304.3

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Problem. Suppose $T \in \mathcal{L}(V)$ has the same matrix with respect to every basis of V. Prove that T is a scalar multiple of the identity operator.

Let $v_1, \dots, v_n$ be a basis of V, and suppose $1 \le i < j \le n$. Next, consider the matrix of T with respect to $v_1, \dots, v_n$:

$$\mathcal{M}(T, (v_1, \dots, v_n)) = \begin{pmatrix} & \vdots & & \vdots & \\ \dots & A_{ii} & \dots & A_{ij} & \dots \\ & \vdots & \ddots & \vdots & \\ \dots & A_{ji} & \dots & A_{jj} & \dots \\ & \vdots & & \vdots & \end{pmatrix}.$$

For convenience, let's relabel:

$$\mathcal{M}(T, (v_1, \dots, v_n)) = \begin{pmatrix} & \vdots & & \vdots & \\ \dots & a & \dots & b & \dots \\ & \vdots & \ddots & \vdots & \\ \dots & c & \dots & d & \dots \\ & \vdots & & \vdots & \end{pmatrix}.$$

Recall that the i^{th} column corresponds to Tv_i, so

$$Tv_i = \dots + av_i + \dots + cv_j + \cdots,$$

and the j^{th} column corresponds to Tv_j, so

$$Tv_j = \dots + bv_i + \dots + dv_j + \cdots.$$

Claim. $a = d$.

Proof. If we swap v_i and v_j in the basis, the i^{th} column will correspond to

$$Tv_j = \dots + dv_j + \dots + bv_i + \cdots,$$

while the j^{th} column will correspond to

$$Tv_i = \dots + cv_j + \dots + av_i + \cdots.$$

Thus, under the new basis,

$$\begin{pmatrix} & \vdots & & \vdots & \\ \cdots & a & \cdots & b & \cdots \\ & \vdots & \ddots & \vdots & \\ \cdots & c & \cdots & d & \cdots \\ & \vdots & & \vdots & \end{pmatrix} \text{ will become } \begin{pmatrix} & \vdots & & \vdots & \\ \cdots & d & \cdots & c & \cdots \\ & \vdots & \ddots & \vdots & \\ \cdots & b & \cdots & a & \cdots \\ & \vdots & & \vdots & \end{pmatrix}.$$

Since T has the same matrix with respect to every basis of V, we see that $a = d$. □

Claim. $b = c = 0$.

Proof. This time, double v_i instead of swapping it with v_j. After we double v_i, our new basis will be $(\ldots, 2v_i, \ldots, v_j, \ldots)$. Under this new basis, the i^{th} column will correspond to

$$T(2v_i) = \cdots + a(2v_i) + \cdots + 2c(v_j) + \cdots ,$$

and the j^{th} column will correspond to

$$T(v_j) = \cdots + \frac{1}{2}b(2v_i) + \cdots + dv_j + \cdots .$$

Thus, under the new basis,

$$\begin{pmatrix} & \vdots & & \vdots & \\ \cdots & a & \cdots & b & \cdots \\ & \vdots & \ddots & \vdots & \\ \cdots & c & \cdots & d & \cdots \\ & \vdots & & \vdots & \end{pmatrix} \text{ will become } \begin{pmatrix} & \vdots & & \vdots & \\ \cdots & a & \cdots & \frac{1}{2}b & \cdots \\ & \vdots & \ddots & \vdots & \\ \cdots & 2c & \cdots & d & \cdots \\ & \vdots & & \vdots & \end{pmatrix}.$$

Since T has the same matrix with respect to every basis of V, we see that

$$b = \frac{1}{2}b \quad \text{and} \quad c = 2c.$$

Thus, $b = c = 0$. □

Claim. T is a scalar multiple of the identity operator.

Proof. Our first claim implies that entries on the diagonal of $\mathcal{M}(T, (v_1, \ldots, v_n))$ are all pairwise equal. Thus, they are all equal. Our second claim implies all other entries are zero. Thus,

$$\mathcal{M}(T, (v_1, \ldots, v_n)) = \begin{pmatrix} \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda \end{pmatrix} = \mathcal{M}(\lambda I)$$

for some $\lambda \in \mathbf{F}$. □

Note. You can view the source code for this solution here.