An algorithm for computing the maximal rectangles in a binary relation

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Maximal Rectangular Relations

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Introduction. Let us establish some notation which will be employed throughout this paper. If R is a relation from X to Y, i.e. $R \subseteq X \times Y$, then we say that $A \times B$ is an R-rectangle if $\emptyset \neq A \times B \subseteq R$. An R-rectangle is maximal if it is properly contained in no other R-rectangle. It is a consequence of Zorn's lemma that every R-rectangle is contained in a maximal R-rectangle. If $a \in X$, put $aR = \{y \in Y\colon (a,y) \in R\}$; dually put $Rb = \{x \in X\colon (x,b) \in R\}$ for $b \in Y$. For sets $A \subseteq X$ and $B \in Y$, put $\phi(A) = \{aR\colon a \in A\}$ and $\psi(B) = \{Rb; b \in B\}$. We adhere to the usual convention for the empty set to write $\phi(\emptyset) = Y$ and $\psi(\emptyset) = X$. One may establish that the mappings ϕ and ψ are a Galois connection between the lattices P (X) and P (Y) of all subsets of X and Y, respectively. As a consequence, $A \subseteq \psi\phi(A)$ and $B \subseteq \phi\psi(B)$ for all $A \subseteq X$ and all $B \subseteq Y$.

If now ϕ and ψ are any Galois connection between P (X) and P (Y) and if we call a subset $A \subseteq X$ (respectively $B \subseteq Y$) closed if $A = \psi\phi(A)$ (respectively, $B = \phi\psi(B)$), then we may summarize the classical results connecting Galois correspondences and relations as follows (the first edition of Birkhoff [3] or Szasz [11] serve as general references):

Theorem 1. Let R be a relation from X to Y and let ϕ and ψ be as above. Then ϕ and ψ are mutually inverse anti-isomorphisms between the lattices of closed subsets of X and of Y, where the lattice ordering is just set containment. Moreover, if (ϕ', ψ') is any Galois connection from P (X) to P (Y) with L_1 and L_2 denoting the closed subsets of X and Y respectively, and if $R = \cup\{L \times \phi'(L)\colon L \in L_1\}$ then (i) $R = \cup\{\phi'(L) \times L\colon L \in L_2\}$ and (ii) $A \times B$ is a maximal R-rectangle if and only if $\emptyset \neq A \in L_2$, $\emptyset \neq B \in L_2$ and $B = \phi(A)$.

One algorithm for computing these antiisomorphisms when X and Y are finite appeared in [6]. In this paper we present another algorithm and its supporting theorems, which describe how a maximal R-rectangle "grows" out of a maximal R'-rectangle where $R' \subseteq R$. We have encoded our algorithm for computer operation.

In Simpson's paper [10], maximal rectangular relations were employed to study the semigroup of all relations on a set, and Fay's paper [6] has its origins in mathematical brain modeling and computer science. Application of Simpson's very interesting work is unfortunately hampered by his not having a method for computing the maximal rectangles in R, and it is very likely that use of such a computational tool will lead to new insights into the nature of semigroups of relations on finite sets.

The idea for studying such a growth process comes from a study of a paper of Bednarek and Taulbee [2]. Osteen [9] calls algorithms of this type "point-sequence" algorithms. The paper [2] gives an algorithm for finding the maximal R-chains of a relation $R \subseteq X \times X$, i.e. sets $C \subseteq X$ which are maximal with respect to the property that each pair x,y of distince elements of C satisfy either $(x,y) \in R$ or $(y,x) \in R$. Letting $\Delta = \{(x,x): x \in X\}$, it is easily established that C is a maximal R-chain if and only if $C \times C$ is a maximal $(R \cup R^{-1} \cup \Delta)$-rectangle. In graph-theoretic terms a maximal R-chain is a maximal complete subgraph and a maximal R-rectangle is a complete bipartite graph, except that $A \cap B$ need not be empty. For graphs without loops, our algorithm will find all maximal complete bipartite subgraphs.

In the next section we state the supporting theorems for our algorithm. The full proofs of these theorems will appear in [7,8]. The following section describes our algorithm and states some open questions of a combinatorial nature. We conclude with an application to a proposed model for animal brain function.

Maximal Rectangles. Throughout this section, we suppose R^* to be a nonvoid relation from a set X^* to a set Y^* and (a,b) to be a fixed element of $X^* \times Y^*$, and we let $X = X\setminus\{a\}$ and $Y = Y^*\setminus\{b\}$. $R = R^* \cap (X \times Y)$ is the restriction of R^* to $X \times Y$. We denote the set of maximal R^*-rectangles by M^* and the set of maximal R-rectangles by M. A maximal R-rectangle is said to be degenerate with respect to (a,b) if $A = \{a\}$, and nondegenerate otherwise. We may suppose without loss of generality that $R^*b = \emptyset$; for if $R^*y \neq \emptyset$ for all $y \in Y^*$, let b be anything not in Y^*, replace Y^* by $Y^* \cup \{b\}$, keeping X and R^* as before. (The reason for supposing $R^*b = 0$ is that it will allow us to establish the supporting theorems for our algorithm very economically, without having to study all the possible ways that a member of M^* can "grow" from a member of M. A complete study of this growth process will appear in [7].) We give first a necessary condition that a rectangle be a nondegenerate member of M^*. We adopt the convention that any singleton set {a} will be written without brackets.

Theorem 2.1. If $A^* \times B^*$ is a nondegenerate member of M^*, then there is some $C \times D$ in M for which one of the following holds:

(i) if $a \notin A^*$ then $A^* \times B^* = C \times D$

(ii) if $a \in A^*$ then $A^* \cap B^* = C \cup a \times D \cap aR^*$.

The rectangles $C \times D$ and $C \cup a \times D \cap aR^*$ on the right side of these equations are all possible candidates for R^*-maximality. The next theorem allows us to determine which of these are actually in M^*. For this purpose, let $T = \{A \times B \cap aR^*: A \times B \in M\}$. We use T' to denote the set of members of T which are maximal with respect to set containment.

Theorem 2.2. Suppose $C \times D \in M$. Then

(i) if $D \subseteq aR^*$ then $C \cup a \times D \in M^*$

(ii) if $D \nsubseteq aR^*$ then $C \times D \in M^*$, and $C \cup a \times D \cap aR^* \in M^*$ if and only if $C \times D \cap aR^* \in T'$.

We have thus far determined how the nondegenerate maximal R^*-rectangles are produced from Members of M. We have still to consider how the degenerate members of M^* arise. These rectangles, one easily sees, are of the form $a \times aR^*$.

Theorem 2.3. $a \times aR^* \in M^*$ if and only if $\emptyset \neq aR^*$ and $aR^* \nsubseteq D$ for all $C \times D \in M$.

The Algorithm. Suppose $X = \{x_1,\ldots,x_m\}$, $Y = \{y_1,\ldots,y_n\}$ and $R \subseteq X \times Y$. It is no loss to suppose $m \leq n$, for otherwise we can replace R by R^{-1} and re-name X and Y. We consider $Y^* = Y \cup b$, where b is anything not in Y and "enlarge" R by declaring that b is not related to anything. (We may suppose R to have been given by an m-by-n matrix of 0's and 1's to which we add an n+1st column of zeros, so that the n+1st column represents the fictitious element b and consequently $Rb = \emptyset$. Let $R_1 = M_1 = x_1 \times x_1R$, and put $T_1 = \emptyset$; for each k, $2 \leq k \leq m$, let $R_k = \{\{x_1,\ldots,x_k\} \times Y\} \cap R$, let M_k be the set of R_k-maximal members and $T_k = \{C \times D \cap x_kR\colon\ C \times D \in M_{k-1}\}$; T_k is the set of containment-maximal members of T_k. Suppose M_{k-1} has been computed. Then:

1) Find x_kR from the matrix of R.
2) Compute the members of the collection $T_k = \{A \times B \cap x_kR\colon\ A \times B \in M_{k-1}\}$ and eliminate the members of T_k which are proper subsets of other members of T_k. The remaining sets are the members of T_k.
   For each $C \times D \in M_{k-1}$,
   (a) if $D \subseteq x_kR$ then $C \cup x_k \times D \in M_k$.
   (b) if $D \not\subseteq x_kR$ then $C \times D \in M_k$, and $(C \cup x_k) \times (D \cap x_kR) \in M_k$ if and only if $\emptyset \neq C \times (D \cap x_kR) \in T'_k$. (This step computes all the nondegenerate members of M_k.)
3) $x_k \times x_kR \in M_k$ if and only if $\emptyset \neq x_kR \not\subseteq D$ for all $C \times D \in M_{k-1}$.
4) Replace k by k+1 and return to step (1) until k = m. The collection M_m is the collection of R-maximal relations.

It will be observed that the adjoined element b does not appear explicitly in any computations. Its purpose is to allow application of Theorems 2.2 and 2.3 to justify steps (2) and (3) of the algorithm.

When b is not adjoined, the computation is complicated by the increased number of positions of $C \times D \in M_{k-1}$ relative to aR* and R*b, as well as by the possibility that (a,b) might be in R* itself, and many more theorems would be needed to support the algorithm. The more general growth process is not itself without interest, however. The reader is referred to [7] for a comprehensive study of the general process.

<u>An Example</u>. Since the correspondence between binary relations and 0-1 matrices is well known, let R be given by the following matrix:

R	1	2	3	4	5
1	0	1	1	0	0
2	1	0	1	1	0
3	1	1	0	1	0
4	0	0	1	0	1

Adjoining b to the set 1,2,3,4,5, we summarize our computations in the table below.

k	x_kR	T_k	T'_k	M_k
1	23			1×23
2	134	1×3	1×3	1×23, 12×3 2×134
3	124	1×2 2×14	1×2 2×14	1×23, 13×2 12×3, 3×124 2×134, 23×14

Table continued

k	$x_k R$	T_k	T'_k	M_k
4	35	1×3, 12×3 2×3	12×3	1×23, 13×2 124×3 3×124, 2×134 23×14, 4×35

There is much repetition evident in this table that one can learn to avoid in practice. In going from step 3 to step 4 in the table, for example, one need only replace 12 × 3 by 124 × 3. So the final column grows from 1 × 23 at the beginning to the entire collection of maximal rectangles at the end by either replacement (step 2a of the algorithm) or addition of lines (steps 2b and 3).

On the basis of a few dozen computer runs with relation matrices of up to 32 rows and columns, we conjecture that the expected number of rectangles for relation matrices with n rows and colums is n^2/k, where k is the number of 1's in the matrix, i.e. the number of points in the relation. A probabilistic answer to this question, in terms of these parameters and perhaps such others as $\max|xR|$ would be useful for matrices with several hundred or more rows and columns.
Since relational data bases for large interactive computer systems are coming into use, one might ask if there is an analog of this algorithm for n-ary relations in general. We have been unable to find an analog of Theorem 2.2 for ternary relations. The problem is interesting and seems quite difficult.

A Model for Neural Activity. In order to make this paper self-contained, we include a sketch of some information concerning the physiology of nerve cells, drawn largely from Stevens [12]. The human brain contains about 10^{10} specialized cells called neurons. A neuron consists of a cell body, called the soma, from which extend in treelike fashion its dendrites and axon. The interconnections of neurons are important. Points of near-contact of the axon of one cell with the soma or dendrites of another are called synapses and are thepoints at which information is thought to flow from one neuron to another. Virtually all of the soma and dendrites of one cell are covered with synapses, the number being as high as 60,000; perhaps 10^4 is a good round figure. There are, therefore about 10^{14} synaptic connections in the human brain. The soma periodically generates spikes of electrical potential which travel along the axon and are transmitted, probably by chemical means, across the synaptic cleft, a small (2×10^{-2} microns) gap at the synapse, where the spikes are then integrated with other incoming spikes to enhance or inhibit the next neuron's production of spikes. The electrical potential arises from an imbalance of sodium and potassium ions across the cell membrane.

The connection between relations and neurons is easily stated. Let X be a set of neurons and define a relation R on X by requiring $(u,v) \in R$ if and only if neuron u's axon synapses with neuron v. Stanislaw Ulam has called the associated graph the most interesting graph of all; it seems to be the only graph that thinks it is interesting, at any rate.

So neurons are interconnected into neural networks, the properties of which determine the behavior of the nervous system and the organism. The relation R described above gives the interconnection scheme of a neural network, and R of course determines a collection of maximal rectangular relations. In his 1949 book [13] Hebb anticipated the idea of a maximal R-rectangle in his definition of a cell assembly, i.e. a pair of collections of neurons X,Y with the property that each neuron of X is connected to every neuron of Y. Hebb postulated that the cell assembly is the fundamental unit in a neural network in which learning, recognition and soforth occur. I am obligated to Professor Fay for his paper [6] and its bibliography for establishing the relation-theoretic aspect of neural networks.

Associated with each neuron u_i is a number f_i which is the deviation of the neuron's spiking rate from its so-called "rest rate," the number of spikes per second produced during a state of drowsy, nonalert consciousness or an early stage of sleep. Anderson, Cooper [1,5] and others have investigated a "brain" consisting of a single rectangular relation $M \times N$ with $|M| = |N|$ as a prototypical learning unit. The m neurons $u_1,\ldots,u_n$ in M determine a vector $(f_1,\ldots,f_n)$ with each f_i as above. Similarly the vectors $v_j \in N$ determine a vector $(g_1,\ldots,g_n)$. By means of a biologically based argument, they suppose that the correspondence between the vectors is linear. Hence, in this scheme, the rectangle $M \times N$ has associated with it a linear transformation A. dropping the requirement that $|M| = |N|$, and passing to a more general relation R, we have thus associated with each maximal R-rectangle $M \times N$ a linear transformation A: $R^m \rightarrow R^n$, where m and n are respectively the cardinal numbers of M and N, and R^k is a real k-dimensional space. (Since each transformation is associated with a vector space itself, the lattice of closed sets of neurons determined by R orders these vector spaces. We shall not pursue this idea here.) One can construct many schemes for modifying these linear transformations. We refer the reader to the papers of Anderson [1] and Cooper [5] for full details. The idea here is that if one can learn the neural interconnections of a biological organism, one can test theories of learning, association, recognition and so forth with a computer simulation. There is of course no hope of doing this for the human brain. Networks for various simple organisms have begun to be sorted out, however. The abdominal ganglion of the sea hare, genus Apleysia, contains only about 1500 neurons and is one of the likely candidates for this sort of simulation.

Description of Proposed Simulation. Here is a brief description of the procedure we intend to follow.

1) Input a relation matrix R with the properties
   (a) There are "input" neurons x_i with the property $Rx_i = \emptyset$.
   (b) There are "output" neurons x_j with the property $x_jR = \emptyset$.
2) Compute all the maximal R-rectangles.
3) For each maximal R-rectangle, input a linear transformation.
4) Input a "learning algorithm" for modification of the transformations.

5) Input a set of spiking frequencies, one for each input neuron.

6) Measure the output spiking frequencies for various choices of input.

7) Apply the learning algorithm of step (4) and repeat step (6).

Having caused the model to learn to respond to certain stimuli, one can input similar stimuli (ones making a small dot product with previous inputs) to test for recognition, generalisation, etc.

We remark that the number of synaptic connection in the human brain, about 10^{14}, is surely too large to be genetically determined. Brindley [4] cites measurements of the amount of DNA in the brain and concludes on information theoretic grounds that only a very small fraction of the synapses could be genetically determined. The rest may be entirely random or formed early in the brain's existence as a response to external stimuli. One need not, therefore, know all the synaptic connections for a neural network if one is allowed to grow new neurons and thus modify the set of maximal rectangles while the simulation is in progress. The maximal rectangle algorithm can handle this sort of growth.

REFERENCES

1. Anderson, J.A., What is a distinctive feature? Tech. Report No. 74-1 from The Center for Neural Studies, Brown University, Providence, Rhode Island (1974).
2. Bednarek, A.R. and O.E. Taulbee, On maximal chains, Rev. Roum. Math. Pures App. XI (1966), 23-25.
3. Birkhoff, G., Lattice Theory, First ed., Colloquium publication XXXV, American Mathematical Society, Providence, 1948.
4. Brindley, G.S., Nerve net models of plausible size that perform many simple learning tasks, Proc. Royal Soc. London, B. 174 (1969), 173-191.
5. Cooper, L.N., A possible organization of animal memory and learning, Proceedings of the Nobel symposium on Collective Properties of Physical Systems, Aspenasgarden, Sweden - June 12-16, 1973.
6. Fay, G., An algorithm for finite galois connections, Technical Report from The Institute for Industrial Economy, Organization and Computation Technique, Budapest (1973).
7. Norris, Eugene M., An algorithm for computing the maximal rectangles in a binary relation, Technical Report No. 08A05-1, Department of Mathematics and Computer Science, University of South Carolina.
8. Norris, E.M., Computing the Maximal Rectangles Contained in a Binary Relation (submitted for publication).
9. Osteen, Robert E., Clique detection algorithms based on line addition and removal, SIAM J. Appl. Math. 26 (1974), 126-135.
10. Simpson, Robert J., The application of rectangular relations to the study of binary relations on a set, Ph.D. dissertation, University of Tennessee, 1972.
11. Szasz, Gabor, Introduction to Lattice Theory, Third edition, Academic Press, New York (1963).
12. Stevens, C.F., Neurophysiology: a Primer, Wiley, New York (1966).
13. Hebb, D.O. The Organization of Behavior, Wiley, New York, (1949).