179.31

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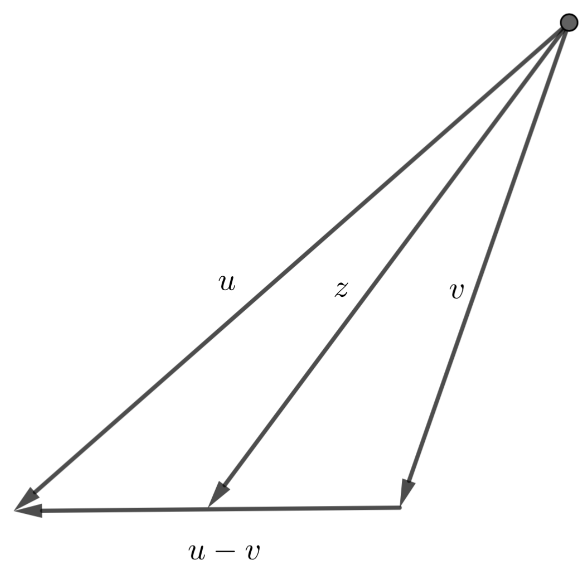
Problem. Use inner products to prove Apollonius's Identity: In a triangle with sides of length a, b, and c, let d be the length of the line segment from the midpoint of the side of length c to the opposite vertex. Then

$$a^2 + b^2 = \frac{1}{2}c^2 + 2d^2.$$

Let the triangle be formed by $u, v, u - v \in \mathbf{R}^2$ as shown in the diagram ($z \in \mathbf{R}^2$ is a vector to the "midpoint" of $u - v$):



Note that, by construction,

$$\begin{aligned} \|u\| &= a \\ \|v\| &= b \\ \|u - v\| &= c \\ \|z\| &= d. \end{aligned}$$

Claim. $z = \frac{1}{2}(u+v)$.

Proof.

$$\begin{aligned} z &= v + \frac{u-v}{2} && \textit{(by construction)} \\ &= \frac{2v+u-v}{2} \\ &= \frac{v+u}{2}. \end{aligned}$$

□

Now that we know what z is, we can prove the identity.

Claim. $a^2 + b^2 = \frac{1}{2}c^2 + 2d^2$.

Proof. Starting with the right-hand side,

$$\begin{aligned} \frac{1}{2}c^2 + 2d^2 &= \frac{1}{2}\|u-v\|^2 + 2\|z\|^2 \\ &= \frac{1}{2}\|u-v\|^2 + 2\left\|\frac{1}{2}(u+v)\right\|^2 \\ &= \frac{1}{2}\|u-v\|^2 + \frac{1}{2}\|u+v\|^2 \\ &= \frac{1}{2}\big(\langle u-v, u-v\rangle + \langle u+v, u+v\rangle\big) \\ &= \frac{1}{2}\big(\langle u,u\rangle - \cancel{\langle u,v\rangle} - \cancel{\langle v,u\rangle} + \langle v,v\rangle + \langle u,u\rangle + \cancel{\langle u,v\rangle} + \cancel{\langle v,u\rangle} + \langle v,v\rangle\big) \\ &= \frac{1}{2}\big(2\langle u,u\rangle + 2\langle v,v\rangle\big) \\ &= \langle u,u\rangle + \langle v,v\rangle \\ &= \|u\|^2 + \|v\|^2 \\ &= a^2 + b^2. \end{aligned}$$

□

Note. You can view the source code for this solution here.