PDE II Project 2

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1 Question 1

This question involved using MATLAB's built-in `pdepe` function to approximate solutions to partial differential equations (PDEs). My solution to this question is based on the set of function files `pdefunc777.m`, `pdebc777.m` and `pdeic777.m` which were solutions to Exercise 1 of Practical 6. My function file `pde_1_ic.m` covers the initial condition of the PDE, as specified in the question, and the function file `pde_1_function.m` gives our PDE in the required form. To use the `pdepe` solver, we must arrange our PDE into a particular form, that is :

$$c(x,t,u,u_x)u_t = x^{-m}\left[x^m f(x,t,u,u_x)\right]_x + s(x,t,u,u_x)$$

Our PDE can be re-arranged from:

$$u_t = 0.47\,u_{xx} - 1.55\,u\,u_x \quad \text{to} \quad u_t = \left[0.47\,u_x\right]_x - 1.55\,u\,u_x \tag{1}$$

This gives us:

$$c(x,t,u,u_x) = 1, \quad m = 0, \quad s(x,t,u,u_x) = -1.55\,u\,u_x, \quad \text{and} \quad f(x,t,u,u_x) = 0.47u_x$$

I modified the file `pdefunc777.m` so that it represented the PDE for this question and saved it as `pde_1_function.m`. To use `pdepe`, the boundary conditions must also be of the form:

$$p(x,t,u) + q(x,t)f(x,t,u,u_x) = 0$$

For $x = -10$, our left-hand side boundary, we have:

$$u(-10,t) = 0.5, \implies p(x,t,u) = u - 0.5,\; q(x,t) = 0$$

For $x = 10$, our right-hand side boundary, we have:

$$u_x(10,t) = 0, \implies p(x,t,u) = 0,\; q(x,t) = 1$$

I modified the function file `pdebc777.m` to reflect the boundary conditions of this question and named it `pde_1_bc.m`. Finally, I modified the file `pdeic777.m` so that it suited the initial condition supplied in the question, saving it as `pde_1_ic.m`. To obtain our approximation of the PDE and the graphs required, I typed the following into the Command Window:

```
>>> x = (-50:50)/5; % [-10,10] interval given in the question
>>> t = (0:50)/10; % [0,5] interval for time (from question)
>>> m = 0;
>>> soln = pdepe(m, @pde_1_function, @pde_1_ic, @pde_1_bc, x, t);
>>> u = soln(:,:,1);
>>> plot_u_on_single_figure(x, u);
```

Here, `plot_on_a_single_figure.m` is a function file that I created to plot all the required graphs on one set of axis.

2 Question 2

Question 2 asks us to approximate the following PDE.

$$
\begin{aligned}
&u_t = 0.52(u_{xx} + u_{yy}) + 24x^2\left(x^2 + y^2\right) \qquad 0 \le x, y \le 1,\ 0 \le t \le 0.5 \\
&u(x, y, 0) = x^2y^2 - 2x - 2y, \qquad u(0, y, t) = -2y,\ u(1, y, t) = y^2 - 2y - 2 \\
&u(x, 0, t) = -2x,\ u(x, 1, t) = x^2 - 2x - 2
\end{aligned}
$$

To solve this problem, we split the solution into two problems using the linearity of the PDE. The first problem will be solving the non-homogeneous PDE but with zero boundary conditions; $v(x, y, t)$ denotes our solution to this problem . The second problem will involve solving the homogeneous PDE with non-zero boundary conditions, with a solution denoted by $w(x, y, t)$. Summing these solutions together gives us a solution to the required PDE. That is:

$$u(x, y, t) = v(x, y, t) + w(x, y, t)$$

To approximate the solution to this PDE, I created a function file `ADI_method.m` which is based off the function file `HEAT_2D_ADI_NEW.m` from Example 1 of practical 7. By considering the PDE in terms of v and w, we get:

$$v_t = 0.52(v_{xx} + v_{yy}) - w_t + 0.52(w_{xx} + w_{yy}) + 24x^2\left(x^2 + y^2\right) \tag{2}$$

We choose $w(x, y, t)$ so that is satisfies the boundary conditions given in the question. By looking at our boundary conditions, we can see that they are satisfied by the function:

$$w(x, t) = x^2\, y^2 - 2x - 2y$$

This therefore gives us a PDE for $v(x, t)$ which we must now approximate.

$$v_t = 0.52(v_{xx} + v_{yy}) + 0.52(2y^2 + 2x^2) + 24x^2\left(x^2 + y^2\right) \tag{3}$$

Which has initial condition:

$$v(x, y, 0) = u(x, y, 0) - w(x, y, 0) = x^2y^2 - 2x - 2y - x^2y^2 - 2x - 2y = 0$$

Therefore we consider the following problem for $v(x, y, t)$ to be solved by the Alternating Direction Implicit(ADI) method.

$$
\begin{aligned}
&v_t = 0.52(v_{xx} + v_{yy}) + 0.52(2y^2 + 2x^2) + 24x^2\left(x^2 + y^2\right) \\
&v(x, y, 0) = 0, \qquad v(0, y, t) = v(1, y, t) = 0 \text{ and } v(x, 0, t) = v(x, 1, t) = 0
\end{aligned}
$$

To approximate the solution to $v(x, y, t)$, the ADI method with a double sweep is used. I modified the previously mentioned function file to reflect the new PDE that needed to be solved and the zero initial condition specified for $v(x, y, t)$. The code for implementing the ADI method in this section was supplied by the original function file. After approximating $v(x, y, t)$, we evaluate $w(x, y, t)$ at every point in our $x - y$ grid. Since $w(x, y, t)$ has no time dependence this matrix is repeated and added to the approximation of $v(x, y, t)$ at every time step, this occurs in lines 71 - 83 of the function file . To obtain our approximation to $u(x, y, t)$ and the appropriate graphs, I typed the following into the Command Window:

```
>>> K = 0.52; % the K specified in the question
>>> T = 0.5; % end of time interval
>>> N = 50; % number of intervals in both x and y directions
>>> M = 50;  % number of time intervals
>>> [u,x,y,t] = ADI_method(T, K, N, M);
>>> surf(x,y,u(:,:,11)) % surface plot at time t = 0.1
>>> surf(x,y,u(:,:,51)) % surface plot at time t = 0.5
```