153.7

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> **Problem.** Suppose $T \in \mathcal{L}(V)$. Prove that 9 is an eigenvalue of T^2 if and only if 3 or -3 is an eigenvalue of T.

> **Claim 1.** If $\lambda \in \mathbf{F}$ is an eigenvalue of T, then λ^2 is an eigenvalue of T^2.

Proof. Since λ is an eigenvalue of T, there exists a $v \in V$ such that $v \neq 0$ and

$$Tv = \lambda v.$$

For this v,

$$\begin{aligned} T^2(v) &= T(Tv) \\ &= T(\lambda v) \\ &= \lambda^2 v. \end{aligned}$$

Thus, λ^2 is an eigenvalue of T^2. □

> **Claim 2.** If $\mu \in \mathbf{F}$ is an eigenvalue of T^2, then $\sqrt{\mu}$ or $-\sqrt{\mu}$ is an eigenvalue of T (assuming that $\sqrt{\mu} \in \mathbf{F}$).

Proof. Since μ is an eigenvalue of T^2, we know that $T^2 - \mu I$ is not injective. Thus, there exists a $v \in V$ such that $v \neq 0$ and

$$(T^2 - \mu I)v = 0.$$

Since

$$\begin{aligned} (T - \sqrt{\mu} I)(T + \sqrt{\mu} I) &= T^2 + T \cdot \sqrt{\mu} I - \sqrt{\mu} I \cdot T + (\sqrt{\mu} I)^2 \\ &= T^2 + \sqrt{\mu} T - \sqrt{\mu} T + (\sqrt{\mu})^2 I \\ &= T^2 + \mu I, \end{aligned}$$

it follows that

$$(T - \sqrt{\mu} I) \underbrace{(T + \sqrt{\mu} I) v}_{w} = 0.$$

Consider $w = (T + \sqrt{\mu} I)v$.

- If $w = 0$, then $(T + \sqrt{\mu} I)v = 0 \implies T + \sqrt{\mu} I$ is not injective. Thus, $-\sqrt{\mu}$ is an eigenvalue of T.

- If $w \neq 0$, then $(T - \sqrt{\mu} I)w = 0 \implies T - \sqrt{\mu} I$ is not injective. Thus, $\sqrt{\mu}$ is an eigenvalue of T. $\square$

These two claims suffice. It follows from **Claim 1** that if 3 or -3 is an eigenvalue of T, then 9 is an eigenvalue of T^2. It follows from **Claim 2** that if 9 is an eigenvalue of T^2, then 3 or -3 is an eigenvalue T.

> **Note.** You can view the source code for this solution here.