

140.16

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**Problem:** Suppose  $V$  is a complex vector space,  $T \in \mathcal{L}(V)$ , and the matrix of  $T$  with respect to a basis of  $V$  contains only real entries. Show that if  $\lambda$  is an eigenvalue of  $T$ , then so is  $\bar{\lambda}$ .

**Solution:** Let  $v_1, \dots, v_n$  be a basis of  $V$  for which

$$\mathcal{M}(T) = \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{pmatrix}$$

where  $A_{ij} \in \mathbf{R}$ . Next, suppose that  $v$  is an eigenvector of  $\lambda$  and that

$$\mathcal{M}(v) = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

where  $c_i \in \mathbf{C}$ . It follows that  $Tv = \lambda v$ . Or, equivalently,

$$\begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} \lambda c_1 \\ \vdots \\ \lambda c_n \end{pmatrix}.$$

Thus,

$$\begin{aligned} A_{11}c_1 + \cdots + A_{1n}c_n &= \lambda c_1 \\ A_{21}c_1 + \cdots + A_{2n}c_n &= \lambda c_2 \\ &\vdots \\ A_{n1}c_1 + \cdots + A_{nn}c_n &= \lambda c_n \end{aligned}$$

Taking the complex conjugate on both sides, since  $A_{ij} \in \mathbf{R}$ ,

$$\begin{aligned} A_{11}\bar{c}_1 + \cdots + A_{1n}\bar{c}_n &= \bar{\lambda}\bar{c}_1 \\ A_{21}\bar{c}_1 + \cdots + A_{2n}\bar{c}_n &= \bar{\lambda}\bar{c}_2 \\ &\vdots \\ A_{n1}\bar{c}_1 + \cdots + A_{nn}\bar{c}_n &= \bar{\lambda}\bar{c}_n \end{aligned}$$

It follows that,

$$\begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{pmatrix} \begin{pmatrix} \overline{c_1} \\ \vdots \\ \overline{c_n} \end{pmatrix} = \begin{pmatrix} \overline{\lambda c_1} \\ \vdots \\ \overline{\lambda c_n} \end{pmatrix}.$$

Letting

$$\overline{v} = \begin{pmatrix} \overline{c_1} \\ \vdots \\ \overline{c_n} \end{pmatrix},$$

we see that

$$T\overline{v} = \overline{\lambda}\overline{v}.$$

Since  $v \neq 0$ , we know that  $\overline{v} \neq 0$ . Thus,  $\overline{\lambda}$  is an eigenvalue of  $V$ . □

**Source code:** [Click here to open the source code on GitHub.](#)