223.9

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Problem. Suppose V is a complex inner product space. Prove that every normal operator on V has a square root. (An operator $S \in \mathcal{L}(V)$ is called a *square root* of $T \in \mathcal{L}(V)$ if $S^2 = T$.)

Suppose T is a normal operator on V. By the *Complex Spectral Theorem*, T has an orthonormal basis of eigenvectors $v_1, \ldots, v_n$. Suppose $Tv_i = \lambda_i v_i$, where $\lambda_i \in \mathbf{C}$.

Recall that a linear map is uniquely defined by where it sends a basis of V. Let S be the linear map defined by $Sv_i = \lambda_i' v_i$, where $(\lambda_i')^2 = \lambda_i$ (at least one λ_i' exists because $\mathbf{F} = \mathbf{C}$).

Claim. $S^2 = T$.

Proof. $S^2 v_i = (\lambda_i')^2 v_i = \lambda_i v_i = Tv_i$ for each basis vector v_i, so $S^2 = T$. □

Note. You can view the source code for this solution here.