216.21

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Problem. Fix a positive integer n. In the inner product space of continuous real-valued functions on $[-\pi, \pi]$ with inner product

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)g(x)\,\mathrm{d}x,$$

let

$$V = \operatorname{span}(1, \sin x, \sin 2x, \ldots, \sin nx, \cos x, \cos 2x, \ldots, \cos nx).$$

(a) Define $D \in \mathcal{L}(V)$ by $Df = f'$. Show that $D^* = -D$. Conclude that D is normal but not self-adjoint.

(b) Define $T \in \mathcal{L}(V)$ by $Tf = f''$. Show that T is self-adjoint.

Claim. For all $f, g \in V$,

$$f(\pi)g(\pi) = f(-\pi)g(-\pi).$$

Proof. Suppose $f, g \in V$. Then,

$$\begin{aligned} f(x) &= a_0 + a_1 \sin x + \cdots + a_n \sin nx + a_{n+1} \cos x + \cdots + a_{2n} \cos nx \\ g(x) &= b_0 + b_1 \sin x + \cdots + b_n \sin nx + b_{n+1} \cos x + \cdots + b_{2n} \cos nx. \end{aligned}$$

Since all terms of $f(\pi)g(\pi)$ with $\sin(k\pi)$ will be zero,

$$\begin{aligned} f(\pi)g(\pi) = a_0 b_0 &+ a_0 \sum_{i=1}^{n} b_{n+i} \cos(i\pi) + b_0 \sum_{i=1}^{n} a_{n+i} \cos(i\pi) \\ &+ \sum_{1 \le i,j \le n} a_{n+i} \cos(i\pi) b_{n+j} \cos(j\pi). \end{aligned}$$

Similarly,

$$\begin{aligned} f(-\pi)g(-\pi) = a_0 b_0 &+ a_0 \sum_{i=1}^{n} b_{n+i} \cos(-i\pi) + b_0 \sum_{i=1}^{n} a_{n+i} \cos(-i\pi) \\ &+ \sum_{1 \le i,j \le n} a_{n+i} \cos(-i\pi) b_{n+j} \cos(-j\pi). \end{aligned}$$

But since $\cos(-x) = \cos(x)$, we see that $f(\pi)g(\pi) = f(-\pi)g(-\pi)$. $\square$

Claim. For all $f, g \in V$,
$$\int_{-\pi}^{\pi} f'(x)g(x)\,\mathrm{d}x = -\int_{-\pi}^{\pi} f(x)g'(x)\,\mathrm{d}x$$

Proof. Starting with the previous claim,
$$\begin{aligned}
0 &= f(\pi)g(\pi) - f(-\pi)g(-\pi) \\
&= (f \cdot g)(x)\Big|_{-\pi}^{\pi} \\
&= \int_{-\pi}^{\pi} (f \cdot g)'(x)\,\mathrm{d}x \\
&= \int_{-\pi}^{\pi} f'(x)g(x)\,\mathrm{d}x + \int_{-\pi}^{\pi} f(x)g'(x)\,\mathrm{d}x.
\end{aligned}$$
Thus,
$$\int_{-\pi}^{\pi} f'(x)g(x)\,\mathrm{d}x = -\int_{-\pi}^{\pi} f(x)g'(x)\,\mathrm{d}x$$
as desired.

□

Recall that $D \in \mathcal{L}(V)$ is defined by $Tf = f'$.

Claim. $D^* = -D$.

Proof. Since D^* is unique, all we need to do is show that for all $f, g \in V$,
$$\langle Df, g\rangle = \langle f, -Dg\rangle.$$
Indeed,
$$\begin{aligned}
\langle Df, g\rangle &= \langle f', g\rangle \\
&= \int_{-\pi}^{\pi} f'(x)g(x)\,\mathrm{d}x \\
&= -\int_{-\pi}^{\pi} f(x)g'(x)\,\mathrm{d}x && \text{(previous claim)} \\
&= -\langle f, Dg\rangle \\
&= \langle f, -Dg\rangle
\end{aligned}$$
as desired. □

Claim. D is normal.

Proof.
$$\begin{aligned}
DD^* &= (D)(-D) \\
&= (-D)(D) \\
&= D^*D,
\end{aligned}$$
and thus D is normal. □

Claim. D is not self-adjoint.

Proof. Clearly $D \neq D^*$ since $D^* = -D$. ☐

Recall that $T \in \mathcal{L}(V)$ is defined by $Tf = f''$.

Claim. T is self-adjoint.

Proof. Since $T = DD$,

$$\begin{aligned} T^* &= (DD)^* \\ &= D^* D^* \\ &= (-D)(-D) \\ &= DD \\ &= T. \end{aligned}$$

Thus, T is self-adjoint. ☐

Note. You can view the source code for this solution here.