1 Integrating linear arithmetic into ArchSAT

The general simplex algorithm is a well known algorithm that decides the satisfiability of systems of linear inequalities and it serves as the basis of ArchSAT's arithmetic theory. Apart from using the simplex algorithm, the theory maintains two states, Γ and Δ. Γ is backtracked—any information gained inside a branch is lost when ArchSAT backjumps—while Δ retains all information added to it, which will allow the theory to attempt instantiation of metavariables whenever ArchSAT finds a model. $\Gamma^n, \Delta^m \parallel M \parallel F$ represents an ArchSAT state with the partial model M and Γ^n, Δ^m as the state of the arithmetic theory.

Each time a literal is propagated or decided the appropriate learn rule is applied which will either generate new clauses or augment the state of the arithmetic theory. Each time the arithmetic state is changed, it updates the underlying simplex and runs it until either it finds UNSAT and generates a new clause or it finds SAT in which case ArchSAT will resume. When ArchSAT reaches a state $\Gamma^n, \Delta^m \parallel M \parallel S$ where M is a model of S, a list l of all literals in M that contain a metavariable is appended to Δ^m and the theory attempts to find an instantiation metavariables that will negate at least one literal in each list in Δ^m.

1.1 Theory state

Note that $\bowtie \in \{<, \leq, >, \geq\}$ and $\overline{\underline{\bowtie}} \in \{<, \leq, >, \geq, =, \neq\}$ Additionally we define $\overline{\bowtie}$ and $\underline{\bowtie}$ in the following way :

$\bowtie$	$\overline{\bowtie}$	$\underline{\bowtie}$
$<$	$\geq$	$>$
$\leq$	$>$	$\geq$
$>$	$\leq$	$<$
$\geq$	$<$	$\leq$

Definiton. $\Gamma = (\Omega, \Sigma, \Theta)$ is a 3-tuple representing the backtracked portion of the arithmetic theory's state.

— Ω is a set of bounds (an upper bound would be the expression $x \bowtie_x u_x$ where $\bowtie_x \in \{<, \leq\}$ and u_x is a rational number representing the upper bound of the variable x while a lower bound would be the expression $y \bowtie_y l_y$ where $\bowtie_y \in \{>, \geq\}$).

— Σ is the set of equalities $x_i = \sum_j^m a_{ij} * x_{ij} \mid i \in [n]$ which define the additional variables in the simplex ArchSAT

— Θ is a partial function $x \in \Omega \to f \in \{e \bowtie e', e = e', e \neq e'\}$ which sends the original literal propagated or decided by ArchSATfrom which any given bound was deduced

Definiton. $\Delta = [\delta_1, \ldots, \delta_n]$ where each δ_i is a set of literals representing the partial model at the end of each branch explored by ArchSAT.

When we have a state $\Gamma, W = \{e_1, \ldots, e_n\}$ this means our working literals, W, are not yet integrated into Γ because they have note yet been normalized. The normalization rules will handle transforming W into a form the simplex algorithm can handle at which point whatever information was stored in W will be integrated into Γ.

1.2 Literal Propagation

Most propagation rules add expressions to the current state of the theory, Γ. The two exceptions are 1) when the expression contains only constants and is trivially false (this saves time wasted adding to the current simplex state) and 2) when ArchSATsends $\lfloor e \neq e' \rfloor$ because this contains an implicit "or" which the simplex algorithm cannot handle. When the simplex state is updated, we start by canonizing everything so that it is in a format the simplex algorithm can understand i.e. :

1. A bound : $x \bowtie k$

2. An equation : $x_a = \sum_i^n a_i * x_i$

The Normalization rules serve the following purposes :

1. **Eq** split an equality into a conjunction of weak inequalities
2. **Neg** removes $\neg$, which the simplex algorithm cannot understand and substitutes the appropriate negated inequality operator
3. **Canon** moves all monomials to the left and all constants to the right of an inequality operator
4. **Var** since the simplex algorithm can only understand bounds on individual variables, this rule introduces an additional variable that can be used to give a bound, originally applied to a linear expression, to the simplex algorithm

Once the working literals are all normalized, they are added to the simplex. If the simplex returns SAT ArchSAT resumes its course. However if it reaches a state $(\{\Omega, x \bowtie_1 k, x \bowtie_2 k'\}, \Sigma, \{\Theta, x \bowtie_1 k \mapsto f, x \bowtie_2 \mapsto g\})$ where $x \bowtie_1 k$ and $x \bowtie_2 k'$ are conflicting, it will reach UNSAT and generate the clause $\neg f \vee \neg g$.

1.3 Instantiation

When ArchSAT reaches a state $\Gamma^n, \Delta^m \parallel M \parallel S$ where M is a model of S, we add M to Δ^m ($\Delta^m := \Delta^m \cup \{M\}$). In order to find an instantiation that would render false at least one literal in each branch, we take the cross product $P = \{\delta_1 \times \cdots \times \delta_n\}$ of all the sets in $\Delta^m = \{\delta_1, \ldots, \delta_n\}$ and use the simplex algorithm to find one set $I = \{l_1, \ldots, l_m\} \in P$ where there is an assignment that satisfies $\neg l_1 \wedge \cdots \wedge \neg l_m$. Then, for all assignments $\alpha(X_i)$, we send the substitution $X_i \mapsto \alpha(X_i)$ and ArchSAT will generate the corresponding instantiations.

1.4 Rules

Learn rules

$$
\begin{aligned}
[\![\lfloor a \overline{\underline{\bowtie}} b \rfloor]\!] &= \lfloor \neg(a \overline{\underline{\bowtie}} b) \rfloor \text{ when } a \overline{\underline{\bowtie}} b \text{ is trivially false} \\
[\![\lfloor e \neq e' \rfloor]\!] &= \lfloor e < e' \rfloor \vee \lfloor e > e' \rfloor \\
[\![\lfloor e \bowtie e' \rfloor]\!] &= \Gamma := \Gamma, e \bowtie e', e \bowtie e' \mapsto e \bowtie e' \\
[\![\lfloor \neg(e \bowtie e') \rfloor]\!] &= \Gamma := \Gamma, \neg(e \bowtie e') \\
[\![\lfloor e = e' \rfloor]\!] &= \Gamma := \Gamma, e = e'
\end{aligned}
$$

Normalization Rules

$$
\text{Eq}\ \frac{\Gamma, W, e = e'}{\Gamma, W, e \leq e', e \geq e', e \leq e' \mapsto e = e', e \geq e' \mapsto e = e'} \qquad \text{Neg}\ \frac{\Gamma, W, \neg(e \bowtie e')}{\Gamma, W, e\ \overline{\bowtie}\ e', e\ \overline{\bowtie}\ e' \mapsto \neg(e \bowtie e')}
$$

$$
\text{Canon} \frac{\Gamma, W, e = \sum_i a_i * x_i + c \bowtie \sum_j b_j * y_j + d, e \mapsto f}{\Gamma, W, e' = \sum_i a_i * x_i - \sum_j b_j * y_j \bowtie d - c, e' \mapsto f}
$$

$$
\text{Var} \frac{\Gamma, W, e = \sum_i^n a_i * x_i \bowtie b, e \mapsto f}{\Gamma \cup (\{s \bowtie b\}, \{s = \sum_i^n a_i * x_i\}, \{s \bowtie b \mapsto f\}), W}\ s \text{ fresh},\ n > 1
$$

$$
\text{Bound}_\alpha \frac{\Gamma, W, x \bowtie k}{\Gamma \cup (\{x \bowtie k\}, \{\}, \{x \bowtie k \mapsto x \bowtie k\}), W} \qquad \text{Bound}_\beta \frac{\Gamma, W, x \bowtie k, x \bowtie k \mapsto f}{\Gamma \cup (\{x \bowtie k\}, \{\}, \{x \bowtie k \mapsto f\}), W}
$$

Simplex rules

$$\text{Leq}\frac{\Gamma = \left(\begin{array}{c}\{\Omega, e_i = (x_i \leq u_{x_i}) \mid i \in N^+, e_j = (x_j \geq l_{x_j}) \mid j \in N^-\},\\ \{\Sigma, e_s = (x = \sum_{j \in N^+ \cup N^-}^{n} a_k * x_k)\}, \{\Theta, e_i \mapsto f_i \mid i \in N^+ \cup N^-, e_s \mapsto f_s\})\end{array}\right)}{\Gamma \cup \left(\begin{array}{c}\{e = (x \leq \sum_{i \in N^+} a_i * u_i + \sum_{j \in N^-} a_j * l_j)\},\\ \{\}, \{e \mapsto f_1 \wedge \cdots \wedge f_n \wedge f_s \mid \{1, \ldots, n\} = N^+ \cup N^-\}\end{array}\right)} {\scriptstyle \begin{array}{c} a_i > 0, i \in N^+ \\ a_j < 0, j \in N^- \end{array}}$$

$$\text{Geq}\frac{\Gamma = \left(\begin{array}{c}\{\Omega, e_i = (x_i \geq l_{x_i}) \mid i \in N^+, e_j = (x_j \leq u_{x_j}) \mid j \in N^-\},\\ \{\Sigma, e_s = (x = \sum_{j \in N^+ \cup N^-}^{n} a_k * x_k)\}, \{\Theta, e_i \mapsto f_i \mid i \in N^+ \cup N^-, e_s \mapsto f_s\})\end{array}\right)}{\Gamma \cup \left(\begin{array}{c}\{e = (x \geq \sum_{i \in N^+} a_i * l_i + \sum_{j \in N^-} a_j * u_j)\},\\ \{\}, \{e \mapsto f_1 \wedge \cdots \wedge f_n \wedge f_s \mid \{1, \ldots, n\} = N^+ \cup N^-\}\end{array}\right)} {\scriptstyle \begin{array}{c} a_i > 0, i \in N^+ \\ a_j < 0, j \in N^- \end{array}}$$

$$\text{Less}\frac{\Gamma = \left(\begin{array}{c}\{\Omega, e_i = (x_i \bowtie_i u_{x_i}) \mid i \in N^+, e_j = (x_j \bowtie_j l_{x_j}) \mid j \in N^-\},\\ \{\Sigma, e_s = x = \sum_{j \in N^+ \cup N^-}^{n} a_k * x_k\}, \{\Theta, e_i \mapsto f_i \mid i \in N^+ \cup N^-, e_s \mapsto f_s\})\end{array}\right)}{\Gamma \cup \left(\begin{array}{c}\{e = (x < \sum_{i \in N^+} a_i * u_i + \sum_{j \in N^-} a_j * l_j)\},\\ \{\}, \{e \mapsto f_1 \wedge \cdots \wedge f_n \wedge f_s \mid \{1, \ldots, n\} = N^+ \cup N^-\}\end{array}\right)} {\scriptstyle \begin{array}{c} a_i > 0, i \in N^+ \\ a_j < 0, j \in N^- \\ \bowtie_i \in \{<, \leq\}, i \in N^+ \\ \bowtie_j \in \{>, \geq\}, j \in N^- \\ \exists i \in N^+ \cup N^- . \bowtie_i \in \{<, >\} \end{array}}$$

$$\text{Greater}\frac{\Gamma = \left(\begin{array}{c}\{\Omega, e_i = (x_i \bowtie_i l_{x_i}) \mid i \in N^+, e_j = (x_j \bowtie_j u_{x_j}) \mid j \in N^-\},\\ \{\Sigma, x = \sum_{j \in N^+ \cup N^-}^{n} a_k * x_k\}, \{\Theta, e_i \mapsto f_i \mid i \in N^+ \cup N^-, e_s \mapsto f_s\})\end{array}\right)}{\Gamma \cup \left(\begin{array}{c}\{e = (x > \sum_{i \in N^+} a_i * l_i + \sum_{j \in N^-} a_j * u_j)\},\\ \{\}, \{e \mapsto f_1 \wedge \cdots \wedge f_n \wedge f_s \mid \{1, \ldots, n\} = N^+ \cup N^-\}\end{array}\right)} {\scriptstyle \begin{array}{c} a_i > 0, i \in N^+ \\ a_j < 0, j \in N^- \\ \bowtie_i \in \{>, \geq\}, i \in N^+ \\ \bowtie_j \in \{<, \leq\}, j \in N^- \\ \exists i \in N^+ \cup N^- . \bowtie_i \in \{<, >\} \end{array}}$$

$$\text{Conflict}_\alpha \frac{(\{\Omega, x \leq k, x \geq k'\}, \Sigma, \{\Theta, x \leq k \mapsto f, x \geq k' \mapsto g\})}{\odot} k < k'$$

$$\text{Conflict}_\beta \frac{(\{\Omega, x \bowtie_1 k, k' \bowtie_2 x\}, \Sigma, \{\Theta, x \bowtie_1 k \mapsto f, k' \bowtie_2 x \mapsto g\})}{\odot} {\scriptstyle \begin{array}{c} k \leq k', \bowtie_1, \bowtie_2 \in \{<, \leq\}, \\ \bowtie_1 \in \{<\} \vee \bowtie_2 \in \{<\} \end{array}}$$

$$\text{Sat}\frac{\Gamma = (\{e_i = x_i \bowtie_i k_i \mid i \in N\}, \Sigma, \Theta), \alpha(x_i) \bowtie_i k_i \mid i \in N}{\Gamma}$$

<u>Instantiation</u>

$$\text{Neq}\frac{\Gamma, \{\Delta, \{\delta, e \neq e'\}\}}{\Gamma, \{\Delta, \{\delta, e < e', e > e'\}\}} \qquad \text{Choose}\frac{\Gamma, \{\delta_1, \ldots, \delta_n\}}{\Gamma, \neg f_1, \ldots, \neg f_n} \{f_1, \ldots, f_n\} \in \delta_1 \times \cdots \times \delta_n$$

2 Examples

$\forall x \ . \ x \geq 0 \Rightarrow x + 1 \geq 1$

$$\begin{array}{rcrr}
\emptyset \parallel \emptyset \parallel \neg A & \longrightarrow & \text{unit prop} & (1)\\
\emptyset \parallel \neg A \parallel \neg A & \longrightarrow & \delta_{\neg\forall} & (2)\\
\emptyset \parallel \neg A \parallel \neg A, A \vee \neg B & \longrightarrow & \text{unit prop} & (3)\\
\emptyset \parallel \neg A, \neg B \parallel \neg A, A \vee \neg B & \longrightarrow & \alpha_{\neg\Rightarrow} & (4)\\
\emptyset \parallel \neg A, \neg B \parallel \neg A, A \vee \neg B, B \vee C, B \vee \neg D & \longrightarrow & \text{unit prop} & (5)\\
\emptyset \parallel \neg A, \neg B, C \parallel \neg A, A \vee \neg B, B \vee C, B \vee \neg D & \longrightarrow & \text{Learn} & (6)\\
\Gamma^1 \parallel \neg A, \neg B, C \parallel \neg A, A \vee \neg B, B \vee C, B \vee \neg D & \longrightarrow & \text{unit prop} & (7)\\
\Gamma^1 \parallel \neg A, \neg B, C, \neg D \parallel \neg A, A \vee \neg B, B \vee C, B \vee \neg D & \longrightarrow & \text{Learn} & (8)\\
\Gamma^2 \parallel \neg A, \neg B, C, \neg D \parallel \neg A, A \vee \neg B, B \vee C, B \vee \neg D, \neg C \vee D & \longrightarrow & \text{UNSAT} & (9)
\end{array}$$

$$
\begin{aligned}
&A \equiv \lfloor \forall x . x \geq 0 \Rightarrow x+1 \geq 1 \rfloor \\
&B \equiv \lfloor \epsilon_x \geq 0 \Rightarrow \epsilon_x + 1 \geq 1 \rfloor \\
&C \equiv \lfloor \epsilon_x \geq 0 \rfloor \\
&D \equiv \lfloor \epsilon_x + 1 \geq 1 \rfloor \\
&\epsilon_x = \epsilon(x) . \neg(x \geq 0 \Rightarrow x+1 \geq 1)
\end{aligned}
$$

SMT Learn step (6) $\lfloor \epsilon_x \geq 0 \rfloor$

$$
\dfrac{\dfrac{\emptyset, \epsilon_x \geq 0}{\Gamma^1 = (\{\epsilon_x \geq 0\}, \emptyset, \{\epsilon_x \geq 0 \mapsto \epsilon_x \geq 0\}), \alpha(\epsilon_x) = 0}\ \text{Bound}_\alpha}{\Gamma^1}\ \text{SAT}
$$

SMT Learn step (8) $\lfloor \neg(\epsilon_x + 1 \geq 1) \rfloor$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\Gamma^1, \neg(\epsilon_x + 1 \geq 1)}{\Gamma^1, \epsilon_x + 1 < 1, \epsilon_x + 1 < 1 \mapsto \epsilon_x + 1 \geq 1}\ \text{Neg}}{\Gamma^1, \epsilon_x < 0, \epsilon_x < 0 \mapsto \epsilon_x + 1 \geq 1}\ \text{Canon}}{\Gamma^2 = \Gamma^1 \cup (\{\epsilon_x < 0\}, \{\}, \{\epsilon_x < 0 \mapsto \epsilon_x + 1 \geq 1\})}\ \text{Bound}_\beta}{\odot}\ \text{Conflict}_\beta
$$

$\forall x, y \ . \ x \geq 0 \wedge y \geq x \Rightarrow y \geq 0$

$$
\begin{aligned}
\emptyset \parallel \emptyset \parallel \neg A &\longrightarrow \text{unit prop} && (10)\\
\emptyset \parallel \neg A \parallel \neg A &\longrightarrow \delta_{\neg\forall} && (11)\\
\emptyset \parallel \neg A \parallel \neg A, A \vee \neg B &\longrightarrow \text{unit prop} && (12)\\
\emptyset \parallel \neg A, \neg B \parallel \neg A, A \vee \neg B &\longrightarrow \delta_{\neg\forall} && (13)\\
\emptyset \parallel \neg A, \neg B \parallel \neg A, A \vee \neg B, B \vee \neg C &\longrightarrow \text{unit prop} && (14)\\
\emptyset \parallel \neg A, \neg B, \neg C \parallel \neg A, A \vee \neg B, B \vee \neg C &\longrightarrow \alpha_{\neg\Rightarrow} && (15)\\
\emptyset \parallel \neg A, \neg B, \neg C \parallel \neg A, A \vee \neg B, B \vee \neg C, C \vee D, C \vee \neg E &\longrightarrow \text{unit prop} && (16)\\
\emptyset \parallel \neg A, \neg B, \neg C, D \parallel \neg A, A \vee \neg B, B \vee \neg C, C \vee D, C \vee \neg E = \Gamma &\longrightarrow \alpha_{\wedge} && (17)\\
\emptyset \parallel \neg A, \neg B, \neg C, D \parallel \Gamma, \neg D \vee F, \neg D \vee G = \Gamma' &\longrightarrow \text{unit prop} && (18)\\
\emptyset \parallel \neg A, \neg B, \neg C, D, \neg E \parallel \Gamma' &\longrightarrow \text{Learn} && (19)\\
\Gamma^1 \parallel \neg A, \neg B, \neg C, D, \neg E \parallel \Gamma' &\longrightarrow \text{unit prop} && (20)\\
\Gamma^1 \parallel \neg A, \neg B, \neg C, D, \neg E, F \parallel \Gamma' &\longrightarrow \text{Learn} && (21)\\
\Gamma^2 \parallel \neg A, \neg B, \neg C, D, \neg E, F \parallel \Gamma' &\longrightarrow \text{unit prop} && (22)\\
\Gamma^2 \parallel \neg A, \neg B, \neg C, D, \neg E, F, G \parallel \Gamma' &\longrightarrow \text{Learn} && (23)\\
\Gamma^3 \parallel \neg A, \neg B, \neg C, D, \neg E, F, G \parallel \Gamma', E \vee \neg F \vee \neg G &\longrightarrow \text{UNSAT} && (24)
\end{aligned}
$$

$$
\begin{aligned}
&A \equiv \lfloor \forall x, y \, . \, x \geq 0 \land y \geq x \Rightarrow y \geq 0 \rfloor \\
&B \equiv \lfloor \forall y \, . \, \epsilon_x \geq 0 \land y \geq \epsilon_x \Rightarrow y \geq 0 \rfloor \\
&C \equiv \lfloor \epsilon_x \geq 0 \land \epsilon_y \geq \epsilon_x \Rightarrow \epsilon_y \geq 0 \rfloor \rfloor \\
&D \equiv \lfloor \epsilon_x \geq 0 \land \epsilon_y \geq \epsilon_x \rfloor \\
&E \equiv \lfloor \epsilon_y \geq 0 \rfloor \\
&F \equiv \lfloor \epsilon_x \geq 0 \rfloor \\
&G \equiv \lfloor \epsilon_y \geq \epsilon_x \rfloor \\
&\epsilon_x = \epsilon(x).\neg(\forall y \, . \, x \geq 0 \land y \geq x \Rightarrow y \geq 0) \\
&\epsilon_y = \epsilon(y).\neg(\epsilon_x \geq 0 \land y \geq \epsilon_x \Rightarrow y \geq 0)
\end{aligned}
$$

SMT Learn step (19) $\lfloor \neg(\epsilon_y \geq 0) \rfloor$

$$
\text{SAT}\,\dfrac{\text{Bound}_\beta\,\dfrac{\text{Neg}\,\dfrac{\emptyset, \neg(\epsilon_y \geq 0)}{\emptyset, \epsilon_y < 0, \epsilon_y < 0 \mapsto \neg(\epsilon_y \geq 0)}}{\Gamma^1 = (\{\epsilon_y < 0\}, \{\}, \{\epsilon_y < 0 \mapsto \neg(\epsilon_y \geq 0)\}), \alpha(\epsilon_y) = -1}}{\Gamma^1}
$$

SMT Learn step (21) $\lfloor \epsilon_x \geq 0 \rfloor$

$$
\text{SAT}\,\dfrac{\text{Bound}_\alpha\,\dfrac{\Gamma^1, \epsilon_x \geq 0}{\Gamma^2 = \Gamma^1 \cup (\{\epsilon_x \geq 0\}, \emptyset, \{\epsilon_x \geq 0 \mapsto \epsilon_x \geq 0\}), \alpha(\epsilon_x) = 0, \alpha(\epsilon_y) = -1}}{\Gamma^2}
$$

SMT Learn step (23) $\lfloor \epsilon_y \geq \epsilon_x \rfloor$

$$
\text{Conflict}_\beta\,\dfrac{\text{Less}\,\dfrac{\text{Var}\,\dfrac{\text{Canon}\,\dfrac{\Gamma^2, \epsilon_y \geq \epsilon_x}{\Gamma^2, \epsilon_y - \epsilon_x \geq 0, \epsilon_y - \epsilon_x \geq 0 \mapsto \epsilon_y \geq \epsilon_x}}{\Gamma^3 = \Gamma^2 \cup (\{s_1 \geq 0\}, \{s_1 = \epsilon_y - \epsilon_x\}, \{s_1 \geq 0 \mapsto \epsilon_y \geq \epsilon_x\})}}{\Gamma^3 = \Gamma^3 \cup (\{\epsilon_x < 0\}, \{\}, \{\epsilon_x < 0 \mapsto \epsilon_y \geq \epsilon_x \land \neg(\epsilon_x \geq 0)\})}}{\odot}
$$

$\forall x, y \,.\, (x \geq 34 \wedge y < 22) \Rightarrow (x > y \wedge y < 34)$

$$\begin{aligned}
\emptyset \parallel \emptyset \parallel \neg A && \longrightarrow && \text{unit prop} && (25)\\
\emptyset \parallel \neg A \parallel \neg A && \longrightarrow && \delta_{\neg\forall} && (26)\\
\emptyset \parallel \neg A \parallel \neg A, A \vee \neg B && \longrightarrow && \text{unit prop} && (27)\\
\emptyset \parallel \neg A, \neg B \parallel \neg A, A \vee \neg B && \longrightarrow && \delta_{\neg\forall} && (28)\\
\emptyset \parallel \neg A, \neg B \parallel \neg A, A \vee \neg B, B \vee \neg C && \longrightarrow && \text{unit prop} && (29)\\
\emptyset \parallel \neg A, \neg B, \neg C \parallel \neg A, A \vee \neg B, B \vee \neg C && \longrightarrow && \alpha_{\neg\Rightarrow} && (30)\\
\emptyset \parallel \neg A, \neg B, \neg C \parallel \neg A, A \vee \neg B, B \vee \neg C, C \vee D, C \vee \neg E && \longrightarrow && \text{unit prop} && (31)\\
\emptyset \parallel \neg A, \neg B, \neg C, D \parallel \neg A, A \vee \neg B, B \vee \neg C, C \vee D, C \vee \neg E = \Gamma && \longrightarrow && \alpha_{\wedge} && (32)\\
\emptyset \parallel \neg A, \neg B, \neg C, D \parallel \Gamma, \neg D \vee F, \neg D \vee G = \Gamma' && \longrightarrow && \text{unit prop} && (33)\\
\emptyset \parallel \neg A, \neg B, \neg C, D, \neg E \parallel \Gamma' && \longrightarrow && \beta_{\neg\wedge} && (34)\\
\emptyset \parallel \neg A, \neg B, \neg C, D, \neg E \parallel \Gamma', E \vee \neg H \vee \neg J && \longrightarrow && \text{unit prop} && (35)\\
\emptyset \parallel \neg A, \neg B, \neg C, D, \neg E, F \parallel \Gamma', E \vee \neg H \vee \neg J && \longrightarrow && \text{Learn} && (36)\\
\Gamma^1 \parallel \neg A, \neg B, \neg C, D, \neg E, F \parallel \Gamma', E \vee \neg H \vee \neg J && \longrightarrow && \text{unit prop} && (37)\\
\Gamma^1 \parallel \neg A, \neg B, \neg C, D, \neg E, F, G \parallel \Gamma', E \vee \neg H \vee \neg J && \longrightarrow && \text{Learn} && (38)\\
\Gamma^2 \parallel \neg A, \neg B, \neg C, D, \neg E, F, G \parallel \Gamma', E \vee \neg H \vee \neg J && \longrightarrow && \text{decide} && (39)\\
\Gamma^2 \parallel \neg A, \neg B, \neg C, D, \neg E, F, G, \neg H^d \parallel \Gamma', E \vee \neg H \vee \neg J && \longrightarrow && \text{Learn} && (40)\\
\Gamma^3 \parallel \neg A, \neg B, \neg C, D, \neg E, F, G, \neg H^d \parallel \Gamma', E \vee \neg H \vee \neg J && \longrightarrow && \text{Backjump} && (41)\\
\Gamma^2 \parallel \neg A, \neg B, \neg C, D, \neg E, F, G, H \parallel \Gamma', E \vee \neg H \vee \neg J && \longrightarrow && \text{Learn} && (42)\\
\Gamma^4 \parallel \neg A, \neg B, \neg C, D, \neg E, F, G, H \parallel \Gamma', E \vee \neg H \vee \neg J && \longrightarrow && \text{unit prop} && (43)\\
\Gamma^4 \parallel \neg A, \neg B, \neg C, D, \neg E, F, G, H, \neg J \parallel \Gamma', E \vee \neg H \vee \neg J && \longrightarrow && \text{Learn} && (44)\\
\Gamma^5 \parallel \neg A, \neg B, \neg C, D, \neg E, F, G, H, \neg J \parallel \Gamma', E \vee \neg H \vee \neg J && \longrightarrow && \text{UNSAT} && (45)\\
&& && && (46)
\end{aligned}$$

$$\begin{aligned}
A &\equiv \lfloor \forall x, y \,.\, (x \geq 34 \wedge y < 22) \Rightarrow (x > y \wedge y < 34) \rfloor\\
B &\equiv \lfloor \forall y \,.\, (\epsilon_x \geq 34 \wedge y < 22) \Rightarrow (\epsilon_x > y \wedge y < 34) \rfloor\\
C &\equiv \lfloor (\epsilon_x \geq 34 \wedge \epsilon_y < 22) \Rightarrow (\epsilon_x > \epsilon_y \wedge y < 34) \rfloor \rfloor\\
D &\equiv \lfloor \epsilon_x \geq 34 \wedge \epsilon_y < 22 \rfloor\\
E &\equiv \lfloor \epsilon_x > \epsilon_y \wedge y < 34 \rfloor\\
F &\equiv \lfloor \epsilon_x \geq 34 \rfloor\\
G &\equiv \lfloor \epsilon_y < 22 \rfloor\\
H &\equiv \lfloor \epsilon_x > \epsilon_y \rfloor\\
J &\equiv \lfloor \epsilon_y < 34 \rfloor\\
\epsilon_x &= \epsilon(x).\neg(\forall y \,.\, (\epsilon_x \geq 34 \wedge y < 22) \Rightarrow (\epsilon_x > y \wedge y < 34))\\
\epsilon_y &= \epsilon(y).\neg((\epsilon_x \geq 34 \wedge \epsilon_y < 22) \Rightarrow (\epsilon_x > \epsilon_y \wedge y < 34))
\end{aligned}$$

SMT Learn step (36) $\lfloor \epsilon_x \geq 34 \rfloor$

$$\mathrm{SAT}\,\dfrac{\mathrm{Bound}_\alpha\,\dfrac{\emptyset, \epsilon_x \geq 34}{\Gamma^1 = (\{\epsilon_x \geq 34\}, \{\}, \{\epsilon_x \geq 34 \mapsto \epsilon_x \geq 34\}), \alpha(\epsilon_x) = 34}}{\Gamma^1}$$

SMT Learn step (38) $\lfloor \epsilon_x \geq 34 \rfloor$

$$\mathrm{SAT}\,\dfrac{\mathrm{Bound}_\alpha\,\dfrac{\Gamma^1, \epsilon_y < 22}{\Gamma^2 = \Gamma^1 \cup (\{\epsilon_y < 22\}, \{\}, \{\epsilon_y < 22 \mapsto \epsilon_y < 22\}), \alpha(\epsilon_y) = 0}}{\Gamma^2}$$

SMT Learn step (40) $\lfloor \neg(\epsilon_x > \epsilon_y) \rfloor$

$$\mathrm{Conflict}_\beta\,\dfrac{\mathrm{Greater}\,\dfrac{\mathrm{Var}\,\dfrac{\mathrm{Canon}\,\dfrac{\mathrm{Neg}\,\dfrac{\Gamma^2, \neg(\epsilon_x > \epsilon_y)}{\Gamma^2, \epsilon_x \leq \epsilon_y, \epsilon_x \leq \epsilon_y \mapsto \neg(\epsilon_x > \epsilon_y)}}{\Gamma^2, \epsilon_x - \epsilon_y \leq 0, \epsilon_x - \epsilon_y \leq 0 \mapsto \neg(\epsilon_x > \epsilon_y)}}{\Gamma^3 = \Gamma^2 \cup (\{s_1 \leq 0\}, \{s_1 = \epsilon_x - \epsilon_y\}, \{s_1 \leq 0 \mapsto \neg(\epsilon_x > \epsilon_y)\})}}{\Gamma^3 = \Gamma^3 \cup (\{s_1 > 56\}, \{\}, \{s_1 > 56 \mapsto \epsilon_x \geq 34 \wedge \epsilon_y < 22 \wedge \epsilon_x > \epsilon_y\})}}{\odot}$$

SMT Learn step (42) $\lfloor \epsilon_x > \epsilon_y \rfloor$

$$\mathrm{SAT}\,\dfrac{\mathrm{Var}\,\dfrac{\mathrm{Canon}\,\dfrac{\Gamma^2, \epsilon_x > \epsilon_y}{\Gamma^2, \epsilon_x - \epsilon_y > 0, \epsilon_x - \epsilon_y > 0 \mapsto \epsilon_x > \epsilon_y}}{\Gamma^4 = \Gamma^2 \cup (\{s_1 \leq 0\}, \{s_1 = \epsilon_x - \epsilon_y\}, \{s_1 > 0 \mapsto \neg(\epsilon_x > \epsilon_y)\}), \alpha(s_1) = 34}}{\Gamma^4}$$

SMT Learn step (44) $\lfloor \neg(\epsilon_y < 34) \rfloor$

$$\mathrm{Conflict}\,\dfrac{\mathrm{Bound}_\beta\,\dfrac{\mathrm{Neg}\,\dfrac{\Gamma^4, \neg(\epsilon_y < 34)}{\Gamma^4, \epsilon_y \geq 34, \epsilon_y \geq 34 \mapsto \neg(\epsilon_y < 34)}}{\Gamma^5 = \Gamma^4 \cup (\{\epsilon_y \geq 34\}, \{\}, \{\epsilon_y \geq 34 \mapsto \neg(\epsilon_y < 34)\}), \alpha(s_1) = 34}}{\odot}$$

$\exists x, y \,.\, x \geq y$

$$\begin{array}{rcrr}
\emptyset \parallel \emptyset \parallel \neg A & \longrightarrow & \text{unit prop} & (47) \\
\emptyset \parallel \neg A \parallel \neg A & \longrightarrow & \gamma_{\neg\exists M} & (48) \\
\emptyset \parallel \neg A \parallel \neg A, A \vee \neg B & \longrightarrow & \text{unit prop} & (49) \\
\emptyset \parallel \neg A, \neg B \parallel \neg A, A \vee \neg B & \longrightarrow & \gamma_{\neg\exists M} & (50) \\
\emptyset \parallel \neg A, \neg B \parallel \neg A, A \vee \neg B, B \vee \neg C & \longrightarrow & \text{unit prop} & (51) \\
\emptyset \parallel \neg A, \neg B, \neg C \parallel \neg A, A \vee \neg B, B \vee \neg C & \longrightarrow & \text{Learn} & (52) \\
\Gamma^1 \parallel \neg A, \neg B, \neg C \parallel \neg A, A \vee \neg B, B \vee \neg C & \longrightarrow & \text{Learn } \{X \mapsto 0, Y \mapsto 0\} & (53) \\
\Gamma^1, \Delta^1 \parallel \neg A, \neg B, \neg C \parallel \neg A, A \vee \neg B, B \vee \neg C, A \vee \neg D & \longrightarrow & \text{unit prop} & (54) \\
\Gamma^1, \Delta^1 \parallel \neg A, \neg B, \neg C, \neg D \parallel \neg A, A \vee \neg B, B \vee \neg C, A \vee \neg D & \longrightarrow & \text{Learn} & (55) \\
\Gamma^1, \Delta^1 \parallel \neg A, \neg B, \neg C, \neg D \parallel \neg A, A \vee \neg B, B \vee \neg C, A \vee \neg D, D & \longrightarrow & \text{UNSAT} & (56)
\end{array}$$

$$
\begin{aligned}
A &\equiv \lfloor \exists x, y \,.\, x \geq y \rfloor \\
B &\equiv \lfloor \exists y \,.\, X_{\neg \exists x,y \,.\, x \geq y} \geq y \rfloor \\
C &\equiv \lfloor X_{\neg \exists x,y \,.\, x \geq y} \geq Y_{\neg X_{\neg \exists x,y \,.\, x \geq y} \geq y} \rfloor \\
D &\equiv \lfloor 0 \geq 0 \rfloor
\end{aligned}
$$

SMT Learn step (52) $\lfloor \neg(X \geq Y) \rfloor$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\emptyset, \neg(X \geq Y)}{\emptyset, X < Y, X < Y \mapsto \neg(X \geq Y)}\,\text{Neg}}{\emptyset, X - Y < 0, X - Y < 0 \mapsto \neg(X \geq Y)}\,\text{Canon}}{\Gamma^1 = (\{s < 0\}, \{s = X - Y\}, \{s < 0 \mapsto \neg(X \geq Y)\}), \alpha(X) = 0, \alpha(Y) = 1}\,\text{Var}}{\Gamma^1}\,\text{SAT}
$$

SMT Learn step (53)

$$
\dfrac{\dfrac{\dfrac{\dfrac{\Delta^1 = \{\{\neg(X \geq Y)\}\}}{X \geq Y}\,\text{Choose}\ \{\neg(X \geq Y)\} \in \{\neg(X \geq Y)\} \times \emptyset}{\emptyset, X - Y \geq 0, X - Y \geq 0 \mapsto X \geq Y}\,\text{Canon}}{\Gamma = (\{s \geq 0\}, \{s = X - Y\}, \{s \geq 0 \mapsto X \geq Y\}), \alpha(X) = 0, \alpha(Y) = 0}\,\text{Var}}{\Gamma}\,\text{SAT}
$$