232.7

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Problem. Suppose T is a positive operator on V. Prove that T is invertible if and only if

$$\langle Tv, v\rangle > 0$$

for all nonzero $v \in V$.

Claim. If $\langle Tv, v\rangle > 0$ for all nonzero $v \in V$, then T is invertible.

Proof. Suppose $\langle Tv, v\rangle > 0$ for all nonzero $v \in V$. For any $v \in V$ where $Tv = 0$,

$$\langle Tv, v\rangle = 0,$$

which implies $v = 0$, since $\langle Tv, v\rangle > 0$ for all nonzero v. Since $Tv = 0$ implies $v = 0$, T is injective and thus invertible. □

Claim. If T is invertible, then $\langle Tv, v\rangle > 0$ for all nonzero $v \in V$.

Proof. Since T is positive, $\langle Tv, v\rangle \geq 0$ for all $v \in V$. Thus, it suffices to prove that $\langle Tv, v\rangle \neq 0$ for all nonzero $v \in V$.

Suppose $v \in V$ is nonzero. Then, $Tv \neq 0$ because T is invertible. If $\sqrt{T}v = 0$ then $Tv = 0$, so $\sqrt{T}v \neq 0$. Thus, by the definiteness of the inner product,

$$\left\langle \sqrt{T}v, \sqrt{T}v \right\rangle > 0.$$

But

$$\begin{aligned}
\left\langle \sqrt{T}v, \sqrt{T}v \right\rangle &= \left\langle \sqrt{T}v, \sqrt{T}^{*}v \right\rangle \\
&= \left\langle \sqrt{T}\sqrt{T}v, v \right\rangle \\
&= \langle Tv, v\rangle,
\end{aligned}$$

so $\langle Tv, v\rangle > 0$ for nonzero v. □

Note. You can view the source code for this solution here.