239.14

THOMAS BREYDO

Problem. Suppose $T \in \mathcal{L}(V)$. Prove that $\dim \operatorname{range} T$ equals the number of nonzero singular values of T.

Claim. $\operatorname{null} T = E(0, \sqrt{T^*T})$

Proof. Decompose T into $S\sqrt{T^*T}$. Suppose $v \neq 0$ and $Tv = 0$. Then,

$$S\sqrt{T^*T}v = 0,$$

which means $\sqrt{T^*T}v = 0$, since S preserves norms. Thus, $\operatorname{null} T = \operatorname{null} \sqrt{T^*T}$. Finally, since $\operatorname{null} \sqrt{T^*T} = E(0, \sqrt{T^*T})$, we get

$$\operatorname{null} T = E(0, \sqrt{T^*T})$$

$\square$

Suppose T has k nonzero singular values. Since $k = \dim V - \dim E(0, \sqrt{T^*T})$. Finally,

$$\begin{aligned}
\dim \operatorname{range} T &= \dim V - \dim \operatorname{null} T \\
&= \dim V - \dim E(0, \sqrt{T^*T}) \\
&= k
\end{aligned}$$

Note. You can view the source code for this solution here.