

215.13

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Problem. Give an example of an operator $T \in \mathcal{L}(\mathbf{C}^4)$ such that T is normal but not self-adjoint.

Define T by $Tv = iv$ for all $v \in \mathbf{C}^4$.

Claim. $T^*v = -iv$.

Proof. Fix $u \in \mathbf{C}^4$. Then, for all $v \in \mathbf{C}^4$,

$$\begin{aligned}\langle u, T^*v \rangle &= \langle Tu, v \rangle && (\text{definition of } T^*) \\ &= \langle iu, v \rangle \\ &= i\langle u, v \rangle \\ &= \langle u, -iv \rangle\end{aligned}$$

Thus, $T^*v = -iv$ as desired. \square

Claim. T is normal.

Proof. For all $v \in \mathbf{C}^4$,

$$\begin{aligned}TT^*v &= (i)(-i)v \\ &= (-i)(i)v \\ &= T^*Tv.\end{aligned}$$

Thus, $TT^* = T^*T$, which means that T is normal. \square

Claim. T is not self-adjoint.

Proof. $T(1) = i$ while $T^*(1) = -i$. Thus, $T \neq T^*$. \square

Note. You can view the source code for this solution [here](#).