Computationally Efficient Nonlinear Dynamic Analysis For Stress/Strain Applications

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ABSTRACT

Dynamic response of analytical models is commonly studied to determine strain induced on systems for normal design loadings. Due to the complexity of the analytical models combined with the possibility of nonlinear effects encountered, traditional methods for computing the time response can be extremely computationally intensive.

In this work, several recently developed techniques are combined to get a more computationally efficient calculation of full field stress and strain on a finite element model (FEM). The full space FEM is reduced to an intermediate space using a combination of reduction techniques. The time response is calculated using an efficient nonlinear response approach – Modal Modification Response Technique (MMRT) or Equivalent Reduced Model Technique (ERMT). The displacement results for the nonlinear system are expanded back to the full space FEM to determine time-dependent full field stress and strain. Results are presented for the nonlinear response of a system involving multiple contact elements to show the accuracy and computational efficiency of the process.

Keywords: Nonlinear, Model Reduction/Expansion, Dynamic Strain

INTRODUCTION

Detailed finite element models are often generated for the simulation of dynamic response on structural systems. With the current technology available, models are becoming increasingly larger and prominently more detailed to accurately capture the precise dynamics experienced throughout the structure. Such sizeable models require significant computational resources to solve the time response using conventional methods; the computations become further lengthened when nonlinearities are introduced to the system and a nonlinear solution scheme must be employed.

A nonlinear solution scheme is required to accurately capture the dynamics of a system where discrete nonlinear connection elements (bilinear springs, gaps, slip-stick, or contact elements) are added to a linear model, although the response of the original linear model will dominate the response of the system. When the response of these types of nonlinear systems is studied, the nonlinear response can be broken into a piecewise linear solution where different configurations occur within the time block. This work focuses on a system with gap-spring contact elements, as shown in Figure 1, to demonstrate the principles at hand on a system with a severe kind of nonlinear connection; the principles also apply to systems with other types of nonlinear connection elements. The system consists of two components with two possible spring contacts as an example of this type of a nonlinear system. An analysis of the dynamic response of the system in Figure 1 will generally be dominated by the linear response of the individual components, and the response will only become nonlinear when the springs come into contact between the beams. The traditional response calculation requires a nonlinear solution scheme that can be extremely computationally expensive but must be employed to capture the nonlinear response even though the nonlinearities occur at discrete time points and at discrete locations.

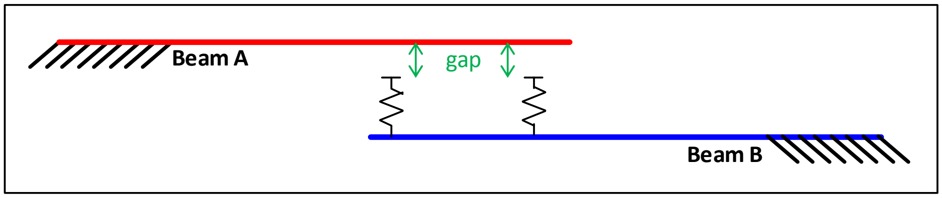


Figure 1. System containing two components and springs contacts

The proposed approach for efficiently calculating full field strain is shown in Figure 2 and further detailed by Harvie [1]. The approach involves reducing the full space model to a more manageable size and then calculating the response using an efficient response calculation, Modal Modification Response Technique (MMRT) or Equivalent Reduced Modeling Technique (ERMT). The highly reduced order models are used to accurately approximate the full field dynamic characteristics while significantly reducing the computation time; the traditional approach, outlined on the left of Figure 2, requires an extremely lengthy computation time due to the nonlinear solution scheme.

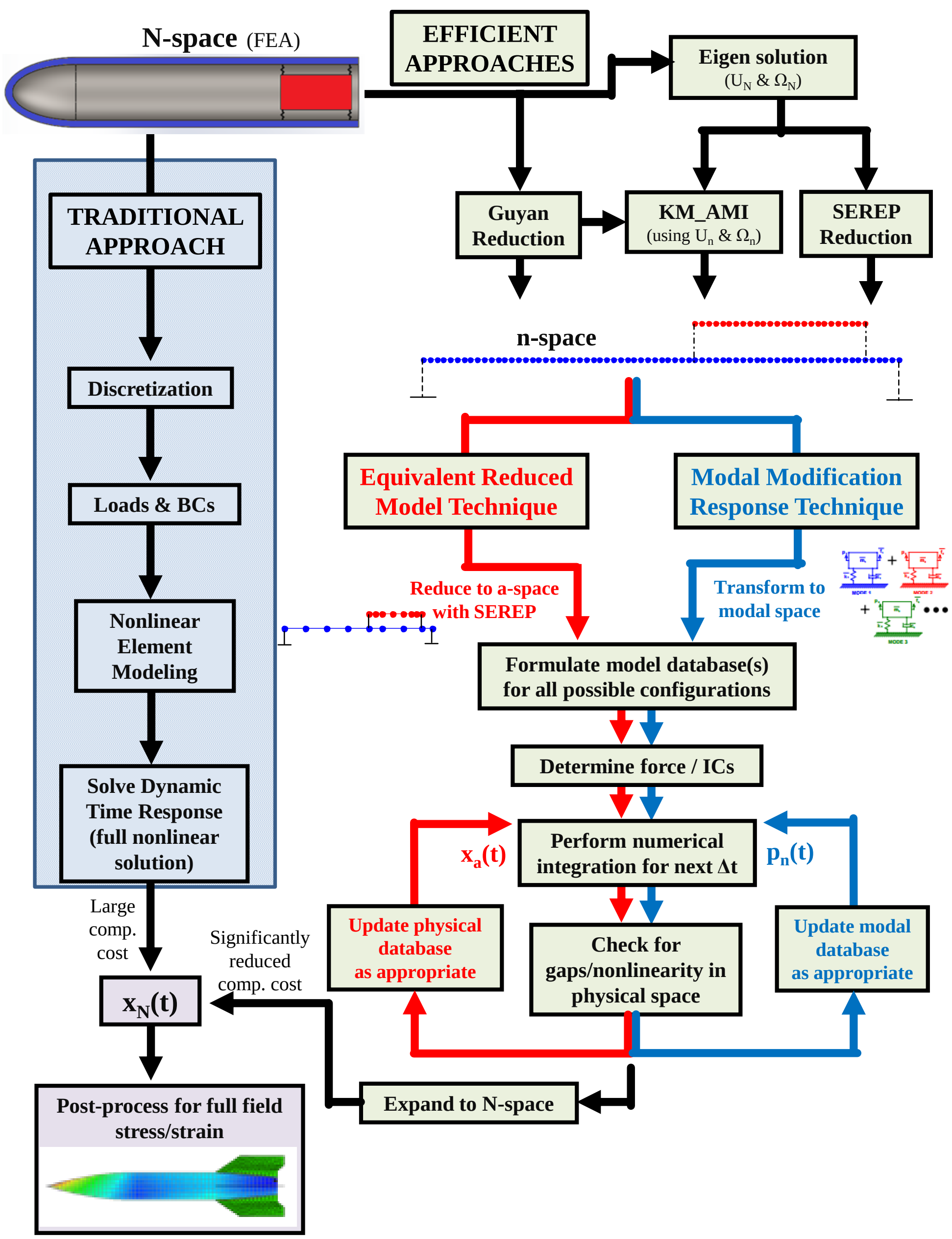


Figure 2. Traditional vs. efficient approaches to calculate stress and strain

As seen in Figure 2, there are several possible paths that can be used to accurately calculate the full field strain at a fraction of the computational cost. This work focuses on the results obtained using a SEREP reduction in conjunction with ERMT; however the other techniques will produce similar results and are explored further by Harvie [1] but are not presented in this paper due to space restrictions.

Several pieces of the proposed approach have already been investigated in depth. Recent work has been performed by Marinone [2-4] on MMRT and Thibault [5-7] on ERMT to confirm the accuracy and efficiency of the nonlinear time response calculations for displacement results. Pingle [8-14] and Carr [15-18] have explored the application for full field strain on linear systems using expansion of limited sets of data. Nonis [19-20] and others [21-22] have proven that a reduced order system assembly can be expanded to full space using information from the uncoupled component modes. The focus of this work is to combine the theory behind these separate sets of work to determine full field strain from the expansion of the highly reduced order models used to determine the response of systems with nonlinear component interactions.

THEORY

Basic Equations Of Motion

For a multiple degree of freedom system subjected to an external force {F}, the equation of motion can be written as

$$[M]\{\ddot{x}\}+[C]\{\dot{x}\}+[K]\{x\}=\{F\} \tag{1}$$

The [M], [C], and [K] matrices are the square symmetric mass, damping, and stiffness matrices, respectively. These matrices are generally coupled and can contain hundreds of thousands to millions of degrees of freedom, depending on the size of the finite element model. To uncouple the matrices, an eigensolution can be performed on the mass and stiffness matrices using

$$[[K]-\lambda[M]]\{x\}=\{0\} \tag{2}$$

The eigensolution of Equation (2) provides the eigenvalues (natural frequencies) of the system, along with an eigenvector (mode shape) for each frequency. The eigenvalues and eigenvectors are generally arranged as

$$\begin{bmatrix} \ddots & & \\ & \Omega^2 & \\ & & \ddots \end{bmatrix}=\begin{bmatrix} \omega_1{}^2 & & \\ & \omega_2{}^2 & \\ & & \ddots \end{bmatrix} \text{ and } [U]=[\{u_1\}\{u_2\}\ldots] \tag{3}$$

If a limited set of eigenvectors are used to describe the system, as is typical, then the resulting [U] matrix will be rectangular with ‘n’ DOF and ‘m’ modes. The relationship between the coupled, physical set of equations and uncoupled, modal set of equations is defined as

$$\{x\}=[U]\{p\}=[\{u_1\}\{u_2\}\ldots]\begin{Bmatrix} p_1 \\ p_2 \\ \vdots \end{Bmatrix} \tag{4}$$

This transformation to modal space allows for the equation of motion to be written in normal form as

$$[U]^T[M][U]\{\ddot{p}\}+[U]^T[C][U]\{\dot{p}\}+[U]^T[K][U]\{p\}=[U]^T\{F\} \tag{5}$$

This equation transforms the highly coupled system into a set of simple, uncoupled systems due to the orthogonality condition. The mode shapes are linearly independent and orthogonal with respect to the mass and stiffness matrices, yet the damping matrix may or may not be proportional. The damping matrix is typically not linearly independent and orthogonal with respect to the mode shapes, but a proportional damping matrix is often approximated to satisfy the orthogonality condition. The proportional damping matrix can be generated using several techniques, and the Wilson-Penzien damping formulation is utilized in this work. Thus the diagonal mass, damping, and stiffness matrices can be written in modal space as

$$\begin{aligned}
\begin{bmatrix} \ddots & & \\ & \overline{M} & \\ & & \ddots \end{bmatrix} &= [U]^T[M][U] \\
\begin{bmatrix} \ddots & & \\ & \overline{C} & \\ & & \ddots \end{bmatrix} &= [U]^T[C][U] \\
\begin{bmatrix} \ddots & & \\ & \overline{K} & \\ & & \ddots \end{bmatrix} &= [U]^T[K][U]
\end{aligned} \tag{6}$$

With a modal matrix [U] of size *n x m* (“n” DOF and “m” modes), the modal properties calculated in (6) will be square and symmetric of size *m x m*. If “m” is less than “n”, then the physical mass, damping, and stiffness matrices of size *n x n* will be reduced to the modal mass, damping, and stiffness matrices of size *m x m*.

Model Reduction and Expansion

Solving dynamic response using the equation of motion in (1) can become computationally intensive when a full space finite element model contains millions of DOF. The computation can be simplified by reducing the FEM with model reduction/expansion techniques. Using model reduction, the size of a model can be reduced to a limited ‘a’ DOF and ‘m’ modes while retaining the full space dynamic characteristics. Using model expansion, a full field solution can be obtained from a subset of points; expansion is commonly used to expand test data to locations that were not included in the reduced set of DOF. Several techniques are available for reduction and expansion, each with their own benefits and drawbacks. Guyan Condensation is one of the more commonly used techniques and is available in most commercially available software, but the Guyan transformation matrix is stiffness based and therefore does not preserve inertial characteristics exactly. Additional processing of the Guyan reduced matrices using the KM_AMI model reduction process [23-28] allows for exact reduced dynamic characteristics if target vectors from the full space model are used. SEREP [29] also preserves the exactness of eigenvalues and eigenvectors in the reduction/expansion process because the full space eigensolution is used in the transformation matrix calculation. While a multitude of reduction/expansion techniques could be used to transform the matrices, the results shown in the case studies were obtained using the accurate SEREP reduction and therefore only SEREP is detailed here. For further details on the various reduction techniques, see Reference 1.

The main concept behind model reduction and expansion involves the mapping between a large model containing ‘n’ DOF and a reduced model containing a limited ‘a’ set of DOF. The general transformation between the full model and reduced DOF is given as

$$\{x_n\} = \begin{Bmatrix} x_a \\ x_d \end{Bmatrix} = [T]\{x_a\} \tag{7}$$

where the full ‘n’ space model is made up of ‘a’ master DOF and ‘d’ deleted DOF. The transformation matrix, [T], contains the appropriate mapping information to relate the full and reduced models. By employing energy conservation principles, the mass and stiffness of the reduced ‘a’ DOF can be calculated using the transformation matrix with

$$[M_a] = [T]^T[M_n][T] \tag{8}$$

$$[K_a] = [T]^T [K_n][T] \tag{9}$$

Once the mass and stiffness matrices are determined at the reduced DOF, the equations of motion are written in the same form as (1) and an eigensolution is performed using (2). The eigenvalues and eigenvectors of the reduced matrices are ideally preserved as accurately as possible when compared to the full space solution, but the accuracy is dependent on the specific technique employed. The reduction/expansion technique utilized in this work is discussed next.

The SEREP transformation matrix can be used to exactly replicate eigenvalues and eigenvectors between a full and reduced model [29]. SEREP involves partitioning the modal equations representing a physical system using

$$\{x_n\} = \begin{Bmatrix} x_a \\ x_d \end{Bmatrix} = \begin{bmatrix} U_a \\ U_d \end{bmatrix} \{p\} = [U_n]\{p\} \tag{10}$$

where the active 'a' set can be described using

$$\{x_a\} = [U_a]\{p\} \tag{11}$$

In Equation (11), the $[U_a]$ matrix contains mode shapes for 'a' DOF and 'm' modes. For a case where the number of modes retained in the reduced model is less than or equaled to the number of reduced DOF ($m \leq a$), a least squares solution provides

$$\{p\} = \left([U_a]^T [U_a]\right)^{-1} [U_a]^T \{x_a\} = [U_a]^g \{x_a\} \tag{12}$$

Substituting (12) into (10) gives

$$\{x_n\} = [U_n][U_a]^g \{x_a\} \tag{13}$$

Hence the SEREP transformation matrix is written as

$$[T_U] = [U_n][U_a]^g \tag{14}$$

The development of this transformation matrix leads to vast computational advantages. Using unit modal mass scaling, Equations (8) and (14) can be combined to efficiently compute the mass matrix as

$$[M_a] = [T_U]^T [M_n][T_U] = \left[{U_a}^g\right]^T \left[{U_a}^g\right] \tag{15}$$

Similarly, Equations (9) and (14) can be used to efficiently compute the stiffness matrix as

$$[K_a] = [T_U]^T [K_n][T_U] = \left[{U_a}^g\right]^T \left[\Omega^2\right] \left[{U_a}^g\right] \tag{16}$$

again assuming unit modal mass scaling is used. Therefore the mass and stiffness matrices can be computed using only the mode shapes and frequencies of a system due to the properties of SEREP.

The reduced mass and stiffness matrices calculated in (15) and (16) are of size 'a', but rank 'm'. Therefore when the number of retained modes is less than the number of retained DOF ($m < a$), the reduced matrices will be rank deficient. Care must be taken when dealing with the rank deficient matrices. For the cases studied here, the rank deficiency is avoided by retaining the same number of modes and DOF ($a = m$) in the reduced model and is sometimes referred to as SEREPa. Such cases contain matrices that are fully ranked and well-conditioned.

System Modeling and Mode Contribution

Various techniques are available for the coupling of several component models into a single system model. These system modeling techniques are used to define the various states that the system will undergo when the different contact connections are considered in the nonlinear response of the system. The system modeling can be performed in physical space, modal space, or a combination of both physical and modal space. Consider two beams that are completely independent of one another, as illustrated in Figure 3.

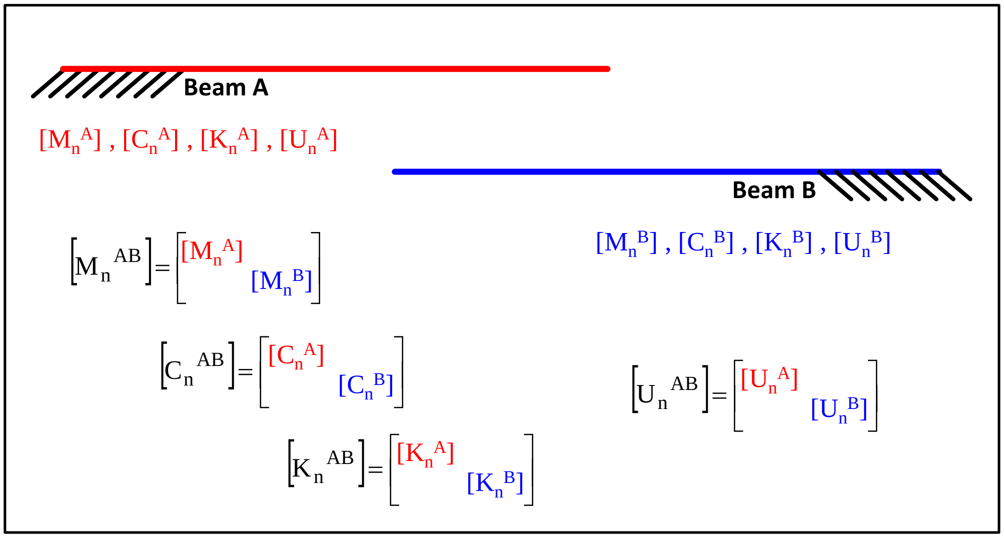


Figure 3. Sample components arranged into common matrix space

The two beams shown in Figure 3 are completely uncoupled and will respond independent of one another when excited. A system model of the uncoupled components is generated by simply writing the variables in common matrix space, as shown in the diagram. To generate a coupled system model, specific coupling terms must be introduced at the desired locations. To include the spring(s) in the system modeling, either a modal or physical approach can be employed. The modal approach involves using Structural Dynamic Modification (SDM) and Component Mode Synthesis (CMS). The physical approach involves using a physical tie matrix to couple the beams. Both approaches involve the use of a mode contribution matrix to determine the appropriate number of component modes that contribute to the system modes. For the results presented here, physical system modeling techniques were used to generate databases for the various configurations.

The equation of motion of a system with a physical change in mass or stiffness can be written at full space as

$$\begin{aligned} &\left[\begin{bmatrix} [M_n{}^A] & \\ & [M_n{}^B] \end{bmatrix} + [\Delta M_{12}]\right]\begin{Bmatrix} \ddot{x}_n{}^A \\ \ddot{x}_n{}^B \end{Bmatrix} \\ &+ \left[\begin{bmatrix} [K_n{}^A] & \\ & [K_n{}^B] \end{bmatrix} + [\Delta K_{12}]\right]\begin{Bmatrix} x_n{}^A \\ x_n{}^B \end{Bmatrix} = \begin{Bmatrix} f_n{}^A \\ f_n{}^B \end{Bmatrix} \end{aligned} \tag{17}$$

Once the physical system matrices are generated at full space, the system is reduced to a set of 'a' DOF and 'm' modes in this work. Truncation can be avoided in the reduction by including enough modes to span the space of the problem. To determine the appropriate modes of the original components that contribute to the final system modes, a mode contribution matrix is used. The mode contribution matrix calculated using this technique is equivalent to the $[U_{12}]$ matrix calculated in SDM/CMS [30-32]. As presented by Thibault [6], the mode contribution matrix in physical system modeling is calculated using vectors from the original system state, 1, and final system state, 2, using

$$[U_{12}] = [U_1]^T [M_2][U_2] \tag{18}$$

The mode contribution matrix is important because it identifies which modes of the unmodified system are necessary to construct the modified system modes. This transformation matrix contains scaling coefficients that specify which of the original state $[U_1]$ vectors are necessary to construct the final state $[U_2]$ vectors. A physical representation of the multiplication in (18) is shown in Figure 4.

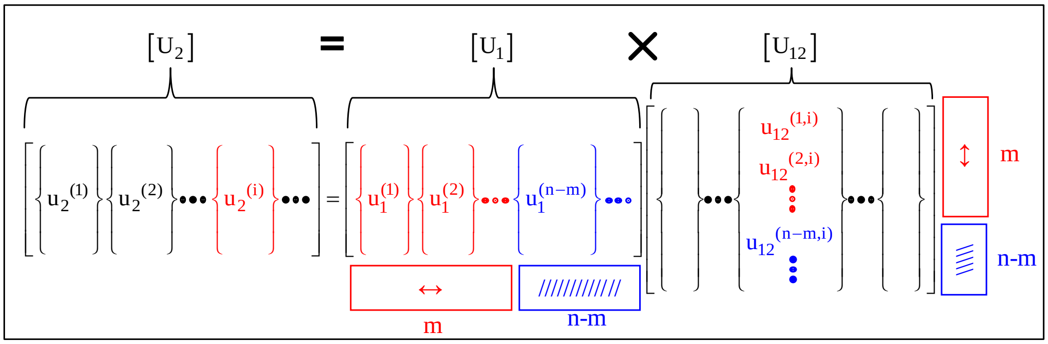


Figure 4. Use of $[U_{12}]$ matrix to determine final system mode shapes

The mode contribution matrix is used to determine which original mode shapes are necessary for the accurate reconstruction of each of the desired final mode shapes. If a dynamic response involves multiple system states, then the $[U_{12}]$ matrix must be computed for each configuration to determine the number of original system modes to appropriately span the space of the solution and avoid truncation. If a component mode has a high value in the contribution matrix for a certain desired system mode and that component is not included in the reduction or modes retained, then the system mode will be in error and is said to be truncated. Conversely, components modes with low contribution values for desired system modes do not participate significantly in the system modes and therefore are not necessary to include in the solution. The mode contribution matrix is important because it can identify the minimal set of component modes that are necessary to include in a system model; retaining fewer modes in a reduced model can result in higher computational savings.

Uncoupled Expansion of Coupled Systems

A breakthrough in the expansion of coupled reduced order components using uncoupled component information has recently been established. The main idea behind the work is that if a coupled reduced order system is comprised of uncoupled components, the full space coupled system modes can be generated using the uncoupled component transformation information. SDM principles show how the mode shapes of a coupled system can often be constructed using appropriate combinations of the uncoupled system mode shapes. The $[U_{12}]$ matrix is used to determine the necessary number of modes of an uncoupled system that must be included in system modeling activities to accurately represent the coupled system mode shapes. Recent analyses by Nonis [19-20] and others [21-22] have shown how these SDM principles can be extended to model reduction and expansion, and the transformation matrix of an uncoupled system can be used to expand a coupled system if enough modes of the uncoupled system are included.

Figure 5 shows a simple example where model expansion is used on a cantilever beam. In Case A, no modifications have been made to the beam and a straightforward expansion is performed using the SEREP transformation matrix calculated using Equation (14). For Case B, the reduced model contains a spring modification near the tip. Traditionally, a new transformation matrix would need to be computed to relate the reduced and expanded mode shapes with the spring modification in place. However, SDM principles show that the mode shapes for the modified beam in Case B are made up of combinations of the mode shapes for the original beam in Case A. Therefore given that enough modes are included in the expansion, the same transformation matrix that is used to expand the mode shapes in Case A on the original, unmodified beam can be used to expand the modified system in Case B.

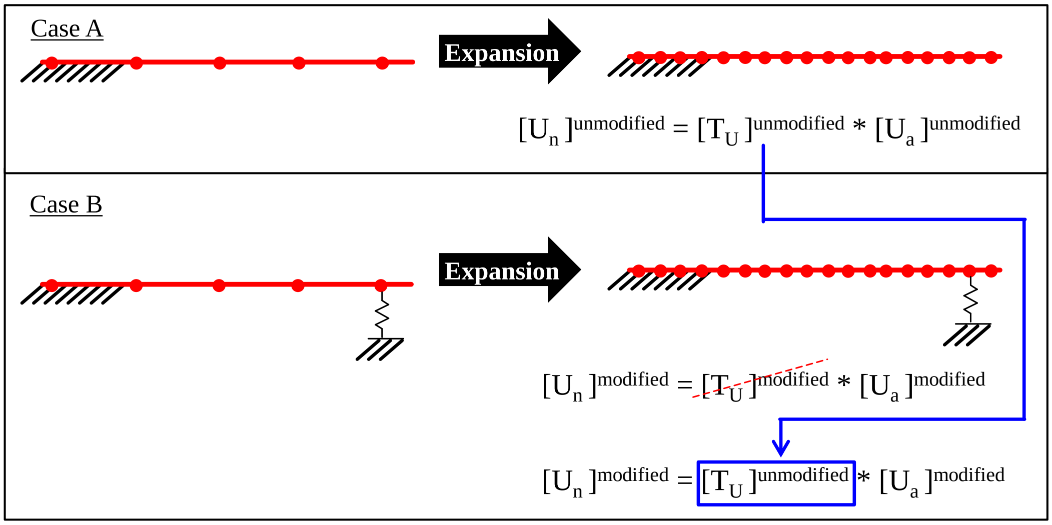


Figure 5. Simple beam with and without spring modification

The results obtained using the technique outlined in Figure 5 will yield exactly the same results as a full space SDM model containing the same number of modes as the reduced model [20]. Therefore in a case where not enough modes are included in the solution, the SDM and uncoupled expansion techniques will contain the same level of truncation. The only possible source of error inherent to the expansion could be truncation if not enough modes are included to span the space of the problem. However if enough modes are included in the reduced model to accurately represent the coupled system modes, the uncoupled transformation matrix will be able to accurately expand the coupled system mode shapes.

The expansion methodology is not limited to a single component such as the one in Figure 5. For a coupled system comprised of multiple components, only the uncoupled transformation matrices of the individual components are needed to expand mode shapes of a system where the components are joined. The coupled system can be expanded to full space using the uncoupled transformation matrices as long as the modes required to accurately produce the coupled system modes are included from the uncoupled system. Furthermore the components are expanded independent from one another, and therefore both components do not need to be expanded to full space if full space information is not desired on all components. The necessary modes to include in the reduced model and transformation matrix can be determined using the mode contribution matrix $[U_{12}]$.

Chipman [33-36] and others [8-18] have shown that a transformation matrix can be used to expand not only mode shapes, but dynamic time response data. Therefore the same principles presented above can be extended to the expansion of coupled response using uncoupled component information. The response of the coupled system at reduced space can be expanded to full space using the transformation matrices of the individual, unconnected beams because the dynamic characteristics of the system are directly related to the dynamic characteristics of the uncoupled components. If the $[U_{12}]$ matrix is evaluated to include enough modal information in the reduced model to accurately represent the system with the coupling elements present, then the time response of the coupled system is accurately expanded to full space using information from the uncoupled components.

For this work, nonlinear systems are analyzed where several possible configurations can exist during response. If all possible configurations are made up of linear combinations of the component mode shapes, then the expansion of nonlinear response of a reduced model can be expanded using the original transformation matrix regardless of the configurations encountered. As long as enough modal information is included in the reduced order models to accurately represent all possible configurations, the expansion of nonlinear coupled response can be performed using information from the uncoupled components. Figure 6 shows the process for determining full space dynamic response of a nonlinear system using the original transformation matrices. Although several configurations exist within the time block shown, all modified states can be generated based on the mode shapes of the original system. Therefore only the transformation matrices of the original, uncoupled components are necessary to expand the nonlinear dynamic response of the system regardless of the configurations encountered.

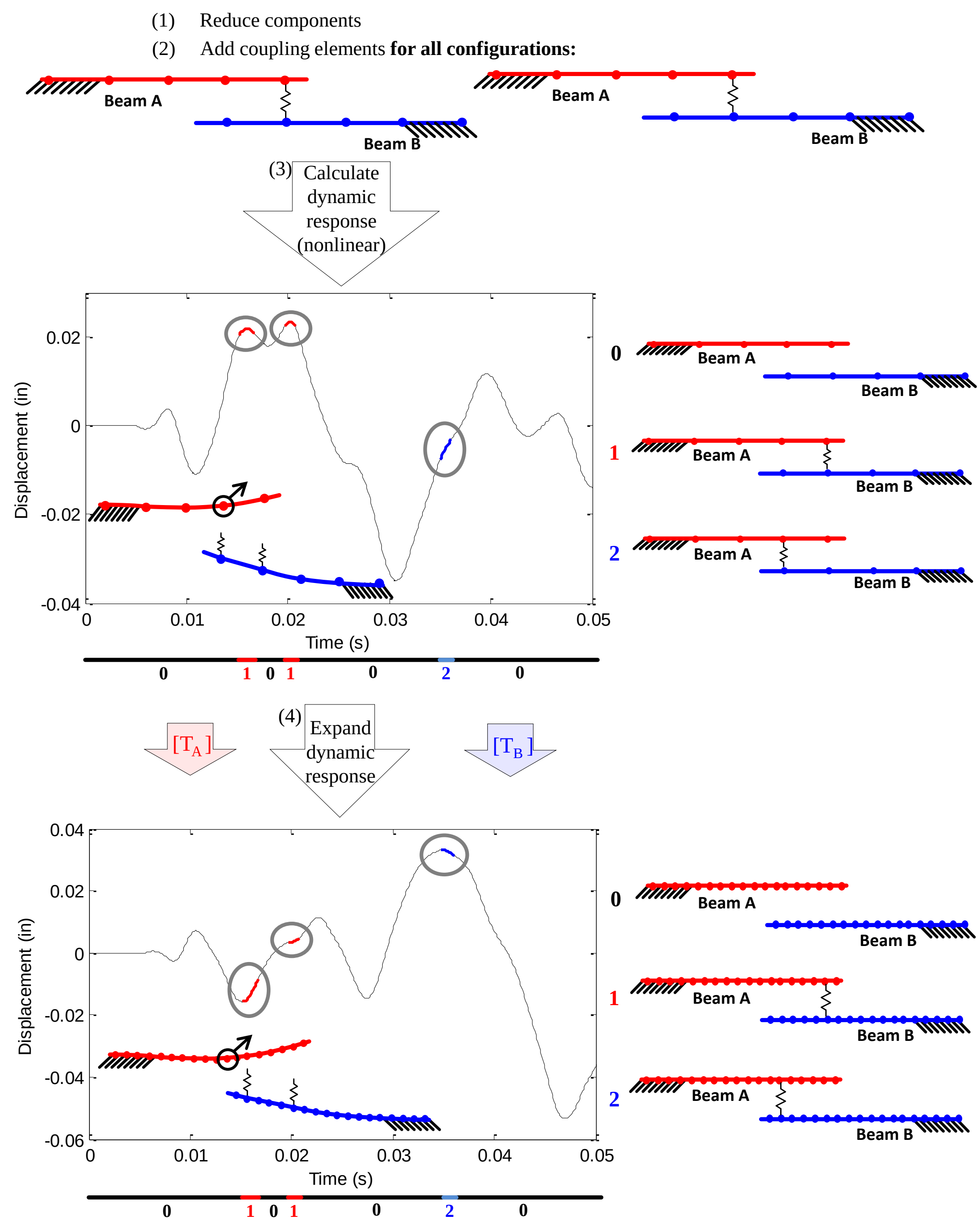


Figure 6. Expansion of nonlinear response using component mode shapes

Although the system in Figure 6 experiences multiple changes of state, all configurations are made up of the mode shapes from the original, uncoupled system. Therefore the nonlinear response of the system at reduced space is accurately expanded to full space using only the mode shapes from the original system, as long as enough mode shapes of the unmodified system are included in the solution.

Efficient Time Response Calculation

Several techniques are available to calculate the time response of a linear or nonlinear system more efficiently than solving the full space solution. Two techniques, the Modal Modification Response Technique (MMRT) and the Equivalent Reduced Model Technique (ERMT), were presented by Avitabile and O'Callahan [37] and analyzed in detail by Marinone [2-4] and Thibault [5-7], respectively. MMRT involves integrating the equations of motion in modal space using a subset of modes to reduce computation time. ERMT involves integrating the equations of motion in physical space using reduced order models to increase efficiency. Both techniques are able to further reduce computational costs in nonlinear response by avoiding a lengthy nonlinear solution scheme; both techniques treat the nonlinear response as piecewise linear response. ERMT provides a slightly more efficient calculation because MMRT requires a projection to physical space at each time step to determine if contacts are engaged, while the ERMT calculation is performed without projection to a different domain. However, in terms of each technique, the ERMT works well with physical models are are reduced from a full finite element model where as the MMRT works well with modal data which can come from either a finite element model or from an experimental test. The ERMT process is described next and was used in the following case studies due to the slightly more efficient calculation.

The Equivalent Reduced Model Technique can be used to compute forced response for both linear and nonlinear systems. An outline of the ERMT process is shown in Figure 2. First, databases are generated at reduced space for all possible configurations of the system using system modeling techniques as necessary. The forcing function and initial conditions are identified and input to the system. The equations of motion are numerically integrated for the next time step. At this point, the displacements at locations where interaction is possible are monitored to determine whether contact has occurred. At each time step the appropriate system matrices are utilized for the given configuration. The integration is performed for each time step through the desired time period. Therefore the total forced response of a system, regardless of linearity, can be computed with traditional integration using this technique. Once the time response is computed at the reduced DOF, the results are expanded to the full model. As presented in References 21 and 22, only the transformation matrix for the original system is needed to expand the time response back to n-space, regardless of the types of configurations that are encountered.

Correlation Tools

Several correlation tools are available for the comparison of a system's time response. The main tools used in this work are the Modal Assurance Criterion (MAC) and Time Response Assurance Criterion (TRAC).

MAC is used to compare two shapes at a single instance in time and is calculated using

$$\mathrm{MAC} = \frac{\left[\{x_{n1}\}^T\{x_{n2}\}\right]^2}{\left[\{x_{n1}\}^T\{x_{n1}\}\right]\left[\{x_{n2}\}^T\{x_{n2}\}\right]} \tag{19}$$

where MAC values approaching 1 indicate high level of correlation.

TRAC is used to compare two time response plots at a single degree of freedom and is calculated similarly using

$$\mathrm{TRAC} = \frac{\left[\{x_{n2}\}^T\{x_{n1}\}\right]^2}{\left[\{x_{n2}\}^T\{x_{n2}\}\right]\left[\{x_{n1}\}^T\{x_{n1}\}\right]} \tag{20}$$

where once again values approaching 1 indicate good correlation.

In this work, the MAC is calculated between the shapes of the full space reference solution and estimated solution obtained from the reduced order model at each time step. Similarly the TRAC is used to compare the time response from the reduced order model to the time response from the full space finite element solution at each degree of freedom. A diagram detailing the two comparison techniques is shown in Figure 7.

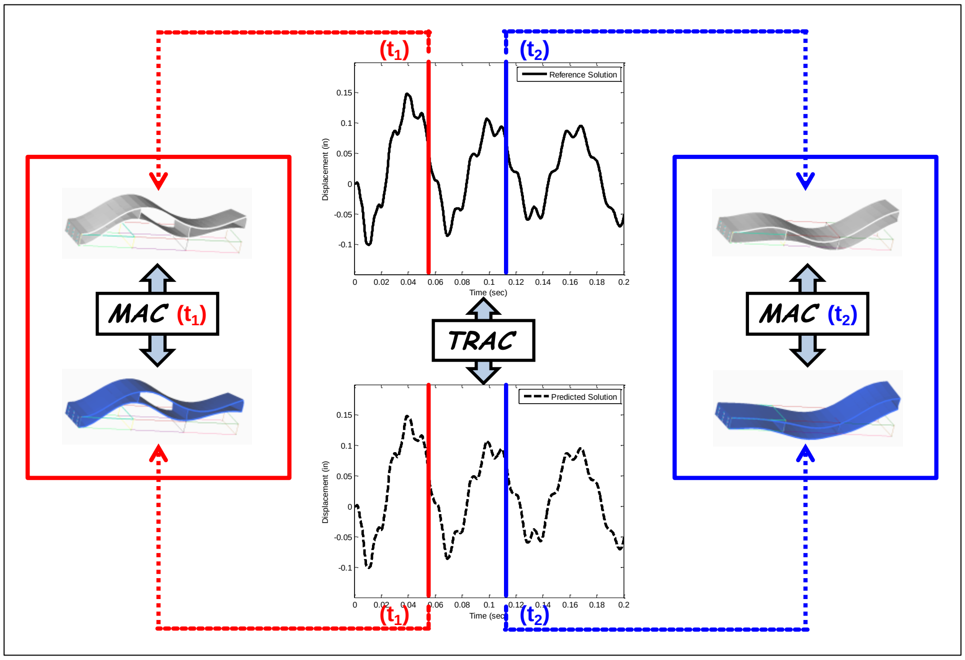


Figure 7. Physical interpretation of MAC and TRAC

CASES STUDIED

For this study, a simple two-beam system was used to demonstrate the proposed approach for calculating full field strain at a fraction of the full space computational cost. Two cases were studied to show the influence of higher order modes when hard nonlinear contact springs are present in a system. Additional cases and further details on all analyses performed can be found in [1]. The cases presented here are summarized as:

Case 1 – Nonlinear Solution with Smaller Reduced Model
Case 2 – Nonlinear Solution with Reduced Model Containing Additional Modes

Structure Description & General Modeling/Testing Performed

A finite element model was generated using Abaqus/CAE [38]. The two-beam system, as illustrated in Figure 8, was generated to imitate a large, complicated model to accurately demonstrate the principles at hand while maintaining a feasible model size on which reference calculations are performed; note that the diagram shown in Figure 8 is not to scale. The main beam, Beam B, is 140 inches in length and joined to the smaller Beam A, 50 inches in length, using 10,000,000 lb/in translation springs; the main beam is grounded using 10,000 lb/in translation springs.

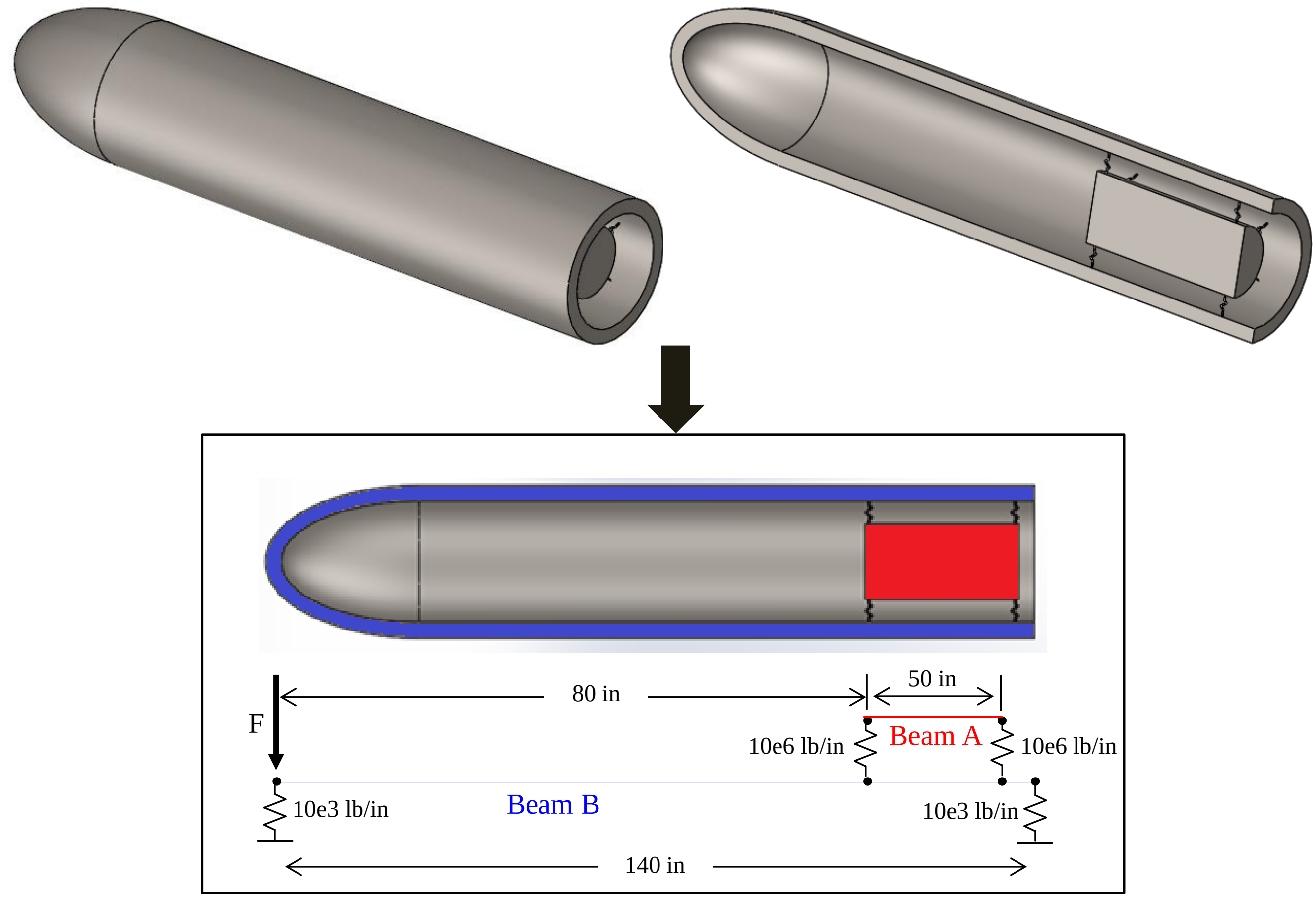


Figure 8. Physical representation of two-beam system

The full space model contains nodes with 0.2 inch spacing on each beam; therefore there are 251 nodes on Beam A and 701 nodes on Beam B. Each node contains a shear DOF and a rotational DOF to capture planar beam bending only. Details on the properties of the structure can be found in Table 1; note that the modeling properties are only applicable for the large N-space model.

Table 1. Geometric, modeling, and material properties of two-beam system

	Property	**Beam A**	**Beam B**
Geometric	Beam width (in)	2	2
	Beam height (in)	1	1
	Length (in)	50	140
Modeling	# of elements	250	700
	# of nodes	251	701
	# of DOF	502	1402
	Node Spacing (in)	0.2	0.2
Material	Material	Aluminum	Aluminum
	Density (lb/in^3)	2.538e-4	2.538e-4
	Young's Modulus (psi)	10e6	10e6

To introduce discrete nonlinearities to the system, a gap-spring interface was used to simulate a contact; the stiffness of the spring contact is either set to a predefined stiffness value when the specified gap distance is closed, or set to zero when the specified gap distance is open. The nonlinear cases have two contact locations between the beams, as shown in Figure 9; these contact locations were chosen so that both contact springs could engage during the response. The initial gap distance was set to 0.003 inches for the nonlinear cases; once again this value was chosen merely so that both contact springs would engage during the response. The three possible configurations that the beams can encounter with the springs engaged are also shown in Figure 9. An eigensolution was performed on the model to determine natural frequencies and mode shapes for the original system and with the possible spring attachments. The frequencies up to 1000 Hz for all configurations are listed in Table 2.

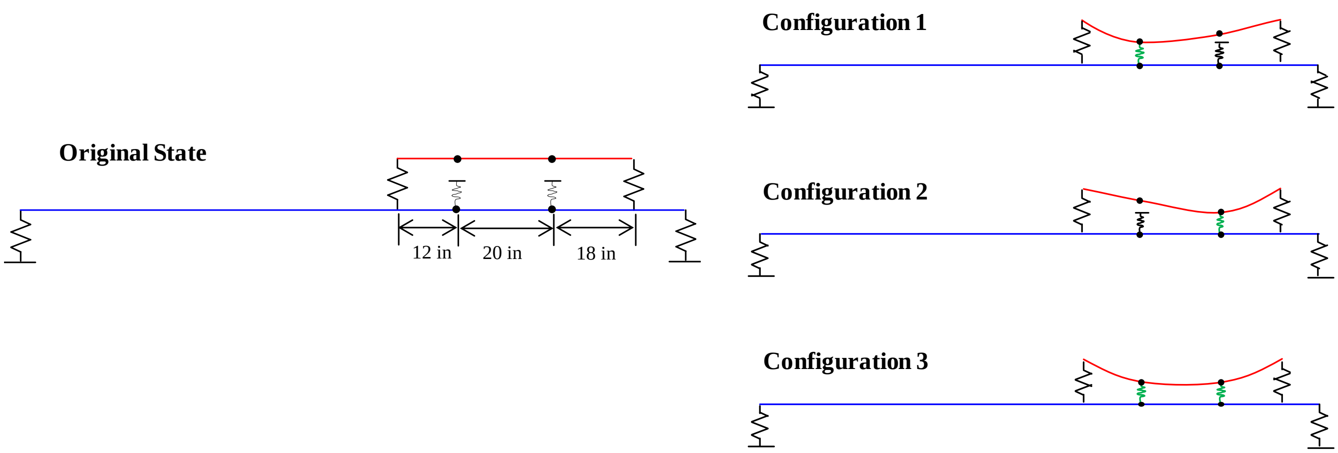


Figure 9. Contact locations and configurations for nonlinear cases

Table 2. Natural frequencies through 1000 Hz of full-space system with and without hard contacts

Mode	Frequency (Hz)			
	Original	Config 1	Config 2	Config 3
1	3.88	4.14	4.07	4.14
2	14.61	15.01	15.05	15.08
3	26.91	27.18	26.97	27.34
4	38.37	43.42	43.11	43.49
5	44.03	67.08	66.73	67.10
6	67.10	76.99	92.65	98.80
7	96.85	99.19	113.84	144.79
8	141.42	145.76	144.79	187.79
9	153.93	190.90	189.58	208.79
10	190.93	227.85	229.99	234.73
11	255.93	262.04	264.22	269.05
12	322.30	325.28	328.99	329.20
13	335.25	383.76	338.36	391.88
14	404.54	404.55	406.64	407.48
15	496.67	499.47	497.10	499.56
16	584.91	585.47	590.07	591.58
17	590.09	591.95	615.35	615.52
18	707.96	707.97	708.67	708.68
19	814.95	815.99	816.16	816.98
20	901.54	907.93	909.59	915.61
21	949.60	950.01	950.13	950.71
22	1078.50	1078.53	1078.71	1078.73

For all cases, a frequency band-limited analytical force pulse was utilized to excite a frequency range of roughly 400 Hz; the frequency range excited by the force pulse includes roughly 13 modes in all configurations, as listed in Table 2. The use of such a force pulse allows for minimal excitation of higher modes and controls the number of modes that substantially participate in the system response due to the impulse. The force was applied at the left-most node of Beam B, and the force was applied perpendicular to the beam to excite modes along the weak axis; this approximation was made to demonstrate the principles at hand and could be extended to different forcing functions and locations.

The large N-space model was reduced from 1904 modes and DOF to an n-space model with 194 modes and DOF by retaining every 10th node from the full space model. A comparison of model sizes for this reduction is shown in Figure 10. All nodes in both models contain both shear and rotary DOF. This reduction was performed to produce a more reasonable sized model.

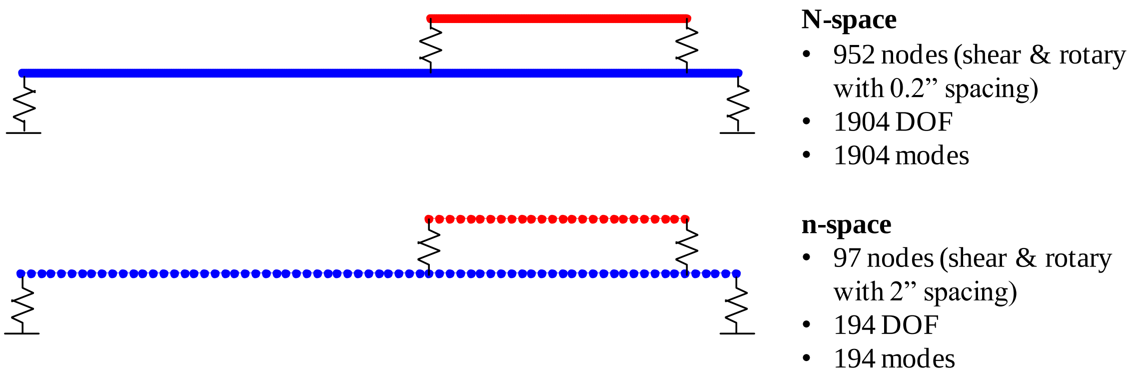


Figure 10. Description of model sizes from initial reduction

The n-space model was reduced further for use with ERMT. Two model sizes, a-space and aa-space, were generated for the various analyses. The a-space model contains 24 modes and DOF while the aa-space model contains only 13 modes and DOF, as outlined in Figure 11. Details on the levels of modeling are shown in Table 3, including the specific DOF and modes that remain active in each model. The a-space model includes modes of the original system up to approximately 1300 Hz while the aa-space model includes modes up to only 400 Hz. Without any additional information, the aa-space model with 13 modes and DOF would seem appropriate to represent the system in all configurations due to the applied force pulse.

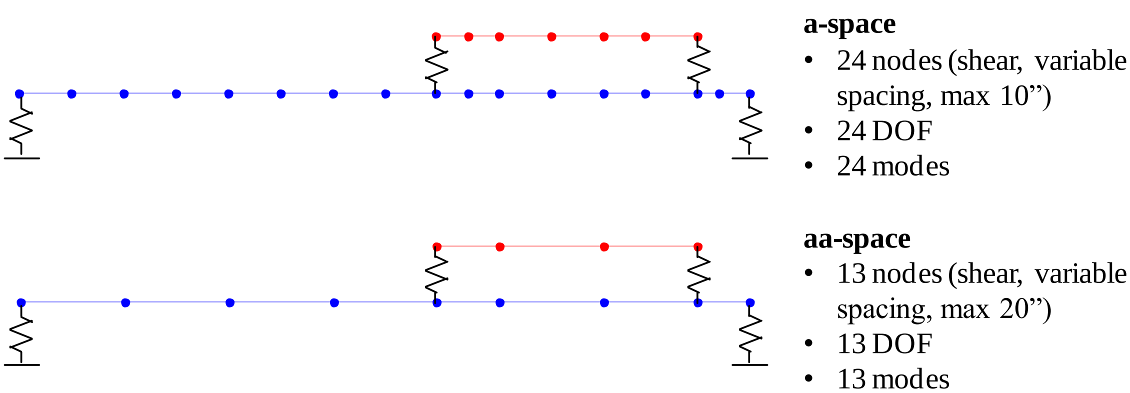


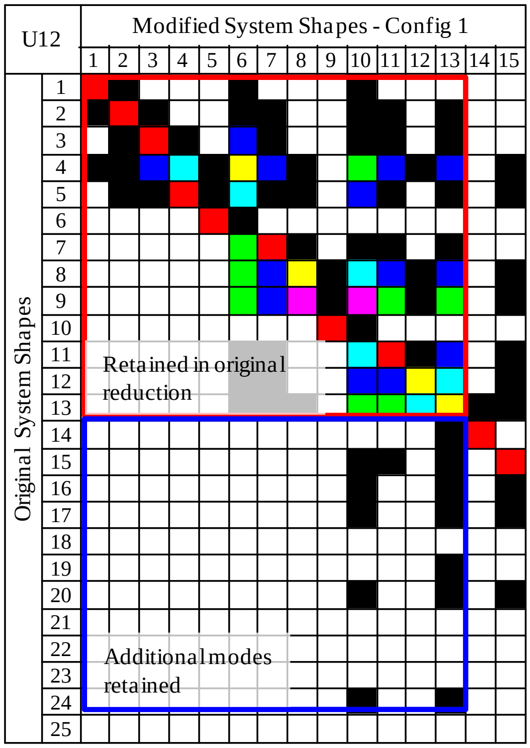
Figure 11. Description of model sizes for smaller, reduced order models

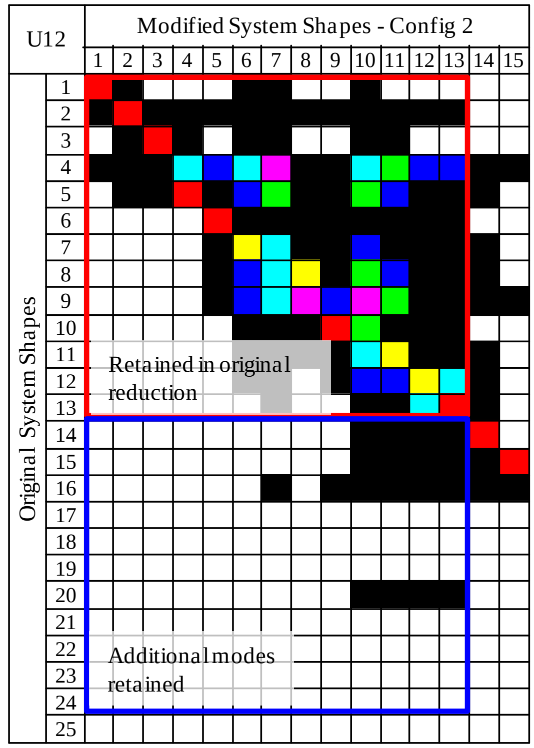
Table 3. Specific modes and DOF active in each size model

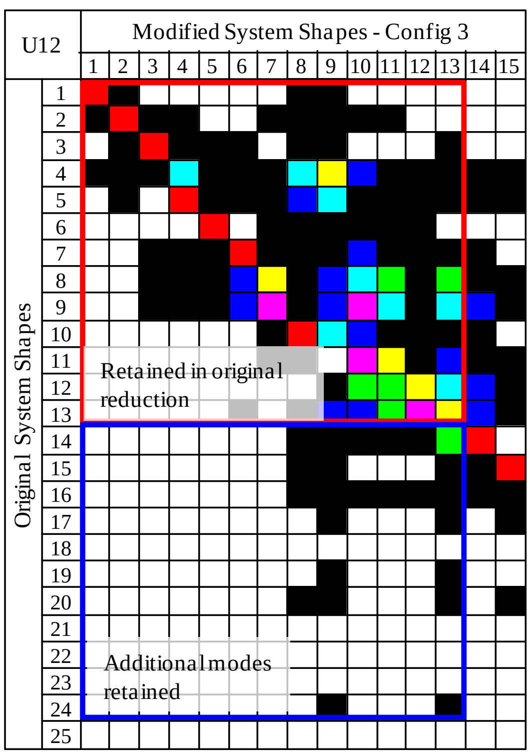
Model	# of DOF	Modes	Active DOF
N-space	952	1-1904	1-1904
n-space	194	1-194	1-2, 21-22, 41-42, ….1903-1904
a-space	24	1-24	1,101,201,301,401,501,601,701,801, 861,921,1021,1121,1301,1341,1401, 1403,1463,1523,1623,1723,1803,1903
aa-space	13	1-13	1,201,401,601,801,921,1121, 1301,1401,1403,1523,1723,1903

The mode contribution matrices were calculated at full space for the modified system in all configurations, and the matrices are shown in Figure 12. As identified previously, the force pulse excites roughly 13 modes of the system in all configurations, and those modes are highlighted in the matrices. The modes of the original system retained in the smallest aa-space model are outlined in red, while the additional modes of the original system retained in the a-space model are outlined in blue. The aa-space model, which includes 13 modes and DOF, includes many of the original system modes that make up the first 13 system modes in the various configurations. However, there are also some minor contributions from higher order modes for the system modes with the contacts engaged. The a-space model with 24 modes contains enough modal information to span the space of the problem for all configurations encountered.

Direct integration of the equations of motion was performed using Newmark time integration [39] to compute the time response for all cases; details regarding the integration technique are shown in Reference 1. Newmark integration was utilized for similarity to the solver used in Abaqus [38], where the Hilber-Hughes-Taylor (HHT) variation of the Newmark method is used. The damping of the system was approximated using a Wilson-Penzien formulation, as detailed in [1], with one percent of critical damping for all modes. Proportional damping was assumed to keep a straightforward solution procedure, but a state space solution could be used to solve systems with nonproportional damping.







Bar Color		**Min Value**	**Max Value**
	Red	0.90	1.00
	Yellow	0.70	0.90
	Magenta	0.50	0.70
	Cyan	0.30	0.50
	Green	0.20	0.30
	Blue	0.10	0.20
	Black	0.01	0.10
	White	0.00	0.01

Figure 12. Mode contribution matrix for beams in all configurations

Case 1 – Nonlinear Solution with Smalller Reduced Model

This case utilizes a reduction to the smallest aa-space model with 13 modes and DOF. A reference full space displacement calculation was computed and compared to the solution obtained using the proposed efficient techniques; the comparison is shown in Table 4. The MAC and TRAC values show that acceptable results are achieved using the technique, but there are some differences between the full and approximate solutions. The reduced solution took less than a second to compute, while the full space solution took over 12 minutes to compute. This shows a reduction in computation time of over 3000 times, while the reduced model was only two orders of magnitude smaller than the full space model. As the model size is increased to the much larger FEM used in industrial applications, the reduction in computation time is expected to be further improved.

Table 4. Comparison of reduced and full solution for case with hard contacts

Model	# of DOF	Solution Time (sec)	Average MAC	Average TRAC
Full Space	1904	740.18	0.9892	0.9918
Reduced	13	0.23		

The dynamic displacement at an arbitrary location on Beam A is plotted in Figure 13; both the time response and corresponding Fast Fourier Transform (FFT) are shown. The initial transient compares very accurately between the full and approximate solutions because no springs have come into contact during that period of time. Following the linear region, some differences in amplitude are observed throughout the remainder of the time trace. Similarly, the FFTs of the response compare rather well at low frequencies, but notable differences are observed at higher frequencies. This trend is unsurprising when the $[U_{12}]$ matrices in Figure 12 are taken into account because not enough modes of the system were retained in the reduced order model to accurately capture the excited system modes of the various configurations.

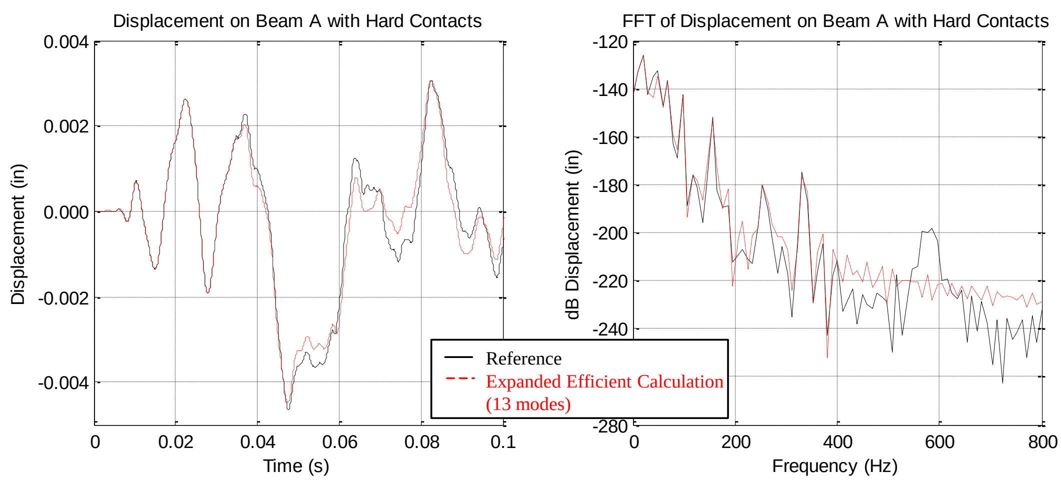


Figure 13. Time response corresponding FFT on Beam A for hard contact case

The same dynamic displacement is shown alongside the dynamic strain in Figure 14. The solution produced using the efficient technique follows the same trends as the full solution, but higher frequency content is missing from the efficient solution. The differences associated with mode truncation are apparent in the displacement solution, but are more noticeable in the amplitudes of the dynamic strain. The predicted solution provides a reasonable approximation, but rapid changes in displacement and strain are not captured using the efficient calculation with the aa-space model.

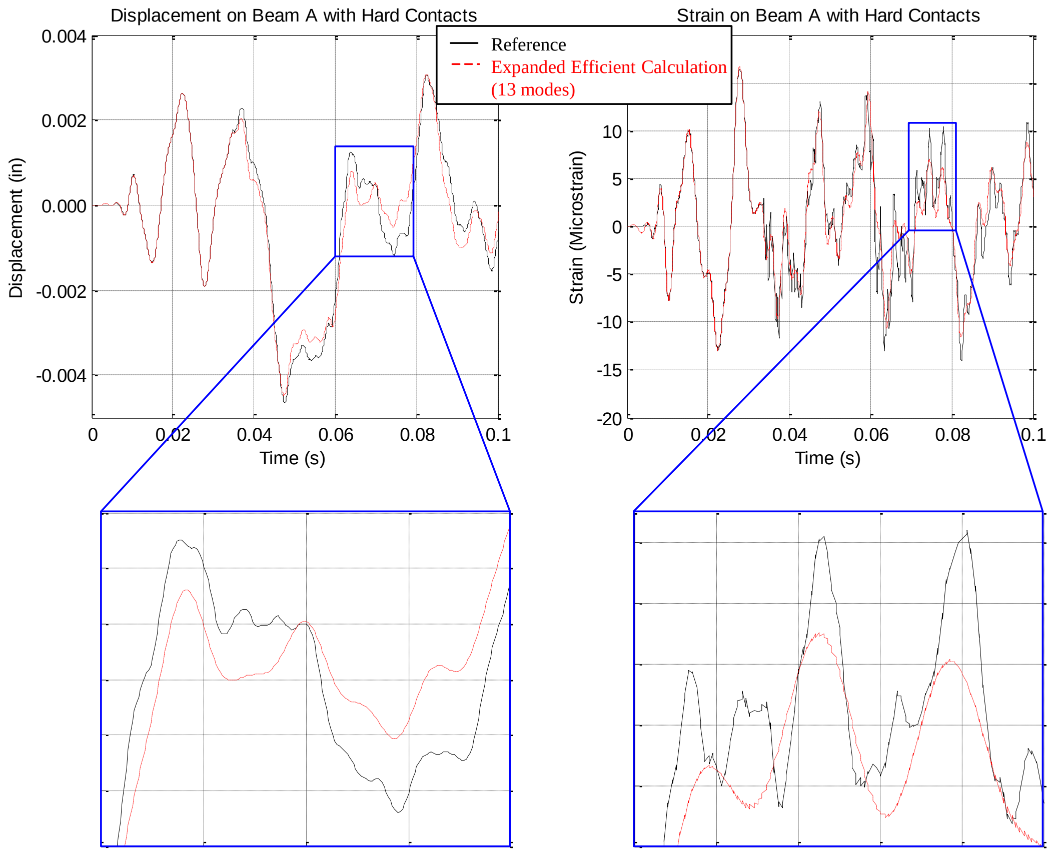


Figure 14. Displacement and corresponding strain on Beam A for hard contact case

The approximation in this case may be appropriate for certain applications. If only the general contour of the response curves is desired, then this solution will be appropriate. However many peak values were not calculated very accurately due to the lack of necessary modes included in the reduced solution. One important point to make is that while the applied force appears to only require the 13 modes used in this solution, the differences occur mainly because the nonlinear contact occurs with a hard gap stiffness which tends to excite higher modes. Had the contact spring been a lower contact stiffness, then the results would have matched much better (as shown by Harvie [1]). Due to the hardness of the contact, more modes are needed to properly span the space of the actual response of the system. To overcome these issues presented here, the next case contains a reduced order model with more modes.

Case 2 – Nonlinear Solution with Reduced Model Containing Additional Modes

Upon examination of the $[U_{12}]$ matrices in Figure 12, there are several modes of the original system higher than mode 13 that have a substantial contribution to the modified system modes within the excited frequency range. Therefore this case was generated where the reduced a-space model contains 24 modes and DOF. As seen in the matrices, all three configurations require at least 20 modes to accurately represent the first 13 final system modes, and certain configurations have notable contribution from up to 24 modes.

The full space displacements were determined using both a full space solution and the proposed efficient technique. Comparisons of the computation time and accuracy for this case are shown in Table 5. Yet again the solution time was reduced significantly by utilizing such a smaller model to solve for the response. The accuracy of this model is higher than that of the previous case because including more modes in the reduced order model allows for the observation of higher frequency content excited by the hard contact springs. For this case, very high accuracy was obtained using an extremely reduced model.

Table 5. Comparison of reduced and full solution for hard contact case with more modes

Model	# of DOF	Solution Time (sec)	Average MAC	Average TRAC
Full Space	1904	740.18	0.9998	0.9999
Reduced	24	0.28		

The displacement results for the full and predicted displacement are shown in the time and frequency domains in Figure 15. For this case, the comparison of displacement overlays nearly perfectly in both domains between the two response calculations. The frequency range of the original system modes included in the reduction extends over 1300 Hz, so the energy distribution of the response is accurately captured in the frequency domain.

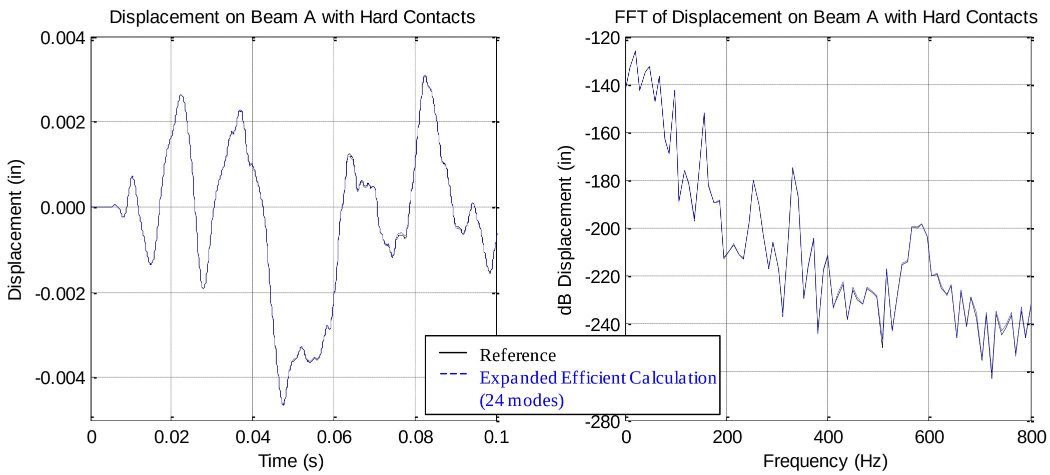


Figure 15. Time response and corresponding FFT on Beam A for hard contact case with more modes

The displacement is also shown in Figure 16 with the corresponding dynamic strain at the location of interest. Both the displacement and strain are predicted more accurately using the reduced order model implemented in this case. The model used in the efficient technique is able to better predict the higher order curvature in the dynamic displacement and strain response because more modes were included in the reduction. While the efficient calculation of the dynamic displacement matches nearly perfectly to the reference solution, some high frequency content is present in the reference strain solution that is not captured using the efficient calculation. The slight differences in the strain calculations can be further reduced by including even more modes in the reduced order model. The strain and displacement results calculated efficiently compare very accurately to the full space model for this case, yet the solution time for the reduced order model is substantially lower than the time required for a full space calculation.

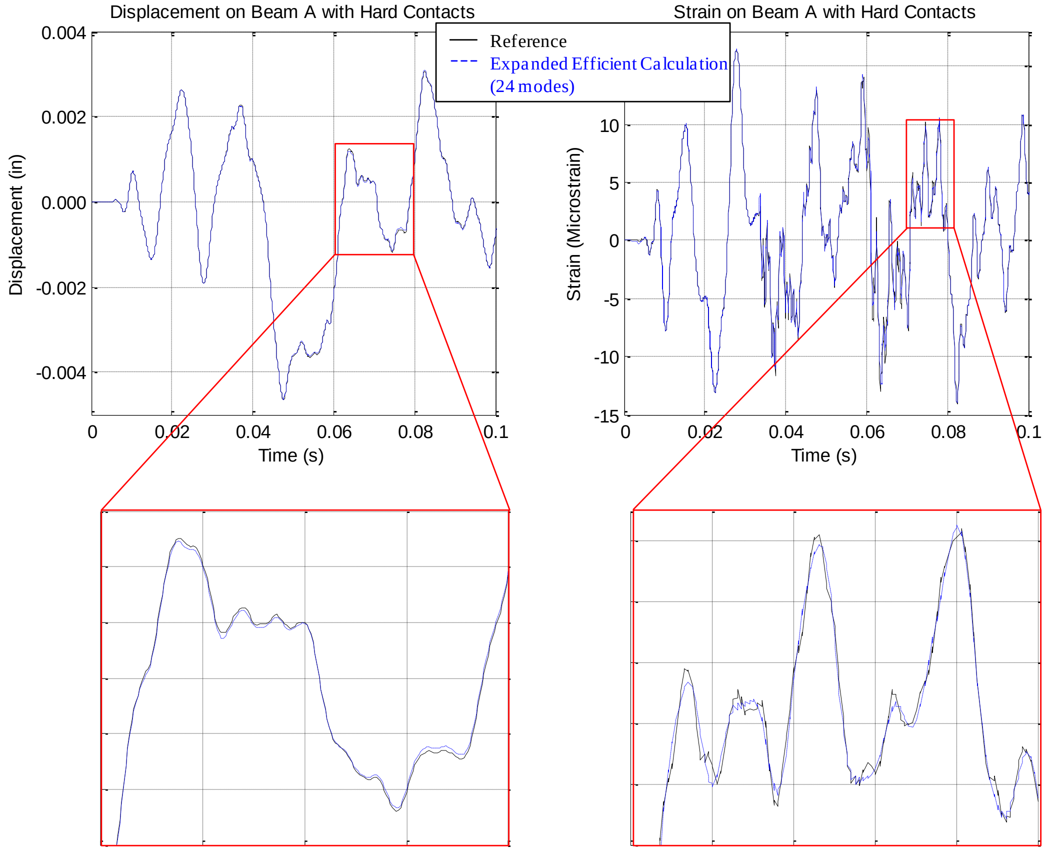


Figure 16. Displacement and corresponding strain on Beam A for hard contact case with more modes

The results obtained from this case compared more accurately to the full space solution because more modes were retained in the reduced solution to accurately represent the system dynamics for all possible configurations. The full solution took over 12 minutes to compute, but accurate strain and displacement data could be obtained in less than a second without compromising accuracy. Although the addition of hard contact springs in the system causes the necessity for additional modes to be retained in a reduced order model, the full space model can still be substantially reduced to retain only the dynamic characteristics that are necessary to the response.

CONCLUSIONS

This paper demonstrates the usefulness of a proposed approach for calculating full field strain on nonlinear systems at a fraction of the computational cost. The proposed approach involves using highly reduced order models to solve the piecewise linear response of systems with discrete nonlinear elements rather than utilizing a nonlinear solution scheme on a large finite element model. The accurate prediction of dynamic displacement requires including enough modes of the original system in the reduced model to accurately capture the dynamics of the system in all possible configurations. If not enough modes are included in the reduced model, as dictated by the mode contribution matrix, then truncation will be present in the predicted nonlinear response. To accurately predict dynamic strain in a system, more modes are generally required in the reduced model than for just the prediction of dynamic displacement. Overall, the full space dynamic strain and displacement were computed accurately and efficiently on a system involving nonlinear contact elements using highly reduced order models.

NOMENCLATURE

Symbols:

$\{X_n\}$	Full Set Displacement Vector	$\{p\}$	Modal Displacement Vector
$\{X_a\}$	Reduced Set Displacement Vector	$[M]$	Physical Mass Matrix
$\{X_d\}$	Deleted Set Displacement Vector	$[C]$	Physical Damping Matrix
$[M_a]$	Reduced Mass Matrix	$[K]$	Physical Stiffness Matrix
$[M_n]$	Expanded Mass Matrix	$\{F\}$	Physical Force Vector
$[K_a]$	Reduced Stiffness Matrix	$\{\ddot{x}\}$	Physical Acceleration Vector
$[K_n]$	Expanded Stiffness Matrix	$\{\dot{x}\}$	Physical Velocity Vector
$[U_a]$	Reduced Set Shape Matrix	$\{x\}$	Physical Displacement Vector
$[U_a^G]$	Guyan Reduced Set Shape Matrix	α	Parameter for Newmark Integration
$[U_n]$	Full Set Shape Matrix	β	Parameter for Newmark Integration
		Δt	Time Step
$[U_a]^g$	Generalized Inverse	$[U_{12}]$	Mode Contribution Matrix
$[T]$	Transformation Matrix		

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