Uncertainty Quantification of Nonlinear Stiffness Coefficients in Non-Intrusive Reduced Order Models

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ABSTRACT

Reduced order modeling of structures for geometric nonlinear vibration has been an active research subject due to its advantage of reducing the computational cost associated with using traditional numerical methods. One type of reduced order models are the non-intrusive ones, for which the ROM is built from data obtained from commercial finite element software. Their advantage is the capability to handle a broad set of complex geometries and boundary conditions experienced in practice. Due to inconsistencies between the ROM and commercial software formulations, however, the constructed ROM cannot be an exact match of the finite element model. Accordingly, epistemic uncertainty exists in the ROM modeling beside issues of truncation of the number of modes.

In the present study, the epistemic uncertainty associated with nonlinear stiffness coefficients of the non-intrusive ROM is quantified using a clamped-clamped straight beam as a demonstration example. A Monte Carlo simulation strategy is used first. Random samples of the optimal displacement level at which the identification is performed are generated with their probability distribution consistent with those used in the construction of the deterministic ROM. Random ROM samples are then identified at these random optimal levels, and the corresponding predicted response for a particular dynamic excitation are computed. Uncertainty bands on these predictions are then calculated to quantify the uncertainty of the ROM predictions. Furthermore, the nonparametric stochastic approach is considered as an alternative strategy to generate these uncertainty bands. The mean ROM used is the one identified by the multiple-level identification method, and the dispersion parameter (a measure of uncertainty level in the approach) of the corresponding stochastic ROM is determined by the maximum likelihood principle. A number of random samples of the mean ROM are generated, and their predictions of the same dynamic validation cases are computed, from which the uncertainty bands of the predictions are obtained. The uncertainty band results are shown to match the Monte Carlo simulation results very well.

Keywords: uncertainty quantification, nonparametric stochastic approach, nonlinear stiffness coefficients, non-intrusive reduced order modeling

INTRODUCTION

Reduced order modeling of structures for geometric nonlinear vibration has been an active research subject due to its advantage of reducing the computational cost associated with using traditional numerical methods [1-14]. One particular class of applications are hypersonic aircraft structures in extreme aerodynamic-thermal environments, e.g., large aerodynamic, thermal, and mechanical loadings with nonlinear interactions [15-17].

One type of reduced order models are the non-intrusive ones, for which the ROM is built from data obtained from commercial finite element software (e.g., Nastran, Abaqus). Their advantage is the capability to handle a broad set of complex geometries and boundary conditions experienced in practice. Such reduced order models have been constructed for a large number of structures, and their strong potentials have been demonstrated. The details can be found in a recent review [18].

In a non-intrusive ROM [7], a modal basis is constructed to represent a given nonlinear displacement, $\underline{u}(\underline{X},t)$,

$$\underline{u}(\underline{X},t) = \sum_{n=1}^{N} q_n(t)\, \underline{\phi}^{(n)}(\underline{X}), \tag{1}$$

where $\underline{u}(\underline{X},t)$ denotes the vector of physical displacements defined on the finite element degrees of freedom. $\underline{\phi}^{(n)}$ are constant basis functions and $q_n(t)$ are the time dependent generalized coordinates.

The governing equation of the ROM is derived from the elasticity theory by the Galerkin approach, expressed as

$$M_{ij}\ddot{q}_j + D_{ij}\dot{q}_j + K_{ij}^{(1)}\, q_j + K_{ijl}^{(2)} q_j q_l + K_{ijlp}^{(3)} q_j q_l q_p = F_i \,, \tag{2}$$

where M_{ij} denotes the elements of the mass matrix, $K_{ij}^{(1)}$, $K_{ijl}^{(2)}$, $K_{ijlp}^{(3)}$ are the linear, quadratic, and cubic stiffness coefficients, and F_i are the modal forces. The viscous damping matrix { D_{ij} } is added to collectively represent various dissipation mechanisms following standard practice.

The above formulation of the ROM is not necessarily the same as the commercial finite element software, hence the ROM built upon the data from the commercial software may show some variation from the finite element model even if a large number of modes are taken. One example is the nonlinear stiffness coefficients. In a non-intrusive ROM, they are identified from either nonlinear forces or tangent stiffness matrices corresponding to enforced displacements [19]. The stiffness coefficients are supposed to be independent of the displacement level enforced, but this is not the case, and a multi-level identification method has been developed to find the optimal displacement level for each nonlinear stiffness coefficient [20]. The optimal displacement levels show variability from one coefficient to another, thus the stiffness coefficients identified at these displacement levels. This variation can be considered as an epistemic uncertainty, that is, uncertainty due to the modeling.

In the present study, this epistemic uncertainty associated with the nonlinear stiffness coefficients is quantified. The Monte Carlo simulation is firstly carried out as follows. Random samples of optimal displacement levels will be generated, whose probability distribution is consistent with that observed in the multi-level identification. Random samples of nonlinear stiffness coefficients will be identified at these displacement-level samples, and structural responses to a dynamic load are computed. The uncertainty bands of the responses are then computed to give quantitative measure of the epistemic uncertainty. Afterwards, the nonparametric stochastic approach [21-24], which uses a dispersion parameter determined by a small number of samples from the Monte Carlo simulation to directly generate the large number of random samples, is employed to repeat the same uncertainty quantification and compared to the Monte Carlo simulation results.

QUANTIFICATION OF EPISTEMIC UNCERTAINTY: MONTE CARLO SIMULATION

As a demonstration example, a clamped-clamped straight beam is considered. The beam is of rectangular cross section and its geometric and material properties of the straight beam are given in Table 1. A finite element model of the beam was constructed with Nastran using 40 CBEAM elements.

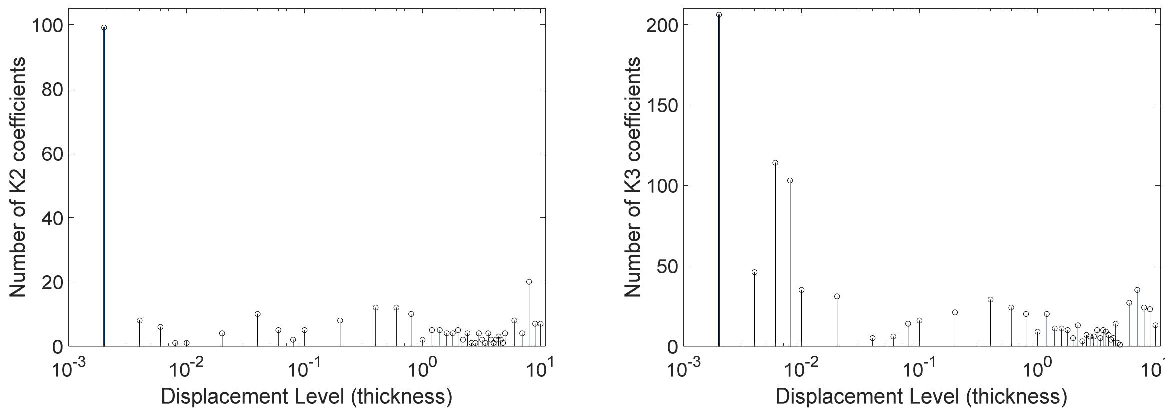
Table 1. Beam Properties

Beam Length	0.2286 m
Cross-section Width	0.0127 m
Cross-section Thickness	7.75 10^{-4} m
Mass per unit length	2763 kg/m^3
Young's Modulus	73,000 MPa
Shear Modulus	27,700 MPa

A nonlinear ROM of the beam has been constructed, including the first 4 symmetric linear modes and 4 associated duals (ROM4L4D). The stiffness coefficients were identified by the multi-level identification method [3]. Static and dynamic validations were carried out, and good matching between the ROM and the Nastran results is obtained (the results will be shown later along with the uncertainty quantification results).

Nevertheless, there is still some discrepancy, especially for the dynamic results at higher load levels. One observation from the multi-level identification results is that the optimal displacement level is not the same for all the coefficients. This is not consistent with the ROM formulation and suggests that epistemic uncertainty exist in the data used in the identification, which leads to the variation of the optimal displacement level. The uncertainty propagates further to affect the identified stiffness coefficients and eventually the ROM predictions.

This epistemic uncertainty can be seen from the distribution of the optimal displacement levels for the quadratic and the cubic coefficients, respectively, as shown in Fig.1. For the quadratic coefficients, the optimal displacement levels are concentrated at a low displacement level but there are a small number of coefficients whose optimal displacement levels distribute widely. For the cubic coefficients, the optimal displacement levels are concentrated at a range of higher displacement level while there are a number of coefficients whose optimal displacement levels distribute widely.



(a) (b)

Figure 1. Distribution of optimal displacement levels for the quadratic and the cubic coefficients. (a) Quadratic coefficients; (b) Cubic coefficients.

The cubic coefficients whose optimal displacement levels are at the highest displacement level are found to be the coefficients which are small and negligible thus difficult to be identified with high accuracy. This is indicated by the distribution of the optimal displacement levels for the remained cubic coefficients after “cleaning up” the ROM, that is, zeroing out those negligible coefficients, as shown in Fig.2. The distribution for the quadratic coefficients essentially does not change.

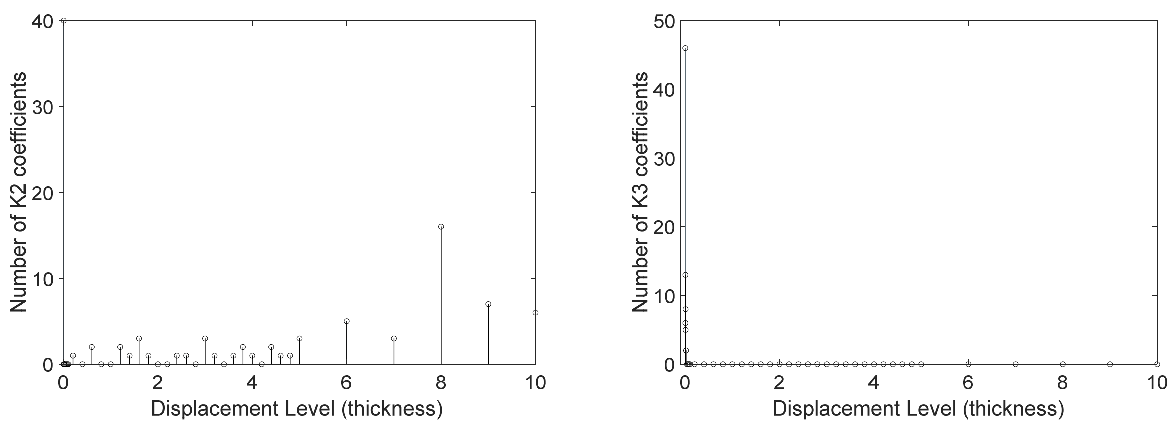


Figure 2. Distribution of optimal displacement levels for the remained quadratic and the cubic coefficients after “cleaning up” the ROM. (a) Quadratic coefficients; (b) Cubic coefficients.

In the present study, this epistemic uncertainty is firstly quantified using the Monte Carlo simulation:

(1) Two sets of random samples of optimal displacement levels are generated for the quadratic and the cubic coefficients, respectively, termed as quadratic and cubic random samples thereafter. The statistic distribution features of the two sets of random samples are made consistent with the cumulative distribution functions of the original data by using the inverse transform sampling method. The cumulative distribution functions of the original data and the generated random samples are shown in Fig.3, and it can be seen that the random samples represent the original data quite well.

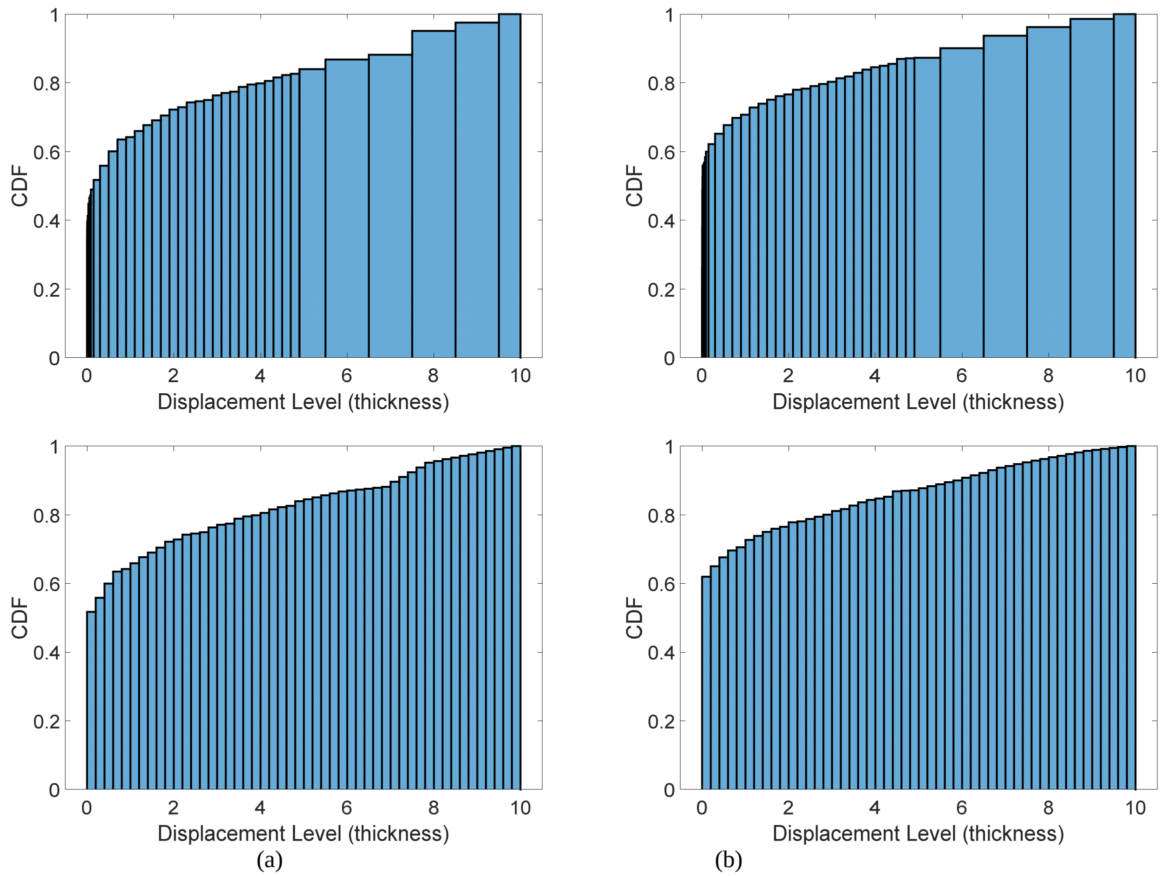
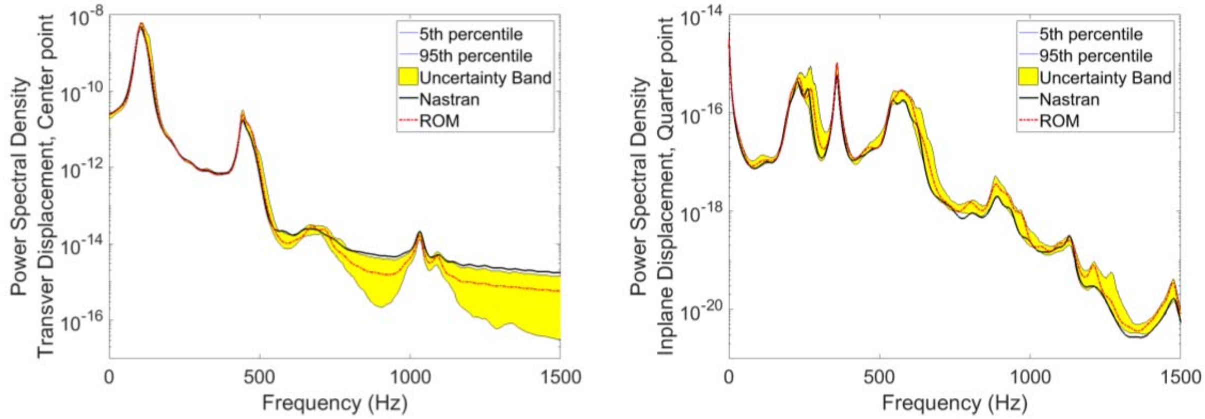


Figure 3. Cumulative distribution functions of optimal displacement levels for the quadratic and the cubic coefficients. Top: original data from multi-level identification. Bottom: random samples generated by the inverse transform sampling method. (a) Quadratic coefficients; (b) Cubic coefficients.

(2) 100 random samples are then taken. For each random sample (a displacement level) in the two sets, the stiffness coefficients are identified by the regular single-level identification method. The quadratic coefficients identified using a quadratic random sample and the cubic coefficients identified from a cubic random sample are combined and considered as a random sample of stiffness coefficients.

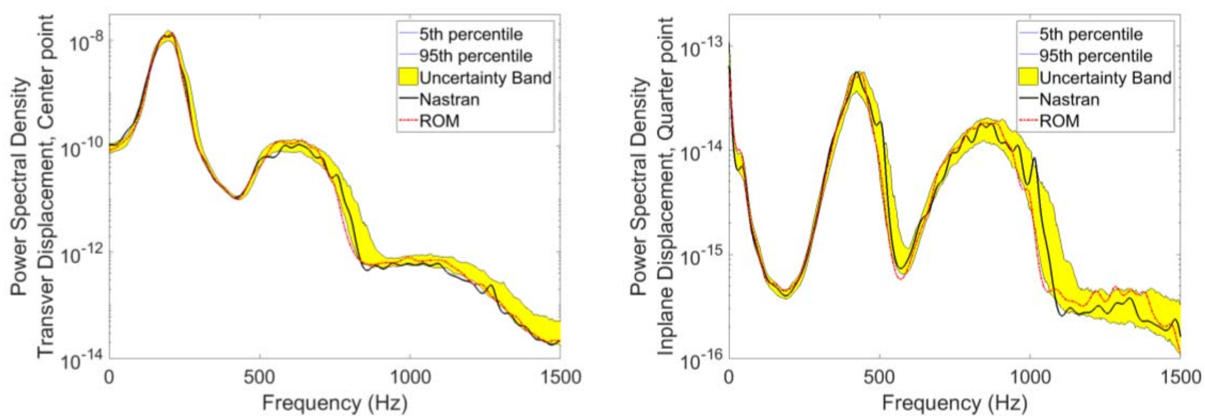
(3) Using the set of random samples of stiffness coefficients obtained in (2), a Monte Carlo simulation is carried out to compute the responses of the random samples to the static and dynamic loads used in the validation. The uncertainty bands are computed from the response data.

In Figs. 4 and 5 are shown the uncertainty band results, along with the validation results of ROM versus Nastran, for a dynamic load case at two load levels, respectively. The dynamic load case is a uniformly distributed force along the span, time-variant as modeled by a white noise with the cut-off frequency of 1000Hz. Firstly, the ROM predictions match Nastran results at dominant frequency peaks very well. Secondly, the uncertainty band encloses Nastran PSD curve in almost the whole range of frequency 0 to 1500 Hz, suggesting the epistemic uncertainty is properly quantified.



(a) (b)

Figure 4. Uncertainty band of the ROM predictions for the dynamic load case. Load level 130dB (OASPL) which gives standard deviation of center transverse displacement at about 0.75 thicknesses. (a) Transverse displacement, center point; (b) Inplane displacement, quarter point.



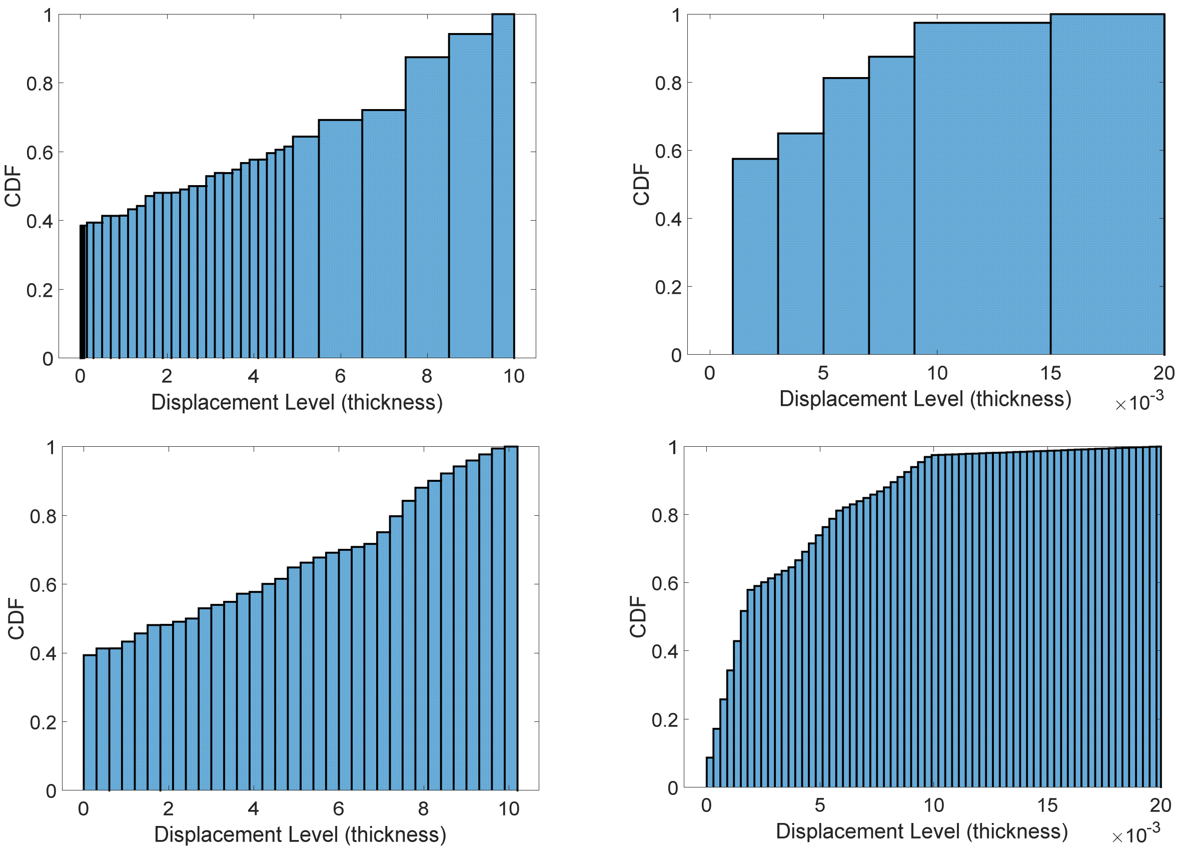
(a) (b)

Figure 5. Uncertainty band of the ROM predictions for the dynamic load case. Load level 145dB (OASPL) which gives standard deviation of center transverse displacement at about 1.9 thicknesses. (a) Transverse displacement, center point; (b) Inplane displacement, quarter point.

Uncertainty band of "cleaned" ROM

The above uncertainty quantification procedure is then applied to the cleaned ROM, to further understand the behavior of epistemic uncertainty.

The sets of quadratic and cubic random samples are generated according to the cumulative distribution functions of the data of the remained coefficients after cleaning up, using the inverse transform sampling method. The cumulative distribution functions of the cleaned data and the generated random samples are shown in Fig.6. A zoomed view is shown for the cubic coefficients since its CDF reaches 1 at a small value. It can be seen that the random samples represent the data quite well.

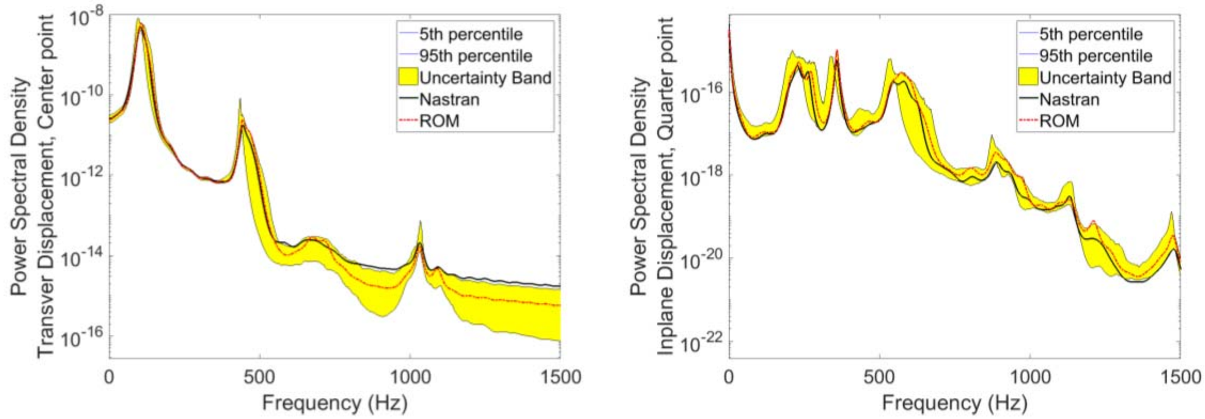


(a) (b)

Figure 6. Cumulative distribution functions of optimal displacement levels for the quadratic and the cubic coefficients remained after cleaning up. Top: cleaned data from multi-level identification. Bottom: random samples generated by the inverse transform sampling method. (a) Quadratic coefficients; (b) Cubic coefficients, zoomed view.

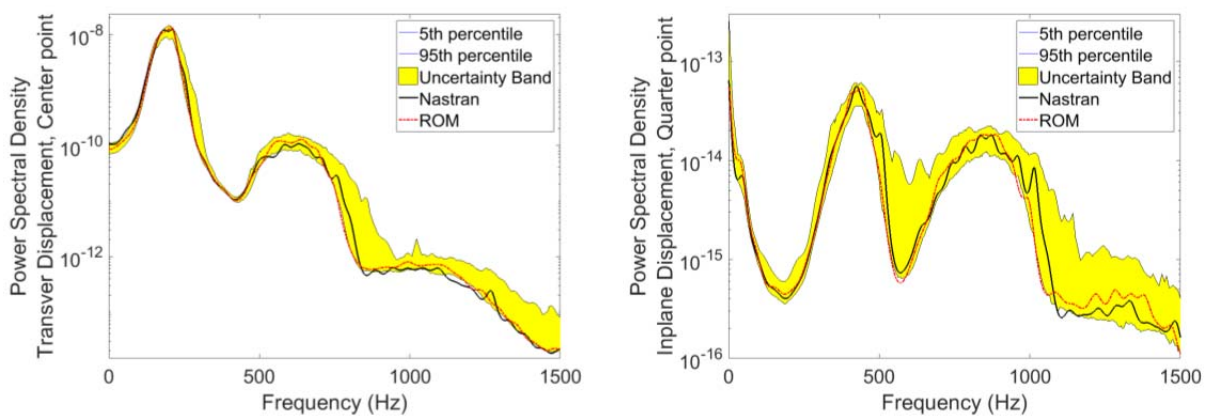
Similar to the uncleaned ROM, random samples of stiffness coefficients are identified at displacement levels given by the above random samples. Again the quadratic coefficients identified using a quadratic random sample and the cubic coefficients identified from a cubic random sample are combined and considered as a random sample of stiffness coefficients. Using the set of random samples of stiffness coefficients obtained, a Monte Carlo simulation is carried out to compute the responses of the random samples to the dynamic loads used in the validation. The uncertainty bands are computed from the response data.

In Figs. 7 and 8 are shown the uncertainty band results for the save dynamic load case at two load levels, respectively. Compared to the uncleaned ROM, the uncertainty band of the cleaned ROM becomes broader. The coefficients remained after cleaning up are usually considered better identified, so the broader uncertainty band implies that the ROM has larger variation than what the uncleaned ROM has shown.



(a) (b)

Figure 7. Uncertainty band of the cleaned ROM predictions for the dynamic load case. Load level 130dB (OASPL) which gives standard deviation of center transverse displacement at about 0.75 thicknesses. (a) Transverse displacement, center point; (b) Inplane displacement, quarter point.



(a) (b)

Figure 8. Uncertainty band of the cleaned ROM predictions for the dynamic load case. Load level 145dB (OASPL) which gives standard deviation of center transverse displacement at about 1.9 thicknesses. (a) Transverse displacement, center point; (b) Inplane displacement, quarter point.

QUANTIFICATION OF EPISTEMIC UNCERTAINTY: NONPARAMETRIC APPROACH

As shown above, in order to evaluate the uncertainty band, a number of Monte Carlo simulations with the ROM have to be done. This requires the identification of random ROM samples, which could be time consuming when the number of samples is large.

The nonparametric approach [21-23] has been developed to directly generate random ROM samples. It can be accomplished using the maximum entropy approach as demonstrated in [24] for an elastic nonlinear ROM. A key observation is that the linear, quadratic, and cubic stiffnesses cannot be varied independently, rather a bigger matrix

$$\underline{\underline{K}}_B = \begin{bmatrix} \underline{\underline{K}}^{(1)} & \underline{\underline{\tilde{K}}}^{(2)} \\ \underline{\underline{\tilde{K}}}^{(2)T} & 2\underline{\underline{\tilde{K}}}^{(3)} \end{bmatrix} \tag{3}$$

must remain positive definite for all realizations. In this equation, the matrices $\underline{\underline{\tilde{K}}}^{(2)}$ and $\underline{\underline{\tilde{K}}}^{(3)}$ are obtained by a reshaping of the quadratic and cubic stiffness tensors $K_{mnl}^{(2)}$ and $K_{mnlp}^{(3)}$, see [10]. This property was instrumental in the uncertainty modeling which proceeds from the matrix $\underline{\underline{K}}_B$ of the mean model first decomposed as (e.g., Cholesky decomposition)

$$\underline{\underline{K}}_B = \overline{\underline{\underline{L}}}_K \, \overline{\underline{\underline{L}}}_K^T \, . \tag{4}$$

Next, lower triangular matrices $\breve{\underline{\underline{H}}}_K$ are generated according to the following, see Fig. 1:

(i) the elements $\breve{H}_{il}$, $i>l$, are all independent of each other and independent of the elements $\breve{H}_{ii}$. Further, they are normally distributed with mean 0 and standard deviation $\sigma_{il} = 1/\sqrt{2\mu_{ii}}$.

(ii) the elements $\breve{H}_{ii}$ are all independent of each other and can be expressed as

$$\breve{H}_{ii} = \sqrt{\frac{\breve{Y}_{ii}}{\mu_{ii}}} \, , \tag{5}$$

where $\breve{Y}_{ii}$ are Gamma random variables, and μ_{ii} is given by

$$\mu_{ii} = \frac{\bar{n} + 2\lambda - 1}{2} \, , \tag{6}$$

where $\bar{n}$ is the size of the matrix $\underline{\underline{K}}_B$ and λ is the free parameter of the distribution which can be used to specify a level of uncertainty on $\underline{\underline{K}}_B$. Finally, random $\underline{\underline{K}}_B$ matrices can be obtained as

$$\breve{\underline{\underline{K}}}_B = \overline{\underline{\underline{L}}}_K \, \breve{\underline{\underline{H}}}_K \, \breve{\underline{\underline{H}}}_K^T \, \overline{\underline{\underline{L}}}_K^T \, . \tag{7}$$

from which random linear, quadratic, and cubic stiffness parameters can be extracted given the form of $\underline{\underline{K}}_B$.

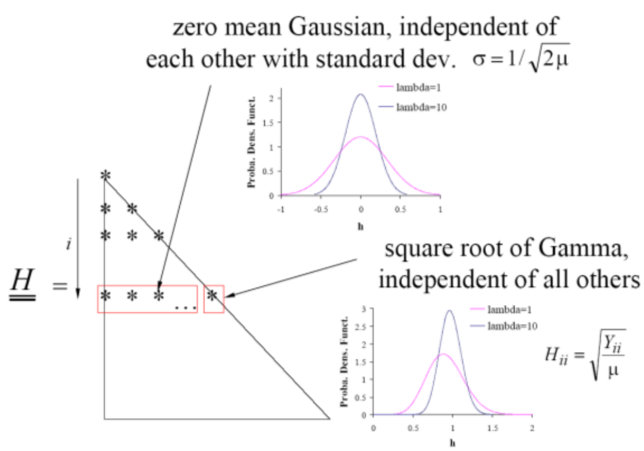


Figure 9. Structure of the random $\breve{\underline{\underline{H}}}_K$ matrices with $\bar{n} = 8$, i =2, and λ_0=1 and 10.

In the nonparametric approach, the parameter λ needs to be determined. In practice, a dispersion parameter δ is usually used, and its relation to λ is given by

$$\delta^2 = \frac{\bar{n} + 1}{\bar{n} + 2\lambda - 1} \, . \tag{8}$$

The parameter δ can be determined using the maximum likelihood principle. To this end, a set of δ values are taken. For each of them, 100 $\underline{\underline{K}}_B$ samples are generated, and the PSD of the ROM responses to the dynamic load at 145dB are computed. For each δ value, the likelihood function of the PSD results from these $\underline{\underline{K}}_B$ samples is evaluated with respect to 10 samples from the Monte Carlo simulation as the true observations. The log of the likelihood function value as function of δ is shown in Fig. 10, from which $\delta = 0.04$ is the optimal value for the nonparametric model.

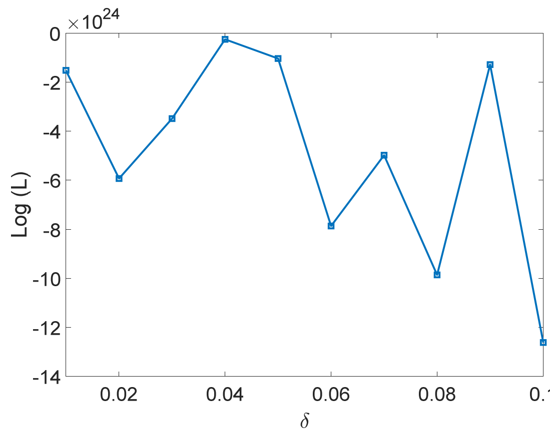
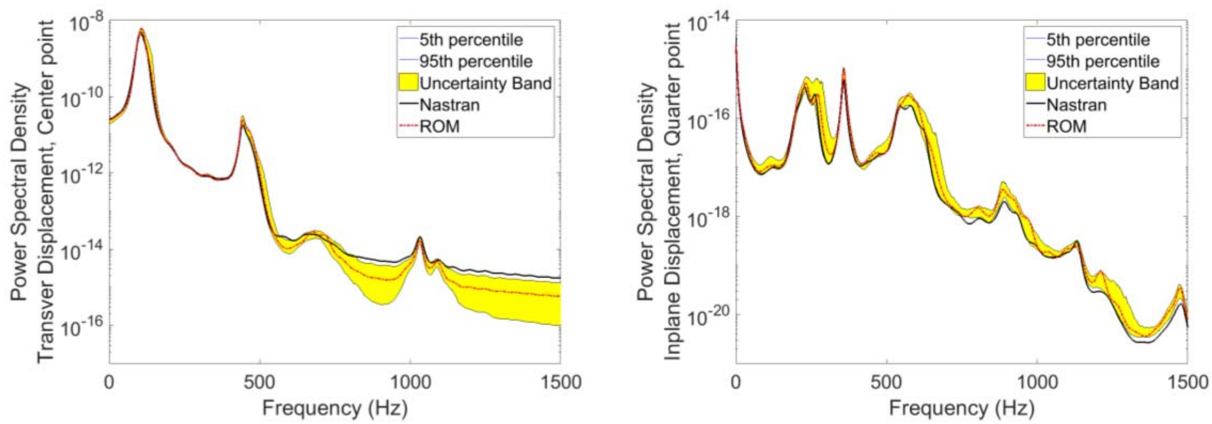


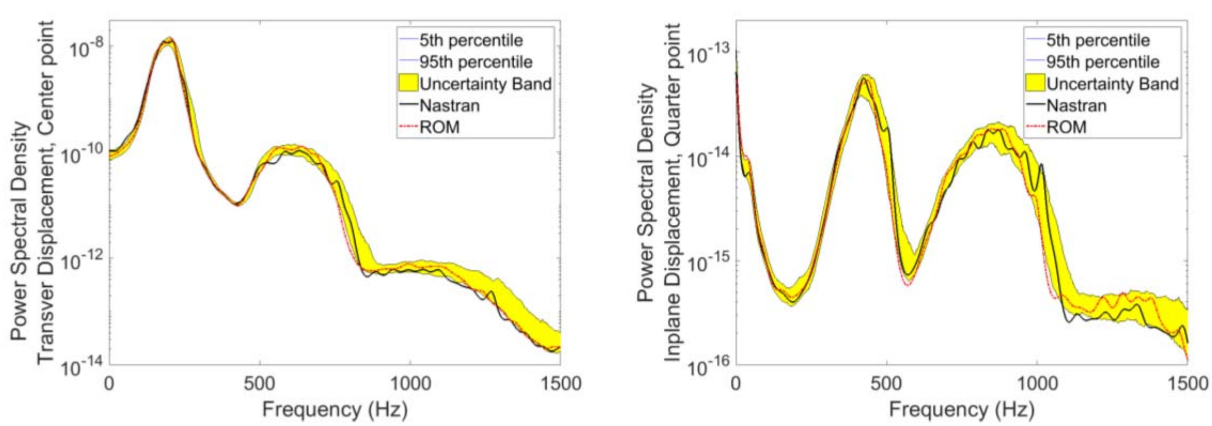
Figure 10. Log of likelihood function value as function of δ.

The 100 $\underline{\underline{K}}_B$ samples corresponding to δ = 0.04 are used to compute the responses to the dynamic load at 130dB and 145dB, then the uncertainty band of the PSD result is computed and shown in Figs. 10 and 11, respectively. The uncertainty band results from the nonparametric approach are almost the same as the results from the Monte Carlo simulation.



(b) (b)

Figure 10. Uncertainty band of the ROM predictions by the nonparametric approach. Dynamic load level 130dB (OASPL) which gives standard deviation of center transverse displacement at about 0.75 thicknesses. (a) Transverse displacement, center point; (b) Inplane displacement, quarter point.



(a) (b)

Figure 11. Uncertainty band of the ROM predictions by the nonparametric approach. Dynamic load level 145dB (OASPL) which gives standard deviation of center transverse displacement at about 1.9 thicknesses. (a) Transverse displacement, center point; (b) Inplane displacement, quarter point.

CONCLUSION

The epistemic uncertainty associated with the nonlinear stiffness coefficients of the reduced order model (ROM) as evaluated by the multi-level identification method is quantified using both the Monte Carlo simulation and the nonparametric approach. A clamped-clamped straight beam is used as an example for demonstration. In the Monte Carlo simulation, random ROM samples are identified at random optimal displacement levels whose probability distribution is consistent with that observed in the multi-level identification. The responses of ROM samples to a dynamic load at two load levels are computed, and the uncertainty bands of the power spectral density (PSD) results are computed for uncertainty quantification. It is demonstrated that the uncertainty bands properly account for the uncertainty of the stiffness coefficients. For the nonparametric approach, the dispersion parameter is determined by using the maximum likelihood principle, then random ROM samples are directly generated. These ROM samples are used to obtain the uncertainty band results for the same dynamic load. The uncertainty bands from the nonparametric approach match the Monte Carlo simulation results very well.

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