142.32

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Problem. Suppose $\lambda_1, \ldots, \lambda_n$ is a list of distinct real numbers. Prove that the list $e^{\lambda_1 x}, \ldots, e^{\lambda_n x}$ is linearly independent in the vector space of real-valued functions on $\mathbf{R}$ (also known as $\mathbf{R}^{\mathbf{R}}$).

As suggested in the hint, let $V = \operatorname{span}\left(e^{\lambda_1 x}, \ldots, e^{\lambda_n x}\right)$ and define $T \in \mathcal{L}(V)$ by $T(f) = f'$.

Claim. The eigenvalues of T are $\lambda_1, \ldots, \lambda_n$, with corresponding eigenvectors $e^{\lambda_1 x}, \ldots, e^{\lambda_n x}$.

Proof. First, note that

$$\begin{aligned} T\left(e^{\lambda_i x}\right) &= \left(e^{\lambda_i x}\right)' \\ &= \lambda_i e^{\lambda_i x} \end{aligned}$$

and thus $\lambda_1, \ldots, \lambda_n$ are all eigenvalues of T. Since V is at most n-dimesional, there can be no other eigenvalues of T. $\square$

Claim. The list $e^{\lambda_1 x}, \ldots, e^{\lambda_n x}$ is linearly independent in V.

Proof. Since $e^{\lambda_1 x}, \ldots, e^{\lambda_n x}$ are eigenvectors corresponding to distinct eigenvaues, they must be linearly independent. $\square$

Since V is a subspace of $\mathbf{R}^{\mathbf{R}}$, it follows that $e^{\lambda_1 x}, \ldots, e^{\lambda_n x}$ is linearly independent in $\mathbf{R}^{\mathbf{R}}$.

Note. You can view the source code for this solution here.