MATH REVIEW

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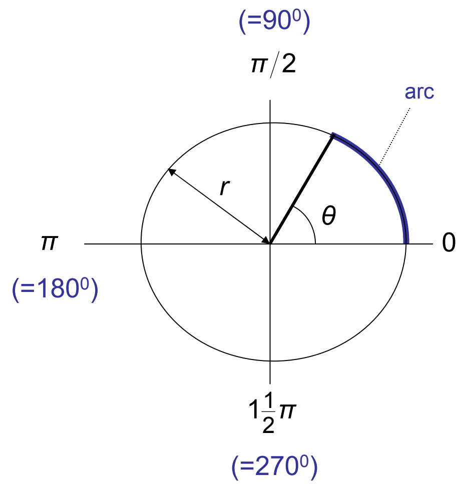
Aims

- To review the theory of
    - trigonometry
    - complex numbers
    - differentiation
    - some useful formulas

Purpose: To prepare for the mathematics involved in physical modelling

Angle Measurement

$$\text{angle in radians} = \frac{\text{arc length}}{\text{radius}}$$



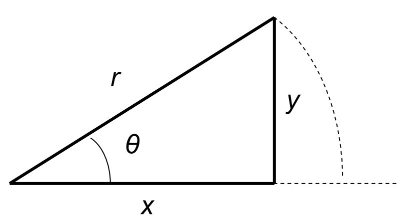
Angle	Degrees	Radians
	90°	$\pi/2$
	60°	$\pi/3$
	45°	$\pi/4$
	30°	$\pi/6$

Sine, Cosine, and Tangent

$$\cos(\theta) = \frac{x}{r}$$

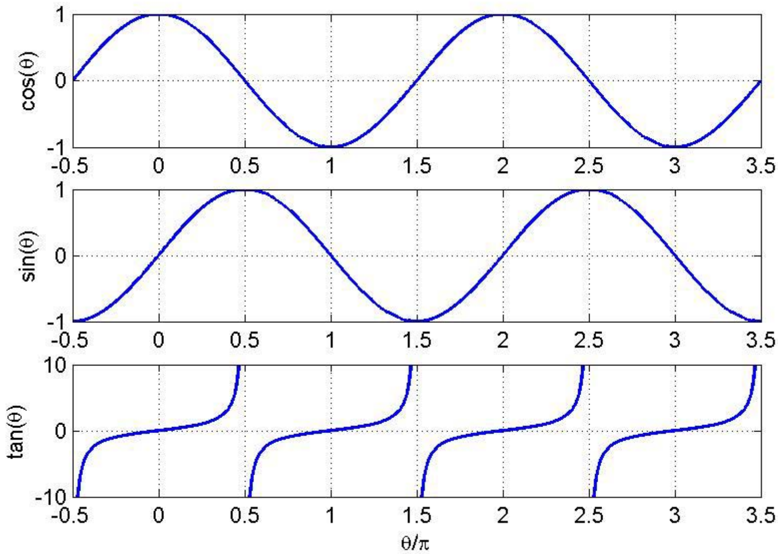
$$\sin(\theta) = \frac{y}{r}$$

$$\tan(\theta) = \frac{y}{x}$$



θ	0	π	$\pi/2$	$\pi/3$	$\pi/4$	$\pi/6$
$\cos(\theta)$	1	-1	0	$\frac{1}{2}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{3}$
$\sin(\theta)$	0	0	1	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}$
$\tan(\theta)$	0	0	$+\infty$	$\sqrt{3}$	1	$1/\sqrt{3}$

Sine, Cosine, and Tangent Plots



Some Trigonometric Relationships

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$

$$\cos^2(\theta) + \sin^2(\theta) = 1$$

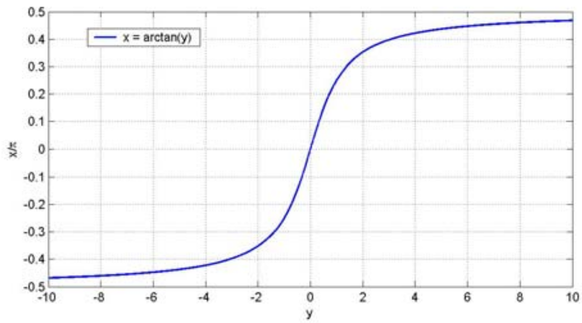
$$\cos(\theta) = \sin(\pi - \theta)$$
$$\sin(\theta) = \cos(\pi - \theta)$$

$$\cos(-\theta) = \cos(\theta)$$
$$\sin(-\theta) = -\sin(\theta)$$
$$\tan(-\theta) = -\tan(\theta)$$

Inverse Functions

$$\left.\begin{aligned} y = \cos(x) &\Leftrightarrow x = \arccos(y) \\ y = \sin(x) &\Leftrightarrow x = \arcsin(y) \\ y = \tan(x) &\Leftrightarrow x = \arctan(y) \end{aligned}\right\}$$

since sin(x), cos(x) and tan(x) are peridodic, output (y) is restricted to the range $[-\pi, \pi]$ or for arctan(y) to $[-\pi/2, \pi/2]$



Complex Numbers: Extension of Number Domain

Inversion of arithmetic operations can force us to define new types of numbers. For example, given two positive numbers *a* and *b,* we can perform addition, and get

$$c = a + b$$

The inverse operation is subtraction:

$$b = c - a$$

This works fine, and results into a positive number as long as $a \leq c$. However, if *a* > *c*, then the outcome becomes **negative**.

This forces us to extend the domain of numbers with **negative numbers**. The real-word meaning of negative numbers is a little bit less obvious than positive numbers; one way of thinking about it is in terms of 'debt' to the bank.

Imaginary Numbers

Consider now the operation squaring, i.e.:

$$y = x^2$$

Let's now invert this operation, i.e. take the square root

$$x = \sqrt{y}$$

This works fine as long as *y* is positive. But what happens if *y* is negative? Such a scenario occurs very often in physics!

It is thus necessary to extend the domain of numbers with numbers which if squared, result into a negative number.

These are called **imaginary** numbers**.**

The Imaginary Unit

Let's start with simplest case, the square root of -1, and define this the **imaginary unit**:

$$j = \sqrt{-1}$$

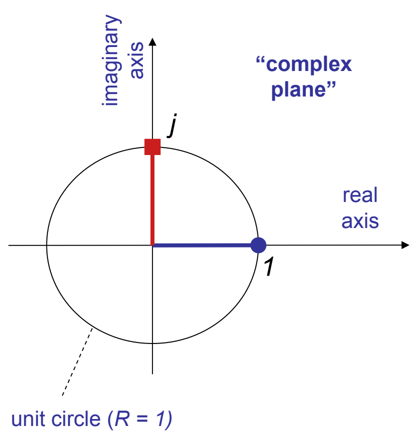
Thus

$$j^2 = -1$$

and for a positive number *a:*

$$\sqrt{-a} = \sqrt{-1 \cdot a} = \sqrt{-1} \cdot \sqrt{a} = j\sqrt{a}$$

Imaginary numbers are orthogonal to real numbers, which can be represented graphically:



Complex Numbers

We are now ready define complex numbers, which are a linear combination of real (normal numbers) and imaginary numbers:

$$z = a + jb$$

a is called the **real part** of *z* and *b* is called the **imaginary part** of *z*:

$$\mathrm{Re}\{z\} = a$$
$$\mathrm{Im}\{z\} = b$$

We can refer to *a* and *b* as the **Cartesian coordinates**.

Note that the imaginary part is thus a real number!

We may treat complex numbers exactly as real numbers, i.e. all standard mathematical rules apply to them.

Complex Conjugate

If we have a complex number

$$z = a + jb$$

Then the **complex conjugate** of that number is:

$$z^* = a - jb$$

In other words, the imaginary part has the opposite sign.

Examples:

$$z = 1 + 2j \Rightarrow z^* = 1 - 2j$$

$$z = 5 - 3j \Rightarrow z^* = 5 + 3j$$

$$z = 0.6 + \sqrt{2}j \Rightarrow z^* = 0.6 - \sqrt{2}j$$

Arithmetic Operations in the Complex Domain

Consider two complex numbers:

$$z_1 = a_1 + jb_1$$
$$z_2 = a_2 + jb_2$$

The we can perform complex addition:

$$\begin{aligned} z_1 + z_2 &= (a_1 + jb_1) + (a_2 + jb_2) \\ &= \underbrace{(a_1 + a_2)}_{real} + j\underbrace{(b_1 + b_2)}_{imaginary} \end{aligned}$$

and complex multiplication:

$$\begin{aligned} z_1 \cdot z_2 &= (a_1 + jb_1)(a_2 + jb_2) \\ &= a_1a_2 + a_1b_2 j + a_2b_1 j + j^2 b_1b_2 \\ &= \underbrace{(a_1a_2 - b_1b_2)}_{real} + j\underbrace{(a_1b_2 + a_2b_1)}_{imaginary} \end{aligned}$$

Arithmetic Operations (cont.)

Complex division:

$$\begin{aligned} z_1 / z_2 &= \frac{a_1 + jb_1}{a_2 + jb_2} \\ &= \frac{(a_1 + jb_1)(a_2 - jb_2)}{(a_2 + jb_2)(a_2 - jb_2)} \\ &= \frac{(a_1a_2 + b_1b_2) + j(a_2b_1 - a_1b_2)}{a_2^2 + b_2^2} \\ &= \underbrace{\left(\frac{a_1a_2 + b_1b_2}{a_2^2 + b_2^2}\right)}_{real} + j\underbrace{\left(\frac{a_2b_1 - a_1b_2}{a_2^2 + b_2^2}\right)}_{imaginary} \end{aligned}$$

Multiply both numerator and denominator with the complex conjugate of the denominator

Polar Coordinates

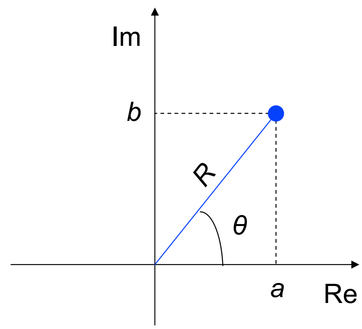
If we represent a complex number z graphically (see right picture), then we can derive an alternative representation of z in terms of polar coordinates:

$$\begin{aligned} z &= a + jb \\ &= R[\cos(\theta) + j\sin(\theta)] \end{aligned}$$

Where we have

the **modulus** of z: $R = \sqrt{a^2 + b^2}$

the **argument** of z: $\theta = \arctan\left(\frac{b}{a}\right)$



a and *b* are the **Cartesian** coordinates

R and *θ* are the **polar** coordinates

Euler’s Equation

Euler found the relationship between the natural exponential function and cosines and sines:

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

Using this for complex numbers, we can write:

$$\begin{aligned} z &= a + jb \\ &= R[\cos(\theta) + j\sin(\theta)] \\ &= R\,e^{j\theta} \end{aligned}$$

So for example, we have:

$$\begin{aligned} e^{j\pi} &= \cos(\pi) - j\sin(\pi) \\ &= -1 \end{aligned}$$

thus:

$$e^{j\pi} + 1 = 0$$

This equation expresses a relationship between the five most important universal constants!

Multiplication and Division with Polar Coordinates

Complex multiplication:

$$z_1 \cdot z_2 = R_1 e^{j\theta_1} R_2 e^{j\theta_2}$$
$$= R_1 R_2 e^{j(\theta_1+\theta_2)}$$

Complex division:

$$z_1 / z_2 = \frac{R_1 e^{j\theta_1}}{R_2 e^{j\theta_2}}$$
$$= \left(\frac{R_1}{R_1}\right) e^{j(\theta_1-\theta_2)}$$

Cartesian coordinates are convenient for addition and subtraction,

Polar coordinates are convenient for multiplication and division.

Powers and Roots with Polar Coordinates

Raising a complex number to the power n means raising the modulus to the power n and multiplying the argument with n :

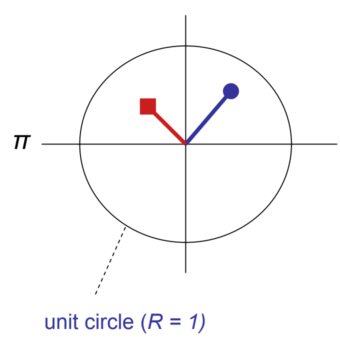
$$z^n = \left(\mathrm{R}e^{\theta}\right)^n = R^n e^{n\theta}$$

Taking the nth-order root of a complex number means taking the nth-order root of the modulus and dividing the argument by n :

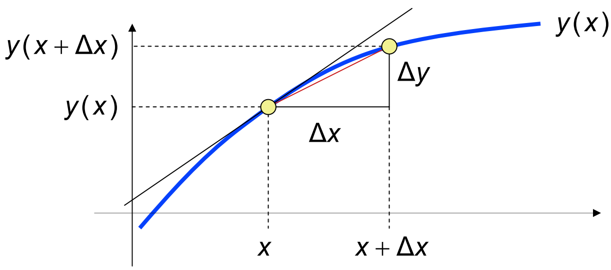
$$\sqrt[n]{z} = \sqrt[n]{\mathrm{R}e^{\theta}} = \sqrt[n]{R}e^{\theta/n}$$

● $z_1 = 0.8e^{j\pi/3}$

■ $z_2 = z_1^2 = 0.64e^{j2\pi/3}$



Differentiation



The **derivative** of a function f as a function of *x* can be considered as the slope of that function at *x*. This can be expressed mathematically as

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{y(x+\Delta x) - y(x)}{\Delta x}$$

Differentiating a Quadratic Function

$$y = x^2 \quad \Rightarrow \quad \begin{aligned} \frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{y(x+\Delta x) - y(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + \Delta x^2 - x^2}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{2x\Delta x + \Delta x^2}{\Delta x} \\ &= \lim_{\Delta x \to 0} 2x + \Delta x \\ &= 2x \end{aligned}$$

Differentiating Polynomial Functions

Using the same strategy, we can obtain a more general result for functions of the type:

$$y = x^n$$

The derivative of which is:

$$\frac{dy}{dx} = nx^{n-1}$$

For example:

$$y = x^3 \Rightarrow \frac{dy}{dx} = 3x^2$$

$$y = 2x^5 \Rightarrow \frac{dy}{dx} = 10x^4$$

$$y = -3x^{-2} \Rightarrow \frac{dy}{dx} = 6x^{-3}$$

$$y = -4x \Rightarrow \frac{dy}{dx} = -4$$

$$y = 3 \Rightarrow \frac{dy}{dx} = 0$$

Trigonometric and Natural Exponential Functions

y	$\frac{dy}{dx}$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\frac{1}{\cos^2(x)}$

$$y = e^x$$

$$\frac{dy}{dx} = e^x$$

Thus for this function, the derivative equals the function itself!

Linearity

For a function which is a linear combination of two (or indeed more) functions of one variable (*x*):

$$y = a \cdot u + b \cdot v$$

The derivative is simply the same linear combination of the derivatives, i.e.

$$\frac{dy}{dx} = a\frac{du}{dx} + b\frac{dv}{dx}$$

For example:

$$y = 2x^2 - 3x$$

$$\Rightarrow \frac{dy}{dx} = 4x - 3$$

$$y = a\sin(x) + bx^{-2}$$

$$\Rightarrow \frac{dy}{dx} = a\cos(x) - 2bx^{-3}$$

Product Rule

For a function that is the product of two functions of the same variable:

$$y = u \cdot v$$

The derivative is:

$$\frac{dy}{dx} = \frac{du}{dx} \cdot v + \frac{dv}{dx} \cdot u$$

For example:

$$y = 5x^2 \sin(x)$$

$$\Rightarrow \frac{dy}{dx} = 10x\sin(x) + 5x^2\cos(x)$$

$$y = \left[x^3 - 2\right]\tan(x)$$

$$\Rightarrow \frac{dy}{dx} = 3x\tan(x) + \frac{x^3 - 2}{\cos^2(x)}$$

Quotient Rule

For a function that is the quotient of two functions of the same variable:

$$y = \frac{u}{v}$$

The derivative is:

$$\frac{dy}{dx} = \frac{\frac{du}{dx} \cdot v - \frac{dv}{dx} \cdot u}{v^2}$$

For example:

$$y = \frac{x^2}{\sin(x)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2x\sin(x) - x^2\cos(x)}{\sin^2(x)}$$

Chain Rule

For a function that is composite function, i.e. the function of a function':

$$y(x) = u(v(x))$$

The derivative is:

$$\frac{dy}{dx} = \frac{dv}{dx} \cdot \frac{du}{dv}$$

For example:

$$y = (3x+2)^2$$

$$u = v^2, v = 3x+2$$

$$\Rightarrow \frac{dv}{dx} = 3$$

$$\Rightarrow \frac{du}{dv} = 2v = 2(3x+2) = 6x+4$$

$$\Rightarrow \frac{dy}{dx} = 3(6x+4) = 18x+12$$

Partial Differentiation

For a function of two different variables x and y:

$$y(x,t)$$

One can define two different derivatives, which are called **partial derivatives**.

The derivative with respect to x and t are respectively denoted as

$$\frac{\partial y}{\partial x} \quad \text{and} \quad \frac{\partial y}{\partial t}$$

For example:

$$y = x^2 \sin(at)$$

$$\frac{\partial y}{\partial x} = 2x\sin(at)$$

$$\frac{\partial y}{\partial t} = ax^2 \cos(at)$$

Differentiation of Complex Functions

Differentiation of functions involving complex numbers is done exactly the same as for real functions. For example:

$$y = j\sin(at) \Rightarrow \frac{dy}{dx} = aj\sin(ax)$$

$$y = ae^{j\omega x} \Rightarrow \frac{dy}{dx} = j\omega ae^{j\omega x}$$

Solutions to Quadratic Equations

Quadratic functions are of the form:

$$y = ax^2 + bx + c$$

A common task is to find the roots of *y*, i.e. the values for *x* for which *y=0*.

A general formula for this exists:

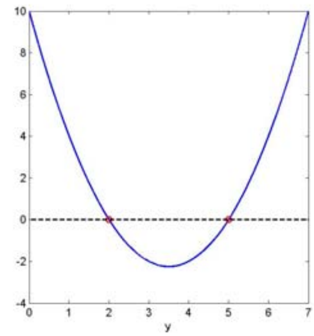
$$x_{\pm} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Hence there are always two solutions,

$$x_{+} \text{ and } x_{-}$$

Although when the expression under the square root equals zero, they are the same.

The roots are the zero-crossings of the function:



Euler Identities

From Euler's equation

$$e^{jx} = \cos(x) + j\sin(x)$$

We can derive immediately:

$$\begin{aligned} e^{jx} + e^{-jx} &= \cos(x) + j\sin(x) + \cos(x) - j\sin(x) \\ &= 2\cos(x) \end{aligned}$$

$$\begin{aligned} e^{jx} - e^{-jx} &= \cos(x) + j\sin(x) - [\cos(x) + j\sin(x)] \\ &= 2j\sin(x) \end{aligned}$$

Summary

We have reviewed:

- Trigonometry (angles, functions, relationships, inverse functions)
- Complex numbers (definitions, coordinates, arithmetic operations)
- Differentiation (first principles, standard functions, rules)

Plus some generally useful formulas.