141.30

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Problem. Suppose $T \in \mathcal{L}(\mathbf{R}^3)$ and $-4, 5$, and $\sqrt{7}$ are eigenvalues of T. Prove that there exists $x \in \mathbf{R}^3$ such that $Tx - 9x = (-4, 5, \sqrt{7})$.

Claim. 9 is not an eigenvalue of T.

Proof. T can have at most $\dim \mathbf{R}^3 = 3$ eigenvalues by *5.13*, and we already have three distinct eigenvalues: $-4, 5$, and $\sqrt{7}$. □

Claim. $T - 9I$ is invertible.

Proof. Otherwise, 9 would be an eigenvalue of T by *5.6*. □

Claim. $x = (T - 9I)^{-1}\big((-4, 5, \sqrt{7})\big)$ works.

Proof. Note that

$$\begin{aligned} Tx - 9x &= (T - 9I)x \\ &= (-4, 5, \sqrt{7}) \end{aligned}$$

as desired. □

Note. You can view the source code for this solution here.

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