286.18

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Problem. Suppose V is a real vector space and $T \in \mathcal{L}(V)$. Prove that the following are equivalent:

(a) All the eigenvalues of $T_{\mathbf{C}}$ are real.
(b) There exists a basis of V with respect to which T has an upper-triangular matrix.
(c) There exists a basis of V consisting of generalized eigenvectors.

Claim. (c) implies (b)

Proof. Suppose there exists a basis of V consisting of generalized eigenvectors. By *8.13*, each vector corresponds to just one eigenvalue. Reorder the vectors so that all vectors with eigenvalue λ_1 come first, then λ_2, and so on. Within each λ group, sort the vectors v_i in increasing value of k_i, where

$$k_i := \text{smallest } m \text{ for which } (T - \lambda_{v_i} I)^m v_i = 0.$$

Suppose that the reordered basis is $v_1, \ldots, v_n$. We will now show that the matrix of T with respect thomas $v_1, \ldots, v_n$ is upper-triangular. Suppose v_i has eigenvalue λ and $k_i = k$, and that

$$Tv_i = a_1 v_1 + \cdots + a_n v_n.$$

We must show that $a_{i+1} = \cdots = a_n = 0$. Since v_i is a generalized eigenvector with eigenvalue λ,

$$\begin{aligned}
0 &= (T - \lambda I)^{k_i+1} v_i \\
&= (T - \lambda I)^{k_i} (Tv_i - \lambda v_i) \\
&= (T - \lambda I)^{k_i} \left(a_1 v_1 + \cdots + a_n v_n - \lambda v_i\right) \\
&= \sum_{j \neq i} (T - \lambda I)^{k_i} (a_j v_j) && \text{see note[1]} \\
&= \sum_{j \neq i} a_j (T - \lambda I)^{k_i} v_j
\end{aligned}$$

But $v_1, \ldots, v_n$ is linearly independent, and so is the list

Since $v_1, \ldots, v_n$ is a basis of V with respect to which T has an upper-triangular matrix, (c) implies (b). $\square$

[1] Note, $(T - \lambda I)^{k_i} (a_i v_i) = (T - \lambda I)^{k_i} (-\lambda v_i) = 0$

> **Note.** You can view the source code for this solution here.