294.5

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Problem. Prove the Real Spectral Theorem via complexification and the Complex Spectral Theorem.

Suppose V is a real inner product space, and $T \in \mathcal{L}(V)$ is self-adjoint. To prove the Real Spectral Theorem, it suffices to show that T has an eigenvalue. From there, the proof is identical to our original proof.

Define the complex inner product on $V_{\mathbf{C}}$ to be

$$\langle u + iv, x + iy \rangle = \langle u, x \rangle + \langle v, y \rangle + (\langle v, x \rangle - \langle u, y \rangle)i$$

for all $u, v, x, y \in V$. Then, by the previous problem, *9.B.4*, we have that $T_{\mathbf{C}}$ is self-adjoint.

Claim. $\langle T_{\mathbf{C}}(u + iv), u + iv \rangle \in \mathbf{R}$ for all $u, v \in V$.

Proof. We begin with a lemma: if $z \in \mathbf{C}$ and $\overline{z} = z$ then $z \in \mathbf{R}$.[1] Next, note that

$$\begin{aligned}
\overline{\langle T_{\mathbf{C}}(u + iv), u + iv \rangle} &= \overline{\langle Tu + iTv, u + iv \rangle} \\
&= \overline{\langle Tu, u \rangle + \langle Tv, v \rangle + (\langle Tv, u \rangle - \langle Tu, v \rangle)i} \\
&= \overline{\langle Tu, u \rangle} + \overline{\langle Tv, v \rangle} + (\overline{\langle Tv, u \rangle} - \overline{\langle Tu, v \rangle})(-i) \\
&= \langle u, Tu \rangle + \langle v, Tv \rangle + (\langle u, Tv \rangle - \langle v, Tu \rangle)(-i) \\
&= \langle u, Tu \rangle + \langle v, Tv \rangle + (\langle v, Tu \rangle - \langle u, Tv \rangle)i \\
&= \langle u + iv, Tu + iTv \rangle \\
&= \langle u + iv, T_{\mathbf{C}}(u + iv) \rangle \\
&= \langle T_{\mathbf{C}}(u + iv), u + iv \rangle. && (T_C \text{ is self-adjoint})
\end{aligned}$$

Thus, by our lemma, $\langle T_{\mathbf{C}}(u + iv), u + iv \rangle \in \mathbf{R}$. □

Claim. If $T_{\mathbf{C}}(u + iv) = \lambda(u + iv)$ for some $u, v \in V$ and $\lambda \in \mathbf{C}$ then $\lambda \in \mathbf{R}$.

[1] Proof: let $z = a + bi$ for $a, b \in \mathbf{R}$. Since $\overline{z} = z$, we have $a - bi = a + bi$, so $-b = b$, so $b = 0$. Thus, $z \in \mathbf{R}$.

Proof. By the previous claim, the following inner product is real:

$$\begin{aligned}\langle T_{\mathbf{C}}(u+iv), u+iv\rangle &= \langle \lambda(u+iv), u+iv\rangle \\ &= \lambda\langle u+iv, u+iv\rangle \\ &= \lambda\|u+iv\|^2.\end{aligned}$$

Thus, $\lambda\|u+iv\|^2 \in \mathbf{R}$. Since $\|u+iv\|^2 \in \mathbf{R}$, we have $\lambda \in \mathbf{R}$. □

Claim. T has an eigenvalue.

Proof. By the Complex Spectral Theorem, $T_{\mathbf{C}}$ has an eigenvalue. Suppose

$$T_{\mathbf{C}}(u+iv) = \lambda(u+iv)$$

for some $u, v \in V$ and $\lambda \in \mathbf{C}$. By the previous claim, $\lambda \in \mathbf{R}$. Furthermore,

$$Tu + iTv = \lambda u + i\lambda v.$$

Since $\lambda \in \mathbf{R}$, we can compare real/imaginary parts to get

$$Tu = \lambda u \text{ and } Tv = \lambda v.$$

Since $u + iv \neq 0$, at least one of u, v is nonzero. Thus, λ is an eigenvalue of T. □

Note. You can view the source code for this solution here.