306.21

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Problem. Suppose V is an inner product space. Suppose $T \in \mathcal{L}(V)$ and

$$\|T^*v\| \leq \|Tv\|$$

for every $v \in V$. Prove that T is normal.

Note. The problem above fails on infinite-dimensional inner product spaces, leading to what are called hyponormal operators, which have a well-developed theory.

We can apply the Gram–Schmidt procedure to a basis of V to produce an orthonormal basis of V: $e_1, \ldots, e_n$.

Claim. $\|T^*e_1\|^2 + \cdots + \|T^*e_n\|^2 \leq \|Te_1\|^2 + \cdots + \|Te_n\|^2$

Proof. Damn it Liam posted it first. □

Note. You can view the source code for this solution here.