202.13

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Problem. Find $p \in \mathcal{P}_5(\mathbf{R})$ that makes
$$cost = \int_{-\pi}^{\pi} |\sin x - p(x)|^2 \, \mathrm{d}x$$
as small as possible.

Consider the vector space V of all continuous functions from $-\pi$ to π, with the inner product
$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x) \, \mathrm{d}x.$$
Let $U = \mathcal{P}_5(\mathbf{R})$, and note that U is a subspace of V.

Claim. The optimal $p \in U$ is that which minimizes $\|\sin x - p(x)\|$.

Proof. Note that
$$\begin{aligned}
\|\sin x - p(x)\| &= \sqrt{\langle \sin x - p(x), \sin x - p(x) \rangle} \\
&= \sqrt{\int_{-\pi}^{\pi} (\sin x - p(x))^2 \, \mathrm{d}x} \\
&= \sqrt{cost},
\end{aligned}$$
so minimizing $\|\sin x - p(x)\|$ minimizes the cost. $\square$

Claim. The optimal $p \in U$ is $P_U \sin x$.

Proof. By *6.56*, we know that for any $p \in U$
$$\|\sin x - P_U \sin x\| \leq \|\sin x - p(x)\|,$$
and that $p(x) = P_U \sin x$ is the only p for which equality holds, minimizing the right-hand side, and in turn minimzing *cost*. $\square$

Claim. The optimal $p \in U$ is

$$p(x) = \frac{-72765\pi^2 + 693\pi^4 + 654885}{8\pi^{10}} x^5 + \frac{-363825 - 315\pi^4 + 39375\pi^2}{4\pi^8} x^3 + \frac{-16065\pi^2 + 105\pi^4 + 155925}{8\pi^6} x$$

Click here to check out the approximation on Desmos.

Proof. We can apply the Gram–Schmidt Procedure to $1, x, x^2, x^3, x^4, x^5$ to obtain an orthonormal basis of U. Then, we can compute

$$\begin{aligned} p(x) &= P_U \sin x \\ &= \langle e_1, \sin x\rangle e_1 + \cdots \langle e_6, \sin x\rangle e_6. \end{aligned}$$

I expanded my `pylinearalg` Python package to use `sympy`, which can handle "symbolic integration" (instead of floating-point approximations). You can see the solution script here. □

Note. You can view the source code for this solution here.