215.13

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Problem. Give an example of an operator $T \in \mathcal{L}(\mathbf{C}^4)$ such that T is normal but not self-adjoint.

Define T by $Tv = iv$ for all $v \in \mathbf{C}^4$.

Claim. $T^*v = -iv$.

Proof. Fix $u \in \mathbf{C}^4$. Then, for all $v \in \mathbf{C}^4$,

$$\begin{aligned} \langle u, T^*v \rangle &= \langle Tu, v \rangle && \textit{(definition of } T^*\textit{)} \\ &= \langle iu, v \rangle \\ &= i\langle u, v \rangle \\ &= \langle u, -iv \rangle \end{aligned}$$

Thus, $T^*v = -iv$ as desired. □

Claim. T is normal.

Proof. For all $v \in \mathbf{C}^4$,

$$\begin{aligned} TT^*v &= (i)(-i)v \\ &= (-i)(i)v \\ &= T^*Tv. \end{aligned}$$

Thus, $TT^* = T^*T$, which means that T is normal. □

Claim. T is not self-adjoint.

Proof. Let $v = (1, 1, 1, 1)$. Notice that

$$Tv = (i, i, i, i),$$

while

$$T^*v = (-i, -i, -i, -i),$$

so $T \neq T^*$. □

> **Note.** You can view the source code for this solution here.