CS2102 - Homework 7

Thomas Castleman

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Problem 1 *Coin Flips*

1. The probability that the coin lands heads *exactly* three times is

$$\frac{\binom{8}{3}}{2^8} = 0.21875 \tag{1}$$

because the total number of ways to flip a coin (2 options) 8 times is 2^8, and the size of the set of events where exactly three were chosen to be heads can be represented by $\binom{8}{3}$ (3 of the eight tosses were chosen).

2. The probability the coin lands heads *at least* three times is

$$1 - \frac{\binom{8}{2} + \binom{8}{1} + \binom{8}{0}}{2^8} = 0.85546875 \tag{2}$$

This can be thought of as taking the complement of the probability that the coin lands heads *less than* three times (because $P(\overline{E}) = 1 - P(E)$). Again, the number of possible ways to flip a coin 8 times is 2^8, and now, in the numerator, we cover three cases: the case where only 2 are heads $\binom{8}{2}$, the case where 1 is heads $\binom{8}{1}$, and the case where none are heads $\binom{8}{0}$.

3. The probability the coin lands the same number of times on both sides (that is, 4 heads and four tails), is

$$\frac{\binom{8}{4}}{2^8} = 0.2734375 \tag{3}$$

because the universe 2^8 is once again the same, but now you are choosing four of the tosses to be heads $\binom{8}{4}$. Because the only options for coin state are heads or tails, the remaining four must be tails and therefore there are the same number of heads and tails.

Problem 2 *Circular Table*

Consider that we are trying to arrange the boys in such a way that no two boys have one spot in between them, as this middle spot would have two boy neighbors. Putting all the boys in even spots or all in odd spots could not work, as there are only 25 of each, so there would definitely be boys spaced two apart.

Instead, we want to spread them out across even and odd, as evenly as possible. However, 25 does not divide evenly, so 12 would have to go in even spots and 13 in odd (the case where 12 go in odd and 13 in even is the same, as you can just rotate the table and then it is the same as the previous case with 13 in odds).

Consider the placement of 13 boys in the odd spots. We cannot put boys in consecutive odd spots (i.e. spot 3 and spot 5), because they are two apart and would therefore give a person with two boy neighbors. This means that we must space them in *every other* odd spot. Between 1 and 50, there are exactly 13 of these every-other-odd spots:

$$1, 5, 9, 13, 17, 21, 25, 29, 33, 37, 41, 45, 49 \tag{4}$$

However, we are unable to put a boy in both 1 and 49, as, because this table is circular, they are two spots away from each other. This means that, having placed 12 of the 13, there will always be 1 left over which has no spot it can be placed in. We are also unable to place the extra in one of the even spots, because there are only 12 every-other-even spots:

$$2, 6, 10, 14, 18, 22, 26, 30, 34, 38, 42, 48 \tag{5}$$

which have all been taken by the 12 boys we allotted for the evens. Therefore, there is no way to place this last boy without causing some spot to have two boy neighbors.

Problem 3 *Committee*

The probability that the women would have a majority is

$$\frac{\binom{15}{4}\binom{10}{2} + \binom{15}{5}\binom{10}{1} + \binom{15}{6}\binom{10}{0}}{\binom{25}{6}} \approx 0.544664031620553359 \tag{6}$$

The universe is all possible ways to choose six people from a group of 25 (10 men and 15 women), so $\binom{25}{6}$. The numerator enumerates each possible case where women have the majority: when 4 women and 2 men are chosen $\binom{15}{4}\binom{10}{2}$, when 5 women and 1 man are chosen $\binom{15}{5}\binom{10}{1}$, and when 6 women and no men are chosen $\binom{15}{6}\binom{10}{0}$.

Problem 4 *Bayes Theorem*

1.

$$\begin{aligned}
P(E_2|E_1) \cdot \frac{P(E_1)}{P(E_2)} &= P(E_2|E_1) \cdot \frac{\frac{|E_1|}{|S|}}{\frac{|E_2|}{|S|}} && \text{(Def. probability)} \\
&= P(E_2|E_1) \cdot \frac{|E_1|}{|E_2|} \\
&= \frac{|E_2 \cap E_1|}{|E_1|} \cdot \frac{|E_1|}{|E_2|} && \text{(Def. conditional probability)} \\
&= \frac{|E_2 \cap E_1|}{|E_2|} \cdot \frac{|E_1|}{|E_1|} && \text{(Commutativity of mult.)} \\
&= \frac{|E_2 \cap E_1|}{|E_2|} \\
&= \frac{|E_1 \cap E_2|}{|E_2|} && \text{(Commutativity of intersection)} \\
&= P(E_1|E_2) && \text{(Def. conditional probability)}
\end{aligned}$$

2.

$$\begin{aligned}
P(E_1|E_2)P(E_2) + P(E_1|\overline{E_2})P(\overline{E_2}) &= \frac{|E_1 \cap E_2|}{|E_2|} \cdot \frac{|E_2|}{|S|} + \frac{|E_1 \cap \overline{E_2}|}{|\overline{E_2}|} \cdot \frac{|\overline{E_2}|}{|S|} && \text{(Def. cond. prob.)} \\
&= \frac{|E_1 \cap E_2|}{|S|} + \frac{|E_1 \cap \overline{E_2}|}{|S|} \\
&= \frac{|E_1 \cap E_2| + |E_1 \cap \overline{E_2}|}{|S|}
\end{aligned}$$

Because E_2 and $\overline{E_2}$ are disjoint, the intersections $E_1 \cap E_2$ and $E_1 \cap \overline{E_2}$ are also disjoint. This means that the cardinality of their union is the same as the sum of their cardinalities. That is,

$$
\begin{aligned}
\frac{|E_1 \cap E_2| + |E_1 \cap \overline{E_2}|}{|S|} &= \frac{|(E_1 \cap E_2) \cup (E_1 \cap \overline{E_2})|}{|S|} \\
&= \frac{|E_1 \cap (E_2 \cup \overline{E_2})|}{|S|} && \text{(Un-distribute $\cap$ over $\cup$)} \\
&= \frac{|E_1 \cap S|}{|S|} && \text{(Complement law)} \\
&= \frac{|E_1|}{|S|} && \text{(Identity law)} \\
&= P(E_1) && \text{(Def. probability)}
\end{aligned}
$$

Problem 5 *False Positive Paradox*

Let E_1 be the set of all possibilities in which you have the disease. Let E_2 be the set of all possibilities in which you test positive. From the information given, we know that

$$P(E_2|E_1) = 1 \tag{7}$$

because the test will *always* return positive when you do have the disease. We also know that

$$P(E_2|\overline{E_1}) = 0.05 \tag{8}$$

because the test has a 0.05 chance of returning positive when you don't actually have the disease.

What we want is $P(E_1|E_2)$, the probability that you actually have the disease given you test positive. We know, by the definition of conditional probability, that

$$P(E_1|E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)} \tag{9}$$

Because the test *always* returns positive when you do have the disease, we know that $E_1 \subseteq E_2$, because having the disease implied that the test returned positive. This means that $P(E_1 \cap E_2) = P(E_1)$.

We also know, because of Bayes Theorem from Problem 4, that

$$P(E_2) = P(E_2|E_1)P(E_1) + P(E_2|\overline{E_1})P(\overline{E_1}) \tag{10}$$

So our expression for $P(E_1|E_2)$ becomes

$$\frac{P(E_1)}{P(E_2|E_1)P(E_1) + P(E_2|\overline{E_1})P(\overline{E_1})} \tag{11}$$

Using the given value of $P(E_1) = 0.02$, we know $P(\overline{E_1}) = 0.98$ and are able to evaluate this expression:

$$\frac{0.02}{1 \cdot 0.02 + 0.05 \cdot 0.98} \approx 0.2898550724637681 \tag{12}$$

Problem 6 *Connected Components*

To be an equivalence relation, the "connected" relation must be reflexive, symmetric, and transitive. Consider the undirected graph $G = (V, E)$:

0.1 Reflexive

Given any node v from any undirected graph, there is always a path from v to itself (the empty sequence). Therefore, $\forall v \in V$, $(v, v) \in R$, which is the definition of reflexive.

0.2 Symmetric

Consider arbitrary vertices $v, w \in V$. $(v, w) \in R$ if and only if there exists some path $(v, v_1, v_2, ...v_n, w)$ from node v to node w. The existence of this path means that there is also a path from w to v, constructed by reversing the path from v to w as follows: $(w, v_n, ...v_2, v_1, v)$. Therefore, $(w, v) \in R$.

Because $(v, w) \in R \implies (w, v) \in R$, we know that R is symmetric.

0.3 Transitive

Consider arbitrary vertices $v, w, u \in V$. If $(v, w), (w, u) \in R$, then there exist paths $P_{v,w} = (v, v_1, v_2, ...v_n, w)$ and $P_{w,u} = (w, w_1, w_2, ...w_m, u)$ from v to w and w to u. Given the existence of $P_{v,w}$ and $P_{w,u}$, we can conclude that there exists a path $P_{v,u}$ from v to u which can be constructed by taking the vertices from $P_{v,w}$ to get from v to w and then following the vertices from $P_{w,u}$ to get to u. That is,

$$P_{v,u} = (v, v_1, v_2, ...v_n, w, w_1, w_2, ...w_m, u) \tag{13}$$

which means that $(v, u) \in R$.

Because $(v, w), (w, u) \in R \implies (v, u) \in R$, we know that R is transitive.

If "connected components" are disjoint subgraphs which are connected, then we can define them using R as follows:

For some non-connected simple graph $G = (V, E)$, subgraphs $H_1, H_2, ...H_n$ where any $H_i = (V_i, E_i)$ are connected components of G provided that

$$(\forall v, w \in V_i, (v, w) \in R) \ \land (\nexists u \in V_k, k \neq i, (v, u) \in R) \tag{14}$$

provided that every $V_i \subseteq V$ and every $E_i \subseteq E$, and that their unions equal V and E, respectively (that is, the set of all V_i's form a partition of V, and the set of all E_i's form a partition of E).

Problem 7 *Complete graphs and Eularian graphs*

Eularian Cycle: Given that a graph can only have an Eularian cycle provided that every vertex has even degree, and that every node in K_n has degree $n - 1$, there can only be Eularian cycles on complete graphs where $n - 1$ is even. That is, any K_n where n is odd.

Eularian Path: Under the belief that an Eularian cycle *is* an Eularian path, we can conclude that there exists an Eularian path on any complete graph that has an Eularian cycle, so any odd n.

There cannot be an Eularian path on any K_n where n is even, because every node will have an odd degree, and only exactly two nodes can have odd degree to satisfy the conditions for an Eularian path. However, 2 is the only even n for which there is exactly two nodes of odd degree, and therefore there exists an Eularian path on K_2.

So, there can be an Eularian path for $n = 2$, and any odd n.

Problem 8 *Cycle Start Nodes*

Given that there exists a cycle starting from v, including w, and ending in v, we can write this cycle down as

$$(v, v_1, v_2, ...v_n, w, u_1, u_2, ...u_m, v) \tag{15}$$

where any v_i or u_i are arbitrary nodes in the cycle.

Using this path we can construct the subpath $(w, u_1, u_2, ...u_m, v)$ from w to v, as well as the subpath $(v, v_1, v_2, ...v_n, w)$ from v to w. Then, by chaining these paths together, we can create a new path

$$(w, u_1, u_2, ...u_m, v, v_1, v_2, ...v_n, w) \tag{16}$$

which goes from w to w, meaning it is a cycle.

Collaboration: I worked with Bihong Hu on this assignment.