223.9

THOMAS BREYDO

Problem. Suppose V is a complex inner product space. Prove that every normal operator on V has a square root.

Note. An operator $S \in \mathcal{L}(V)$ is called a *square root* of $T \in \mathcal{L}(V)$ if $S^2 = T$.

Suppose T is a normal operator on V. By the *Complex Spectral Theorem,* T has an orthonormal basis of eigenvectors $v_1, \ldots, v_n$. Suppose $Tv_i = \lambda_i v_i$, where $\lambda_i \in \mathbf{C}$.

Recall that a linear map is uniquely defined by where it sends a basis of V. Let S be the linear map defined by $Sv_i = \sqrt{\lambda_i} v_i$.

Claim. $S^2 = T$.

Proof. $S^2 v_i = Tv_i$ for each basis vector v_i, so $S^2 = T$. $\square$

Note. You can view the source code for this solution here.