

153.7

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Problem. Suppose $T \in \mathcal{L}(V)$. Prove that 9 is an eigenvalue of T^2 if and only if 3 or -3 is an eigenvalue of T .

Claim 1. If $\lambda \in \mathbf{F}$ is an eigenvalue of T , then λ^2 is an eigenvalue of T^2 .

Proof. Since λ is an eigenvalue of T , there exists a $v \in V$ such that $v \neq 0$ and

$$Tv = \lambda v.$$

For this v ,

$$\begin{aligned} T^2(v) &= T(Tv) \\ &= T(\lambda v) \\ &= \lambda^2 v. \end{aligned}$$

Thus, λ^2 is an eigenvalue of T^2 . □

Claim 2. If $\mu \in \mathbf{F}$ is an eigenvalue of T^2 , then $\sqrt{\mu}$ or $-\sqrt{\mu}$ is an eigenvalue of T (assuming that $\sqrt{\mu} \in \mathbf{F}$).

Proof. Since μ is an eigenvalue of T^2 , we know that $T^2 - \mu I$ is not injective. Thus, there exists a $v \in V$ such that $v \neq 0$ and

$$(T^2 - \mu I)v = 0.$$

Since

$$\begin{aligned} (T - \sqrt{\mu}I)(T + \sqrt{\mu}I) &= T^2 + T \cdot \sqrt{\mu}I - \sqrt{\mu}I \cdot T + (\sqrt{\mu}I)^2 \\ &= T^2 + \sqrt{\mu}T - \sqrt{\mu}T + (\sqrt{\mu})^2 I \\ &= T^2 + \mu I, \end{aligned}$$

it follows that

$$(T - \sqrt{\mu}I) \underbrace{(T + \sqrt{\mu}I)v}_w = 0.$$

Consider $w = (T + \sqrt{\mu}I)v$.

- If $w = 0$, then $(T + \sqrt{\mu}I)v = 0 \implies T + \sqrt{\mu}I$ is not injective. Thus, $-\sqrt{\mu}$ is an eigenvalue of T .

- If $w \neq 0$, then $(T - \sqrt{\mu}I)w = 0 \implies T - \sqrt{\mu}I$ is not injective. Thus, $\sqrt{\mu}$ is an eigenvalue of T . \square

These two claims suffice. It follows from **Claim ??** that if 3 or -3 is an eigenvalue of T , then 9 is an eigenvalue of T^2 . It follows from **Claim ??** that if 9 is an eigenvalue of T^2 , then 3 or -3 is an eigenvalue T .

Note. You can view the source code for this solution [here](#).