142.32

THOMAS BREYDO

Problem. Suppose $\lambda_1, \ldots, \lambda_n$ is a list of distinct real numbers. Prove that the list $e^{\lambda_1 x}, \ldots, e^{\lambda_n x}$ is linearly independent in the vector space of real-valued functions on \mathbf{R} (also known as $\mathbf{R}^{\mathbf{R}}$).

As suggested in the hint, let $V = \operatorname{span}\left(e^{\lambda_1 x}, \dots, e^{\lambda_n x}\right)$ and define $T \in \mathcal{L}(V)$ by T(f) = f'.

Claim. The eigenvalues of T are $\lambda_1, \ldots, \lambda_n$, with corresponding eigenvetors $e^{\lambda_1 x}, \ldots, e^{\lambda_n x}$.

Proof. First, note that

$$T\left(e^{\lambda_i x}\right) = \left(e^{\lambda_i x}\right)'$$
$$= \lambda_i e^{\lambda_i x}$$

and thus $\lambda_1, \ldots, \lambda_n$ are all eigenvalues of T. Since V is at most n-dimesional, there can be no other eigenvalues of T.

Claim. The list $e^{\lambda_1 x}, \dots, e^{\lambda_n x}$ is linearly independent in V.

Proof. Since $e^{\lambda_1 x}, \dots, e^{\lambda_n x}$ are eigenvectors corresponding to distinct eigenvalues, they must be linearly independent.

Since V is a subspace of $\mathbf{R}^{\mathbf{R}}$, it follows that $e^{\lambda_1 x}, \dots, e^{\lambda_n x}$ is linearly independent in $\mathbf{R}^{\mathbf{R}}$.

Note. You can view the source code for this solution here.