175.5

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Problem. Suppose $T \in \mathcal{L}(V)$ is such that

$$||Tv|| \le ||v||$$

for every $v \in V$. Prove that $T - \sqrt{2}I$ is invertible.

First, we prove the following claim:

Claim. If $\lambda > 1$ is an eigenvalue of T then there exists a $v \in V$ for which

$$||Tv|| > ||v||.$$

Proof. Take $v \in V$ to be an eigenvector with eigenvalue λ . Notice that

$$v \neq 0 \implies ||v|| \neq 0$$

Multiplying both sides of $\lambda > 1$ by ||v||, we get that

$$\lambda ||v|| > ||v||.$$

Then, since $Tv = \lambda v$,

$$\begin{split} \|Tv\| &= \|\lambda v\| \\ &= \lambda \|v\| \\ &> \|v\|. \end{split}$$

Claim. $T - \sqrt{2}I$ is invertible.

Proof. Suppose the contrary, that is not invertible. Then, $\lambda = \sqrt{2}$ is an eigenvalue of T. Since $\lambda > 1$, the previous claim tells us that there exists a $v \in V$ for which

$$||Tv|| > ||v||.$$

Thus it is not true that

$$||Tv|| \le ||v||$$

for every $v \in V$, which is a contradiction. So, $T - \sqrt{2}I$ is not invertible.

Note. You can view the source code for this solution here.