Efficient Computational Nonlinear Dynamic Analysis Using Modal Modification Response Technique

Tim Marinone, Peter Avitabile
Structural Dynamics and
Acoustic Systems Laboratory
University of Massachusetts Lowell
One University Avenue
Lowell, Massachusetts 01854

Jason Foley, Janet Wolfson
Air Force Research Laboratory
Munitions Directorate
Fuzes Branch, Eglin Air Force Base
306 W. Eglin Blvd., Bldg 432
Eglin AFB, FL 32542-5430

ABSTRACT

Generally, structural systems contain nonlinear characteristics in many cases. These nonlinear systems require significant computational resources for solution of the equations of motion. Much of the model, however, is linear where the nonlinearity results from discrete local elements connecting different components together. Using a component mode synthesis approach, a nonlinear model can be developed by interconnecting these linear components with highly nonlinear connection elements.

The approach presented in this paper, the Modal Modification Response Technique (MMRT), is a very efficient technique that has been created to address this specific class of nonlinear problem. By utilizing a Structural Dynamics Modification (SDM) approach in conjunction with mode superposition, a significantly smaller set of matrices are required for use in the direct integration of the equations of motion. The approach will be compared to traditional analytical approaches to make evident the usefulness of the technique for a variety of test cases.

INTRODUCTION

Generally, any nonlinear response analysis involves significant computation, especially if the full analytical model system matrices are used for the forced response problem. These nonlinearities can be broadly broken down into two categories: global and local, with significant effort expended on research of both. Due to the significant computational time required for these nonlinear cases, the analyst may often be unable to investigate the nonlinearities in depth, especially if a set of performance characteristics related to temperature, preload, deflection, etc. characterize the nonlinear connection elements. Thus, there is significant motivation to develop a set of reduced order models that can accurately predict nonlinear response at a substantially reduced computation time.

One area of interest involves the dynamic response of systems with nonlinear connections. These systems are typically linear, but the introduction of the local nonlinearity causes the system to become highly nonlinear. The approach taken in this paper is to treat the system as a linear system, but to model the nonlinear component as a change in the linear system by utilizing a Structural Dynamic Modification. Accordingly, direct integration schemes in conjunction with mode superposition are employed in order to maximize the efficiency of the model.

This approach was first presented by Avitabile and O'Callahan [1] where a detailed overview of the theory and a simple analytical example were provided. Friswell *et al.* [2] looked at reducing models with local nonlinearities with several different reduction schemes for a periodic solution. Lamarque and Janin [3] looked at modal superposition using a 1-DOF and 2-DOF system with impact and concluded that modal superposition had limitations due to difficulties in modeling impact. Ozguven [4] converted non-linear ODE's into a set of non-linear algebraic equations, which could be reduced by using linear modes. This technique was found to provide the best reduction in computational time when the structure was excited at a forcing frequency that corresponded to a resonance of the structure. An alternative approach that has been studied uses Non-linear Normal Modes (NNMs) which are formulated by Ritz vectors [5-6]. This approach seeks to extend the concept of linear orthogonal modes to nonlinear systems.

This paper presents the analytical time results of a beam system subjected to a force impulse. Four cases are studied: single beam with no contact, single beam with contact, multiple beams with single contact, and multiple beams with multiple contacts. For each of these cases two types of contact stiffness will be studied; a soft contact representing a rubber/isolation material, and a hard contact representing a metal on metal contact. For all cases, the time results of the full space model will be used as a reference.

THEORY

Modal Modification Response Technique (MMRT)

The MMRT technique is based on the Structural Dynamic Modification process and mode superposition method. From the modal data base of an unmodified system, structural changes can be explored using the modal transformation to project changes from physical space to modal space of the unmodified system; this results in a heavily coupled set of equations which are drastically less than those of the physical model and can be written as

$$\left[\begin{bmatrix}\ddots & & \\ & \bar{M}_1 & \\ & & \ddots\end{bmatrix}+\left[\Delta\bar{M}_{12}\right]\right]\{\ddot{p}_1\}+\left[\begin{bmatrix}\ddots & & \\ & \bar{K}_1 & \\ & & \ddots\end{bmatrix}+\left[\Delta\bar{K}_{12}\right]\right]\{p_1\}=[0] \tag{1}$$

where

$$\begin{aligned}\left[\Delta\bar{M}_{12}\right]&=[U_1]^T[\Delta M_{12}][U_1]\\ \left[\Delta\bar{K}_{12}\right]&=[U_1]^T[\Delta K_{12}][U_1]\end{aligned} \tag{2}$$

The solution of the modified set of modal equations in modal space produces an eigensolution and the resulting eigenvectors of this are the $[U_{12}]$ matrix; this contains the scaling coefficients necessary to form the final modal vectors from the starting modal vectors. This $[U_{12}]$ matrix is the key to identifying the necessary set of modal vectors to accurately predict the final modified set of modes [7]. Figure 1 illustrates the contribution of the $[U_{12}]$ matrix in forming the final modified set of modes $[U_2]$, where "m" modes of the $[U_{12}]$ matrix are used, and "n-m" modes are excluded.

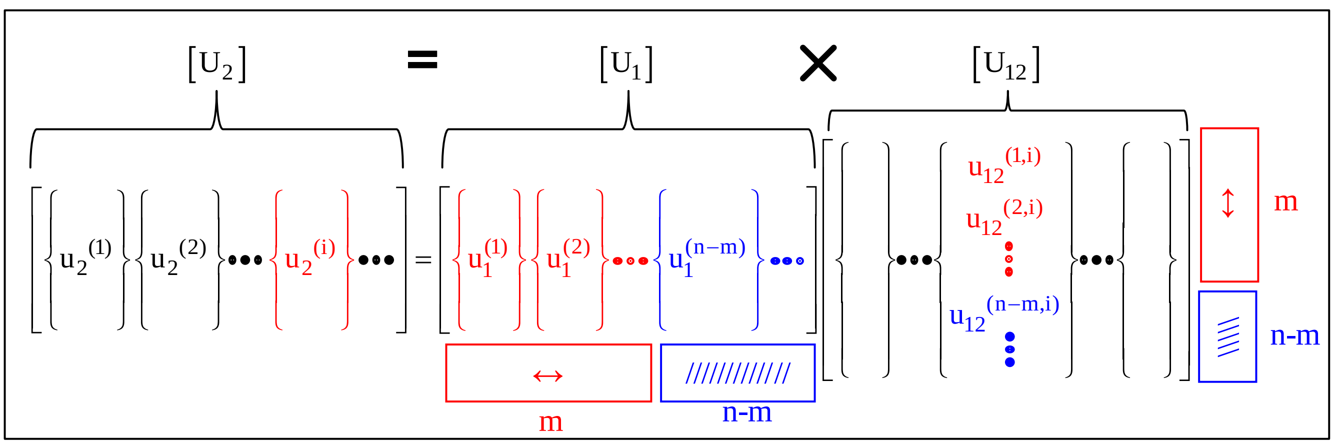


Figure 1 – Schematic for SDM process using $[U_{12}]$

Mode superposition is executed in a piecewise linear fashion depending on the "state" of the nonlinear connection element. Once the linear state changes, a structural modification is performed to update the characteristics of the system along with updated initial conditions to proceed on with the numerical integration. This equation is written as

$$\begin{bmatrix} \left[I_m^A\right] & \\ & \left[I_m^B\right] \end{bmatrix} \begin{Bmatrix} \left\{\ddot{p}^A\right\} \\ \left\{\ddot{p}^B\right\} \end{Bmatrix} + \left[\begin{bmatrix} \left[\Omega_m^{2A}\right] & \\ & \left[\Omega_m^{2B}\right] \end{bmatrix} + [U]^T [K_{TIE}][U] \right] \begin{Bmatrix} \left\{p^A\right\} \\ \left\{p^B\right\} \end{Bmatrix} = \begin{Bmatrix} \left[U^A\right]^T \left\{f^A\right\} \\ \left[U^B\right]^T \left\{f^B\right\} \end{Bmatrix} \tag{3}$$

where the initial conditions for the updated state are

$$\begin{aligned} \{p\}_j &= [U]_j^g \{x\}^{(i-1)} \\ \{\dot{p}\}_j &= [U]_j^g \{\dot{x}\}^{(i-1)} \end{aligned} \tag{4}$$

and the generalized inverse can be written as either a pseudo inverse or mass weighted inverse

$$\begin{aligned} [U]_j^g &= \left[\left[U_j\right]^T \left[U_j\right]\right]^{-1} \left[U_j\right]^T \quad \text{or} \\ [U]_j^g &= \left[\bar{M}_j\right]\left[U_j\right]^T \left[M_j\right] \end{aligned} \tag{5}$$

Figure 2 shows the schematic of this technique where the state of the system is checked at each time step by computing the physical response from mode superposition.

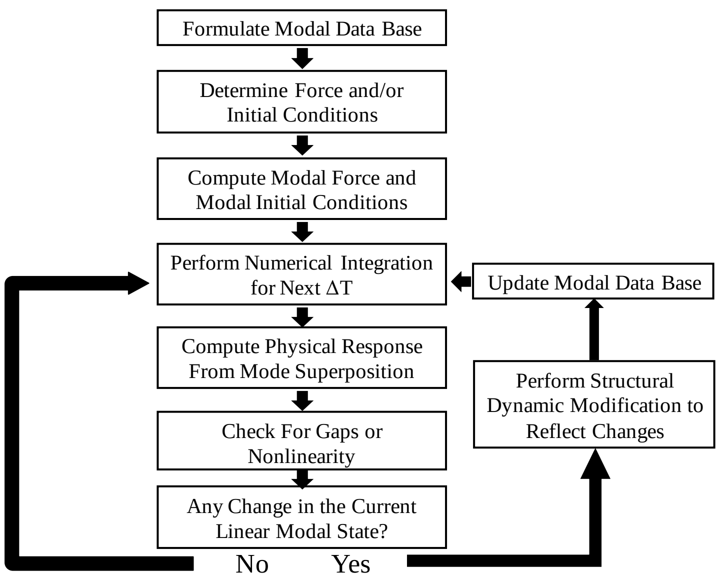


Figure 2 – Schematic for MMRT

<u>Direct Integration of the Equations of Motion</u>

The direct integration of the equations of motion used here are that of the Newmark [8] method commonly used. From the known initial conditions for displacement and velocity, the initial acceleration is

$$\ddot{\vec{x}}_0 = [M]^{-1}\left(\vec{F}_0 - [C]\dot{\vec{x}}_0 - [K]\vec{x}_0\right) \tag{6}$$

where $\ddot{x}_0$ = Initial Acceleration Vector

$\dot{x}_0$ = Initial Velocity Vector

x_0 = Initial Displacement Vector

$\vec{F}_0$ = Initial Force Vector

Choosing an appropriate Δt, α and β (the values chosen for the analytical case studies were 0.0001 s, 0.25 and 0.5 in order to satisfy sampling parameters and to give constant acceleration), the displacement vector is

$$\begin{aligned}\vec{x}_{i+1} = \left[\frac{1}{\alpha(\Delta t)^2}[M] + \frac{\beta}{\alpha\Delta t}[C] + [K]\right]^{-1} \Bigg\{ \vec{F}_{i+1} + [M]\left(\frac{1}{\alpha(\Delta t)^2}\vec{x}_i + \frac{1}{\alpha\Delta t}\dot{\vec{x}}_i + \left(\frac{1}{2\alpha} - 1\right)\ddot{\vec{x}}_i\right) \\ + [C]\left(\frac{\beta}{\alpha\Delta t}\vec{x}_i + \left(\frac{\beta}{\alpha} - 1\right)\dot{\vec{x}}_i + \left(\frac{\beta}{\alpha} - 2\right)\frac{\Delta t}{2}\ddot{\vec{x}}_i\right)\Bigg\}\end{aligned} \tag{7}$$

With the displacement vector known, the acceleration and velocity vectors are

$$\dot{\vec{x}}_{i+1} = \dot{\vec{x}}_i + (1-\beta)\Delta t\ddot{\vec{x}}_i + \beta\Delta t\ddot{\vec{x}}_{i+1} \tag{8}$$

$$\ddot{\vec{x}}_{i+1} = \frac{1}{\alpha(\Delta t)^2}(\vec{x}_{i+1} - \vec{x}_i) - \frac{1}{\alpha\Delta t}\dot{\vec{x}}_i - \left(\frac{1}{2\alpha} - 1\right)\ddot{\vec{x}}_i \tag{9}$$

Normal rules regarding integration of the equations of motion are utilized here and are not further discussed.

<u>Time Response Correlation Tools</u>

In order to quantitatively compare two different time solutions, two correlation tools (TRAC and MAC) will be used [9]. Xn_i and Xn_j are the two compared displacement vectors.

TRAC (Time Response Assurance Criterion) – Correlates single DOF across all instances in time.

$$TRAC_{ji} = \frac{\left[\{Xn_j\}^T\{Xn_i\}\right]^2}{\left[\{Xn_j\}^T\{Xn_j\}\right]\left[\{Xn_i\}^T\{Xn_i\}\right]} \tag{10}$$

MAC (Modal Assurance Criterion) – Correlates all DOF at single instance in time.

$$MAC_{ij} = \frac{\left[\{Xn_i\}^T\{Xn_j\}\right]^2}{\left[\{Xn_i\}^T\{Xn_i\}\right]\left[\{Xn_j\}^T\{Xn_j\}\right]} \tag{11}$$

Both MAC and TRAC values close to 1.0 indicate strong similarity between vectors, where values close to 0.0 indicate minimal or no similarity.

MODEL DESCRIPTION AND CASES STUDIED

This section presents the analytical models developed as well as the cases studied. The full-space time solution is used as the reference solution for all cases.

Model: Beam A and Beam B

Two planar element beam models created using MAT_SAP [10] (a FEM program developed for MATLAB [11]) are used for all the cases studied. Figure 3 shows the two beams assembled into the linear system, where the red points are the accelerometer measurements, and the blue location is the point of force applied to the system (point 14). Note that 3 inches of each beam are clamped for the cantilevered boundary condition.

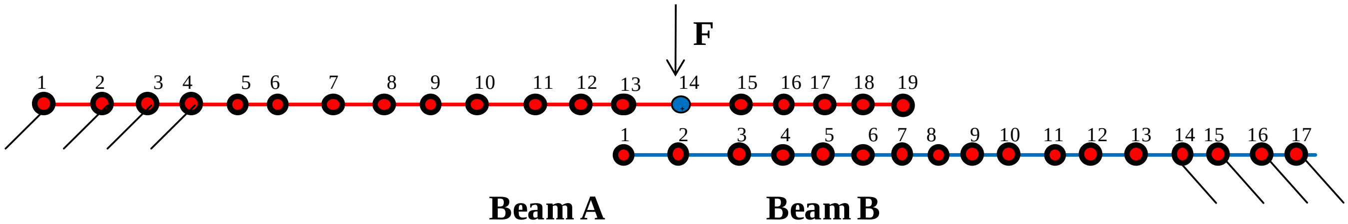


Figure 3 – Schematic of Impact and Response Points for Beam System

Table 1 lists the beam characteristics, while Table 2 lists the natural frequencies for the first 10 modes.

Table 1 – Model Beam Characteristics

Beam	Length (in)	Width (in)	Thickness (in)	Num. of Elements	Num. of Nodes	Num. of DOF	Node Spacing (in)	Material	Density (lb/in^3)	Young's Modulus (Msi)
A	18	2	0.123	72	73	146	0.25	Aluminum	2.54E-04	10
B	16	4	0.123	64	65	130	0.25	Aluminum	2.54E-04	10

Table 2 – Model Beam Frequencies

Mode #	1	2	3	4	5	6	7	8	9	10
Beam A	12.91	84.12	252.34	519.59	806.16	1256.55	1682.96	2201.36	2755.52	3510.01
Beam B	22.62	141.56	396.6	776.92	1284.71	1918.28	2678.33	3563.89	4572.7	5707.04

The force pulse is an analytic force pulse designed to be frequency band-limited, exciting modes up to 1000 Hz while minimally exciting the higher order modes. Using this force pulse, the number of modes that will be involved in the response can be determined easily, as modes above 1000 Hz can be considered of negligible importance in the response. Modes 1 to 5 will be primarily excited in Beam A, while modes 1 to 4 will be primarily excited for Beam B. Figures 4a and 4b show the force pulse in the time and frequency domain. Damping was assumed 1% of critical damping for all modes (both unmodified and system).

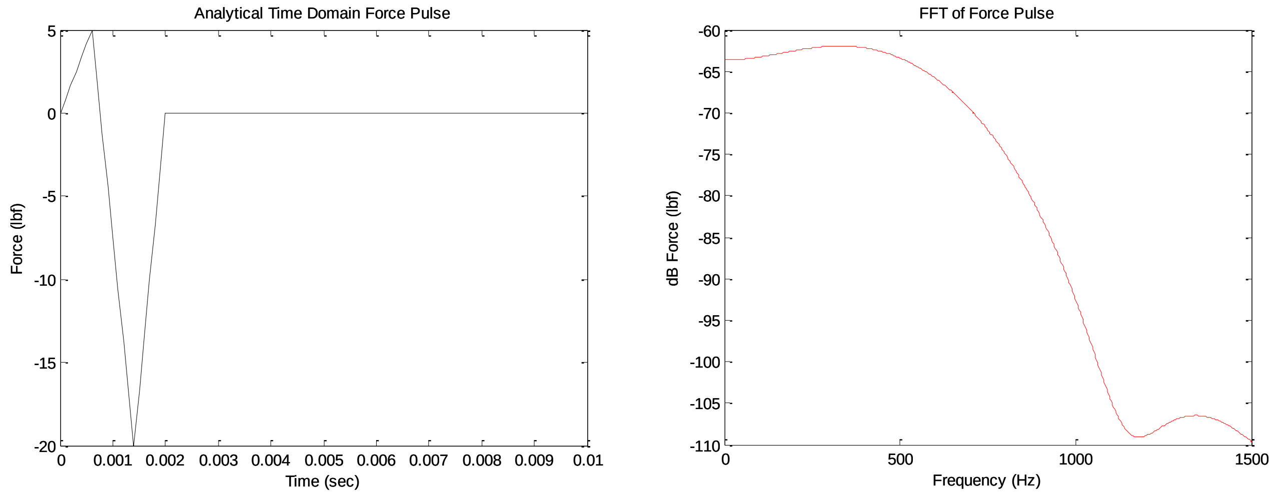


Figure 4 – Analytical Force Pulse in the Time (a) and Frequency (b) Domain

<u>Case 1: Single Beam with No Contact</u>

For the first case, the system is a single beam (Beam A) with no contact. As this system is linear and because there is no SDM performed, the number of modes needed is only a function of the input excitation. Based on the input force spectrum seen previously in Figure 4b, the time response needs five modes. In order to determine if this assumption is accurate, the FFT of the response of system was plotted using an increasing number of modes and is compared to the full solution (using all 146 modes) in Figure 5.

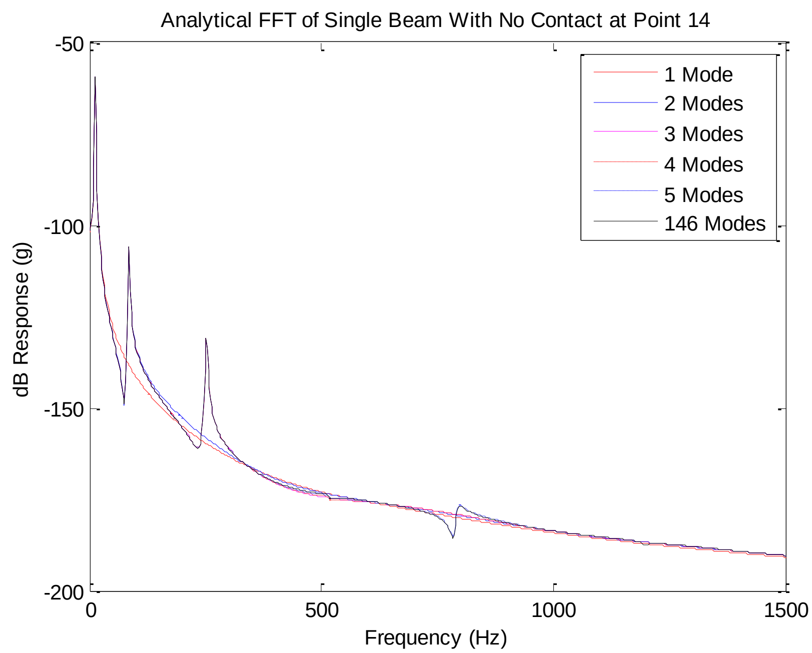


Figure 5 – FFT of time response for single beam no contact using varying number of modes.

The FFT in Figure 5 shows that modes 1 to 3 have a large magnitude, in contrast to modes 4 and 5, which are minimally excited. Figure 5 further shows that a much smaller mode set is required for the time response, as the force pulse has only excited the first few modes of the system. Based on these results, a minimum of three modes should be used in order to approximate the full space model accurately. In order to confirm this, the time response at point 14 will be computed using modes 1 through 5 and will be compared to the full time solution in Figures 6 a-e. In addition, Figure 6f will be computed using all modes (excluding mode 1) to show the effect if a key mode (one with significant magnitude) is not included.

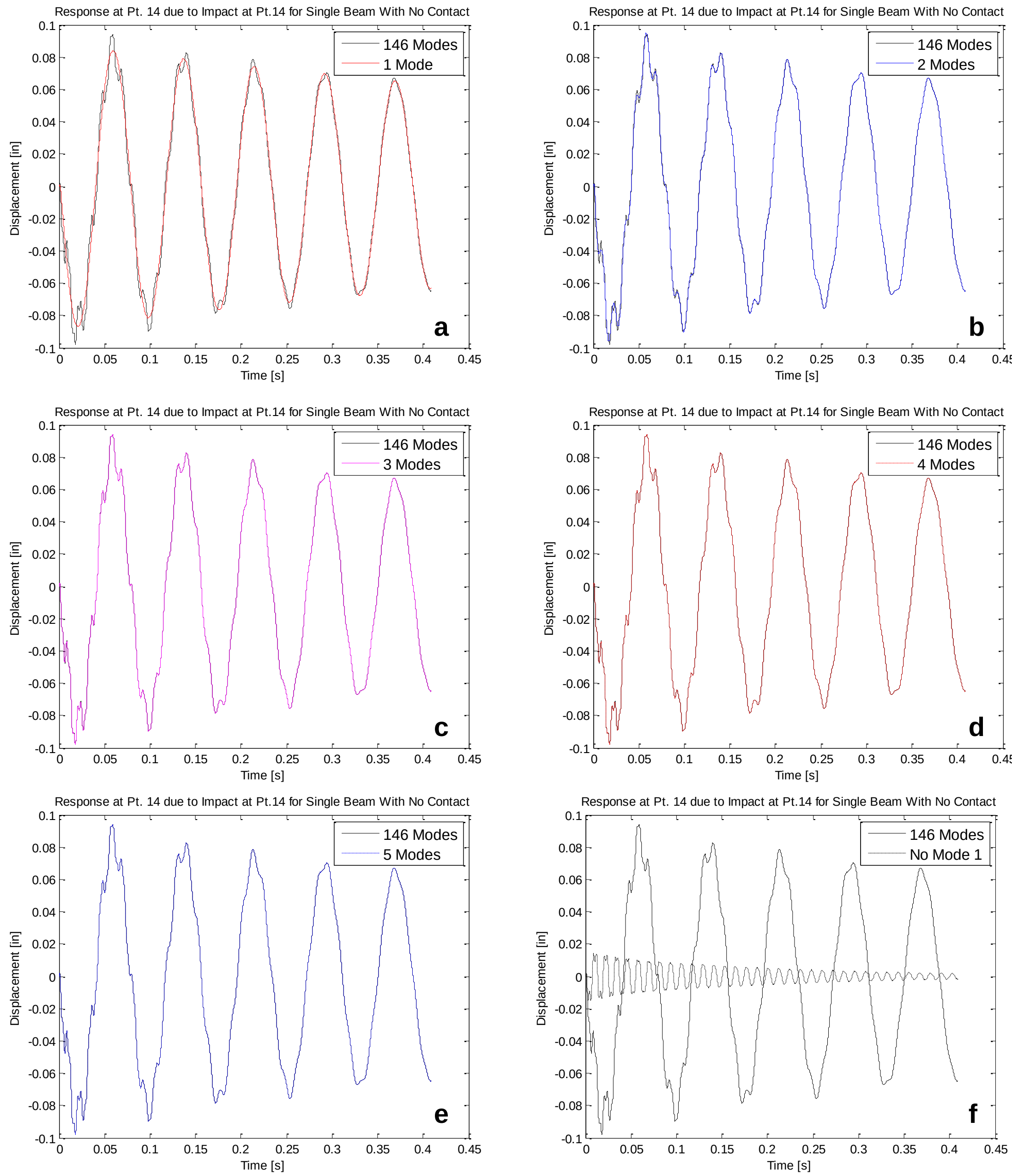


Figure 6 – Comparison of full solution results for single beam with no contact to results using modes 1 (a), 1-2 (b), 1-3 (c), 1-4 (d), 1-5 (e) and all modes but mode 1 (f).

As seen in Figures 6a and 6b, using only one or two modes is not sufficient to accurately reproduce the full space model, as the input force spectrum excited additional higher order modes. Once the third mode was added, however, the response overlays almost perfectly and the further addition of modes 4 and 5 have a negligible effect. Finally, even though 145 of the 146 modes were used in Figure 6f, the exclusion of mode 1 prevents the accurate reconstruction of the full time response. In order to further show the effect of mode truncation, the MAC and TRAC of the time responses were averaged as listed in Table 3.

Table 3 – Average MAC and TRAC vs. # of modes used for single beam no contact solution.

# of Modes	Average MAC	Average TRAC	Solution Time (s)
1	0.9439	0.8098	1.67
2	0.9942	0.9044	2.13
3	0.9985	0.9862	2.81
4	0.9986	0.9876	3.52
5	0.9994	0.9991	4.51
146	1.0000	1.0000	99.25

Table 3 indicates that TRAC is more sensitive than MAC for determining the accuracy of time responses, as MAC is weighted by DOFs with large responses. Table 3 also provides further confirmation that the first three beam modes are the most critical modes for this time response, as the TRAC improves significantly when adding modes 2 and 3, but has a minimal improvement when adding the higher order modes. Finally, the solution time for the full system is close to 100 seconds in contrast to the modal model, which is less than 5 seconds. MMRT is shown to have noticeable improvements in computational time even for the linear system.

In order to demonstrate that the number of modes needed is controlled by the forcing function, the same beam configuration was run where the analytical force input was changed to excite the first 250 Hz. As fewer modes are excited over this new frequency bandwidth, the response is affected by fewer modes. Comparing the previous results for using 1 and 2 modes in Table 3, the TRAC improves to 0.826 when using 1 mode, and improves to 0.993 when using 2 modes. These results show that decreasing the bandwidth of the forcing function reduces the effect of mode truncation, as the impact spectrum does not distribute energy to the higher frequency modes.

This first case demonstrated that an accurate time solution could be obtained using a limited number of modes, if the primary modes excited by the structure are included in the modal database. If the wrong selection of modes is used, the analyst will not obtain the correct time response, regardless of how many other modes are used.

The following cases will show the application of the MMRT when there is contact and the system becomes nonlinear. In addition, both soft and hard contacts will be studied to show the effect of different contact stiffness on the accuracy of the modal models. The soft stiffness case will be studied first, as this contact stiffness is unlikely to excite a high frequency range.

CASE A: SOFT CONTACT

Case A-2: Single Beam with Contact

This case consists of the tip of Beam A coming into contact with a fixed object once the beam has displaced a known gap distance (0.05 inches) shown in Figure 7.

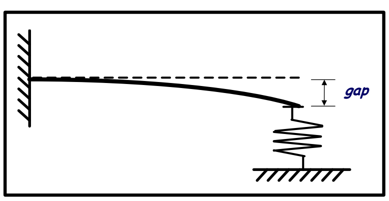
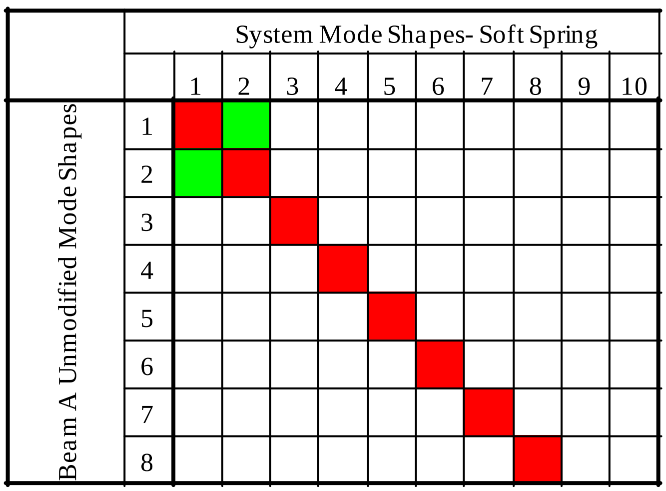


Figure 7 – Diagram of Single Beam with Contact

The contact is represented as an additional spring stiffness that is added as soon as Beam A closes the gap. The contact stiffness of 10 lb/in will be used to represent a soft contact, such as typically seen in a damper or isolation mount. For the purposes of MMRT, the spring stiffness was applied to the original model as a SDM. In addition, the contact is only represented as a spring stiffness in the translational dof, not in the rotational dof as well. Table 4 lists the frequency values of the SDM system, while Figure 8 shows the [U_{12}] matrix, or the contribution of the original modes in the SDM (a highlighted box indicates that the mode of the original model contributes to the SDM model with a magnitude greater than 1.2% **-** the actual range for each color is shown below).

Table 4 – Frequencies for Single Beam with Contact for Soft Spring

Mode #	**Unmodified**	**Soft Spring - SDM**
1	12.91	26.09
2	84.12	86.02
3	252.34	252.54
4	519.59	519.65
5	806.16	806.18
6	1256.55	1256.55
7	1682.96	1682.96
8	2201.36	2201.36



Bar Color		**Min. Value**	**Max Value**
	Black	0.0120	0.0200
	Blue	0.0200	0.0500
	Green	0.0500	0.1000
	Cyan	0.1000	0.2000
	Magenta	0.2000	0.5000
	Yellow	0.5000	0.8000
	Red	0.8000	1.0000

Figure 8 – Mode Contribution Matrix for single beam with soft single contact

Table 4 and Figure 8 shows that the addition of the spring has a pronounced effect on the lower order modes, with higher order modes remaining relatively unaffected. The soft spring has a noticeable effect on the first two modes, as modes 3 and up are approximately at the same frequency and require no additional mode shape to be added to form the SDM modal matrix (as indicated by the red main diagonal). From Case 1 (single beam with no contact), the modes needed in the MMRT process are the modes which are excited by the input force spectrum. For the forcing function used, these should be the first 3 modes of the system at a minimum, with additional higher order modes having minimal effect. Examining Figure 8 shows that only modes 1 and 2 are used in the SDM process, so using only 3 modes should provide a high correlation to the full response solution.

In order to confirm this, the time response at point 14 will be computed using modes 1 through 5 and will be compared to the full time solution in Figures 9 a-e for the soft spring. In addition, Figure 9f will show the time response when mode 3 is not included when forming the SDM modal database.

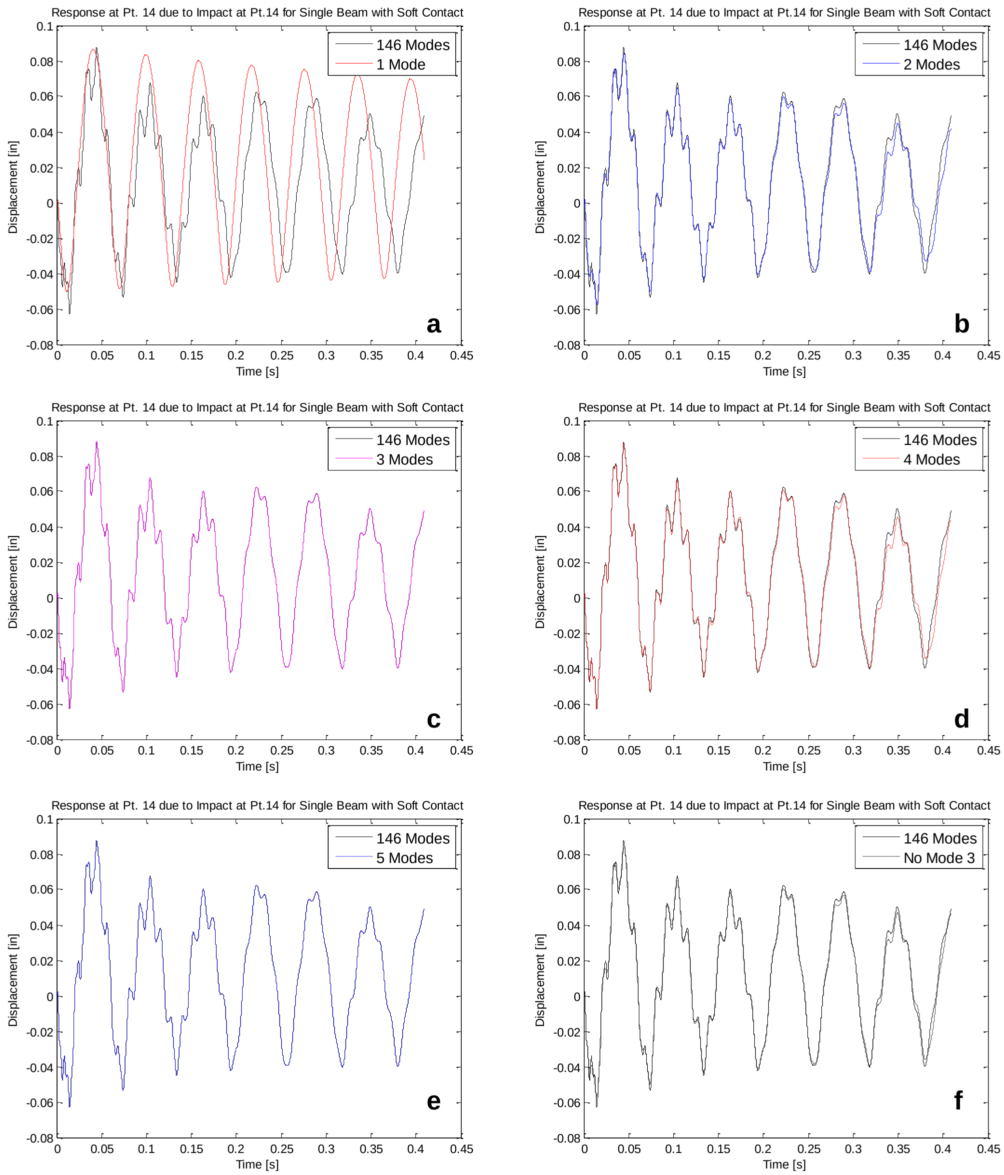


Figure 9 – Comparison of full solution results for single beam with soft contact to results using modes 1 (a), 1-2 (b), 1-3 (c), 1-4 (d), 1-5 (e) and all modes but mode 3 (f).

As the first three modes of the SDM database (the primary modes excited by the forcing function) only required the first 3 modes of the unmodified modal database with the soft spring, three modes were adequate to represent the full system model with a high degree of accuracy. As seen previously, the addition of modes 4 and 5 had minimal effect due to the input force spectrum drops off at the higher frequencies. Figure 9f showed that the exclusion of mode 3 from the modal database had a small negative effect. As mode 3 did not contribute to any SDM modes other than mode 3, however, the response was reasonable comparable to the full space solution (average TRAC = 0.884).

The MAC and TRAC of the time responses were averaged as listed in Table 5 to show the effect of the additional modes.

Table 5 – Average MAC and TRAC vs. # of modes used for single beam contact solution.

# of Modes	Soft		
	Average MAC	**Average TRAC**	**Solution Time (s)**
1	0.8816	0.2884	1.68
2	0.9670	0.8858	1.98
3	0.9910	0.9699	2.54
4	0.9659	0.9677	3.78
5	0.9893	0.9860	4.63
Full	1.0000	1.0000	186.25

As explained previously, the first 3 modes were sufficient to approximate the soft spring solution with a high degree of accuracy. Finally, using MMRT causes a drastic reduction in solution time as seen comparing the solution time from the full solution to the time using only a few modes.

This second case demonstrated that the analyst should examine the [U_{12}] matrix to determine the number of modes that are used in the SDM modal. Even though the forcing input may only excite a few modes, the SDM modal database may need additional modes. Failure to include these modes will produce an incorrect time response regardless of how many other modes are used.

Case A-3: Multiple Beams with Single Contact

This case consists of the tip of Beam A coming into contact with Beam B once Beam A has displaced a known gap distance (0.05 inches) shown in Figure 10.

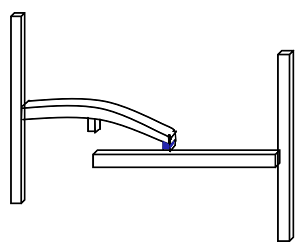
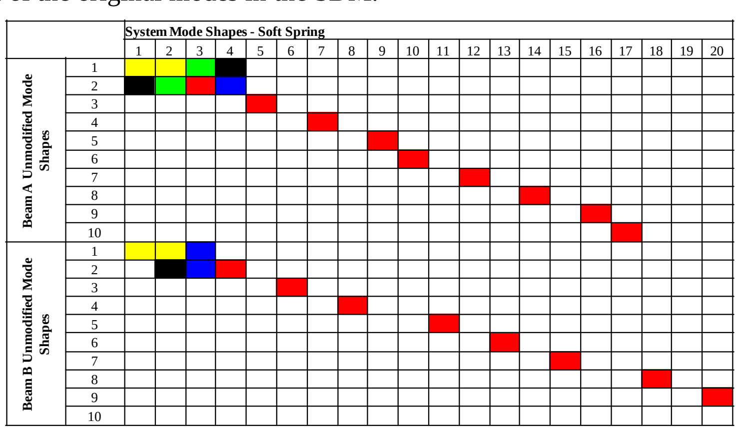


Figure 10 – Diagram of Multiple Beam with Single Contact

The model uses the same soft stiffness contact of 10 lb/in to represent the contact of the beams as explained in Case A-2. Table 6 lists the frequency values of the SDM system along with which beam is excited, while Figure 11 shows the contribution of the original modes in the SDM.



Bar Color	Min. Value	Max Value
Black	0.0120	0.0200
Blue	0.0200	0.0500
Green	0.0500	0.1000
Cyan	0.1000	0.2000
Magenta	0.2000	0.5000
Yellow	0.5000	0.8000
Red	0.8000	1.0000

Figure 11 – Mode Contribution Matrix for multiple beams with single contact for soft spring

Table 6 – Frequencies for Multiple Beams with Single Contact for Soft Spring

Mode #	Unmodified	SDM - Soft Spring
1	Beam A - 12.91	20.35
2	Beam B - 22.62	29.58
3	Beam A - 84.12	86.01
4	Beam B - 141.56	142.43
5	Beam A - 252.34	252.54
6	Beam B - 396.6	396.71
7	Beam A - 519.59	519.65
8	Beam B - 776.92	777.00
9	Beam A - 806.16	806.18
10	Beam A - 1256.55	1256.55
11	Beam B - 1284.71	1284.80
12	Beam A - 1682.96	1682.96
13	Beam B - 1918.28	1918.28
14	Beam A - 2201.36	2201.36
15	Beam B - 2678.33	2678.39
16	Beam A - 2755.52	2755.53
17	Beam A - 3510.01	3510.02
18	Beam B - 3563.89	3563.89
19	Beam A - 3948.54	3948.55
20	Beam B - 4572.70	4572.72

Due to the contact between the two beams, the first mode of the SDM system now consists of modes from both beams, which increases the number of modes needed for the SDM modal database. Based on the force input, which excites up to 1000 Hz, the first 10 modes of the system should be excited. Examining Figure 11 shows that for the first 10 modes of the SDM system, the soft spring modal database requires modes 1 through 6 of Beam A and modes 1 through 4 of Beam B. Note that the number of modes used in the time response is controlled by the frequency bandwidth of the input forcing function, while the number of modes used in the SDM database is controlled by the $[U_{12}]$ matrix.

The time solution using the expected number of modes in the time response (10) with the required number of modes for the SDM (6 from Beam A and 4 from Beam B) will be computed and compared to the full solution in Figure 12a. In addition, the effect of including incorrect modes in the time response will be shown by using the same modes in the time response, where the SDM modal database contains all modes of Beam A but no modes of Beam B in Figure 12b.

Figure 12a shows that including the necessary modes (6 from Beam A and 4 from Beam B) needed to form the $[U_2]$ matrix based on the number of modes (10) excited by the input forcing function is enough to accurately reproduce the time solution. If not all of the necessary modes are included as seen in Figure 12b where no modes from Beam B were included, using all of the modes of Beam A is not sufficient to accurately compute the response.

In contrast to the model of the single beam, more than 5 modes were needed in order to obtain an accurate time solution of the system, due to the need for mode shapes from both beams. The MAC and TRAC of the time responses were averaged as listed in Table 7.

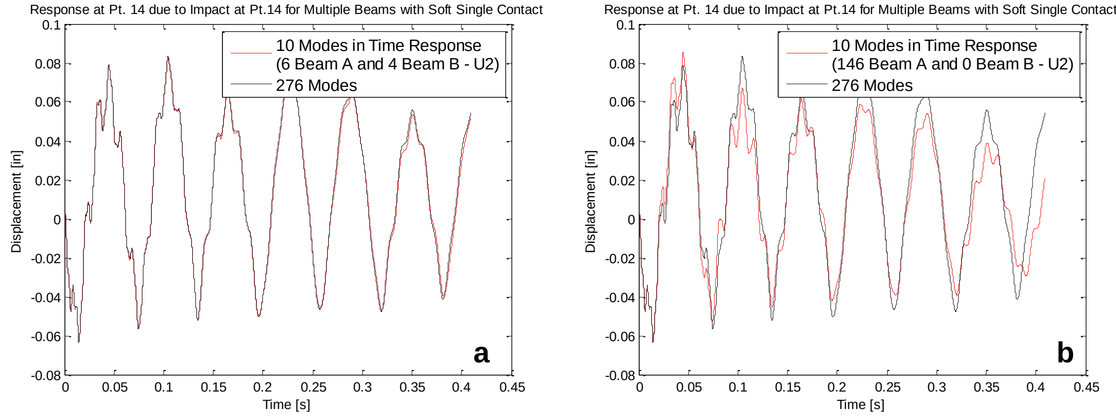


Figure 12 – Comparison of results using 10 modes in the time response with the [U_2] matrix formed from 6 Beam A and 4 Beam B modes (a) and 146 Beam A and 0 Beam B modes (b) for multiple beam single contact with soft spring.

Table 7 – Average MAC and TRAC vs. # of modes used for multiple beam single contact solution.

Soft Spring			
# of Modes	**Average MAC**	**Average TRAC**	**Solution Time (s)**
10 – Time (6 Beam A+ 4 Beam B) – [U_2]	0.9998	0.9916	6.78
10 – Time (146 Beam A + 0 Beam B) – [U_2]	0.8146	0.6167	6.78
276	1.0000	1.0000	1137.74

The third case demonstrated that the mode shapes of both components affect the modeling of nonlinear systems. The forcing function bandwidth governs the number of modes needed to compute the time response and may be different from the number of modes needed to compute the SDM modal database. The [U_{12}] matrix is critical in order to determine the number of modes needed to form the system mode shapes, and the exclusion of modes used in the matrix can noticeably degrade the correlation. For the soft spring contact, using the predicted number of mode shapes based on the forcing function and [U_{12}] matrix provided an accurate analytical time solution.

Case A-4: Multiple Beams with Multiple Contacts

This case consists of the tip of Beam A coming into contact with the tip of Beam B shown in Figures 13a, b, and c. Note that each system is a potential configuration of the beam depending on the relative displacements of the two beams.

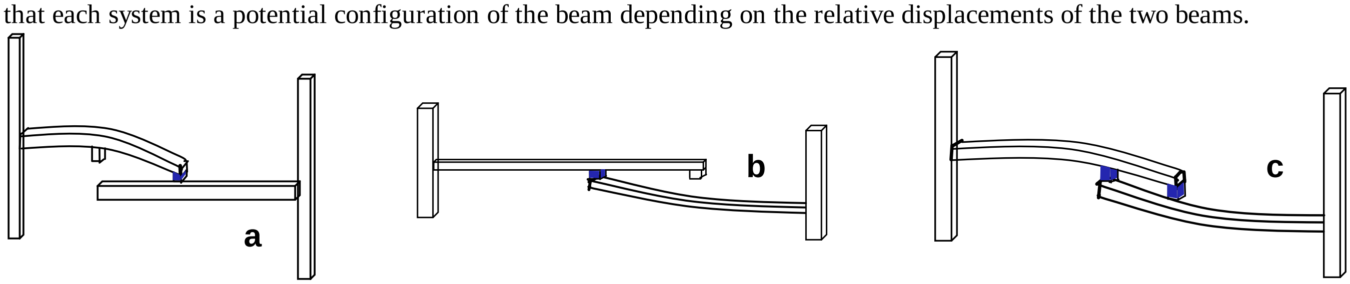
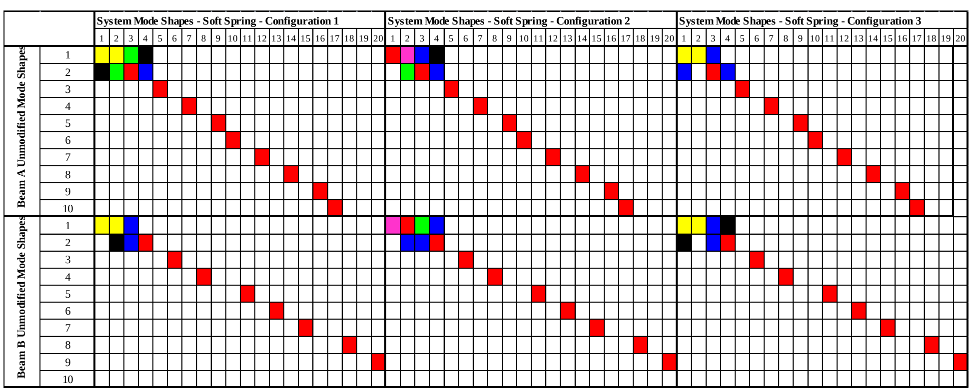


Figure 13 – Diagram of Multiple Beam with Multiple Contact for Configurations 1(a), 2 (b) and 3 (c)

The model uses the same soft spring stiffness value as described in Case A-3. Because the system has multiple possible configurations, multiple SDMs are required in order to represent the different possible configurations. The modal database must therefore contain all of the necessary modes for all of the configurations in order to obtain an accurate model. Table 8 lists the frequency values of the SDM system, while Figure 14 shows the contribution of the original modes in the SDM.

Table 8 – Frequencies for Multiple Beams with Single Contact for Soft Spring

Mode #	Unmodified	Soft Spring		
		Config. 1	Config. 2	Config. 3
1	Beam A - 12.91	20.35	14.78	21.08
2	Beam B - 22.62	29.58	33.04	39.23
3	Beam A - 84.12	86.01	85.71	87.24
4	Beam B - 141.56	142.43	142.98	143.82
5	Beam A - 252.34	252.54	252.61	252.82
6	Beam B - 396.6	396.71	396.96	397.07
7	Beam A - 519.59	519.65	519.61	519.68
8	Beam B - 776.92	777	777.04	777.12
9	Beam A - 806.16	806.18	806.28	806.3
10	Beam A - 1256.55	1256.55	1256.56	1256.56
11	Beam B - 1284.71	1284.80	1284.76	1284.85
12	Beam A - 1682.96	1682.96	1682.99	1682.99
13	Beam B - 1918.28	1918.28	1918.29	1918.30
14	Beam A - 2201.36	2201.36	2201.37	2201.37
15	Beam B - 2678.33	2678.39	2678.34	2678.39
16	Beam A - 2755.52	2755.53	2755.53	2755.53
17	Beam A - 3510.01	3510.02	3510.02	3510.03
18	Beam B - 3563.89	3563.89	3563.89	3563.89
19	Beam A - 3948.54	3948.55	3948.54	3948.55
20	Beam B - 4572.70	4572.72	4572.70	4572.72

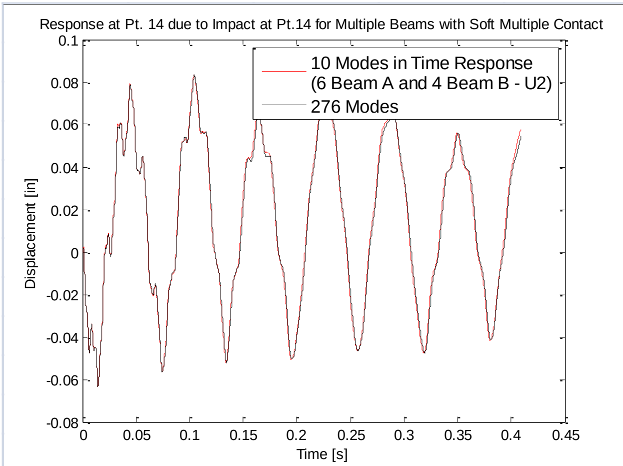


Bar Color	Min. Value	Max Value
Black	0.0120	0.0200
Blue	0.0200	0.0500
Green	0.0500	0.1000
Cyan	0.1000	0.2000
Magenta	0.2000	0.5000
Yellow	0.5000	0.8000
Red	0.8000	1.0000

Figure 14 – Mode Contribution Matrix for multiple beams with multiple contacts for soft spring

Table 8 and Figure 14 shows that not only does the stiffness of the spring affect the number of modes needed, but the location of the spring affects the modes as well. Depending on whether the spring is at the tip of Beam A or tip of Beam B, the mode shapes and frequencies change noticeably. For example, the second mode of Beam A is used for the first mode of the soft spring SDM for configurations 1 and 3, but is not needed in configuration 2. Thus, if only configuration 2 came into play this mode could be neglected in the modal database; due to also using configurations 1 and 3 that require this second mode, this mode should be used in the database.

Examining Figure 14 shows that for a forcing function which excites the first 10 modes of the system, the soft spring configuration requires modes 1-6 of Beam A and modes 1-4 of Beam B. Figure 15 compares the full solution to the solution using 10 modes for the time solution with 6 modes from Beam A and 4 modes from Beam B for the $[U_2]$ matrix for the soft spring.



Soft Spring			
# of Modes	Average MAC	Average TRAC	Solution Time (s)
10 – Time (6 Beam A + 4 Beam B) – [U2]	0.9884	0.9774	11.75
276	1	1	915.21

Figure 15 – Comparison of results in the time domain for multiple beams with multiple contacts with soft spring.

As seen in Case 3, the $[U_{12}]$ matrix provides a good understanding of the soft spring model. Even with a more complex model with multiple possible configurations, the computational savings gained from the MMRT technique are substantial. This case used all of the lessons learned from the previous three cases in order to compute an accurate time solution using MMRT with a subset of the modes and demonstrate the usefulness of the technique on a complex structure with multiple potential interactions.

CASE B: HARD CONTACT

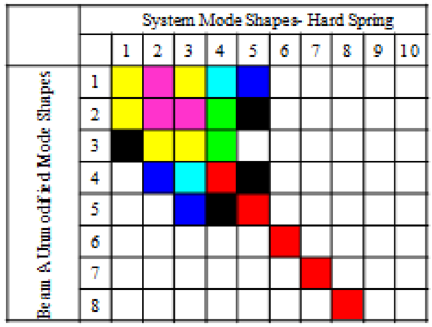
For the soft stiffness contact cases shown above, the contact stiffness was soft enough that the contact did not excite a frequency bandwidth beyond the excitation force spectrum. Thus, the forcing function controls the number of modes needed in the time response, and the number of modes needed in the SDM modal database is based on those modes. As long as the excitation bandwidth due to the contact stiffness impact is below the input bandwidth, the procedure to identify the number of modes needed works well as seen in Cases A-2 through A-4.

For a harder stiffness contact case, however, there is a possibility that the contact stiffness would excite a frequency bandwidth beyond the excitation force spectrum. Under this scenario, the number of modes needed in the time response would be a function of not only the forcing function, but also of the contact stiffness. In order to examine this possible scenario in detail, the contact stiffness of 1000 lb/in is used for the same cases.

<u>Case B-2: Single Beam with Contact</u>

This case consists of the tip of Beam A coming into contact with a fixed object once the beam has displaced a known gap distance (0.05 inches) as explained previously in Case A-2. Figure 16 shows the $[U_{12}]$ matrix and lists the natural frequencies of the first 8 modes.

Mode #	Unmodified	Hard Spring - SDM
1	12.91	66.68
2	84.12	223.55
3	252.34	326.04
4	519.59	529.68
5	806.16	808.16
6	1256.55	1256.73
7	1682.96	1682.97
8	2201.36	2201.52



Bar Color		Min. Value	Max Value
	Black	0.0120	0.0200
	Blue	0.0200	0.0500
	Green	0.0500	0.1000
	Cyan	0.1000	0.2000
	Magenta	0.2000	0.5000
	Yellow	0.5000	0.8000
	Red	0.8000	1.0000

Figure 16 – Mode Contribution Matrix for single beam with single contact for soft (a) and hard (b) spring

Figure 16 shows that in order to accurately obtain the first 5 system modes, the first 5 modes of the unmodified system are required. This is expected, as the mode shapes with the harder spring attached look less like the original model and therefore require more modes in order to form the SDM. As Figure 16 shows that modes 1 to 5 of the unmodified system are involved in the first 3 modes of the SDM, 5 modes should be used to provide a high correlation to the full response solution. In order to confirm this, the time response at point 14 will be computed using 1 through 5 modes and compared to the full time solution in Figures 17 a-e for the hard spring. In addition, Figure 17f will show the time response when mode 3 is not included when forming the SDM modal database.

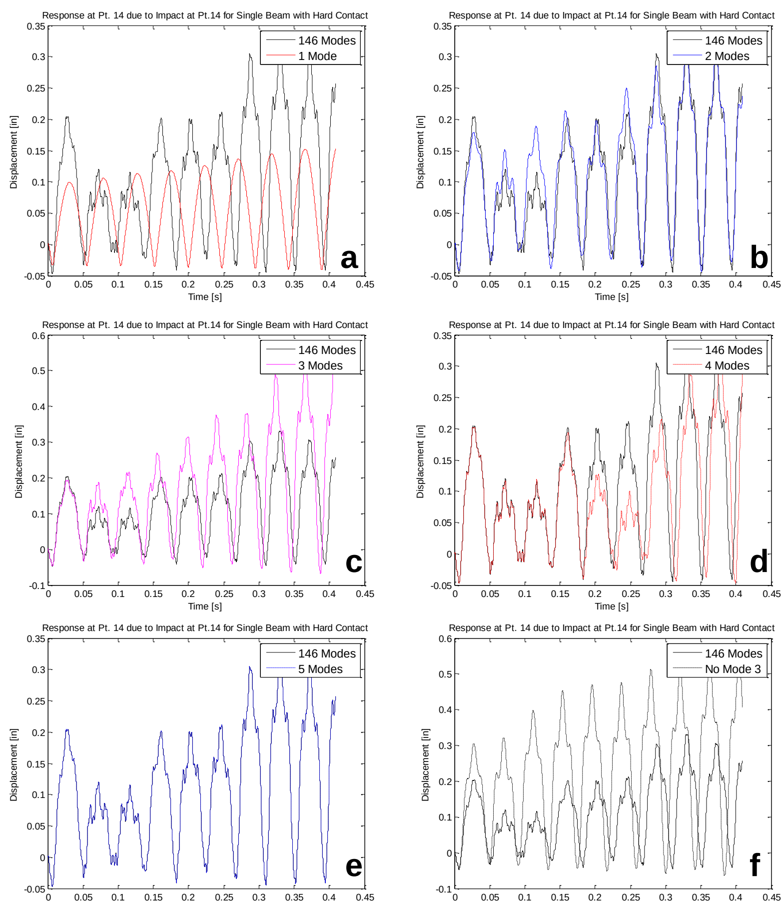


Figure 17 – Comparison of full solution results for single beam with hard contact to results using modes 1 (a), 1-2 (b), 1-3 (c), 1-4 (d), 1-5 (e) and all modes but mode 3 (f).

In contrast to Case A-2, using the hard spring produced poor results using modes 1 to 4. As seen in Figure 16, modes 2 and 3 of the SDM database require modes 1 to 5 of the unmodified modal database and were not able to be adequately represented using only modes 1 to 4. Even though the forcing function was limited primarily to the first 3 modes of the system, the first 3 modes of the SDM required more than 3 modes of the unmodified modal database. Figure 17f showed that the exclusion of mode 3 from the modal database decreased the accuracy of the results. Examining Figure 16 shows that mode 3 had a substantial contribution to SDM modes 1 to 4, and thus the SDM modes could not be fully represented without the missing mode. The response is poor compared to the full space solution (average TRAC = 0.439)

In order to show the effect of the modes, the MAC and TRAC of the time responses were averaged as listed in Table 11.

Table 11 – Average MAC and TRAC vs. # of modes used for single beam contact solution.

# of Modes	Hard		
	Average MAC	**Average TRAC**	**Solution Time (s)**
1	0.8672	0.3591	1.65
2	0.8586	0.5056	2.29
3	0.8569	0.5479	3.01
4	0.8935	0.6615	3.73
5	0.9992	0.9949	4.27
Full	1.0000	1.0000	298.91

As the hard spring solution required five modes of the unmodified modal database, the response remained poor until all five of the modes were included as seen in the sudden increase in TRAC (0.6615 to 0.9949).

Finally, the FFT of the time response for both the soft and hard spring is shown in Figure 18. Note that the hard spring FFT is more nonlinear in contrast to the soft spring, where the main frequencies of the system can still be identified clearly. The hard spring does not contain significant energy beyond 1000 Hz, showing that for this case, the forcing function and not the contact stiffness govern the number of modes needed in the time response.

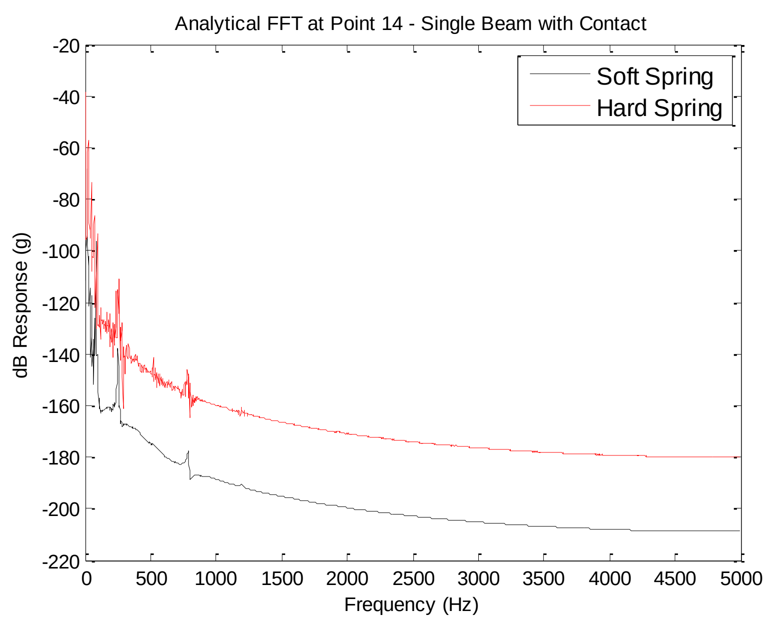


Figure 18 – FFT comparison between the soft and hard spring for single beam with contact

Case B-3: Multiple Beams with Single Contact

This case consists of the tip of Beam A coming into contact with the tip of Beam B once Beam A has displaced a known gap distance (0.05 inches) as explained previously in Case A-3. Table 12 lists the frequency values of the SDM system along with which beam is excited, while Figure 19 shows the $[U_{12}]$ matrix.

Table 12 – Frequencies for Multiple Beams with Single Contact for Hard Spring

Mode #	Unmodified	SDM - Hard Spring
1	Beam A - 12.91	21.25
2	Beam B - 22.62	67.74
3	Beam A - 84.12	127.04
4	Beam B - 141.56	236.02
5	Beam A - 252.34	338.31
6	Beam B - 396.6	424.26
7	Beam A - 519.59	532.04
8	Beam B - 776.92	786.59
9	Beam A - 806.16	809.25
10	Beam A - 1256.55	1256.69
11	Beam B - 1284.71	1294.66
12	Beam A - 1682.96	1682.97
13	Beam B - 1918.28	1918.72
14	Beam A - 2201.36	2201.53
15	Beam B - 2678.33	2683.90
16	Beam A - 2755.52	2755.78
17	Beam A - 3510.01	3510.46
18	Beam B - 3563.89	3564.14
19	Beam A - 3948.54	3948.79
20	Beam B - 4572.70	4575.19

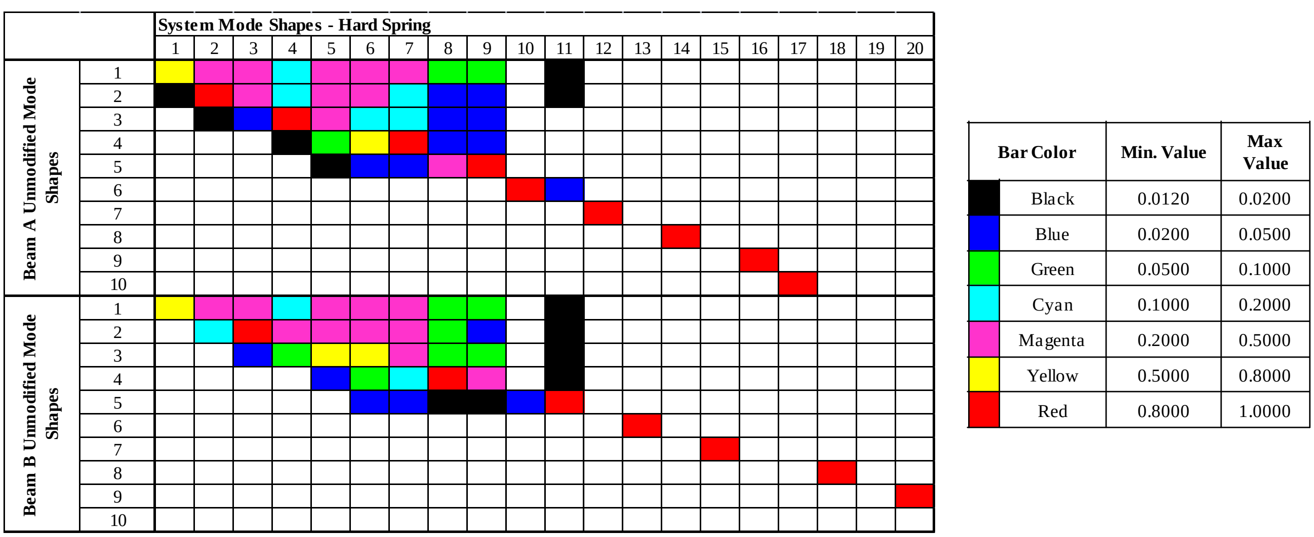


Figure 19 – Mode Contribution Matrix for multiple beams with single contact for hard spring

As seen in Case A-3, the forcing function excited the first 10 modes of the system. Examining Figure 19 shows that for the first 10 modes of the SDM system, the hard spring modal database requires modes 1-6 of Beam A and modes 1-5 of Beam B. The time solution using the expected number of modes in the time response (10) with the required number of modes for the SDM (6 from Beam A and 5 from Beam B) will be computed and compared to the full solution in Figure 20.

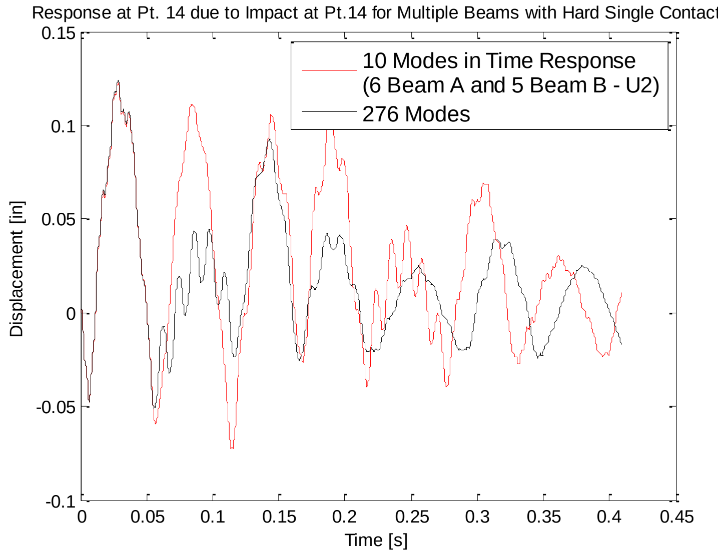


Figure 20 – Comparison of results using 10 modes in the time response with the [U_2] matrix formed from 6 Beam A and 5 Beam B modes for multiple beam single contact with hard spring.

The expected number of modes used for the time response (10) with the modes used in the SDM modal database (6 from Beam A and 5 from Beam B), does not produce an accurate solution as seen in Figure 20. In contrast, Case A-3 with the soft spring did produce an accurate solution using the same mode set as seen in Figure 12a. Figure 21 shows the time response using 23 modes for the time response, where 20 modes from Beam A and 15 modes from Beam B formed the SDM database.

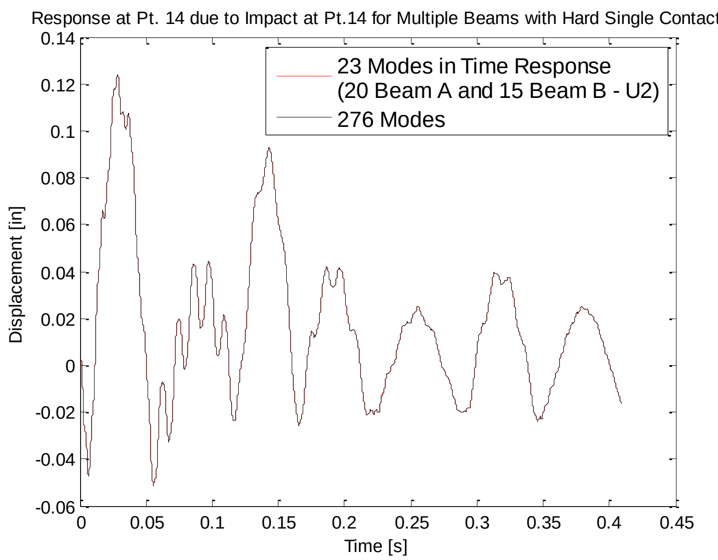


Figure 21 – Comparison of results using 23 modes in the time response with the [U_2] matrix formed from 20 Beam A and 15 Beam B modes for multiple beam single contact with hard spring.

Figure 21 shows that additional modes are needed beyond the expected number of modes (based only on the input force) in order to compute the accurate time response of the hard spring system. To investigate why using the expected number of modes did not produce an accurate time response for the hard spring stiffness, the FFT of the time response for both springs is shown in Figure 22.

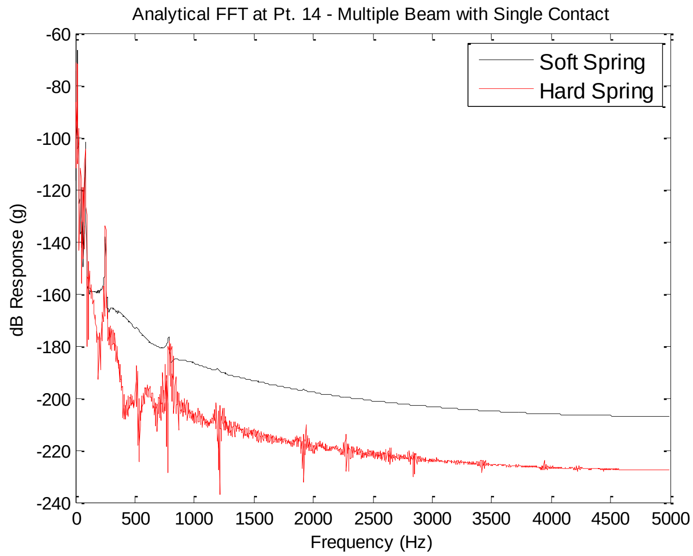


Figure 22 – FFT comparison between the soft and hard spring for multiple beam with single contact

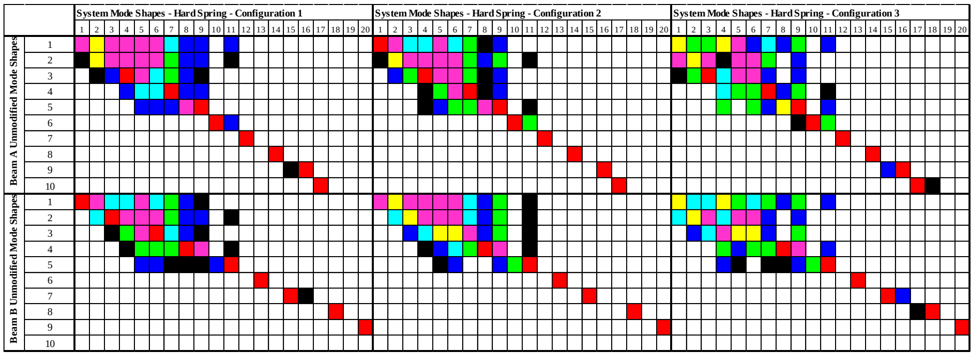
Note that in contrast to the soft spring, the hard spring contact excited modes above 1000 Hz (almost up to 3000 Hz), which is above the input force spectrum. Accordingly, the number of modes needed in the time response is no longer governed by the forcing function bandwidth, but by the contact stiffness bandwidth. This explains why Figure 21 required additional modes in order to obtain an accurate answer, as the contact stiffness was exciting modes that were not included when the first 10 modes of the system formed the database. Since these unused higher order modes were excited by the contact stiffness, they needed to be included to obtain a correct time solution. In order to show the effect of the modes, the MAC and TRAC of the time responses were averaged as listed in Table 13.

Table 13 – Average MAC and TRAC vs. # of modes used for multiple beam single contact solution.

Hard Spring			
# of Modes	**Average MAC**	**Average TRAC**	**Solution Time (s)**
10 – Time (6 Beam A + 5 Beam B) – U2	0.7888	0.5232	6.87
23 – Time (20 Beam A + 15 Beam B) – U2	1.0000	0.9988	17.11
276	1.0000	1.0000	1240.13

Case B-4: Multiple Beams with Multiple Contacts

This case consists of the tip of Beam A coming into contact with the tip of Beam B once the relative displacements of the beams are within a specified contact tolerance (0.001 inch) as described previously in Case A-4. Table 14 lists the frequency values of the SDM system, while Figure 17 shows the contribution of the original modes in the SDM.



Bar Color	Min. Value	Max Value
Black	0.0120	0.0200
Blue	0.0200	0.0500
Green	0.0500	0.1000
Cyan	0.1000	0.2000
Magenta	0.2000	0.5000
Yellow	0.5000	0.8000
Red	0.8000	1.0000

Figure 23 – Mode Contribution Matrix for multiple beams with multiple contacts for hard spring

Table 14 – Frequencies for Multiple Beams with Single Contact for Soft and Hard Springs

Mode #	Unmodified	Hard Spring		
		Config. 1	Config. 2	Config. 3
1	Beam A - 12.91	21.25	15.5	36.22
2	Beam B - 22.62	67.74	69.52	113.69
3	Beam A - 84.12	127.04	119.79	228.57
4	Beam B - 141.56	236.02	234.14	312.68
5	Beam A - 252.34	338.31	325.85	339.4
6	Beam B - 396.6	424.26	467.09	504.56
7	Beam A - 519.59	532.04	527.49	532.41
8	Beam B - 776.92	786.59	786.04	790.33
9	Beam A - 806.16	809.25	828.78	833.74
10	Beam A - 1256.55	1256.69	1257.17	1257.22
11	Beam B - 1284.71	1294.66	1290.03	1300.10
12	Beam A - 1682.96	1682.97	1686.00	1686.01
13	Beam B - 1918.28	1918.72	1920.05	1920.48
14	Beam A - 2201.36	2201.53	2202.67	2202.84
15	Beam B - 2678.33	2683.90	2678.85	2684.38
16	Beam A - 2755.52	2755.78	2756.01	2756.28
17	Beam A - 3510.01	3510.46	3511.34	3511.79
18	Beam B - 3563.89	3564.14	3563.98	3564.23
19	Beam A - 3948.54	3948.79	3948.54	3948.79
20	Beam B - 4572.70	4575.19	4572.70	4575.19

Figure 23 shows that the hard spring configuration requires modes 1-6 of Beam A and modes 1-5 of Beam B. As Case B-3 showed that additional modes beyond the modes excited by the forcing function are required when using the hard spring, only the time solution where accurate results are obtained is shown. Using the number of modes based on the forcing function provides an average TRAC of 0.3625, showing that there are an insufficient number of modes used as expected. Figure 24 compares the full solution to the solution using 23 modes for the time solution with 20 modes from Beam A and 15 modes from Beam B for the $[U_2]$ matrix for the hard spring.

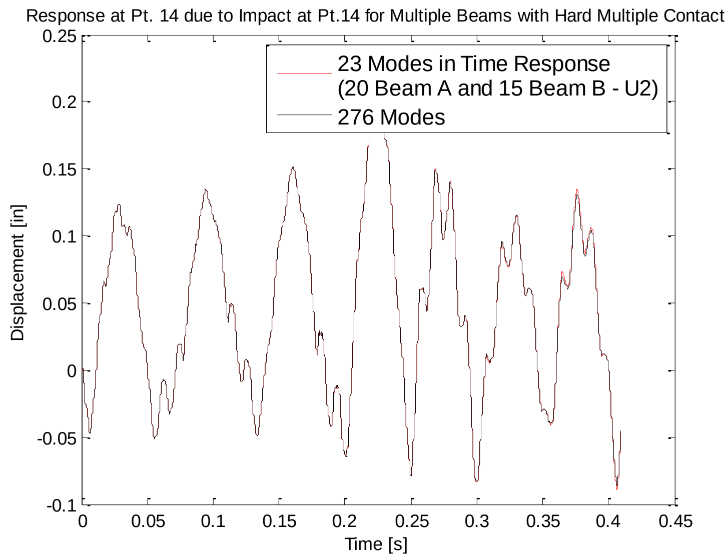


Figure 24 – Comparison of results in the time domain for multiple beams with multiple contacts with hard spring.

As seen in Cases A-3 and B-3, the [U_{12}] matrix provides a good understanding of the soft spring model, but a poor understanding of the hard spring model. In order to show the effect of the modes, the MAC and TRAC of the time responses were averaged as listed in Table 15.

Table 15 – Average MAC and TRAC vs. # of modes used for multiple beams multiple contact solution.

Hard Spring			
# of Modes	**Average MAC**	**Average TRAC**	**Solution Time (s)**
23 – Time (20 Beam A + 15 Beam B) – U2	0.9999	0.9962	17.11
276	1.0000	1.0000	1196.69

Finally, the FFT of the time response was computed for both the soft and hard spring shown in Figure 25.

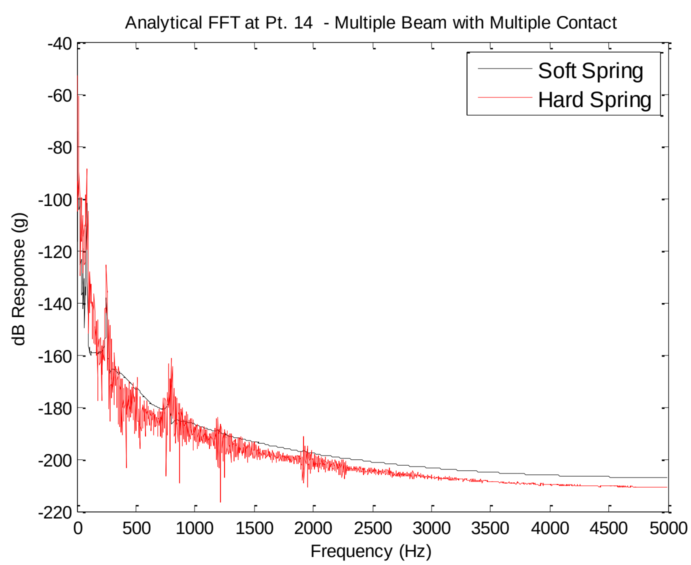


Figure 25 – FFT comparison between the soft and hard spring for multiple beam with multiple contact

As seen in Case B-3, the hard contact has excited modes above the forcing function bandwidth of 1000 Hz. Since the contact stiffness now governs the number of modes needed in the time response, more than the 10 modes excited by the forcing function are required.

Comparing the soft and hard spring cases shows that the number of modes needed can be predicted accurately as long as the forcing function bandwidth defines the frequency range that is excited as seen in the soft spring cases. Once that condition is no longer true, however, determining the number of modes needed requires some knowledge of the frequency bandwidth of contact ahead of time. For both cases, however, the analyst can obtain an accurate modal solution with significant computational savings provided the used modes satisfy the force spectrum and the contact stiffness spectrum.

CONTACT TIME STEP STUDY

For the chosen Δt of 0.0001 seconds, Raleigh Criteria and Shannon's Sampling Theorem state that the maximum frequency range that can be observed is 5 kHz, well above the forcing function frequency range. Although this time resolution may be fine enough to accurately capture the time response for the linear system, this resolution may not be adequate when the response becomes nonlinear. Since the Δt chosen affects when the system first comes into contact, the system may respond differently depending on whether the impact is slow or abrupt (ie. soft or hard spring). For a slow spring, the system remains in contact for a longer period of time, and the time response should remain consistent regardless of the time step chosen. For the hard spring, however, the system may experience a high frequency impact chatter, which may not be seen with a large time step. Figures 26a and 26b show the time response for the first 0.1 seconds for the soft and hard spring respectively, where a time step of 0.00005s and the previously used time step of 0.0001s are compared.

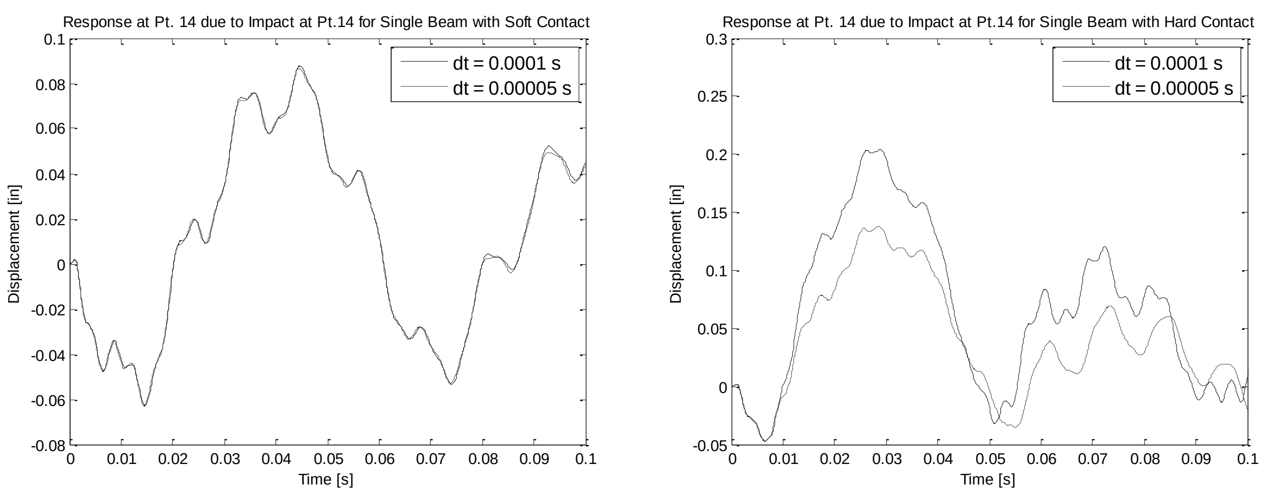


Figure 26 – Comparison of results with two different time steps for soft (a) and hard (b) contact

Figure 26a shows that for the soft spring system, the reduced time step had minimal effect on the time solution. Since the system remains in contact with the soft spring for an extended period, the results compare very well. For the hard spring in Figure 26b, the system comes into contact and then immediately bounces off. Since the time step duration directly affects the acceleration, there is a divergence in the solution.

Although this study shows that the chosen time step was not sufficiently fine for the hard spring, the time step of 0.0001 seconds was used for all of the previous cases in order to demonstrate the main principles of MMRT. Further study is required to determine the required time step when the contact stiffness dominates the response.

COMPARISON TO LARGE DOF MODELS

Although the computational timesavings using MMRT is significant, the effect is not dramatically seen until comparing to a large FEM model as typically seen in industry. Figure 27 shows the FEM of a missile rack system with a fine mesh resolution.

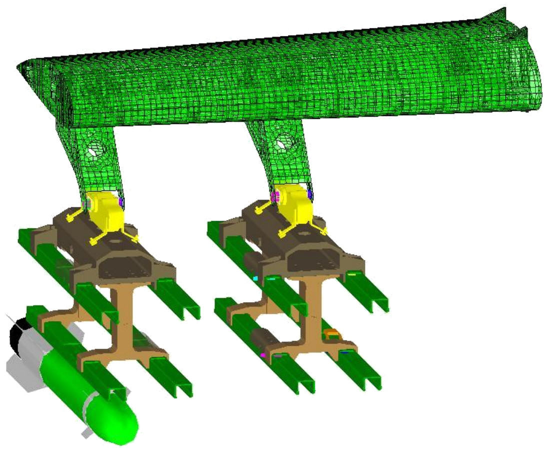


Figure 27: Typical large FEM model.

Analysts use this model in dynamic simulations to compute the transient dynamics during a missile firing under a variety of condition. As the system is a complex system with many detailed components, the time response models require significant computation time (typically days). The computational time can be dramatically reduced by using the mode shapes and frequencies of the various components. This allows analysts to study multiple configurations in detail, providing further information in order to improve the design.

CONCLUSION

A proposed technique – Modal Modification Response Technique (MMRT) **-** for computing the time response of a nonlinear system is described. For linear systems that are connected by a nonlinear local connection, mode superposition and structural dynamic modification can be used to approximate the nonlinear response of the system at significant computational savings. An analytical study of this technique is shown using a linear beam system with contact due to impact. Four cases of increasing complexity are studied; observations on the number of modes needed for each case are made based on the forcing function and SDM mode contribution matrix. For all cases, MMRT yields accurate results with substantially less computation time.

FURTHER WORK

Further work will be done using this technique to demonstrate the usefulness of this approach. For comparison to experimental results, several additional items will need to be studied in order to yield accurate results.

First, the underlying linear system will need to be a very high accuracy model in order to accurately predict the time responses. Care will need to be taken in modeling and updating the model to test data to ensure that both the model and measurements are reflective of the physical structure. Second, the damping will have to be measured experimentally for the linear system for as many modes as possible in order to have high time correlation. In addition, efforts will be needed to determine the correct damping for the SDM models, as the damping may change once the beam(s) are in contact. Third, the stiffness of the contact will have to be determined in order to accurately calculate the SDM. In addition, the impact force of the beam contact may need to be included in the analytical model as this is an additional force input to the structure that affects the time response. Finally, variation in the time step used was found to affect the results obtained and will require further study.

ACKNOWLEDGEMENTS

Some of the work presented herein was partially funded by Air Force Research Laboratory Award No. FA8651-10-1-0009 "Development of Dynamic Response Modeling Techniques for Linear Modal Components". Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the particular funding agency. The authors are grateful for the support obtained.

NOMENCLATURE

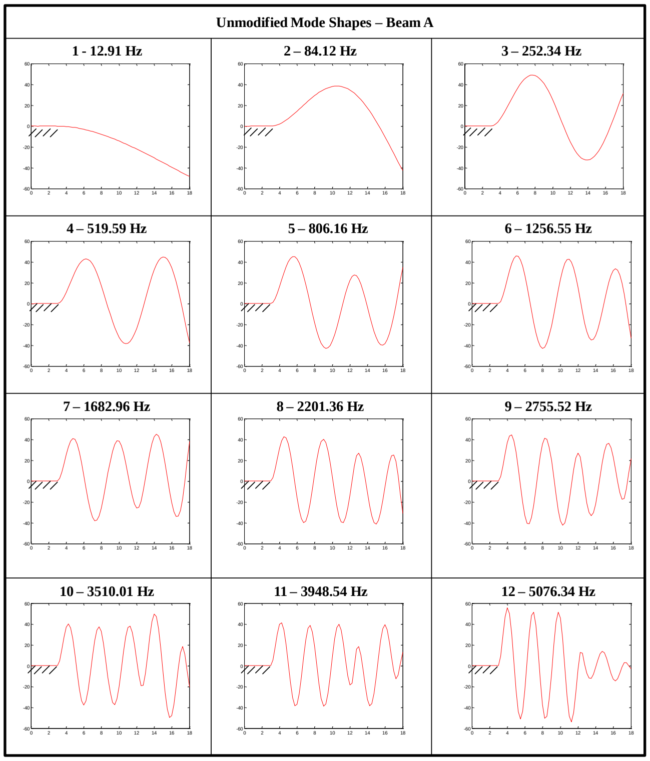
Symbols:

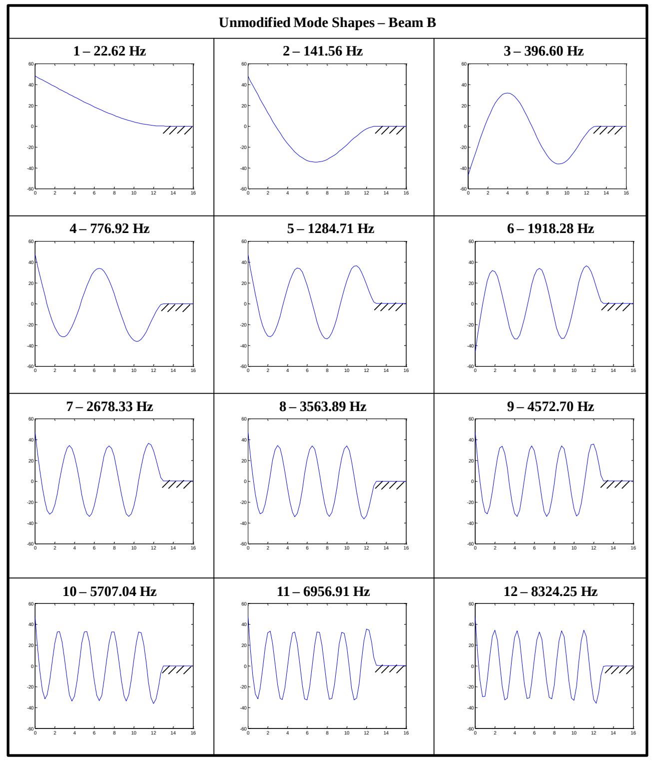
$[M]$	Physical Mass Matrix	$[U_1]$	Mode Shapes for State 1
$[C]$	Physical Damping Matrix	$[U_{12}]$	Mode Contribution Matrix
$[K]$	Physical Stiffness Matrix	$[U_2]$	Mode Shapes for State 2
$[\bar{M}_1]$	Modal Mass Matrix for State 1	$\{F\}$	Physical Force
$[\Delta\bar{M}_{12}]$	Modal Mass Change Matrix	$\{\ddot{p}_1\}$	Modal Acceleration
$[\bar{K}_1]$	Modal Stiffness Matrix for State 1	$\{\dot{p}_1\}$	Modal Velocity
$[\Delta\bar{K}_{12}]$	Modal Stiffness Change Matrix	$\{p_1\}$	Modal Displacement
$[\Delta M_{12}]$	Physical Mass Change Matrix	$\{\ddot{x}_1\}$	Physical Acceleration
$[\Delta K_{12}]$	Physical Stiffness Change Matrix	$\{\dot{x}_1\}$	Physical Velocity
$[U]^g$	Generalized Inverse	$\{x_1\}$	Physical Displacement

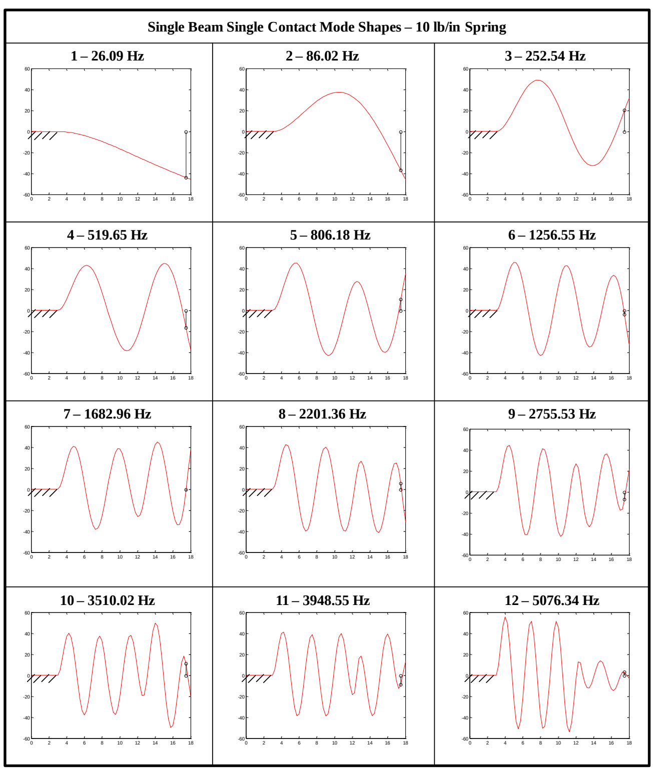
REFERENCES

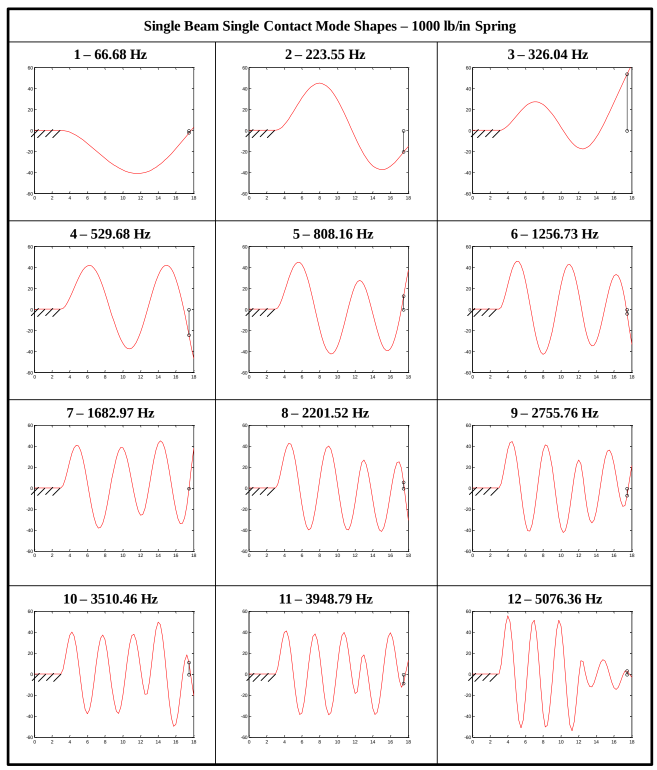
1. Avitabile, P., and O'Callahan, J. "Efficient techniques for forced response involving linear modal components interconnected by discrete nonlinear connection elements." Mechanical Systems and Signal Processing. 23.1 (2009): 45-67. Print.
2. Friswell, M.I., Penney, J.E.T., Garvey, S.D., "Using Linear Model Reduction to Investigate the Dynamics of Structures with Local Non-Linearities" Mechanical Systems and Signal Processing (1995) 9(3), 317 – 328
3. Lamarque C., Janin O. "Modal Analysis of Mechanical Systems with Impact Non-Linearities: Limitations to a Modal Superposition" Journal of Sound and Vibration (2000) 235(4), 567-609.
4. H. Ozguven, B.Kuran "A Modal Superposition Method for Non-Linear Structures" Journal of Sound and Vibration (1996) 189(3), 315-339
5. Al-Shudeifat, Mohammad, Eric Butcher, and Thomas Burton. "Enhanced Order Reduction of Forced Nonlinear Systems Using New Ritz Vectors." Proceedings of the IMAC-XXVIII. 2010. Print.
6. Rhee W "Linear and Nonlinear Model Reduction in Structural Dynamics with Application to Model Updating" PHD Dissertation-Texas Tech University
7. Avitabile, P. "Twenty Years of Structural Dynamic Modification- A Review." Sound and Vibration. Jan (2003): 14-27. Print.
8. Rao, S. *Mechanical Vibrations*, 4th ed. Prentice Hall, New Jersey, 2004, pg. 834-843.
9. Van Zandt, T., "Development of Efficient Reduced Models for Multi-Body Dynamics Simulations of Helicopter Wing Missile Configuration," Master's Thesis, University of Massachusetts Lowell, April 2006.
10. MAT_SAP/MATRIX, A general linear algebra operation program for matrix analysis, Dr. John O'Callahan, University of Massachusetts Lowell, 1986.
11. MATLAB R2010a, The MathWorks Inc., Natick, MA

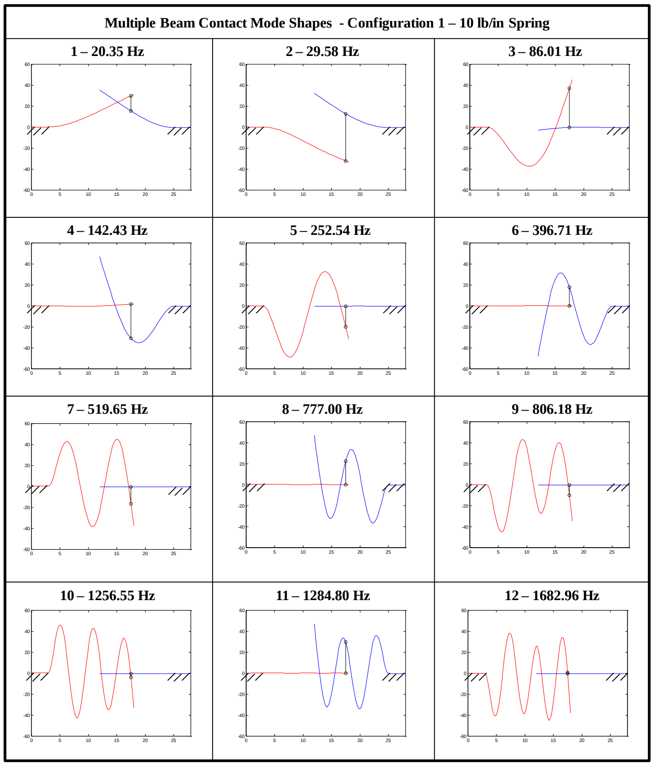
APPENDIX A: COMPONENT AND SYSTEM MODE SHAPES

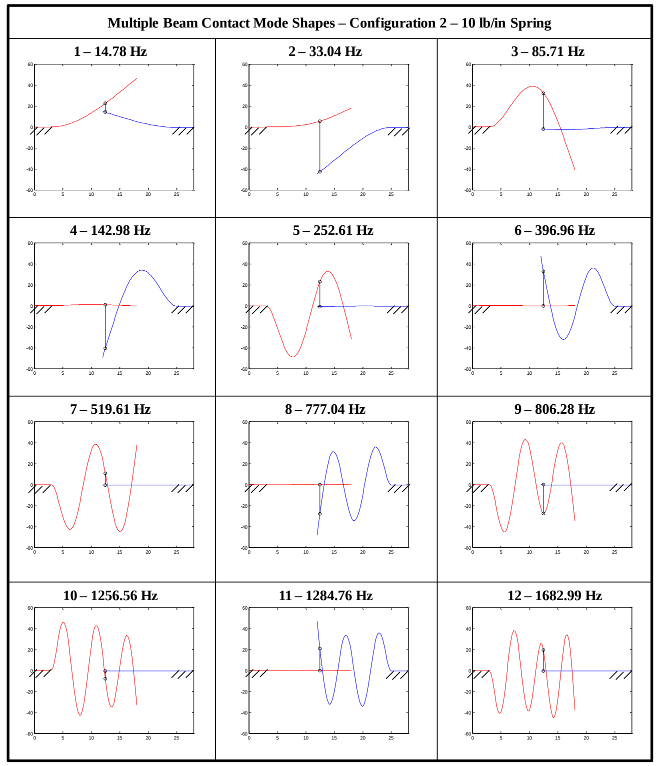


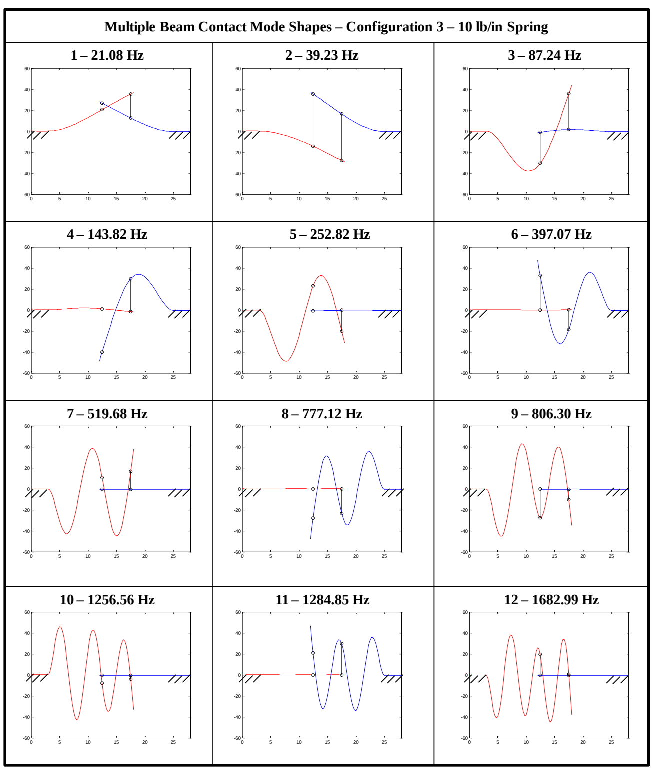




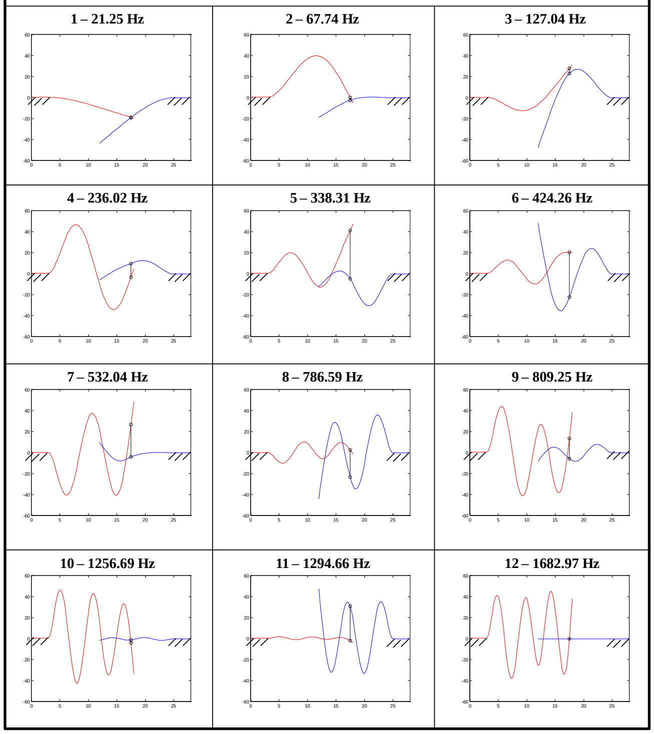



Multiple Beam Contact Mode Shapes - Configuration 1 – 10 lb/in Spring
1 – 20.35 Hz
2 – 29.58 Hz
3 – 86.01 Hz
4 – 142.43 Hz
5 – 252.54 Hz
6 – 396.71 Hz
7 – 519.65 Hz
8 – 777.00 Hz
9 – 806.18 Hz
10 – 1256.55 Hz
11 – 1284.80 Hz
12 – 1682.96 Hz

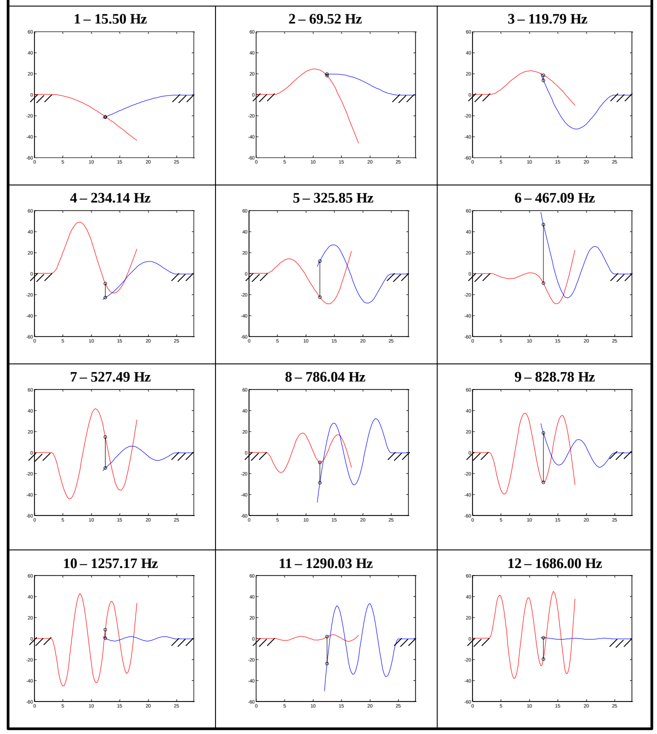

Multiple Beam Contact Mode Shapes – Configuration 2 – 10 lb/in Spring
1 – 14.78 Hz
2 – 33.04 Hz
3 – 85.71 Hz
4 – 142.98 Hz
5 – 252.61 Hz
6 – 396.96 Hz
7 – 519.61 Hz
8 – 777.04 Hz
9 – 806.28 Hz
10 – 1256.56 Hz
11 – 1284.76 Hz
12 – 1682.99 Hz


Multiple Beam Contact Mode Shapes – Configuration 3 – 10 lb/in Spring
1 – 21.08 Hz
2 – 39.23 Hz
3 – 87.24 Hz
4 – 143.82 Hz
5 – 252.82 Hz
6 – 397.07 Hz
7 – 519.68 Hz
8 – 777.12 Hz
9 – 806.30 Hz
10 – 1256.56 Hz
11 – 1284.85 Hz
12 – 1682.99 Hz

Multiple Beam Contact Mode Shapes – Configuration 1 – 1000 lb/in Spring



Multiple Beam Contact Mode Shapes – Configuration 2 – 1000 lb/in Spring



Multiple Beam Contact Mode Shapes – Configuration 3 – 1000 lb/in Spring