frac{2}{3} \delta_{kl} \overline{b}_{ij} \tag{6.16}$$

Now,

$$\frac{\partial b_{kk}}{\partial F_{kM}} = \frac{\partial F_{kM}}{\partial F_{kM}} F_{kM} + F_{kM} \frac{\partial F_{kM}}{\partial F_{kM}} = 2 F_{kM} \tag{6.17}$$

leads to

$$\frac{\partial \overline{b}_{kk}}{\partial F_{kM}} = \frac{1}{J^{2/3}} \frac{\partial b_{kk}}{\partial F_{kM}} - \frac{2}{3} b_{kk} J^{-5/3} J F^{T}_{Mk} = \frac{2}{J^{2/3}} \left(F_{kM} - \frac{2 b_{kk}}{3} F^{T}_{Mk} \right)$$
$$= \frac{2}{J^{2/3}} F_{kM} - \frac{2}{3} \overline{b}_{kk} F^{T}_{Mk} \tag{6.18}$$

and

$$\frac{\partial \overline{b}_{kk}}{\partial F_{kM}} F_{lM} = 2 \overline{b}_{kl} - \frac{2}{3} \overline{b}_{kk} F_{Mk} F_{lM} = 2 \overline{b}_{kl} - \frac{2}{3} \overline{b}_{kk} \delta_{kl} \tag{6.19}$$

Also,

$$\frac{\partial J}{\partial F_{kM}} F_{lM} = J F^{T}_{Mk} F_{lM} \tag{6.20}$$

and

$$\frac{\partial(J-1)}{\partial F_{kM}} = \frac{\partial(J-1)}{\partial J}\frac{\partial J}{\partial F_{kM}} = JF^{T}_{Mk} \tag{6.21}$$

Then, summation of the appropriate terms yields

$$D^{\tau}_{ijkl} = G\left[\delta_{ik}\overline{b}_{jl} + \overline{b}_{il}\delta_{jk} - \frac{2}{3}\left(\delta_{kl}\overline{b}_{ij} + \overline{b}_{kl}\delta_{ij}\right) + \frac{2}{3}\left(\frac{\overline{b}_{qq}}{3}\right)\delta_{ij}\delta_{kl}\right]$$
$$+ K(2J-1)J\delta_{ij}\delta_{kl} \tag{6.22}$$

thus giving the tangent stiffness in terms of the left Cauchy-Green deformation tensor. Now, simply modifying this formulation by writing

$$D^{\tau}_{ijkl} = \frac{1}{2}\left(\frac{\partial\tau_{ij}}{\partial F_{kM}}F_{lM} + \frac{\partial\tau_{ij}}{\partial F_{lM}}F_{kM}\right) \tag{6.23}$$

instead of $D^{\tau}_{ijkl} = \frac{\partial\tau_{ij}}{\partial F_{kM}}F_{lM}$ is consistent with the fact that D^{τ}_{ijkl} is a symmetric tensor and ensures that it will have the correct symmetries.

6.1 Example 7: Tangent Moduli

Again considering the simple shear model, with $G = 300$ psi and $K = 300{,}000$ psi, we have

$$\left[\overline{F}\right] = \begin{bmatrix} 1 & 0 & 0.1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{6.24a}$$

$$\left[\overline{F}\right]^{T} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0.1 & 0 & 1 \end{bmatrix} \tag{6.24b}$$

$$\left[\overline{b}\right] = \left[\overline{F}\right]\left[\overline{F}\right]^{T} = \begin{bmatrix} 1.01 & 0 & 0.1 \\ 0 & 1 & 0 \\ 0.1 & 0 & 1 \end{bmatrix} \tag{6.25a}$$

$$\left[\overline{b}\right]^{-1} = \begin{bmatrix} 1 & 0 & -0.1 \\ 0 & 1 & 0 \\ -0.1 & 0 & 1.01 \end{bmatrix} \tag{6.25b}$$

From Equations (5.10) and (5.27), along with the Voigt-Mandel transformation, we get

$$t_{11} = stress(1) = 300(1.01 - 3.01/3) + 0 = 2 \text{ psi} \tag{6.26a}$$

$$t_{22} = stress(2) = 300(1 - 3.01/3) + 0 = -1 \text{ psi} \tag{6.26b}$$

$$t_{33} = stress(3) = 300(1 - 3.01/3) + 0 = -1 \text{ psi} \tag{6.26c}$$

$$t_{23} = t_{32} = stress(4) = 300(0) = 0 \tag{6.26d}$$

$$t_{31} = t_{13} = stress(5) = 300(0.1) = 30 \text{ psi} \tag{6.26e}$$

$$t_{12} = t_{21} = stress(6) = 300(0) = 0 \tag{6.26f}$$

or, in matrix format, the Cauchy stress tensor is then

$$[t] = \begin{bmatrix} 2 & 0 & 30 \\ 0 & -1 & 0 \\ 30 & 0 & -1 \end{bmatrix} \text{ psi} \tag{6.27}$$

Then, evaluating Equation (6.22) and applying Equation (6.23), we have, in Voigt-Mandel form,

$$\left[D^{\tau}_{ijkl}\right] = \begin{bmatrix} 300402.6667 & 299798.6667 & 299798.6667 & 0 & 20 & 0 \\ 299798.6667 & 300400.6667 & 299800.6667 & 0 & -40 & 0 \\ 299798.6667 & 299800.6667 & 300400.6667 & 0 & 20 & 0 \\ 0 & 0 & 0 & 600 & 0 & 30 \\ 20 & -40 & 20 & 0 & 603 & 0 \\ 0 & 0 & 0 & 30 & 0 & 603 \end{bmatrix} \text{psi} \tag{6.28}$$

We note that this is also the expression we should generate from the push-forward relationship of Equation (6.8) if we had D^{SE}_{IJKL} *a priori*.

We now want to further define constitutive equations of hyperelastic materials. The rate form can be obtained by taking the material time-derivative, referring to Equation (6.5):

$$\dot{\mathbf{S}} = \frac{\partial^2 W}{\partial \mathbf{E} \partial \mathbf{E}} : \dot{\mathbf{E}} = \mathbf{D}^{SE} : \frac{\dot{\mathbf{C}}}{2} = \mathbf{D}^{SE} : \dot{\mathbf{E}} \tag{6.29a}$$

or

$$\dot{S}_{IJ} = D^{SE}_{IJKL} \dot{E}_{KL} \tag{6.29b}$$

where $(\dot{\ })$ indicates $\partial/\partial t$; and, we note that,

$$D^{SE}_{IJKL} = D^{SE}_{JIKL} = D^{SE}_{IJLK} \tag{6.30}$$

which demonstrates *minor symmetry*, and

$$D^{SE}_{IJKL} = D^{SE}_{KLIJ} \tag{6.31}$$

which demonstrates *major symmetry*. The fourth-order tensor D^{SE}_{IJKL} is expressly specified as the *tangent modulus tensor*, and its elements called the *tangent moduli*, by some researchers.

Many engineering applications involve small strains and large rotations. A material having this response is called a *Saint Venant-Kirchhoff material*. It is path-independent and possesses an elastic strain-energy potential. The strain energy per unit volume can be expressed as

$$W = \int S_{IJ} dE_{IJ} = \int D^{SE}_{IJKL} E_{KL} dE_{IJ} = \frac{1}{2} D^{SE}_{IJKL} E_{IJ} E_{KL} = \frac{1}{2} \mathbf{E} : \mathbf{D}^{SE} : \mathbf{E} \tag{6.32}$$

We can write the expression

$$N_{Ij} = \frac{\partial W}{\partial F^T_{jI}} \quad \text{or} \quad \mathbf{N} = \frac{\partial W}{\partial \mathbf{F}^T} \tag{6.33}$$

since N_{Ij} is *conjugate in power* to $\dot{F}^T_{jI}$. The concept of conjugate pairs is presented/discussed in Chap. 7. Taking the time-derivative of the nominal stress and referring to Equation (6.29a),

$$\dot{\mathbf{N}} = \frac{\partial^2 W}{\partial \mathbf{F}^T \partial \mathbf{F}^T} : \dot{\mathbf{F}}^T = \mathbf{A}^{(1)} : \dot{\mathbf{F}}^T \tag{6.34a}$$

or

$$\dot{N}_{Ij} = A^{(1)}_{IjKl} \dot{F}^T_{lK} \tag{6.34b}$$

where

$$A^{(1)}_{IjKl} = D^{SE}_{INPK} F_{jN} F_{lP} + S_{IK} \delta_{lj} \tag{6.35}$$

where $A^{(1)}_{IjKl}$ is the *first elasticity tensor*. The *second elasticity tensor* $A^{(2)}_{IJKl}$ is defined by

$$A^{(2)}_{IJKL} \equiv D^{SE}_{IJKL} \tag{6.36}$$

The *third elasticity tensor* $A^{(3)}_{ijkl}$ is defined by

$$A^{(3)}_{ijkl} = F_{iM}F_{jN}F_{kP}F_{lQ}D^{SE}_{MNPQ} + \tau_{ik}\delta_{jl} \tag{6.37}$$

and the *fourth elasticity tensor* is defined by

$$A^{(4)}_{ijkl} = F_{iM}F_{jN}F_{kP}F_{lQ}D^{SE}_{MNPQ} \tag{6.38}$$

And, we can note that,

$$A^{(4)}_{ijkl} = F_{iI}F_{jJ}F_{kK}F_{lL}A^{(2)}_{IJKL} \tag{6.39}$$

We can also write, for the rate of the Piola stress,

$$\dot{\mathbf{P}} = \frac{\partial^2 W}{\partial \mathbf{F} \partial \mathbf{F}} : \dot{\mathbf{F}} = \hat{\mathbf{A}}^{(1)} : \dot{\mathbf{F}} \tag{6.40a}$$

or

$$\dot{P}_{iJ} = \hat{A}^{(1)}_{iJkL}\dot{F}_{kL} \tag{6.40b}$$

where

$$\hat{A}^{(1)}_{iJkL} = A^{(1)}_{JiLk} \tag{6.41}$$

The first elasticity tensor $A^{(1)}_{IjKl}$ has major symmetry,

$$A^{(1)}_{IjKl} = A^{(1)}_{KlIj} \tag{6.42}$$

but does not have minor symmetries (Belytschko et al. 2000), i.e.,

$$A^{(1)}_{IjKl} \neq A^{(1)}_{jIKl} \neq A^{(1)}_{IjlK} \tag{6.43}$$

Accordingly, since

$$\hat{A}^{(1)}_{jIlK} = A^{(1)}_{IjKl} \tag{6.44}$$

we can write, regarding symmetries,

$$\hat{A}^{(1)}_{jIlK} = \hat{A}^{(1)}_{lKjI} \tag{6.45}$$

and

$$\hat{A}^{(1)}_{jIlK} \neq \hat{A}^{(1)}_{IjlK} \neq \hat{A}^{(1)}_{jIKl} \tag{6.46}$$

The proof of the rate relationship shown in Equations (6.40a) and (6.40b) will now be given. We start with

$$\mathbf{P} = \frac{\partial W}{\partial \mathbf{F}} \tag{6.47}$$

then, taking the time-derivative

$$\dot{\mathbf{P}} = \frac{\partial}{\partial t}\left(\frac{\partial W}{\partial \mathbf{F}}\right) = \frac{\partial}{\partial \mathbf{F}}\left(\frac{\partial W}{\partial t}\right) = \frac{\partial}{\partial \mathbf{F}}\left(\frac{\partial W}{\partial \mathbf{F}} : \dot{\mathbf{F}}\right) \tag{6.48}$$

and

$$\dot{\mathbf{P}} = \frac{\partial^2 W}{\partial \mathbf{F} \partial \mathbf{F}} : \dot{\mathbf{F}} + \frac{\partial W}{\partial \mathbf{F}} : \frac{\partial^2 \mathbf{F}}{\partial \mathbf{F} \partial t} = \frac{\partial^2 W}{\partial \mathbf{F} \partial \mathbf{F}} : \dot{\mathbf{F}} + \frac{\partial W}{\partial \mathbf{F}} : \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{F}}{\partial \mathbf{F}}\right) \tag{6.49}$$

therefore

$$\dot{\mathbf{P}} = \frac{\partial^2 W}{\partial \mathbf{F} \partial \mathbf{F}} : \dot{\mathbf{F}} = \hat{\mathbf{A}}^{(1)} : \dot{\mathbf{F}} \tag{6.50}$$

Now, we want an expression for $\hat{A}^{(1)}_{jIlK}$, where

$$\hat{A}^{(1)}_{jIlK} = \frac{\partial^2 W}{\partial F_{jI} \partial F_{lK}} = \frac{\partial P_{jI}}{\partial F_{lK}} \quad \text{or} \quad \hat{\mathbf{A}}^{(1)} = \frac{\partial^2 W}{\partial \mathbf{F} \partial \mathbf{F}} = \frac{\partial \mathbf{P}}{\partial \mathbf{F}} \tag{6.51}$$

Given the following strain-energy function,

$$W(F_{iJ}) = \tilde{W}(\lambda_1, \lambda_2, \lambda_3) = \breve{W}(I_1, I_2, I_3) = \overset{\frown}{W}\left(\bar{I}_1, \bar{I}_2, J\right) \tag{6.52}$$

where

$$\overset{\frown}{W} = C_{10}\left(\bar{I}_1 - 3\right) + C_{01}\left(\bar{I}_2 - 3\right) + D_1 (J - 1)^2 \tag{6.53}$$

$$I_1 = b_{ii} \tag{6.54a}$$

$$I_2 = \frac{1}{2}\left(I_1^2 - b_{ij} b_{ji}\right) \tag{6.54b}$$

$$I_3 = \det b_{ij} = J^2 \tag{6.54c}$$

$$\frac{\partial I_1}{\partial F_{jI}} = 2F_{jI} \tag{6.55a}$$

$$\frac{\partial I_2}{\partial F_{jI}} = 2\left(I_1 F_{jI} - b_{jk} F_{kI}\right) \tag{6.55b}$$

$$\frac{\partial I_3}{\partial F_{jI}} = 2I_3 F_{Ij}^T \tag{6.55c}$$

$$b_{ij} = F_{iI} F_{jI}^T \tag{6.56a}$$

$$\overline{b}_{ij} = J^{-2/3} b_{ij} \tag{6.56b}$$

and

$$\bar{I}_1 = \frac{I_1}{J^{2/3}} = \frac{b_{ii}}{J^{2/3}} \tag{6.57a}$$

$$\bar{I}_2 = \frac{I_2}{J^{4/3}} = \frac{1}{2}\left(\bar{I}_1^2 - \frac{b_{ij} b_{ji}}{J^{4/3}}\right) \tag{6.57b}$$

$$J = \sqrt{\det b_{ij}} \tag{6.57c}$$

$\bar{I}_1$ and $\bar{I}_2$ being, as indicated, the first and second invariants of $\overline{b}_{ij}$. Employing the chain rule we can write

$$\frac{\partial W}{\partial F_{jI}} = \frac{\partial \widehat{W}}{\partial \bar{I}_1} \frac{\partial \bar{I}_1}{\partial F_{jI}} + \frac{\partial \widehat{W}}{\partial \bar{I}_2} \frac{\partial \bar{I}_2}{\partial F_{jI}} + \frac{\partial \widehat{W}}{\partial J} \frac{\partial J}{\partial F_{jI}} \tag{6.58}$$

where

$$\frac{\partial \bar{I}_1}{\partial F_{jI}} = \frac{1}{J^{2/3}} \frac{\partial I_1}{\partial F_{jI}} - \frac{2I_1}{3J^{5/3}} \frac{\partial J}{\partial F_{jI}} = \frac{2}{J^{2/3}}\left(F_{jI} - \frac{I_1}{3} F_{Ji}^T\right) = \frac{2}{J^{2/3}} F_{jI} - \frac{2}{3}\bar{I}_1 F_{Ij}^T \tag{6.59a}$$

$$\begin{aligned} \frac{\partial \bar{I}_2}{\partial F_{jI}} &= \frac{1}{J^{4/3}} \frac{\partial I_2}{3F_{jI}} - \frac{4I_2}{3J^{7/3}} \frac{\partial J}{\partial F_{jI}} = \frac{2}{J^{4/3}}\left(I_1 F_{jI} - b_{jk} F_{kI} - \frac{2I_2}{3} F_{Ij}^T\right) \\ &= \frac{2}{J^{2/3}} \bar{I}_1 F_{jI} - \frac{2}{J^{4/3}} b_{jk} F_{kI} - \frac{4\bar{I}_2}{3} F_{Ij}^T \end{aligned} \tag{6.59b}$$

and, from the definition of the derivative of a scalar-valued function of a tensor,

$$\frac{\partial J}{\partial F_{jI}} = JF_{Ij}^T \tag{6.60}$$

where this relationship is derived in Appendix C.

The first Piola-Kirchhoff stress tensor is given by

$$P_{jI} = \frac{\partial W}{\partial F_{jI}} \tag{6.61}$$

Then

$$P_{jI} = \frac{\partial \widehat{W}}{\partial \bar{I}_1}\left(\frac{2}{J^{2/3}}F_{jI} - \frac{2}{3}\bar{I}_1 F_{Ij}^T\right) + \frac{\partial \widehat{W}}{\partial \bar{I}_2}\left(\frac{2}{J^{2/3}}\bar{I}_1 F_{jI} - \frac{2}{J^{4/3}}b_{jk}F_{kI} - \frac{4\bar{I}_2}{3}F_{Ij}^T\right) + \frac{\partial \widehat{W}}{\partial J}\left(JF_{Ij}^T\right) \tag{6.62}$$

where

$$\frac{\partial \widehat{W}}{\partial \bar{I}_1} = C_{10}, \tag{6.63a}$$

$$\frac{\partial \widehat{W}}{\partial \bar{I}_2} = C_{01}, \tag{6.63b}$$

$$\frac{\partial \widehat{W}}{\partial J} = 2D_1(J-1) \tag{6.63c}$$

giving

$$P_{jI} = C_{10}\left(\frac{2}{J^{2/3}}F_{jI} - \frac{2}{3}\bar{I}_1 F_{Ij}^T\right) + C_{01}\left(\frac{2}{J^{2/3}}\bar{I}_1 F_{jI} - \frac{2}{J^{4/3}}b_{jk}F_{kI} - \frac{4\bar{I}_2}{3}F_{Ij}^T\right) + 2D_1(J-1)JF_{Ij}^T \tag{6.64}$$

With

$$C_{10} = \frac{G}{2}, \tag{6.65a}$$

$$C_{01} = 0 \tag{6.65b}$$

and

$$D_1 = \frac{K}{2} \tag{6.65c}$$

we have

$$P_{jI} = \frac{G}{2}\left(\frac{2}{J^{2/3}}F_{jI} - \frac{2}{3}\bar{I}_1 F_{Ij}^{T}\right) + K(J-1)JF_{Ij}^{T} \tag{6.66}$$

Using the simple shear model again, this time as a numerical test:

$$G = 300$$

$$[F] = \begin{bmatrix} 1 & 0 & 0.1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[F]^{-T} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -0.1 & 0 & 1 \end{bmatrix}$$

$$[b] = \begin{bmatrix} 1 & 0 & 0.1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0.1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1.01 & 0 & 0.1 \\ 0 & 1 & 0 \\ 0.1 & 0 & 1 \end{bmatrix}$$

$$\bar{I}_1 = 1.01 + 1 + 1 = 3.01$$

$$[P] = 300\left(\begin{bmatrix} 1 & 0 & 0.1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{3.01}{3}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -0.1 & 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 & 30 \\ 0 & -1 & 0 \\ 30.1 & 0 & -1 \end{bmatrix}$$

which is the correct answer.

Now, getting back to the equation development, we can write

$$\hat{A}^{(1)}_{jIlK} = \frac{\partial P_{jI}}{\partial F_{lK}} \tag{6.67}$$

where

$$P_{jI} = G\left(J^{-2/3}F_{jI} - \frac{1}{3}\bar{I}_1 F_{Ij}^{T}\right) + K(J-1)JF_{Ij}^{T} \tag{6.68}$$

Then, differentiating Equation (6.68), term by term, we get

first term:

$$\begin{aligned}\frac{\partial\left(J^{-2/3}F_{jI}\right)}{\partial F_{lK}} &= \frac{\partial\left(J^{-2/3}\right)}{\partial F_{lK}}F_{jI}+J^{-2/3}\frac{\partial F_{jI}}{\partial F_{lK}} = \frac{\partial\left(J^{-2/3}\right)}{\partial J}\frac{\partial J}{\partial F_{lK}}F_{jI}+J^{-2/3}\frac{\partial F_{jI}}{\partial F_{lK}}\\ &= -\frac{2}{3}J^{-5/3}\frac{\partial J}{\partial F_{lK}}F_{jI}+J^{-2/3}\frac{\partial F_{jI}}{\partial F_{lK}} = -\frac{2}{3}J^{-5/3}JF_{Kl}^{T}F_{jI}+J^{-2/3}\delta_{jl}\delta_{KI}\\ &= J^{-2/3}\left(\delta_{jl}\delta_{KI}-\frac{2}{3}F_{Kl}^{T}F_{jI}\right)\end{aligned} \tag{6.69}$$

second term:

$$\begin{aligned}\frac{\partial\left(\bar{I}_1F_{Ij}^{T}\right)}{\partial F_{lK}} &= \frac{\partial\bar{I}_1}{\partial F_{lK}}F_{Ij}^{T}+\bar{I}_1\frac{\partial F_{Ij}^{T}}{\partial F_{lK}} = \left(\frac{2}{J^{2/3}}F_{lK}-\frac{2}{3}\bar{I}_1F_{Kl}^{T}\right)F_{Ij}^{T}+\bar{I}_1\frac{\partial F_{Ij}^{T}}{\partial F_{lK}}\\ &= \bar{I}_1\left(\frac{\partial F_{Ij}^{T}}{\partial F_{lK}}-\frac{2}{3}F_{Kl}^{T}F_{Ij}^{T}\right)+2J^{-2/3}F_{lK}F_{Ij}^{T}\end{aligned} \tag{6.70}$$

third term:

$$\begin{aligned}\frac{\partial\left[(J-1)JF_{Ij}^{T}\right]}{\partial F_{lK}} &= \frac{\partial[(J-1)J]}{\partial F_{lK}}F_{Ij}^{T}+(J-1)J\frac{\partial F_{Ij}^{T}}{\partial F_{lK}} = \frac{\partial[(J-1)J]}{\partial J}\frac{\partial J}{\partial F_{lK}}F_{Ij}^{T}+(J-1)J\frac{\partial F_{Ij}^{T}}{\partial F_{lK}}\\ &= (2J-1)\frac{\partial J}{\partial F_{lK}}F_{Ij}^{T}+(J-1)J\frac{\partial F_{Ij}^{T}}{\partial F_{lK}} = (2J-1)JF_{Kl}^{T}F_{Ij}^{T}+(J-1)J\frac{\partial F_{Ij}^{T}}{\partial F_{lK}}\end{aligned} \tag{6.71}$$

Then,

$$\hat{A}_{jIlK}^{(1)} = G(\mathit{first\ term})-\frac{G}{3}(\mathit{second\ term})+K(\mathit{third\ term}) \tag{6.72}$$

where the *first term* is

$$J^{-2/3}\left(\delta_{jl}\delta_{KI}-\frac{2}{3}F_{Kl}^{T}F_{jI}\right)$$

the *second term* is

$$\bar{I}_1\left(\frac{\partial F^T_{Ij}}{\partial F_{lK}} - \frac{2}{3}F^T_{Kl}F^T_{Ij}\right) + 2J^{-2/3}F_{lK}F^T_{Ij}$$

and the *third term* is

$$(2J-1)JF^T_{Kl}F^T_{Ij} + (J-1)J\,\frac{\partial F^T_{Ij}}{\partial F_{lK}}$$

Given that

$$F^T_{Ij} = \frac{1}{2J}\hat{\varepsilon}_{jbk}\,\hat{\varepsilon}_{IDJ}F_{bD}F_{kJ} \tag{6.73}$$

we can write

$$\frac{\partial}{\partial F_{lK}}\left(\frac{1}{2}\hat{\varepsilon}_{\,jbk}\hat{\varepsilon}_{\,IDJ}F_{bD}F_{kJ}\right) = \hat{\varepsilon}_{jlk}\hat{\varepsilon}_{\,IKJ}F_{kJ} \tag{6.74}$$

since

$$\frac{\partial}{\partial F_{bD}}(F_{bD}F_{cF}) = \frac{\partial F_{bD}}{\partial F_{bD}}F_{cF} + F_{bD}\frac{\partial F_{cF}}{\partial F_{bD}} = F_{cF} + F_{bD}\delta_{cb}\delta_{DF} = 2F_{cF} \tag{6.75}$$

Combining Equations (6.60) and (6.73), we see also that

$$\frac{\partial J}{\partial F_{jI}} = \frac{1}{2}\hat{\varepsilon}_{\,jkl}\,\hat{\varepsilon}_{IJK}F_{kJ}F_{lK} \tag{6.76}$$

Now, continuing

$$\begin{aligned}
\frac{\partial\left(F^T_{Ij}\right)}{\partial F_{lK}} &= \frac{1}{2}\left(\frac{\partial J^{-1}}{\partial F_{lK}}\hat{\varepsilon}_{jbk}\hat{\varepsilon}_{\,IDJ}F_{bD}F_{kJ} + J^{-1}\frac{\partial}{\partial F_{lK}}\hat{\varepsilon}_{\,jbk}\hat{\varepsilon}_{\,IDJ}F_{bD}F_{kJ}\right)\\
&= \frac{1}{2}\left(\frac{\partial J^{-1}}{\partial J}\frac{\partial J}{\partial F_{lk}}\hat{\varepsilon}_{jbk}\hat{\varepsilon}_{\,IDJ}F_{bD}F_{kJ} + J^{-1}\frac{\partial}{\partial F_{lK}}\hat{\varepsilon}_{jbk}\hat{\varepsilon}_{\,IDJ}F_{bD}F_{kJ}\right)\\
&= -\frac{1}{2}\left(J^{-2}\frac{\partial J}{\partial F_{lK}}\hat{\varepsilon}_{jbk}\hat{\varepsilon}_{\,IDJ}F_{bD}F_{kJ} - J^{-1}\frac{\partial}{\partial F_{lK}}\hat{\varepsilon}_{jbk}\hat{\varepsilon}_{\,IDJ}F_{bD}F_{kJ}\right)\\
&= -\frac{1}{2J}\left(F^T_{Kl}\hat{\varepsilon}_{jbk}\hat{\varepsilon}_{\,IDJ}F_{bD}F_{kJ} - 2\hat{\varepsilon}_{jlk}\hat{\varepsilon}_{\,IKJ}F_{kJ}\right)\\
&= \frac{1}{J}\hat{\varepsilon}_{jlk}\hat{\varepsilon}_{\,IKJ}F_{kJ} - F^T_{Kl}F^T_{Ij}
\end{aligned} \tag{6.77}$$

Then, the *first term* is

$$J^{-2/3}\left(\delta_{jl}\delta_{KI} - \frac{2}{3}F^T_{Kl}F_{jI}\right)$$

the *second term* is

$$\begin{aligned}
&\bar{I}_1\left(\frac{1}{J}\hat{\varepsilon}_{jlk}\hat{\varepsilon}_{IKJ}F_{kJ} - F^T_{Kl}F^T_{Ij} - \frac{2}{3}F^T_{Kl}F^T_{Ij}\right) + 2J^{-2/3}F_{lK}F^T_{Ij}\\
&\quad = \bar{I}_1\left(\frac{1}{J}\hat{\varepsilon}_{jlk}\hat{\varepsilon}_{IKJ}F_{kJ} - \frac{5}{3}F^T_{Kl}F^T_{Ij}\right) + 2J^{-2/3}F_{lK}F^T_{Ij}
\end{aligned}$$

and the *third term* is

$$\begin{aligned}
&(2J-1)JF^T_{Kl}F^T_{Ij} - \frac{1}{2}(J-1)J\left(\frac{1}{J}\hat{\varepsilon}_{jlk}\hat{\varepsilon}_{IKJ}F_{kJ} - F^T_{Kl}F^T_{Ij}\right)\\
&\quad = (5J-3)\frac{J}{2}F^T_{Kl}F^T_{Ij} - \frac{1}{2}(J-1)\hat{\varepsilon}_{jlk}\hat{\varepsilon}_{IKJ}F_{kJ}
\end{aligned}$$

Thus, the expression for $\hat{A}^{(1)}_{jIlK}$ is obtained by substitution into Equation (6.72):

$$\begin{aligned}
\hat{A}^{(1)}_{jIlK} = G\Bigg[&J^{-2/3}\left(\delta_{jl}\delta_{KI} - \frac{2}{3}F^T_{Kl}F_{jI}\right) - \frac{\bar{I}_1}{3}\left(\frac{1}{J}\hat{\varepsilon}_{jlk}\hat{\varepsilon}_{IKJ}F_{kJ} - \frac{5}{3}F^T_{Kl}F^T_{Ij}\right) - \frac{2}{3}J^{-2/3}F_{lK}F^T_{Ij}\Bigg]\\
&+ K\left[(5J-3)\frac{J}{2}F^T_{Kl}F^T_{Ij} - \frac{1}{2}(J-1)\hat{\varepsilon}_{jlk}\hat{\varepsilon}_{IKJ}F_{kJ}\right]
\end{aligned} \tag{6.78}$$

Now, we will take a different and, as will be demonstrated, improved approach to the determination of an expression for $\hat{A}^{(1)}_{jIlK}$. Given that,

$$\frac{\partial}{\partial \mathbf{F}}\left(\mathbf{F}^{-T}\mathbf{F}^{T}\right) : \mathbf{G} = 0 \tag{6.79}$$

where $\mathbf{F}$ is the standard deformation gradient and $\mathbf{G}$ is an arbitrary second-order two-point tensor, we can expand Equation (6.79) and write

$$\frac{\partial}{\partial \mathbf{F}}\left(\mathbf{F}^{-T}\mathbf{F}^{T}\right) : \mathbf{G} = \left(\frac{\partial \mathbf{F}^{-T}}{\partial \mathbf{F}} : \mathbf{G}\right)\mathbf{F}^{T} + \mathbf{F}^{-T}\left(\frac{\partial \mathbf{F}^{T}}{\partial \mathbf{F}} : \mathbf{G}\right) = 0 \tag{6.80}$$

from which we obtain the relationship,

$$\left(\frac{\partial \mathbf{F}^{-T}}{\partial \mathbf{F}} : \mathbf{G}\right)\mathbf{F}^T = -\mathbf{F}^{-T}\mathbf{G}^T \tag{6.81}$$

since

$$\frac{\partial \mathbf{F}^{-T}}{\partial \mathbf{F}} : \mathbf{G} = \mathbf{G}^T \tag{6.82}$$

Therefore, we can write

$$\frac{\partial \mathbf{F}^{-T}}{\partial \mathbf{F}} : \mathbf{G} = -\mathbf{F}^{-T}\mathbf{G}^T\mathbf{F}^{-T} \tag{6.83}$$

yielding, when we invoke the arbitrariness of **G**,

$$\frac{\partial \mathbf{F}^{-T}}{\partial \mathbf{F}} = -\mathbf{F}^{-T}\mathbf{F}^{-T} \tag{6.84}$$

and, in index, or component, notation we can write

$$\frac{\partial F^T_{Ji}}{\partial F_{kL}} G_{kL} = -F^T_{Jk}G^T_{kL}F^T_{Li} \Rightarrow \frac{\partial F^T_{Ji}}{\partial F_{kL}} = -F^T_{Li}F^T_{Jk} \tag{6.85}$$

Therefore, can we say that

$$\frac{\partial F^T_{Ij}}{\partial F_{lK}} = \frac{1}{J}\hat{\varepsilon}_{jlk}\hat{\varepsilon}_{IKJ}F_{kJ} - F^T_{Kl}F^T_{Ij} = -F^T_{Kj}F^T_{Il} \tag{6.86}$$

Yes, we can, because

$$\frac{1}{J}\hat{\varepsilon}_{jlk}\hat{\varepsilon}_{IKJ}F_{kJ} - F^T_{Kl}F^T_{Ij} + F^T_{Kj}F^T_{Il} = 0 \tag{6.87}$$

Thus, we can write the terms in the expression for $\hat{A}^{(1)}_{jlIK}$, Equation (6.72),

first term:

$$J^{-2/3}\left(\delta_{jl}\delta_{KI} - \frac{2}{3}F^T_{Kl}F_{jI}\right) \tag{6.88a}$$

second term:

$$\frac{\partial\left(\bar{I}_1 F^T_{Ij}\right)}{\partial F_{lK}} = \frac{\partial \bar{I}_1}{\partial F_{lK}} F^T_{Ij} + \bar{I}_1 \frac{\partial F^T_{Ij}}{\partial F_{lK}} = \left(\frac{2}{J^{2/3}} F_{lK} - \frac{2}{3}\bar{I}_1 F^T_{Kl}\right) F^T_{Ij} + \bar{I}_1 \frac{\partial F^T_{Ij}}{\partial F_{lK}}$$
$$= \bar{I}_1 \left(\frac{\partial F^T_{Ij}}{\partial F_{lK}} - \frac{2}{3} F^T_{Kl} F^T_{Ij}\right) + 2J^{-2/3} F_{lK} F^T_{Ij} \tag{6.88b}$$
$$= -\bar{I}_1 \left(F^T_{Kj} F^T_{Il} + \frac{2}{3} F^T_{Kl} F^T_{Ij}\right) + 2J^{-2/3} F_{lK} F^T_{Ij}$$

third term:

$$\frac{\partial\left[(J-1)JF^T_{Ij}\right]}{\partial F_{lK}} = \frac{\partial[(J-1)J]}{\partial F_{lK}} F^T_{Ij} + (J-1)J \frac{\partial F^T_{Ij}}{\partial F_{lK}} = \frac{\partial[(J-1)J]}{\partial J} \frac{\partial J}{\partial F_{lK}} F^T_{Ij} + (J-1)J \frac{\partial F^T_{Ij}}{\partial F_{lK}}$$
$$= (2J-1) \frac{\partial J}{\partial F_{lK}} F^T_{Ij} + (J-1)J \frac{\partial F^T_{Ij}}{\partial F_{lK}} = (2J-1)JF^T_{Kl} F^T_{Ij} + (J-1)J \frac{\partial F^T_{Ij}}{\partial F_{lK}}$$
$$= (2J-1)JF^T_{Kl} F^T_{Ij} - (J-1)JF^T_{Kj} F^T_{Il} \tag{6.88c}$$

Therefore, we see that we can write the important derivative expression from Equation (6.86) in the simplest form possible which can be easily understood by all,

$$\frac{\partial F^{-T}_{iJ}}{\partial F_{kL}} = -F^{-T}_{iL} F^{-T}_{kJ} \tag{6.89}$$

Now, shifting focus slightly, we can also write for the current (Eulerian) configuration

$$D^{te}_{ijkl} = \frac{1}{J} F_{iA} F_{jB} F_{kC} F_{lD} D^{SE}_{ABCD} \tag{6.90}$$

which is the "push-forward" transformation.

Using the chain rule, we can write

$$\frac{\partial S_{IJ}}{\partial F_{kL}} = \frac{\partial S_{IJ}}{\partial C_{DE}} \frac{\partial C_{DE}}{\partial F_{kL}} = \frac{1}{2} D^{SE}_{IJDE} \frac{\partial C_{DE}}{\partial F_{kL}} \tag{6.91}$$

since

$$D^{SE}_{IJDE} = 2 \frac{\partial S_{IJ}}{\partial C_{DE}} \tag{6.92}$$

Now, let

$$Q^{SF}_{IJbB} = \frac{\partial S_{IJ}}{\partial F_{bB}} = \frac{\partial S_{IJ}}{\partial C_{DE}} \frac{\partial C_{DE}}{\partial F_{bB}} \tag{6.93}$$

which leads to

$$Q^{SF}_{IJbB} = \frac{1}{2} D^{SE}_{IJDE} (\delta_{BD} F_{bE} + F_{bD} \delta_{BE}) \tag{6.94}$$

or

$$Q^{SF}_{IJbB} = \frac{1}{2} \left(D^{SE}_{IJBE} F_{bE} + D^{SE}_{IJDB} F_{bD} \right) \tag{6.95}$$

or

$$Q^{SF}_{IJkL} = \frac{1}{2} \left(D^{SE}_{IJLP} F_{kP} + D^{SE}_{IJPL} F_{kP} \right) \tag{6.96}$$

Then, we can write

$$\dot{S}_{IJ} = Q^{SF}_{IJkL} \dot{F}_{kL} \tag{6.97}$$

We can also write

$$R^{NF}_{IjkL} = Q^{SF}_{IPkL} F_{jP} + S_{IL} \delta_{jk} \tag{6.98}$$

where

$$\dot{N}_{Ij} = R^{NF}_{IjkL} \dot{F}_{kL} \tag{6.99}$$

Quite obviously, the expressions formulated in Equations (6.1) through (6.99) can only realistically be numerically evaluated through the utilization of computer programming or codes such as, for example, MATLAB®.

Chapter 7
Conjugate Pairs

Abstract Whatever measures are chosen to represent the stress and strain (or rate of strain) couplets, their product provides a measure of the work done (or the power spent). Therefore, it is imperative that the stress and strain tensors be conjugate. The notion of conjugation in this context was introduced in the last century. As an example of conjugate pairs, the mechanical work produced by combining second Piola-Kirchhoff stress with Green-Lagrange strain must match that obtained by combining Cauchy stress with Almansi strain. The matching of conjugate pairs requires that mass in an infinitesimal volume be conserved. Therefore, the initial (Lagrangian) differential volume is multiplied by the Jacobian to convert into the current (Eulerian) differential volume in the matching of conjugate integrals. Another example of conjugate pairs matches the power conjugates of second Piola-Kirchhoff stress and the rate of Green-Lagrange strain with Cauchy stress and the rate of deformation. Conjugation in power is helpful in the development of weak forms; measures of stress and rate which are conjugate in power can be used to construct principles of virtual work or power, i.e., weak forms of the momentum equation.

Whatever measures we choose to use to represent the stress and strain (or rate of strain) couplets, their product gives us a measure of the work done (or the power spent). The notion of conjugation in this context was introduced by Hill (1968). Now, considering conjugate pairs, we know that the mechanical work produced by combining second Piola-Kirchhoff stress with Green-Lagrange strain must match that obtained by combining Cauchy stress with Almansi strain.

Conservation of mass in an infinitesimal volume requires that

$$\rho d\upsilon = \rho_0 dV \tag{7.1}$$

where ρ_0 is the initial density, ρ is the current density, V is the initial volume, and υ is the current volume, and we know that

$$\frac{\rho_0}{\rho} = \frac{d\upsilon}{dV} = \det F_{iJ} = J \tag{7.2}$$

R. M. Hackett, *Hyperelasticity Primer*,
https://doi.org/10.1007/978-3-319-73201-5_7

Therefore,

$$dv = JdV \tag{7.3}$$

Then, utilizing Equation (7.3), we can write

$$\int_{v} t_{ij}e_{ij}dv = \int_{V} J\left(\frac{1}{J}F_{iJ}S_{JK}F_{jK}\right)e_{ij}dV = \int_{V} S_{JK}\left(F_{iJ}e_{ij}F_{jK}\right)dV = \int_{V} S_{JK}E_{JK}dV \tag{7.4a}$$

or

$$\int_{v} \mathbf{t} : \mathbf{e}\, dv = \int_{V} J\left(\frac{1}{J}\mathbf{FSF}^{T}\right) : \mathbf{e}dV = \int_{V} \mathbf{S} : \left(\mathbf{F}^{T}\mathbf{eF}\right)dV = \int_{V} \mathbf{S} : \mathbf{E}\, dV \tag{7.4b}$$

which demonstrates that $\mathbf{S}:\mathbf{E}$ and $\mathbf{t}:\mathbf{e}$ form *corresponding work conjugates*.

Given that

$$d_{ij} = \frac{1}{2}F_{Ji}\dot{C}_{JK}F_{Kj} \quad \text{or} \quad \mathbf{d} = \frac{1}{2}\mathbf{F}^{-T}\dot{\mathbf{C}}\mathbf{F}^{-1} \tag{7.5}$$

where d_{ij} is the spatial *rate-of-deformation tensor* (Simo and Hughes 1998), we can write

$$\begin{aligned}\int_{v} t_{ij}d_{ij}dv &= \int_{V} J\left(\frac{1}{J}F_{iJ}S_{JK}F_{jK}\right)d_{ij}dV = \int_{V} S_{JK}\left(F_{iJ}d_{ij}F_{jK}\right)dV \\ &= \frac{1}{2}\int_{V} S_{JK}\dot{C}_{JK}dV = \int_{V} S_{JK}\dot{E}_{JK}dV\end{aligned} \tag{7.6a}$$

or

$$\int_{v}\mathbf{t} : \mathbf{d}\, dv = \int_{V} J\left(\frac{1}{J}\mathbf{FSF}^{T}\right) : \mathbf{d}dV = \int_{V}\mathbf{S} : \left(\mathbf{F}^{T}\mathbf{dF}\right)dV = \frac{1}{2}\int_{V}\mathbf{S} : \dot{\mathbf{C}}dV = \int_{V}\mathbf{S} : \dot{\mathbf{E}}\, dV \tag{7.6b}$$

which demonstrates that $\mathbf{S}:\dot{\mathbf{E}}$ and $\mathbf{t}:\mathbf{d}$ form *corresponding power conjugates*. Conjugation in power is helpful in the development of weak forms; measures of stress and rate which are conjugate in power can be used to construct principles of virtual work or power, i.e., weak forms of the momentum equation. Variables which are conjugate in power are also conjugate in work (Belytschko et al. 2000).

We can write

$$\begin{aligned}
\mathbf{N} : \dot{\mathbf{F}}^T &\equiv N_{Ij}\dot{F}_{jI} = S_{IJ}F_{jJ}\dot{F}_{jI} = \left(\mathbf{S}\mathbf{F}^T\right) : \dot{\mathbf{F}}^T = \mathbf{S} : \left(\mathbf{F}^T\dot{\mathbf{F}}\right) \\
&= \mathbf{S} : \left[\frac{1}{2}\left(\mathbf{F}^T\dot{\mathbf{F}} + \dot{\mathbf{F}}^T\mathbf{F}\right) + \frac{1}{2}\left(\mathbf{F}^T\dot{\mathbf{F}} - \dot{\mathbf{F}}^T\mathbf{F}\right)\right] \text{ decomposing} \\
&\quad \left(\mathbf{F}^T\dot{\mathbf{F}}\right) \text{ into } \textit{symmetric} \text{ and } \textit{antisymmetric} \text{ parts} \\
&= \mathbf{S} : \frac{1}{2}\left(\mathbf{F}^T\dot{\mathbf{F}} + \dot{\mathbf{F}}^T\mathbf{F}\right) \quad \text{since contraction of symmetric and} \\
&\qquad\qquad\qquad\qquad\qquad \text{antisymmetric tensors vanishes} \\
&= \mathbf{S} : \frac{1}{2}\dot{\mathbf{C}} = \mathbf{S} : \dot{\mathbf{E}} \equiv S_{IJ}\dot{E}_{IJ}
\end{aligned} \tag{7.7}$$

which shows that $\mathbf{N} : \dot{\mathbf{F}}^T$ and $\mathbf{t} : \mathbf{d}$ also form corresponding power conjugates. Now, employing

$$\mathbf{A} : \mathbf{B} \equiv \mathrm{tr}\left[\mathbf{A}^T\mathbf{B}\right] \equiv \mathrm{tr}\left[\mathbf{A}\mathbf{B}^T\right] \tag{7.8}$$

we can write

$$\mathbf{P} : \dot{\mathbf{F}} = \mathrm{tr}\left[\mathbf{N}\dot{\mathbf{F}}\right] = \mathrm{tr}\left[\mathbf{P}\dot{\mathbf{F}}^T\right] \tag{7.9a}$$

and

$$\mathbf{N} : \dot{\mathbf{F}}^T = \mathrm{tr}\left[\mathbf{P}\dot{\mathbf{F}}^T\right] = \mathrm{tr}\left[\mathbf{N}\dot{\mathbf{F}}\right] \tag{7.9b}$$

Therefore,

$$\mathbf{P} : \dot{\mathbf{F}} = \mathbf{N} : \dot{\mathbf{F}}^T \quad \text{or} \quad P_{iJ}\dot{F}_{iJ} = N_{Ji}\dot{F}_{iJ} \tag{7.10}$$

which shows that $\mathbf{P} : \dot{\mathbf{F}}$ and $\mathbf{t} : \mathbf{d}$ too form corresponding power conjugates.

As was noted in Chap. 5, the Biot stress tensor $\mathbf{T}$ is power conjugate to the rate of the right stretch tensor $\dot{\mathbf{U}}$. We can demonstrate that with the following manipulation. Since

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} = [\mathbf{R}\mathbf{U}]^T[\mathbf{R}\mathbf{U}] = \mathbf{U}^T\left[\mathbf{R}^T\mathbf{R}\right]\mathbf{U} = \mathbf{U}^T\mathbf{U} \tag{7.11}$$

and, thus

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{U}^T\mathbf{U} - \mathbf{1}\right) \tag{7.12}$$

we see that

$$\dot{\mathbf{E}} = \frac{1}{2}\left(\dot{\mathbf{U}}^T\mathbf{U} + \mathbf{U}^T\dot{\mathbf{U}}\right) \tag{7.13}$$

Then,

$$\begin{aligned} S_{IJ}\dot{E}_{IJ} \equiv \mathbf{S}{:}\dot{\mathbf{E}} &= \mathbf{S} : \frac{1}{2}\left(\dot{\mathbf{U}}^T\mathbf{U}+\mathbf{U}^T\dot{\mathbf{U}}\right) \\ &= \frac{1}{2}\mathrm{tr}\left(\mathbf{S}\dot{\mathbf{U}}^T\mathbf{U}+\mathbf{S}\mathbf{U}^T\dot{\mathbf{U}}\right) \\ &= \frac{1}{2}\mathrm{tr}\left(\mathbf{S}\mathbf{U}\dot{\mathbf{U}}+\mathbf{U}\mathbf{S}\dot{\mathbf{U}}\right) \\ &= \frac{1}{2}(\mathbf{S}\mathbf{U}+\mathbf{U}\mathbf{S}) : \dot{\mathbf{U}} = \mathbf{T}{:}\dot{\mathbf{U}} \equiv T_{IJ}\dot{U}_{IJ} \end{aligned} \tag{7.14}$$

We also note that in this derivation we have generalized in that we have not specified that $\mathbf{U}$ be symmetric (Hjelmstad 2005).

The *Biot strain tensor* is designated $\overline{\mathbf{U}} = \mathbf{U} - \mathbf{1}$ (Belytschko et al. 2000). We can then write the rate of the Biot strain tensor,

$$\dot{\overline{U}}_{IJ} = \dot{U}_{IJ} \tag{7.15}$$

and note that the Biot stress and the rate of the Biot strain are conjugate in power.

Chapter 8
Incrementation: Part One

Abstract The focus here is on the analysis and solution of transient, or time-dependent, problems. Time-dependent solutions are normally based on time-stepping algorithms wherewith solutions to increments of the solution variables, i.e., deformation gradient, strain and stress, are obtained. Typically, at each point in time where the incremental values are obtained, they are added to the previously accumulated values of the corresponding variables, thus yielding the summed incremental values for the current solution point. This procedure is continued, marching step-by-step timewise as far as it is desired that the solution progresses. The solution algorithm employed must be unconditionally stable and accurate. The procedure described is additive, where the incremental deformation gradient is primary; there is also a multiplicative approach wherein the relative deformation gradient is the primary variable. The relationship between the incremental deformation gradient and the relative deformation gradient is defined. A numerical example demonstrating the relationship between the incremental deformation gradient and the relative deformation gradient is presented.

The formulations presented thus far have dealt with steady-state conditions, with the minor exception of some material related to rates of deformation. Since most of the hyperelastic problems we would expect to encounter, and deal with, are transient in nature, we need to shift our focus to the analysis and solution of transient, or time-dependent, problems. We first note that time-dependent solutions are normally based on time-stepping algorithms wherewith we obtain solutions to increments of the solution variables; for example, we solve for the incremental deformations, incremental strains, and incremental stresses. At each point in time where the incremental values are obtained, we add them to the previously accumulated values of the corresponding variables and thus obtain the accumulated values for the current solution point. We continue this procedure, thus marching step-by-step timewise as far as we wish the solution to progress. Of course, we recognize that the solution algorithm we employ must be stable (unconditionally, hopefully) and accurate. Now, we will see how we apply incremental mechanics in the analysis/solution of hyperelastic problems.

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We start by considering the primary variable F_{iJ}. We can write

$$F_{iJ}^{t+\Delta t} = F_{iJ}^{t} + \delta F_{iJ} \quad \text{or} \quad \mathbf{F}_{t+\Delta t} = \mathbf{F}_t + \delta \mathbf{F} \tag{8.1}$$

where $F_{iJ}^{t+\Delta t}$ is the current deformation gradient, F_{iJ}^{t} is the deformation gradient at the end of the previous time step, δF_{iJ} is the incremental deformation gradient, t is the time variable, and Δt is the time step.

From Simo and Hughes (1998) we have a multiplicative approach for obtaining $F_{iJ}^{t+\Delta t}$,

$$F_{iJ}^{t+\Delta t} = f_{ik}^{t+\Delta t} F_{kJ}^{t} \quad \text{or} \quad \mathbf{F}_{t+\Delta t} = \mathbf{f}_{t+\Delta t} \mathbf{F}_t \tag{8.2}$$

where $f_{ij}^{t+\Delta t}$, the *relative deformation gradient*, is defined by

$$f_{ij}^{t+\Delta t} = \delta_{ij} + e_j^t e_i^t u_{i,j} \quad \text{or} \quad \mathbf{f}_{t+\Delta t} = \mathbf{1} + \nabla^t \mathbf{u} \tag{8.3}$$

where ∇ is the del operator, $\nabla = e_i \partial_i$. So, we can write

$$F_{iJ}^{t+\Delta t} = F_{iJ}^{t} + \left(e_j^t e_i^t u_{i,j}\right) F_{iJ}^{t} \quad \text{or} \quad \mathbf{F}_{t+\Delta t} = \mathbf{F}_t + \left(\nabla^t \mathbf{u}\right) \mathbf{F}_t \tag{8.4}$$

and

$$f_{ij}^{t+\Delta t} = \delta_{ij} + \delta F_{iJ} F_{Jj}^{t} \quad \text{or} \quad \mathbf{f}_{t+\Delta t} = \mathbf{1} + \delta \mathbf{F}\, \mathbf{F}_t^{-1} \tag{8.5}$$

We can also appropriately define $f_{ij}^{t+\Delta t}$, using the chain rule, as

$$f_{ij}^{t+\Delta t} = F_{iJ}^{t+\Delta t} F_{Jj}^{t} = \frac{\partial x_i^{t+\Delta t}}{\partial X_J} \frac{\partial X_J}{\partial x_j^t} = \frac{\partial x_i^{t+\Delta t}}{\partial x_j^t} \tag{8.6}$$

The concept of the relative deformation gradient $f_{ij}^{t+\Delta t}$ is shown graphically in Fig. 8.1.

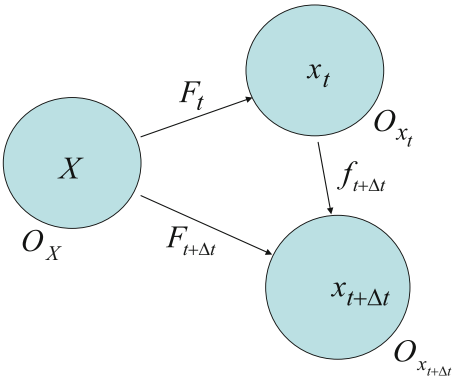

Fig. 8.1 Deformation gradient and relative deformation gradient mapping neighborhoods O_X, O_{x_t}, and $O_{x_{t+\Delta t}}$

We can write the expression for the volume-preserving part of the relative deformation gradient as

$$\overline{f}_{ij}^{t+\Delta t} = \overline{F}_{iJ}^{t+\Delta t} \overline{F}_{Jj}^{t} \equiv \left(\frac{J_{t+\Delta t}}{J_t} \right)^{-\frac{1}{n}} f_{ij}^{t+\Delta t} \tag{8.7}$$

and we can also write

$$\delta \overline{F}_{iJ} = \overline{F}_{iJ}^{t+\Delta t} - \overline{F}_{iJ}^{t} \tag{8.8}$$

We should, at this point, consider a numerical example that demonstrates the formulations contained in Equations (8.1) through (8.8).

8.1 Example 8: Relative Deformation Gradient

Given the two-dimensional (for convenience) deformation gradient defined at time t,

$$[F_t] = \begin{bmatrix} -\frac{1}{2} & -1 \\ 1 & \frac{1}{2} \end{bmatrix} \tag{8.9}$$

then

$$[F_t]^{-1} = \begin{bmatrix} \frac{2}{3} & \frac{4}{3} \\ -\frac{4}{3} & -\frac{2}{3} \end{bmatrix} \tag{8.10}$$

Let

$$[F_{t+\Delta t}] = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \tag{8.11}$$

then, by re-arranging Equation (8.2), we get

$$[f_{t+\Delta t}] = [F_{t+\Delta t}][F_t]^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{2}{3} & \frac{4}{3} \\ -\frac{4}{3} & -\frac{2}{3} \end{bmatrix} = \begin{bmatrix} \frac{4}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} \end{bmatrix} \tag{8.12}$$

and, by re-arranging Equation (8.3),

$$\left[\nabla^t u\right] = \left[f_{t+\Delta t}\right] - [I] = \begin{bmatrix} \frac{4}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & -\frac{1}{3} \end{bmatrix} \tag{8.13}$$

Then, combining Equations (8.1) and (8.4), we have

$$[\delta F] = \left[\nabla^t u\right][F_t] = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & -1 \\ 1 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \tag{8.14}$$

and, from Equation (8.5),

$$\begin{aligned} \left[f_{t+\Delta t}\right] = [I] + [\delta F][F_t]^{-1} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{2}{3} & \frac{4}{3} \\ -\frac{4}{3} & -\frac{2}{3} \end{bmatrix} \\ &= \begin{bmatrix} \frac{4}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} \end{bmatrix} \end{aligned} \tag{8.15}$$

Now, considering the relative deformation gradient $f_{ij}^{t+\Delta t}$, we can write

$$J_t = \det \begin{bmatrix} -\frac{1}{2} & -1 \\ 1 & \frac{1}{2} \end{bmatrix} = \begin{vmatrix} -\frac{1}{2} & -1 \\ 1 & \frac{1}{2} \end{vmatrix} = \frac{3}{4} \tag{8.16}$$

and

$$J_{t+\Delta t} = \det \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} = 1 \tag{8.17}$$

along with

$$\left[\overline{F}_t\right] = J_t^{-\frac{1}{2}}[F_t] = \frac{2}{\sqrt{3}} \begin{bmatrix} -\frac{1}{2} & -1 \\ 1 & \frac{1}{2} \end{bmatrix} \tag{8.18}$$

$$\det\left[\overline{F}_t\right] = \begin{vmatrix} -\dfrac{1}{\sqrt{3}} & -\dfrac{2}{\sqrt{3}} \\ \dfrac{2}{\sqrt{3}} & \dfrac{1}{\sqrt{3}} \end{vmatrix} = 1 \tag{8.19}$$

and

$$\left[\overline{F}_{t+\Delta t}\right] = J_{t+\Delta t}^{-\frac{1}{2}}[F_{t+\Delta t}] = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \tag{8.20}$$

$$\det\left[\overline{F}_{t+\Delta t}\right] = \begin{vmatrix} 0 & -1 \\ 1 & 1 \end{vmatrix} = 1 \tag{8.21}$$

Then, from Equation (8.7), we can solve for the volume-preserving part of the relative deformation gradient,

$$\left[\overline{f}_{t+\Delta t}\right] = \left[\overline{F}_{t+\Delta t}\right]\left[\overline{F}_t\right]^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \dfrac{1}{\sqrt{3}} & \dfrac{2}{\sqrt{3}} \\ -\dfrac{2}{\sqrt{3}} & -\dfrac{1}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} \dfrac{2}{\sqrt{3}} & \dfrac{1}{\sqrt{3}} \\ -\dfrac{1}{\sqrt{3}} & \dfrac{1}{\sqrt{3}} \end{bmatrix} \tag{8.22}$$

or

$$\left[\overline{f}_{t+\Delta t}\right] = \left(\frac{J^{t+\Delta t}}{J^t}\right)^{-\frac{1}{2}}\left[f_{t+\Delta t}\right] = \frac{1}{\sqrt{3}}\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \tag{8.23}$$

$$\det\left[\overline{f}_{t+\Delta t}\right] = \begin{vmatrix} \dfrac{2}{\sqrt{3}} & \dfrac{1}{\sqrt{3}} \\ -\dfrac{1}{\sqrt{3}} & \dfrac{1}{\sqrt{3}} \end{vmatrix} = 1 \tag{8.24}$$

We can also write, from Equation (8.8),

$$\left[\delta\overline{F}\right] = \left[\overline{F}_{t+\Delta t}\right] - \left[\overline{F}_t\right] = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} -\dfrac{1}{\sqrt{3}} & -\dfrac{2}{\sqrt{3}} \\ \dfrac{2}{\sqrt{3}} & \dfrac{1}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{\sqrt{3}} & \dfrac{2-\sqrt{3}}{\sqrt{3}} \\ \dfrac{\sqrt{3}-2}{\sqrt{3}} & \dfrac{\sqrt{3}-1}{\sqrt{3}} \end{bmatrix} \tag{8.25}$$

We have thus defined and studied the incremental deformation gradient δF_{iJ} and the relative deformation gradient $f_{ij}^{t+\Delta t}$ and the relationship between them.

Chapter 9
Incrementation: Part Two

Abstract While rate equations are exact, the matching (linearized) incremental equations are approximations. It is very important that approximations do not lead to results that differ very much from exact solutions. Incremental formulations for both Green-Lagrange and Almansi strain are given, based upon their corresponding rate equations. Incremental Green-Lagrange and Almansi deviatoric strain expressions are determined, also based upon their corresponding rate equations. The respective error expressions, for incremental Green-Lagrange and Almansi strains and for incremental Green-Lagrange and Almansi deviatoric strains, based upon comparisons with exact expressions are given. A numerical example demonstrating errors associated with incremental strain formulations is presented.

We consider in this chapter on incrementation the definition of incremental measures of strain, both Lagrangian and Euleran. We begin with the Green-Lagrange strain expression,

$$E_{IJ} = \frac{1}{2}(C_{IJ} - \delta_{IJ}) \tag{9.1}$$

where

$$C_{IJ} = F_{kI}F_{kJ} \tag{9.2}$$

We can write the rate of Green-Lagrange strain expression,

$$\dot{E}_{IJ} = \frac{1}{2}\left(F_{kI}\dot{F}_{kJ} + \dot{F}_{kI}F_{kJ}\right) \tag{9.3}$$

and a corresponding incremental strain expression,

$$\delta E_{IJ} = \frac{1}{2}\left[F_{kI}^{t+\Delta t}(\delta F_{kJ}) + (\delta F_{kI})F_{kJ}^{t+\Delta t}\right] \tag{9.4}$$

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Whereas rate equations are exact, the matching (linearized) incremental equations are approximations, the error in Equation (9.4) being

$$\frac{1}{2}(\delta F_{kI})(\delta F_{kJ})$$

Also, from Belytschko et al. (2000), we have

$$\delta E_{IJ} = \frac{1}{2}\left(F_{kI}^{t+\Delta t} F_{kJ}^{t+\Delta t} - F_{kI}^{t} F_{kJ}^{t}\right) \tag{9.5}$$

which is exact, not an approximation.

Next we consider the incremental Almansi strain. We know that

$$e_{ij} = \frac{1}{2}\left(\delta_{ij} - b_{ij}^{-1}\right) \tag{9.6}$$

where

$$b_{ij}^{-1} = F_{Ki} F_{Kj} \tag{9.7}$$

then, we can write the rate of Almansi strain expression,

$$\dot{e}_{ij} = -\frac{1}{2}\left(F_{Ki} \dot{F}_{Kj} + \dot{F}_{Ki} F_{Kj}\right) \tag{9.8}$$

The corresponding approximate incremental strain expression is

$$\delta e_{ij} = -\frac{1}{2}\left[F_{Ki}^{t+\Delta t}\left(\delta F_{Kj}\right) + (\delta F_{Ki}) F_{Kj}^{t+\Delta t}\right] \tag{9.9}$$

with the error in Equation (9.9) being

$$\frac{1}{2}(\delta F_{Ki})\left(\delta F_{Kj}\right)$$

Also, we have

$$\delta e_{ij} = -\frac{1}{2}\left(F_{Ki}^{t+\Delta t} F_{Kj}^{t+\Delta t} - F_{Ki}^{t} F_{Kj}^{t}\right) \tag{9.10}$$

which is exact.

Now, we consider incremental deviatoric strain. We can write from Equation (2.12),

$$\overline{E}_{IJ} = \frac{1}{2}\left(\overline{F}_{kI} \overline{F}_{kJ} - \delta_{IJ}\right) \tag{9.11}$$

the corresponding rate of deviatoric Green-Lagrange strain expression,

$$\dot{\overline{E}}_{IJ} = \frac{1}{2}\left(\overline{F}_{kI}\dot{\overline{F}}_{kJ} + \dot{\overline{F}}_{kI}\overline{F}_{kJ}\right) \tag{9.12}$$

and a corresponding incremental deviatoric strain expression,

$$\delta\overline{E}_{IJ} = \frac{1}{2}\left[\overline{F}_{kI}^{t+\Delta t}\left(\delta\overline{F}_{kJ}\right) + \left(\delta\overline{F}_{kI}\right)\overline{F}_{kJ}^{t+\Delta t}\right] \tag{9.13}$$

The error in Equation (9.13) is

$$\frac{1}{2}\left(\delta\overline{F}_{kI}\right)\left(\delta\overline{F}_{kJ}\right)$$

Writing the exact expression, we have

$$\delta\overline{E}_{IJ} = \frac{1}{2}\left(\overline{F}_{kI}^{t+\Delta t}\overline{F}_{kJ}^{t+\Delta t} - \overline{F}_{kI}^{t}\overline{F}_{kJ}^{t}\right) \tag{9.14}$$

Now, considering the incremental deviatoric Almansi strain, we write

$$\overline{b}_{ij}^{-1} = \overline{F}_{Ki}\overline{F}_{Kj} \tag{9.15}$$

then, we can write the rate of deviatoric Almansi strain expression,

$$\dot{\overline{e}}_{ij} = -\frac{1}{2}\left(\overline{F}_{Ki}\dot{\overline{F}}_{Kj} + \dot{\overline{F}}_{Ki}\overline{F}_{Kj}\right) \tag{9.16}$$

The corresponding approximate incremental deviatoric strain expression is

$$\delta\,\overline{e}_{ij} = -\frac{1}{2}\left[\overline{F}_{Ki}^{t+\Delta t}\left(\delta\overline{F}_{Kj}\right) + \left(\delta\overline{F}_{Ki}\right)\overline{F}_{Kj}^{t+\Delta t}\right] \tag{9.17}$$

with the error in Equation (9.17) being

$$\frac{1}{2}\left(\delta\overline{F}_{Ki}\right)\left(\delta\overline{F}_{Kj}\right)$$

Also, we have

$$\delta\,\overline{e}_{ij} = -\frac{1}{2}\left(\overline{F}_{Ki}^{t+\Delta t}\overline{F}_{Kj}^{t+\Delta t} - \overline{F}_{Ki}^{t}\overline{F}_{Kj}^{t}\right) \tag{9.18}$$

which is exact.

At this point, with our formulation of incremental strain expressions completed, it will be useful to again consider an example.

9.1 Example 9: Incremental Strain

We use the same simple shear formulation as was used earlier, shown in Fig. 3.2, but now in an incremental format. We write

$$[F_t] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{9.19a}$$

$$[F_{t+\Delta t}] = \begin{bmatrix} 1 & 0 & 0.1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{9.19b}$$

and

$$[F_t]^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{9.20a}$$

$$[F_{t+\Delta t}]^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0.1 & 0 & 1 \end{bmatrix} \tag{9.20b}$$

and, with $J = 1$,

$$[\overline{F}_t] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{9.21a}$$

$$[\overline{F}_{t+\Delta t}] = \begin{bmatrix} 1 & 0 & 0.1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{9.21b}$$

and

$$[\overline{F}_t]^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{9.22a}$$

$$\left[\overline{F}_{t+\Delta t}\right]^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0.1 & 0 & 1 \end{bmatrix} \tag{9.22b}$$

Then, from Equation (9.14), we have

$$\begin{aligned} \left[\delta\overline{E}\right] &= \frac{1}{2}\left\{\left[\overline{F}_{t+\Delta t}\right]^T\left[\overline{F}_{t+\Delta t}\right] - \left[\overline{F}_t\right]^T\left[\overline{F}_t\right]\right\} \\ &= \frac{1}{2}\left\{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0.1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0.1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\} \\ &= \begin{bmatrix} 0 & 0 & 0.05 \\ 0 & 0 & 0 \\ 0.05 & 0 & 0.005 \end{bmatrix} \end{aligned} \tag{9.23}$$

We can also write

$$\left[\overline{F}_t\right]^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{9.24a}$$

$$\left[F_{t+\Delta t}\right]^{-1} = \begin{bmatrix} 1 & 0 & -0.1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{9.24b}$$

and

$$\left[\overline{F}_t\right]^{-T} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{9.25a}$$

$$\left[F_{t+\Delta t}\right]^{-T} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -0.1 & 0 & 1 \end{bmatrix} \tag{9.25b}$$

and then, from Equation (9.18), we get

$$
\begin{aligned}
[\delta\overline{e}] &= -\frac{1}{2}\left\{\left[\overline{F}_{t+\Delta t}\right]^{-T}\left[\overline{F}_{t+\Delta t}\right]^{-1} - \left[\overline{F}_{t}\right]^{-T}\left[\overline{F}_{t}\right]^{-1}\right\} \\
&= -\frac{1}{2}\left\{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -0.1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -0.1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\} \\
&= \begin{bmatrix} 0 & 0 & 0.05 \\ 0 & 0 & 0 \\ 0.05 & 0 & -0.005 \end{bmatrix}
\end{aligned}
\tag{9.26}
$$

Chapter 10
Incrementation: Part Three

Abstract Expressions relative to incremental polar decomposition are given. The relationships between the deformation gradient and the orthogonal rotation tensor and the right and left stretches are given. From this, the expression for the incremental deformation gradient, in terms of the rotation tensor and the right stretch tensor, and their respective increments, is obtained. Since the obtained expression is approximate, correct to the first order, a correction term, the product of the two incremental terms, is added to the expression. An expression for the exact value is also given. An expression for the incremental Green-Lagrange strain, in terms of the right stretch tensor and the incremental right stretch tensor, correct to the first order, is given. A correction term, expressed as a function of the incremental right stretch tensor, is provided. A numerical example is presented to demonstrate the application of the developed incremental polar decomposition formulation.

Next we consider incremental polar decomposition. We can write, as we have seen,

$$F_{iJ} \equiv R_{iK}U_{KJ} \equiv v_{ik}R_{kJ} \tag{10.1}$$

Then, from Ogden (1997),

$$\delta F_{iJ} = \delta R_{iK}U_{KJ} + R_{iK}\delta U_{KJ} \tag{10.2}$$

correct to the *first order*, with a correction term of

$$-\delta R_{iK}\delta U_{Kj}$$

For the exact value, we have the expression

$$\delta F_{iJ} = R_{iK}^{t+\Delta t}U_{KJ}^{t+\Delta t} - R_{iK}^{t}U_{KJ}^{t} \tag{10.3}$$

And, considering the incremental Lagrangian strain, we have

$$\delta E_{IJ} = \frac{1}{2}(U_{IK}\delta U_{KJ} + \delta U_{IK}U_{KJ}) \tag{10.4}$$

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correct to the *first order*, with a correction term of

$$-\frac{1}{2}\delta U_{IK}\delta U_{KJ}$$

We again employ the simple shear model, shown in Fig. 3.2, to demonstrate the polar decomposition formulations in Equations (10.1) through (10.4).

10.1 Example 10: Incremental Polar Decomposition

We, from earlier developed expressions, can write

$$[R] = \begin{bmatrix} 0.998753 & 0 & 0.049936 \\ 0 & 1 & 0 \\ -0.049938 & 0 & 0.998753 \end{bmatrix} \tag{10.5a}$$

$$[U] = \begin{bmatrix} 0.998753 & 0 & 0.049939 \\ 0 & 1 & 0 \\ 0.049938 & 0 & 1.003747 \end{bmatrix} \tag{10.5b}$$

and

$$[\delta R] = \begin{bmatrix} -0.001247 & 0 & 0.049936 \\ 0 & 0 & 0 \\ -0.049938 & 0 & -0.001247 \end{bmatrix} \tag{10.6a}$$

$$[\delta U] = \begin{bmatrix} -0.001247 & 0 & 0.049939 \\ 0 & 0 & 0 \\ 0.049938 & 0 & 0.003747 \end{bmatrix} \tag{10.6b}$$

the values in $[\delta R]$ and $[\delta U]$ are obtained by subtracting $[I]$ from $[R]$ and $[U]$, respectively. Then

$$[\delta R][U] = \begin{bmatrix} 0.001248 & 0 & 0.050061 \\ 0 & 0 & 0 \\ -0.049938 & 0 & -0.003746 \end{bmatrix} \tag{10.7}$$

$$[R][\delta U] = \begin{bmatrix} 0.001248 & 0 & 0.050064 \\ 0 & 0 & 0 \\ 0.049938 & 0 & 0.001248 \end{bmatrix} \tag{10.8}$$

and, from Equation (10.2),

$$[\delta F] = \begin{bmatrix} 0.002496 & 0 & 0.100125 \\ 0 & 0 & 0 \\ 0 & 0 & -0.002498 \end{bmatrix} \tag{10.9}$$

If we add the correction term of $-[\delta R][\delta U]$, i.e.,

$$-[\delta R][\delta U] = -\begin{bmatrix} 0.002495 & 0 & 0.000125 \\ 0 & 0 & 0 \\ 0 & 0 & -0.002499 \end{bmatrix} \tag{10.10}$$

we get the correct result,

$$[\delta F] = \begin{bmatrix} 0 & 0 & 0.1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{10.11}$$

Still from Ogden (1997), considering the incremental Lagrangian strain, from Equation (10.4),

$$[\delta E] = \frac{1}{2}([U][\delta U] + [\delta U][U]) \tag{10.12}$$

we have

$$[U] = \begin{bmatrix} 0.998753 & 0 & 0.049939 \\ 0 & 1 & 0 \\ 0.049938 & 0 & 1.003747 \end{bmatrix} \tag{10.13a}$$

$$[\delta U] = \begin{bmatrix} -0.001247 & 0 & 0.049939 \\ 0 & 0 & 0 \\ 0.049938 & 0 & 0.003747 \end{bmatrix} \tag{10.13b}$$

and

$$[U][\delta U] = \begin{bmatrix} 0.001248 & 0 & 0.050064 \\ 0 & 0 & 0 \\ 0.050063 & 0 & 0.006255 \end{bmatrix} \tag{10.14a}$$

$$[\delta U][U] = \begin{bmatrix} 0.001248 & 0 & 0.050064 \\ 0 & 0 & 0 \\ 0.050063 & 0 & 0.006255 \end{bmatrix} \tag{10.14b}$$

yielding

$$[\delta E] = \begin{bmatrix} 0.001248 & 0 & 0.050064 \\ 0 & 0 & 0 \\ 0.050063 & 0 & 0.006255 \end{bmatrix} \tag{10.15}$$

correct to the first order. The correction term is

$$-\frac{1}{2}[\delta U][\delta U] = -\begin{bmatrix} 0.001248 & 0 & 0.000063 \\ 0 & 0 & 0 \\ 0.000063 & 0 & 0.001254 \end{bmatrix} \tag{10.16}$$

applying it gives

$$[\delta E] = \begin{bmatrix} 0 & 0 & 0.05 \\ 0 & 0 & 0 \\ 0.05 & 0 & 0.005 \end{bmatrix} \tag{10.17}$$

which we know to be correct.

Chapter 11
Incrementation: Part Four

Abstract Incremental constitutive relationships are considered. Given rate relationships, corresponding incremental constitutive relationships can be written. While the rate equations are exact, the incremental equations are approximate and correct only to the first order. The increment of a second-order tensor function of the deformation gradient can be written formally as a Taylor series and expanded to yield, respectively, fourth, sixth, ... order tensors with Cartesian components. This procedure is employed to develop the sixth-order "first elasticity tensor." A very significant numerical example is presented to demonstrate the utilization of the first elasticity tensor. A solution of the simple shear problem, in which the incremental first Piola-Kirchhoff stress tensor is evaluated as the product of the first elasticity tensor and the incremental deformation gradient and summed at each time step, is obtained, first using the fourth-order first elasticity tensor and then using the sixth-order first elasticity tensor. The enhanced convergence resulting from the use of the sixth-order first elasticity tensor is demonstrated.

Next, we consider incremental constitutive relationships. Given the rate relationships shown in Equations (6.24b) and (6.29b), we can write respective corresponding incremental constitutive relationships, correct to the first order,

$$\delta S_{IJ} = D^{SE}_{IJKL}\delta E_{KL} \quad \text{or} \quad \delta \mathbf{S} = \mathbf{D}^{SE} : \delta \mathbf{E} \tag{11.1a}$$

and

$$\delta N_{Ij} = A^{(1)}_{IjKl}\delta F_{lK} \quad \text{or} \quad \delta \mathbf{N} = \mathbf{A}^{(1)} : \delta \mathbf{F}^{T} \tag{11.1b}$$

and, correspondingly,

$$\delta P_{jI} = \hat{A}^{(1)}_{jIlK}\delta F_{lK} \quad \text{or} \quad \delta \mathbf{P} = \hat{\mathbf{A}}^{(1)} : \delta \mathbf{F} \tag{11.1c}$$

While observing that the rate equations are exact in formulation, we note that the incremental Equations (11.1a, 11.1b, and 11.1c) are approximate and correct only

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to the first order. The increment of a second-order tensor function of $\mathbf{F}$, $\mathbf{H}(\mathbf{F})$, at $\mathbf{F}$ may be written formally as a Taylor series (Ogden 1997),

$$\delta\mathbf{H} = \mathbf{A}^1 : \mathbf{F} + \frac{1}{2}\mathbf{A}^2 : [\delta\mathbf{F}, \delta\mathbf{F}] + \cdots \tag{11.2}$$

where $\delta\mathbf{F}$ is the incremental deformation gradient and where

$$\mathbf{A}^1 = \frac{\partial\mathbf{H}}{\partial\mathbf{F}} \tag{11.3a}$$

$$\mathbf{A}^2 = \frac{\partial^2\mathbf{H}}{\partial\mathbf{F}^2}, \cdots \tag{11.3b}$$

are respectively fourth, sixth, ... order tensors with Cartesian components,

$$\mathrm{A}^1_{iJkL} = \frac{\partial H_{iJ}}{\partial F_{kL}} \tag{11.4a}$$

$$\mathrm{A}^2_{iJkLmN} = \frac{\partial^2 H_{iJ}}{\partial F_{kL}\partial F_{mN}}, \cdots \tag{11.4b}$$

So, we can write

$$\hat{A}^{(2)}_{jIlKnM} = \frac{\partial P_{jI}}{\partial F_{lK}\partial F_{nM}} = \frac{\partial \hat{A}^{(1)}_{jIlK}}{\partial F_{nM}} \tag{11.5}$$

and, expanding in the Taylor series, we have

$$\delta P_{jI} = \hat{A}^{(1)}_{jIlK}\delta F_{lK} + \frac{1}{2}\hat{A}^{(2)}_{jIlKnM}\delta F_{nM}\delta F_{lK} + \cdots \tag{11.6}$$

We now need to develop the expression for $\hat{A}^{(2)}_{jIlKnM}$ by taking $\frac{\partial \hat{A}^{(1)}_{jIlK}}{\partial F_{nM}}$, and getting

fourth term:

$$\begin{aligned} &\frac{\partial}{\partial F_{nM}}\left[J^{-2/3}\left(\delta_{jl}\delta_{KI} - \frac{2}{3}F^T_{Kl}F_{jI}\right)\right] \\ &\quad = \frac{2}{3}J^{-2/3}\left(F^T_{Ml}F^T_{Kn}F_{jI} - F^T_{Mn}\delta_{jl}\delta_{KI} + \frac{2}{3}F^T_{Mn}F^T_{Kl}F_{jI} - F^T_{Kl}\delta_{jn}\delta_{MI}\right) \end{aligned} \tag{11.7}$$

fifth term:

$$
\begin{aligned}
&\frac{\partial}{\partial F_{nM}}\left[2J^{-2/3}F_{lK}F_{Ij}^{T}-\bar{I}_1\left(\frac{2}{3}F_{Kl}^{T}F_{Ij}^{T}+F_{Kj}^{T}F_{Il}^{T}\right)\right]\\
&\quad=-\left(2J^{-2/3}F_{nM}-\frac{2}{3}\bar{I}_1F_{Mn}^{T}\right)F_{Kj}^{T}F_{Il}^{T}+\bar{I}_1\left(F_{Mj}^{T}F_{Kn}^{T}F_{Il}^{T}+F_{Kj}^{T}F_{Ml}^{T}F_{In}^{T}\right)\\
&\qquad-\frac{2}{3}\left[2J^{-2/3}F_{nM}F_{Kl}^{T}F_{Ij}^{T}-\bar{I}_1\left(\frac{2}{3}F_{Mn}^{T}F_{Kl}^{T}F_{Ij}^{T}+F_{Ml}^{T}F_{Kn}^{T}F_{Ij}^{T}+F_{Kl}^{T}F_{Mj}^{T}F_{In}^{T}\right)\right]\\
&\qquad-2J^{-2/3}\left[\frac{2}{3}F_{Mn}^{T}F_{lK}F_{Ij}^{T}-\delta_{nl}\delta_{MK}F_{Ij}^{T}+F_{lK}F_{Mj}^{T}F_{In}^{T}\right]
\end{aligned}
\tag{11.8}
$$

sixth term:

$$
\begin{aligned}
&\frac{\partial}{\partial F_{nM}}\left[(2J-1)JF_{Kl}^{T}F_{Ij}^{T}-(J-1)JF_{Kj}^{T}F_{Il}^{T}\right]=(4J-1)JF_{Mn}^{T}F_{Kl}^{T}F_{Ij}^{T}\\
&\quad-(2J-1)J\left(F_{Ml}^{T}F_{Kn}^{T}F_{Ij}^{T}+F_{Kl}^{T}F_{Mj}^{T}F_{In}^{T}+F_{Mn}^{T}F_{Kj}^{T}F_{Il}^{T}\right)\\
&\quad+(J-1)J\left(F_{Mj}^{T}F_{Kn}^{T}F_{Il}^{T}+F_{Kj}^{T}F_{Ml}^{T}F_{In}^{T}\right)
\end{aligned}
\tag{11.9}
$$

Then, we can write

$$
\hat{A}^{(2)}_{jIlKnM}=G(\textit{fourth term})-\frac{G}{3}(\textit{fifth term})+K(\textit{sixth term})
\tag{11.10}
$$

and the second-order expression for the incremental value of P_{jI},

$$
\delta P_{jI}=\hat{A}^{(1)}_{jIlK}\delta F_{lK}+\frac{1}{2}\hat{A}^{(2)}_{jIlKnM}\delta F_{nM}\delta F_{lK}
\tag{11.11}
$$

Now we will formulate, and obtain the numerical solution for, an incremental hyperelasticity problem, once again employing the simple shear model shown in Fig. 3.2.

11.1 Example 11: Incremental Analysis

We will solve for the Piola stress tensor for the simple shear model incrementally, given that $G = 300$ psi and $K = 300{,}000$ psi and using, of course, a computer program—in this case a FORTRAN® program—a listing of which is given in Appendix F. For the starting point, we set

$$[F_0] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{11.12}$$

and the corresponding expression for $\hat{A}^{(1)}_{jIlK}$, converted to a Voigt-Mandel format, is

$$\left[\hat{A}^{(1)}\right] = \begin{bmatrix} 300400 & 299800 & 299800 & 0 & 0 & 0 \\ 299800 & 300400 & 299800 & 0 & 0 & 0 \\ 299800 & 299800 & 300400 & 0 & 0 & 0 \\ 0 & 0 & 0 & 600 & 0 & 0 \\ 0 & 0 & 0 & 0 & 600 & 0 \\ 0 & 0 & 0 & 0 & 0 & 600 \end{bmatrix} \tag{11.13}$$

Then, for a first-order incremental formulation, applying $\delta\mathbf{F} = 0.01$, with Equation (11.1c), we get for the Piola stress tensor, at the end of the first increment

$$[P_1] = \begin{bmatrix} 0.00000 & 0.00000 & 3.00000 \\ 0.00000 & 0.00000 & 0.00000 \\ 3.00000 & 0.00000 & 0.00000 \end{bmatrix} \tag{11.14}$$

The units of this, and subsequent stress and constitutive tensors in this example, are psi. Then, for the second increment, with the updated deformation gradient,

$$[F_2] = \begin{bmatrix} 1 & 0 & 0.01 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{11.15}$$

and the corresponding expression for $\hat{A}^{(1)}_{jIlK}$, again converted to a Voigt-Mandel format,

$$\left[\hat{A}^{(1)}\right] = \begin{bmatrix} 300400.0167 & 299800.0067 & 299800.0067 & 0 & -3003 & 0 \\ 299800.0067 & 300400.0167 & 299800.0067 & 0 & -3000 & 0 \\ 299800.0067 & 299800.0067 & 300400.0167 & 0 & -3003 & 0 \\ 0 & 0 & 0 & 600 & 0 & -3 \\ -3003 & -3000 & -3003 & 0 & 660.10 & 0 \\ 0 & 0 & 0 & -3 & 0 & 600 \end{bmatrix} \tag{11.16}$$

we have for the summed Piola stress tensor, again with $\delta\mathbf{F} = 0.01$,

$$[P_2] = \begin{bmatrix} -0.02000 & 0.00000 & 6.00000 \\ 0.00000 & -0.02000 & 0.00000 \\ 6.00030 & 0.00000 & -0.02000 \end{bmatrix} \tag{11.17}$$

Then we have for the updated deformation gradient,

$$[F_3] = \begin{bmatrix} 1 & 0 & 0.02 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{11.18}$$

and the resulting Piola stress tensor,

$$[P_3] = \begin{bmatrix} -0.06000 & 0.00000 & 9.00000 \\ 0.00000 & -0.06000 & 0.00000 \\ 9.00150 & 0.00000 & -0.06000 \end{bmatrix} \tag{11.19}$$

We continue this procedure and obtain the following values of the summed Piola stress tensor, through a total of 10 summed incremental values of deformation gradient, $\delta\mathbf{F} = 0.01$,

$$[P_4] = \begin{bmatrix} -0.12000 & 0.00000 & 12.00000 \\ 0.00000 & -0.12000 & 0.00000 \\ 12.00420 & 0.00000 & -0.12000 \end{bmatrix} \tag{11.20a}$$

$$[P_5] = \begin{bmatrix} -0.20000 & 0.00000 & 15.00000 \\ 0.00000 & -0.20000 & 0.00000 \\ 15.00900 & 0.00000 & -0.20000 \end{bmatrix} \tag{11.20b}$$

$$[P_6] = \begin{bmatrix} -0.30000 & 0.00000 & 18.00000 \\ 0.00000 & -0.30000 & 0.00000 \\ 18.01650 & 0.00000 & -0.30000 \end{bmatrix} \tag{11.20c}$$

$$[P_7] = \begin{bmatrix} -0.42000 & 0.00000 & 21.00000 \\ 0.00000 & -0.42000 & 0.00000 \\ 21.02730 & 0.00000 & -0.42000 \end{bmatrix} \tag{11.20d}$$

$$[P_8] = \begin{bmatrix} -0.56000 & 0.00000 & 24.00000 \\ 0.00000 & -0.56000 & 0.00000 \\ 24.04200 & 0.00000 & -0.56000 \end{bmatrix} \tag{11.20e}$$

$$[P_9] = \begin{bmatrix} -0.72000 & 0.00000 & 27.00000 \\ 0.00000 & -0.72000 & 0.00000 \\ 27.06120 & 0.00000 & -0.72000 \end{bmatrix} \tag{11.20f}$$

$$[P_{10}] = \begin{bmatrix} -0.90000 & 0.00000 & 30.00000 \\ 0.00000 & -0.90000 & 0.00000 \\ 30.08550 & 0.00000 & -0.90000 \end{bmatrix} \tag{11.20g}$$

The final incrementally obtained expression, $[P_{10}]$, corresponds to

$$[P] = \begin{bmatrix} -1 & 0 & 30 \\ 0 & -1 & 0 \\ 30.1 & 0 & -1 \end{bmatrix} \tag{11.21}$$

Now, we follow the same incremental procedure, with $\delta\mathbf{F} = 0.01$, but this time applying the second-order formulation, Equation (11.11). The resulting summed values of the Piola stress tensor are then,

$$[P_1] = \begin{bmatrix} -0.01000 & 0.00000 & 3.00000 \\ 0.00000 & -0.01000 & 0.00000 \\ 3.00000 & 0.00000 & -0.01000 \end{bmatrix} \tag{11.22a}$$

$$[P_2] = \begin{bmatrix} -0.04000 & 0.00000 & 6.00000 \\ 0.00000 & -0.04000 & 0.00000 \\ 6.00060 & 0.00000 & -0.04000 \end{bmatrix} \tag{11.22b}$$

$$[P_3] = \begin{bmatrix} -0.09000 & 0.00000 & 9.00000 \\ 0.00000 & -0.09000 & 0.00000 \\ 9.00240 & 0.00000 & -0.09000 \end{bmatrix} \tag{11.22c}$$

$$[P_4] = \begin{bmatrix} -0.16000 & 0.00000 & 12.00000 \\ 0.00000 & -0.16000 & 0.00000 \\ 12.00600 & 0.00000 & -0.16000 \end{bmatrix} \tag{11.22d}$$

$$[P_5] = \begin{bmatrix} -0.25000 & 0.00000 & 15.00000 \\ 0.00000 & -0.25000 & 0.00000 \\ 15.01200 & 0.00000 & -0.25000 \end{bmatrix} \tag{11.22e}$$

$$[P_6] = \begin{bmatrix} -0.36000 & 0.00000 & 18.00000 \\ 0.00000 & -0.36000 & 0.00000 \\ 18.02100 & 0.00000 & -0.36000 \end{bmatrix} \tag{11.22f}$$

$$[P_7] = \begin{bmatrix} -0.49000 & 0.00000 & 21.00000 \\ 0.00000 & -0.49000 & 0.00000 \\ 21.03360 & 0.00000 & -0.49000 \end{bmatrix} \tag{11.22g}$$

$$[P_8] = \begin{bmatrix} -0.64000 & 0.00000 & 24.00000 \\ 0.00000 & -0.64000 & 0.00000 \\ 24.05040 & 0.00000 & -0.64000 \end{bmatrix} \tag{11.22h}$$

$$[P_9] = \begin{bmatrix} -0.81000 & 0.00000 & 27.00000 \\ 0.00000 & -0.81000 & 0.00000 \\ 27.07200 & 0.00000 & -0.81000 \end{bmatrix} \tag{11.22i}$$

$$[P_{10}] = \begin{bmatrix} -1.00000 & 0.00000 & 30.00000 \\ 0.00000 & -1.00000 & 0.00000 \\ 30.09900 & 0.00000 & -1.00000 \end{bmatrix} \tag{11.22j}$$

We thus see a much improved correspondence between $[P_{10}]$ and the exact expression for $[P]$.

In Chap. 12, we define the meaning of objectivity and consider how it determines the suitability of certain tensors, e.g., stress and strain, to describe material response, and for the development of constitutive laws. The developed constitutive laws must adhere to defined objectivity requirements in order to assure that the response of the material is independent of the observer. We will verify in Chap. 12 that the solution developed here meets the requirements of objectivity.

Chapter 12
Objectivity

Abstract An objective stress measure is one that ensures that stress-strain responses are not affected by superposed rigid-body rotations. This means that they should be invariant to observers in different frames of reference. For example, if frame one is fixed, while frame two is rotating with respect to frame one, the stress response obtained in both frames using the same constitutive equation should obey the transformation that rotates frame one to the orientation of frame two. Single-based second-order deformation tensors and strain tensors are objective. Two-point second-order tensors such as the deformation gradient are also objective, even though they transform like vectors and not like second-order tensors, because one of the indices of the tensor describes the material coordinates which are independent of the observer. Objective tensors are suitable for describing material response and for the development of incremental constitutive laws, since they are independent of the observer; however, objective tensors usually do not preserve their objectivity through time differentiation. A frequently encountered non-objective tensor is the spatial velocity gradient tensor, while the rate-of-deformation tensor is objective. It consequently can be used in the formulation of spatial rate-constitutive laws. A strain-energy function is objective if and only if the balance of angular momentum condition holds. A numerical example is presented in order to illustrate some of the important objectivity determinations.

Any significant study of deformation tensors must include material related to observer transformations. Observer motion consists of translation and rotation; hence it can only modify those kinematical tensors which also relate to translation and rotation. Deformation tensors such as U_{IJ}, v_{ij}, C_{IJ}, and b_{ij} and strain tensors such as E_{IJ} and e_{ij} are termed *objective*. Two-point second-order tensors such as F_{iJ} are also objective because one of the indices of the tensor describes the material coordinates.

An objective stress measure is one that ensures that the stress-strain responses are not affected by any superposed rigid-body rotations. This means that they should be invariant to observers in different frames of reference. For example, if frame one is fixed, while frame two is rotating with respect to frame one, the stress response obtained in both frames using the same constitutive equation should obey the following transformation,

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$$\mathbf{t}^* = \mathbf{OtO}^T \tag{12.1}$$

where **t** is the Cauchy stress, the superscript (*) indicates the rotated tensor, and **O** is the orthogonal tensor which rotates frame one to the orientation of frame two (Bonet and Wood 2008). The same would be true for any objective stress measure, or corresponding single-base second-order tensor. We can write the transformations

$$\mathbf{F}^* = \frac{\partial \mathbf{x}^*}{\partial \mathbf{x}} \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \mathbf{OF} \tag{12.2a}$$

and

$$\mathbf{b}^* = \mathbf{ObO}^T \tag{12.2b}$$

and then proceed to solve for **t***, or a different rotated stress tensor, in the standard manner. Then, if the computed stress tensor satisfies Equation (12.1), or if it is a two-point tensor, Equation (12.2a), it is objective. This is true in the latter case, even though it transforms like a vector and not like a second-order tensor, because one index describes the material coordinates which are independent of the observer (Holzapfel 2000). Tensors that are not objective are termed *relative*.

In order to illustrate this important determination, we employ a numerical example.

12.1 Example 12: Objectivity

We once again utilize the simple shear model,

$$[F] = \begin{bmatrix} 1 & 0 & \gamma \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{12.3}$$

where $\gamma = 0.1$. Then, from $\mathbf{b} = \mathbf{FF}^T$, we compute

$$[b] = \begin{bmatrix} 1 & 0 & 0.1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0.1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1.01 & 0 & 0.1 \\ 0 & 1 & 0 \\ 0.1 & 0 & 1 \end{bmatrix} \tag{12.4}$$

We define a rotation by

$$[O] = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{12.5}$$

and thus have, where $\alpha = 45$ degrees,

$$[O] = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix} \tag{12.6}$$

We then apply the transformation relationship $\mathbf{F}^* = \mathbf{OF}$ and obtain

$$\begin{aligned} [F^*] = [O][F] &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0.1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 & 0.1 \\ 1 & 1 & 0.1 \\ 0 & 0 & \sqrt{2} \end{bmatrix} \end{aligned} \tag{12.7}$$

and

$$[F^*]^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & -0.1\sqrt{2} \\ -1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix} \tag{12.8}$$

We note that $J^* \equiv J$. From the relationship $\mathbf{b}^* = \mathbf{F}^*\mathbf{F}^{*T}$, we get

$$\begin{aligned} [b^*] &= \frac{1}{2} \begin{bmatrix} 1 & -1 & 0.1 \\ 1 & 1 & 0.1 \\ 0 & 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0.1 & 0.1 & \sqrt{2} \end{bmatrix} \\ &= \begin{bmatrix} 1.005 & 0.005 & 0.05\sqrt{2} \\ 0.005 & 1.005 & 0.05\sqrt{2} \\ 0.05\sqrt{2} & 0.05\sqrt{2} & 1 \end{bmatrix} \end{aligned} \tag{12.9}$$

and from the orthogonal transformation $\mathbf{b}^* = \mathbf{ObO}^T$ we obtain

$$
\begin{aligned}
[b^*] &= \frac{1}{2}\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} 1.01 & 0 & 0.1 \\ 0 & 1 & 0 \\ 0.1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix} \\
&= \begin{bmatrix} 1.005 & 0.005 & 0.05\sqrt{2} \\ 0.005 & 1.005 & 0.05\sqrt{2} \\ 0.05\sqrt{2} & 0.05\sqrt{2} & 1 \end{bmatrix}
\end{aligned} \tag{12.10}
$$

demonstrating the objectivity of the left Cauchy-Green deformation tensor **b**.

We compute **t**, the Cauchy stress, from

$$t_{ij} = \frac{G}{J}\left(\overline{b}_{ij} - \frac{\overline{b}_{kk}}{3}\delta_{ij}\right) + K(J-1)\delta_{ij} \tag{12.11}$$

where

$$\overline{b}_{ij} = J^{-2/3} b_{ij} \tag{12.12a}$$

$$J = 1 \tag{12.12b}$$

$$G = 300\,\text{psi} \tag{12.12c}$$

thus getting

$$
\begin{aligned}
[t] &= 300\left(\begin{bmatrix} 1.01 & 0 & 0.1 \\ 0 & 1 & 0 \\ 0.1 & 0 & 1 \end{bmatrix} - \frac{3.01}{3}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \\
&= \begin{bmatrix} 2 & 0 & 30 \\ 0 & -1 & 0 \\ 30 & 0 & -1 \end{bmatrix}
\end{aligned} \tag{12.13}
$$

where the units of this stress tensor and the subsequent ones are psi. We compute **t***, the rotated Cauchy stress tensor, from

$$t^*_{ij} = \frac{G}{J}\left(\overline{b}^*_{ij} - \frac{\overline{b}^*_{kk}}{3}\delta_{ij}\right) + K(J-1)\delta_{ij} \tag{12.14}$$

and get

$$\left[t^*\right] = 300\left(\begin{bmatrix} 1.005 & 0.005 & 0.05\sqrt{2} \\ 0.005 & 1.005 & 0.05\sqrt{2} \\ 0.05\sqrt{2} & 0.05\sqrt{2} & 1 \end{bmatrix} - \frac{3.01}{3}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$$
$$= \begin{bmatrix} 0.5 & 1.5 & 15\sqrt{2} \\ 1.5 & 0.5 & 15\sqrt{2} \\ 15\sqrt{2} & 15\sqrt{2} & -1 \end{bmatrix} \tag{12.15}$$

and from Equation (12.1), we get

$$\left[t^*\right] = \frac{1}{2}\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}\begin{bmatrix} 2 & 0 & 30 \\ 0 & -1 & 0 \\ 30 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$$
$$= \begin{bmatrix} 0.5 & 1.5 & 15\sqrt{2} \\ 1.5 & 0.5 & 15\sqrt{2} \\ 15\sqrt{2} & 15\sqrt{2} & -1 \end{bmatrix} \tag{12.16}$$

thus demonstrating the objectivity of **t**.

Now, we consider the objectivity of **P**, the first Piola-Kirchhoff stress tensor. We compute **P** from

$$P_{iJ} = G\left(J^{-2/3}F_{iJ} - \frac{1}{3}\bar{I}_1 F_{Ji}^T\right) + K(J-1)JF_{Ji}^T \tag{12.17}$$

where

$$\bar{I}_1 = \frac{b_{ii}}{J^{2/3}} \tag{12.18}$$

and get

$$[P] = 300\left(\begin{bmatrix} 1 & 0 & 0.1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{3.01}{3}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -0.1 & 0 & 1 \end{bmatrix}\right)$$
$$= \begin{bmatrix} -1 & 0 & 30 \\ 0 & -1 & 0 \\ 30.1 & 0 & -1 \end{bmatrix} \tag{12.19}$$

From the rotated first Piola-Kirchhoff stress tensor,

$$P_{iJ}^* = G\left(J^{-2/3}F_{iJ}^* - \frac{1}{3}\bar{I}_1^*\left(F_{iJ}^*\right)^{-T}\right) + K(J-1)J\left(F_{iJ}^*\right)^{-T} \tag{12.20}$$

where

$$\bar{I}_1^* = \frac{b_{ii}^*}{J^{2/3}} \tag{12.21}$$

we get

$$\begin{aligned}\left[P^*\right] &= 300\left(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 & 0.1 \\ 1 & 1 & 0.1 \\ 0 & 0 & \sqrt{2} \end{bmatrix} - \frac{3.01}{6}\begin{bmatrix} \sqrt{2} & -\sqrt{2} & 0 \\ \sqrt{2} & \sqrt{2} & 0 \\ -0.2 & 0 & 2 \end{bmatrix}\right) \\ &= \begin{bmatrix} -0.5\sqrt{2} & 0.5\sqrt{2} & 15\sqrt{2} \\ -0.5\sqrt{2} & -0.5\sqrt{2} & 15\sqrt{2} \\ 30.1 & 0 & -1 \end{bmatrix}\end{aligned} \tag{12.22}$$

and from the orthogonal transformation $\mathbf{P}^* = \mathbf{OP}$ we get

$$\begin{aligned}\left[P^*\right] &= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}\begin{bmatrix} -1 & 0 & 30 \\ 0 & -1 & 0 \\ 30.1 & 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} -0.5\sqrt{2} & 0.5\sqrt{2} & 15\sqrt{2} \\ -0.5\sqrt{2} & -0.5\sqrt{2} & 15\sqrt{2} \\ 30.1 & 0 & -1 \end{bmatrix}\end{aligned} \tag{12.23}$$

which demonstrates that **P** is objective.

The tensors **t** and **P**, as well as the second Piola-Kirchhoff stress tensor **S**, are objective, therefore they are deemed suitable for describing material response and for the development of constitutive laws, since they are independent of the observer. However, objective tensors usually do not preserve their objectivity through time differentiation. This is demonstrated through

$$\dot{\mathbf{F}}^* = \mathbf{O}\dot{\mathbf{F}} + \dot{\mathbf{O}}\mathbf{F} \tag{12.24a}$$

and

$$\dot{\mathbf{b}}^* = \mathbf{O}\dot{\mathbf{b}}\mathbf{O}^T + \dot{\mathbf{O}}\mathbf{b}\mathbf{O}^T + \mathbf{O}\mathbf{b}\dot{\mathbf{O}}^T \tag{12.24b}$$

which can be contrasted with Equations (12.2a and 12.2b).

A frequently encountered relative (non-objective) tensor is the spatial *velocity gradient tensor*

$$\mathbf{l} = \dot{\mathbf{F}}\mathbf{F}^{-1} \tag{12.25}$$

which is derived in Appendix E. The rotated velocity gradient

$$\mathbf{l}^* = \dot{\mathbf{F}}^* \left(\mathbf{F}^*\right)^{-1} \tag{12.26}$$

can be evaluated to give

$$\mathbf{l}^* = \mathbf{OlO}^T + \dot{\mathbf{O}}\mathbf{O}^T \tag{12.27}$$

and we see that the presence of the second term in this equation renders the velocity gradient non-objective. From Appendix E, we have the derived relationship

$$\mathbf{d} = \frac{1}{2}\left(\mathbf{l} + \mathbf{l}^T\right) \tag{12.28}$$

where $\mathbf{d}$ is the *rate-of-deformation tensor*. It turns out, fortuitously, we might say, that the rate-of-deformation tensor is objective. Writing the expression for the evaluated $\mathbf{d}^*$ we have

$$\mathbf{d}^* = \mathbf{OdO}^T + \frac{1}{2}\left(\dot{\mathbf{O}}\mathbf{O}^T + \mathbf{O}\dot{\mathbf{O}}^T\right) \tag{12.29}$$

The fact that the term in parentheses in this equation is the time derivative of $\mathbf{OO}^T = \mathbf{1}$ and thus is equal to zero shows that the rate-of-deformation tensor satisfies the condition of objectivity, as defined by Equation (12.1), and is therefore objective. It consequently can be, and is, used in the formulation of spatial rate-constitutive laws (Simo and Hughes 1998).

The strain-energy function $W(\mathbf{X}, \mathbf{F})$ is said to be objective, or frame indifferent, if

$$W(\mathbf{X}, \mathbf{OF}) = W(\mathbf{X}, \mathbf{F}) \tag{12.30}$$

(Simo and Hughes 1998). In fact, one can show that $W(\mathbf{X}, \mathbf{F})$ is objective if and only if the balance of angular momentum condition $\mathbf{PF}^T = \mathbf{FP}^T$ holds. We can thus note that the strain-energy function considered in Example 12 is objective.

Chapter 13
Finite Viscoelasticity

Abstract Finite viscoelasticity plays a major role in defining the behavior of polymeric material systems which exhibit finite strains. An extension of small strain linear viscoelasticity to finite strains is directly accomplished when the generalized Maxwell model is chosen as the underlying analogous viscoelastic material structure. The rheological elements in parallel preserve the linear structure of the formulation even at finite strains. This formulation is based upon the Mooney-Rivlin strain-energy function and the second Piola-Kirchhoff stress tensor. The internal variables are approximated by a recursive expression. The efficient solution of the hereditary integral is crucial for the incremental numerical implementation. The viscoelastic solution is for the incremental nominal stress tensor which is equal to the product of the formulated fourth-order elasticity tensor and the incremental deformation gradient. For the special case of a finite elastic material, the formulation appropriately reduces to yield the elastic value of the incremental nominal stress tensor.

In an introductory sense, it is appropriate to present enough of the fundamental aspects of small strain linear viscoelasticity to make anyone studying this material feel sufficiently comfortable with that topic, with reference to Hackett and Bennett (2000). We begin by noting a standard version of the linear, small strain, isotropic constitutive relationship, i.e., the material Jacobian,

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} K+\frac{4G}{3} & K-\frac{2G}{3} & K-\frac{2G}{3} & 0 & 0 & 0 \\ K-\frac{2G}{3} & K+\frac{4G}{3} & K-\frac{2G}{3} & 0 & 0 & 0 \\ K-\frac{2G}{3} & K-\frac{2G}{3} & K+\frac{4G}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2G & 0 & 0 \\ 0 & 0 & 0 & 0 & 2G & 0 \\ 0 & 0 & 0 & 0 & 0 & 2G \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{31} \\ \varepsilon_{12} \end{Bmatrix} \tag{13.1}$$

where σ_{ij} and ε_{ij} are the stress and strain components, respectively, and G and K are the shear and bulk moduli, respectively, related through the expression

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$G \equiv E/(3 - E/3K)$, where E is the Young's modulus. As is customary, we will assume that dilatation is completely elastic and, therefore, viscoelastic behavior is based upon deviatoric response. This is true for both small and finite strains. We proceed to decompose the stress tensor into deviatoric and volumetric parts. Using Cartesian tensor index notation, the strain is given by the kinematic relationship

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{j,i} + u_{i,j} + u_{k,i}u_{k,j}\right), \quad i, j, k = 1, 2, 3 \tag{13.2}$$

It can be decomposed into deviatoric and mean components,

$$\varepsilon_{ij} = \overline{\varepsilon}_{ij} + \varepsilon_m \delta_{ij} \tag{13.3}$$

where $\overline{\varepsilon}_{ij}$ is the deviatoric strain and δ_{ij} is the Kronecker delta. The mean strain is given by

$$\varepsilon_m = \frac{1}{3}\varepsilon_{ii} \tag{13.4}$$

Correspondingly, the stress can be decomposed into deviatoric and mean components,

$$\sigma_{ij} = \overline{\sigma}_{ij} + \sigma_m \delta_{ij} \tag{13.5}$$

where $\overline{\sigma}_{ij}$ is the deviatoric stress. The mean stress is given by

$$\sigma_m = \frac{1}{3}\sigma_{ii} \tag{13.6}$$

The mean stress and strain are related through the expression

$$\sigma_m = 3K\varepsilon_m \tag{13.7}$$

In a deviatoric Maxwell model, a single spring and dashpot in series, the relationship between stress and *elastic* strain is

$$\overline{\sigma}_{ij} = 2G\overline{\varepsilon}_{ij}^{e} \tag{13.8}$$

while the relationship between stress and *viscous* strain rate is

$$\overline{\sigma}_{ij} = 2\eta\dot{\overline{\varepsilon}}_{ij}^{v} \tag{13.9}$$

where η is the viscosity. Combining Equations (13.8) and (13.9) gives

$$\dot{\overline{\sigma}}_{ij} = 2G\dot{\overline{\varepsilon}}_{ij} - \frac{\overline{\sigma}_{ij}}{\tau} \tag{13.10}$$

where $\tau \equiv \eta/G$, τ being the *relaxation time*.

With the same prescribed deviatoric strain, $\overline{\varepsilon}_{ij}$, or deviatoric strain rate, $\dot{\overline{\varepsilon}}_{ij}$, applied to each individual element of a generalized Maxwell model, comprised of several Maxwell models in parallel, the resulting deviatoric stress is the sum of the individual contributions,

$$\overline{\sigma}_{ij} = \sum_{n=1}^{N} \overline{\sigma}_{ij}^{(n)} \tag{13.11}$$

where N is the number of elements in the generalized Maxwell model. The stress relaxation of a generalized Maxwell model, under constant deviatoric strain $\overline{\varepsilon}_{ij}^{0}$, can be expressed as

$$\overline{\sigma}_{ij} = \overline{\varepsilon}_{ij}^{0} \sum_{n=1}^{N} G^{(n)} \exp\left(-\frac{t}{\tau^{(n)}}\right) \tag{13.12}$$

where t is the time variable, $G^{(n)}$ is the shear modulus of element n and $\tau^{(n)}$ is its

corresponding relaxation time, i.e., $\tau^{(n)} \equiv \eta^{(n)}/G^{(n)}$, where $\eta^{(n)}$ is the viscosity of element n. The generalized Maxwell model representing the response of a solid, with $\tau^{(1)} \to \infty$,

$$\overline{\sigma}_{ij} = \overline{\varepsilon}_{ij}^{0} \left[G^{(1)} + \sum_{n=2}^{N} G^{(n)} \exp\left(-\frac{t}{\tau^{(n)}}\right) \right] \tag{13.13}$$

or

$$\overline{\sigma}_{ij} = \phi(t) \overline{\varepsilon}_{ij}^{0} \tag{13.14}$$

where

$$\phi(t) = G^{(1)} + \sum_{n=2}^{N} G^{(n)} \exp\left(-\frac{t}{\tau^{(n)}}\right) \tag{13.15}$$

is the well-known Prony series.

Now, for each element in the generalized Maxwell model, from Equation (13.10),

$$\dot{\overline{\sigma}}_{ij}^{(n)} = 2G^{(n)}\dot{\overline{\varepsilon}}_{ij} - \frac{\overline{\sigma}_{ij}^{(n)}}{\tau^{(n)}} \tag{13.16}$$

then, summing over all of the elements,

$$\sum_{n=1}^{N} \dot{\overline{\sigma}}_{ij}^{(n)} = 2\sum_{n=1}^{N} G^{(n)}\dot{\overline{\varepsilon}}_{ij} - \sum_{n=1}^{N} \frac{\dot{\overline{\sigma}}_{ij}^{(n)}}{\tau^{(n)}} \tag{13.17}$$

Knowing that

$$G = \sum_{n=1}^{N} G^{(n)} \tag{13.18}$$

and that

$$\dot{\overline{\sigma}}_{ij} = \sum_{n=1}^{N} \dot{\overline{\sigma}}_{ij}^{(n)} \tag{13.19}$$

from Equation (13.11), and substituting Equations (13.18) and (13.19) into Equation (13.17) gives

$$\dot{\overline{\sigma}}_{ij} = 2G\dot{\overline{\varepsilon}}_{ij} - \sum_{n=1}^{N} \frac{\overline{\sigma}_{ij}^{(n)}}{\tau^{(n)}} \tag{13.20}$$

which is the governing equation for the linear viscoelastic material. Substituting Equations (13.3) through (13.7) into Equation (13.20) gives

$$\dot{\sigma}_{ij} = 2G\dot{\varepsilon}_{ij} + \left(K - \frac{2}{3}G\right)\dot{\varepsilon}_{ii} - \sum_{n=1}^{N} \frac{\overline{\sigma}_{ij}^{(n)}}{\tau^{(n)}} \tag{13.21}$$

A simple, stable integration operator for this equation is the central difference operator:

$$\dot{f}_{t+\frac{\Delta t}{2}} = \frac{\Delta f}{\Delta t} \tag{13.22a}$$

$$f_{t+\frac{\Delta t}{2}} = f_t + \frac{\Delta f}{2} \tag{13.22b}$$

where f is some function, f_t is its value at the beginning of the increment, Δf is the change in the function over the increment, and Δt is the time increment. Applying this operator gives

$$\Delta\sigma_{ij} = 2G\Delta\varepsilon_{ij} + \left(K - \frac{2}{3}G\right)\Delta\varepsilon_{ii} - \Delta t\sum_{n=1}^{N} \frac{\overline{\sigma}_{ij}^{(n)}}{\tau^{(n)}} \tag{13.23}$$

where the summation term is from the preceding increment.

By operating on Equation (13.23) we can obtain the terms in the material Jacobian matrix of Equation (13.1), i.e.,

$$\frac{\partial\Delta\sigma_{xx}}{\partial\Delta\varepsilon_{xx}} = K + \frac{4}{3}G \tag{13.24a}$$

$$\frac{\partial\Delta\sigma_{xx}}{\partial\Delta\varepsilon_{yy}} = K - \frac{2}{3}G \tag{13.24b}$$

$$\frac{\partial\Delta\sigma_{xy}}{\partial\Delta\varepsilon_{xy}} = 2G \tag{13.24c}$$

etc.

An extension of small strain linear viscoelasticity to finite strains can be directly accomplished when the generalized Maxwell model is chosen as the underlying analogous viscoelastic material structure. The rheological elements in parallel preserve the linear structure of the formulation even at finite strains. We take up the topic of finite viscoelasticity because it plays a major role in defining the behavior of polymeric material systems which exhibit finite strains. Considerable formidable research has been conducted on this topic, notably Lubliner (1985) (see Appendix E), Kaliske and Rothert (1997), Simo and Hughes (1998), Reese and Govindjee (1998), Aboudi (2000) and Bonet (2001). The formulation presented here probably draws most heavily from that of Aboudi (2000). It is based upon the strain-energy function presented in Chap. 4,

$$\widehat{W} = C_{10}\left(\bar{I}_1 - 3\right) + C_{01}\left(\bar{I}_2 - 3\right) + W_H(I_3) \tag{13.25}$$

where

$$C_{10} = \frac{G}{2}, \tag{13.26a}$$

$$C_{01} = 0 \tag{13.26b}$$

and

$$W_H = \frac{K}{2}(J - 1)^2 \tag{13.26c}$$

or, we can write

$$\widehat{W} = \overline{W}_D + \frac{K}{2}(J - 1)^2 \tag{13.27}$$

where, by analogy with the Maxwell-model-based formulation,

$$\overline{W}_D = \overline{W}^{\infty} + \sum_{n=1}^{N} \overline{W}^{(n)} \exp\left(-\frac{t}{\tau^{(n)}}\right) \tag{13.28}$$

or

$$\overline{W}\left(\mathbf{E}, \mathbf{H}^{(n)}\right) = \overline{W}^{\infty}(\mathbf{E}) + \sum_{n=1}^{N} \mathbf{H}^{(n)}\mathbf{E} \tag{13.29}$$

where $\overline{W}^{\infty}$ is the elastic strain energy for long-term deformations, $\mathbf{E}$ is the Green-Lagrange strain tensor and $\mathbf{H}^{(n)}$ is a set of N internal variables. The description in the reference configuration is necessary in order to preserve the principle of objectivity (frame indifference, Chap. 12). The internal variables $\mathbf{H}^{(n)}$ at time t can be expressed in terms of convolution integrals,

$$\mathbf{H}^{(n)}(t) = \int_0^t \exp\left(-\frac{t-\tau}{\tau^{(n)}}\right) \frac{\partial \overline{\mathbf{S}}^{(n)}}{\partial \tau} d\tau \tag{13.30}$$

where $\overline{\mathbf{S}}^{(n)}$ are second Piola-Kirchhoff stresses obtained from the strain-energy functions

$$\overline{\mathbf{S}}^{(n)} = \frac{\partial \overline{W}^{(n)}}{\partial \mathbf{E}} \tag{13.31}$$

Next, it is assumed that each term $\overline{W}^{(n)}$ is simply a multiplier of $\overline{W}^{(0)}$,

$$\overline{W}^{(n)} = \delta^{(n)} \overline{W}^{(0)} \tag{13.32}$$

where $\overline{W}^{(0)}$ is the short-term elastic energy and the material parameter $\delta^{(n)}$ is employed as a scalar quantity—in the general case, e.g., when orthotropic material is considered, it is represented by an unsymmetric fourth-order tensor $\boldsymbol{\delta}^{(n)}$. We thus have the relationship

$$\overline{W}^{\infty} = \overline{W}^{(0)} \left[1 - \sum_{n=1}^{N} \delta^{(n)}\right] \tag{13.33}$$

Consequently, the second Piola-Kirchhoff stress tensor can be written as

$$\overline{\mathbf{S}}(t) = \overline{\mathbf{S}}^{\infty}(t) + \sum_{n=1}^{N} \mathbf{H}^{(n)}(t) \tag{13.34}$$

where

$$\overline{\mathbf{S}}^{\infty}(t) = \frac{\partial \overline{W}^{\infty}}{\partial \mathbf{E}} = \left[1 - \sum_{n=1}^{N} \delta^{(n)}\right] \frac{\partial \overline{W}^{(0)}}{\partial \mathbf{E}} \tag{13.35a}$$

and

$$\mathbf{H}^{(n)}(t) = \int_0^t \delta^{(n)} \exp\left(-\frac{t-\tau}{\tau^{(n)}}\right) \frac{\partial \overline{\mathbf{S}}^{(0)}}{\partial \tau} d\tau \tag{13.35b}$$

The efficient solution of this hereditary integral is crucial for a numerical implementation. It can be shown that the internal variables $\mathbf{H}^{(n)}$ can be approximated by the recursive expression

$$\mathbf{H}^{(n)}_{t+\Delta t} = \exp\left(-\frac{\Delta t}{\tau^{(n)}}\right)\mathbf{H}^{(n)}_t + \delta^{(n)} \frac{\Delta \overline{\mathbf{S}}^{(0)}}{\Delta t} \tau^{(n)} \left[1 - \exp\left(-\frac{\Delta t}{\tau^{(n)}}\right)\right] \tag{13.36}$$

which is derived in Appendix D. Thus the following expression for the increment of $\mathbf{H}^{(n)}$ can be obtained,

$$\Delta \mathbf{H}^{(n)} = \beta^{(n)} \delta^{(n)} \Delta \overline{\mathbf{S}}^{(0)} - \alpha^{(n)} \mathbf{H}^{(n)} \tag{13.37}$$

where

$$\alpha^{(n)} = 1 - \exp\left(-\frac{\Delta t}{\tau(n)}\right) \tag{13.38a}$$

and

$$\beta^{(n)} = \frac{\alpha^{(n)} \tau^{(n)}}{\Delta t} \tag{13.38b}$$

Then, using Equations (13.34) and (13.35a) the incremental expression for the second Piola-Kirchhoff stress tensor is obtained

$$\Delta\overline{\mathbf{S}}^{(n)} = \left[1 - \sum_{n=1}^{N}\left(1-\beta^{(n)}\right)\delta^{(n)}\right]\Delta\overline{\mathbf{S}}^{(0)} - \sum_{n=1}^{N}\alpha^{(n)}\mathbf{H}^{(n)} \tag{13.39}$$

At this point, we shift our attention from the second Piola-Kirchhoff stress tensor to the nominal stress tensor N_{Ji}. We have the relationship, from Equation (6.97),

$$\dot{\overline{\mathbf{S}}} = \mathbf{Q} : \dot{\mathbf{F}} \quad \text{or} \quad \dot{\overline{S}}_{IJ} = Q^{SF}_{IJkL}\dot{F}_{kL} \tag{13.40}$$

which we have considered earlier and, accordingly, to the first order,

$$\Delta\overline{\mathbf{S}} = \mathbf{Q} : \Delta\mathbf{F} \quad \text{or} \quad \Delta\overline{S}_{IJ} = Q^{SF}_{IJkL}\Delta F_{kL} \tag{13.41}$$

We also know that

$$\overline{\mathbf{N}} = \overline{\mathbf{S}}\mathbf{F}^{T} \quad \text{or} \quad \overline{N}_{Ij} = \overline{S}_{IK}F^{T}_{jK} \tag{13.42}$$

From this expression we can write

$$\dot{\overline{\mathbf{N}}} = \dot{\overline{\mathbf{S}}}\mathbf{F}^{T} + \overline{\mathbf{S}}\dot{\mathbf{F}}^{T} \quad \text{or} \quad \dot{\overline{N}}_{Ij} = \dot{\overline{S}}_{IK}F^{T}_{jK} + \overline{S}_{IK}\dot{F}^{T}_{jK} \tag{13.43}$$

Then, to the first order, we have

$$\Delta\overline{\mathbf{N}} = \Delta\overline{\mathbf{S}}\mathbf{F}^{T} + \overline{\mathbf{S}}\Delta\mathbf{F}^{T} \quad \text{or} \quad \Delta\overline{N}_{Ij} = \Delta\overline{S}_{IK}F^{T}_{jK} + \overline{S}_{IK}\Delta F^{T}_{jK} \tag{13.44}$$

It follows that the desired incremental constitutive relationship for this material is, from Equation (6.99),

$$\Delta\overline{\mathbf{N}} = \mathbf{R} : \Delta\mathbf{F} \quad \text{or} \quad \Delta\overline{N}_{Ij} = R^{NF}_{IjkL}\Delta F_{kL} \tag{13.45}$$

where

$$R^{NF}_{IjkL} = Q^{SF}_{IPkL}F_{jP} + S_{IL}\delta_{jk} \tag{13.46}$$

We can note that in both Equations (13.37) and (13.39), $\Delta\overline{\mathbf{S}}^{(0)}_{t+\Delta t}$ is determined at any instant from the deformation gradient increment by Equation (13.41).

The increment of the nominal stress tensor can be obtained from Equation (13.44) which, using Equations (13.34), (13.35a), and (13.39), provides the expression

$$\Delta\overline{\mathbf{N}} = \left[\xi\Delta\overline{\mathbf{S}}^{(0)} - \sum_{n=1}^{N}\alpha^{(n)}\mathbf{H}^{(n)}\right]\mathbf{F}^T + \left[\eta\overline{\mathbf{S}}^{(0)} + \sum_{n=1}^{N}\mathbf{H}^{(n)}\right]\Delta\mathbf{F}^T \tag{13.47}$$

where

$$\xi = 1 - \sum_{n=1}^{N}\left(1 - \beta^{(n)}\right)\delta^{(n)} \tag{13.48a}$$

and

$$\eta = 1 - \sum_{n=1}^{N}\delta^{(n)} \tag{13.48b}$$

while

$$\Delta\overline{\mathbf{S}} = \xi\Delta\overline{\mathbf{S}}^{(0)} - \sum_{n=1}^{N}\alpha^{(n)}\mathbf{H}^{(n)} \tag{13.49a}$$

and

$$\overline{\mathbf{S}} = \eta\overline{\mathbf{S}}^{(0)} + \sum_{n=1}^{N}\mathbf{H}^{(n)} \tag{13.49b}$$

We can use Equation (13.41) to rewrite this increment as

$$\Delta\overline{N}_{Ij} = \left[\xi Q^{SF}_{IPkL}F_{jP} + \eta\delta_{jk}\overline{S}^{(0)}_{IL}\right]\Delta F_{kL} + \sum_{n=1}^{N}H^{(n)}_{IP}\Delta F_{jP} - \sum_{n=1}^{N}\alpha^{(n)}H^{(n)}_{IP}F_{jP} \tag{13.50}$$

so that

$$\Delta\overline{N}_{Ij} = \left[\xi Q^{SF}_{IPkL}F_{jP} + \eta\delta_{jk}\overline{S}^{(0)}_{IL} + \sum_{n=1}^{N}H^{(n)}_{IL}\right]\Delta F_{kL} - \sum_{n=1}^{N}\alpha^{(n)}H^{(n)}_{IP}F_{jP} \tag{13.51}$$

Consequently, the final form of the nominal stress tensor increment, which is expressed in terms of the deformation gradient increment and deformation history, is

$$\Delta\overline{\mathbf{N}} = \mathbf{V} : \Delta\mathbf{F} - \overline{\mathbf{Y}} \quad \text{or} \quad \Delta\overline{N}_{Ij} = V_{IjkL}\Delta F_{kL} - \overline{Y}_{Ij} \tag{13.52}$$

where the *viscoelastic tangent tensor* **V** is given by

$$V_{IjkL} = \xi Q^{SF}_{IPkL} F_{jP} + \eta \delta_{jk} \overline{S}^{(0)}_{IL} + \delta_{jk} \sum_{n=1}^{N} H^{(n)}_{IL} \tag{13.53}$$

and

$$\overline{\mathbf{Y}} = \sum_{n=1}^{n} \alpha^{(n)} \mathbf{H}^{(n)} \mathbf{F}^{T} \quad \text{or} \quad \overline{Y}_{Ij} = \sum_{n=1}^{N} \alpha^{(n)} H^{(n)}_{IL} F^{T}_{jL} \tag{13.54}$$

accounts for the deformation history. In the special case of a finite elastic material, $\delta^{(n)} = \alpha^{(n)} = 0$ so that Equation (13.52) reduces to Equation (13.45), with equal tangent tensors, namely, $\mathbf{V} = \mathbf{R}$.

Chapter 14
Finite Element Implementation

Abstract The state-of-the-art methodology for modeling nonlinear geometrical and material response in the field of solid mechanics is the finite element method. Embedded in any finite element computational model is a material model. The solution of a set of nonlinear equilibrium equations in the finite element analysis of finite deformations is typically achieved through the employment of a Newton-Raphson iteration procedure. This requires the linearization of the equilibrium equations, which necessitates an understanding of the directional derivative. A relatively simple numerical technique for solving nonlinear equations in computational finite elasticity consists of employing the so-called incremental/iterative solution technique of Newton's type. It is an efficient method with the desirable feature of a quadratic convergence rate near the solution point. It requires a consistent linearization of all of the quantities associated with the nonlinear problem, generating efficient recurrence update expressions. The nonlinear problem is then replaced by a sequence of easily solved linear equations at each iteration. The element stiffness formulation, based upon the "first elasticity tensor," is carried out.

The obvious reason for the development of the theory presented in this monograph is so that it can be incorporated in the accurate modeling of material systems that exhibit large deformations when subjected to mechanical loading conditions. The state-of-the-art methodology for modeling the nonlinear geometrical and material response of these systems is the *finite element method*. It should go without saying that anyone who would read and digest this monograph would already be aware of the finite element method, its capabilities, and the range of its employment. It would certainly be beneficial if one were also already familiar with finite element theory and with one or more of the commercial (or other) finite element computational codes that are widely employed to analyze a broad range of problems that fall under the heading of solid mechanics.

Embedded in any finite element computational model is a material model. It might be as simple as a small (infinitesimal) deformation, linear elastic model. Or, it might be as complicated as a large (finite) deformation, viscoplastic material model exhibiting mechanistic internal damage. In any case, the material model must obviously be incorporated in the code correctly in order to simulate the response of the system that is being modeled. Commercial finite element codes contain material models already programmed to simulate viscoelasticity, plasticity, etc.

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These codes also provide for the user to input his/her own formulated material model by way of a UMAT format. These are widely utilized as more and more material models are developed.

The solution of a set of nonlinear equilibrium equations in the finite element analysis of finite deformations is typically achieved through the employment of a Newton-Raphson iteration procedure. This requires the *linearization* of the equilibrium equations, which necessitates an understanding of the *directional derivative* (Holzapfel 2000).

A commonly applied and relatively simple numerical technique for solving nonlinear equations in computational finite elasticity consists of employing the so-called *incremental/iterative solution technique* of *Newton's type*, which was introduced in the pioneering work of Hughes and Pister (1978). It is an efficient method with the desirable feature of a quadratic convergence rate near the solution point. It requires a *consistent linearization* of all of the quantities associated with the nonlinear problem, generating efficient *recurrence update expressions*. The nonlinear problem is then replaced by a sequence of easily solved linear equations at each iteration.

Since primary emphasis in any finite element code is upon employing the appropriate element stiffness formulation, we demonstrate in the following material how the element tangent stiffness may be developed. We can express the linearized internal virtual work for a body in equilibrium, in terms of the directional derivative (Bonet and Wood 2008), as

$$D\delta w(\phi, \delta\mathbf{v})[\mathbf{u}] = \int_V \left(\nabla_0 \delta\mathbf{v}\right) : \hat{\mathbf{A}}^{(1)} : (\nabla_0 \mathbf{u}) dV \tag{14.1a}$$

$$\hat{\mathbf{A}}^{(1)} = \frac{\partial \mathbf{P}}{\partial \mathbf{F}} \tag{14.1b}$$

where δw is the virtual work, ϕ denotes the deformed configuration, $\delta\mathbf{v}$ is an arbitrary virtual velocity relative to the current position of the body, $\mathbf{u}$ is the displacement vector, $\mathbf{P}$ is the first Piola-Kirchhoff stress tensor, and $\mathbf{F}$ is the deformation gradient. The symbol $Df(x_0)[\mathbf{u}]$ denotes a (directional) derivative, formed at x_0, that operates on $\mathbf{u}$. The directional derivative $D\delta w(\phi_k, \delta\mathbf{v})[\mathbf{u}]$ is simply the change in δw due to ϕ_k changing to $\phi_k + \mathbf{u}$, where ϕ_k is a trial solution configuration.

The discretization of the continuum can be established in the initial configuration using isoparametric finite elements to interpolate the initial geometry in terms of the particles X_a defining the initial position of the element nodes where $N_a(\xi_1, \xi_2, \xi_3)$ are the standard shape functions and n denotes the number of nodes per element, i.e.,

$$\mathbf{X} = \sum_{a=1}^{n} N_a(\xi_1, \xi_2, \xi_3)\mathbf{X}_a \tag{14.2}$$

The subsequent motion is fully described in terms of the current position $\mathbf{x}_a(t)$ of the nodal particles

$$\mathbf{x} = \sum_{a=1}^{n} N_a \mathbf{x}_a(t) \tag{14.3}$$

Differentiating Equation (14.2) with respect to time gives the velocity interpolation as

$$\mathbf{v} = \sum_{a=1}^{n} N_a \mathbf{v}_a \tag{14.4}$$

and the virtual velocity interpolation as

$$\delta\mathbf{v} = \sum_{a=1}^{n} N_a \delta\mathbf{v}_a \tag{14.5}$$

Likewise, consistency with Equation (14.3) implies that the displacement $\mathbf{u}$ is interpolated as

$$\mathbf{u} = \sum_{b=1}^{n} N_b \mathbf{u}_b \tag{14.6}$$

The deformation gradient tensor is interpolated over an element,

$$\mathbf{F} = \nabla_0 \mathbf{x} = \nabla_0 \sum_{a=1}^{n} N_a \mathbf{x}_a = \sum_{a=1}^{n} \mathbf{x}_a \otimes \nabla_0 N_a \tag{14.7a}$$

or

$$F_{iJ} = \frac{\partial x_i}{\partial X_J} = \frac{\partial \sum_{a=1}^{n} N_a x_{a,i}}{\partial X_J} = \sum_{a=1}^{n} x_{a,i} \frac{\partial N_a}{\partial X_J} \tag{14.7b}$$

Then, applying the chain rule we get

$$\nabla_0 N_a \equiv \frac{\partial N_a}{\partial \mathbf{X}} = \frac{\partial N_a}{\partial \boldsymbol{\xi}} \frac{\partial \boldsymbol{\xi}}{\partial \mathbf{X}} = \left(\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{X}}\right)^T \frac{\partial N_a}{\partial \boldsymbol{\xi}} = \left(\frac{\partial \mathbf{X}}{\partial \boldsymbol{\xi}}\right)^{-T} \frac{\partial N_a}{\partial \boldsymbol{\xi}} \tag{14.8a}$$

and

$$\frac{\partial \mathbf{X}}{\partial \boldsymbol{\xi}} = \sum_{a=1}^{n} \mathbf{X}_a \otimes \nabla_\xi N_a \tag{14.8b}$$

or

$$\frac{\partial X_I}{\partial \xi_\alpha} = \sum_{a=1}^{n} X_{a,I} \frac{\partial N_a}{\partial \xi_\alpha}, \quad \alpha = 1,2,3 \quad I = 1,2,3 \tag{14.8c}$$

where $X_{a,I}$ is the I-th coordinate of node a. Following the same steps, we can write

$$\nabla_0 \delta \mathbf{v} = \nabla_0 \left[\sum_{a=1}^{n} N_a \delta \mathbf{v}_a\right] = \sum_{a=1}^{n} \delta \mathbf{v}_a \otimes \nabla_0 N_a \tag{14.9a}$$

or

$$(\nabla_0 \delta \mathbf{v})_{iJ} = \frac{\partial(\delta v_i)}{\partial X_J} = \sum_{a=1}^{n} \delta v_{a,i} \frac{\partial N_a}{\partial X_J} \tag{14.9b}$$

And, recalling Equation (14.8a), we can write

$$\frac{\partial N_a}{\partial X_J} = \left(\frac{\partial \mathbf{X}}{\partial \boldsymbol{\xi}}\right)^{-T}_{kJ} \frac{\partial N_a}{\partial \xi_k} \tag{14.10}$$

Similarly,

$$\nabla_0 \mathbf{u} = \nabla_0 \left[\sum_{b=1}^{n} N_b \mathbf{u}_b\right] = \sum_{b=1}^{n} \mathbf{u}_b \otimes \nabla_0 N_b \tag{14.11a}$$

or

$$(\nabla_0 \mathbf{u})_{iJ} = \frac{\partial u_i}{\partial X_J} = \sum_{b=1}^{n} u_{b,i} \frac{\partial N_b}{\partial X_J} \tag{14.11b}$$

Then, substituting Equations (14.5), (14.6), (14.9a), and (14.11a) into Equation (14.1a) we have

$$D\delta w^{(e)}(\phi_k, N_a \delta \mathbf{v}_a)[N_b \mathbf{u}_b] = \delta \mathbf{v}_a \mathbf{K}_{ab}^{(e)} \mathbf{u}_b \tag{14.12}$$

where $\mathbf{K}_{ab}^{(e)}$ is the tangent stiffness. Thus we can write, using indicial notation,

$$(\nabla_0 \delta \mathbf{v})_{iJ} \hat{A}\,^{(1)}_{iJkL} (\nabla_0 \mathbf{u})_{kL} = \sum_a^n \delta v_{a,i} \left(\frac{\partial N_a}{\partial X_J} \right) \hat{A}\,^{(1)}_{iJkL} \sum_b^n \left(\frac{\partial N_b}{\partial X_L} \right) u_{b,k} \tag{14.13}$$

which yields the expression

$$\delta \mathbf{v}_a \mathbf{K}_{ab}^{(e)} \mathbf{u}_b = \sum_{i,J,k,L=1}^{3} \delta v_{a,i} \left(\int_{v^{(e)}} \frac{\partial N_a}{\partial X_J} \hat{A}\,^{(1)}_{iJkL} \frac{\partial N_b}{\partial X_L} dv \right) u_{b,k} \tag{14.14}$$

which gives us

$$\left[K_{ab}^{(e)} \right]_{ik} = \int_{v^{(e)}} \sum_{J,L=1}^{3} \frac{\partial N_a}{\partial X_J} \hat{A}\,^{(1)}_{iJkL} \frac{\partial N_b}{\partial X_L} dv \tag{14.15}$$

and ultimately the element stiffness expression

$$K_{ik}^{(e)} \equiv \hat{A}\,^{(1)}_{iJkL} \frac{\partial N_a}{\partial X_J} \frac{\partial N_b}{\partial X_L} = \frac{\partial N_a}{\partial X_J} \hat{A}\,^{(1)}_{iJkL} \frac{\partial N_b}{\partial X_L} \tag{14.16}$$

The resulting formulation thus provides a demonstrated appropriate stiffness model which can be incorporated into a finite element code to model/analyze hyperelastic material systems.

Chapter 15
Model Parameters from Test Data

Abstract The establishment of a meaningful hyperelastic model displaying an appropriate strain-energy function is of primary importance for accurate prediction of the response of highly deformable bodies. This can be accomplished with the employment of classic models, a number of which have been presented earlier, or possibly through the formulation of a completely new and different strain-energy function; the Gent model is specifically emphasized in this chapter. Selection of parameters that appropriately characterize the response/behavior of the material to be modeled is obviously extremely important; identifying the methodology for determining the actual values of the parameters is also of great importance. In order to decide whether a specific model accurately characterizes the range of response of a particular real material, it is necessary to compare predictions made with the model to results obtained from laboratory tests on the real material. Different tests are described and their appropriateness for producing data for particular model parameters is discussed. In addition to engineering materials, soft biological tissue is an area of focus in this chapter.

First and foremost in establishing a meaningful hyperelastic model for the prediction of the behavior/response of deformable bodies that exhibit finite elastic strain is the selection of an appropriate strain-energy function. This can be made from established models, a number of which have been presented in Chap. 4, or even possibly by the formulation of an altogether different strain-energy function. This includes the consideration of model parameters that appropriately characterize the response/behavior of the material to be modeled. Secondly, identifying the means of acquiring the actual values of the model parameters is also of extreme importance. Certainly, the development of more advanced and accurate testing apparatuses and data acquisition systems, and thus enhanced laboratory procedures, over recent decades has greatly benefited the effort to acquire good and reliable data.

To decide whether a specific model accurately characterizes the response of a particular real material, one must compare predictions made with the model with results obtained from laboratory tests conducted on the real material. The industry-wide terminology used in this regard speaks to the "validation" of the model.

Hyperelastic constitutive laws are used to model materials that respond elastically when subjected to very large strains. They account for nonlinear material behavior as well as for large shape changes. The main applications of the theory are

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(1) to model rubber materials and the rubbery behavior of polymeric materials; (2) to model polymeric foams that are subjected to large reversible shape changes, e.g., sponges; and (3) to model soft biological tissue, primarily that which constitutes the organs of the human body.

15.1 Rubber

Behavioral characteristics of solid rubber are (a) the material is close to ideally elastic, i.e., when deformed at constant temperature, stress is a function only of current strain and is independent of history, or rate, of loading, and the behavior is reversible; (b) the material strongly resists volume changes and volumetric behavior is essentially linear; (c) the material is very compliant in shear; (d) the material is isotropic; (e) the shear modulus is temperature-dependent, the material becoming stiffer as it is heated; (f) the material gives off heat when stretched. If rubber is subjected to large hydrostatic stress ($>$100 MPa), its volumetric and shear responses are strongly coupled. Compression increases the shear modulus, thus under the condition of large hydrostatic stress, material constants must be determined by testing under combined hydrostatic and shear loading (Bower 2010). In this case, analysis will require the use of one of the foam models since all of the other hyperelastic models decouple volumetric and shear effects. If the pressure is ambient, the following standard tests are usually quite adequate for acquiring the values of the material properties of the rubber that are to be used in the hyperelastic models.

15.1.1 Uniaxial Tension Test

The specimen is the classical uniaxial tension bar mounted onto a tensile testing machine. The strain is measured in the thinner area of the bar, for example by optical scanning (video extensometry); the thicker portions of the bar, which are clamped, must not effect the straining of the bar.

15.1.2 Biaxial Tension Test

The specimen is a disk under equibiaxial tension mounted onto a “scissor” fixture. Data is typically recorded with the use of a video extensometer.

15.1.3 Volumetric Compression Test

A cylindrical elastomeric specimen, having a small slenderness ratio and constrained in a stiff fixture, is compressed in a volumetric test configuration. The actual displacement during compression is very small and therefore great care must be taken to measure only the specimen compliance and not the stiffness of the instrument itself. The initial slope of the resulting stress-strain curve is the bulk modulus. For dense elastomers this value is typically 2–3 orders of magnitude greater than that of the shear modulus.

It is important that tests carried out to obtain different model parameters for a given material have consistent interrelated characteristics, otherwise a physically impossible material model might result when the parameters obtained from the different tests are combined in analysis software. The best way to avoid this potential problem is to cut specimens for simple tension, pure shear, and equal biaxial extension from the same initially undisturbed slab of material. Of course, it is understood that all of the tests must be performed under the same controlled environmental conditions (Venkatesh and Srinivasa Murthy 2012).

It is notable that rubbery materials exhibit a particular damage phenomenon referred to as the *Mullins effect* (Mullins 1969). This effect can be observed when cyclic tension tests are performed on a specimen, with increasing values of deformation at each cycle: the material is deformed up to a given strain value, then unloaded, then reloaded to a higher strain value. When the reload is applied it is possible to detect/observe a stress-softening effect.

15.2 Polymers

In general, the response of a typical polymer is highly dependent on loading rate, strain history, temperature, and humidity. The latter two, when combined, are termed hygrothermal (moisture and heat) conditions. Polymers exhibit different regimens of mechanical behavior. The regimens are classified as *glassy*, *viscoelastic*, *rubbery*, and *melt*, and they are strongly temperature-dependent. The different regimens for a particular polymer can be identified by applying a sinusoidal variation of shear stress to a test specimen of the polymer and measuring the resulting corresponding shear strain amplitude. Typically, the ratio of stress amplitude to strain amplitude defines the corresponding apparent shear modulus, as a function of temperature.

At a critical temperature known as the *glass transition temperature*, T_g, a polymeric material undergoes a dramatic change in mechanical response. Below T_g it behaves like a glass, having a stiff response. Below but near T_g the stress depends strongly on the strain rate. At T_g there is a dramatic drop in the modulus. Above T_g there is a regime in which the polymer shows rubbery behavior: the response is elastic, the stress does not depend strongly on strain rate or history, and the modulus increases. All polymers exhibit these trends, but the extent of each regime, and the detailed behavior within each regime, depend on the molecular structure of the solid. Heavily cross-linked polymers (elastomers) are the most likely to show ideal rubbery behavior. It is this rubbery behavior that hyperelastic constitutive laws are intended to simulate (Bower 2010).

Dynamic mechanical analysis (DMA) is one of the primary techniques employed in the determination of viscoelastic functions. It is based upon the application of transverse oscillations of displacement to a point in the central area of a test piece clamped at both ends. The size of the displacement is variable, and the oscillatory frequencies usually cover an interval of around four decades, between 0.01 and 100 Hz. The frequencies employed are limited by the resonance of the apparatus. The sample should of course be isothermally thermostated. The applied forces are determined by the electric current, which enters the oscillatory system through a defined system of calibration. It should be noted that although the most common way of using the DMA equipment is through flection, small modifications in the design allow measurements to be made in elongation or shear. The selection of the type of measurement depends on the type of material and its characterization. For rubber, a shear experiment is normally done. The range of measurement, in terms of the storage relaxation modulus, varies between 0.0001 and 200 GPa. A Fourier transformation of the recorded modulus data is carried out to transform it from the frequency domain to the time domain (Riande et al. 2000).

15.3 Foams

Polymeric foams (e.g., sponges) are close to reversible and show little rate- or history dependence. They are highly compressible in most cases, with comparable bulk and shear moduli, and have a complicated true stress-true strain response, the response in compression being quite different from that in tension because of buckling in the cell walls. They can be anisotropic, depending on their cell structure, those with a random cell structure being isotropic (Bower 2010).

Vital throughout many industries, solid foam is a lightweight cellular material classified into two types based on pore structure: open and closed cell. Open cell structured foam is soft due to broken pores adjacent to each other forming an open interconnected network. Closed cell foam is dense with intact pores and no interconnectivity. Closed cell foam has a higher compressive strength due to its structure. It is this variance in density that can determine foam type, its intended use, and its successful deployment.

Laboratory tests show a significant dependency of the mechanical behavior of soft foams on temperature and humidity, thus tests are necessarily carried out under constant climatic conditions. Additionally, the material shows a combination of elastic and inelastic behavior (Schrodt et al. 2005). Specimens are typically tested in uniaxial compression. They are cubes having a height significantly less than the lateral dimensions to minimize buckling. A test employing an indenter, a cylinder with a spherical calotte at its end, is also utilized to measure hardness and modulus values. To minimize the Mullins effect, a preprocess procedure consisting of repeated strain-controlled cyclic deformation followed by load discharge can be employed. Soft foams typically show a compressible viscoelastic material behavior (Schrodt et al. 2005).

15.4 Soft Biological Tissue

A great amount of effort has been exerted by researchers working in established laboratories/facilities in the medical field to employ analytical and experimental techniques to reveal the biomechanical mechanisms of, for example, cardiovascular disease, pulmonary/respiratory disorders, ocular disease, and tissue regeneration, as well as to develop methodologies aimed at identifying the behavior of diseased tissue within organs of the body, with a purpose being to distinguish its characteristics from those of normal healthy tissue. As an outgrowth, it has thus been determined that diseased, e.g., cancerous, human tissue is generally stiffer than healthy tissue—a malignant breast tumor can be as much as 2–3 times as stiff as a benign one (Low et al. 2016).

Elastography is an *in vivo* medical imaging nondestructive evaluation technique in which the deformation of soft biological tissue is imaged for the purpose of detecting or classifying tumors (Ophir et al. 2002). Ultrasound-based transient elastography (TE) and shear wave elastography (SWE), and magnetic resonance elastography (MRE), are alternative imaging modalities, each having its own merits and limitations (Low et al. 2016)—the basic idea being that pathological changes to soft tissue often trigger substantial/significant stiffness changes. From measured displacement fields, spatial distributions of material properties can be recovered/calculated and employed in analytical models (Oberai et al. 2009).

At the core of elastography-based techniques is the *inverse problem* of stiffness parameter *reconstruction* for the analytical models. Reconstruction techniques are based on the elasticity constitutive models that are used to model the *forward problem* (see Fig. 15.1). Given that most soft tissues exhibit nonlinear characteristics under the mechanical stimulation of elastography procedures, a hyperelastic model formulation can be employed to characterize the soft tissue (Gokhale et al. 2008).

Of course the determination of the model parameters for the reconstruction from the inverse problem solution is not straightforward and is usually difficult (Beatty (1987), requiring minimization and Newton-type optimization techniques normally couched in a finite element (FE) framework (Oberai et al. 2003, 2004; Abbasi et al. 2016). But, these procedures, applied by researchers in the field, have yielded hyperelastic, or *pseudoelastic*, models that have proven to be of much value in identifying, and distinguishing between, malignant tumors and benign tumors.

An *inverse problem*, in the scientific field, is the process of calculating from a set of observations the factors that produced the observations. It is called an inverse

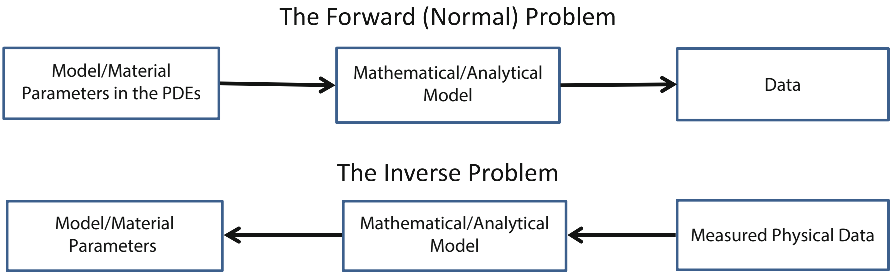

Fig. 15.1 Schematic of the inverse problem

problem because it starts with the results and calculates the causes. The procedure is based on (a) a forward theory: the mathematical process of calculating data based on a mathematical or an analytical model with a given set of model parameters (and perhaps some other appropriate information such as geometry) and (b) the inverse theory: the mathematical process of calculating the numerical values (and associated statistics) of parameters of the defined mathematical or analytical model, based on a set of physical data observations.

Let us now consider a tensile test of a square bar (Dill 2007). The x_1 axis is in the direction of loading. If l is the deformed length and L is the original length, the longitudinal stretch is $\lambda_1 = l/L$ and the extension is $\delta = (l - L)/L = \lambda_1 - 1$. If h is the deformed width and H is the original width, the transverse stretch is $\lambda_2 = \lambda_3 = h/H$. The area of the original cross section is $A = H^2$.

The area of the cross section of the deformed bar is $a = h^2$. The ratio of the areas is therefore $a/A = \lambda_2\lambda_3$. Apart from rigid motion, the deformation is

$$x_1 = \lambda_1 X_1 \tag{15.1a}$$

$$x_2 = \lambda_2 X_2 \tag{15.1b}$$

$$x_3 = \lambda_3 X_3 \tag{15.1c}$$

The non-zero components of the deformation gradient are

$$F_{11} = \lambda_1 \tag{15.2a}$$

$$F_{22} = \lambda_2 \tag{15.2b}$$

$$F_{33} = \lambda_3 \tag{15.2c}$$

The non-zero components of the Cauchy-Green deformation tensor are

$$C_{11} = \lambda_1^2 \tag{15.3a}$$

$$C_{22} = \lambda_2^2 \tag{15.3b}$$

$$C_{33} = \lambda_3^2 \tag{15.3c}$$

If f is the resultant force on a cross-section, the non-zero components of the stress tensors are

$$N_{11} = \frac{f}{A} \tag{15.4a}$$

$$t_{11} = \frac{f}{a} = \frac{N_{11}A}{a} = \frac{N_{11}}{\lambda_2\lambda_3} \tag{15.4b}$$

$$S_{11} = \frac{N_{11}}{\lambda_1} \tag{15.4c}$$

The component N_{11} is the nominal stress or "engineering stress." The component t_{11} is the Cauchy stress or "true stress." The component S_{11} is the second Piola-Kirchhoff stress. Then, we can see that

$$t_{11} = \frac{\lambda_1}{\lambda_2\lambda_3}S_{11} \tag{15.5}$$

which is also found, in a generalized format, from the following expressions,

$$S_{IJ} = N_{Ik}F_{Jk} \tag{15.6a}$$

$$\tau_{ij} = F_{iI}S_{IJ}F_{jJ} \tag{15.6b}$$

$$[F] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \tag{15.7a}$$

$$[F]^T = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \tag{15.7b}$$

and

$$[t] = \frac{1}{J}[\tau] = \frac{1}{J}[F][S][F]^T \tag{15.8}$$

Continuing along these lines, we consider the well-known Gent model for hyperelastic materials, an empirical two-parameter expression for $\widetilde{W}$, suitable for use over the entire range of strains (Gent 1996). For an incompressible material model, the strain-energy function form is

$$\widetilde{W} = -G\frac{J_m}{2}\ln\left(1 - \frac{I_1 - 3}{J_m}\right) \tag{15.9}$$

where J_m is the maximum value of J_1, where

$$J_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3 \tag{15.10}$$

where $\lambda_1, \lambda_2, \lambda_3$ are the principal stretches. The value of J_m is on the order of 10^2 for unfilled rubber vulcanizates (Gent 1996); it corresponds to a maximum extension ratio λ_m of approximately 10 and represents the limit state of the material.

For a network of molecular chains, this would be the fully stretched state. The resulting stress-strain law is

$$t_{ij} = -p + \lambda_i \frac{\partial \widetilde{W}}{\partial \lambda_i} = -p + \frac{GJ_m\lambda_i^2}{J_m - I_1 + 3} \tag{15.11}$$

where p is an undefined hydrostatic pressure, arising from the incompressibility. For the tensile (uniaxial) test $\left(\lambda_2 = \lambda_3 = \lambda_1^{-1/2}, J_1 = \lambda_1^2 + 2\lambda_1^{-1} - 3, t_{22} = t_{33} = 0\right)$,

$$t_{11} = \frac{GJ_m\left(\lambda_1^2 - \lambda_1^{-1}\right)}{J_m - I_1 + 3} = \frac{GJ_m\left(\lambda_1^2 - \lambda_1^{-1}\right)}{J_m - J_1} = \frac{G\left(\lambda_1^2 - \lambda_1^{-1}\right)}{1 - \frac{J_1}{J_m}} \tag{15.12}$$

and

$$N_{11} = \frac{G\left(\lambda_1 - \lambda_1^{-2}\right)}{1 - \frac{J_1}{J_m}} \tag{15.13}$$

This model is extended to compressible materials by introducing the reduced invariant $\bar{I}_1$ (Dill 2007),

$$W = -G\frac{J_m}{2}\ln\left(1 - \frac{\bar{I}_1 - 3}{J_m}\right) + \frac{K}{2}\left(\frac{1}{2}\left(J^2 - 1\right) + \ln J\right) \tag{15.14}$$

where

$$\bar{I}_1 = I_1 I_3^{-1/3} \tag{15.15a}$$

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \tag{15.15b}$$

$$I_3 = \lambda_1^2 \lambda_2^2 \lambda_3^2 \tag{15.15c}$$

Then,

$$\tau_{ij} = \frac{GJ_m\left(3\lambda_1^2 - I_1\lambda_1^{-1}\right)}{3\left(J_m - \bar{I}_1 + 3\right)} + \frac{KJ_m}{2}\left(J^2 - 1\right)\lambda_i^{-2} \tag{15.16}$$

Finally, the design of new experimental procedures and equipment is fertile territory. A knowledge of the theory combined with an understanding of laboratory protocol can lead to the synthesis of new equipment that will yield data that are less subject to experimental error.

Exercises

1. What effect prevents the normal stresses of a material from being proportional to the normal strains?
2. Determine whether or not the displacement field

$$u_1 = x_1 - 2x_2, \qquad u_2 = 3x_1 + 2x_2, \qquad u_3 = 5x_3$$

   is admissible in an elastic material and state why.
3. Prove that $\mathrm{tr}(\mathbf{u} \otimes \mathbf{v}) = \mathbf{u} \cdot \mathbf{v}$
4. There is a relationship between the permutation symbol and the Kronecker delta that is often referred to as the $\hat{\varepsilon} - \delta$ identity. Given that $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$, where $\mathbf{e}_i$ is the basis vector, prove the $\hat{\varepsilon} - \delta$ identity:

$$\hat{\varepsilon}_{ijk}\hat{\varepsilon}_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$$

5. For a uniaxial isotropic elastic material specimen with $\lambda = 0.1$, compare true strain, logarithmic strain, engineering strain and Green-Lagrange strain.
6. What does $\hat{\varepsilon}_{ijk}a_j b_k$ represent?
7. Does a polar decomposition apply to stresses, strains, deformation gradients, or stiffness tensors?
8. Given the deformation gradient

$$[F] = \begin{bmatrix} 0.9 & 0.3 & 0.2 \\ -0.2 & 1.1 & 0.1 \\ -0.3 & 0.1 & 1.2 \end{bmatrix}$$

   Determine $\mathbf{R}$ and $\mathbf{U}$ in $\mathbf{F} = \mathbf{R}\mathbf{U}$.

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9. Given $\mathbf{F}$ in exercise 8, evaluate $\overline{\mathbf{C}}$ and $\overline{\mathbf{b}}$ and their respective invariants. Comment on the results.

10. Derive the expression for the strain-energy function for a uniaxial model of a linearly elastic material having a Young's modulus E.

11. Given the Green-Lagrange strain tensor

$$[E] = \begin{bmatrix} 0.15 & 0.30 & 0.20 \\ 0.30 & 0.25 & 0.10 \\ 0.20 & 0.10 & x \end{bmatrix}$$

What must x be equal to if the material is incompressible?

12. Pure shear in an isotropic elastic material is defined by the relationship $\tau = G\gamma$ where τ is the shear stress, γ is the shear strain, and G is the shear modulus. Write, in terms of the stretch λ, in matrix form, the deformation gradient representing the condition of pure shear.

13. Given the right Cauchy-Green deformation tensor

$$[C] = \begin{bmatrix} 1 & 0 & 0.1 \\ 0 & 1 & 0 \\ 0.1 & 0 & 1.01 \end{bmatrix}$$

Obtain the corresponding second Piola-Kirchhoff stress tensor $\mathbf{S}$ where the material is neo-Hookean with $G = 300$ psi.

14. The strain-energy function for a hyperelastic material is given by

$$W(\mathbf{E}) = \frac{1}{2}\, aE_{II}E_{JJ} + \frac{1}{2}bE_{IJ}E_{JI} + \frac{1}{3}\, cE_{IJ}E_{JK}E_{KI}$$

where a, b, and c are material constants and $\mathbf{E}$ is the Green-Lagrange strain tensor. Find the stress tensor $\mathbf{S}$ as a function of the strain $\mathbf{E}$ implied by the strain-energy function.

15. A material frequently encountered in the literature is defined by a strain-energy function having the form

$$\Psi(\lambda_1, \lambda_2, \lambda_3) = \overline{\Psi}\left(\overline{\lambda}_1, \overline{\lambda}_2, \overline{\lambda}_3\right) + \frac{K}{2}(\ln J)^2$$

where $\overline{\lambda}_\alpha = J^{-1/3}\lambda_\alpha$ and where $\overline{\Psi}$, the distortional component, is given by

$$\overline{\Psi}\left(\overline{\lambda}_1, \overline{\lambda}_2, \overline{\lambda}_3\right) = G\left[\left(\ln\overline{\lambda}_1\right)^2 + \left(\ln\overline{\lambda}_2\right)^2 + \left(\ln\overline{\lambda}_3\right)^2\right]$$

Derive an expression for the Cauchy stress tensor for this material.

16. A modified St. Venant-Kirchhoff constitutive behavior is defined by its corresponding strain-energy function

$$\Psi(\mathbf{E}, J) = \mu \mathrm{tr}\left(\mathbf{E}^2\right) + \frac{\kappa}{2}(\ln J)^2$$

where $\mathbf{E}$ is the Green-Lagrange strain tensor, J is the Jacobian of the deformation gradient and μ and κ are positive material constants.

(a) Obtain an expression for the second Piola-Kirchhoff stress tensor $\mathbf{S}$ as a function of the right Cauchy-Green deformation tensor $\mathbf{C}$.
(b) Obtain an expression for the Kirchhoff stress tensor $\boldsymbol{\tau}$ as a function of the left Cauchy-Green deformation tensor $\mathbf{b}$.
(c) Determine the material elasticity tensor.

17. Given the internal strain-energy expression $W = a\ln(I_1 - 2)$, where a is a constant, determine what the stress-strain equation would be for uniaxial tension. Make a sketch of it for $a = 1.0$ to see how straight (or not) it is. What type of material might this represent?

18. Derive the relationship

$$\dot{\mathbf{E}} = \mathbf{F}^T \mathbf{d} \mathbf{F}$$

19. For the two-dimensional simple shear test, obtain the expressions for the two principal stretches λ_1 and λ_2 in terms of γ. Hint: the eigenvalues of $\mathbf{b}$ are the squared principal stretches λ_α.

20. Show that the principal stresses for a two-dimensional simple shear test are

$$\sigma_1 = -\sigma_2 = 2G \sinh^{-1}\frac{\gamma}{2}$$

21. In a plane stress situation, the right Cauchy-Green deformation tensor $\mathbf{C}$ is given by

$$[C] = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{21} & C_{22} & 0 \\ 0 & 0 & C_{33} \end{bmatrix}; \quad C_{33} = \frac{h^2}{H^2}$$

where H and h are the initial and current thicknesses, respectively. Show that incompressibility implies that

$$C_{33} = \frac{1}{\det\hat{\mathbf{C}}}; \quad \left[\hat{C}\right] = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$$

Using these equations, show that for an incompressible neo-Hookean material the plane stress condition $S_{33} = 0$ enables the pressure to be explicitly evaluated as

$$p = \frac{1}{3}\mu(C_{11} + C_{22} - 2C_{33})$$

and therefore the in-plane second Piola-Kirchhoff stress tensor is given by

$$\hat{\mathbf{S}} = \mu\left(\hat{\mathbf{1}} - C_{33}\hat{\mathbf{C}}^{-1}\right)$$

where the caret hat, in this problem, indicates the 2×2 in-plane components of a tensor.

22. Prove that the Biot stress and the rate of the Biot strain are conjugate in power.
23. Given that

$$c_{ijkl} = J^{-1}F_{iI}F_{jJ}F_{kK}F_{lL}C_{IJKL}$$

show that

$$\dot{\mathbf{E}} : \mathbf{C} : \dot{\mathbf{E}} = J\mathbf{d} : \mathbf{c} : \mathbf{d}$$

for any arbitrary motion.

24. Demonstrate whether or not the Biot stress tensor is objective.
25. Given the orthogonal rotation tensor

$$[O] = \begin{bmatrix} \sin 5\alpha & \cos 6\alpha & -\sin 3\alpha \\ -\sin 3\alpha & \sin 5\alpha & \cos 6\alpha \\ \cos 6\alpha & -\sin 3\alpha & \sin 5\alpha \end{bmatrix}$$

where $\alpha = 6°$. Using the value of the untransformed Cauchy stress tensor in Example 12, determine the transformed second Piola-Kirchhoff stress tensor $\mathbf{S}^*$ and comment on its frame indifference.

26. A scalar function $f(x)$ of a k -dimensional vector variable $x = \{x_1, x_1, \ldots, x_k\}^T$ is said to be homogeneous of order n if for any arbitrary constant α, $f(\alpha x) = \alpha^n f(x)$. The general type of incompressible hyperelastic material model developed by Ogden is defined by the following strain-energy function

$$\Psi = \sum_{p=1}^{N} \frac{\mu_p}{\alpha_p}\left(\lambda_1^{\alpha_p} + \lambda_2^{\alpha_p} + \lambda_3^{\alpha_p} - 3\right)$$

Derive the expression for the homogeneous counterpart of this function, $\hat{\Psi}$. Hint: $\overline{\mathbf{C}} = (\det\mathbf{C})^{-1/3}\mathbf{C}$, because $n = 0$ (incompressible).

27. Demonstrate why the engineering, or small, strain tensor $\boldsymbol{\varepsilon}$ is not a valid measure of strain when the rigid body rotation is large, but the Green-Lagrange strain tensor $\mathbf{E}$ is a valid measure regardless of the magnitude of rotation, where

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}$$

28. Derive the relationship

$$\left(\frac{\partial W(\mathbf{F})}{\partial \mathbf{F}}\right)^T = 2\mathbf{F}^T \frac{\partial W(\mathbf{b})}{\partial \mathbf{b}}$$

29. For a thin sheet of incompressible hyperelastic material which is in a state of plane stress, i.e., $\sigma_{23} = \sigma_{31} = \sigma_{33} = 0$, show that

$$\sigma_1 = 2\left(\lambda_1^2 - \lambda_1^{-2}\lambda_2^{-2}\right)\left(\frac{\partial \breve{W}}{\partial I_1} + \lambda_2^2 \frac{\partial \breve{W}}{\partial I_2}\right)$$

$$\sigma_2 = 2\left(\lambda_2^2 - \lambda_1^{-2}\lambda_2^{-2}\right)\left(\frac{\partial \breve{W}}{\partial I_1} + \lambda_1^2 \frac{\partial \breve{W}}{\partial I_2}\right)$$

where σ_1 and σ_2 are the principal stresses, $I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_1^{-2}\lambda_2^{-2}$ and $I_2 = \lambda_1^2\lambda_2^2 + \lambda_1^{-2} + \lambda_2^{-2}$. What is the value of I_3?

30. Consider the strain-energy function $\overline{W}\left[\overline{I}_1\left(\overline{\mathbf{C}}\right), \overline{I}_2\left(\overline{\mathbf{C}}\right)\right]$ in terms of the reduced invariants $\overline{I}_1 = J^{-2/3} I_1$ and $\overline{I}_2 = J^{-4/3} I_2$. Show that the derivatives of $\overline{I}_1$ and $\overline{I}_2$ with respect to the tensor $\overline{\mathbf{C}}$ are

$$\frac{\partial \overline{I}_1}{\partial \overline{\mathbf{C}}} = \mathbf{1}$$

and

$$\frac{\partial \overline{I}_2}{\partial \overline{\mathbf{C}}} = \overline{I}_1 \mathbf{1} - \overline{\mathbf{C}}$$

31. Consider the incompressible hyperelastic material $(\lambda_1\lambda_2\lambda_3 = 1)$ characterized by a strain-energy function in terms of principal stretches according to

$$\Psi = \frac{G}{\alpha}\left(\lambda_1^\alpha + \lambda_2^\alpha + \lambda_3^\alpha - 3\right)$$

Show that for $\alpha = 2$ we can obtain a version of the neo-Hookean model.

32. Derive the Mooney-Rivlin model from the Ogden model, Equation (4.16).

33. Derive the expression for the Cauchy stress tensor, Equation (5.43), from the Arruda-Boyce model, Equation (4.19).

34. Derive the expression for the Cauchy stress tensor, Equation (5.44), from the Ogden-Storakers foam model strain-energy function, Equation (4.26).

35. Determine the relationship between the constants C_{10} and C_{01} in Equation (5.37) in terms of the shear modulus G. Hint: think Ogden model.

36. Determine the numerical values of the components of the stiffness tensor in Example 6, with $\lambda_\Phi = 1$ and $\lambda_R = \lambda_Z \simeq 1$.

37. Determine the expression for the second Piola-Kirchhoff stress tensor, and for the Cauchy stress tensor, for a compressible, soft, biological, transversely isotropic tissue—for example, for an artery.

Appendix A: Tensor Derivatives

It is extremely helpful to have available a number of tensorial relationships when undertaking to understand, or to formulate, pertinent expressions in the study of finite elasticity. In this monograph, we adhere to the convention, for deformation gradients, that

$$\mathbf{F}^{-1} \equiv F_{Ji} \tag{A.1a}$$

$$\mathbf{F}^{T} \equiv F_{iJ}^{T} \tag{A.1b}$$

$$\mathbf{F}^{-T} \equiv F_{Ji}^{T} \tag{A.1c}$$

We use different indices for the two coordinate systems because the corresponding bases, the Lagrangian (material or reference) basis (e_I) and the Eulerian (displaced or current) basis (e_i), are, in principle, independent of each other. It is usually unnecessary to write the superscript (^{T}) to denote a transpose in the case of the product of two deformation gradients. We know that we write the expression for the *right Cauchy-Green deformation tensor*,

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} \Rightarrow C_{IJ} = F_{jI}F_{jJ} \tag{A.2}$$

and for the *left Cauchy-Green deformation tensor*,

$$\mathbf{b} = \mathbf{F}\mathbf{F}^T \Rightarrow b_{ij} = F_{iI}F_{jI} \tag{A.3}$$

It is often useful to have beforehand derivative relationships that greatly benefit the operation on second-order tensors of the same basis, such as C_{IJ} and b_{ij} (as opposed to the dual basis second-order tensor F_{iJ}). Some of these derivative relationships, using A_{ij} as the representative single-basis second-order tensor and taking the derivative with respect to a single-basis second-order tensor, are

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$$\frac{\partial A_{ij}}{\partial A_{kl}} = \delta_{ik}\delta_{lj} \tag{A.4a}$$

$$\frac{\partial A_{ij}}{\partial A^T_{kl}} = \frac{\partial A_{ij}}{\partial A_{lk}} = \delta_{il}\delta_{kj} \tag{A.4b}$$

$$\frac{\partial A^T_{ij}}{\partial A_{kl}} = \frac{\partial A_{ji}}{\partial A_{kl}} = \delta_{jk}\delta_{li} \tag{A.4c}$$

or if the second-order tensor is *symmetric* this is replaced by the symmetrized form

$$\frac{\partial A_{ij}}{\partial A_{kl}} = \frac{1}{2}\left(\delta_{ik}\delta_{lj} + \delta_{il}\delta_{kj}\right) \tag{A.4d}$$

where δ_{ij} is the Kronecker delta, and

$$\frac{\partial}{\partial \underline{\underline{A}}}\left(\underline{\underline{A}}^{-1}\right) \Rightarrow \frac{\partial A^{-1}_{ij}}{\partial A_{kl}} = -A^{-1}_{ik}A^{-1}_{lj} \tag{A.5a}$$

$$\frac{\partial}{\partial \underline{\underline{A}}}\left(\underline{\underline{A}}^{-T}\right) \Rightarrow \frac{\partial A^{-1}_{ji}}{\partial A_{kl}} = -A^{-1}_{li}A^{-1}_{jk} \tag{A.5b}$$

$$\frac{\partial}{\partial \underline{\underline{A}}^T}\left(\underline{\underline{A}}^{-1}\right) \Rightarrow \frac{\partial A^{-1}_{ij}}{\partial A_{lk}} = -A^{-1}_{kj}A^{-1}_{il} \tag{A.5c}$$

$$\frac{\partial}{\partial \underline{\underline{A}}^T}\left(\underline{\underline{A}}^{-T}\right) \Rightarrow \frac{\partial A^{-1}_{ji}}{\partial A_{lk}} = -A^{-1}_{jl}A^{-1}_{ki} \tag{A5.d}$$

$$\frac{\partial}{\partial \underline{\underline{A}}}\det\underline{\underline{A}} = \left(\det\underline{\underline{A}}\right)\underline{\underline{A}}^{-T} \Rightarrow \frac{\partial}{\partial A_{ij}}\det A_{ij} = \left(\det A_{ij}\right)A^{-1}_{ji} \tag{A5.e}$$

or if the second-order tensor is symmetric we have

$$\frac{\partial A^{-1}_{ij}}{\partial A_{kl}} = -\frac{1}{2}\left(A^{-1}_{ik}A^{-1}_{jl} + A^{-1}_{il}A^{-1}_{jk}\right) \tag{A5.f}$$

Also, since

$$A^{-1}_{ji} = \frac{1}{2}\,\hat{\varepsilon}_{ibc}\hat{\varepsilon}_{jdf}A_{bd}A_{cf} \tag{A.6}$$

we can write

$$\frac{\partial A^{-1}_{ji}}{\partial A_{lk}} = \hat{\varepsilon}_{ilc}\hat{\varepsilon}_{jkf}A_{cf} \tag{A.7}$$

where $\hat{\varepsilon}_{ijk}$ is the permutation symbol, or alternator. Also, applying the product rule, we have

$$\frac{\partial\left(A_{ij}A_{ij}\right)}{\partial A_{kl}} = \delta_{ik}\delta_{lj}A_{ij} + A_{ij}\delta_{ik}\delta_{lj} = 2A_{kl} \tag{A.8}$$

Now, considering derivatives with respect to the deformation gradient, we can write

$$\frac{\partial F_{iJ}}{\partial F_{kL}} = \delta_{ik}\delta_{LJ} \tag{A.9}$$

The derivative of a second-order tensor, e.g., C_{IJ} or b_{ij}, with respect to the deformation gradient F_{iJ}, which is a two-point tensor, is, not unexpectedly, not as straightforward as in the above cases involving only single-basis second-order tensors. The existence of dual bases, Eulerian and Lagrangian, provides some more-complicated features. In these cases we have, for example,

$$\frac{\partial C_{IJ}}{\partial F_{bB}} = \delta_{BI}F_{bJ} + \delta_{BJ}F_{bI} \tag{A.10a}$$

and

$$\frac{\partial b_{ij}}{\partial F_{bB}} = \delta_{ib}F_{jB} + \delta_{jb}F_{iB} \tag{A.10b}$$

We can also note the relationships

$$F_{iJ}F_{Jk} = \delta_{ik} \tag{A.11a}$$

$$F^{T}_{jI}F^{T}_{Kj} = \delta_{IK} \tag{A.11b}$$

$$F_{iJ}F_{Ji} = \delta_{ii} = 3 \tag{A.11c}$$

and that

$$\frac{\partial J}{\partial F_{iJ}} = JF^{T}_{Ji} = \frac{1}{2}\hat{\varepsilon}_{ibc}\hat{\varepsilon}_{JDF}F_{bD}F_{cF} \tag{A.12}$$

and

$$\frac{\partial(\hat{\varepsilon}_{ibc}\hat{\varepsilon}_{JDF}F_{bD}F_{cF})}{\partial F_{kL}} = 2\hat{\varepsilon}_{ikc}\hat{\varepsilon}_{JLF}F_{cF} \tag{A.13}$$

The standard *principal invariants* of a second-order tensor A_{ij} are given by

$$I_1 = \mathrm{tr}\left(A_{ij}\right) = A_{ii} \tag{A.14a}$$

$$I_2 = \frac{1}{2}\left[I_1^2 - \mathrm{tr}\left(A_{ij}\right)^2\right] = \frac{1}{2}\left(I_1^2 - A_{ij}A_{ji}\right) \tag{A.14b}$$

and

$$I_3 = \det A_{ij} = \hat{\varepsilon}_{ijk}A_{i1}A_{j2}A_{k3} \tag{A.14c}$$

Derivatives of the principal invariants of a second-order tensor with respect to the tensor itself are given by

$$\frac{\partial I_1}{\partial A_{ij}} = \delta_{ij} \tag{A.15a}$$

$$\frac{\partial I_2}{\partial A_{ij}} = A_{kk}\delta_{ij} - A_{ji} \tag{A.15b}$$

$$\frac{\partial I_3}{\partial A_{ij}} = I_3 A_{ji}^{-1} \tag{A.15c}$$

Also, since it is often necessary to find the derivative of the principal invariants of the two Cauchy-Green deformation tensors with respect to the deformation gradient F_{iJ}, for useful reference we note that,

$$\frac{\partial I_1}{\partial F_{iJ}} = 2F_{iJ} \tag{A.16a}$$

$$\frac{\partial I_2}{\partial F_{iJ}} = 2(I_1F_{iJ} - F_{iI}F_{kI}F_{kJ}) \tag{A.16b}$$

$$\frac{\partial I_3}{\partial F_{iJ}} = 2I_3F_{Ji}^T \tag{A.16c}$$

Now, considering the *reduced principal invariants*,

$$\bar{I}_1 = \frac{I_1}{J^{2/3}} \tag{A.17a}$$

$$\bar{I}_2 = \frac{I_2}{J^{4/3}} \tag{A.17b}$$

$$J = \sqrt{\det A_{ij}} \tag{A.17c}$$

their derivatives with respect to the second-order tensor A_{ij} are,

$$\frac{\partial \bar{I}_1}{\partial A_{ij}} = I_3^{-1/3}\left(\delta_{ij} - \frac{I_1}{3} A_{ji}^{-1}\right) \tag{A.18a}$$

$$\frac{\partial \bar{I}_2}{\partial A_{ij}} = I_3^{-2/3}\left(A_{kk}\delta_{ij} - A_{ji} - \frac{2I_2}{3}\ A_{ji}^{-1}\right) \tag{A.18b}$$

and

$$\frac{\partial J}{\partial A_{ij}} = \frac{1}{2} I_3^{1/2} A_{ji}^{-1} \tag{A.18c}$$

and their derivatives with respect to the deformation gradient F_{iJ} are,

$$\frac{\partial \bar{I}_1}{\partial F_{iJ}} = \frac{2}{J^{2/3}} F_{iJ} - \frac{2}{3} \bar{I}_1 F_{Ji}^T \tag{A.19a}$$

$$\frac{\partial \bar{I}_2}{\partial F_{iJ}} = \frac{2}{J^{2/3}} \bar{I}_1 F_{iJ} - \frac{2}{J^{4/3}} F_{iI} F_{kI} F_{kJ} - \frac{4\bar{I}_2}{3} F_{Ji}^T \tag{A.19b}$$

and

$$\frac{\partial J}{\partial \mathbf{F}} = \frac{\partial J}{\partial \mathbf{C}} \frac{\partial \mathbf{C}}{\partial \mathbf{F}} = 2\mathbf{F} \frac{\partial J}{\partial \mathbf{C}} = 2\mathbf{F}\left(\frac{J}{2} \mathbf{F}^{-1} \mathbf{F}^{-T}\right) = J\mathbf{F}^{-T} \ \text{ or } \ \frac{\partial J}{\partial F_{iJ}} = J F_{Ji}^T \tag{A.19c}$$

This relationship is derived in Appendix C.

Other valuable tensor relationships frequently utilized in finite deformation analyses include

$$A : \underline{\underline{B}} \ \equiv \ \operatorname{tr}\left(\underline{\underline{A}}^T \underline{\underline{B}}\right) \ \equiv \ \operatorname{tr}\left(\underline{\underline{A}}\,\underline{\underline{B}}^T\right) \tag{A.20}$$

Voigt-Mandel Transformation:

The Voigt rule depends on whether a tensor is a *kinetic* quantity, such as stress, or a *kinematic* quantity, such as strain. The Voigt rule for kinetic tensors, such as the symmetric tensor $\underline{\underline{\sigma}}$, is

$$[\sigma] \equiv \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} \sigma_1 & \sigma_6 & \sigma_5 \\ \sigma_6 & \sigma_2 & \sigma_4 \\ \sigma_5 & \sigma_4 & \sigma_3 \end{bmatrix} \to \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} \equiv \{\sigma\} \tag{A.21}$$

The Voigt rule for kinematic tensors, such as the strain tensor $\underline{\underline{\varepsilon}}$, is

$$[\varepsilon] \equiv \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ & \varepsilon_{22} & \varepsilon_{23} \\ sym & & \varepsilon_{33} \end{bmatrix} \to \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{13} \\ \varepsilon_{12} \end{Bmatrix} \equiv \{\varepsilon\} \tag{A.22}$$

In the case of higher-order tensors, e.g., C_{ijkl}, we can write the Voigt matrix form of the elastic constitutive matrix in *plane strain* as

$$[C] = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1112} \\ C_{2211} & C_{2222} & C_{2212} \\ C_{1211} & C_{1222} & C_{1212} \end{bmatrix} \tag{A.23}$$

We can use the following simple MATLAB® program (subroutine) to convert from fourth-order tensor format to Voigt-Mandel format:

```
kk= [1,6,5;6,2,4;5,4,3];
   VM(6,6) =0;
   for i=1:3;
     for j=1:3;
        m=kk(i,j);
        for k=1:3;
           for l=1:3;
             n=kk(k,l);
    VM(m,n) =C(i,j,k,l);
       end
     end
    end
  end
```

where C(i,j,k,l) is the fourth-order tensor and VM(m,n) is the 6×6 Voigt-Mandel matrix.

Additional useful derivative relationships involving finite deformation tensors are

$$\frac{\partial \bar{I}_1}{\partial C_{IJ}} = I_3^{-1/3}\frac{\partial I_1}{\partial C_{IJ}} - \frac{1}{3}I_1 I_3^{-4/3}\frac{\partial I_3}{\partial C_{IJ}} = I_3^{-1/3}\left(\delta_{IJ} - \frac{1}{3}I_1 C_{JI}^{-1}\right) \tag{A.24a}$$

$$\frac{\partial \bar{I}_2}{\partial C_{IJ}} = I_3^{-2/3}\frac{\partial I_2}{\partial C_{IJ}} - \frac{2}{3}I_2 I_3^{-5/3}\frac{\partial I_3}{\partial C_{IJ}} = I_3^{-2/3}\left(C_{KK}\delta_{IJ} - C_{JI} - \frac{2}{3}I_2 C_{JI}^{-1}\right) \tag{A.24b}$$

and

$$\frac{\partial J}{\partial \mathbf{C}} = \frac{1}{2}\ J\mathbf{F}^{-1}\mathbf{F}^{-T} = \frac{1}{2}\ J\mathbf{C}^{-T} \ \text{ or } \ \frac{\partial J}{\partial C_{IJ}} = \frac{1}{2}\ JC_{JI}^{-1} \tag{A.24c}$$

which is derived in Appendix C.

Appendix B: Second Elasticity Tensor Derivation

In this Appendix, we present a derivation of the second elasticity tensor for a neo-Hookean material. This derivation is independent of the Sussman and Bathe (1987) formulation presented in Chaps. 5 and 6, and some steps in it differ from theirs. It was done somewhat as a means of verifying their formulation, as well as to demonstrate a slightly different approach.

Given the following strain-energy function,

$$W(F_{iJ}) = \tilde{W}(\lambda_1, \lambda_2, \lambda_3) = \breve{W}(I_1, I_2, I_3) = \widehat{W}\left(\bar{I}_1, \bar{I}_2, J\right) \tag{B.1}$$

where

$$\widehat{W} = C_{10}\left(\bar{I}_1 - 3\right) + C_{01}\left(\bar{I}_2 - 3\right) + D_1(J-1)^2 \tag{B.2}$$

where $C_{10},\ C_{01}$, and D_1 are material constants.

Our formulation is based on the right Cauchy-Green deformation tensor

$$C_{IJ} = F_{jI}F_{jJ} \tag{B.3}$$

from which we obtain the right Cauchy-Green distortion tensor

$$\overline{C}_{IJ} = J^{-2/3}C_{IJ} \tag{B.4}$$

$$\frac{\partial C_{IJ}}{\partial C_{KL}} = \delta_{IK}\delta_{LJ} \tag{B.5a}$$

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$$\begin{aligned}\frac{\partial \overline{C}_{IJ}}{\partial C_{KL}} &= \frac{\partial J^{-2/3}}{\partial J}\frac{\partial J}{\partial C_{KL}}C_{IJ} + J^{-2/3}\frac{\partial C_{IJ}}{\partial C_{KL}} \\ &= -\frac{2}{3}J^{-5/3}\frac{1}{2}JC_{LK}^{-1}C_{IJ} + J^{-2/3}\delta_{IK}\delta_{LJ} \\ &= -J^{-2/3}\left(\frac{1}{3}C_{LK}^{-1}C_{IJ} - \delta_{IK}\delta_{LJ}\right)\end{aligned} \tag{B.5b}$$

$$\frac{\partial C_{IJ}}{\partial C_{LK}} = \delta_{IL}\delta_{KJ} \tag{B.6a}$$

$$\frac{\partial \overline{C}_{IJ}}{\partial C_{LK}} = \frac{\partial J^{-2/3}}{\partial J}\frac{\partial J}{\partial C_{LK}}C_{IJ} + J^{-2/3}\frac{\partial C_{IJ}}{\partial C_{LK}} \tag{B.6b}$$

$$\frac{\partial}{\partial C_{IJ}}(C_{IJ}C_{JI}) = \frac{\partial C_{IJ}}{\partial C_{IJ}}C_{JI} + C_{IJ}\frac{\partial C_{JI}}{\partial C_{IJ}} = C_{JI} + C_{IJ}\delta_{JI}\delta_{JI} = 2C_{JI} \tag{B.7}$$

$$\frac{\partial}{\partial \underline{\underline{C}}}\left(\underline{\underline{C}}^{-1}\right) \Rightarrow \frac{\partial C_{IJ}^{-1}}{\partial C_{KL}} = -C_{IK}^{-1}C_{LJ}^{-1} \tag{B.8}$$

$$\frac{\partial}{\partial \underline{\underline{C}}}\left(\underline{\underline{C}}^{-T}\right) \Rightarrow \frac{\partial C_{JI}^{-1}}{\partial C_{KL}} = -C_{LI}^{-1}C_{JK}^{-1} \tag{B.9}$$

$$\frac{\partial}{\partial \underline{\underline{C}}^{T}}\left(\underline{\underline{C}}^{-1}\right) \Rightarrow \frac{\partial C_{IJ}^{-1}}{\partial C_{LK}} = -C_{KJ}^{-1}C_{IL}^{-1} \tag{B.10}$$

$$\frac{\partial}{\partial \underline{\underline{C}}^{T}}\left(\underline{\underline{C}}^{-T}\right) \Rightarrow \frac{\partial C_{JI}^{-1}}{\partial C_{LK}} = -C_{JL}^{-1}C_{KI}^{-1} \tag{B.11}$$

$$\frac{\partial}{\partial \underline{\underline{C}}}\det \underline{\underline{C}} = \left(\det \underline{\underline{C}}\right)\underline{\underline{C}}^{-T} \Rightarrow \frac{\partial}{\partial C_{IJ}}\det C_{IJ} = (\det C_{IJ})C_{JI}^{-1} \tag{B.12}$$

$$I_1 = trC_{IJ} = C_{II} \tag{B.13a}$$

$$\frac{\partial I_1}{\partial F_{iJ}} = 2F_{iJ}^{T} \tag{B.13b}$$

$$I_2 = \frac{1}{2}\left[(trC_{IJ})^2 - tr(C_{IJ})^2\right] = \frac{1}{2}\left[(C_{II})^2 - C_{IJ}C_{JI}\right] \tag{B.14a}$$

$$\frac{\partial I_2}{\partial F_{iJ}} = 2(I_1\delta_{IJ} - F_{iI}F_{iJ})F_{iJ}^{T} \tag{B.14b}$$

$$I_3 = \det C_{IJ} = \hat{\varepsilon}_{IJK} C_{I1} C_{J2} C_{K3} = J^2 \tag{B.15a}$$

$$\frac{\partial I_3}{\partial F_{iJ}} = 2 I_3 F^T_{Ji} \tag{B.15b}$$

$$\frac{\partial I_1}{\partial C_{IJ}} = \delta_{IJ} \tag{B.16a}$$

$$\frac{\partial I_2}{\partial C_{IJ}} = C_{KK}\delta_{IJ} - C_{JI} \tag{B.16b}$$

$$\frac{\partial I_3}{\partial C_{IJ}} = I_3 C^{-1}_{JI} \tag{B.16c}$$

Also,

$$C^{-1}_{JI} = \frac{1}{2I_3} \hat{\varepsilon}_{IBC} \hat{\varepsilon}_{JDF} C_{BD} C_{CF} \tag{B.17}$$

$$\bar{I}_1 = J^{-2/3} C_{kk} \tag{B.18a}$$

$$\bar{I}_2 = \frac{1}{2}\left(\bar{I}_1^2 - J^{-4/3} C_{IJ} C_{IJ}\right) \tag{B.18b}$$

$$J = \sqrt{\det C_{IJ}} \tag{B.18c}$$

$$\frac{\partial \bar{I}_1}{\partial C_{IJ}} = I_3^{-1/3} \frac{\partial I_1}{\partial C_{IJ}} - \frac{1}{3} I_1 I_3^{-4/3} \frac{\partial I_3}{\partial C_{IJ}} = I_3^{-1/3}\left(\delta_{IJ} - \frac{1}{3} I_1 C^{-1}_{JI}\right) \tag{B.19}$$

$$\frac{\partial \bar{I}_2}{\partial C_{IJ}} = I_3^{-2/3} \frac{\partial I_2}{\partial C_{IJ}} - \frac{2}{3} I_2 I_3^{-5/3} \frac{\partial I_3}{\partial C_{IJ}} = I_3^{-2/3}\left(C_{KK}\delta_{IJ} - C_{JI} - \frac{2}{3} I_2 C^{-1}_{JI}\right) \tag{B.20}$$

$$\frac{\partial J}{\partial C_{IJ}} = \frac{1}{2} I_3^{-1/2} \frac{\partial I_3}{\partial C_{IJ}} = \frac{1}{2} I_3^{1/2} C^{-1}_{JI} = \frac{1}{2} J C^{-1}_{JI} \tag{B.21}$$

This relationship is derived in Appendix C.

Referring to Sussman and Bathe (1987):

$$S_{IJ} = \frac{1}{2}\left(\frac{\partial \overline{W}}{\partial E_{IJ}} + \frac{\partial \overline{W}}{\partial E_{JI}}\right) \tag{B.22}$$

$$D_{IJKL} = \frac{1}{2}\left(\frac{\partial S_{IJ}}{\partial E_{KL}} + \frac{\partial S_{IJ}}{\partial E_{LK}}\right) \tag{B.23}$$

The use of the operator

$$(\,)^{*}_{IJ} = \frac{1}{2}\left(\frac{\partial}{\partial E_{IJ}} + \frac{\partial}{\partial E_{JI}}\right) \tag{B.24}$$

instead of $\partial/\partial E_{IJ}$ is consistent with the fact that E_{IJ} is a symmetric tensor and ensures that S_{IJ} and D_{IJKL} have the correct symmetries.

$$E_{IJ} = \frac{1}{2}(C_{IJ} - \delta_{IJ}) \tag{B.25a}$$

yielding

$$C_{IJ} = 2E_{IJ} + \delta_{IJ} \tag{B.25b}$$

and

$$\frac{\partial C_{IJ}}{\partial E_{IJ}} = 2 \tag{B.25c}$$

$$(\,)^{*}_{IJ} = \frac{1}{2}\left(\frac{\partial}{\partial C_{IJ}}\frac{\partial C_{IJ}}{\partial E_{IJ}} + \frac{\partial}{\partial C_{JI}}\frac{\partial C_{JI}}{\partial E_{JI}}\right) = \left(\frac{\partial}{\partial C_{IJ}} + \frac{\partial}{\partial C_{JI}}\right) \tag{B.26}$$

Then,

$$(I_1)^{*}_{IJ} = 2\delta_{IJ} \tag{B.27a}$$

$$(I_2)^{*}_{IJ} = 2I_1\delta_{IJ} - (C_{IJ} + C_{JI}) \tag{B.27b}$$

$$(I_3)^{*}_{IJ} = \left(C^{-1}_{JI} + C^{-1}_{IJ}\right)I_3 \tag{B.27c}$$

and, also,

$$(I_3)^{*}_{IJ} = \frac{1}{2}(\hat{\varepsilon}_{IBC}\hat{\varepsilon}_{JDF} + \hat{\varepsilon}_{JBC}\hat{\varepsilon}_{IDF})C_{BD}C_{CF} \tag{B.28}$$

Moving on to the operations on the reduced invariants,

$$\left(\bar{I}_1\right)^*_{IJ} = I_3^{-1/3}(I_1)^*_{IJ} - \frac{1}{3}I_1 I_3^{-4/3}(I_3)^*_{IJ} \tag{B.29}$$

$$\left(\bar{I}_2\right)^*_{IJ} = I_3^{-2/3}(I_2)^*_{IJ} - \frac{2}{3}I_2 I_3^{-5/3}(I_3)^*_{IJ} \tag{B.30}$$

$$(J)^*_{IJ} = \frac{1}{2}I_3^{-1/2}(I_3)^*_{IJ} \tag{B.31}$$

B.1 Second Piola-Kirchhoff Stress Tensor

$$S_{IJ} = C_{10}\left(\bar{I}_1\right)^*_{IJ} + C_{01}\left(\bar{I}_2\right)^*_{IJ} + K(J-1)(J)^*_{IJ} \tag{B.32}$$

$$(I_1)^{**}_{IJKL} = \left(\frac{\partial}{\partial C_{KL}} + \frac{\partial}{\partial C_{LK}}\right)(I_1)^*_{IJ} = 2\left(\frac{\partial}{\partial C_{KL}} + \frac{\partial}{\partial C_{LK}}\right)\delta_{JK} = 0 \tag{B.33}$$

$(I_2)^{**}_{IJKL}$ is not used here since $C_{01} = 0$

$$\begin{aligned}
(I_3)^{**}_{IJKL} &= \left(\frac{\partial}{\partial C_{KL}} + \frac{\partial}{\partial C_{LK}}\right)(I_3)^*_{IJ} = \left(\frac{\partial}{\partial C_{KL}} + \frac{\partial}{\partial C_{LK}}\right)\left(C^{-1}_{IJ} + C^{-1}_{JI}\right)I_3 \\
&= \frac{\partial\left(C^{-1}_{IJ}I_3\right)}{\partial C_{KL}} + \frac{\partial\left(C^{-1}_{IJ}I_3\right)}{\partial C_{LK}} + \frac{\partial\left(C^{-1}_{JI}I_3\right)}{\partial C_{KL}} + \frac{\partial\left(C^{-1}_{JI}I_3\right)}{\partial C_{LK}} \\
&= \left[\left(C^{-1}_{IJ} + C^{-1}_{JI}\right)\left(C^{-1}_{KL} + C^{-1}_{LK}\right) - \left(C^{-1}_{IK}C^{-1}_{LJ} + C^{-1}_{KJ}C^{-1}_{IL} + C^{-1}_{LI}C^{-1}_{JK} + C^{-1}_{JL}C^{-1}_{KI}\right)\right]I_3
\end{aligned} \tag{B.34}$$

$$\begin{aligned}
\left(\bar{I}_1\right)^{**}_{IJKL} &= \left(\frac{\partial}{\partial C_{KL}} + \frac{\partial}{\partial C_{LK}}\right)\left(\bar{I}_1\right)^*_{IJ} \\
&= \left(\frac{\partial}{\partial C_{KL}} + \frac{\partial}{\partial C_{LK}}\right)\left[I_3^{-1/3}(I_1)^*_{IJ} - \frac{1}{3}I_1 I_3^{-4/3}(I_3)^*_{IJ}\right]
\end{aligned} \tag{B.35}$$

$$\begin{aligned}
\left(\bar{I}_2\right)^{**}_{IJKL} &= \left(\frac{\partial}{\partial C_{KL}} + \frac{\partial}{\partial C_{LK}}\right)\left(\bar{I}_2\right)^*_{IJ} \\
&= \left(\frac{\partial}{\partial C_{KL}} + \frac{\partial}{\partial C_{LK}}\right)\left[I_3^{-2/3}(I_2)^*_{IJ} - \frac{2}{3}I_2 I_3^{-5/3}(I_3)^*_{IJ}\right]
\end{aligned} \tag{B.36}$$

$$(J)^{**}_{IJKL} = \left(\frac{\partial}{\partial C_{KL}} + \frac{\partial}{\partial C_{LK}}\right)(J)^*_{IJ} = \left(\frac{\partial}{\partial C_{KL}} + \frac{\partial}{\partial C_{LK}}\right)\left[\frac{1}{2}I_3^{-1/2}(I_3)^*_{IJ}\right] \tag{B.37}$$

B.2 Second Elasticity Tensor

$$D_{IJKL} = C_{10}\left(\bar{I}_1\right)^{**}_{IJKL} + C_{01}\left(\bar{I}_2\right)^{**}_{IJKL} + K\left[(J)^{*}_{IJ}(J)^{*}_{KL} + (J-1)(J)^{**}_{IJKL}\right] \tag{B.38}$$

$$\begin{aligned}
\left(\bar{I}_1\right)^{**}_{IJKL} &= \left(\frac{\partial}{\partial C_{KL}} + \frac{\partial}{\partial C_{LK}}\right)\left[I_3^{-1/3}(I_1)^{*}_{IJ}\right] - \frac{1}{3}\left(\frac{\partial}{\partial C_{KL}} + \frac{\partial}{\partial C_{LK}}\right)\left[I_1 I_3^{-4/3}(I_3)^{*}_{IJ}\right] \\
&= \left(\frac{\partial}{\partial C_{KL}} + \frac{\partial}{\partial C_{LK}}\right)\left(I_3^{-1/3}\right)(I_1)^{*}_{IJ} + I_3^{-1/3}(I_1)^{**}_{IJKL} \\
&\quad - \frac{1}{3}\left[\left(\frac{\partial}{\partial C_{KL}} + \frac{\partial}{\partial C_{LK}}\right)(I_1) I_3^{-4/3}(I_3)^{*}_{IJ} + I_1\left(\frac{\partial}{\partial C_{KL}} + \frac{\partial}{\partial C_{LK}}\right)\right. \\
&\quad \left.\times\left(I_3^{-4/3}\right)(I_3)^{*}_{IJ} + I_1 I_3^{-4/3}(I_3)^{**}_{IJKL}\right]
\end{aligned} \tag{B.39}$$

$$(I_1)^{**}_{IJKL} = 0 \tag{B.40}$$

$$\begin{aligned}
\left(\bar{I}_1\right)^{**}_{IJKL} &= \left(\frac{\partial I_3^{-1/3}}{\partial C_{KL}} + \frac{\partial I_3^{-1/3}}{\partial C_{LK}}\right)(I_1)^{*}_{IJ} - \frac{1}{3}\left[(2\delta_{KL}) I_3^{-4/3}(I_3)^{*}_{IJ}\right. \\
&\quad \left. + I_1\left(\frac{\partial I_3^{-4/3}}{\partial C_{KL}} + \frac{\partial I_3^{-4/3}}{\partial C_{LK}}\right)(I_3)^{*}_{IJ} + I_1 I_3^{-4/3}(I_3)^{**}_{IJKL}\right] \\
&= \left(\frac{\partial I_3^{-1/3}}{\partial I_3}\frac{\partial I_3}{\partial C_{KL}} + \frac{\partial I_3^{-1/3}}{\partial C_{LK}}\frac{\partial I_3}{\partial C_{Lk}}\right)(I_1)^{*}_{IJ} - \frac{1}{3}\left[(2\delta_{KL}) I_3^{-4/3}(I_3)^{*}_{IJ}\right. \\
&\quad \left. + I_1\left(\frac{\partial I_3^{-4/3}}{\partial I_3}\frac{\partial I_3}{\partial C_{KL}} + \frac{\partial I_3^{-4/3}}{\partial I_3}\frac{\partial I_3}{\partial C_{LK}}\right)(I_3)^{*}_{IJ} + I_1 I_3^{-4/3}(I_3)^{**}_{IJKL}\right] \\
&= -\frac{1}{3} I_3^{-4/3} C^{-1}_{LK} I_3 (I_1)^{*}_{IJ} - \frac{1}{3} I_3^{-4/3} C^{-1}_{KL} I_3 (I_1)^{*}_{IJ} - \frac{1}{3}\left[(2\delta_{KL}) I_3^{-4/3}(I_3)^{*}_{IJ}\right. \\
&\quad \left. + I_1\left(-\frac{4}{3}\right) I_3^{-7/3} C^{-1}_{LK} I_3 (I_3)^{*}_{IJ} + I_1\left(-\frac{4}{3}\right) I_3^{-7/3} C^{-1}_{KL} I_3 (I_3)^{*}_{IJ} + I_1 I_3^{-4/3}(I_3)^{**}_{IJKL}\right] \\
&= -\frac{1}{3} I_3^{-1/3}\left(C^{-1}_{KL} + C^{-1}_{LK}\right)(I_1)^{*}_{IJ} - \frac{1}{3}(2\delta_{KL}) I_3^{-4/3}(I_3)^{*}_{IJ} \\
&\quad + \frac{4}{9} I_1 I_3^{-4/3} C^{-1}_{LK}(I_3)^{*}_{IJ} + \frac{4}{9} I_1 I_3^{-4/3} C^{-1}_{KL}(I_3)^{*}_{IJ} - \frac{1}{3} I_1 I_3^{-4/3}(I_3)^{**}_{IJKL} \\
&= -\frac{1}{3} I_3^{-1/3}\left(C^{-1}_{KL} + C^{-1}_{LK}\right)(I_1)^{*}_{IJ} + \frac{4}{9} I_1 I_3^{-4/3}\left(C^{-1}_{KL} + C^{-1}_{LK}\right)(I_3)^{*}_{IJ} \\
&\quad - \frac{1}{3}(2\delta_{KL}) I_3^{-4/3}(I_3)^{*}_{IJ} - \frac{1}{3} I_1 I_3^{-4/3}(I_3)^{**}_{IJKL}
\end{aligned} \tag{B.41}$$

$$\left(\bar{I}_1\right)^{**}_{IJKL} = -\frac{1}{3}I_3^{-4/3}\left[(I_3)^{*}_{KL}(I_1)^{*}_{IJ} + (I_1)^{*}_{KL}(I_3)^{*}_{IJ} + I_1(I_3)^{**}_{IJKL}\right] + \frac{4}{9}I_1I_3^{-7/3}(I_3)^{*}_{KL}(I_3)^{*}_{IJ} \tag{B.42}$$

$$\begin{aligned}\left(\frac{\partial}{\partial C_{KL}} + \frac{\partial}{\partial C_{LK}}\right)\left[(J-1)(J)^{*}_{IJ}\right] &= \frac{\partial(J-1)}{\partial J}\frac{\partial J}{\partial C_{KL}}(J)^{*}_{IJ} + \frac{\partial(J-1)}{\partial J}\frac{\partial J}{\partial C_{LK}}(J)^{*}_{IJ} \\ &\quad + (J-1)(J)^{**}_{IJKL} \\ &= \left(\frac{1}{2}I_3^{1/2}C^{-1}_{LK} + \frac{1}{2}I_3^{1/2}C^{-1}_{KL}\right)(J)^{*}_{IJ} + (J-1)(J)^{**}_{IJKL} \\ &= (J)^{*}_{KL}(J)^{*}_{IJ} + (J-1)(J)^{**}_{IJKL}\end{aligned} \tag{B.43}$$

$$\begin{aligned}(J)^{**}_{IJKL} &= \left(\frac{\partial}{\partial C_{KL}} + \frac{\partial}{\partial C_{LK}}\right)(J)^{*}_{IJ} = \left(\frac{\partial}{\partial C_{KL}} + \frac{\partial}{\partial C_{LK}}\right)\left[\frac{1}{2}I_3^{-1/2}(I_3)^{*}_{IJ}\right] \\ &= \frac{1}{2}\left[\frac{\partial I_3^{-1/2}}{\partial I_3}\frac{\partial I_3}{\partial C_{KL}}(I_3)^{*}_{IJ} + \frac{\partial I_3^{-1/2}}{\partial I_3}\frac{\partial I_3}{\partial C_{LK}}(I_3)^{*}_{IJ} + I_3^{-1/2}(I_3)^{**}_{IJKL}\right] \\ &= \frac{1}{4}\left[-\frac{1}{2}I_3^{-3/2}I_3C^{-1}_{LK} - \frac{1}{2}I_3^{-3/2}I_3C^{-1}_{KL}\right](I_3)^{*}_{IJ} + \frac{1}{2}I_3^{-1/2}(I_3)^{**}_{IJKL} \\ &= -\frac{1}{4}I_3^{-3/2}(I_3)^{*}_{KL}(I_3)^{*}_{IJ} + \frac{1}{2}I_3^{-1/2}(I_3)^{**}_{IJKL}\end{aligned} \tag{B.44}$$

This formulation lends itself fairly directly to programming, employing either FORTRAN® or MATLAB®.

Appendix C: Derivative Expressions

In this Appendix we develop a number of different derivative expressions that are important in the field of finite elasticity. First, we want to derive the expression for $\partial_{\mathbf{F}} J$. We write

$$\frac{\partial J}{\partial \mathbf{F}} : \mathbf{A} = \frac{d}{d\alpha} \det(\mathbf{F} + \alpha \mathbf{A}) \Big|_{\alpha=0} \tag{C.1a}$$

$$= \frac{d}{d\alpha} \det\left[\alpha \mathbf{F}\left(\frac{1}{\alpha}\mathbf{1} + \mathbf{F}^{-1}\mathbf{A}\right)\right]\Bigg|_{\alpha=0} \tag{C.1b}$$

$$= \frac{d}{d\alpha}\left[\alpha^3 J \det\left(\frac{1}{\alpha}\mathbf{1} + \mathbf{F}^{-1}\mathbf{A}\right)\right]\Bigg|_{\alpha=0} \tag{C.1c}$$

where $\mathbf{1}$ is the second-order identity tensor, $\mathbf{F}$ is the deformation gradient tensor, $J = \det(\mathbf{F})$ and $\mathbf{A}$ is an arbitrary second-order tensor. Expanding the determinant of a tensor in the form of a characteristic equation in terms of the invariants I_1, I_2, I_3 we have (note the sign of λ)

$$\det(\lambda \mathbf{1} + \mathbf{F}) = \lambda^3 + I_1(\mathbf{F})\lambda^2 + I_2(\mathbf{F})\lambda + I_3(\mathbf{F}) \tag{C.2}$$

Using this expression, we can write

$$\frac{\partial J}{\partial \mathbf{F}} : \mathbf{A} = \frac{d}{d\alpha}\left[\alpha^3 J\left(\frac{1}{\alpha^3} + I_1\left(\mathbf{F}^{-1}\mathbf{A}\right)\frac{1}{\alpha^2} + I_2\left(\mathbf{F}^{-1}\mathbf{A}\right)\frac{1}{\alpha} + I_3\left(\mathbf{F}^{-1}\mathbf{A}\right)\right)\right]\Bigg|_{\alpha=0} \tag{C.3a}$$

$$= J\frac{d}{d\alpha}\left[\mathbf{1} + I_1\left(\mathbf{F}^{-1}\mathbf{A}\right)\alpha + I_2\left(\mathbf{F}^{-1}\mathbf{A}\right)\alpha^2 + I_3\left(\mathbf{F}^{-1}\mathbf{A}\right)\alpha^3\right]\Bigg|_{\alpha=0} \tag{C.3b}$$

$$= J\left[I_1\left(\mathbf{F}^{-1}\mathbf{A}\right) + 2I_2\left(\mathbf{F}^{-1}\mathbf{A}\right)\alpha + 3I_3\left(\mathbf{F}^{-1}\mathbf{A}\right)\alpha^2\right]\Big|_{\alpha=0} \tag{C.3c}$$

$$= J\left[I_1\left(\mathbf{F}^{-1}\mathbf{A}\right)\right] \tag{C.3d}$$

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Recalling that

$$I_1(\mathbf{F}) \equiv tr\mathbf{F} \tag{C.4}$$

we can write,

$$\frac{\partial J}{\partial \mathbf{F}} : \mathbf{A} = J\mathrm{tr}\left(\mathbf{F}^{-1}\mathbf{A}\right) = J\left(\mathbf{F}^{-T} : \mathbf{A}\right) \tag{C.5}$$

Invoking the arbitrariness of **A** we then have

$$\frac{\partial J}{\partial \mathbf{F}} = J\mathbf{F}^{-T} \tag{C.6}$$

Next, we want to derive the expression for $\partial_{\mathbf{C}} J$, where **C** is the right Cauchy-Green deformation tensor. We begin by writing

$$\begin{aligned}\frac{\partial J}{\partial \mathbf{F}} : \mathbf{1} &= \frac{\partial J}{\partial \mathbf{C}} : \frac{\partial \mathbf{C}}{\partial \mathbf{F}} = \frac{\partial J}{\partial \mathbf{C}} : \left(\mathbf{1}^T\mathbf{F} + \mathbf{F}^T\mathbf{1}\right) \\ &= \left[\mathbf{F}\frac{\partial J}{\partial \mathbf{C}}\right] : \mathbf{1} + \left[\mathbf{F}\left(\frac{\partial J}{\partial \mathbf{C}}\right)^T\right] : \mathbf{1}\end{aligned} \tag{C.7}$$

Given that

$$\left(\frac{\partial J}{\partial \mathbf{C}}\right)^T = \frac{\partial J}{\partial \mathbf{C}} \tag{C.8}$$

then,

$$\frac{\partial J}{\partial \mathbf{F}} = 2\mathbf{F}\frac{\partial J}{\partial \mathbf{C}} \tag{C.9}$$

giving

$$\frac{\partial J}{\partial \mathbf{C}} = \frac{1}{2}\mathbf{F}^{-1}\frac{\partial J}{\partial \mathbf{F}} = \frac{1}{2}J\mathbf{F}^{-1}\mathbf{F}^{-T} = \frac{1}{2}J\mathbf{C}^{-1} \tag{C.10}$$

Now, we move on to the more involved derivation of the expression for $\partial_{\overline{\mathbf{C}}} J$. Given that

$$\overline{\mathbf{C}} = J^{-2/3}\mathbf{F}^T\mathbf{F} \tag{C.11}$$

we can write

$$\frac{\partial J}{\partial \mathbf{F}} : \mathbf{1} = \frac{\partial J}{\partial \overline{\mathbf{C}}} : \frac{\partial \overline{\mathbf{C}}}{\partial \mathbf{F}} = \frac{\partial J}{\partial \overline{\mathbf{C}}} : \frac{\partial}{\partial \mathbf{F}} \left(J^{-2/3} \mathbf{F}^T \mathbf{F} \right) \tag{C.12a}$$

$$= \frac{\partial J}{\partial \overline{\mathbf{C}}} : \left[\frac{\partial J^{-2/3}}{\partial J} \frac{\partial J}{\partial \mathbf{F}} \mathbf{F}^T \mathbf{F} + J^{-2/3} \frac{\partial \mathbf{F}^T}{\partial \mathbf{F}} \mathbf{F} + J^{-2/3} \mathbf{F}^T \frac{\partial \mathbf{F}}{\partial \mathbf{F}} \right] \tag{C.12b}$$

$$= \frac{\partial J}{\partial \overline{\mathbf{C}}} : \left[-\frac{2}{3} J^{-5/3} J \mathbf{F}^{-T} \left(\mathbf{F}^T \mathbf{F} \right) + J^{-2/3} \mathbf{1}^T \mathbf{F} + J^{-2/3} \mathbf{F}^T \mathbf{1} \right] \tag{C.12c}$$

$$= \frac{\partial J}{\partial \overline{\mathbf{C}}} : \left[-\frac{2}{3} J^{-2/3} \mathbf{1} \mathbf{F} + J^{-2/3} \mathbf{1}^T \mathbf{F} + J^{-2/3} \mathbf{F}^T \mathbf{1} \right] \tag{C.12d}$$

$$= J^{-2/3} \left\{ \left[-\frac{2}{3} \mathbf{F} \frac{\partial J}{\partial \overline{\mathbf{C}}} \right] : \mathbf{1}^T + \left[\mathbf{F} \frac{\partial J}{\partial \overline{\mathbf{C}}} \right] : \mathbf{1} + \left[\mathbf{F} \left(\frac{\partial J}{\partial \overline{\mathbf{C}}} \right)^T \right] : \mathbf{1} \right\} \tag{C.12e}$$

We know that

$$\mathbf{1}^T = \mathbf{1} \tag{C.13a}$$

and

$$\left(\frac{\partial J}{\partial \overline{\mathbf{C}}} \right)^T = \frac{\partial J}{\partial \overline{\mathbf{C}}} \tag{C.13b}$$

So,

$$\frac{\partial J}{\partial \mathbf{F}} = \frac{4}{3} J^{-2/3} \mathbf{F} \frac{\partial J}{\partial \overline{\mathbf{C}}} \tag{C.14}$$

or

$$\begin{aligned} \frac{\partial J}{\partial \overline{\mathbf{C}}} &= \frac{3}{4} J^{2/3} \mathbf{F}^{-1} \frac{\partial J}{\partial \mathbf{F}} = \frac{3}{4} J^{2/3} \mathbf{F}^{-1} \left(J \mathbf{F}^{-T} \right) = \frac{3}{4} J^{5/3} \mathbf{F}^{-1} \mathbf{F}^{-T} \\ &= \frac{3}{4} J^{5/3} \mathbf{C}^{-1} = \frac{3}{4} J \overline{\mathbf{C}}^{-1} \end{aligned} \tag{C.15}$$

Now, we further move on to the derivation of the expression for $\partial_{\overline{\mathbf{C}}} \mathbf{C}$. Given that

$$\mathbf{C} = J^{2/3} \overline{\mathbf{C}} \tag{C.16}$$

we can write

$$\frac{\partial \mathbf{C}}{\partial \overline{\mathbf{C}}} = \frac{\partial}{\partial \overline{\mathbf{C}}}\left(J^{2/3}\overline{\mathbf{C}}\right) = \frac{\partial J^{2/3}}{\partial \overline{\mathbf{C}}}\overline{\mathbf{C}} + J^{2/3}\frac{\partial \overline{\mathbf{C}}}{\partial \overline{\mathbf{C}}} = \frac{\partial J^{2/3}}{\partial J}\frac{\partial J}{\partial \overline{\mathbf{C}}}\overline{\mathbf{C}} + J^{2/3}\mathbf{I} \tag{C.17a}$$

$$= \frac{2}{3}J^{-1/3}\frac{3}{4}J\overline{\mathbf{C}}^{-1} \otimes \overline{\mathbf{C}} + J^{2/3}\mathbf{I} \tag{C.17b}$$

$$= J^{2/3}\left(\mathbf{I} + \frac{1}{2}\overline{\mathbf{C}}^{-1} \otimes \overline{\mathbf{C}}\right) \tag{C.17c}$$

where $\mathbf{I}$ is the fourth-order identity tensor.

C.1 Additional Formulation

We can say that

$$\frac{\partial}{\partial \mathbf{F}}\left(\mathbf{F}^{-1}\mathbf{F}\right) : \mathbf{A} = \left(\frac{\partial \mathbf{F}^{-1}}{\partial \mathbf{F}} : \mathbf{A}\right)\mathbf{F} + \mathbf{F}^{-1}\left(\frac{\partial \mathbf{F}}{\partial \mathbf{F}} : \mathbf{A}\right) = 0 \tag{C.18}$$

and, thus, that,

$$\left(\frac{\partial \mathbf{F}^{-1}}{\partial \mathbf{F}} : \mathbf{A}\right)\mathbf{F} = -\mathbf{F}^{-1}\mathbf{A} \tag{C.19}$$

Then,

$$\frac{\partial \mathbf{F}^{-1}}{\partial \mathbf{F}} : \mathbf{A} = -\mathbf{F}^{-1}\mathbf{A}\mathbf{F}^{-1} \Rightarrow \frac{\partial \mathbf{F}^{-1}}{\partial \mathbf{F}} = -\mathbf{F}^{-1}\mathbf{F}^{-1} \tag{C.20}$$

and, in index notation,

$$\frac{\partial F_{Ji}}{\partial F_{kL}}A_{kL} = -F_{Jk}A_{kL}F_{Li} \Rightarrow \frac{\partial F_{Ji}}{\partial F_{kL}} = -F_{Jk}F_{Li} \tag{C.21}$$

or

$$\frac{\partial F^{-1}_{iJ}}{\partial F_{kL}} = -F^{-1}_{kJ}F^{-1}_{iL} \tag{C.22}$$

and, also

$$\frac{\partial F_{Ji}^{T}}{\partial F_{kL}} A_{kL} = -F_{Jk}^{T} A_{kL} F_{Li}^{T} \Rightarrow \frac{\partial F_{Ji}^{T}}{\partial F_{kL}} = -F_{Li}^{T} F_{Jk}^{T} \tag{C.23}$$

or

$$\frac{\partial F_{iJ}^{-T}}{\partial F_{kL}} = -F_{iL}^{-T} F_{kJ}^{-T} \tag{C.24}$$

since

$$F_{iJ}^{-1} \equiv F_{Ji} \tag{C.25}$$

Appendix D: Derivation of Recursive Formula

The derivation of the recursive relationship for the internal variables in the incremental finite viscoelastic formulation is presented here. The internal variables $\mathbf{H}^{(n)}$ at time t can be expressed in terms of convolution integrals,

$$\mathbf{H}^{(n)}(t) = \int_0^t \delta^{(n)} \exp\left(-\frac{t-\tau}{\tau^{(n)}}\right) \frac{\partial \overline{\mathbf{S}}^{(0)}(\tau)}{\partial \tau} d\tau \tag{D.1}$$

Considering the time interval $[t, t+\Delta t]$ we employ the multiplicative split of the exponential expression in Equation (D.1),

$$\exp\left(-\frac{t+\Delta t}{\tau^{(n)}}\right) = \exp\left(-\frac{t}{\tau^{(n)}}\right) \exp\left(-\frac{\Delta t}{\tau^{(n)}}\right) \tag{D.2}$$

and the separation of the deformation history into a period $0 \leq \tau \leq t$ when the result is known and into the current unknown time step $t \leq \tau \leq t + \Delta t$ yields

$$\begin{aligned} \mathbf{H}^{(n)}_{t+\Delta t} &= \delta^{(n)} \int_0^{t+\Delta t} \exp\left(-\frac{t+\Delta t-\tau}{\tau^{(n)}}\right) \frac{d\overline{\mathbf{S}}^{(0)}(\tau)}{d\tau} d\tau \\ &= \exp\left(-\frac{\Delta t}{\tau^{(n)}}\right) \delta^{(n)} \int_0^t \exp\left(-\frac{t-\tau}{\tau^{(n)}}\right) \frac{d\overline{\mathbf{S}}^{(0)}(\tau)}{d\tau} d\tau \end{aligned} \tag{D.3a}$$

$$+ \delta^{(n)} \int_t^{t+\Delta t} \exp\left(-\frac{t+\Delta t-\tau}{\tau^{(n)}}\right) \frac{d\overline{\mathbf{S}}^{(0)}(\tau)}{d\tau} d\tau \tag{D.3b}$$

$$= \exp\left(-\frac{\Delta t}{\tau^{(n)}}\right) \mathbf{H}^{(n)}_t + \delta^{(n)} \int_t^{t+\Delta t} \exp\left(-\frac{t+\Delta t-\tau}{\tau^{(n)}}\right) \frac{d\overline{\mathbf{S}}^{(0)}(\tau)}{d\tau} d\tau \tag{D.3c}$$

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which is an exact recursive formula for the current value of the internal variable $\mathbf{H}^{(n)}$. The transition from differential coefficient to discrete time steps,

$$\frac{d\overline{\mathbf{S}}^{(0)}(\tau)}{d\tau} = \lim_{\Delta t \to 0} \frac{\Delta\overline{\mathbf{S}}(\tau)}{\Delta\tau} = \lim_{\Delta t \to 0} \frac{\overline{\mathbf{S}}^{(0)}_{t+\Delta t} - \overline{\mathbf{S}}^{(0)}_{t}}{\Delta t} \tag{D.4}$$

introduces a time approximation of second order into the formula which was exact up to that point. We integrate the remaining expression

$$\begin{aligned}\mathbf{H}^{(n)}_{t+\Delta t} &= \exp\left(-\frac{\Delta t}{\tau^{(n)}}\right)\mathbf{H}^{(n)}_{t} \\ &\quad + \delta^{(n)}\left(\frac{\overline{\mathbf{S}}^{(0)}_{t+\Delta t} - \overline{\mathbf{S}}^{(0)}_{t}}{\Delta t}\right)\int_{t}^{t+\Delta t}\exp\left(-\frac{t+\Delta t-\tau}{\tau^{(n)}}\right)d\tau\end{aligned} \tag{D.5}$$

analytically, and obtain the recursive formula

$$\mathbf{H}^{(n)}_{t+\Delta t} = \exp\left(-\frac{\Delta t}{\tau^{(n)}}\right)\mathbf{H}^{(n)}_{t} + \delta^{(n)}\left(\frac{1-\exp\left(-\frac{\Delta t}{\tau^{(n)}}\right)}{\frac{\Delta t}{\tau^{(n)}}}\right)\left(\overline{\mathbf{S}}^{(0)}_{t+\Delta t} - \overline{\mathbf{S}}^{(0)}_{t}\right) \tag{D.6}$$

The recursive determination of the current variables $\mathbf{H}^{(n)}_{t+\Delta t}$ requires the quantities $\overline{\mathbf{S}}^{(0)}_{t}, \mathbf{H}^{(n)}_{t}$, where $n = 1, \ldots, N$, of the preceding time step, therefore they must be stored in a database.

Appendix E: Lubliner Finite Viscoelasticity Formulation

We first derive the important *rate-of-deformation tensor* and its companion, the *spin tensor*, and then present important aspects of the finite viscoelasticity formulation by Lubliner (1985), which is based on a multiplicative decomposition of the deformation gradient into elastic and inelastic components. To begin, we write

$$[F] = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix} \tag{E.1a}$$

and

$$[l] = \begin{bmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} & \frac{\partial v_1}{\partial x_3} \\ \frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial x_2} & \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_3}{\partial x_1} & \frac{\partial v_3}{\partial x_2} & \frac{\partial v_3}{\partial x_3} \end{bmatrix} \tag{E.1b}$$

where $\mathbf{F}$ is the well-known deformation gradient, and, correspondingly, $\mathbf{l}$ is the *velocity gradient tensor* which can also be expressed as

$$l_{ij} \equiv \frac{\partial v_i}{\partial x_j} \equiv v_{i,j} \tag{E.2}$$

Expanding $\mathbf{l}$ into symmetric and antisymmetric parts, we have

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$$\mathbf{l} = \frac{1}{2}\left(\mathbf{l} + \mathbf{l}^T\right) + \frac{1}{2}\left(\mathbf{l} - \mathbf{l}^T\right) \tag{E.3}$$

or

$$\mathbf{l} = \mathbf{d} + \mathbf{w} \tag{E.4}$$

where the symmetric part is

$$\mathbf{d} = \frac{1}{2}\left(\mathbf{l} + \mathbf{l}^T\right) \tag{E.5}$$

with **d** being the *rate-of-deformation tensor*, i.e.,

$$[d] = \begin{bmatrix} \dfrac{\partial v_1}{\partial x_1} & \dfrac{1}{2}\left(\dfrac{\partial v_1}{\partial x_2} + \dfrac{\partial v_2}{\partial x_1}\right) & \dfrac{1}{2}\left(\dfrac{\partial v_1}{\partial x_3} + \dfrac{\partial v_3}{\partial x_1}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial v_2}{\partial x_1} + \dfrac{\partial v_1}{\partial x_2}\right) & \dfrac{\partial v_2}{\partial x_2} & \dfrac{1}{2}\left(\dfrac{\partial v_2}{\partial x_3} + \dfrac{\partial v_3}{\partial x_2}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial v_3}{\partial x_1} + \dfrac{\partial v_1}{\partial x_3}\right) & \dfrac{1}{2}\left(\dfrac{\partial v_3}{\partial x_2} + \dfrac{\partial v_2}{\partial x_3}\right) & \dfrac{\partial v_3}{\partial x_3} \end{bmatrix} \tag{E.6}$$

and the antisymmetric part is given by

$$\mathbf{w} = \frac{1}{2}\left(\mathbf{l} - \mathbf{l}^T\right) \tag{E.7}$$

where **w** is the *spin tensor*, i.e.,

$$[w] = \begin{bmatrix} 0 & \dfrac{1}{2}\left(\dfrac{\partial v_1}{\partial x_2} - \dfrac{\partial v_2}{\partial x_1}\right) & \dfrac{1}{2}\left(\dfrac{\partial v_1}{\partial x_3} - \dfrac{\partial v_3}{\partial x_1}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial v_2}{\partial x_1} - \dfrac{\partial v_1}{\partial x_2}\right) & 0 & \dfrac{1}{2}\left(\dfrac{\partial v_2}{\partial x_3} - \dfrac{\partial v_3}{\partial x_2}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial v_3}{\partial x_1} - \dfrac{\partial v_1}{\partial x_3}\right) & \dfrac{1}{2}\left(\dfrac{\partial v_3}{\partial x_2} - \dfrac{\partial v_2}{\partial x_3}\right) & 0 \end{bmatrix} \tag{E.8}$$

Now, we can write

$$\mathbf{l} = \dot{\mathbf{F}}\mathbf{F}^{-1} \tag{E.9a}$$

and

$$\bar{\mathbf{l}} = \dot{\bar{\mathbf{F}}}\,\bar{\mathbf{F}}^{-1} \tag{E.9b}$$

along with

$$\dot{\mathbf{F}} = \mathbf{lF} \quad \text{or} \quad \frac{\partial \mathbf{F}}{\partial t} = \frac{\partial \mathbf{v}}{\partial \mathbf{x}} \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \frac{\partial \mathbf{v}}{\partial \mathbf{X}} \tag{E.10}$$

We know that

$$J^{-1}\dot{J} \equiv \mathrm{tr}\left(\dot{\mathbf{F}}\mathbf{F}^{-1}\right) \equiv \mathrm{tr}(\mathbf{d}) \tag{E.11a}$$

and

$$\dot{\mathbf{C}} \equiv 2\mathbf{F}^T \mathbf{dF} \tag{E.11b}$$

and, it can be shown that

$$\dot{\overline{\mathbf{C}}} = 2\overline{\mathbf{F}}^T (\mathrm{dev}\,\mathbf{d})\overline{\mathbf{F}} \tag{E.12a}$$

and

$$\bar{\mathbf{l}} = \mathrm{dev}\,\mathbf{l} \tag{E.12b}$$

where "dev" is the deviatoric operator. We also know that

$$\frac{\partial J}{\partial \mathbf{C}} = \frac{1}{2} J \mathbf{C}^{-1} \tag{E.13}$$

which is derived in Appendix C, and that

$$\frac{\partial \overline{\mathbf{C}}}{\partial \mathbf{C}} = J^{-2/3}\left(\mathbf{1} \otimes \mathbf{1} - \frac{1}{3}\mathbf{C} \otimes \mathbf{C}^{-1}\right) = J^{-2/3}\left(\mathbf{1} \otimes \mathbf{1} - \frac{1}{3}\overline{\mathbf{C}} \otimes \overline{\mathbf{C}}^{-1}\right) \tag{E.14}$$

where $\mathbf{C}^{-1}$ is called the *Piola deformation tensor*. Given the strain-energy function $W\left(J, \overline{\mathbf{C}}\right)$, we have

$$\frac{\partial W}{\partial \mathbf{C}} = \frac{1}{2} J \frac{\partial W}{\partial J} \mathbf{C}^{-1} + J^{-2/3}\left(\frac{\partial W}{\partial \overline{\mathbf{C}}} - \frac{1}{3} \overline{\mathbf{C}}^{-1} \frac{\partial W}{\partial \overline{C}} \overline{\mathbf{C}} : \overline{\mathbf{C}}\right) \tag{E.15}$$

The Cauchy stress (true stress) is then given by

$$\mathbf{t} = 2J^{-1}\mathbf{F}\frac{\partial W}{\partial \mathbf{C}}\mathbf{F}^T = \frac{\partial W}{\partial J}\mathbf{1} + 2J^{-5/3}\mathrm{dev}\left(\mathbf{F}\frac{\partial W}{\partial \overline{\mathbf{C}}}\mathbf{F}^T\right) \tag{E.16}$$

where

$$S_{IJ} = JF_{Ij}t_{jk}F^T_{Jk} \ \text{ or } \ \mathbf{S} = J\mathbf{F}^{-1}\mathbf{t}\mathbf{F}^{-T} \tag{E.17}$$

the tensor S_{IJ} being the second Piola-Kirchhoff stress. Thus, the dependence of W on J yields the hydrostatic stress, while the dependence of W on $\overline{\mathbf{C}}$ yields the deviatoric stress. Using $\overline{\mathbf{F}}$, the stress deviator may be expressed in the form

$$\operatorname{dev}\mathbf{t} = 2J^{-1}\operatorname{dev}\left(\overline{\mathbf{F}}\frac{\partial W}{\partial\overline{\mathbf{C}}}\overline{\mathbf{F}}^{T}\right) \tag{E.18}$$

If the strain-energy function has the form

$$W\left(J,\overline{\mathbf{C}}\right) = W_0(J) + J\overline{W}\left(\overline{\mathbf{C}}\right) \tag{E.19}$$

then the deviator of the Cauchy stress is obviously determined entirely by $\overline{\mathbf{F}}$. We now combine this additive decomposition with the multiplicative decomposition of $\mathbf{F}$ into elastic and inelastic components, i.e.,

$$\mathbf{F} = \mathbf{F}_e\mathbf{F}_i \tag{E.20}$$

The elastic deformation is

$$\mathbf{C}_e = \mathbf{F}_e^T\mathbf{F}_e = \mathbf{F}_i^{-T}\mathbf{F}^T\mathbf{F}\mathbf{F}_i^{-1} = \mathbf{F}_i^{-T}\mathbf{C}\mathbf{F}_i^{-1} \tag{E.21}$$

and

$$\mathbf{C}_i^{-1} = \mathbf{F}_i^{-1}\mathbf{F}_i^{-T} \tag{E.22}$$

In general we would have

$$J = J_eJ_i \tag{E.23a}$$

and

$$\overline{\mathbf{F}} = \overline{\mathbf{F}}_e\overline{\mathbf{F}}_i \tag{E.23b}$$

It is common, however, to assume that in polymers the volume deformation is purely elastic and that viscoelasticity appears in the distortional response only, i.e.,

$$J_i \equiv 1 \rightarrow J_e \equiv J \tag{E.24}$$

therefore

$$\overline{\mathbf{F}}_i = \mathbf{F}_i \tag{E.25}$$

We assume that the free energy (per reference volume) can be expressed by

$$\psi\left(T,J,\overline{\mathbf{C}}_e,\mathbf{A}\right) = \psi_0(T,J) + J\overline{\psi}_e\left(T,\overline{\mathbf{C}}_e\right) + \psi_i(T,\mathbf{A}) \tag{E.26}$$

where T is the temperature and $\mathbf{A}$ is an internal-variable tensor (symmetric and positive definite) which relaxes to $\mathbf{C}^{-1}$, so that at equilibrium

$$\frac{\partial \psi}{\partial \mathbf{A}} = 0 \tag{E.27}$$

as required by the second law of thermodynamics, with the dependence on $\overline{\mathbf{C}}_e$ being isotropic. It is the "distortional elastic" free energy $\overline{\psi}_e$ that yields the stress deviator and thus embodies the specifically rubber-like qualities of the behavior of the material. A fairly general form for $\overline{\psi}_e$, due to Mooney and Rivlin (reference Chap. 4), can be written as

$$\overline{\psi}_e\left(T, \overline{\mathbf{C}}_e\right) = \sum_{m,\, n \geq 0} C_{mn}(T)(I_1 - 3)^m (I_2 - 3)^n \tag{E.28}$$

and the stress deviator can be written as

$$\operatorname{dev} \mathbf{t} = 2 \operatorname{dev}\left[C_{10}(T)\overline{\mathbf{F}}\mathbf{A}\overline{\mathbf{F}}^T + 2C_{01}(T)\overline{\mathbf{F}}\mathbf{A}\overline{\mathbf{C}}\mathbf{A}\overline{\mathbf{F}}^T\right] \tag{E.29}$$

In the initial state, with $\mathbf{A} = \mathbf{1}$, we have

$$\operatorname{dev} \mathbf{t}_{in} = \operatorname{dev}\left[2C_{10}(T)\overline{\mathbf{b}} + 4C_{01}(T)\overline{\mathbf{b}}^2\right] \tag{E.30}$$

where

$$\overline{\mathbf{b}} \equiv \overline{\mathbf{F}}\,\overline{\mathbf{F}}^T \equiv J^{-2/3}\mathbf{b} \tag{E.31}$$

is the distortional factor of the left Cauchy-Green deformation tensor $\mathbf{b}$, and for the relaxation,

$$\operatorname{dev} \mathbf{t}_{rel} = J^{-1}\operatorname{dev}\left[2C_{10}(T)\overline{\mathbf{F}}\mathbf{A}\overline{\mathbf{F}}^T + 4C_{01}(T)\overline{\mathbf{F}}\mathbf{A}\overline{\mathbf{C}}\mathbf{A}\overline{\mathbf{F}}^T\right] \tag{E.32}$$

This formulation applies to standard-solid-like behavior for large deformations and is an extension of the representation consisting of a spring in series with a Kelvin element. The alternative representation, consisting of a spring in parallel with a Maxwell element, would have as its extension to large deformations, the assumption that the distortional elastic-free energy consists of two parts:

$$\overline{\psi}_e = \overline{\psi}_e^{(0)}\left(T, \overline{\mathbf{C}}\right) + \overline{\psi}_e^{(1)}\left(T, \overline{\mathbf{C}}\mathbf{A}\right) \tag{E.33}$$

with $\mathbf{A}$ now interpreted as being related to the inelastic (viscous) strain of the Maxwell element; its rate equation would be

$$\dot{\mathbf{A}} = \frac{1}{\tau}\left(\mathbf{C}^{-1} - \mathbf{A}\right) \tag{E.34}$$

where τ is the relaxation time. If $\overline{\psi}_e^{(0)}$ and $\overline{\psi}_e^{(1)}$ are both of Mooney-Rivlin type, then the stress deviator is given by

$$\operatorname{dev}\mathbf{t} = J^{-1}\operatorname{dev}\left[2C_{10}^{(0)}(T)\overline{\mathbf{b}} + 4C_{01}^{(0)}(T)\overline{\mathbf{b}}^2 + 2C_{10}^{(1)}(T)\overline{\mathbf{F}}\mathbf{A}\overline{\mathbf{F}}^T + 4C_{01}^{(1)}(T)\overline{\mathbf{F}}\mathbf{A}\overline{\mathbf{C}}\mathbf{A}\overline{\mathbf{F}}^T\right] \tag{E.35}$$

We can extend this formulation into one with more than one relaxation time, by analogy with the generalized Maxwell model of linear (infinitesimal deformation) viscoelasticity, i.e.,

$$\overline{\psi}_e = \overline{\psi}_e^{(0)}\left(T, \overline{\mathbf{C}}\right) + \sum_i \overline{\psi}_e^{(i)}\left(T, \overline{\mathbf{C}}\mathbf{A}^{(i)}\right) \tag{E.36}$$

with each $\mathbf{A}^{(i)}$ governed by the rate equation:

$$\dot{\mathbf{A}}^{(i)} = \frac{1}{\tau^{(i)}}\left(\mathbf{C}^{-1} - \mathbf{A}^{(i)}\right) \tag{E.37}$$

Solving the rate equation, we obtain

$$\mathbf{A}_n^{(i)} = \mathbf{C}^{-1} - \left(\mathbf{C}^{-1} - \mathbf{A}_{n-1}^{(i)}\right)\exp\left(-\frac{t_n}{\tau^{(i)}}\right) \tag{E.38}$$

Appendix F: Computer Program Listing

```
!  Mooney-Rivlin (modified, i.e., near-incompressible)
!  Material Model - Computes the First Elasticity Tensor and the First
!  Piola Stress Tensor (First and Second Order)

   IMPLICIT REAL*8 A-H, O-Z

   DIMENSION F(3,3),Ft(3,3),Fit(3,3),b(3,3),P(3,3),
  +          Delta(3,3),Ahat1_VM(6,6),TEMP(3,3,3,3),DF(3,3),
  +          Term1(3,3,3,3),Term2(3,3,3,3),Term3(3,3,3,3),
  +          Ahat1(3,3,3,3),Term4(3,3,3,3,3,3),Term5(3,3,3,3,3,3),
  +          Term6(3,3,3,3,3,3),Ahat2(3,3,3,3,3,3),
  +          DPIOLA1(3,3),DPIOLA2(3,3),DPIOLA(3,3)

   PARAMETER(ZERO=0.0D0,ONE=1.0D0,TWO=2.0D0,THREE=3.0D0,
             FOUR=4.0D0)

   OPEN(UNIT=2, FILE='OUTPUT2.DAT', STATUS='UNKNOWN')

   SHEAR =    300. ! Modulus
   BULK = 300000. ! Modulus

!  F is the deformation gradient

!  INPUT DATA

   F(1,1)=1.
   F(1,2)=0.
   F(1,3)=0.01  ! Simple Shear
   F(2,1)=0.
   F(2,2)=1.
   F(2,3)=0.
   F(3,1)=0
   F(3,2)=0.
   F(3,3)=1.
```


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```
!   *****************************************************
!   DETERMINANT OF DEFORMATION GRADIENT TENSOR

    DET=F(1,1)*F(2,2)*F(3,3)+F(2,1)*F(3,2)*F(1,3)+
  +      F(3,1)*F(2,3)*F(1,2)-F(3,1)*F(2,2)*F(1,3)-
  +      F(2,1)*F(1,2)*F(3,3)-F(3,2)*F(2,3)*F(1,1)
!   *****************************************************

!   FINITE DEFORMATION TENSORS

    DO i=1,3
         DO J=1,3
              Ft(i,J)=F(J,i)
         ENDDO
    ENDDO

!   Ft is the transpose of F
!   Fit is the inverse of Ft
    CALL INVERSE(Ft,Fit)
    DO I=1,3                  ! Kronecker Delta
         DO J=1,3
              IF(I.EQ.J)THEN
                   Delta(I,J)=1.
              ELSE
                   Delta(I,J)=0.
         ENDIF
      ENDDO
    ENDDO

    CALL MATMULT(F,Ft,b)

    bI1=b(1,1)+b(2,2)+b(3,3)
    bI1bar=bI1/DET

!   --------------------Piola Stress---------------------
    DO j=1,3
        DO I=1,3
            P(j,I)=SHEAR*(DET**(-TWO/THREE))*F(j,I)-
  +                  (SHEAR/THREE)*bI1bar*Fit(j,I)+
  +                   BULK*(DET-ONE)*DET*Fit(j,I)
        ENDDO
```

```
   ENDDO
!  -----------------------Term1------------------------
   DO j=1,3
        DO I=1,3
           DO l=1,3
               DO K=1,3
          Term1(j,I,l,K)=(DET**(-TWO/THREE))*(Delta(j,l)*
  +                         Delta(K,I)-(TWO/THREE)*Fit(l,K)*F(j,I))
               ENDDO
           ENDDO
        ENDDO
   ENDDO
!  -----------------------Term2------------------------
   DO j=1,3
        DO I=1,3
           DO l=1,3
               DO K=1,3
          Term2(j,I,l,K)=-bI1bar*(Fit(j,K)*Fit(l,I)+
  +                         (TWO/THREE)*Fit(l,K)*Fit(j,I))+
  +                         2*(DET**(-TWO/THREE))*F(l,K)*Fit(j,I)
               ENDDO
           ENDDO
        ENDDO
   ENDDO
!  -----------------------Term3------------------------
   DO j=1,3
        DO I=1,3
           DO l=1,3
               DO K=1,3
          Term3(j,I,l,K)=(2*DET-ONE)*DET*Fit(l,K)*Fit(j,I)-
  +                         (DET-ONE)*DET*Fit(j,K)*Fit(l,I)
               ENDDO
           ENDDO
        ENDDO
   ENDDO
!  -----------------------Ahat1------------------------
   DO j=1,3
        DO I=1,3
           DO l=1,3
               DO K=1,3
          Ahat1(j,I,l,K)=SHEAR*Term1(j,I,l,K)-
  +                         (SHEAR/THREE)*Term2(j,I,l,K)+
  +                         BULK*Term3(j,I,l,K)
               ENDDO
           ENDDO
        ENDDO
```

```
   ENDDO
   CALL CONVERTVM4(Ahat1,Ahat1_VM)
!  -----------------------Term4-------------------------
   DO j=1,3
      DO I=1,3
         DO l=1,3
            DO K=1,3
               DO n=1,3
                  DO M=1,3
        Term4(j,I,l,K,n,M)=(TWO/THREE)*(DET**(-TWO/THREE))*
  +                          (((TWO/THREE)*Fit(n,M)*Fit(l,K)*F(j,I)+
  +                             Fit(l,M)*Fit(n,K))*F(j,I)-
  +                            (Fit(n,M)*Delta(j,l)*Delta(K,I)+
  +                             Fit(l,K)*Delta(j,n)*Delta(M,I)))
                  ENDDO
               ENDDO
            ENDDO
         ENDDO
      ENDDO
   ENDDO
!  -----------------------Term5-------------------------
   DO j=1,3
      DO I=1,3
         DO l=1,3
            DO K=1,3
               DO n=1,3
                  DO M=1,3
   Term5(j,I,l,K,n,M)=-(TWO*DET**(-TWO/THREE)*F(n,M)-(TWO/THREE)*
  +                      bI1bar*Fit(n,M))*Fit(j,K)*Fit(l,I)+
  +                      bI1bar*(Fit(j,M)*Fit(n,K)*Fit(l,I)+
  +                      Fit(j,K)*Fit(l,M)*Fit(n,I))-

  +                     (TWO/THREE)*(TWO*(DET**(-TWO/THREE))*
  +                      F(n,M)*Fit(l,K)*Fit(j,I)-
  +                      bI1bar*(Fit(n,M)*Fit(l,K)*Fit(j,I)+
  +                      Fit(l,M)*Fit(n,K)*Fit(j,I)+
  +                      Fit(l,k)*Fit(j,M)*Fit(n,I)))-

  +                      TWO*(DET**(-TWO/THREE))*((TWO/THREE)*
  +                      Fit(n,M)*F(l,K)*Fit(j,I)+
  +                      F(l,K)*Fit(j,M)*Fit(n,I)-
  +                      Delta(n,l)*Delta(M,K)*Fit(j,I))

                  ENDDO
               ENDDO
            ENDDO
         ENDDO
      ENDDO
```

```
     ENDDO
!   -----------------------Term6------------------------
    DO j=1,3
       DO I=1,3
          DO l=1,3
             DO K=1,3
                DO n=1,3
                   DO M=1,3
    Term6(j,I,l,K,n,M)=(4*DET-ONE)*DET*Fit(n,M)*Fit(l,K)*Fit(j,I)-
   +                   (TWO*DET-ONE)*DET*(Fit(l,M)*
   +                    Fit(n,K)*Fit(j,I)+Fit(l,K)*Fit(j,M)*
   +                    Fit(n,I)+Fit(n,M)*Fit(j,K)*Fit(l,I))+
   +                   (DET-ONE)*DET*(Fit(j,M)*Fit(n,K)*
   +                    Fit(l,I)+Fit(j,K)*Fit(l,M)*Fit(n,I))
                   ENDDO
                ENDDO
             ENDDO
          ENDDO
       ENDDO
    ENDDO
!   -----------------------Ahat2------------------------
    DO j=1,3
       DO I=1,3
          DO l=1,3
             DO K=1,3
                DO n=1,3
                   DO M=1,3
         Ahat2(j,I,l,K,n,M)=SHEAR*Term4(j,I,l,K,n,M)-
   +                        (SHEAR/3)*Term5(j,I,l,K,n,M)+
   +                         BULK*Term6(j,I,l,K,n,M)
                   ENDDO
                ENDDO
             ENDDO
          ENDDO
       ENDDO
    ENDDO
!   COMPUTE INCREMENTAL PIOLA STRESS
    DO j=1,3
       DO I=1,3
          DF(j,I)=0.
       ENDDO
```

```
ENDDO
DF(1,3)=0.01
DO j=1,3
   DO I=1,3
       SUM=0.
          DO l=1,3
             DO K=1,3
                SUM=SUM+Ahat1(j,I,l,K)*DF(l,K)
             ENDDO
          ENDDO
       DPIOLA1(j,I)=SUM ! first order
   ENDDO
ENDDO

DO j=1,3
   DO I=1,3
      DO l=1,3
         DO K=1,3
            SUM=0.
              DO n=1,3
                 DO M=1,3
            SUM=SUM+Ahat2(j,I,l,K,n,M)*DF(n,M)
                 ENDDO
              ENDDO
            TEMP(j,I,l,K)=SUM
         ENDDO
      ENDDO
   ENDDO
ENDDO
DO j=1,3
     DO I=1,3
        SUM=0.
           DO l=1,3
              DO K=1,3
        SUM=SUM+TEMP(j,I,l,K)*DF(l,K)
              ENDDO
           ENDDO
        DPIOLA2(j,I)=SUM
     ENDDO
ENDDO
DO j=1,3
     DO I=1,3
  DPIOLA(j,I)=DPIOLA1(j,I)+(ONE/TWO)*DPIOLA2(j,I) ! second order
     ENDDO
ENDDO
```

```
!  ***********************************************************
!   OUTPUT:

37  FORMAT(3(4X,F15.7))

    WRITE(2,*) "Deformation Gradient:"
    DO i=1,3
       WRITE(2,37) (F(i,J), J=1,3)
    ENDDO

    WRITE(2,*) "Deformation Gradient Transpose:"
    DO i=1,3
       WRITE(2,37) (Ft(i,J), J=1,3)
    ENDDO

    WRITE(2,*) "Deformation Gradient Inverse Transpose:"
    DO I=1,3
       WRITE(2,37) (Fit(I,j), j=1,3)
    ENDDO

    WRITE(2,*) "Left Cauchy-Green Tensor:"
    DO i=1,3
       WRITE(2,37) (b(i,j), j=1,3)
    ENDDO

    WRITE(2,*) "Incremental Piola Stress Tensor - First Order:"
    DO i=1,3
       WRITE(2,37) (DPIOLA1(i,J), J=1,3)
    ENDDO

    WRITE(2,*) "Incremental Piola Stress Tensor - Second Order:"
    DO i=1,3
       WRITE(2,37) (DPIOLA(i,J), J=1,3)
    ENDDO

    WRITE(2,*) "Ahat1:"
    DO I=1,6
      WRITE(2,38) (Ahat1_VM(I,J), J=1,6)
    ENDDO

38  FORMAT(6(4X,F12.4))

    STOP
    END

!   ********************************************************

    SUBROUTINE MATMULT(A,B,C)
! This subroutine computes the inner product of two 2nd-order tensors
```

```
      IMPLICIT REAL*8 A-H, O-Z
!     C==AxB

      DIMENSION A(3,3),B(3,3),C(3,3)

      DO I=1,3
         DO J=1,3
               C(I,J)=0.
         ENDDO
      ENDDO

      DO I=1,3
         DO J=1,3
            DO K=1,3
                  C(I,J)=C(I,J)+A(I,K)*B(K,J)
            ENDDO
         ENDDO
      ENDDO

      RETURN
      END

!     *****************************************************
      SUBROUTINE INVERSE(X,XINV)

! This subroutine inverts a 3x3 matrix

      IMPLICIT REAL*8 A-H, O-Z

      DIMENSION X(3,3),XINV(3,3)

      DETX= X(1,1)*X(2,2)*X(3,3)+X(1,2)*X(2,3)*X(3,1)+X(2,1)*X(3,2)*
     +      X(1,3)-X(3,1)*X(2,2)*X(1,3)-X(2,1)*X(1,2)*X(3,3)-X(1,1)*
     +      X(3,2)*X(2,3)

      XINV(1,1)=(X(2,2)*X(3,3)-X(2,3)*X(3,2))/DETX
      XINV(2,1)=-(X(2,1)*X(3,3)-X(3,1)*X(2,3))/DETX
      XINV(3,1)=(X(2,1)*X(3,2)-X(2,2)*X(3,1))/DETX
      XINV(1,2)=-(X(1,2)*X(3,3)-X(3,2)*X(1,3))/DETX
      XINV(2,2)=(X(1,1)*X(3,3)-X(1,3)*X(3,1))/DETX
      XINV(3,2)=-(X(1,1)*X(3,2)-X(3,1)*X(1,2))/DETX
      XINV(1,3)=(X(1,2)*X(2,3)-X(1,3)*X(2,2))/DETX
      XINV(2,3)=-(X(1,1)*X(2,3)-X(1,3)*X(2,1))/DETX
      XINV(3,3)=(X(1,1)*X(2,2)-X(1,2)*X(2,1))/DETX

      RETURN
      END
```

```
!    ******************************************************
     SUBROUTINE CONVERTVM4(TEN4,VM2)

! This subroutine converts a symmetric 4th-order (3x3x3x3) tensor into
! the Voigt-Mandel(VM) form (6x6 matrix)

     REAL*8 TEN4(3,3,3,3),VM2(6,6)

!    LOCALS
     INTEGER KK(3,3)
     DATA KK/1,6,5,6,2,4,5,4,3/

     INTEGER I,J,K,L,M,N
     REAL C

     C=1.0
     DO I=1,3
         DO J=1,3
             M = KK(I,J)
                DO K=1,3
                   DO L=1,3
                       N = KK(K,L)
                     IF (M.LE.3) THEN
                       IF (N.LE.3) THEN
                          VM2(M,N) = TEN4(I,J,K,L)
                       ELSE
                          VM2(M,N) = TEN4(I,J,K,L)*C
                          VM2(M,N) = TEN4(I,J,K,L)+TEN4(I,J,K,L)
                       ENDIF
                     ELSE
                       IF (N.LE.3) THEN
                          VM2(M,N) = TEN4(I,J,K,L)*C
                       ELSE
!    Taking into consideration that TEN4(I,J,K,L) may not be symmetric
!    about K and L
                          VM2(M,N) = TEN4(I,J,K,L)+TEN4(I,J,K,L)
                       ENDIF
                     ENDIF
                   ENDDO
                ENDDO
         ENDDO
     ENDDO

     RETURN
     END
```

Glossary

A, $\bar{\mathrm{A}}$	Any tensor of rank n, reduced
$\boldsymbol{A}^{(1)}_{IjKl}, \hat{\boldsymbol{A}}^{(1)}_{iJkL}, \boldsymbol{A}^{(2)}_{IJKL}, \boldsymbol{A}^{(3)}_{ijkl}, \boldsymbol{A}^{(4)}_{ijkl}$	Fourth-order elasticity tensors
$\mathbf{A}_{iJkL}$	Fourth-order tensor
$\mathbf{A}_{iJkLmN}$	Sixth-order tensor
$\boldsymbol{A}$	Reference area
$\boldsymbol{a}$	Current area
$\mathrm{a}_{\mathbf{0}}$	Unit vector in reference configuration designating preferred direction
$\boldsymbol{a}^{*}_{ij}$ **(or** a*)	Rotated second-order tensor
$\boldsymbol{B}_{ij}$ **(or** B)	Second-order tensor
$\overline{B}_{iJ}$	Distortion gradient
$\boldsymbol{b}_{ij}$ **(or** b)	Left Cauchy-Green deformation tensor
$\overline{b}_{ij}$	Left Cauchy-Green distortion tensor
$\boldsymbol{c}$	Material property
$\boldsymbol{C}_{10}, \boldsymbol{C}_{01}$	Mooney-Rivlin model coefficients
$\boldsymbol{C}_{IJ}$	Right Cauchy-Green deformation tensor
$\overline{C}_{IJ}$	Right Cauchy-green distortion tensor
$\boldsymbol{Df}(\boldsymbol{x})[]$	Directional derivative
$\boldsymbol{D}_{1}$	Mooney-Rivlin model coefficient
$\boldsymbol{D}_{IJKL}$	Material tangent stiffness tensor
$\boldsymbol{D}_{ijkl}$	Spatial tangent stiffness tensor
$\boldsymbol{d}_{ij}$ **(or** d)	Spatial rate-of-deformation tensor
E	Young's modulus
$\boldsymbol{E}_{IJ}$	Green-Lagrange strain tensor
$\overline{E}_{IJ}$	Reduced Green-Lagrange strain tensor

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$\boldsymbol{e_{ij}}$	Almansi strain tensor
$\boldsymbol{e_i}$	Unit vector
$\boldsymbol{F_{iJ}}$ **(or** $\boldsymbol{F}$)	Deformation gradient
$\boldsymbol{\delta F_{iJ}}$	Incremental deformation gradient
$\overline{F}_{iJ}$	Distortion gradient
$\boldsymbol{f}$	Function
$\boldsymbol{f_{ij}}$	Relative deformation gradient
$\boldsymbol{G}$	Shear modulus
$\boldsymbol{G_{iJ}}$, $\boldsymbol{H_{iJ}}$	Second-order two-point tensors
$\boldsymbol{H_{IJ}}$ **(or** H)	Finite viscoelasticity internal variable
$\boldsymbol{H}$	Reference width
$\boldsymbol{h}$	Current width
$\boldsymbol{I_1, I_2, I_3}$	Invariants
$\boldsymbol{\bar{I}_1, \bar{I}_2, \bar{I}_3}$	Reduced invariants
$\bar{I}_{4(i)}$	Reduced pseudo-invariant
$\boldsymbol{J}$	Jacobian
$\boldsymbol{J^{el}}$	Elastic volume ratio
$\boldsymbol{K}$	Bulk modulus
$\boldsymbol{k_1}$	Material property
$\boldsymbol{k_2}$	Material parameter
$\mathrm{K}_{ab}^{(e)}$	Tangent stiffness
$\boldsymbol{L}$	Reference length
$\boldsymbol{l}$	Current length
$\boldsymbol{l_i, m_i, n_i}$	Direction cosines
$\boldsymbol{l_{ij}}$	Spatial velocity gradient tensor
$\boldsymbol{N_{Ij}}$ **(or** N)	Nominal stress tensor
$\boldsymbol{N}$	Number of viscoelastic elements, number of fiber families
$\boldsymbol{N_a}$	Shape function
$\boldsymbol{n}$	Tensor rank; counter
$\mathrm{n}^{(i)}$	Eigenvector of the left stretch tensor
O	Orthogonal rotation tensor
$\boldsymbol{P_{iJ}}$ **(or P)**	First Piola-Kirchhoff, or Piola, stress tensor
$\boldsymbol{p}$	Hydrostatic pressure
$\boldsymbol{Q}$	Fung model exponential argument
$\boldsymbol{Q_{ij}}$	Orthogonal tensor
$\boldsymbol{Q}_{IJkL}^{SF}$	Tangent stiffness tensor
$\boldsymbol{R_{iJ}}$	Rotation tensor
$\boldsymbol{R}_{IjkL}^{NF}$	Tangent stiffness tensor
$\boldsymbol{S_{IJ}}$ **(or** S)	Second Piola-Kirchhoff stress tensor
$\boldsymbol{T_{IJ}}$ **(or** T)	Biot stress tensor
$\boldsymbol{t_{ij}}$ **(or** t)	Cauchy stress tensor
$\boldsymbol{t}$	Time variable
$\boldsymbol{\Delta t}$	Time increment

$\boldsymbol{u_i}$ (or $\boldsymbol{u}$)	Eulerian deformation
$\boldsymbol{U_{IJ}}$ (or $\boldsymbol{U}$)	Right stretch tensor
$\bar{\boldsymbol{U}}_{\boldsymbol{IJ}}$	Biot strain tensor
$\boldsymbol{V}$	Initial volume
$\boldsymbol{\upsilon}$	Current volume
$\boldsymbol{v}$	velocity
$\boldsymbol{V_{IjkL}}$	Viscoelastic tangent tensor
$\boldsymbol{v_{ij}}$ (or v)	Left stretch tensor
$W, \breve{W}, \frown{W}, \widetilde{W}$	Internal strain-energy functions
$\overline{W}_D$	Deviatoric part of $\breve{W}$
$\boldsymbol{W_H}$	Hydrostatic work term
$\boldsymbol{w}$	Strain energy per unit volume; work
w	Spin tensor
$\boldsymbol{x_i}$	Eulerian coordinate system
$\boldsymbol{X_I}$	Lagrangian coordinate system
$\boldsymbol{Y_{Ij}}$ (or $\boldsymbol{Y}$)	Finite viscoelasticty tensor
$\boldsymbol{\alpha}$	Viscoelastic coefficient
$\boldsymbol{\alpha_{IJKL}}, \boldsymbol{\beta_{IJKL}}, \boldsymbol{\delta_{IJKL}}$	Fung model parameters
$\boldsymbol{\beta}$	Degree of compressibility; viscoelastic coefficient; Degree of nonlinearity
$\boldsymbol{\beta_0}, \boldsymbol{\gamma_{IJ}}$	Fung model parameters
$\boldsymbol{\xi}$	Viscoelastic coefficient; element coordinate
$\boldsymbol{\gamma}$	Shear strain; viscoelastic coefficient
$\boldsymbol{\delta}$	Incremental prefix; virtual prefix
δ	Fourth-order tensor
$\underline{\underline{\varepsilon}}$ (or $\boldsymbol{\varepsilon}$)	Strain tensor
$\boldsymbol{\varepsilon_{ij}}$	Strain tensor
$\boldsymbol{\varepsilon_m}$	Mean strain
$\bar{\varepsilon}_{ij}$	Deviatoric strain tensor
$\boldsymbol{\eta}$	Viscosity
$\boldsymbol{\kappa}$	Dispersion parameter; material constant
$\boldsymbol{\lambda}, \boldsymbol{\mu}$	Material constants
$\boldsymbol{\lambda}$	Stretch
$\boldsymbol{\lambda_1}, \boldsymbol{\lambda_2}, \boldsymbol{\lambda_3}$	Principal stretches
$\boldsymbol{\xi_1}, \boldsymbol{\xi_2}, \boldsymbol{\xi_3}$	Natural coordinates
$\boldsymbol{\rho}$	Current density
$\boldsymbol{\rho_0}$	Initial density
$\boldsymbol{\Theta}$	Tensor determinant
$\underline{\underline{\sigma}}$ (or $\boldsymbol{\sigma}$)	Stress tensor
$\boldsymbol{\sigma_{ij}}$	Stress tensor
$\boldsymbol{\sigma_m}$	Mean stress
$\bar{\sigma}_{ij}$	Deviatoric stress tensor
$\boldsymbol{\tau}$	Relaxation time

$\boldsymbol{\tau_{ij}}$ **(or** τ**)**	Kirchhoff stress tensor
$\boldsymbol{\phi}$	Prony series
$\boldsymbol{\phi}$	Deformed configuration
$\boldsymbol{\phi_k}$	Trial solution configuration
$\boldsymbol{\mu_0}$	Initial shear modulus

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