

## 269.20

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**Problem.** Suppose  $V$  is a complex vector space and  $V_1, \dots, V_m$  are nonzero subspaces of  $V$  such that  $V = V_1 \oplus \dots \oplus V_m$ . Suppose  $T \in \mathcal{L}(V)$  and each  $V_j$  is invariant under  $T$ . For each  $j$ , let  $p_j$  denote the characteristic polynomial of  $T|_{V_j}$ . Prove that the characteristic polynomial of  $T$  equals  $p_1 \cdots p_m$ .

**Claim.** Every eigenvalue of  $T|_{V_j}$  is also an eigenvalue of  $T$ .

*Proof.* Suppose  $\lambda$  is an eigenvalue of  $T|_{V_j}$  with eigenvector  $v \in V_j$ . Then,

$$\begin{aligned} Tv &= T|_{V_j} v \\ &= \lambda v, \end{aligned}$$

so  $\lambda$  is an eigenvalue of  $T$ . □

Suppose the eigenvalues of  $T$  are  $\lambda_1, \dots, \lambda_k$ .

**Claim.** The sum of the multiplicities of some eigenvalue  $\lambda$  over all  $T|_{V_1}, \dots, T|_{V_m}$  is equal to the multiplicity of eigenvalue  $\lambda$  of  $T$ . In other words,

$$\dim G(\lambda, T) = \dim G(\lambda, T|_{V_1}) + \dots + \dim G(\lambda, T|_{V_m}).$$

*Proof.* By 8.21,

$$V_j = G(\lambda_1, T|_{V_j}) \oplus \dots \oplus G(\lambda_k, T|_{V_j}),$$

since the eigenvalues of  $V_j$  are contained within  $\lambda_1, \dots, \lambda_k$ . We use this to go from line two to line three below:

$$\begin{aligned}
 G(\lambda_1, T) \oplus \dots \oplus G(\lambda_k, T) &= V \\
 &= \bigoplus_{j=1}^m V_j \\
 &= \bigoplus_{j=1}^m \left( G(\lambda_1, T|_{V_j}) \oplus \dots \oplus G(\lambda_k, T|_{V_j}) \right) \\
 &= \left( \bigoplus_{j=1}^m G(\lambda_1, T|_{V_j}) \right) \oplus \dots \oplus \left( \bigoplus_{j=1}^m G(\lambda_k, T|_{V_j}) \right).
 \end{aligned}$$

We gather that for each  $\lambda$ ,

$$\begin{aligned}
 G(\lambda, T) &= \bigoplus_{j=1}^m G(\lambda, T|_{V_j}) \\
 &= G(\lambda, T|_{V_1}) \oplus \dots \oplus G(\lambda, T|_{V_m}),
 \end{aligned}$$

and thus

$$\begin{aligned}
 \dim G(\lambda, T) &= \dim \left( G(\lambda, T|_{V_1}) \oplus \dots \oplus G(\lambda, T|_{V_m}) \right) \\
 &= \dim G(\lambda, T|_{V_1}) + \dots + \dim G(\lambda, T|_{V_m}).
 \end{aligned} \quad \square$$

Suppose the characteristic polynomial of  $T$  is  $q$ .

**Claim.**  $p_1 \cdots p_m = q$ .

*Proof.*

$$\begin{aligned}
 p_1 \cdots p_m &= \prod_{j=1}^m p_j \\
 &= \prod_{j=1}^m \left( (z - \lambda_1)^{\dim G(\lambda_1, T|_{V_j})} \dots (z - \lambda_k)^{\dim G(\lambda_k, T|_{V_j})} \right) \\
 &= \left( \prod_{j=1}^m (z - \lambda_1)^{\dim G(\lambda_1, T|_{V_j})} \right) \dots \left( \prod_{j=1}^m (z - \lambda_k)^{\dim G(\lambda_k, T|_{V_j})} \right) \\
 &= \left( (z - \lambda_1)^{\sum_{j=1}^m \dim G(\lambda_1, T|_{V_j})} \right) \dots \left( (z - \lambda_k)^{\sum_{j=1}^m \dim G(\lambda_k, T|_{V_j})} \right) \\
 &= \left( (z - \lambda_1)^{\dim G(\lambda_1, T)} \right) \dots \left( (z - \lambda_k)^{\dim G(\lambda_k, T)} \right) \\
 &= q
 \end{aligned}$$

$\square$

**Note.** You can view the source code for this solution [here](#).