140.16

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Problem: Suppose V is a complex vector space, $T \in \mathcal{L}(V)$, and the matrix of T with respect to a basis of V contains only real entries. Show that if λ is an eigenvalue of T, then so is $\overline{\lambda}$.

Solution: Let $v_1, \ldots, v_n$ be a basis of V for which

$$\mathcal{M}(T) = \begin{pmatrix} A_{11} & & \\ & \ddots & \\ & & A_{nn} \end{pmatrix}$$

where $A_{ij} \in \mathbf{R}$. Next, suppose that v is an eigenvector of λ and that

$$\mathcal{M}(v) = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

where $c_i \in \mathbf{C}$. It follows that $Tv = \lambda v$. Or, equivalently,

$$\begin{pmatrix} A_{11} & & \\ & \ddots & \\ & & A_{nn} \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} \lambda c_1 \\ \vdots \\ \lambda c_n \end{pmatrix}.$$

Thus,

$$\begin{aligned} A_{11}c_1 + \cdots + A_{1n}c_n &= \lambda c_1 \\ A_{21}c_1 + \cdots + A_{2n}c_n &= \lambda c_2 \\ &\vdots \\ A_{n1}c_1 + \cdots + A_{nn}c_n &= \lambda c_n \end{aligned}$$

Taking the complex conjugate on both sides, since $A_{ij} \in \mathbf{R}$,

$$\begin{aligned} A_{11}\overline{c_1} + \cdots + A_{1n}\overline{c_n} &= \overline{\lambda}\overline{c_1} \\ A_{21}\overline{c_1} + \cdots + A_{2n}\overline{c_n} &= \overline{\lambda}\overline{c_2} \\ &\vdots \\ A_{n1}\overline{c_1} + \cdots + A_{nn}\overline{c_n} &= \overline{\lambda}\overline{c_n} \end{aligned}$$

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It follows that,

$$\begin{pmatrix} A_{11} & & \\ & \ddots & \\ & & A_{nn} \end{pmatrix} \begin{pmatrix} \overline{c_1} \\ \vdots \\ \overline{c_n} \end{pmatrix} = \begin{pmatrix} \overline{\lambda}\overline{c_1} \\ \vdots \\ \overline{\lambda}\overline{c_n} \end{pmatrix}.$$

Equivalently,

$$T\overline{v} = \overline{\lambda}\overline{v}$$

Since $v \neq 0$, we know that $\overline{v} \neq 0$. Thus, $\overline{\lambda}$ is an eigenvalue of V. $\square$

Source code: Click here to open the source code on GitHub.