231.4

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Problem. Suppose $T \in \mathcal{L}(V, W)$. Prove that T^*T is a positive operator on V and TT^* is a positive operator on W.

We will show the claim is true for T^*T, and taking $T = S^*$ will complete the proof for SS^*.

Claim. T^*T is self-adjoint.

Proof.

$$\begin{aligned}(T^*T)^* &= T^*(T^*)^* \\ &= T^*T,\end{aligned}$$

as desired. □

Claim. For all $v \in V$, $\langle T^*Tv, v\rangle \geq 0$.

Proof. Let $w = Tv$. Then,

$$\begin{aligned}\langle T^*Tv, v\rangle &= \langle v, (T^*T)^*v\rangle \\ &= \langle v, T^*Tv\rangle && (T^*T \text{ is self-adjoint}) \\ &= \langle v, T^*w\rangle \\ &= \langle Tv, w\rangle \\ &= \langle w, w\rangle \\ &\geq 0, && (\text{definition of inner product})\end{aligned}$$

as desired. □

Note. You can view the source code for this solution here.