250.14

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Problem. Suppose V is an inner product space and $N \in \mathcal{L}(V)$ is nilpotent. Prove that there exists an orthonormal basis of V with respect to which N has an upper-triangular matrix.

By *8.19 Matrix of a nilpotent operator,* there exists a basis $v_1, \ldots, v_n$ of V such that

$$\mathcal{M}\big(N, (v_1, \ldots, v_n)\big) = \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix}.$$

If we apply the Gram—Schmidt procedure to $v_1, \ldots, v_n$, we will obtain a orthonormal basis $e_1, \ldots, e_n$ of V such that $\mathcal{M}\big(N, (e_1, \ldots, e_n)\big)$ is still upper-triangular.

Thus, there exists an orthonormal basis of V with respect to which N has an upper-triangular matrix.

Note. You can view the source code for this solution here.