Timestep Limitations for Generalized Rayleigh Damping in Explicit Finite Element Analysis

Engineers in U.S. industries have begun to use transient dynamic structural analysis programs of the type developed at DOE labs over the last two decades. These finite element programs use a mixed formulation and explicit time integration. As a wider class of problems is addressed with these programs, more capabilities are requested and added to them. Recently, there has been an interest in modeling problems with structural damping. At least one code (DYNA3D [1]) is now available with generalized Rayleigh damping, a linear combination of mass and stiffness proportional damping. Here we demonstrate that using stiffness proportional damping can lead to severe timestep limitations for most problems while mass proportional damping leads to no such limitations but causes extreme damping for low frequency modes (including rigid body modes).

Governing Equations

In most explicit structural dynamics finite element programs, the discretized form of the equation of motion and a central difference time integrator are combined to give the following equations:

$$[M]\{a\}^n = \{f_{\text{int}}\}^n + \{f_{ext}\}^n \tag{1}$$

$$\{v\}^{n+1} = \{v\}^n + \Delta t\{a\}^n \tag{2}$$

$$\{u\}^{n+1} = \{u\}^n + \Delta t\{v\}^{n+1} \tag{3}$$

where $[M]$ is the diagonalized mass matrix, Δt is the time step, and $\{a\}^n$, $\{v\}^n$, and $\{u\}^n$ are the nodal acceleration, velocity, and displacement vectors, respectively, at time step n. The vector $\{f_{ext}\}^n$ is the external nodal force vector at timestep n. In the homogeneous case, which is used for stability analysis, the external force vector is set to zero.

The vector $\{f_{\text{int}}\}^n$ is the internal force vector at timestep n due to the divergence of the stress. It may be expressed in a linear case as

$$\{f_{\text{int}}\}^n = -[C]\{v\}^n - [K]\{u\}^n \tag{4}$$

where $[C]$ is a global damping matrix and $[K]$ is a global stiffness matrix. In the case of stiffness and mass proportional damping and ignoring the nodal force vector, the first equation becomes

$$[M]\{a\}^n = -(\alpha[M]+\beta[K])\{v\}^n - [K]\{u\}^n \tag{5}$$

where $[C] = \alpha[M]+\beta[K]$ and α and β are nonnegative real scalar damping coefficients. Under the assumption of linearity, a modal decomposition may be done. This amounts to solving the generalized eigenvalue problem

$$[K]\{u_m\} = \omega_m^2[M]\{u_m\} \tag{6}$$

where $\{u_m\}$ is the mth mode or eigenvector and ω_m is the corresponding natural frequency (ω_m^2 is the corresponding eigenvalue).
For a given mode, m, the equations (1)-(5) may be written

$$a^n = -(\alpha + \beta\omega_m^2)v^n - \omega_m^2 u^n \tag{7}$$

$$v^{n+1} = v^n + \Delta t a^n \tag{8}$$

$$u^{n+1} = u^n + \Delta t v^{n+1} \tag{9}$$

where u, v, and a are now modal accelerations, velocities, and amplitudes, and ω_m is the natural frequency associated with them.

A more familiar form of equation (7) is

$$a^n = -2\xi_m\omega_m v^n - \omega_m^2 u^n \tag{10}$$

where ξ_m is the modal damping (fraction of critical damping) associated with the mth mode. Comparing equations (10) and (7), one can derive the following relationship between critical damping, frequency, and Rayleigh damping coefficients α and β:

$$\xi_m = \frac{\alpha}{2\omega_m} + \frac{\beta\omega_m}{2} \tag{11}$$

Equations (8)-(10) may be written as:

$$\begin{Bmatrix} v \\ u \end{Bmatrix}^{n+1} = [A]\begin{Bmatrix} v \\ u \end{Bmatrix}^{n} \tag{12}$$

where $[A]$ is the amplification matrix. More explicitly,

$$[A] = \begin{bmatrix} 1 - 2\xi_m\omega_m\Delta t & -\omega_m^2\Delta t \\ \Delta t\left(1 - 2\xi_m\omega_m\Delta t\right) & 1 - \omega_m^2\Delta t^2 \end{bmatrix} \tag{13}$$

For stability, all the eigenvalues of $[A]$ must lie within the range

$$|\lambda_i| \le 1 \tag{14}$$

This is the stability condition. For the above amplification matrix, the eigenvalues are

$$\lambda_i = \frac{2 - \omega_m^2\Delta t^2 - 2\xi_m\omega_m\Delta t \pm \omega_m\Delta t\sqrt{\left(\omega_m\Delta t + 2\xi_m\right)^2 - 4}}{2} \tag{15}$$

Stability in the Absence of Damping

If one sets the damping ratio to zero, the stability condition can be shown to become

$$\Delta t \le \frac{2}{\omega_m} \tag{16}$$

which is the familiar Courant condition. This must hold for all natural frequencies in the problem, including the maximum frequency. An estimate of the maximum frequency that is often used is the maximum of

$$\omega_{crit} = \frac{2c_{dil}}{h} \tag{17}$$

over all the elements, where c_{dil} is the dilatational wave speed in an element and h is the minimum element dimension in that element [2]. This makes the stability condition for an undamped problem

$$\Delta t \le \frac{h}{c_{dil}} \tag{18}$$

It should be noted that equation (17) gives an upper bound on the maximum system frequency. Boundary conditions and contributions from other elements will tend to lower the true maximum system frequency from this value.

Stability with Damping

If damping is present, it can be shown that equations (14) and (15) yield a stability condition that reduces the maximum stable timestep. The damped stability condition is

$$\Delta t_{damped} \le \frac{2\left(\sqrt{\xi_m^2 + 1} - \xi_m\right)}{\omega_m} \tag{19}$$

which must hold for all undamped modes (frequencies). If Δt_c is the maximum stable timestep from the Courant condition for the undamped system, then the relation between the maximum timestep for the damped system is

$$\Delta t_{damped} \le \left(\sqrt{\xi_m^2 + 1} - \xi_m\right)\Delta t_c \tag{20}$$

For high damping, the stable timestep rapidly approaches zero.

In terms of the damping coefficients α and β, the stability criterion is

$$\Delta t_{damped} \le \frac{2}{\omega_m}\left(\sqrt{\left(\frac{\alpha}{2\omega_m} + \frac{\beta\omega_m}{2}\right)^2 + 1} - \left(\frac{\alpha}{2\omega_m} + \frac{\beta\omega_m}{2}\right)\right) \tag{21}$$

At the higher frequencies (those that govern the timestep for the undamped case) the damped stable timestep is limited by the coefficient of stiffness proportional damping.

Consider the case where α is zero. In this case, the stability criterion (21) is

$$\Delta t_{damped} \le \sqrt{\beta^2 + \Delta t_c^2} - \beta \tag{22}$$

β is usually specified to provide a typical value of damping at a given structural frequency. This frequency is normally many orders of magnitude less than the maximum natural frequency of the undamped system. As a result, the damped stable timestep is usually much smaller than that of the undamped system.

Figure 1 shows the effect on stable time step. In this plot, f is the product of the undamped time step and the frequency for which the modal damping is specified.

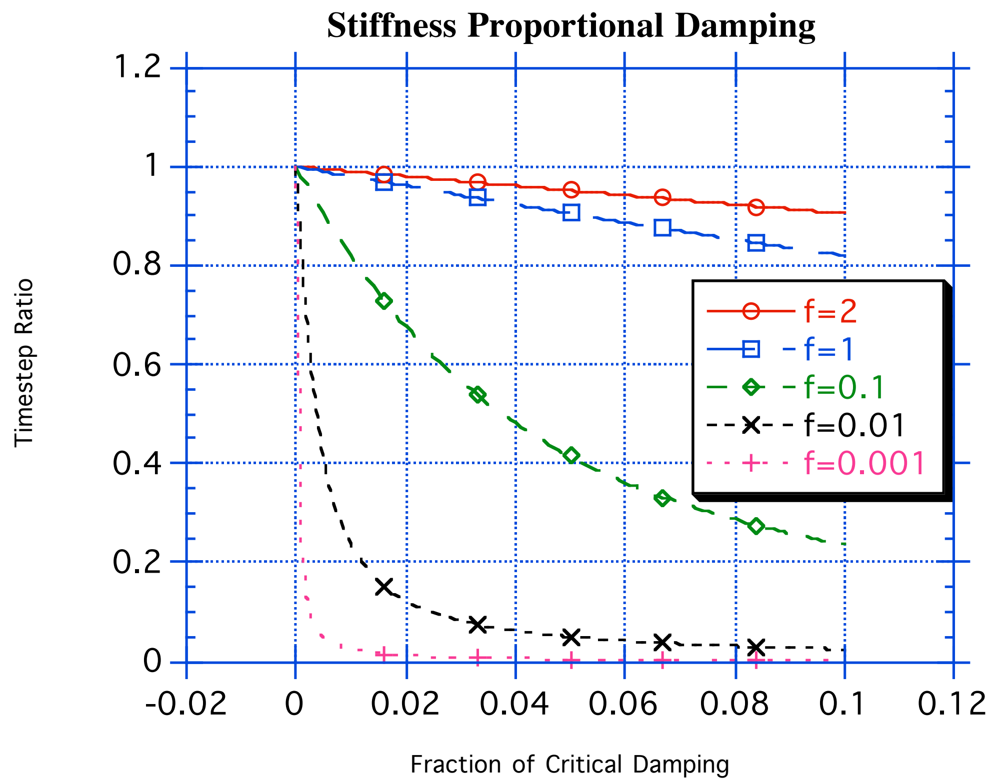

Figure 1. A plot of the timestep ratio ($\Delta t_{damped} / \Delta t_c$) vs. fraction of critical damping at $\omega_m = f / \Delta t_c$.

Now consider the case where β is zero. The stability criterion (21) is

$$\Delta t_{damped} \le \frac{2\left[\sqrt{\left(\frac{\alpha}{2\omega_m}\right)^2 + 1} - \left(\frac{\alpha}{2\omega_m}\right)\right]}{\omega_m} \qquad (23)$$

for all natural frequencies of the system.

One can show that the minimum of the right-hand side of equation (23) occurs at the maximum value of ω_m. For large ω_m, the numerator in equation (23) approaches unity, so the stability condition is approximately that for the undamped system.

One must remember that although mass proportional Rayleigh damping does not usually significantly reduce the stable timestep, it will cause excessive damping of low frequency modes. Particularly if the analysis includes rigid body accelerations, the results may be very inaccurate (the damping at $\omega_m = 0$ is infinite).

References

[1] Whirley, R.G. and Hallquist, J.O.
"DYNA3D A Nonlinear, Explicit, Three-Dimensional Finite Element Code for Solid and Structural Mechanics," Lawrence Livermore National Laboratory, Report UCRL-MA-107254, May 1991.

[2] Taylor, L.M. and Flanagan, D.P.
"PRONTO 3D A Three-Dimensional Transient Solid Dynamics Program," Sandia National Laboratories, Report SAND87-1912, December 1992.