Chapter 2
Subgraphs, Paths, and Connected Graphs

2.1 Subgraphs and Spanning Subgraphs (Supergraphs)

Subgraph: Let H be a graph with vertex set $V(H)$ and edge set $E(H)$, and similarly let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Then, we say that H is a subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. In such a case, we also say that G is a supergraph of H.

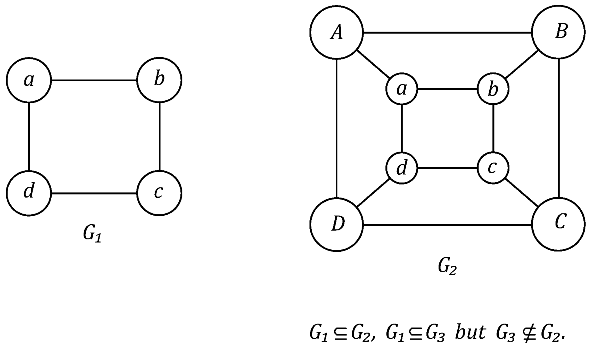


Fig. 2.1 G_1 is a subgraph of G_2 and G_3

In Fig. 2.1, G_1 is a subgraph of both G_2 and G_3 but G_3 is not a subgraph of G_2.

Any graph isomorphic to a subgraph of G is also referred to as a subgraph of G.

If H is a subgraph of G then we write $H \subseteq G$. When $H \subseteq G$ but $H \neq G$, i.e., $V(H) \neq V(G)$ or $E(H) \neq E(G)$, then H is called a proper subgraph of G.

Spanning subgraph (*or Spanning supergraph*): A *spanning subgraph* (or *spanning supergraph*) of G is a *subgraph* (or *supergraph*) H with $V(H) = V(G)$, i.e. H and G have exactly the same vertex set.

It follows easily from the definitions that any simple graph on n vertices is a subgraph of the complete graph K_n. In Fig. 2.1, G_1 is a proper spanning subgraph of G_3.

S. Saha Ray, *Graph Theory with Algorithms and its Applications*,
DOI: 10.1007/978-81-322-0750-4_2, © Springer India 2013

2.2 Operations on Graphs

The *union* of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is another graph $G_3 = (V_3, E_3)$ denoted by $G_3 = G_1 \cup G_2$, where vertex set $V_3 = V_1 \cup V_2$ and the edge set $E_3 = E_1 \cup E_2$.

The *intersection* of two graphs G_1 and G_2 denoted by $G_1 \cap G_2$ is a graph G_4 consisting only of those vertices and edges that are in both G_1 and G_2.

The *ring sum* of two graphs G_1 and G_2, denoted by $G_1 \oplus G_2$, is a graph consisting of the vertex set $V_1 \cup V_2$ and of edges that are either in G_1 or G_2, but not in both.

Figure 2.2 shows union, intersection, and ring sum on two graphs G_1 and G_2.

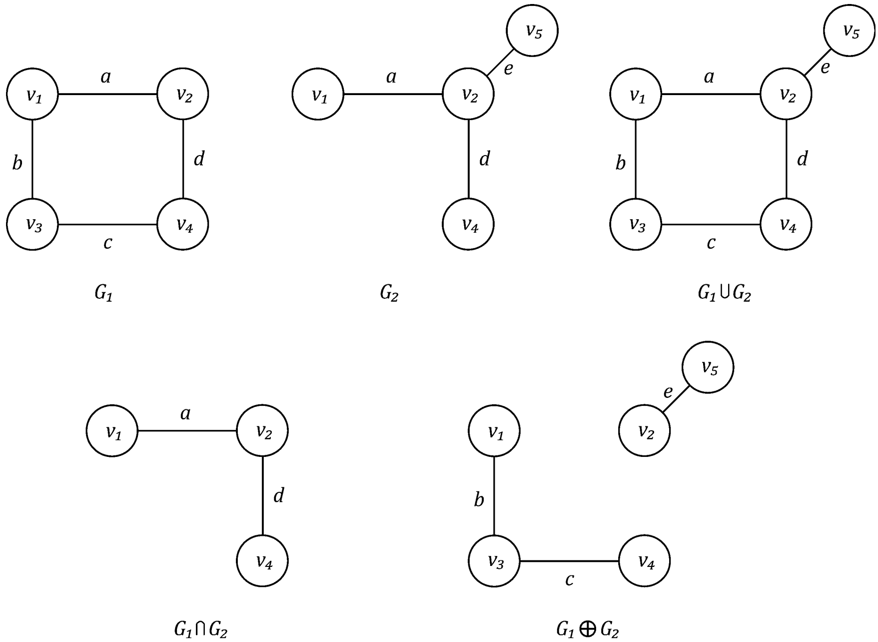


Fig. 2.2 Union, intersection, and ring sum of two graphs

Three operations are *commutative*, i.e.,

$$G_1 \cup G_2 = G_2 \cup G_1, \quad G_1 \cap G_2 = G_2 \cap G_1, \quad G_1 \oplus G_2 = G_2 \oplus G_1$$

If G_1 and G_2 are edge disjoint, then $G_1 \cap G_2$ is a null graph, and $G_1 \oplus G_2 = G_1 \cup G_2$. If G_1 and G_2 are vertex disjoint, then $G_1 \cap G_2$ is empty.

For any graph G, $G \cap G = G \cup G = G$ and $G \oplus G =$ *a null graph*.

If g is a subgraph of G, i.e., $g \subseteq G$, then $G \oplus g = G - g$, and is called a *complement* of g in G.

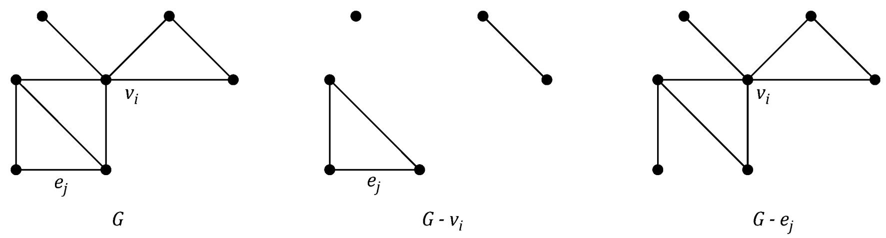


Fig. 2.3 Vertex deletion and edge deletion from a graph G

Decomposition: A graph G is said to be decomposed into two subgraphs G_1 and G_2, if $G_1 \cup G_2 = G$ and $G_1 \cap G_2$ is a *null graph*.

Deletion: If v_i is a vertex in graph G, then $G - v_i$ denotes a subgraph of G obtained by deleting v_i from G. Deletion of a vertex always implies the deletion of all edges incident on that vertex. If e_j is an edge in G, then $G - e_j$ is a subgraph of G obtained by deleting e_j from G. Deletion of an edge does not imply deletion of its end vertices. Therefore, $G - e_j = G \oplus e_j$ (Fig. 2.3).

Fusion: A pair of vertices a, b in a graph G are said to be *fused* if the two vertices are replaced by a single new vertex such that every edge, that was incident on either a or b or on both, is incident on the new vertex. Thus, fusion of two vertices does not alter the number of edges, but reduces the number of vertices by one (Fig. 2.4).

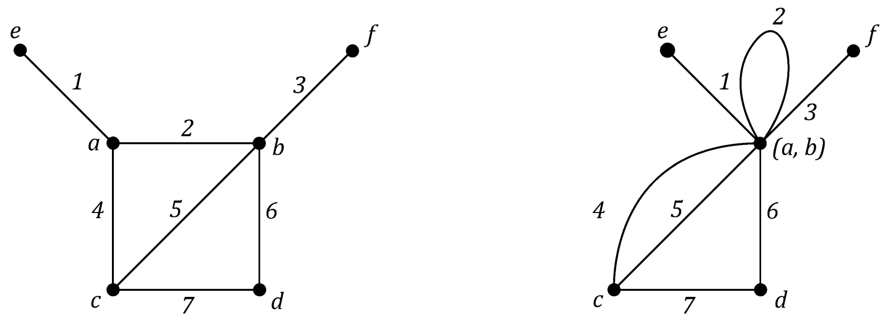


Fig. 2.4 Fusion of two vertices a and b

Induced subgraph: A subgraph $H \subseteq G$ is an induced subgraph, if $E_H = E_G \cap E(V_H)$. In this case, H is induced by its set V_H of vertices. In an induced subgraph $H \subseteq G$, the set E_H of edges consists of all $e \in E_G$, such that $e \in E(V_H)$. To each nonempty subset $A \subseteq V_G$, there corresponds a unique induced subgraph $G[A] = (A, E_G \cap E(A))$ (Fig. 2.5).

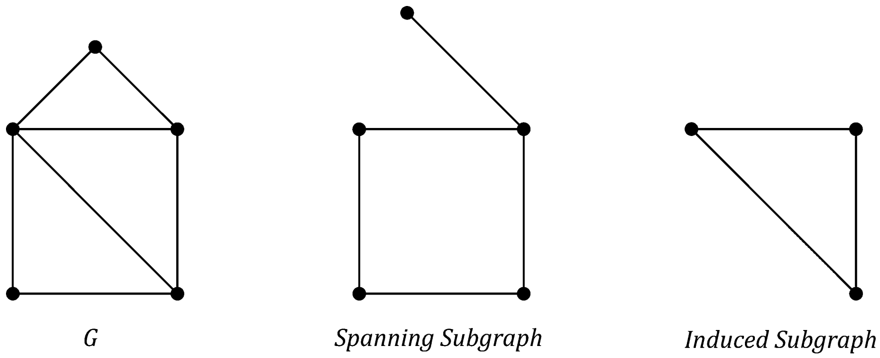


Fig. 2.5 Spanning subgraph and induced subgraph of a graph G

Trivial graph: A graph $G = (V, E)$ is trivial, if it has only one vertex. Otherwise G is nontrivial.

Discrete graph: A graph is called discrete graph if $E_G = \phi$.

Stable: A subset $X \subseteq V_G$ is stable, if $G[X]$ is a discrete graph.

2.3 Walks, Trails, and Paths

Walk: A *walk* in a graph G is a finite sequence

$$W \equiv v_0 e_1 v_1 e_2 \cdots v_{k-1} e_k v_k$$

whose terms are alternately vertices and edges such that for $1 \le i \le k$, the edge e_i has ends v_{i-1} and v_i.

Thus, each edge e_i is immediately preceded and succeeded by the two vertices with which it is incident. We say that W is a $v_0 - v_k$ walk or a walk from v_0 to v_k.

Origin and terminus: The vertex v_0 is the *origin* of the walk W, while v_k is called the *terminus* of W. v_0 and v_k need not be distinct.

The vertices $v_1, v_2, \ldots, v_{k-1}$ in the above walk W are called its *internal vertices*. The integer k, the number of edges in the walk, is called the *length of* W, denoted by $|W|$.

In a walk W, there may be repetition of vertices and edges.

Trivial walk: A *trivial walk* is one containing no edge. Thus for any vertex v of G, $W \equiv v$ gives a trivial walk. It has length 0.

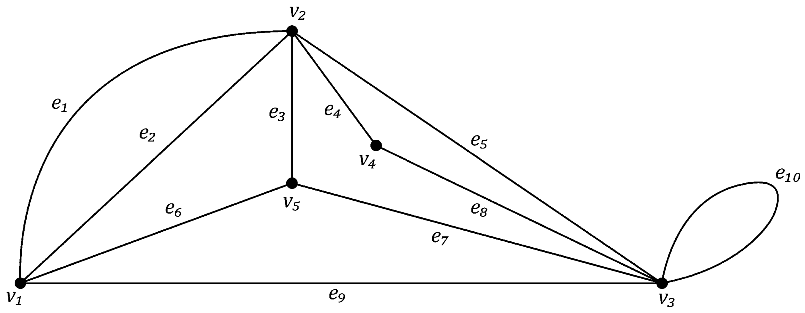


Fig. 2.6 A graph with five vertices and ten edges

In Fig. 2.6, $W_1 \equiv v_1e_1v_2e_5v_3e_{10}v_3e_5v_2e_3v_5$ and $W_2 \equiv v_1e_1v_2e_1v_1e_1v_2$ are both walks of length 5 and 3, respectively, from v_1 to v_5 and from v_1 to v_2 , respectively.

Given two vertices u and v of a graph G, a u–v walk is called *closed* or *open*, depending on whether $u = v$ or $u \neq v$.

Two walks W_1 and W_2 above are both open, while $W_3 \equiv v_1v_5v_2v_4v_3v_1$ is closed in Fig. 2.6.

Trail: If the edges $e_1, e_2, \ldots, e_k$ of the walk $W \equiv v_0e_1v_1e_2v_2 \cdots\cdots e_kv_k$ are distinct then W is called a *trail*. In other words, a trail is a walk in which no edge is repeated. W_1 and W_2 are not trails, since for example e_5 is repeated in W_1, while e_1 is repeated in W_2. However, W_3 is a trail.

Path: If the vertices $v_0, v_1, \ldots, v_k$ of the walk $W \equiv v_0e_1v_1e_2v_2 \cdots e_kv_k$ are distinct then W is called a *path*. Clearly, any two paths with the same number of vertices are isomorphic.

A path with n vertices will sometimes be denoted by P_n.

Note that P_n has length $n - 1$.

In other words, *a path is a walk in which no vertex is repeated*. Thus, in a path no edge can be repeated either, so a every path is a trail. Not every trail is a path, though. For example, W_3 is not a path since v_1 is repeated. However, $W_4 \equiv v_2v_4v_3v_5v_1$ is a path in the graph G as shown in Fig. 2.6.

2.4 Connected Graphs, Disconnected Graphs, and Components

Connected vertices: A vertex u is said to be *connected* to a vertex v in a graph G if there is a path in G from u to v.

Connected graph: A graph G is called *connected* if every two of its vertices are connected.

Disconnected graph: A graph that is not connected is called *disconnected*.

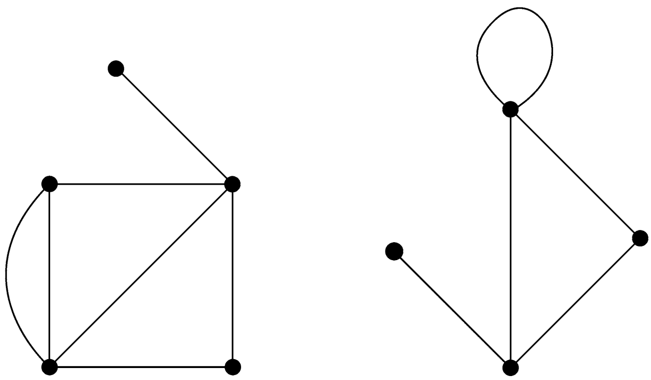


Fig. 2.7 A disconnected graph with two components

It is easy to see that a disconnected graph consists of two or more *connected graphs*. Each of these connected subgraphs is called a component. Figure 2.7 shows a disconnected graph with two components.

Theorem 2.1 *A graph G is disconnected iff its vertex set V can be partitioned into two non-empty, disjoint subsets V_1 and V_2 such that there exists no edge in G whose one end vertex is in subset V_1 and the other in subset V_2.*

Proof Suppose that such a partitioning exists. Consider two arbitrary vertices a and b of G, such that $a \in V_1$ and $b \in V_2$. No path can exist between vertices a and b; otherwise there would be at least one edge whose one end vertex would be in V_1 and the other in V_2. Hence, if a partition exists, G is not connected.

Conversely, let G be a disconnected graph. Consider a vertex a in G. Let V_1 be the set of all vertices that are connected by paths to a. Since G is disconnected, V_1 does not include all vertices of G. The remaining vertices will form a (non-empty) set V_2. No vertex in V_1 is connected to any vertex in V_2 by path. Hence the partition exists. □

Theorem 2.2 *If a graph* (*connected* or *disconnected*) *has exactly two vertices of odd degree, there must be a path joined by these two vertices.*

Proof Let G be a graph with all even vertices except vertices v_1 and v_2, which are odd. From Handshaking lemma, which holds for every graph and therefore for every component of a disconnected graph, no graph can have an odd number of odd vertices. Therefore, in graph G, v_1 and v_2 must belong to the same component, and hence there must be a path between them. □

Theorem 2.3 *A simple graph with n vertices and k components can have at most* $(n - k)(n - k + 1)/2$ *edges.*

Proof Let the number of vertices in each of the k components of a graph G be $n_1, n_2, \ldots, n_k$. Thus, we have

$$n_1 + n_2 + \cdots + n_k = n$$

where $n_i \geq 1 \quad for \quad i = 1, 2, \ldots, k.$

Now, $\sum_{i=1}^{k} (n_i - 1) = n - k$

$$\begin{aligned}
&\Rightarrow \left(\sum_{i=1}^{k}(n_i - 1)\right)^2 = n^2 + k^2 - 2nk \\
&\Rightarrow [(n_1 - 1) + (n_2 - 1) + \cdots + (n_k - 1)]^2 = n^2 + k^2 - 2nk \\
&\Rightarrow \sum_{i=1}^{k}(n_i - 1)^2 + 2\sum_{i,j=1,i\neq j}^{k}(n_i - 1)(n_j - 1) = n^2 + k^2 - 2nk \\
&\Rightarrow \sum_{i=1}^{k}(n_i)^2 - 2\sum_{i=1}^{k} n_i + k + 2\sum_{i,j=1,i\neq j}^{k}(n_i - 1)(n_j - 1) = n^2 + k^2 - 2nk \\
&\Rightarrow \sum_{i=1}^{k} n_i^2 - 2n + k + 2\sum_{i,j=1,i\neq j}^{k}(n_i - 1)(n_j - 1) = n^2 + k^2 - 2nk \\
&\Rightarrow \sum_{i=1}^{n} n_i^2 + 2\sum_{i,j=1,i\neq j}^{k}(n_i - 1)(n_j - 1) = n^2 + k^2 - 2nk + 2n - k.
\end{aligned}$$

Since each $(n_i - 1) \geq 0$.

$$\begin{aligned}
\sum_{i=1}^{n} n_i^2 \leq n^2 + k^2 - 2nk + 2n - k &= n^2 + k(k - 2n) - (k - 2n) \\
&= n^2 - (k - 1)(2n - k)
\end{aligned}$$

Now, the maximum number of edges in the ith component of G is $n_i(n_i - 1)/2$. Since the maximum number of edges in a simple graph with n vertices is $n(n-1)/2$ therefore, the maximum number of edges in G is

$$\begin{aligned}
\frac{1}{2}\sum_{i=1}^{k} n_i(n_i - 1) \quad &= \frac{1}{2}\sum_{i=1}^{n} n_i^2 - \frac{n}{2} \\
&\leq \frac{1}{2}\left[n^2 - (k - 1)(2n - k)\right] - \frac{n}{2} \\
&= \frac{1}{2}\left[n^2 - 2nk + 2n + k^2 - k - n\right] \\
&= \frac{1}{2}\left[(n - k)^2 + (n - k)\right] \\
&= \frac{1}{2}(n - k)(n - k + 1)
\end{aligned}$$

□

2.5 Cycles

Cycle: A nontrivial closed trail in a graph G is called a cycle if its origin and internal vertices are distinct. In detail, the closed trail

$C \equiv v_1 v_2 \cdots v_n v_1$ is a cycle if

1. C has at least one edge and
2. $v_1, v_2, \ldots, v_n$ are n distinct vertices.

*k-**Cycle***: A cycle of length k, , i.e., with k edges, is called a k-cycle. A k-cycle is called odd or even depending on whether k is odd or even.

Figure 2.8 cites C_3, C_4, C_5, and C_6. A 3-cycle is often called a triangle. Clearly, any two cycles of the same length are isomorphic.

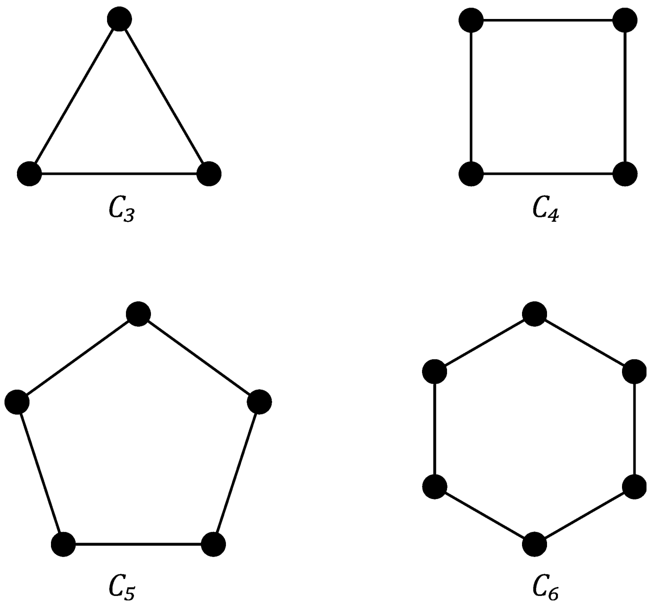
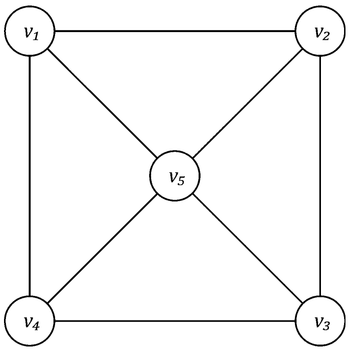


Fig. 2.8 Cycles C_3, C_4, C_5 and C_6

An n-cycle, i.e., a cycle with n vertices, will sometimes be denoted by C_n.

In Fig. 2.9, $C \equiv v_1 v_2 v_3 v_4 v_1$, is a 4-cycle and $T \equiv v_1 v_2 v_5 v_3 v_4 v_5 v_1$ is a non-trivial closed trail which is not a cycle (because v_5 occurs twice as an internal vertex) and $C' \equiv v_1 v_2 v_5 v_1$ is a triangle.

Fig. 2.9 A graph containing 3-cycles and 4-cycles



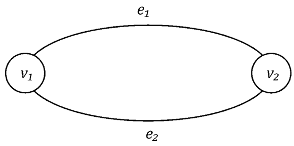


Fig. 2.10 A 2-cycle

Note that, a loop is just a 1-cycle. Also, given parallel edges e_1 and e_2 in Fig. 2.10 with distinct end vertices v_1 and v_2, we can find the cycle $v_1e_1v_2e_2v_1$ of length 2. Conversely, the two edges of any cycle of length 2 are a pair of parallel edges.

Theorem 2.4 *Given any two vertices u and v of a graph G, every u–v walk contains a u–v path.*

Proof We prove the statement by induction on the length l of a u–v walk W.

Basic step: $l = 0$, having no edge, W consists of a single vertex ($u = v$). This vertex is a u–v path of length 0.

Induction step: $l \geq 1$. We suppose that the claim holds for walks of length less than l. If W has no repeated vertex, then its vertices and edges form a u–v path. If W has a repeated vertex w, then deleting the edges and vertices between appearances of w (leaving one copy of w) yields a shorter u-v walk W' contained in W. By the induction hypothesis, W' contains a u–v path P, and this path P is contained in W (Fig. 2.11). This proves the theorem. □

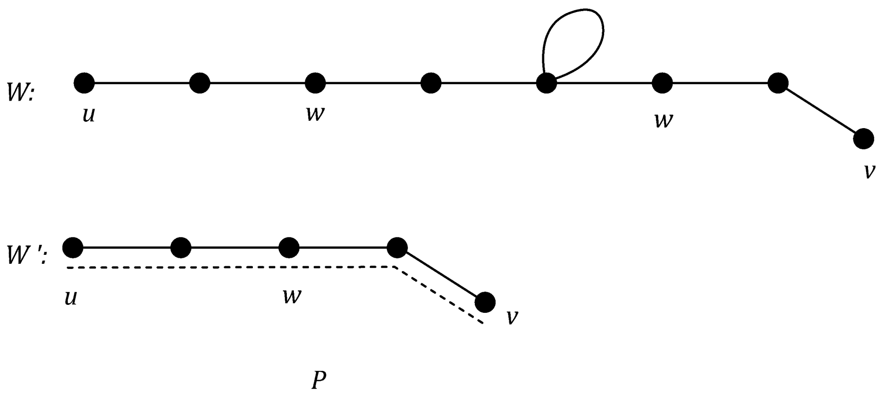


Fig. 2.11 A walk W and a shorter walk W' of W containing a path P

Theorem 2.5 *The minimum number of edges in a connected graph with n vertices is* $n - 1$.

Proof Let m be the number of edges of such a graph. We have to show $m \geq n - 1$. We prove this by method of induction on m. If $m = 0$ then obviously $n = 1$ (otherwise G will be disconnected). Clearly, then $m \geq n - 1$. Let the result be true for $m = 0, 1, 2, 3, \ldots, k$. We shall show that the result is true for $m = k + 1$. Let G be a graph with $k + 1$ edges. Let e be an edge of G. Then the subgraph $G - e$ has

k edges and n number of vertices. If $G - e$ is also connected then by our hypothesis $k \geq n - 1$, i.e., $k + 1 \geq n > n - 1$.

If $G - e$ is disconnected then it would have two connected components. Let the two components have k_1, k_2 number of edges and n_1, n_2 number of vertices, respectively. So, by our hypothesis, $k_1 \geq n_1 - 1$ and $k_2 \geq n_2 - 1$. These two imply that $k_1 + k_2 \geq n_1 + n_2 - 2$, i.e., $k \geq n - 2$ (since, $k_1 + k_2 = k$, $n_1 + n_2 = n$), i.e., $k + 1 \geq n - 1$.

Thus, the result is true for $m = k + 1$. □

Theorem 2.6 *A graph G is bipartite if and only if it has no odd cycles.*

Proof Necessary condition:

Let G be a bipartite graph with bipartition (X, Y), i.e., $V = X \cup Y$.

For any cycle $C : v_1 \to v_2 \cdots \to v_{k+1} (= v_1)$ of length $k, v_1 \in X \Rightarrow v_2 \in Y, v_3 \in X \Rightarrow v_4 \in Y \cdots \Rightarrow v_{2m} \in Y \Rightarrow v_{2m+1} \in X$. Consequently, $k + 1 = 2m + 1$ is odd and $k = |C|$ is even. Hence, G has no odd cycle.

Sufficient condition:

Suppose that, all the cycles in G are even, i.e., G be a graph with no odd cycle.

To show: G is a bipartite graph. It is sufficient to prove this theorem for the connected graph only.

Let us assume that G is connected. Let $v \in G$ be an arbitrary chosen vertex.

Now, we define,

$$X = \{x | d_G(v, x)\,\text{is even}\},$$

i.e., X is the set of all vertices x of G with the property that any shortest $v - x$ path of G has even length and $Y = \{y | d_G(v, y)\,\textit{is odd}\}$, i.e., Y is the set of all vertices y of G with the property that any shortest $v - y$ path of G has odd length.

Here,

$$\begin{aligned} d_G(u, v) &= \text{shortest distance from the vertex } u \text{ to the vertex } v \\ &= \min\left\{k : u \xrightarrow{k} v\right\} \end{aligned}$$

[If the graph G is connected then this shortest distance should be finite, i.e., $d_G(u, v) < \infty$ for $\forall u, v \in G$. Otherwise, G is disconnected]

Then clearly, since the graph G is connected $V = X \cup Y$ and also by definition of distance $X \cap Y = \emptyset$.

Now, we show that $V = X \cup Y$ is a bipartition of G by showing that any edge of G must have one end vertex in X and another in Y.

Suppose that $u, w \in V(G)$ are both either in X or in Y and they are adjacent.

Let $P : v \xrightarrow{*} u$ and $Q : v \xrightarrow{*} w$ be the two shortest paths from v to u and v to w, respectively.

Let x be the last common vertex of the two shortest paths P and Q such that $P = P_1P_2$ and $Q = Q_1Q_2$ where $P_2 : x \xrightarrow{*} u$ and $Q_2 : x \xrightarrow{*} w$ are independent (Fig. 2.12).

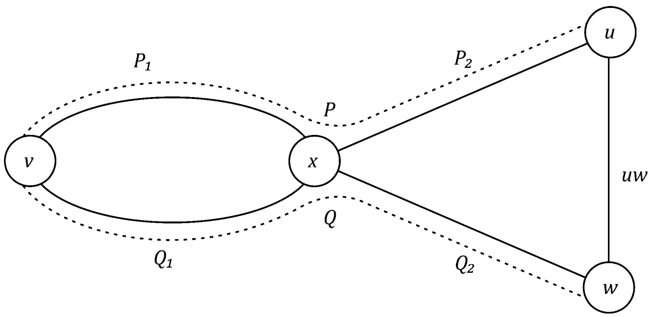


Fig. 2.12 Two shortest paths P and Q

Since P and Q are shortest paths, therefore, $P_1 : v \xrightarrow{*} x$ and $Q_1 : v \xrightarrow{*} x$ are shortest paths from v to x.

Consequently, $|P_1| = |Q_1|$

Now consider the following two cases.

Case 1: $u, w \in X$, then $|P|$ is even and $|Q|$ is even (Also, $|P_1| = |Q_1|$)

Case 2: $u, w \in Y$, then $|P|$ is odd and $|Q|$ is odd (Also, $|P_1| = |Q_1|$)

Therefore, in either case, $|P_2| + |Q_2|$ must be even and so $uw \notin E(G)$. Otherwise, $x \xrightarrow{*} u \rightarrow w \xrightarrow{*} x$ would be an odd cycle, which is a contradiction.

Therefore, X and Y are stable subsets of V. This implies (X, Y) is a bipartition of G. Therefore, $G[X]$ and $G[Y]$ are discrete induced subgraphs of G.

Hence, G is a bipartite graph.

If G is disconnected then each cycle of G will belong to any one of the connected components of G say $G_1, G_2, \ldots, G_p$.

If G_i is bipartite with bipartition (X_i, Y_i), then $\left(X_1 \cup X_2 \cup X_3 \cup \cdots\cdots \cup X_p,\right.$ $\left.Y_1 \cup Y_2 \cup \cdots\cdots \cup Y_p\right)$ is a bipartition of G.

Hence, the disconnected graph G is bipartite. □

Exercises:

1. Show that the following two graphs are isomorphic (Fig. 2.13).

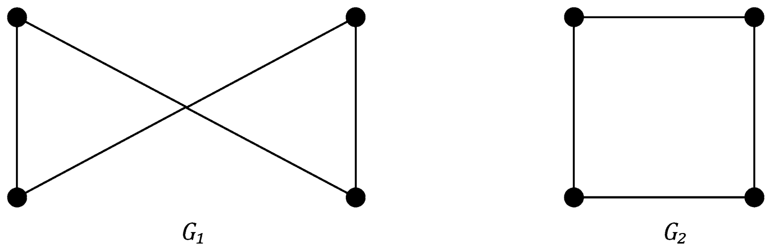


Fig. 2.13

2. Check whether the following two graphs are isomorphic or not (Fig. 2.14).

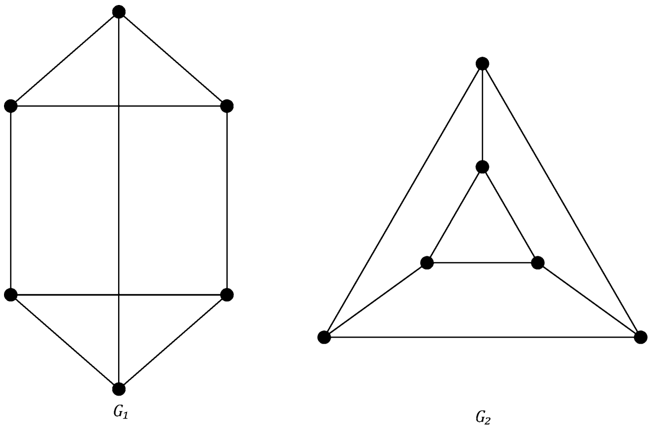


Fig. 2.14

3. Show that the following graphs are isomorphic and each graph has the same bipartition (Fig. 2.15).

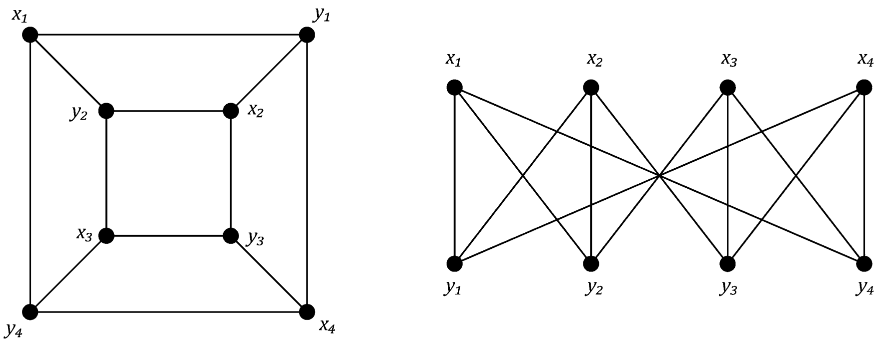


Fig. 2.15

4. What is the difference between a closed trail and a cycle?
5. Are the following graphs isomorphic? (Fig. 2.16).

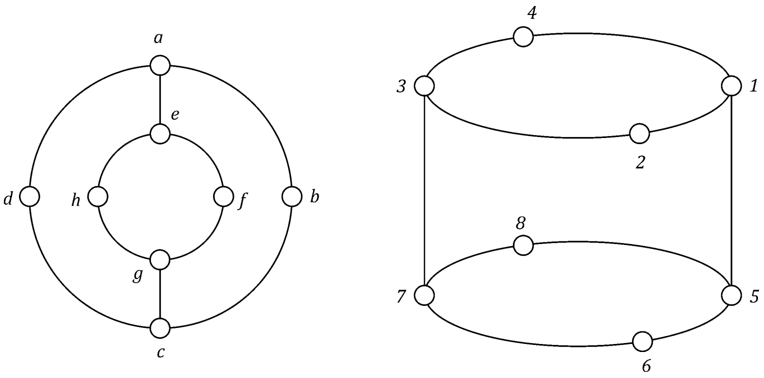


Fig. 2.16

6. Prove that a simple graph having n number of vertices must be connected if it has more than $(n-1)(n-2)/2$ edges.
7. Check whether the following two given graphs G_1 and G_2 are isomorphic or not (Fig. 2.17).

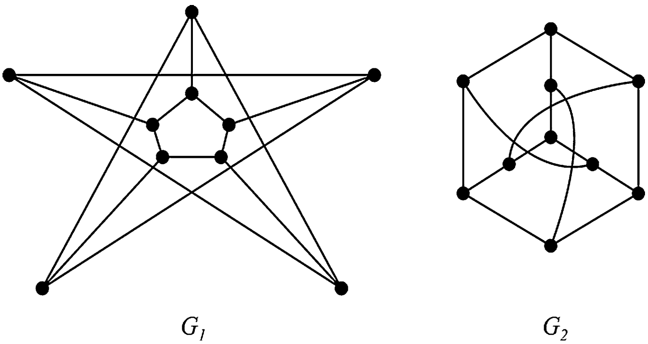


Fig. 2.17

8. Prove that the number of edges in a bipartite graph with n vertices is at most$(n^2/2)$.
9. Prove that there exists no simple graph with five vertices having degree sequence $4, 4, 4, 2, 2$.
10. Find, if possible, a simple graph with five vertices having degree sequence $2, 3, 3, 3, 3$.
11. If a simple regular graph has n vertices and 24 edges, find all possible values of n.

12. If $\delta(G)$ and $\Delta(G)$ be the minimum and maximum degrees of the vertices of a graph G with n vertices and e edges, show that

$$\delta(G) \leq \frac{2e}{n} \leq \Delta(G)$$

13. Show that the minimum number of edges in a simple graph with n vertices is $n - k$, where k is the number of connected components of the graph.
14. Find the maximum number of edges in
    (a) a simple graph with n vertices
    (b) a bipartite graph with bipartition (X, Y) where $|X| = m$ and $|Y| = n$, respectively.

http://www.springer.com/978-81-322-0749-8

Graph Theory with Algorithms and its Applications
In Applied Science and Technology
Saha Ray, S.
2013, XVI, 216 p., Hardcover
ISBN: 978-81-322-0749-8