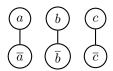
K-SAT reduces to VERTEX-COVER

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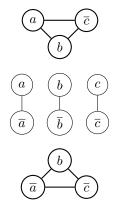
Theorem. For every formula ϕ in K-cnf with C clauses and V variables, we can construct a graph G in polynomial time such that ϕ is satisfiable iff G has a k-vertex cover, where k = C(K-1) + V.

Proof. We begin by constructing G using two gadgets. For all V variables, we construct a *variable gadget*, which consists of the symbols x and \overline{x} for some variable x. For example, if ϕ uses the variables a, b, c then G starts as the following:



This takes $\mathcal{O}(V)$ time, since we construct 2V nodes and V edges.

Next, we construct C clause gadgets (one per clause). These are K-cliques that contain the variables in a certain clause. Continuing the previous example, if C=2, K=3, and $\phi=(a\vee b\vee \overline{c})\wedge(\overline{a}\vee b\vee \overline{c})$ then we would add two 3-cliques to G:

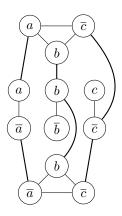


We will call each of these K-cliques a clause gadget. Each of the C clauses takes

$$\mathcal{O}\left(K + \binom{K}{2}\right) = \mathcal{O}\left(K^2\right)$$

time to construct, so it takes $\mathcal{O}(CK^2)$ time to construct all of them.

Finally, we connect each of the $\hat{C}K$ variables to their respective variable gadgets:



This takes $\mathcal{O}(CK)$ time. In total, constructing G takes

$$\mathcal{O}(V) + \mathcal{O}(CK^2) + \mathcal{O}(CK) = \mathcal{O}(V + CK^2)$$

time, which is clearly polynomial time since C, K, and V are bounded by the length of $\langle \phi \rangle$.

- (A) How do we know that ϕ is satisfiable if G has a k-vertex cover, where k = C(K-1) + V? Let Ψ be a k-vertex cover of G.
 - (1) Every clause gadget contributes at least K-1 vertices to Ψ since it is a connected graph with K nodes. In total, the C clause gadgets contribute at least C(K-1) vertices to Ψ .
 - (2) Every variable gadget contributes at least 1 vertex to Ψ since it has an internal edge. In total, the V variable gadgets will contribute at least V vertices to Ψ .

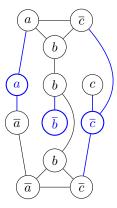
Combining (1) and (2), we see that $exactly \ K-1$ nodes are included from every clause gadget and $exactly \ 1$ is included from each variable gadget—otherwise, we would have more than C(K-1)+V nodes. Note that in each of the C clauses, the one node that was not included will have its symbol included in the corresponding variable gadget. Otherwise, the clause-variable edge will not be covered. Since the V symbols included from variable gadgets are non-conflicting (1 per gadget), we can construct the a satisfying set S which contains these V symbols. The set will satisfy ϕ because each clause has one node whose corresponding symbol was included in Ψ .

- (B) How do we know that G has a k-vertex cover if ϕ is satisfiable? Given a satisfying set S of V symbols, we can construct a k-vertex cover for G by doing the following:
 - (1) For every symbol in S, include in the cover the corresponding symbol from the variable's gadget. (This includes V nodes in the cover.)

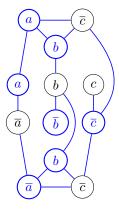
(2) For every K-clique that represents a clause, include K-1 of the nodes in the cover, leaving out a node that is also present in S. There will be at least one such node; otherwise S isn't a satisfying set. (This includes C(K-1) nodes in the cover.)

These two steps create a k-vertex cover, where k = C(K-1) + V. This cover will cover all edges because all edges within the C clause gadgets will be covered by the K-1 nodes included from each gadget; also, since the one node that isn't included from the clause gadget is present in S, its symbol will be included in the corresponding variable gadget, which covers that clause-variable edge; finally, since S is a satisfying set, all variables \star will be present in S (either as \star or $\overline{\star}$), which means the edge within each of the V variable gadgets will be covered

Continuing the previous example, notice that one possible S would be $\{a, \bar{b}, \bar{c}\}$ since $\phi = 1$ when $a = \bar{b} = \bar{c} = 1$. After (1), we would have the following nodes included:



After (2), we would indeed have a k-vertex cover, where k = 2(3-1) + 3 = 7:



Combining (A) and (B) we get that ϕ is satisfiable iff G has a k-vertex cover, where k = C(K - 1) + V.