175.5

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Problem. Suppose $T \in \mathcal{L}(V)$ is such that
$$\|Tv\| \leq \|v\|$$
for every $v \in V$. Prove that $T - \sqrt{2}I$ is invertible.

First, we prove the following claim:

Claim. If $\lambda > 1$ is an eigenvalue of T then there exists a $v \in V$ for which
$$\|Tv\| > \|v\|.$$

Proof. Take $v \in V$ to be an eigenvector with eigenvalue λ. Notice that
$$v \neq 0 \implies \|v\| > 0$$
Multiplying both sides of $\lambda > 1$ by $\|v\|$, we get that
$$\lambda\|v\| > \|v\|.$$
Then, since $Tv = \lambda v$,
$$\begin{aligned}\|Tv\| &= \|\lambda v\| \\ &= \lambda\|v\| \\ &> \|v\|.\end{aligned}$$
□

Claim. $T - \sqrt{2}I$ is invertible.

Proof. Suppose the contrary, that is not invertible. Then, $\lambda = \sqrt{2}$ is an eigenvalue of T. Since $\lambda > 1$, the previous claim tells us that there exists a $v \in V$ for which
$$\|Tv\| > \|v\|.$$
Thus it is not true that
$$\|Tv\| \leq \|v\|$$
for every $v \in V$, which is a contradiction. So, $T - \sqrt{2}I$ is not invertible. □

Note. You can view the source code for this solution here.