190.17

THOMAS BREYDO

Problem. For $u \in V$, let Φu denote the linear functional V defined by

$$(\Phi u)(v) = \langle v, u \rangle$$

for $v \in V$.

(a) Show that if $\mathbf{F} = \mathbf{R}$, then Φ is a linear map from V to V'. (Recall from Section 3.F that $V' = \mathcal{L}(V, \mathbf{F})$ and that V' is called the dual space of V.)

(b) Show that if $\mathbf{F} = \mathbf{C}$ and $V \neq \{0\}$, then Φ is not a linear map.

(c) Show that Φ is injective.

(d) Suppose $\mathbf{F} = \mathbf{R}$ and V is finite-dimensional. Use parts (a) and (c) and a dimension-counting argument (but without using 6.42) to show that Φ is an isomorphism from V onto V'.

Note. Part (d) gives an alternative proof of the Reisz Representation Theorem (6.42) when $\mathbf{F} = \mathbf{R}$. Part (d) also gives a natural isomorphism (meaning that it does not depend on a choice of basis) from a finite-dimensional real inner product space onto its dual space.

Claim 1. If $\mathbf{F} = \mathbf{R}$, then Φ is a linear map from V to V'.

Proof. Suppose $u_1, u_2 \in V$. Then, the functional $\Phi(u_1 + u_2)$ sends $v \in V$ to

$$\begin{aligned}\big(\Phi(u_1 + u_2)\big)(v) &= \langle v, u_1 + u_2 \rangle \\ &= \langle v, u_1 \rangle + \langle v, u_2 \rangle \\ &= (\Phi u_1)(v) + (\Phi u_2)(v).\end{aligned}$$

Thus, Φ is additive.

Next, suppose $\lambda \in \mathbf{F}$ and $u \in V$. Then, the functional $\Phi(\lambda u)$ sends sends $v \in V$ to

$$\begin{aligned}
\big(\Phi(\lambda u)\big)(v) &= \langle v, \lambda u \rangle \\
&= \overline{\lambda}\langle v, u \rangle \\
&= \lambda \langle v, u \rangle && (\mathbf{F} = \mathbf{R}) \\
&= \lambda (\Phi u)(v).
\end{aligned}$$

Thus, Φ is homogenous. Since it is both additive and homogenous, it is linear. □

Claim 2. If $\mathbf{F} = \mathbf{C}$ and $V \neq \{0\}$, then Φ is not a linear map.

Proof. Since $V \neq \{0\}$, there exists a $u \in V$ such that $u \neq 0$. We will show that

$$\Phi(iu) \neq i\Phi(u),$$

and thus Φ is not a linear map. Namely, $\Phi(iu)$ and $i\Phi(u)$ send u to two different values:

$$\begin{aligned}
\big(\Phi(iu)\big)(u) &= \langle u, iu \rangle \\
&= \overline{i}\langle u, u \rangle \\
&= -i\langle u, u \rangle,
\end{aligned}$$

while

$$\begin{aligned}
\big(i\Phi(u)\big)(u) &= i\big(\Phi(u)\big)(u) \\
&= i\langle u, u \rangle.
\end{aligned}$$

These are indeed different because $\langle u, u \rangle \neq 0$ (since $u \neq 0$). □

Claim 3. Φ is injective.

Proof. Suppose $u \in V$ and Φu is the zero map. Then,

$$(\Phi u)(u) = 0.$$

Since $(\Phi u)(u) = \langle u, u \rangle$, the definiteness of the inner product implies $u = 0$. Thus, Φ is injective. □

Claim 4. If $\mathbf{F} = \mathbf{R}$ and V is finite-dimensional, then Φ is an isomorphism from V onto V'.

Proof. By Claim 1 and Claim 3, Φ is an injective linear map from V onto V'. Thus,

$$\begin{aligned} \dim V' &= \dim V \\ &= \dim \operatorname{range} \Phi + \dim \operatorname{null} \Phi \\ &= \dim \operatorname{range} \Phi. \end{aligned}$$

But if $\dim \operatorname{range} \Phi = \dim V'$, then Φ must be surjective. Since it is both injective and surjective, it is invertible. Thus, it is an isomorphism. □

Note. You can view the source code for this solution here.