Some syntactic approaches to the handling of inconsistent knowledge bases: A comparative study*

Part 1: The flat case

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Abstract

This paper presents and discusses several methods for reasoning from inconsistent knowledge bases. A so-called argued consequence relation, taking into account the existence of consistent arguments in favour of a conclusion and the absence of consistent arguments in favour of its contrary, is particularly investigated. Flat knowledge bases, i.e., without any priority between their elements, are studied under different inconsistency-tolerant consequence relations, namely the so-called argumentative, free, universal, existential, cardinality-based, and paraconsistent consequence relations. The syntax-sensitivity of these consequence relations is studied. A companion paper is devoted to the case where priorities exist between the pieces of information in the knowledge base.

Key words: inconsistency, argumentation, nonmonotonic reasoning, syntax-sensitivity.

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1. Introduction

An important problem in the management of knowledge-based systems is the handling of inconsistency. Inconsistency may be present for mainly three reasons:

- The knowledge base includes default rules (e.g., "birds fly", "penguins are birds", "penguins do not fly") and facts (e.g., "Tweety is a bird") and later a new information is received (e.g., "Tweety is a penguin") which contradicts a plausible conclusion which could be previously derived from the knowledge base;
- In model-based diagnosis, where a knowledge base contains a description of the normal behavior of a system, together with observations made on this system. Failure detection occurs when observations conflict with the normal functioning mode of the system and the hypothesis that the components of the system are working well; this leads to diagnose what component(s) fail(s);
- Several consistent knowledge bases pertaining to the same domain, but coming from n different sources of information, are available. For instance, each source is a reliable specialist in some aspect of the concerned domain but is less reliable on other aspects. A straightforward way of building a global base Σ is to concatenate the knowledge bases Σ_i provided by each source. Even if Σ_i is consistent, it is unlikely that $\Sigma_1 \cup \Sigma_2 \ldots \cup \Sigma_n$ will be consistent also.

These three causes of inconsistency are in general the most common ones. There are two attitudes in front of inconsistent knowledge. One is to revise the knowledge base and restore consistency. The other is to accept inconsistency and to cope with it. The first approach meets two difficulties: there are several ways of restoring consistency yielding different results; moreover part of the information is thrown away and we no longer have access to it. This approach may be natural when handling exceptions, as in the above example where it seems more intuitively reasonable to delete $\neg$bird$\vee$fly than $\neg$penguin$\vee\neg$fly from {bird, penguin, $\neg$penguin$\vee\neg$fly, $\neg$bird$\vee$fly, $\neg$penguin$\vee$bird}. Restoring consistency also makes sense in model-based diagnosis, since it comes down to find the reasons for a failure. In the case of multiple sources, restoring consistency looks debatable, since the goal of retaining all available information is quite legitimate in this case. However we must take a step beyond classical logic, since the presence of inconsistency enables anything to be entailed from a set of formulas. Gabbay and Hunter (1991, 1993) claim that inconsistency in a database exists on purpose and may be useful if its presence triggers suitable actions that cope with it. They give the example of overbooking in airline booking systems. They suggest the specification of an "action language" on top of the object language, with a view to trigger external actions that eventually may modify the contents of the database in a suitable way, depending on the environment surrounding the database.

This paper is primarily devoted to the treatment of inconsistency caused by the use of multiple sources of information rather than the one caused by the use of default rules with exceptions. This paper investigates several methods for coping with inconsistency by suitable notions of consequence relations capable of inferring non-trivial conclusions from an inconsistent knowledge base. These consequence relations coincide with the classical definition when the knowledge base is consistent. Knowledge bases considered in this paper are flat, i.e., finite sets of equally reliable propositional formulas. The proposal made by Rescher and Manor (1970) is often used: compute first the set of maximal consistent subsets of the knowledge base; then a formula is accepted as a consequence when it can be classically inferred from all maximal consistent subsets of propositions (this is the so-called universal consequence) or from at least one maximal consistent subset (this is the so-called existential consequence).

It turns out that the universal consequence relation is very conservative hence rather unproductive while the existential one is too permissive and leads to pairs of mutually exclusive conclusions. A mild inference approach is proposed in this paper, that is more productive than the universal consequence but does not lead to conclusions which are pairwise contradictory. It is based on the idea of arguments that goes back to Toulmin (1956), and is related to previous proposals by Poole (1985), Pollock (1987), and Simari and Loui (1992) that were suggested in the framework of defeasible reasoning for handling exceptions. We suggest that a conclusion can be inferred from an inconsistent knowledge base if the latter contains an argument that supports this conclusion, but no argument that supports its negation.

The paper is organized as follows. Section 2 introduces the notion of argument and defines the argued consequence relation. Section 3 compares different notions of consequence relations that are inconsistency-tolerant, including several ones that come from the nonmonotonic logic literature. In Section 4, we study the syntax-sensitivity of these consequence relations (namely to what extent a consequence relation depends on the syntax of the knowledge base) according to the following syntax properties: the insensitiveness to the addition of consequences of consistent sub-bases, including the duplication of formulas, and the insensitiveness with respect to the clausal form. Section 5 contains a thorough analysis of our argument-based inference process.

2. Definition of an argument-based consequence relation

In this paper, for the sake of simplicity, we only consider a finite propositional language denoted by $\mathscr{L}$. We denote the set of classical interpretations by Ω, by $\vdash$ the classical consequence relation, Greek letters $\alpha,\beta,\delta,\ldots$ represent formulas. Let Σ be a multiset of propositional formulas, possibly inconsistent but not deductively closed. Σ is a multiset since the same formula may be present several times. This is why we consider Σ as a multiset. $Cn(\Sigma)$ denotes the deductive closure of Σ, i.e., $Cn(\Sigma)=\{\phi\in\mathscr{L}, \Sigma\vdash\phi\}$. $Cn(\Sigma)$ is a set, not a multiset. The knowledge bases considered in this paper are flat, which means that all formulas in Σ have the same reliability. In the following, sub(multi)sets of Σ are denoted by capital letters A,B,C,…. They will be called *subbases* of Σ.

When the knowledge base Σ is not deductively closed, we call it a "belief base", following Nebel (1991), while bases which are deductively closed are called "belief sets" after Gärdenfors (1988). Our view of a belief base Σ is syntactic in the sense that, for instance, $\Sigma = \{\phi\}$ is not the same as $\Sigma' = \{\phi, \phi\}$. A formula in Σ is called a "belief" because it represents a proposition taken for granted, that does not require justification. In the presence of inconsistency, the approaches developed in this paper must be syntactic in nature, since they explicitly use formulas that appear in the belief base originally, while two inconsistent belief bases over the same language are semantically equivalent (in a trivial way). Moreover, in the context of belief revision, logically equivalent (consistent) belief bases may be revised differently. For example, the two belief bases $\Sigma_1=\{\alpha,\beta\}$ and $\Sigma_2=\{\alpha\wedge\beta\}$ are logically equivalent but can be revised differently if we learn the new information $\{\neg\alpha\}$. In some approaches, we get $Cn(\{\neg\alpha\wedge\beta\})$ in the case of adding $\neg\alpha$ to Σ_1, while with other approaches adding $\neg\alpha$ to Σ_2 we get $Cn(\{\neg\alpha\})$. Other aspects of syntax-sensitivity are discussed in Section 4. This syntactic treatment of inconsistency-handling is very different from the one advocated by Rescher and Brandom (1980). These authors adopt a semantic treatment of inconsistent sets of propositional sentences, in which the classical set of interpretations is imbedded in a larger set of non-standard possible worlds.

In the present paper, the method that copes with inconsistency is to extract from an inconsistent belief base consistent arguments supporting a proposition or refuting it. The following definitions are helpful to formalize this view.

Definition 1: A sub-base A of Σ is said to be *consistent* if it is not possible to deduce a contradiction from A, namely, it is not true that $A \vdash \bot$. A is said to to be *maximally* *consistent* if i) it is consistent and ii) either $A = \Sigma$ or adding any formula ϕ from Σ-A to A entails the inconsistency of $A \cup \{\phi\}$.

Definition 2: A sub-base A of Σ is said to be an *argument* for a formula ϕ, if it satisfies the following conditions:

(i) $A \nvdash \perp$ (A is consistent),

(ii) $A \vdash \phi$, and

(iii) $\forall \psi \in A,\ A - \{\psi\} \nvdash \phi$

An argument for ϕ is then a minimal (for set- inclusion) consistent subset of formulas that implies ϕ. Other authors (for instance Elvang-Goransson et al.(1993), Dung(1993), Cayrol(1995)) call "argument" the pair (A, ϕ). Notice that our notion of argument is identical to the one proposed by Simari and Loui (1992). These authors apply it to default reasoning (arguments are used to determine the notion of specificity which is also very similar to the notion of environment used in the ATMS terminology (De Kleer, 1986)[1]).

Definition 3 (Benferhat, et al.1993)**:** A formula ϕ is said to be an *argued consequence* of Σ, denoted by $\Sigma \vdash_{\mathscr{A}} \phi$, if and only if:

(i) there exists an argument for ϕ in Σ, and

(ii) there is no argument for $\neg\phi$ in Σ.

As a consequence of this definition, if a belief base contains only the two contradictory statements $\{\phi,\neg\phi\}$ then the inference $\phi \wedge \neg\phi \vdash_{\mathscr{A}} \psi$ does not hold for any ψ. In other words, our approach is in agreement with a basic motivation of paraconsistent logics (e.g., Da Costa, 1963), where they reject the principle "ex absurdo quodlibet" which allows the deduction of any formula from an inconsistent base.

The notion of argued consequence used here is rather straightforward and does not involve a comparison between the strength of arguments in favor of ϕ and $\neg\phi$. For instance an argument A in favor of ϕ can be weaker than an argument B in favor of $\neg\phi$ if A contains formulas ψ such that arguments A' against them exist ($A' \vdash \neg\psi$) while this is not true for arguments in B. Fox et al. (1992), Elvang-Goransson et al.(1993) Krause et

[1] An ATMS (assumption-based truth maintenance system) is devoted to hypothetical reasoning. This system uses two kinds of propositional symbols, the assumption ones and the non-assumption ones. An ATMS is able to determine under which set of assumptions a given proposition p is true. This set of assumptions when it is minimal (with respect to the set-inclusion relation) and consistent is called *environment* of the proposition p. Therefore an ATMS can be seen as a way to compute arguments, by considering the formulas of the knowledge base as assumptions, and an environment of a proposition p can therefore be seen as an argument for p.

al. (1994), offer an elaborate strategy for performing a comparison of arguments (as pairs) pertaining to different conclusions, based on ideas of rebuttal and undercutting and define classes of acceptability for such pairs. Their notion of "probable inference" coincides with our argument-based inference. Dung (1993) studies properties of the so-called defeat relation between pairs (A, ϕ) such that A $\vdash \phi$, that is, the relation that describes how one argument may defeat another. These works are carefully analyzed by Cayrol (1995) and related to nonmonotonic reasoning.

It is easy to verify that $\vdash_{\mathcal{A}}$ is nonmonotonic indeed. Let us consider the following example where our belief base Σ contains only the formula ϕ. It is obvious that the formula ϕ is an argued consequence of Σ. Let us add to Σ the information that ϕ is false, then ϕ will no longer be an argued consequence of $\Sigma' = \{\phi,\neg\phi\}$ since there exists also an argument for $\neg\phi$ in Σ'. This nonmonotonicity is only due to the presence of inconsistency, as seen now.

Proposition 1: if Σ is consistent, then $\Sigma \vdash \phi$ iff $\Sigma \vdash_{\mathcal{A}} \phi$

<u>Proof:</u>

• If a formula ϕ is a logical consequence of Σ, then there obviously exists in Σ an argument for ϕ. Since Σ is consistent, then $\neg\phi$ cannot be deduced from Σ, which means that there is no argument for $\neg\phi$ in Σ, and therefore by definition ϕ is also an argued consequence of Σ.

• The second part of the proof goes in a similar way. ■

Proposition 1 means that the argued consequence resorts to what Satoh (1990) calls "lazy nonmonotonic reasoning" because non-monotonicity only appears in the presence of inconsistency, an idea also advocated by Lin (1987).

3. Comparative Study of Inconsistency-Tolerant Consequence Relations

In this sub-section we compare argument based inference relations with other inconsistency-tolerant consequence relations studied in Benferhat et al. (1993). We start this comparative study by presenting the different approaches from the most conservative ones to the most adventurous ones. But first we need some further definitions:

Definition 4: A sub-base A of Σ is said to be *minimally inconsistent* if and only if it satisfies the two following requirements:

- $A \vdash \bot$, and
- $\forall \phi \in A, A - \{\phi\} \nvdash \bot$.

From now on, we denote by Inc(Σ) the set of formulas belonging to at least one minimally inconsistent sub-base of Σ, namely:

$$\text{Inc}(\Sigma) = \{\phi, \exists A \subseteq \Sigma, \text{ such that } \phi \in A \text{ and } A \text{ is minimally inconsistent}\}$$

The set Inc(Σ) can be related to the "base of nogoods" used in the terminology of the ATMS (De Kleer, 1986)[1]. Once Inc(Σ) is computed, and all elements of Inc(Σ) are removed from Σ, the resulting base is called the *free base* of Σ, denoted by Free(Σ) (Benferhat et al., 1992). In other words, the set Free(Σ) contains all formulae which are not involved in any inconsistency of the belief base Σ:

Definition 5: A formula ϕ is said to be *free* iff it does not belong to any minimally inconsistent sub-base of Σ, namely:

$$\phi \text{ is free if and only if } \phi \notin \text{Inc}(\Sigma)$$

We denote by Free(Σ) the set of free formulas in Σ. Now, let us introduce the notion of the free consequence, denoted by $\vdash_{Free}$:

Definition 6: A formula ϕ is said to be a *free consequence* (or a *sound* consequence) of Σ, denoted by $\Sigma \vdash_{Free} \phi$, if and only if ϕ is logically entailed from Free(Σ), namely:

$$\Sigma \vdash_{Free} \phi \quad \text{iff} \quad \text{Free}(\Sigma) \vdash \phi$$

The free inference relation is very conservative as it will be shown later. It corresponds to a maximal revision of Σ, deleting all formulas involved in a conflict. Note that if $\Sigma \vdash_{Free} \phi$, then there is a very safe argument A for ϕ in Free(Σ), since the formulas forming this argument are not involved in any inconsistency of Σ, and are thus not rebutted by any subset of Σ. Moreover there cannot be any argument against ϕ. Hence free consequences are argued consequences, and very safe ones.

[1] A no-good is a minimal set of incompatible assumptions. Links between minimal inconsistent sub-bases and nogoods can be established in the following way: let Σ be a belief base, and let Σ' be a new belief base obtained from Σ by replacing each formula ϕ_i in Σ by $\neg H_i \vee \phi_i$, where H_i is an assumption symbol (all H_i are different). Then we can show that the sub-base $A=\{\phi_i / i=1,m\}$ is minimal consistent sub-base of Σ iff $H_A=\{H_i / \neg H_i \vee \phi_i \in \Sigma', \phi_i \in A\}$ is a nogood.

Let us now recall the approach first proposed by Rescher and Manor (1970), where they define the universal (called also the inevitable) consequence relation in the following way. Let MC(Σ) be the set of maximal consistent sub-bases of Σ.

Definition 7: A formula ϕ is said to be a *Universal consequence* or *MC-consequence or Inevitable consequence* of Σ, denoted by $\Sigma \vdash_{MC} \phi$, if and only if ϕ is entailed from each element of MC(Σ), namely:

$$\Sigma \vdash_{MC} \phi \qquad \text{iff} \qquad \forall A \in MC(\Sigma), A \vdash \phi$$

As mentioned above, the free consequence relation is more conservative than the MC-consequence:

Proposition 2: Each free consequence is also a MC-consequence. The converse is false

Proof:

(i) Let us partition a belief base Σ into a pair (Inc(Σ), Free(Σ)), and let $\Sigma_1, \ldots, \Sigma_n$ be the maximal consistent sub-bases of Inc(Σ). It is obvious that $\Sigma_1 \cup$Free(Σ), …, $\Sigma_n \cup$Free(Σ) form the maximal consistent sub-bases of Σ (since Free(Σ) are outside any conflict). Then each element of MC(Σ) contains Free(Σ), therefore if a formula is a free consequence then it is also a MC-consequence.

(ii) To show that the converse is false, let us consider the following counter-example where our base contains the five formulas:

$$\Sigma = \{\alpha, \neg\alpha\vee\neg\beta, \beta, \neg\alpha\vee\delta, \neg\beta\vee\delta\}$$

The base Σ is inconsistent, and the inconsistency is caused by the first three formulas, which means that the free base of Σ is Free(Σ)= $\{\neg\alpha\vee\delta, \neg\beta\vee\delta\}$. It is clear that δ cannot be entailed from Free(Σ).

In contrast with the MC-consequence, the base contains three maximal sub-bases:

$$A=\{\neg\alpha\vee\neg\beta, \beta, \neg\alpha\vee\delta, \neg\beta\vee\delta\},$$
$$B=\{\alpha, \beta, \neg\alpha\vee\delta, \neg\beta\vee\delta\}, \text{ and}$$
$$C=\{\alpha, \neg\alpha\vee\neg\beta, \neg\alpha\vee\delta, \neg\beta\vee\delta\}$$

corresponding to the case where we remove from Σ each element of Inc(Σ). We see that each sub-base entails δ, therefore δ is a MC-consequence.

In the above example, it is clear that the MC-consequence involves an idea of parsimony with respect to the removal of inconsistency; each maximal consistent sub-base is obtained by removing the least number of formulas sufficient to restore consistency. This is not so when considering Free(Σ). ■

There is another way to find the proof of the previous proposition noticing that:

$$\mathrm{Free}(\Sigma) = \bigcap_{A_i \in MC(\Sigma)} A_i$$

Indeed, if a formula ϕ does not belong to Free(Σ) then there exists a minimally inconsistent sub-base A_k containing ϕ, and therefore there exists at least one maximally consistent sub-base which contains A_k except for ϕ, which means that there exists at least one element of MC(Σ) which does not contain ϕ, and consequently ϕ does not belong to the intersection of the elements of MC(Σ). The converse is also true. Indeed, if $\phi \notin \bigcap_{A_i \in MC(\Sigma)} A_i$ then there exists a maximal *consistent* sub-base A such that $\phi \notin A$, and $A \cup \{\phi\}$ is inconsistent, therefore there exists a minimally inconsistent sub-base of $A \cup \{\phi\}$ containing ϕ, hence ϕ is not free. Then from the properties of Cn, we find:

$$\mathrm{Cn}(\mathrm{Free}(\Sigma)) = \mathrm{Cn}(\bigcap_{A_i \in MC(\Sigma)} A_i) \subseteq \bigcap_{A_i \in MC(\Sigma)} \mathrm{Cn}(A_i).$$

The next propositions compare the MC-consequence to the argued consequence:

Proposition 3: A formula ϕ is an argued consequence of Σ iff $\exists A_i \in MC(\Sigma)$, such that $A_i \vdash \phi$, and $\nexists A_j \in MC(\Sigma)$, such that $A_j \vdash \neg\phi$.

Proof

If ϕ is an argued consequence of Σ, there is an argument A in favour of ϕ. There is a maximal consistent set A_i containing A since A is consistent, hence $A_i \vdash \phi$. Besides, since there is no argument against ϕ, no maximal consistent sub-base will entails $\neg\phi$. Conversely, if $A_i \vdash \phi$ for $A_i \in MC(\Sigma)$ then A_i contains an argument for ϕ. Now if there were an argument A against ϕ, then there would be a maximal consistent sub-set of Σ containing A that would entail $\neg\phi$, but such a maximal subset of Σ does not exist by hypothesis. ■

Proposition 4: Each MC-consequence of Σ is also an argued consequence of Σ. The converse is false

Proof:

- If ϕ is a MC-consequence of Σ, then each element of MC(Σ) enables us to infer ϕ. (hence $\exists A_i \in MC(\Sigma)$, $A_i \vdash \phi$). As, each element of MC(Σ) is consistent and entails ϕ, then it does not exist an element of MC(Σ) which enables us to deduce $\neg\phi$. In other words, $\nexists A_i \in MC(\Sigma)$, $A_i \vdash \neg\phi$.

Then, using the previous proposition, we conclude that ϕ is an argued consequence of Σ.

- The converse is false, indeed let $\Sigma = \{\alpha, \neg\alpha, \alpha \rightarrow \beta\}$. We have:

$$MC(\Sigma) = \{A, B\}$$

where:

$$A=\{\alpha, \alpha \rightarrow \beta\}, \text{ and}$$

$$B=\{\neg\alpha, \alpha \rightarrow \beta\}.$$

In this example, β is an argued consequence of Σ, while it is not a MC-consequence.■

From the above results, the argued inference is less conservative than the MC-consequence.

One of the main difficulty for implementing the MC-consequence is the cardinality of $MC(\Sigma)$ which increases exponentially with the number of conflicts in the base and in general, it is not possible to take into account all the elements of $MC(\Sigma)$. One may think of selecting a non-empty subset of $MC(\Sigma)$, denoted by $L(\Sigma)$, which represents maximal consistent sub-bases that keep as many formulas of Σ as possible. The set $L(\Sigma)$ is computed in the following manner:

$$A \in L(\Sigma) \text{ iff } \quad A \in MC(\Sigma) \text{ and } \forall B \in MC(\Sigma), |A| \geq |B|$$

where $|\Sigma|$ is the cardinality of Σ. The idea of selecting a subset of $MC(\Sigma)$ using a cardinality criterion was used independently in diagnosis problems. It corresponds to the property of parsimony advocated in (Reggia et al., 1985). In model-based diagnosis, the number of diagnoses (sets of faulty components, also called *hitting sets* in (Reiter, 1987)) is very high in general. To select a subset of all possible diagnosis, De Kleer (1990) proposes a probabilistic criterion where he assumes that each component has a very small probability to fail and that all components fail independently. De Kleer (1990) shows that the selected diagnosis are those which contain a small number of failing components. A similar probabilistic justification of $L(\Sigma)$ can be found in (Benferhat et al., 1993). See also (Lang, 1994) for discussions about links between inconsistency handling and diagnosis.

In order to generate a set of consequences from an inconsistent belief base, based on $L(\Sigma)$, a definition similar to the MC-consequence can be used:

Definition 8: A formula ϕ is said to be a *L-consequence* (or *cardinality-based*)[1] of Σ, denoted by $\Sigma \vdash_L \phi$, if and only if it is entailed from each element of $L(\Sigma)$, namely:

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[1]'L-consequence' is short for 'lexicographic consequence'. This name comes from the prioritized version of the consequence notion (studied in the companion paper).

$$\Sigma \vdash_L \phi \qquad \text{iff} \qquad \forall A \in L(\Sigma), A \vdash \phi$$

Proposition 5: Each MC-consequence of Σ is also a L-consequence of Σ. The converse is false.

This is obvious since the L-consequence uses a subset of MC(Σ). However the L-consequence and the argued consequence are not comparable.

Counter-example

(i) Let $\Sigma = \{\alpha, \neg\alpha, \alpha \rightarrow \beta\}$. This belief base is inconsistent and $L(\Sigma)=\{\{\alpha, \alpha \rightarrow \beta\}, \{\neg\alpha, \alpha \rightarrow \beta\}\} = MC(\Sigma)$. It is obvious that β is not a L-consequence of Σ, while β is an argued consequence.

(ii) Let us consider the following example where $\Sigma = \{\alpha, \beta, \neg\alpha, \alpha \rightarrow \beta, \neg\alpha \rightarrow \neg\beta\}$. This base is inconsistent and L(Σ) has only one element which is the sub-base $\{\alpha, \beta, \alpha \rightarrow \beta, \neg\alpha \rightarrow \neg\beta\}$. β is a L-consequence of Σ, while β is not an argued consequence since $\neg\beta$ has an argument $\{\neg\alpha, \neg\alpha \rightarrow \neg\beta\}$ in Σ. ■

The second counter-example indicates that the L-consequence may implicitly delete some useful pieces of knowledge. It may result in destroying some arguments, as well as some rebuttals (i.e., formulas whose presence ensures an argument for $\neg\phi$ that inhibits arguments for ϕ). The argumentative inference looks more respectful of the various points of view that are expressed in the belief base.

An open question is to see in which situation MC-Consequence and L-consequence generate the same conclusions. The presence of duplicated formulas in Σ may prevent the identity beween L-consequence and MC-consequence. Indeed duplicating formulas in a maximal consistent subset A can make this multiset the only element in L(Σ), while MC(Σ) remains the same. The L-consequence may thus appear as an arbitrary selection from MC(Σ) in some contexts. The following example shows that even if we leave apart the question of duplicated pieces of information, L-consequence may always generate more results than MC-consequence. Namely, it is possible to find a belief base Σ without duplicated formulas, and where a L-consequence of Σ is not a MC-consequence of Σ.

Example:

Let $\Sigma = \{\psi\rightarrow\phi, \psi, \neg\psi, \neg\phi\wedge\neg\psi, \neg\phi\}$

This belief base is inconsistent, does not contain duplicated formulas and has three maximal consistent sub-bases:

$A = \{\psi\rightarrow\phi, \psi\}$, $B = \{\psi\rightarrow\phi, \neg\psi, \neg\phi, \neg\phi\wedge\neg\psi\}$, $C = \{\psi, \neg\phi\}$ and $L(\Sigma)=\{B\}$.

It is clear that $\neg\phi\wedge\neg\psi$ is not a MC-consequence while it is a L-consequence. ■

Let us now restrict ourselves to a kind of non-redundant belief base called minimal core set (Goldszmidt et al., 1990), (Goldszmidt, 1992):

Definition 9: A consistent belief base A is said to be a *minimal core set* if $\nexists\phi\in A$, $A-\{\phi\}\vdash\phi$.

A is a minimal core set if it is formed of a set of independent axioms. Even when maximal consistent subsets are minimal core sets, the L-consequence can be more productive than the MC-consequence. Namely it is possible to find a belief base Σ such that each of its maximal consistent sub-base is a minimal core set, and where a L-consequence of Σ is not a MC-consequence of Σ.

Example:

Let $\Sigma = \{\phi\rightarrow\psi, \phi\wedge\xi, \neg\psi\wedge\neg\xi\wedge\phi\}$. There are two maximal consistent sub-bases, which are minimal core sets: $A = \{\phi\rightarrow\psi, \phi\wedge\xi\}$, $B = \{\neg\psi\wedge\neg\xi\wedge\phi\}$and $L(\Sigma)=\{A\}$. It is clear that ψ is not a MC-consequence while it is a L-consequence. ■

In order to let each MC-consequence be also a L-consequence of Σ, it is enough that all maximal consistent sub-bases of Σ have the same cardinality.

Rescher and Manor (1970) have also proposed another definition of the consequence relation, called existential consequence that can be described in the following way:

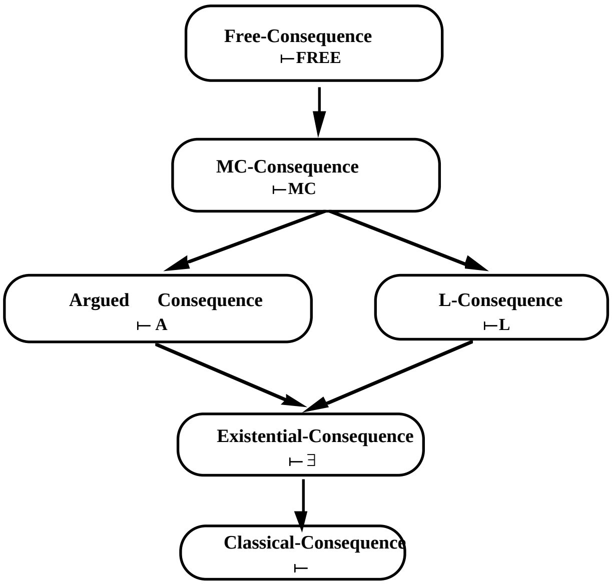
Definition 10: A formula ϕ is said to be an *existential consequence* of Σ, denoted by $\Sigma\vdash_{\exists}\phi$, if and only if there exists at least one element of $MC(\Sigma)$ which entails ϕ, namely:

$$\Sigma \vdash_{\exists} \phi \qquad \text{iff} \qquad \exists\, A \in MC(\Sigma),\ A \vdash \phi$$

It is not hard to see that this approach is the most adventurous one, but unfortunately it has an important drawback, since this approach generally leads to a trivially inconsistent set of results. Indeed, there may exist $A_i \vdash \phi$ and $A_j \vdash \neg\phi$, in which case both ϕ and $\neg\phi$ will be entailed.

Figure 1 summarizes the links existing between the different consequence relations studied here, the edges mean the set inclusion relation between the set of results generated by each consequence relation. The top of the diagram thus corresponds to the most conservative inferences. All inferences reduce to the classical one when Σ is consistent. Cayrol (1994) has shown that most existing argumentation systems come down to one of the consequence relations studied here.

Figure 1: A comparative study of consequence relations



4. Syntax-Sensitivity of the Consequence Relations

All introduced consequence relations are syntax-sensitive in the presence of inconsistency. But syntax-sensitivity can be a matter of degree. For instance the L-consequence can be viewed as very much syntax-sensitive. Namely, the following situation may happen: let $A \in L(\Sigma)$, $B \in MC(\Sigma) - L(\Sigma)$ and define C to be logically equivalent to B but $|C| > |A|$. To have it, it is enough to duplicate formulas in B and not in A a sufficient number of times. This duplication gives another multiset of formulas Σ' where $A \notin L(\Sigma')$ and $B \in L(\Sigma')$, which means that the set of L-consequences of Σ' may be noticeably different from the set of L-consequences of Σ, although they differ only via duplication of formulas. However some inconsistency-tolerant inferences are insensitive to duplication and can thus be considered as less syntax-sensitive. Less syntax-sensitive is the duplication-insensitive inference that is moreover not altered by adding non-trivial consequences of Σ, for instance, adding to Σ a consequence of all consistent subsets of Σ. An even less syntax-sensitive inference would be one that is not affected by adding a consequence of any consistent subset, for instance when all formulas of an inconsistent belief base can be put in clausal form without altering the set of consequences. This would be useful since belief bases are encoded by means of clauses (Horn or other types) in logic programming.

This section provides a formal discussion of the syntax-sensitivity of the consequence relations described above. To this aim, and according to the above discussion, it must be checked if the consequence relations described here satisfy the following properties:

Duplication insensitivity (DI):
An inference relation $\mid\!\sim$ is said to be a *DI relation* if and only if $\forall\phi\in\Sigma$,

$$\Sigma\mid\!\sim\psi \text{ iff } \Sigma\cup\{\phi\}\mid\!\sim\psi.$$

Local consequence insensitivity (LCI):
An inference relation $\mid\!\sim$ is said to be a *LCI relation* if and only if $\forall\phi$ such that there exists $A\in MC(\Sigma)$ and $A\vdash\phi$, $\Sigma\mid\!\sim\psi$ iff $\Sigma\cup\{\phi\}\mid\!\sim\psi$.

Universal consequence insensitivity (UCI):
An inference relation $\mid\!\sim$ is said to be a *UCI relation* if and only if $\forall\phi$ such that $\Sigma\vdash_{MC}\phi$,

$$\Sigma\mid\!\sim\psi \text{ iff } \Sigma\cup\{\phi\}\mid\!\sim\psi.$$

Clausal form insensitivity (CFI):
Let Σ' a new belief base obtained by replacing each formula in Σ by its clausal form. Then, an inference relation $\mid\!\sim$ is said to be *a CFI relation* if and only if: $\Sigma\mid\!\sim\psi$ iff $\Sigma'\mid\!\sim\psi$.

It is clear that if a consequence relation is a LCI relation then it is also a DI relation and a UCI relation. We start discussing the syntax-sensitivity of the MC-consequence relation. First, MC-consequences are not LCI relations, namely there may exist $A\in MC(\Sigma)$ and ϕ such that $A\vdash\phi$ but $\Sigma\vdash_{MC}\psi$ while $\Sigma\cup\{\phi\}\nvdash_{MC}\psi$

Counter-example:

The belief base $\Sigma=\{\psi\to\phi, \psi, \neg\psi, \neg\phi\}$ is inconsistent and has three maximal consistent sub-bases:

$$A=\{\psi\to\phi, \psi\}\equiv\phi\wedge\psi$$
$$B=\{\psi\to\phi, \neg\psi, \neg\phi\}\equiv\neg\phi\wedge\neg\psi$$
$$C=\{\psi, \neg\phi\}\equiv\neg\phi\wedge\psi$$

It is clear that A entails ϕ. Let us now consider the augmented belief base:

$$\Sigma'=\Sigma\cup\{\phi\}=\{\psi\to\phi, \psi, \neg\psi, \neg\phi, \phi\}$$

Σ' has four maximal consistent sub-bases:

$$A'=\{\psi\to\phi, \psi, \phi\}=A\cup\{\phi\}$$
$$B'=\{\psi\to\phi, \neg\psi, \neg\phi\}=B$$

$C' = \{\psi, \neg\phi\} = C$

$D' = \{\psi \rightarrow \phi, \neg\psi, \phi\} \equiv \phi \wedge \neg\psi$

So:

$$\Sigma \vdash_{MC} \psi \vee \neg\phi$$

while:

$\Sigma' \nvdash_{MC} \psi \vee \neg\phi$ (since D' does not entail $\psi \vee \neg\phi$). ■

To study the UCI property some preliminary results are needed.

Proposition 6: Let $A \in MC(\Sigma)$; then:

i) if A is consistent with ϕ then $A \cup \{\phi\} \in MC(\Sigma \cup \{\phi\})$

ii) if A is inconsistent with ϕ then $A \in MC(\Sigma \cup \{\phi\})$.

Proof

i) It is enough to show that:

- $A \cup \{\phi\}$ is consistent, and:
- $\forall \psi \in (\Sigma \cup \{\phi\}) - (A \cup \{\phi\}) = \Sigma - A$, $A \cup \{\phi\} \cup \{\psi\}$ is inconsistent.

The first condition holds since A is a maximal consistent sub-base of Σ therefore is consistent, and since A is consistent with ϕ then adding ϕ to A always yields a consistent sub-base. Moreover A is a maximal consistent subset of Σ, so $\forall \psi \in \Sigma - A$, $A \cup \{\psi\}$ is inconsistent, hence $A \cup \{\phi\} \cup \{\psi\}$ is also inconsistent.

ii) The proof can be shown in a similar way. A is consistent since it is a maximal consistent sub-base of Σ. Adding to A any formula of $\Sigma \cup \{\phi\} - A$ leads to an inconsistent sub-base. Indeed, adding ϕ to A yields an inconsistent sub-base (by hypothesis) and adding $\psi \in \Sigma - A$ to A yields an inconsistent sub-base (since A is maximal consistent sub-base of Σ). ■

Immediate consequences of the previous proposition are:

Corollary 1 Let $A \in MC(\Sigma)$, and ϕ such that $A \vdash \phi$ then $A \cup \{\phi\} \in MC(\Sigma \cup \{\phi\})$.

Corollary 2: $\forall A \in MC(\Sigma)$, there exists $B \in MC(\Sigma \cup \{\phi\})$ such that $A \subseteq B$

Proof

The proof is obvious, since for each maximal consistent sub-base A of Σ, either A belongs to $MC(\Sigma \cup \{\phi\})$ or $A \cup \{\phi\}$ belongs $MC(\Sigma \cup \{\phi\})$ using Proposition 6 (depending on whether A is consistent with ϕ or not). ■

Corollary 3: $|\mathrm{MC}(\Sigma)| \leq |\mathrm{MC}(\Sigma\cup\{\phi\})|$

The previous results show that adding a formula ϕ, which is a consequence of one maximal consistent subset of Σ, may increase the number of maximal consistent sub-bases of Σ. This is not the case when the formula is a consequence of all maximally consistent sub-bases of Σ as seen in the following proposition:

Proposition 7: The MC-consequence is an UCI-relation, more precisely if ϕ is a MC-consequence of Σ then $|\mathrm{MC}(\Sigma)| = |\mathrm{MC}(\Sigma\cup\{\phi\})|$ and $\Sigma\vdash_{MC}\psi$ iff $\Sigma\cup\{\phi\}\vdash_{MC}\psi$

Proof

From Proposition 6, $|\mathrm{MC}(\Sigma\cup\{\phi\})| \geq |\mathrm{MC}(\Sigma)|$. Let us show that the cardinality of $\mathrm{MC}(\Sigma\cup\{\phi\})$ is at most equal to the cardinality of $\mathrm{MC}(\Sigma)$. Let $A\in \mathrm{MC}(\Sigma\cup\{\phi\})$ then A is of the form $B\cup\{\phi\}$. Indeed, if A does not contain ϕ, it means that $A\cup\{\phi\}$ is inconsistent, and consequently $A\in \mathrm{MC}(\Sigma)$; hence A entails ϕ and this is a contradiction. Now, since A is of the form $B\cup\{\phi\}$ then we can check that B is a maximal consistent sub-base of Σ. Indeed, assume that B is not maximal, then there exists $C\in \mathrm{MC}(\Sigma)$ such that $B\subset C$, and since C entails ϕ, we have that $C\cup\{\phi\}$ is also a maximal consistent sub-base of $\Sigma\cup\{\phi\}$ (using Proposition 6). Therefore $B\cup\{\phi\}=A\subset C\cup\{\phi\}$. Mind that the inclusion remains strict because we are not dealing with sets but with multi-sets (e.g., $\{\phi\}$ is strictly included in $\{\phi,\phi\}$). This contradicts the fact that A is a maximal consistent sub-base of $\Sigma\cup\{\phi\}$. So, we have proved that $|\mathrm{MC}(\Sigma\cup\{\phi\})| \leq |\mathrm{MC}(\Sigma)|$, since any belief base A in $\mathrm{MC}(\Sigma\cup\{\phi\})$ is of the form $B\cup\{\phi\}$ where $B\in \mathrm{MC}(\Sigma)$.

Now it is obvious that any $C\cup\{\phi\}\in \mathrm{MC}(\Sigma\cup\{\phi\})$ is logically equivalent to C since C entails ϕ. Hence the set of MC-consequences of Σ is the same as the set of MC-consequences of $\Sigma\cup\{\phi\}$. ■

The previous proposition also shows that the MC-consequence satisfies the so-called cumulativity[1] property proposed in Makinson(1989) and well-known in non-monotonic reasoning; see for instance (Gabbay, 1985), (Kraus et al., 1990). See also Benferhat et al. (1993) for a study of the MC-consequence from the point of view of nonmonotonic reasoning. However the MC- consequence is not a CFI relation.

Counter-example:

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[1]A nonmonotonic inference relation $\mid\!\sim$ satisfies the cumulativity property iff in the presence of $\alpha \mid\!\sim \delta$ $\alpha \mid\!\sim \beta$ is equivalent to $\alpha\wedge\beta \mid\!\sim \delta$; i.e., in the above notations : if $\Sigma\vdash_{MC}\phi$, then $\Sigma\vdash_{MC}\psi$ if and only if $\Sigma\cup\{\phi\}\vdash_{MC}\psi$.

Let $\Sigma = \{\psi\wedge\phi, \neg\psi\}$. Let Σ' a new belief base constructed from Σ by replacing each formula of Σ by its clausal form, namely: $\Sigma'=\{\psi, \phi, \neg\psi\}$.
Σ has two maximal consistent sub-bases: $\{A=\{\psi\wedge\phi\}, B=\{\neg\psi\}\}$.
Σ' has also two maximal consistent sub-bases: $\{A'=\{\psi,\phi\}, B=\{\neg\psi,\phi\}\}$.
It is clear that $\Sigma'\vdash_{MC}\phi$ while $\Sigma\nvdash_{MC}\phi$. ■

Proposition 8: The MC-consequence is a DI relation, namely:

$$\forall\phi\in\Sigma \qquad \forall\psi\in\mathscr{L} \qquad \Sigma\vdash_{MC}\psi \qquad \text{iff } \Sigma\cup\{\phi\}\vdash_{MC}\psi.$$

Proof

Let us first show that for each element A of $MC(\Sigma\cup\{\phi\})$:

- either A does not contain ϕ and A is an element of $MC(\Sigma)$,
- or $A=B\cup\{\phi\}$ and B is an element of $MC(\Sigma)$.

Indeed, consider the first case where A does not contain ϕ. Since $A\in MC(\Sigma\cup\{\phi\})$ then A is consistent and adding any formula from $\Sigma\cup\{\phi\}-A$ (therefore, from $\Sigma-A$) leads to an inconsistent sub-base, therefore A is also a maximal consistent sub-base of Σ.

Now, let us consider the second case: A contains ϕ. Since $B\cup\{\phi\}=A$ is a maximal consistent sub-base of $\Sigma\cup\{\phi\}$, B is consistent. Note that B already contains at least one formula of Σ which is ϕ (by hypothesis $\phi\in\Sigma$ and $B\cup\{\phi\}$ is consistent). Assume now that B is not maximal, then there exists $C\subseteq\Sigma$ such that $B\subset C$ which means that $B\cup\{\phi\}\subset C\cup\{\phi\}$ and this contradicts the fact that $B\cup\{\phi\}$ is a maximal consistent sub-base of $\Sigma\cup\{\phi\}$.

Thus, each element A of $MC(\Sigma\cup\{\phi\})$ is either an element of $MC(\Sigma)$ or is of the form $B\cup\{\phi\}$ and B belongs to $MC(\Sigma)$. Noticing that $B\cup\{\phi\}$ is semantically equivalent to B (since $\phi\in\Sigma$): $\Sigma\cup\{\phi\}\vdash_{MC}\psi \Rightarrow \Sigma\vdash_{MC}\psi$.

The converse is obvious since each maximal consistent sub-base of Σ is semantically equivalent to a maximal consistent sub-base of $\Sigma\cup\{\phi\}$, and using Proposition 6, we conclude that $\Sigma\vdash_{MC}\psi \Rightarrow \Sigma\cup\{\phi\}\vdash_{MC}\psi$. ■

The above properties show that the MC-consequence, although syntax-sensitive, is insensitive to the repetition of formulas, and to the adding of any formula that is already a MC-consequence of Σ. Let us now discuss the syntax sensitivity of the L-consequence:

Proposition 9: Let $A\in L(\Sigma)$, and ϕ such that A is consistent with ϕ then $A\cup\{\phi\}\in L(\Sigma\cup\{\phi\})$

Proof:

Since A is consistent with ϕ (and A is consistent) then $A\cup\{\phi\}$ is also consistent. Suppose that $A\cup\{\phi\} \notin L(\Sigma\cup\{\phi\})$ then there exists a consistent sub-base B of $\Sigma\cup\{\phi\}$ such that $|B| > |A\cup\{\phi\}|$. There are two cases:

i) B contains ϕ, then B is of the form $C\cup\{\phi\}$ where $|C| > |A|$ and C is a consistent sub-base of Σ of maximal cardinality, this contradicts the fact that $A\in L(\Sigma)$.

ii) B does not contain ϕ, this means that B is a sub-base of Σ, and since $|B| > |A\cup\{\phi\}|$ then $|B| > |A|$ and this again contradicts the fact that $A\in L(\Sigma)$. ■

Proposition 10: Let $A,B\in L(\Sigma)$, and ϕ such that A is consistent with ϕ and B is inconsistent with ϕ. Then $B\notin L(\Sigma\cup\{\phi\})$ while $A\cup\{\phi\}\in L(\Sigma\cup\{\phi\})$.

Proof:

The proof is obvious, noticing that if $A,B\in L(\Sigma)$ then $|A|=|B|$. Using the previous proposition, $A\cup\{\phi\}\in L(\Sigma\cup\{\phi\})$, and therefore $|A\cup\{\phi\}|>|B|$ which means that $B\notin L(\Sigma\cup\{\phi\})$. The following example illustrates this case:
Let $\Sigma=\{\phi,\neg\phi\}$, we have: $L(\Sigma) =\{A,B\}$ where $A = \{\phi\}$, $B = \{\neg\phi\}$ .It is obvious that A is consistent with ϕ. Let $\Sigma'=\Sigma\cup\{\phi\}$; then $L(\Sigma') =\{A'\}$ where $A'=\{\phi,\phi\}$. ■

Proposition 6 has shown that for each sub-base $A\in MC(\Sigma)$, and for any formula ϕ, either $A\in MC(\Sigma\cup\{\phi\})$ or $A\cup\{\phi\}\in MC(\Sigma\cup\{\phi\})$. This is not true with L-consequences which are more syntactic and it may happen that $|L(\Sigma\cup\{\phi\})|<|L(\Sigma)|$, contrary to the MC-consequence (see corollary 3), as seen in the example in the proof of Proposition 10 where adding a formula can cause the deletion of some sub-base which belongs to $L(\Sigma)$. Hence The L-consequence is neither a LCI relation nor a DI relation. It is not a CFI relation either (the same counter-example as for the MC-consequence works). However the following result holds:

Proposition 11: An L-consequence is a UCI relation.

Proof:

Notice that if ϕ is a MC-consequence of Σ then ϕ is also a L-consequence of Σ. Let us show that each sub-base A which belongs to $L(\Sigma\cup\{\phi\})$ can be put under the form $B\cup\{\phi\}$ where $B\in L(\Sigma)$. Indeed, if A does not contain ϕ (which means that A is a consistent sub-base of Σ) then A is inconsistent with ϕ, therefore there exists a maximal consistent sub-base C containing A and which is inconsistent with ϕ, and this contradicts the fact that ϕ is a MC-consequence of Σ. Let us now show that $B\in L(\Sigma)$. Indeed if B does not belong to $L(\Sigma)$, then there exists a sub-base C which belongs to $L(\Sigma)$ such that $|C|>|B|$, therefore $C\cup\{\phi\}\in L(\Sigma\cup\{\phi\})$ (since C entails ϕ, see Proposition 17) and $|C\cup\{\phi\}|>|B\cup\{\phi\}|=|A|$, and this contradicts the fact that A belongs to $L(\Sigma\cup\{\phi\})$.

Finally, ψ is a L-consequence of Σ iff $\forall B\in L(\Sigma)$ $B\vdash\psi$ iff $B\cup\{\phi\}\vdash\psi$ (since B entails ϕ by hypothesis) iff $\forall A\in L(\Sigma\cup\{\phi\})$ $A\vdash\psi$ iff ψ is a L-consequence of $\Sigma\cup\{\phi\}$. ■

We now discuss the syntax sensitivity of the argument-based consequence relation:

Proposition 12: An argument-based consequence relation is an DI relation, namely:

$$\forall\phi\in\Sigma,\text{ then}\quad \Sigma\vdash_{\mathcal{A}}\psi \quad\text{iff}\quad \Sigma\cup\{\phi\}\vdash_{\mathcal{A}}\psi$$

Proof:

• The fact that ψ is an argued consequence of Σ means that there exists an argument in favour of ψ in $\Sigma\cup\{\phi\}$. Assume that we have $\Sigma\cup\{\phi\}\nvdash_{\mathcal{A}}\psi$, this means that there exists an argument A in favour of $\neg\psi$ in $\Sigma\cup\{\phi\}$, and there are two cases:

- either A does not contain ϕ, which means that $A\subseteq\Sigma$ and this contradicts the fact that ψ is an argued consequence of Σ,

- or A contains ϕ, and there exists also an argument in favour of $\neg\psi$ in Σ (obtained by replacing in A the added formula ϕ by the one existing in Σ) and this again contradicts the fact that ψ is an argued consequence of Σ.

• To see that the converse is also true, it is enough to show that there exists an argument in favour of ψ in Σ (it is clear that there does not exist an argument in favour of $\neg\psi$ in Σ since it does not exist in $\Sigma\cup\{\phi\}$). Indeed, let A be an argument in favour of ψ in $\Sigma\cup\{\phi\}$, then we have two cases:

- either A does not contain ϕ, which means that $A\subseteq\Sigma$ hence A is also an argument in favour of ψ in Σ,

- or A contains ϕ, and there exists also an argument in favour of ψ in Σ (obtained by replacing in A the added formula ϕ by the existing one in Σ). ■

Proposition 13 Argued consequences are UCI-relations.

Proof:

• Let us show now that an argument-based consequence relation is a UCI relation. Notice that if ϕ is a MC-consequence of Σ then it is also an argued consequence of Σ. Assume that ψ is an argued consequence of Σ but not of $\Sigma\cup\{\phi\}$, then there exists an argument A for $\neg\psi$ in $\Sigma\cup\{\phi\}$ (and of course also an argument for ψ in Σ and therefore in $\Sigma\cup\{\phi\}$). There are two cases: if A does not contain ϕ, it means that A is also an argument for $\neg\psi$ in Σ and this contradicts the fact that ψ is an argued consequence of Σ. Now if A contains ϕ, namely $A=B\cup\{\phi\}$, such that $B\cup\{\phi\}\vdash\neg\psi$, it means that there exists a maximal consistent sub-base C of Σ containing B such that $C\cup\{\phi\}\vdash\neg\psi$, and since ϕ is a MC-consequence of Σ then $C\vdash\phi$ and therefore $C\vdash\neg\psi$ which means that A is also an argument for $\neg\psi$ in Σ and this contradicts the fact that ψ is an argued consequence of Σ.

Now assume that ψ is an argued consequence of $\Sigma\cup\{\phi\}$. It is clear that there is no argument in Σ which supports $\neg\psi$ (since such argument does not exist in $\Sigma\cup\{\phi\}$). Let A be an argument in $\Sigma\cup\{\phi\}$ which supports ψ. Assume A does not contain ϕ. Then A is also an argument for ψ in Σ which means that ψ is also an argued consequence of Σ. Assume now that A contains ϕ, namely $A=B\cup\{\phi\}$, such that $B\cup\{\phi\}\vdash\psi$. The latter means that there exists a maximal consistent sub-base C of Σ containing B such that $C\cup\{\phi\}\vdash\psi$. Since ϕ is a MC-consequence of Σ, $C\vdash\phi$ and therefore $C\vdash\psi$ which means that A is also an argument for ψ in Σ and therefore ψ is an argued consequence of Σ. ■

However, argued consequences are neither LCI relations, nor CFI relations. To see it, let us consider the following counter-example: $\Sigma = \{\phi\wedge\psi, \neg\phi\wedge\xi\}$. It is clear that $MC(\Sigma)=\{\{\phi\wedge\psi\}, \{\neg\phi\wedge\xi\}\}$ and $\Sigma\nvdash_{\mathscr{A}} \psi\wedge\xi$ while:

- $\Sigma\cup\{\psi\}\vdash_{\mathscr{A}} \psi\wedge\xi$, and
- $\Sigma'=\{\phi, \psi, \neg\phi, \xi\}\vdash_{\mathscr{A}} \psi\wedge\xi$ (Σ' is a clausal form of Σ)

Finally, the syntax-sensitivity of the existential-consequence relation is described by the two following propositions:

Proposition 14: An existential consequence relation is a DI relation, namely:

$$\forall\phi\in\Sigma,\text{ then}\quad \Sigma\vdash_{\exists}\psi \quad\text{iff}\quad \Sigma\cup\{\phi\}\vdash_{\exists}\psi$$

Proof:

• The first part is obvious since an existential consequence relation is monotonic.

• The converse is also true, since if there is an argument A in favour of ψ in $\Sigma\cup\{\phi\}$, then:
- either A does not contain ϕ, which means that $A\subseteq\Sigma$, hence ϕ is also an existential consequence of Σ,
- or A contains ϕ, and there exists also an argument in favour of ψ in Σ (obtained by replacing in A the added formula ϕ by the existing one in Σ). ■

Proposition 15: An existential consequence is an UCI relation.

Proof:

Let ϕ be such that $\Sigma \vdash_{MC} \phi$. If ψ is an existential consequence of Σ then it is obviously an existential consequence of $\Sigma \cup \{\phi\}$ ($\vdash_{\exists}$ is a monotonic relation).
Now, let ψ be an existential consequence of $\Sigma \cup \{\phi\}$, then we have an argument A in $\Sigma \cup \{\phi\}$ for ψ. If A does not contain ϕ, then A is also an argument for ψ in Σ which means that ψ is also an existential consequence of Σ. Now if A contains ϕ namely we have $A = B \cup \{\phi\}$, such that $B \cup \{\phi\} \vdash \psi$ which means that there exists a maximal consistent sub-base C of Σ containing B such that $C \cup \{\phi\} \vdash \psi$, and since ϕ is a MC-consequence of Σ, $C \vdash \phi$ and therefore $C \vdash \psi$ which means that A is again an argument for ψ in Σ and therefore ψ is also an existential consequence of Σ. ■

However, an existential consequence is neither a LCI relation nor a CFI relation. The same counterexample as for the argued consequence works.

The following array summarizes the syntax-sensitivity of the considered consequence relations:

	Duplication insensitivity	Local consequence insensitivity	Universal consequence insensitivity	Clausal form insensitivity
MC-consequence	Yes	No	Yes	No
L-consequence	No	No	Yes	No
Argumentative-consequence	Yes	No	Yes	No
Existential- consequence	Yes	No	Yes	No

All the consequence relations, except the L-consequence one, are insensitive to the duplication of formulas in the belief base, and they are all insensitive to the addition of a formula which is a logical consequence of all maximally consistent sub-bases of Σ. In contrast, none of these consequence relations are LCI relations nor CFI relations. The failure of these two properties shows how much these consequence relations are syntax-sensitive.

5. Structuring the Set of Argued Consequences

In this section, some properties of argument-based consequence relation are investigated in greater details. The following proposition shows that even if ϕ and ψ are argued consequence of a belief base Σ, their conjunction is not necessarily so: it may be that $\Sigma \vdash_{\mathscr{A}} \phi$, $\Sigma \vdash_{\mathscr{A}} \psi$, and *not* $\Sigma \vdash_{\mathscr{A}} \phi \wedge \psi$.

Example

$\Sigma = \{\neg\alpha\vee\beta, \alpha\vee\delta, \alpha, \neg\alpha\}$. It is clear that β and δ are both argued consequences of Σ, while there is no argument which supports $\beta\wedge\delta$. ■

The lack of closure under conjunction must not be seen as a major drawback of $\vdash_{\mathscr{A}}$. This property is not desirable since in the present case it may lead to perform the conjunctions of propositions that are supported by antagonist views (as in the previous example). This situation also happens in numerical settings such as evidence theory (Shafer, 1976) since we may have Belief(ϕ)>0, Belief(ψ)>0 and Belief($\phi\wedge\psi$)=0 with Shafer belief functions. The $\vdash_{\mathscr{A}}$ consequence relation captures the cases when we believe, in two mutually consistent propositions which cannot be advocated together because their justifications are conflicting.

The following proposition shows that if a formula ϕ is an argued consequence of Σ then all logical consequences of ϕ are also argued consequences of Σ.

Proposition 16: $\vdash_{\mathscr{A}}$ satisfies the property of Right Weakening, i.e.,

$$\text{If } \phi \vdash \psi \text{ then } \Sigma \vdash_{\mathscr{A}} \phi \text{ implies } \Sigma \vdash_{\mathscr{A}} \psi$$

Proof

Indeed, $\Sigma \vdash_{\mathscr{A}} \phi$ means that there exists an argument A_1 for ϕ in Σ. Since $\phi \vdash \psi$, we conclude that A_1 is also an argument for ψ in Σ. Assume now that there exists also an argument A_2 for $\neg\psi$ in Σ, then since $\phi \vdash \psi$ we conclude that A_2 is also an argument for $\neg\phi$ in Σ (since $\vdash\neg\phi\vee\psi$) and this contradicts the fact that $\Sigma \vdash_{\mathscr{A}} \phi$. ■

An important issue when reasoning with an inconsistent belief base Σ is to characterize the set of argued consequences of Σ, in terms of the classical consequence relation. The two previous propositions are very important to characterise the set of argued consequences of a possibly inconsistent base Σ, denoted by $Cn_{\mathscr{A}}(\Sigma)$, i.e.

$$Cn_{\mathscr{A}}(\Sigma) = \{\phi, \Sigma \vdash_{\mathscr{A}} \phi\}.$$

The fact that the argued consequence is not closed under conjunction means that $Cn_{\mathscr{A}}(\Sigma)$ is generally not equal to its closure under classical inference Cn:

$$Cn_{\mathscr{A}}(\Sigma) \neq Cn(Cn_{\mathscr{A}}(\Sigma))$$

Besides $Cn_{\mathscr{A}}(\Sigma)$ is not closed under $\vdash_{\mathscr{A}}$ either. For instance considering $\Sigma=\{\neg\alpha\vee\beta,\alpha\vee\delta,\alpha,\neg\alpha\}$, $\Sigma\vdash_{\mathscr{A}} \beta$ holds, $\Sigma\vdash_{\mathscr{A}} \beta\wedge\delta$ does not hold, but $\Sigma\cup\{\beta\}\vdash_{\mathscr{A}} \beta\wedge\delta$

holds since $\{\alpha\vee\delta,\neg\alpha,\beta\}\vdash\beta\wedge\delta$ while nothing supports $\neg\beta\vee\neg\delta$. It shows that propositions derived using $\vdash_{\mathcal{A}}$ are not considered as strongly believed as propositions present in the belief base.

For the rest of this section, we assume that only the language is based on the finite set of propositional symbols appearing in the base Σ.

Definition 11: A formula R is said to be a *prime implicate* of Σ with respect to the argument-based consequence relation if and only if:

(i) $\Sigma \vdash_{\mathcal{A}} R$

(ii) $\nexists R'$, such that $R' \vdash R$ and $\Sigma \vdash_{\mathcal{A}} R'$

A prime implicate can be inferred from a maximal consistent subset of Σ. However, if $\Sigma_i \in MC(\Sigma)$, then the conjunction of formulas in Σ_i (also denoted by Σ_i) is not a prime implicate since it is defeated by other maximal consistent subsets of Σ. Indeed, $\forall i \neq j$, $\Sigma_i \vdash \neg\Sigma_j$.

The construction of prime implicates can be more easily achieved from the semantical point view. Indeed, let $[\Sigma_i]$ be the set of models of the maximal consistent sub-base Σ_i. Any model of $[\Sigma_i]$ can be viewed as the formula φ_{Σ_i} made of the conjunction of literals it satisfies. Then it is clear that the following expression is a prime implicate:

$$\phi = \varphi_{\Sigma 1}\vee\ldots\vee\varphi_{\Sigma i\text{-}1}\vee \Sigma_i \vee\varphi_{\Sigma i+1}\vee\ldots\vee\varphi_{\Sigma n}$$

The set of models of ϕ is the union of the set of models of Σ_i and a selection of models of other maximal consistent sub-bases one per base. Indeed, ϕ cannot be defeated by a maximal consistent sub-base. Moreover, if each maximal consistent sub-base is complete (i.e., $\forall a \in \mathcal{L}$, either $a \in \Sigma_i$ or $\neg a \in \Sigma_i$) there exists exactly one prime implicate, and in this case the argued consequence and MC-consequence are equivalent. Indeed, the prime implicate in this case is: $\Sigma_1 \vee \Sigma_2 \vee \ldots \vee \Sigma_n$. Therefore:

$$Cn_{\vdash_{\mathcal{A}}}(\Sigma) = Cn(\Sigma_1 \vee \Sigma_2 \vee \ldots \vee \Sigma_n) = \cap_i Cn(\Sigma_i)$$

But in general, the prime implicates can be numerous.

Let $R_1,\ldots,R_n$ be the set of prime implicates of Σ, then $Cn_{\mathcal{A}}(\Sigma)$ can be seen as the union of the deductive closure of each R_i under Cn, namely (Benferhat, et al.1993):

$$Cn_{\mathscr{A}}(\Sigma)=Cn(R_1)\cup\ldots\cup Cn(R_n)$$

And it is easy to check that $\forall$ i, j = 1,n $\Sigma \vdash_{\mathscr{A}} R_i$, $\Sigma \vdash_{\mathscr{A}} R_j$ and $\Sigma \nvdash_{\mathscr{A}} R_i \wedge R_j$

Examples

(1) let $\Sigma = \{\neg\alpha \vee \beta, \alpha \vee \delta, \alpha, \neg\alpha\}$. There are two maximal consistent sub-bases, $\Sigma_1=\{\neg\alpha\vee\beta, \alpha\vee\delta, \alpha\}$, $\Sigma_2=\{\neg\alpha\vee\beta, \alpha\vee\delta, \neg\alpha\}$. Then:

$$[\Sigma_1]=\{\alpha\wedge\beta\wedge\delta, \alpha\wedge\beta\wedge\neg\delta\}\ [\Sigma_2]=\{\neg\alpha\wedge\beta\wedge\delta, \neg\alpha\wedge\neg\beta\wedge\delta\},$$

Therefore there are four prime implicates:

$R_1=\beta\wedge(\delta\vee\alpha)$, $R_2=(\alpha\wedge\beta)\vee(\neg\beta\wedge\delta\wedge\neg\alpha)$

$R_3=\delta\wedge(\neg\alpha\vee\beta)$, $R_4=(\neg\alpha\wedge\delta)\vee(\neg\delta\wedge\alpha\wedge\beta)$

(2) Consider now $\Sigma = \{\neg\alpha \vee \beta, \alpha \vee \beta, \alpha, \neg\alpha, \delta\}$. We have two maximal consistent sub-bases, $\Sigma_1=\{\neg\alpha\vee\beta, \alpha\vee\beta, \alpha,\delta\}$, $\Sigma_2=\{\neg\alpha\vee\beta, \alpha\vee\beta, \neg\alpha, \delta\}$ Then:

$[\Sigma_1]=\{\alpha\wedge\beta\wedge\delta\}$, $[\Sigma_2]=\{\neg\alpha\wedge\beta\wedge\delta\}$

The maximal consistent sub-bases are complete, therefore we have only one prime implicate: $R=(\alpha\wedge\beta\wedge\delta)\vee(\neg\alpha\wedge\beta\wedge\delta)=\beta\wedge\delta$. Then: $Cn_{\mathscr{A}}(\Sigma) = Cn(\{\beta \wedge \delta\})$. ■

More generally, if there are n maximal consistent belief bases in Σ, and if the number of models of Σ_i is m_i, then the number of prime implicates of Σ with respect to $\vdash_{\mathscr{A}}$ is $\Sigma_{i=1,n} \prod_{j\neq i} m_j$.

The previous definition of prime implicates makes sense only if the language is built only on the propositional symbols appearing in the belief base. For instance in the following example $\Sigma=\{\alpha,\neg\alpha\}$, there is only one prime implicate, the tautology T, if there is only one propositional letter in the language. It is not possible to deduce $\alpha\vee\beta$ from Cn(T), but $\alpha\vee\beta$ is an argued consequence of the belief base.

Proposition 17: $\forall$ R_1, R_2, two prime implicates of Σ, $\{R_1,R_2\}$ is consistent

Proof:

If $\{R_1,R_2\}$ is inconsistent then $R_1 \wedge R_2 \vdash \bot$, or equivalently $R_1 \vdash \neg R_2$. Since R_1 is an argued consequence of Σ, and $\vdash_{\mathscr{A}}$ satisfies Right Weakening, we conclude that $\neg R_2$ is also an argued consequence of Σ, which contradicts the fact that R_2 is an argued consequence of Σ. ■

At the semantic level, there is a close connection between the set $Cn_{\mathscr{A}}(\Sigma)$ of argued consequences of Σ and the notion of "system of important subsets" introduced recently by Schlechta (1995) and that he uses as a semantics of his default logic based on generalized quantifiers. Indeed, the sentences in $Cn_{\mathscr{A}}(\Sigma)$ can be mapped to subsets of the set of interpretations Ω which form such a system à la Schlechta; say $\mathscr{N}(\Sigma)$, such that i) $\Omega \in \mathscr{N}(\Omega)$, ii) $S \subseteq S' \subseteq \Omega$, $S \in \mathscr{N}(\Omega)$ implies $S' \subseteq \Omega$ (this is the Right Weakening property) and iii) $S, S' \subseteq \Omega$ imply $S \cap S' \neq \emptyset$ (due to the consistency of any pair of prime implicates).

The arguments supporting the prime implicates can be viewed as a set of scenarios extracted from Σ, that express different points of views on what is the actual information contained in Σ. These points of view are pairwise compatible but the subsets A_i and A_j supporting two prime implicates R_i and R_j should not be mixed up (even if not inconsistent). Indeed, $Cn_{\mathscr{A}}(\Sigma)$ still reflects conflicts lying in Σ since, although the argument-based inference forbids two prime implicates R_i and R_j to be inconsistent, the set $\{R_1, \ldots, R_n\}$ can be globally inconsistent for $n>2$. Namely one argued consequence of Σ can be defeated by other consequences grouped together.

<u>Example</u>

Consider the set $\Sigma=\{\neg\alpha, \neg\beta, \alpha, \beta, \neg\delta\vee\neg\sigma, \neg\alpha\vee\beta\}$

The maximal consistent subsets of Σ are:

$\Sigma_1 = \{\neg\alpha, \neg\beta, \neg\delta \vee \neg\sigma, \neg\alpha \vee \beta\}$

$\Sigma_2 = \{\neg\alpha, \beta, \neg\delta \vee \neg\sigma, \neg\alpha \vee \beta\}$

$\Sigma_3 = \{\alpha, \beta, \neg\delta \vee \neg\sigma, \neg\alpha \vee \beta\}$

$\Sigma_4 = \{\neg\beta, \alpha, \neg\delta \vee \neg\sigma\}$

Consider the three formulas:

$\phi_1=(\neg\alpha\wedge\neg\beta \wedge (\neg\delta\vee\neg\sigma)) \vee (\neg\delta\wedge\sigma\wedge(\alpha\vee\beta))$

$\phi_2=(\neg\alpha\wedge\beta\wedge(\neg\delta\vee\neg\sigma))\vee(\delta\wedge\neg\sigma \wedge (\alpha \vee \neg\beta))$

$\phi_3=(\alpha\wedge\beta\wedge(\neg\delta\vee\neg\sigma))\vee(\neg\delta\wedge\neg\sigma\wedge(\neg\alpha \vee \neg\beta))$

It is easy to see that $\Sigma_1\vdash\phi_1$, $\Sigma_2\vdash\phi_2$ and $\Sigma_3\vdash\phi_3$, but we never have $\Sigma_i\vdash\neg\phi_j$ for $i\neq j$.

So, $\Sigma\vdash_{\mathscr{A}} \phi_1$, $\Sigma\vdash_{\mathscr{A}} \phi_2$, $\Sigma\vdash_{\mathscr{A}} \phi_3$. However, $\phi_1\wedge\phi_2\wedge\phi_3\vdash\bot$. ■

This result can be viewed as a weakness of the argument-based inference which avoids obvious direct contradictions, but does not escape hidden ones. But the inconsistency of $\{\phi_1, \phi_2, \phi_3\}$ in the example occurs only if two already conflicting sources supporting ϕ_1 and ϕ_2 unite to defeat ϕ_3. And, since the two sources are in conflict with each other it is not clear why one should accept to join them against ϕ_3. Namely, when $\phi_1\wedge\phi_2\wedge\phi_3=\bot$, $\phi_1\wedge\phi_2$ cannot defeat ϕ_3 because $\phi_1\wedge\phi_2$ is not an argued consequence of Σ. It confirms the fact that $Cn_{\mathscr{A}}(\Sigma)$ is an heterogeneous set of properties that pertain to distinct views of the world. This means that a question-answering system whereby a question "is it true that ϕ" is answered by yes or no after computing $\Sigma \vdash_{\mathscr{A}} \phi$ is not really informative enough. The system must also supply the argument for ϕ. This way of coping with inconsistency looks natural, and the arguments for ϕ and ψ should enable the user to decide whether these two plausible conclusions can be accepted together or not. One interesting problem is the combination of arguments, namely: can we construct from the argument A of α and the argument B of β, an argument for $\alpha\wedge\beta$? Several authors suggest $A\cup B$ as the needed argument (Fox et al, 1992), (Darwiche, 1993). The previous results somewhat question this suggestion: this suggestion makes no sense if A and B are supplied by conflicting sources.

6. Conclusion

The proposed notion of argument-based inference is appealing for several reasons. First it is an extension of classical inference that copes with inconsistency in a very mild way. Namely it is rather faithful to the actual contents of the belief base, and does not do away with information contained in it, as opposed to revision approaches that restore consistency. Moreover it is more productive than the approach based on inferring from all maximal consistent subsets, and looks less arbitrary than the selection of consistent subsets of maximal cardinality. Second, it avoids outright contradictory responses (such as ϕ and $\neg\phi$), although several deduced sentences can be globally inconsistent. But as pointed out earlier, the arguments supporting a set of more than two globally contradictory sentences are distinct, so that the reality of this contradiction is debatable, and only reflects the presence of different points of view. Anyway it seems that it is the price to pay in order to remain faithful to an inconsistent belief base. It would be interesting to apply the above result to defeasible reasoning and study in such a framework the argument-based inference as well as the one proposed by Simari and Loui (1992).

Another result of this paper is the study of syntax sensitivity of the consequence relations (namely to what extent the consequence relation depends on the syntax of the belief base) by proposing several syntax-insensitivity properties. We have shown that all the consequence relations, except L-consequence, are insensitive to the duplication of formulas in the belief base, and all of them are insensitive to the addition of a formula which is a logical consequence of all maximally consistent sub-bases of Σ. In contrast, all the consequence relations are sensitive to the addition of a formula which is a logical consequence of some (but not all) consistent sub-bases, and are sensitive to the transformation of the belief base under clausal form.

In a companion paper (Benferhat et al., 1994; see also Benferhat et al., 1993), the approaches developed in this paper are extented to layered belief bases where layers express levels of certainty as in possibilistic logic (Dubois et al., 1994).

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