232.7

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Problem. Suppose T is a positive operator on V. Prove that T is invertible if and only if

$$\langle Tv, v\rangle > 0$$

for all nonzero $v \in V$.

Claim. If $\langle Tv, v\rangle > 0$ for all nonzero $v \in V$, then T is invertible.

Proof. Suppose $\langle Tv, v\rangle > 0$ for all nonzero $v \in V$. For any $v \in V$ where $Tv = 0$,

$$\langle Tv, v\rangle = 0,$$

which implies $v = 0$, since $\langle Tv, v\rangle > 0$ for all nonzero v. Since $Tv = 0$ implies $v = 0$, T is injective and thus invertible. □

Claim. If T is invertible, then $\langle Tv, v\rangle > 0$ for all nonzero $v \in V$.

Proof. Since T is positive, $\langle Tv, v\rangle \geq 0$ for all $v \in V$. Thus, it suffices to prove that $\langle Tv, v\rangle \neq 0$ for all nonzero $v \in V$.

Suppose $v \in V$ is nonzero. Then, $Tv \neq 0$ because T is invertible. Let S be the unique positive square root of T. Then, $Sv \neq 0$ (because otherwise $Tv = S^2v = 0$). Thus, by the definiteness of the inner product,

$$\langle Sv, Sv\rangle > 0.$$

But

$$\begin{aligned}\langle Sv, Sv\rangle &= \langle Sv, S^*v\rangle \\ &= \langle S^2v, v\rangle \\ &= \langle Tv, v\rangle,\end{aligned}$$

so $\langle Tv, v\rangle > 0$ for nonzero v. □

Note. You can view the source code for this solution here.