VIBRATION OF STRINGS AND MEMBRANES

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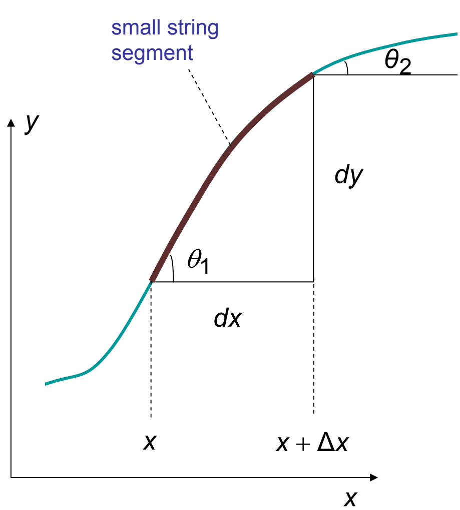
Aims

- to explore the motion of strings and membranes, through the physics.
- to get familiar with the wave equation and its solutions.
- to explore the influence of physical parameters, such as damping, string length, on the string vibrations.
- Learn about modal synthesis as a simulation method.

Content

- Transversal Vibration in Strings
- Wave Equation
- Travelling-Wave Solution
- Boundary Conditions
- String Normal Modes of Vibration
- Damping Effects
- Lossy Wave Equation
- Modal Synthesis
- Transversal Vibration in Membranes

Transversal Vibration in Ideal Strings (No Damping)



Transversal components of the Tension:

$$T_{y,x} = T\sin(\theta_1)$$
$$T_{y,x+dx} = T\sin(\theta_2)$$

Net force on the string segment:

$$F = T_{y,x+dx} - T_{y,x}$$
$$= T\sin(\theta_2) - T\sin(\theta_1)$$

Transversal Vibration in Ideal Strings (cont.)

Let's write the net force on the segment as:

$$F = T\sin(\theta_2) - T\sin(\theta_1) \\ = T\left[g(x+dx) - g(x)\right]$$

$$g(x) = \sin(\theta)_x$$

For an infinitely small *dx*, we have:

$$\frac{\partial g}{\partial x} = \frac{g(x+dx) - g(x)}{dx} \Leftrightarrow g(x+dx) - g(x) = \frac{\partial g}{\partial x}dx$$

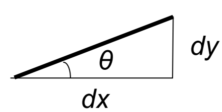
Thus the net force then equals

$$F = T\frac{\partial g}{\partial x}dx$$

Transversal Vibration in Ideal Strings (cont.)

We have the function:

$$g(x) = \sin(\theta)_x$$



For very small angles, we may write

$$\sin(\theta)_x \approx \tan(\theta)_x = \frac{dy}{dx} \approx \frac{\partial y}{\partial x}$$

After substituting that in to $F = T\frac{\partial g}{\partial x}dx$, we obtain

$$F = Tdx\frac{\partial g}{\partial x} = Tdx\frac{\partial\left(\frac{\partial y}{\partial x}\right)}{\partial x} = Tdx\frac{\partial^2 y}{\partial x^2}$$

Transversal Vibration in Ideal Strings (cont.)

Newton's second law:

$$F = m\frac{\partial^2 y}{\partial t^2} = \rho\, dx \frac{\partial^2 y}{\partial t^2} \qquad (m = \rho\, dx)$$

mass per unit length (Kg/m)

Hence the **equation of motion** is:

$$\rho\, dx \frac{\partial^2 y}{\partial t^2} = T dx \frac{\partial^2 y}{\partial x^2}$$

or:

$$\frac{\partial^2 y}{\partial t^2} = \left(\frac{T}{\rho}\right)\frac{\partial^2 y}{\partial x^2}$$

Transversal Vibration in Ideal Strings (cont.)

Dimensional analysis:

$$\frac{T}{\rho} \to \frac{\mathrm{N}}{\mathrm{Kg\,m^{-1}}} = \frac{\mathrm{Kg\,m\,s^{-2}}}{\mathrm{Kg\,m^{-1}}} = \left(\frac{\mathrm{m}}{\mathrm{s}}\right)^2$$

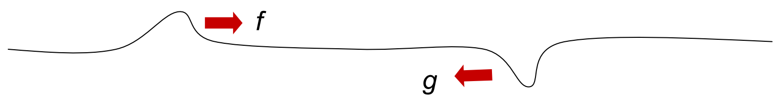
Thus $\sqrt{T/\rho}$ has the dimension of velocity. Indeed, we can define the **wave velocity**

$$c = \sqrt{T/\rho}$$

that indicates how fast waves travel through the string. The **wave equation** can now be written:

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

Travelling-Wave Solution



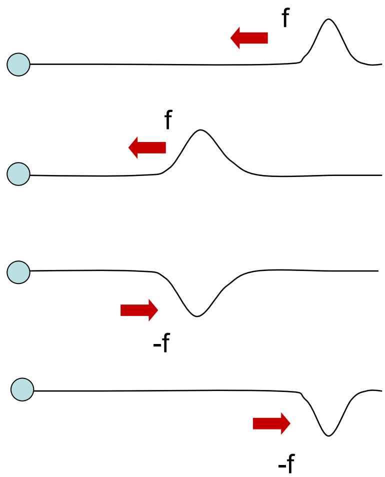
The general solution to the second order PDE is of the form of a sum of two travelling waves:

$$y(x,t) = f(ct - x) + g(ct + x)$$

Where we may also write the travelling components as a function of time:

$$y(x,t) = f(t - x/c) + g(t + x/c)$$

Fixed Boundary Conditions



If we "fix" (or "pin") the string at $x = 0$, the boundary boundary condition is:

$$y(0,t) = f(t - 0/c) + g(t + 0/c) = 0$$

Thus $g(t) + f(t) = 0$

$$\Leftrightarrow$$

$$g(t) = -f(t)$$

In other words, the wave reflects with inverted sign at the boundary. The same reflection with sign-inversion happens at the other end of the string if that is fixed as well.

Forced Vibration of an Infinite String

For a string of infinite length and driven with a sinusoidal force from the left end, there is only a wave travelling in positive direction, hence the solution takes the form:

$$y(x,t) = f(\underbrace{t - x/c}_{t'})$$

A string is a linear system, thus the resulting string vibrations will also be sinusoidal of the same frequency:

$$\begin{aligned} f(t') &= Ae^{j\omega t'} \\ &= Ae^{j\omega(t-x/c)} \\ &= Ae^{j(\omega t - kx)} \end{aligned}$$

$$k = \frac{\omega}{c}$$

wave number

Free Vibration of a Finite Length String

For a string of finite length, fixed at both ends, the general travelling-wave solution for lossless, free vibration at frequency ω takes the form:

$$y(x,t) = Ae^{j(\omega t - kx)} + Be^{j(\omega t + kx)}$$

If we then apply the boundary conditions:

$$\begin{cases} y(0,t) = 0 \\ y(L,t) = 0 \end{cases}$$

One obtains the equations:

$$\begin{cases} Ae^{j(\omega t)} + Be^{j(\omega t)} = 0 \\ Ae^{j(\omega t - kL)} + Be^{j(\omega t + kL)} = 0 \end{cases}$$

Free Vibration of a Finite Length String (cont.)

Omitting the trivial solution, this results into:

$$\begin{cases} A + B = 0 \Leftrightarrow B = -A \\ Ae^{-jkL} + Be^{jkL} = 0 \end{cases}$$

Combining these equations gives:

$$Ae^{-jkL} - Ae^{jkL} = 2Aj\sin(kL) = 0$$

Which holds for all wave numbers for which:

$$kL = n\pi$$

Where n is an integer. In other words, we have a series of harmonically related frequencies at which the string can freely vibrate:

$$f_n = \frac{\omega_n}{2\pi} = \frac{nc}{2L} \qquad n = 1,2,3,4....$$

These are the frequencies of the **normal modes** of the string.

Free Vibration of a Finite Length String (cont.)

The general solution for free vibration at one particular frequency takes the form:

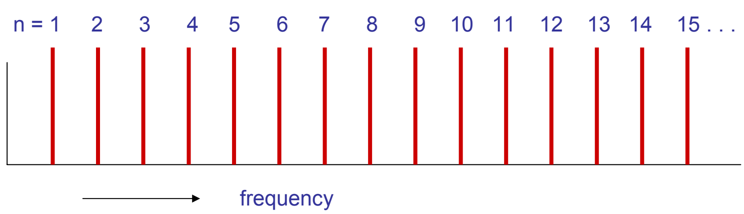
$$y_n(x,t) = A_n e^{j(\omega_n t - k_n x)} + B_n e^{j(\omega_n t + k_n x)}$$

Which, if we substitute $B = -A$ and use Euler's equations, becomes:

$$y_n(x,t) = A_n \sin(k_n x) e^{j\omega_n t}$$

In general, if a string is plucked, all modes tend to be excited.

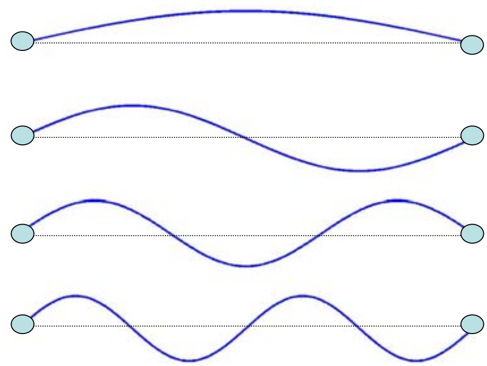
Spectrum of possible frequencies for the string to oscillate at



Normal Modes of the String

The vibration amplitude is position-dependent, and is scaled by the term:

$$\sin(k_n x) = \sin(n\pi x / L)$$



$n = 1$ $\qquad f_1 = \dfrac{c}{2L}$

$n = 2$ $\qquad f_2 = \dfrac{c}{L} = 2f_1$

$n = 3$ $\qquad f_3 = \dfrac{3c}{2L} = 3f_1$

$n = 4$ $\qquad f_4 = \dfrac{2c}{L} = 4f_1$

$x = 0$ $\qquad x = L$

Damping Effects: Air Damping

The lossless wave equation, in 'Newtonian form' is:

$$\rho\, dx \frac{\partial^2 y}{\partial t^2} = T dx \frac{\partial^2 y}{\partial x^2}$$

We may take into account the resistance that the surrounding air exerts on the vibration string by adding a damping force:

$$\rho\, dx \frac{\partial^2 y}{\partial t^2} = T dx \frac{\partial^2 y}{\partial x^2} - r_1 dx \frac{\partial y}{\partial t} \qquad \left(r_1 = \text{air damping per unit length} \right)$$

Now writing the wave equation in classical form, we obtain:

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} - 2b_1 \frac{\partial y}{\partial t} \qquad \left(b_1 = \frac{r_1}{2\rho} \right) \longleftarrow \text{air damping constant}$$

Damping Effects : Internal Friction

Similarly, we can adapt the wave equation in order to take into account internal friction effects, which takes the form:

$$\rho\, dx \frac{\partial^2 y}{\partial t^2} = T dx \frac{\partial^2 y}{\partial x^2} + r_2 dx \frac{\partial^3 y}{\partial t \partial x^2}$$

$r_2 =$ internal damping per unit length

The classical wave equation with internal friction then becomes

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} + 2b_2 \frac{\partial^3 y}{\partial t \partial x^2} \qquad \left(b_2 = \frac{r_2}{2\rho} \right)$$

Internal friction damping constant

Lossy Wave Equation (Air & Internal Damping)

Taking both effects into account, we get the lossy wave equation:

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} - 2b_1 \frac{\partial y}{\partial t} + 2b_2 \frac{\partial^3 y}{\partial t \partial x^2}$$

Let's 'try' a solution of the form:

$$y = e^{st} e^{\pm jkx}$$

where we expect that, just as with the harmonic oscillator, s will contain a real part, representing exponential decay, i.e.

$$s = j\omega - \alpha$$

Lossy String: Free Vibration Solutions

Given a solution for free vibration of the form:

$$y = e^{st} e^{\pm jkx}$$

The relevant derivatives are:

$$\frac{\partial y}{\partial t} = s e^{st} e^{\pm jkx} = sy \qquad \frac{\partial y}{\partial x} = jk e^{st} e^{\pm jkx} = \pm jky$$

$$\frac{\partial^2 y}{\partial t^2} = s^2 e^{st} e^{\pm jkx} = s^2 y \qquad \frac{\partial^2 y}{\partial x^2} = j^2 k^2 e^{st} e^{\pm jkx} = -k^2 y$$

$$\frac{\partial^3 y}{\partial t \partial x^2} = \frac{\partial\left(\dfrac{\partial^2 y}{\partial x^2}\right)}{\partial t} = \frac{\partial\left(-k^2 e^{st} e^{\pm jkx}\right)}{\partial t} = -k^2 s\, e^{st} e^{\pm jkx} = -k^2 sy$$

Lossy String: Free Vibration Solutions (cont.)

Substitution of all the derivatives into the lossy wave equation gives:

$$s^2 y = -k^2 c^2 y - 2b_1 sy - 2b_2 k^2 sy$$

or, omitting the trivial solution *y = 0* :

$$s^2 + 2\left(b_1 + b_2 k^2\right)s + k^2 c^2 = 0$$

so that

$$\begin{aligned} s_k &= \frac{-2\alpha_k \pm \sqrt{4\alpha_k^2 - 4k^2 c^2}}{2} \\ &= -\alpha_k \pm j\sqrt{k^2 c^2 - \alpha_k^2} \\ &= -\alpha_k \pm j\omega_k \end{aligned} \qquad \left(\begin{aligned} \alpha_k &= b_1 + b_2 k^2 \\ \omega_k &= \sqrt{k^2 c^2 - \alpha_k^2} \end{aligned}\right)$$

Lossy String: Free Vibration Solutions (cont.)

So we have the expected solutions of the form:

$$y_k = e^{s_k t} e^{\pm jkx}$$

Where

$$s_k = \pm j\omega_k - \alpha_k \quad \text{with} \quad \begin{aligned} \alpha_k &= b_1 + b_2 k^2 \\ \omega_k &= \sqrt{k^2 c^2 - \alpha_k^2} \end{aligned}$$

Need to know the wave number to compute solutions -> this follows from imposing boundary conditions.

Lossy String: Free Vibration Solutions (cont.)

The general solution takes the form:

$$y_k(x,t) = \left[A e^{s_k t - jkx} + B e^{s_k t + jkx} \right] e^{-\alpha_k t}$$

Imposing the (fixed, fixed) boundary conditions

$$\begin{cases} y(0,t) = 0 \\ y(L,t) = 0 \end{cases}$$

gives

$$\begin{cases} \left[A e^{s_k t} + B e^{s_k t} \right] e^{-\alpha_k t} = 0 \\ \left[A e^{s_k t - jkL} + B e^{s_k t + jkL} \right] e^{-\alpha_k t} = 0 \end{cases}$$

Lossy String: Free Vibration Solutions (cont.)

Since the exponential cannot be zero for finite t, this becomes:

$$\begin{cases} Ae^{s_k t} + Be^{s_k t} = 0 \\ Ae^{s_k t - jkL} + Be^{s_k t + jkL} = 0 \end{cases}$$

Which leads again to

$$\sin(kL) = 0$$

Hence we have again a series of wave numbers at which the string can freely vibrate:

$$k_n = \frac{n\pi}{L} \qquad\qquad n = 1,2,3,4,5....$$

Lossy String: Free Vibration Solutions (cont.)

Hence we can write the possible free-vibration solutions (= modes of vibration) in terms of the mode number n :

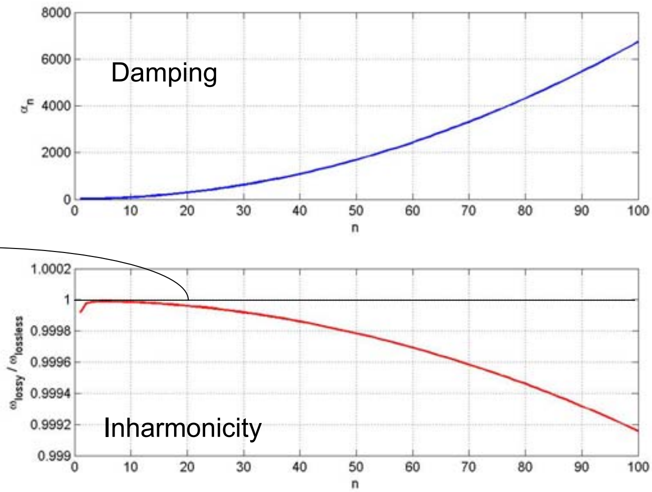
$$y_n(x,t) = A_n \underbrace{\sin(k_n x)}_{\text{position-dependent amplitude scaling}} \underbrace{e^{-\alpha_n t}}_{\text{exponential damping}} \underbrace{e^{j\omega_n t}}_{\text{sinusoidal oscillation}}$$

Where $\quad \alpha_n = b_1 + b_2 k_n^2 \quad \omega_n = \sqrt{k_n^2 c^2 - \alpha_n^2} \quad$ and $\quad k_n = \dfrac{n\pi}{L}$

Modal Frequency and Damping

Example:

L = 0.6;
c = 329
b1 = 20.1
b2 = 2.7e-2



purely harmonically related partials)

Damping introduces slight inharmonicity.

External Force Excitation

The string can be excited by adding a force density term:

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} - 2b_1 \frac{\partial y}{\partial t} + 2b_2 \frac{\partial^3 y}{\partial t \partial x^2} + F(x,t)$$

It can be shown that if we excite the string at $x=x_e$, the resulting motion up at $x = x_p$ is:

$$y(x_p,t) = \sum_{n=1}^{N} \sin\left(\frac{n\pi x_e}{L}\right) \sin\left(\frac{n\pi x_p}{L}\right) F(x_e,t) * h_n(t)$$

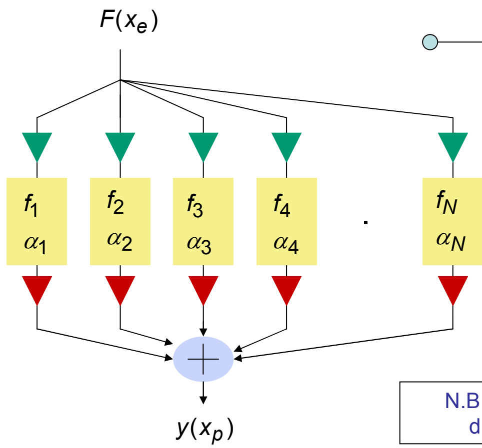
Where we have the individual ‘modal oscillators’ impulse responses:

$$h_n(t) = \left(\frac{e^{-\alpha_n t}}{M\omega_n}\right) \sin(\omega_n t) \qquad \underbrace{M_n = \frac{\sigma L}{2}}_{\text{"modal mass"}}$$

take ω_n out for auralisation, scales better to ‘heard’ air pressure

Modal Synthesis

This can be used for simulation on basis of summing modal contributions, i.e. “modal synthesis”:



modal weights (excitation) $\sin\left(\frac{n\pi x_e}{L}\right)$

modal oscillators $h_n(t)=\left(\frac{e^{-\alpha_n t}}{m\omega_n}\right)\sin(\omega_n t)$

modal weights (pick-up) $\sin\left(\frac{n\pi x_p}{L}\right)$

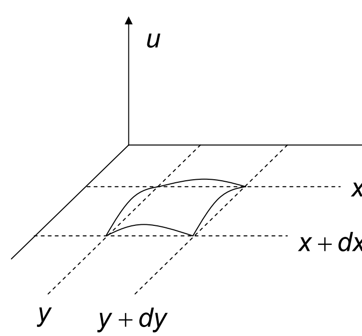
N.B.: implement modal oscillators as discretised harmonic oscillators

2D Vibration: Rectangular Membrane

The (transversal) forces on the membrane surface element *dxdy* are:

$$F_x = -T dx \frac{\partial^2 u}{\partial x^2} dx$$

$$F_y = -T dy \frac{\partial^2 u}{\partial y^2} dx$$



Now aplpying Newton’s 2nd law yields:

$$F = F_x + F_y = \underbrace{(\sigma\, dx\, dy)}_{mass} \cdot \underbrace{\frac{\partial^2 u}{\partial t^2}}_{accelaration}$$

Rectangular Membrane: Equation of Motion

Equating force with mass times acceleration gave

$$T\,dx\,dy\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=(\sigma\,dx\,dy)\frac{\partial^2 u}{\partial t^2}$$

From which we obtain the equation of motion:

$$\frac{\partial^2 u}{\partial t^2}=\left(\frac{T}{\rho}\right)\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=c^2\nabla u$$

Where c is the wave speed:

$$c=\sqrt{T/\rho}$$

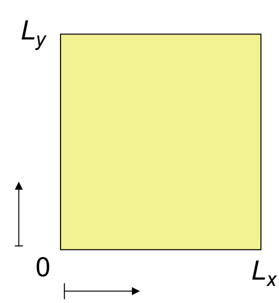
Rectangular Membrane with Fixed Ends: Free Vibrations

Assume solution of a factorised form:

$$u(x,y,z)=X(x)\,Y(y)\,e^{j\omega t}$$

Impose boundary conditions

$$u=0 \;\text{ at }\begin{cases}x=0\\ y=0\\ x=L_x\\ y=L_y\end{cases}$$



This leads to solutions of the form:

$$u_{nm}(x,y,t)=A\sin\left(\frac{m\pi x}{L_x}\right)\sin\left(\frac{n\pi y}{L_y}\right)e^{j\omega_{mn}t}$$

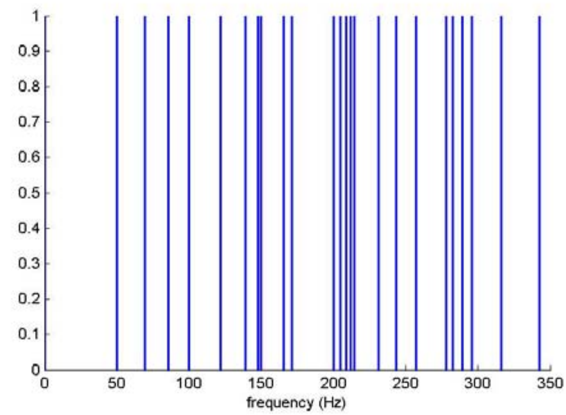
with modal frequencies:

$$\omega_{mn}=\pi\,c\sqrt{\frac{m^2}{L_x^2}+\frac{n^2}{L_y^2}}$$

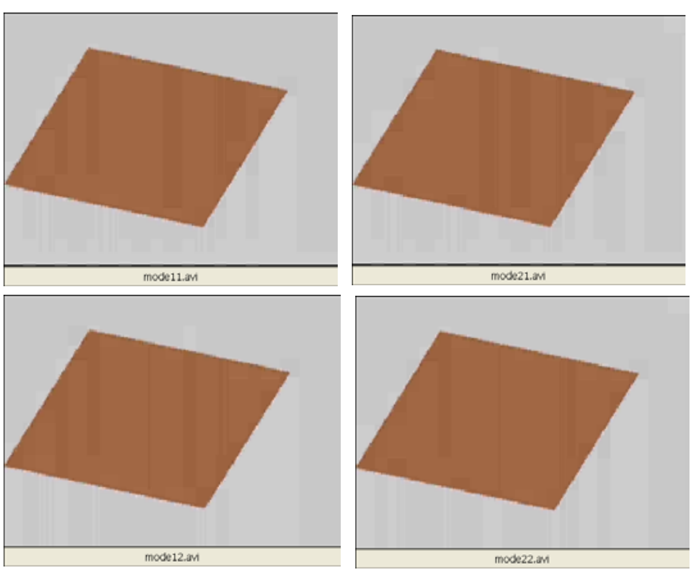
Rectangular Membrane: Modal Frequencies

$$\omega_{mn} = \pi c \sqrt{\frac{m^2}{L_x^2} + \frac{n^2}{L_y^2}}$$

- spectrum is inharmonic
- modal density increases with frequency



Rectangular Membrane: Modes of Vibration



Free, Damped Membrane Vibrations

Damping can be incorporated in a way similar to with the string, leading to so-called "proportional damping":

$$u_{nm}(x,y,t) = A\sin\left(\frac{m\pi x}{L_x}\right)\sin\left(\frac{n\pi y}{L_y}\right)e^{-\alpha_{mn}t}e^{j\omega_{mn}t}$$

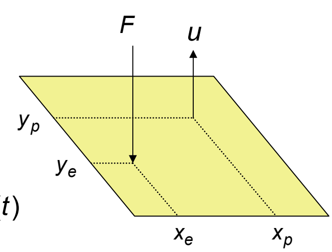
where

$$\alpha_{mn} = b_1 + b_2 k_{mn}^2$$

$$\omega_{mn} = \sqrt{k_{mn}^2 c^2 - \alpha_{mn}^2}$$

Modal Synthesis

As with the string, we can show that the output is a sum of modal contributions:



$$u(x_p,t) = \sum_{m=1}^{M}\sum_{n=1}^{N} A(x_e,y_e,x_p,y_p)\,F(x_e,t) * h_{mn}(t)$$

$$A(x_e,y_e,x_p,y_p) = \sin\left(\frac{n\pi x_e}{L_x}\right)\sin\left(\frac{n\pi y_e}{L_y}\right)\sin\left(\frac{n\pi x_p}{L_x}\right)\sin\left(\frac{n\pi y_p}{L_y}\right)$$

$$h_{mn}(t) = \left(\frac{e^{-\alpha_{mn}t}}{M_{mn}\omega_{mn}}\right)\sin(\omega_{mn}t) \qquad M_{mn} = \frac{\sigma L_x L_y}{4}$$

take ω_n out for auralisation, scales better to 'heard' air pressure

Summary

- Learned how the wave equation is constructed from considering transversal tension forces strings and membranes.
- Understood how damping forces can be added to the wave equation in order to take into account air damped and internal friction.
- Learned how to apply simple boundary conditions
- Understood that the motion at any point along the string is the sum of two travelling waves.
- Learned how to calculate the modes of vibration (frequencies and damping) of string and membranes.
- Learned about modal synthesis.