269.20

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Problem. Suppose V is a complex vector space and $V_1, \ldots, V_m$ are nonzero subspaces of V such that $V = V_1 \oplus \cdots \oplus V_m$. Suppose $T \in \mathcal{L}(V)$ and each V_j is invariant under T. For each j, let p_j denote the characteristic polynomial of $T|_{V_j}$ Prove that the characteristic polynomial of T equals $p_1 \cdots p_m$.

Claim. Every eigenvalue of $T|_{V_j}$ is also an eigenvalue of T.

Proof. Suppose λ is an eigenvalue of $T|_{V_j}$, with corresponding generalized eigenvector $v \in V_j$. Then,

$$\left(T|_{V_j} - \lambda I\right)^{\dim V_j} v = 0.$$

Since $\dim V_j < \dim V$ and T is invariant on V_j,

$$(T - \lambda I)^{\dim V} v = 0.$$

Thus, λ must be one of the eigenvalues of T. $\square$

Suppose the eigenvalues of T are $\lambda_1, \ldots, \lambda_k$.

Claim. The sum of the multiplicities of some eigenvalue λ over all $T|_{V_1}, \ldots, T|_{V_m}$ is equal to the multiplicity of eigenvalue λ of T. In other words,

$$\dim G(\lambda, T) = \dim G(\lambda, T|_{V_1}) + \cdots + \dim G(\lambda, T|_{V_m}).$$

Proof. By *8.21*,

$$V_j = G(\lambda_1, T|_{V_j}) \oplus \cdots \oplus G(\lambda_k, T|_{V_j}),$$

since the eigenvalues of V_j are contained within $\lambda_1, \ldots, \lambda_k$ We use this to go from line two to line three below:

$$\begin{aligned}
G(\lambda_1, T) \oplus \cdots \oplus G(\lambda_k, T) &= V \\
&= \bigoplus_{j=1}^{m} V_j \\
&= \bigoplus_{j=1}^{m} \left(G(\lambda_1, T|_{V_j}) \oplus \cdots \oplus G(\lambda_k, T|_{V_j}) \right) \\
&= \left(\bigoplus_{j=1}^{m} G(\lambda_1, T|_{V_j}) \right) \oplus \cdots \oplus \left(\bigoplus_{j=1}^{m} G(\lambda_k, T|_{V_j}) \right).
\end{aligned}$$

We gather that for each λ,

$$\begin{aligned}
G(\lambda, T) &= \bigoplus_{j=1}^{m} G(\lambda, T|_{V_j}) \\
&= G(\lambda, T|_{V_1}) \oplus \cdots \oplus G(\lambda, T|_{V_m}),
\end{aligned}$$

and thus

$$\begin{aligned}
\dim G(\lambda, T) &= \dim \left(G(\lambda, T|_{V_1}) \oplus \cdots \oplus G(\lambda, T|_{V_m}) \right) \\
&= \dim G(\lambda, T|_{V_1}) + \cdots + \dim G(\lambda, T|_{V_m}).
\end{aligned}$$

□

Suppose the characteristic polynomial of T is q.

Claim. $p_1 \cdots p_m = q$.

Proof.

$$\begin{aligned}
p_1 \cdots p_m &= \prod_{j=1}^{m} p_j \\
&= \prod_{j=1}^{m} \left((z-\lambda_1)^{\dim G(\lambda_1, T|_{V_j})} \right) \cdots \left((z-\lambda_k)^{\dim G(\lambda_k, T|_{V_j})} \right) \\
&= \left(\prod_{j=1}^{m} (z-\lambda_1)^{\dim G(\lambda_1, T|_{V_j})} \right) \cdots \left(\prod_{j=1}^{m} (z-\lambda_k)^{\dim G(\lambda_k, T|_{V_j})} \right) \\
&= \left((z-\lambda_1)^{\sum_{j=1}^{m} \dim G(\lambda_1, T|_{V_j})} \right) \cdots \left((z-\lambda_1)^{\sum_{j=1}^{m} \dim G(\lambda_k, T|_{V_j})} \right) \\
&= \left((z-\lambda_1)^{\dim G(\lambda_1, T)} \right) \cdots \left((z-\lambda_k)^{\dim G(\lambda_k, T)} \right) \\
&= q
\end{aligned}$$

□

> **Note.** You can view the source code for this solution here.