154.14

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Problem. Give an example of an operator whose matrix with respect to some basis contains only 0's on the diagonal, but the operator is invertible.

Claim. In $\mathbf{R}^2$, reflection across $y = x$ is invertible but contains only 0's on the diagonal with respect to the standard basis of $\mathbf{R}^2$.

Proof. Define $T \in \mathcal{L}(\mathbf{R}^2)$ by

$$T\big((x, y)\big) = (y, x).$$

Since $T\big((1, 0)\big) = (0, 1)$ and $T\big((0, 1)\big) = (1, 0)$,

$$\mathcal{M}(T) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

But, T is invertible, since $T^2 = I$:

$$\begin{aligned} T^2\big((x, y)\big) &= T\big((y, x)\big) \\ &= (x, y). \end{aligned}$$

□

Note. You can view the source code for this solution here.