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# Measuring and repairing inconsistency in knowledge bases with graded truth

David Picado Muiño[a,b,*,1]

[a] *Institut für Diskrete Mathematik und Geometrie, Wiedner Hauptstrasse 8/10, 1040 Vienna, Austria*
[b] *European Centre for Soft Computing, Edificio Científico Tecnológico, Gonzalo Gutiérrez Quirós s/n, 33600 Mieres, Spain*

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**Abstract**

In this paper we present a family of measures aimed at determining the amount of inconsistency in knowledge bases with graded truth, i.e., knowledge bases that consist of propositions along with a degree of truth or an interval of possible degrees of truth. Our approach to measuring inconsistency is also graded in the sense that we consider minimal adjustments in the truth degrees of the propositions necessary to make the knowledge base consistent within the frame of Łukasiewicz semantics. The computation of the family of measures we present here, in as much as it yields an adjustment in the truth degrees of each proposition that restores or brings consistency, provides the modeler with possible repairs of the knowledge base. Our motivation and case study for this paper is the fuzzy medical expert system CADIAG2.


*Keywords:* Graded truth; Degrees of truth; Łukasiewicz semantics; Inconsistency; Measures of inconsistency; Fuzzy expert systems; CADIAG2

## 1. Introduction

In the last few years the amount of literature dealing with aspects of inconsistency in knowledge bases has grown considerably and has become central in the field of databases and knowledge-based systems. In this paper we will focus primarily on two particular aspects of inconsistency in what we call knowledge bases with graded truth: the evaluation of inconsistency and possible repair strategies in the presence of inconsistency. By knowledge bases with graded truth we mean collections of propositions that are not necessarily fully true, i.e., that are fuzzy or vague, and that occur in the knowledge base together with a degree of truth or set of possible degrees of truth (in other words, collections of graded propositions—graded knowledge bases—where grades are intended to represent degrees of truth).

The evaluation of inconsistency in a knowledge base helps us understand it better. In particular, a measure of inconsistency allows us to determine how reliable the information contained in a database is and how this information could be used (for example, for inferential purposes) and even modified or adjusted to meet consistency. Our main

* Correspondence address: European Centre for Soft Computing, Edificio Científico Tecnológico, Gonzalo Gutiérrez Quirós s/n, 33600 Mieres, Spain. Tel.: +34 985456545.

*E-mail address:* david.picado@softcomputing.es.

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contribution to the evaluation of inconsistency in knowledge bases with graded truth in this paper reduces to the finite case and consists of the definition of a family of measures aimed at quantifying the amount of inconsistency along with the study and analysis of some of their properties and their computation. Each measure in the family that we present here quantifies the amount of inconsistency by determining *how far* the knowledge base is from consistency with respect to Łukasiewicz semantics based on a notion of distance given by a particular $p$-norm (which we call *p-distance*). This approach to quantifying inconsistency connects to possible repairs of the knowledge base in the sense that the computation of these measures yields an adjustment in the degrees of truth that makes the knowledge base consistent based on the principle of minimal change.

There are several approaches to measuring inconsistency in knowledge bases in the literature, mostly tailored for propositional knowledge bases (we refer to [11] for a survey on some of the most recent and popular approaches). Such measures can be roughly divided into two groups (we follow [10] in this categorization): those based on the size and/or amount of minimal inconsistent subsets of propositions contained in the knowledge base—see for example [14]—and those based on a more *fine-grained* analysis of the *information* contained in the propositions in the knowledge base—see for example [8,15]. Our approach to dealing with inconsistency differs substantially from the ones just mentioned, which are mostly motivated by inconsistency in non-graded knowledge bases: in our approach we take into account the (minimal) amount of adjustment of the truth degrees of the propositions in the knowledge base in order to restore consistency. There is some recent work on graded knowledge bases similar in nature to ours for probabilistic knowledge bases, i.e., for knowledge bases with degrees of belief or frequencies—see [21] and our companion papers [13,20]). The inconsistency measures defined in these papers are similar to ours, with the main difference being that we are dealing with knowledge bases with graded truth and the underlying semantics on which our family of measures defined is Łukasiewicz semantics. There are several repair methods in the presence of inconsistency based on some notion of distance minimization or any other minimization criteria, like the ones that are naturally connected to the computation of our family of measures and that we present in this paper (we refer to [4] and the extensive number of references given in it). The strategy of these approaches, mostly devised for propositional knowledge bases, generally reduces to deletion or modification of some propositions in the knowledge base. Instead, we look for an adjustment of the truth degree of our propositions.

Our paper is structured as follows: in Section 2 we introduce most of the notation we will be using throughout the paper and some preliminary definitions. In Section 3 the notions of $Ł_\eta$-consistency and maximal $Ł_\eta$-consistency for propositional knowledge are defined. The main reason why we introduce them here is because of their usefulness when it comes to proving some results in further sections. In Section 4 we define the family of $p$-measures briefly introduced in previous paragraphs. In Section 5 we give some examples aimed at illustrating how the theory works, stressing particularly some consequences of choosing distinct $p$-measures both for quantifying inconsistency and for repairing the knowledge base. Section 6 is devoted to the analysis of our family of $p$-measures. Some properties and features about them are stated and proved. In Section 7 we deal with the computation of our $p$-measures of inconsistency for the general case. In particular, we show that its computation reduces to the solution of a certain set of convex optimization problems. Section 8 is devoted to the fuzzy medical expert system CADIAG2. As will be explained in more detail in the referred section, the input in a run of the inference engine of CADIAG2 consists of a database that represents possibly vague information about the presence of symptoms, findings, signs and other types of medical entities in a particular patient. The possibility of inconsistency in such databases motivates and helps us justify most of the concepts presented in our paper.

## 2. Notation and preliminary definitions

Throughout this paper we will be working with a finite propositional language $L = \{q_1, \ldots, q_l\}$, for some $l \in \mathbb{N}$. We will denote by *SL* the closure of $L \cup \{\bot\}$ under *implication*, i.e., the connective $\rightarrow$.

We will be using a large number of abbreviations which correspond to other common logical connectives within the context of many-valued logics—see [9]. We consider the following common abbreviations, for $\phi, \theta \in SL$:

- $\bot \rightarrow \bot$ is abbreviated by $\top$.
- $\phi \rightarrow \bot$ by $\neg\phi$.
- $\neg(\phi \rightarrow \neg\theta)$ by $\phi \& \theta$.
- $\neg\phi \rightarrow \theta$ by $\phi \underline{\vee} \theta$.

- $\phi \& (\phi \rightarrow \theta)$ by $\phi \wedge \theta$.
- $((\phi \rightarrow \theta) \rightarrow \theta) \wedge ((\theta \rightarrow \phi) \rightarrow \phi)$ by $\phi \vee \theta$.

Next we define the notion of Ł-valuation.

**Definition 1.** Let $\omega : SL \longrightarrow [0, 1]$. We say that $\omega$ is an Ł-valuation on $L$ if, for $\phi, \theta \in SL$, we have what follows:

- $\omega(\phi \rightarrow \theta) = \min\{1, 1 - \omega(\phi) + \omega(\theta)\}$.
- $\omega(\perp) = 0$.

From these two clauses we can define the behaviour of Ł-valuations for the other connectives introduced above. Let $\phi, \theta \in SL$. We have the following (see [9]):

- $\omega(\top) = 1$.
- $\omega(\neg\phi) = 1 - \omega(\phi)$.
- $\omega(\phi \& \theta) = \max\{0, \omega(\phi) + \omega(\theta) - 1\}$.
- $\omega(\phi \veebar \theta) = \min\{1, \omega(\phi) + \omega(\theta)\}$.
- $\omega(\phi \wedge \theta) = \min\{\omega(\phi), \omega(\theta)\}$.
- $\omega(\phi \vee \theta) = \max\{\omega(\phi), \omega(\theta)\}$.

We will denote by $\Omega$ the set of Ł-valuations on $L$. We will sometimes refer to Ł-valuations on $L$ as $l$-coordinate vectors and we will denote such set by $\mathbb{D}_l$:

$$\mathbb{D}_l = \{\vec{y} \in \mathbb{R}^l | y_i \in [0, 1]\}.$$

Consider $\Gamma = \{\phi_1, \dots, \phi_k\} \subseteq SL$, for some $k \in \mathbb{N}$. We will denote by $\wedge\Gamma$ the sentence $\phi_1 \wedge \cdots \wedge \phi_k$. Similarly $\vee\Gamma$, $\veebar\Gamma$ and $\&\Gamma$ will denote the sentences $\phi_1 \vee \cdots \vee \phi_k$, $\phi_1 \veebar \cdots \veebar \phi_k$ and $\phi_1 \& \cdots \& \phi_k$ respectively.

Sentences of the form $\phi \wedge \cdots \wedge \phi$ where $\phi$ occurs $k$ times, for some $k \in \mathbb{N}$, will be abbreviated by the expression $\bigwedge^k \phi$ (and similarly for the other connectives). It is customary to refer to $\&^k \phi$, the sentence $\phi \& \cdots \& \phi$ where $\phi$ occurs $k$ times, by $\phi^k$ in the literature and we will follow this convention.

Let $\phi \in SL$. We will denote by $L_\phi = \{p_1, \dots, p_k\} \subseteq L$ the set of propositional variables that occur in $\phi$, for some $k \in \mathbb{N}$. We will sometimes use the notation $\phi(p_1, \dots, p_k)$.

Let $\omega$ be an Ł-valuation on $L$. We have that

$$\omega(\phi) = f(\omega(p_1), \dots, \omega(p_k))$$

for some $f : [0, 1]^k \rightarrow [0, 1]$. We will denote this $f$ by $f_\phi$. We will write sometimes $f_\phi(y_1, \dots, y_k)$ or, in order to simplify notation slightly, $f_\phi(\vec{y})$, where $\vec{y}$ is an $l$-coordinate vector, i.e., we consider as many parameters in $f$ as variables in $L$. We will call functions of the form $f_\phi$ for $\phi \in SL$ McNaughton functions—see [17].

We will denote the collection of closed intervals contained in [0,1] by $\mathbb{I}$. Intervals of the form $[\eta, \eta] \in \mathbb{I}$, real point values in the interval [0,1], will normally be denoted by $\eta$ itself.

Throughout we will be working with finite subsets of sentences in *SL*. Unless otherwise stated, even though some results in the paper also apply to infinite subsets of sentences, we will consider finite subsets of *SL*.

For the next definition let $\Gamma \subset SL$.

**Definition 2.** An assignment $v$ on $\Gamma$ is a map from $\Gamma$ to $\mathbb{I}$.

We will say that $v$ is a point-valued assignment on $\Gamma$ whenever we have that $v(\phi) \in [0, 1]$ for all $\phi \in \Gamma$.

We denote the set of assignments on $\Gamma$ by $V_\Gamma$ and set

$$FL = \{[\Gamma, v] | \Gamma \subset SL, \Gamma \neq \emptyset, v \in V_\Gamma\}.$$

We will sometimes write $[\Gamma, v] \in FL$ in extended form, that is to say, as

$$\{v(\phi) = \mathcal{I} | \phi \in \Gamma\},$$

with $\mathcal{I} \in \mathbb{I}$, or as a collection of pairs of the form

$$\{\langle \phi, \mathcal{I} \rangle | \phi \in \Gamma, \quad v(\phi) = \mathcal{I}\},$$

without explicitly mentioning the map $v$ on $\Gamma$.

Knowledge bases of the form $[\Gamma, v] \in FL$ formalize what in this paper we refer to as knowledge bases with graded truth and constitute the *raw* data in our analysis. Thus, $v$ in $[\Gamma, v]$ shall not be seen here as meta-level data aimed at simply manipulating the information in $\Gamma$ but rather as an essential piece of information.

We will generally assume that there are no sentences in $\Gamma$ in a knowledge base of the form $[\Gamma, v] \in FL$ that are *semantically equivalent* with respect to Ł-valuations on $L$.

Let $\Gamma = \Gamma_1 \cup \Gamma_2 \subset SL$ and $[\Gamma, v] \in FL$. We denote by $v_{|\Gamma_1}$ and $v_{|\Gamma_2}$ the restriction of $v$ on $\Gamma_1$ and $\Gamma_2$ respectively.

Consider now $[\Gamma_1, v_1], [\Gamma_2, v_2] \in FL$ and assume that, if $\phi \in \Gamma_1 \cap \Gamma_2$, $v_1(\phi) = v_2(\phi)$. We define the assignment $v = v_1 + v_2$ on $\Gamma_1 \cup \Gamma_2$ as follows: $v(\phi) = v_1(\phi)$ for all $\phi \in \Gamma_1$ and $v(\phi) = v_2(\phi)$ for all $\phi \in \Gamma_2$.

To illustrate these two concepts take for example $[\Gamma, v] = \{\langle \phi, \mathcal{I}_1 \rangle, \langle \psi, \mathcal{I}_2 \rangle\} \in FL$, with $\Gamma_1 = \{\phi\}$ and $\Gamma_2 = \{\psi\}$. We will have that $[\Gamma_1, v_{|\Gamma_1}] = \{\langle \phi, \mathcal{I}_1 \rangle\}$ and that $[\Gamma_2, v_{|\Gamma_2}] = \{\langle \psi, \mathcal{I}_2 \rangle\}$. Notice that $v = v_{|\Gamma_1} + v_{|\Gamma_2}$.

For the next definition let $[\Gamma, v] \in FL$.

**Definition 3.** An Ł-valuation $\omega$ on $L$ satisfies $[\Gamma, v]$ (denoted $\vDash_\omega [\Gamma, v]$) if, for all $\phi \in \Gamma$, we have that $\omega(\phi) \in v(\phi)$.

In that sense we say that $[\Gamma, v]$ is satisfiable or *consistent*[2] if there exists an Ł-valuation $\omega$ on $L$ that satisfies $[\Gamma, v]$.

## 3. The notion of Ł$_\eta$-consistency

In this section we introduce the notions of Ł$_\eta$-consistency and maximal Ł$_\eta$-consistency for subsets of *SL*.

For the next definitions let $\Gamma \subset SL$ and $\eta \in [0, 1]$.

**Definition 4.** We say that $\Gamma$ is Ł$_\eta$-consistent if there exists an Ł-valuation $\omega$ on $L$ such that $\omega(\wedge\Gamma) \geq \eta$.

**Definition 5.** We define the notion of maximal consistency of $\Gamma$—denoted $mc(\Gamma)$—as follows:

$$mc(\Gamma) = \sup\{\eta | \Gamma \text{ is Ł}_\eta\text{-consistent}\}$$

We say that $\Gamma$ is maximally Ł$_\eta$-consistent to mean that $mc(\Gamma) = \eta$.

As an example consider the set $\Gamma = \{\neg q, q\&q\}$. Notice that there exists an Ł-valuation $\omega$ on $\{q\}$ such that $\omega(\wedge\Gamma) = \frac{1}{3}$, with $\omega(q) = \frac{2}{3}$. Therefore, $\Gamma$ is Ł$_{1/3}$-consistent. In fact, $\Gamma$ is Ł$_\eta$-consistent for all $\eta \in [0, \frac{1}{3}]$. Notice that it is not possible to find an Ł-valuation $\omega$ on $\{q\}$ for which $\omega(\wedge\Gamma) > \frac{1}{3}$ and thus $mc(\Gamma) = \frac{1}{3}$.

These definitions resemble those of $\eta$-consistency and maximal $\eta$-consistency presented in [14], the essential difference being that such notions are defined based on probability functions on $L$ instead of Ł-valuations. The notion of maximal Ł$_\eta$-consistency is given in this paper mostly due to its usefulness when it comes to proving some results in further sections—for more on maximal Ł$_\eta$-consistency in general and, in particular, as a measure of inconsistency see [18].

Notice that Ł$_\eta$-consistency of a set of sentences $\Gamma$ is the same as Ł$_\eta$-consistency of the sentence $\wedge\Gamma$. We will talk indistinctively about the consistency of sentences and sets of sentences.

**Proposition 6.** *$mc(\Gamma)$ is attained by some Ł-valuation.*

**Proof.** The result follows from compactness of the space $[0, 1]^{2k+l}$ and the continuity of $f_{\wedge\Gamma}$.

For a more detailed proof, let $mc(\Gamma) = \eta$. We can define an increasing sequence $\{\eta_n\}$ whose limit is $\eta$ such that for all $n \in \mathbb{N}$ there exists an Ł-valuation $\omega_n$ on $L$ with $\omega_n(\wedge\Gamma) \geq \eta_n$. We need to prove that there exists an Ł-valuation $\omega$ on $L$ such that $\omega(\wedge\Gamma) \geq \eta$.

Let $\vec{\omega}_n \in \mathbb{D}_l$ be the vector representation of $\omega$, for all $n \in \mathbb{N}$.

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[2] We use both words interchangeably throughout the article.

We can take a convergent subsequence $\{\vec{\omega}^1_{n_k}\}$ in the first coordinates of $\{\vec{\omega}_n\}$. We know such a convergent subsequence needs to exist and converges in the interval [0,1] by compactness. Next we can pick a convergent subsequence $\{\vec{\omega}^2_{n_k}\}$ in the second coordinates of $\{\vec{\omega}^1_{n_k}\}$. As before, such subsequence needs to exist and converges in the interval [0,1] by compactness. We can proceed in the same way for the other coordinates. The final subsequence, $\{\vec{\omega}^l_{n_k}\}$, will have as limit an Ł-valuation $\omega$ (with vector representation $\vec{\omega}$) on $L$ for which $\omega(\wedge\Gamma) \geq \eta$. □

The two results that follow, although probably known, are given here due to its usefulness for other results in further sections. Their respective proofs are, for the sake of completeness, also given.

**Proposition 7.** *For all $k \in \mathbb{N}$ we can construct a sentence $\phi \in SL$ (which we will denote by $\phi_{1/k}$) that is maximally Ł$_{1/k}$-consistent.*

**Proof.** Let us define $\phi_{1/k}$ as follows:

$$\phi_{1/k} = \neg q \wedge q^{k-1},$$

for $q \in L$.

It can be easily checked that $\phi_{1/k}$ is maximally Ł$_{1/k}$-consistent. Consider the Ł-valuation $\omega$ on $L$ that assigns to $q$ the value $(k-1)/k$. We have that $\omega(\phi_{1/k}) = 1/k$. It is also clear that any other Ł-valuation $\omega'$ on $L$ for which $\omega'(q) < (k-1)/k$ or $\omega'(q) > (k-1)/k$ will be such that $\omega'(\phi_{1/k}) < 1/k$. □

**Proposition 8.** *Let $r \in \mathbb{Q} \cap [0, 1]$. We can construct a sentence $\phi \in SL$ (which we will denote by $\phi_r$) that is maximally Ł$_r$-consistent.*

**Proof.** Let $r = u/v$. Let us define $\phi_r$ as follows:

$$\phi_r = \underline{\bigvee}^{u} \phi_{1/v}$$

By Proposition 7 the sentence $\phi_{1/v}$ is maximally Ł$_{1/v}$-consistent and thus the sentence $\underline{\bigvee}^{u} \phi_{1/v}$ is maximally Ł$_{u/v}$-consistent. □

## 4. Measuring inconsistency: *p*-distance

Our approach to measuring inconsistency for knowledge bases of the form $[\Gamma, v] \in FL$ is based on the quantification of the *minimal* adjustment that one needs to make on the assignment $v$ in order for $[\Gamma, v]$ to be satisfiable.

In order to quantify the minimal adjustment we will rely on the so-called *p-norm* which, for a vector $\vec{x} \in \mathbb{R}^m$, is given by

$$\|\vec{x}\|_p = \left(\sum_{i=1}^{m} |x_i|^p\right)^{1/p},$$

for $p \geq 1$ (not necessarily an integer). The most common $p$-norms are certainly those of order 1 and 2, i.e., 1-norm and 2-norm. We also have

$$\|\vec{x}\|_\infty = \lim_{p\to\infty} \left(\sum_{i=1}^{m} |x_i|^p\right)^{1/p} = \max_i |x_i|$$

among the most common norms, which we will refer to as the $\infty$-norm and regard as a $p$-norm with $p = \infty$. In this context we will abuse notation slightly and will write $[1, \infty]$ for the range of possible values of $p$.

For what follows let $[\Gamma, v] \in FL$ and $\Delta \subset \Gamma$, with

$$\Delta' = \Gamma - \Delta = \{\phi_1, \ldots, \phi_k\},$$

for some $k \in \mathbb{N}$, and $v(\phi_i) = [\underline{\eta}_i, \overline{\eta}_i]$ for all $i \in \{1, \ldots, k\}$. $\Delta$ is intended to represent a set of sentences in $\Gamma$ regarded by the modeler as *correctly* evaluated by $v$ and thus the assignments made by $v$ on $\Delta$ should (arguably) not be subject to any modifications when assessing the $p$-distance (to consistency) of $[\Gamma, v]$ (e.g., the modeler, at the time of assessing the $p$-distance to consistency of the knowledge base or possible adjustments in $v$ in order for $[\Gamma, v]$ to meet consistency, could consider that the information represented by some graded statements in $[\Gamma, v]$—i.e., $[\Delta, v_{|\Delta}]$—is reliable and that therefore it shall not be subject to any adjustments).

We define $[\Gamma, v_{\vec{x}}]$ to be the set

$$\{v_{\vec{x}}(\phi_i) = [\underline{\eta}_i - x_i, \overline{\eta}_i + x_{i+k}] | i \in \{1, \ldots, k\}\} \cup \{v_{\vec{x}}(\theta) = v(\theta) | \theta \in \Delta\},$$

where $x_i, x_{i+k}$ are positive real values satisfying the constraint

$$0 \leq \underline{\eta}_i - x_i \leq \overline{\eta}_i + x_{i+k} \leq 1,$$

for all $i \in \{1, \ldots, k\}$.

For the next definitions let $[\Gamma, v] \in FL$ and consider $[\Gamma, v_{\vec{x}}]$ and $\Delta \subset \Gamma$ as above, with $\vec{x} \in \mathbb{R}^{2k}$ and $p \in [1, \infty]$.

**Definition 9.** We define the real set $S^p_\Delta([\Gamma, v])$ as follows:

$$S^p_\Delta([\Gamma, v]) = \{a \geq 0 | \|\vec{x}\|_p = a, [\Gamma, v_{\vec{x}}] \text{ is satisfiable}\}.$$

Notice that $S^p_\Delta([\Gamma, v])$ is bounded below for all $p \in [1, \infty]$ and thus, if it is not empty, its infimum exists, i.e., $\inf(S^p_\Delta([\Gamma, v]))$ exists. Notice also that $\inf(S^p_\Delta([\Gamma, v]))$ corresponds to the $p$-norm distance between the Cartesian product $\Pi_{\phi \in \Gamma} v(\phi)$ and the set given by the point-valued assignments $u$ on $\Gamma$ for which $[\Gamma, u]$ is satisfiable.

**Definition 10.** We define the $p$-distance (to consistency) of $[\Gamma, v]$ with respect to the set $\Delta$—denoted $DC^p_\Delta([\Gamma, v])$—as follows:

- If $S^p_\Delta([\Gamma, v]) \neq \emptyset$ then $DC^p_\Delta([\Gamma, v]) = \inf(S^p_\Delta([\Gamma, v]))$.
- If $S^p_\Delta([\Gamma, v]) = \emptyset$ then we set $DC^p_\Delta([\Gamma, v]) = \infty$.

That $DC^p_\Delta([\Gamma, v])$ is well defined is clear. We will have that $[\Delta, v_{|\Delta}]$ is not satisfiable if and only if $S^p_\Delta([\Gamma, v]) = \emptyset$ and thus that $DC^p_\Delta([\Gamma, v]) = \infty$. If $[\Delta, v_{|\Delta}]$ is satisfiable (or $\Delta = \emptyset$) and define $\Omega_{[\Delta, v_{|\Delta}]}$ to be the set of Ł-valuations on $L$ that satisfy $[\Delta, v_{|\Delta}]$ (or consider just the set $\Omega$ in case $\Delta = \emptyset$) we have that each $\omega \in \Omega_{[\Delta, v_{|\Delta}]}$ satisfies $[\Gamma, v_{\vec{x}}]$ for some real values $x_i$, with $i \in \{1, \ldots, 2k\}$, and thus, if any such Ł-valuation existed, $S^p_\Delta([\Gamma, v])$ would not be empty.

We will write $DC^p([\Gamma, v])$ instead of $DC^p_\emptyset([\Gamma, v])$ whenever $\Delta = \emptyset$ (i.e., whenever there is no given set $\Delta \subset \Gamma$ with respect to which we are defining the $p$-distance of $[\Gamma, v]$).

Next we define the notion of *normalized* $p$-distance of $[\Gamma, v]$ with respect to $\Delta$ (in a pretty standard way).

**Definition 11.** Let $p \in [1, \infty)$. The normalized $p$-distance (to consistency) of $[\Gamma, v]$ with respect to $\Delta$—denoted $\overline{DC}^p_\Delta([\Gamma, v])$—is defined from $DC^p_\Delta([\Gamma, v])$ as follows:

- If $DC^p_\Delta([\Gamma, v]) = \infty$ then $\overline{DC}^p_\Delta([\Gamma, v]) = \infty$.
- If $DC^p_\Delta([\Gamma, v]) \neq \infty$ then

$$\overline{DC}^p_\Delta([\Gamma, v]) = \frac{DC^p_\Delta([\Gamma, v])}{k^{1/p}}.$$

The normalized $\infty$-distance of $[\Gamma, v]$ with respect to $\Delta$ coincides with its $\infty$-distance:

$$\overline{DC}^\infty_\Delta([\Gamma, v]) = DC^\infty_\Delta([\Gamma, v]).$$

The normalized $p$-distance gives us the *ratio* between the value $DC^p_\Delta([\Gamma, v])$ and the smallest upper bound that could be placed on the $p$-distance of a knowledge base in *FL* of the same size as $[\Gamma, v]$, i.e., $k^{1/p}$ (see Proposition 19). Thus, clearly, $\overline{DC}^p_\Delta([\Gamma, v])$ is at most 1. The normalized $p$-distance brings into play the size of the knowledge base and gives a better ground to compare the amount of inconsistency of distinct knowledge bases in *FL*.

For our next definition let us consider again $[\Gamma, v] \in FL$, with $[\Gamma, v_{\vec{x}}]$ and $\Delta \subset \Gamma$ as above.

**Definition 12.** We define the set $R_\Delta([\Gamma, v])$ of repairs of $[\Gamma, v]$ with respect to $\Delta$ as follows:

$$R_\Delta([\Gamma, v]) = \{\vec{x} \in \mathbb{R}^{2k} | [\Gamma, v_{\vec{x}}] \text{ is satisfiable}\}.$$

For the next definition let $\vec{x} \in \mathbb{R}^{2k}$ be a repair in $R_\Delta([\Gamma, v])$.

**Definition 13.** We say that $\vec{x}$ is $p$-optimal (for $[\Gamma, v]$ with respect to $\Delta$) if $\|\vec{x}\|_p = DC^p_\Delta([\Gamma, v])$.

As is obvious, the adjustment that $\vec{x} \in R_\Delta([\Gamma, v])$ induces on $[\Gamma, v]$ consists of replacing the graded statements of the form $\langle \phi_i, [\underline{\eta}_i, \overline{\eta}_i] \rangle$ in $[\Gamma, v]$ by those of the form $\langle \phi_i, [\underline{\eta}_i - x_i, \overline{\eta}_i + x_{i+k}] \rangle$ in $[\Gamma, v]$, for each $i \in \{1, \ldots, k\}$. Notice that such an adjustment is, if $\vec{x}$ is $p$-optimal for some $p \in [1, \infty]$, *semantically* equivalent to the point-valued alternative that would consist of replacing each $[\underline{\eta}_i, \overline{\eta}_i]$ by $\underline{\eta}_i - x_i$ if $x_i \neq 0$ or $\overline{\eta}_i + x_{i+k}$ if $x_{i+k} \neq 0$ (as is clear, for $\vec{x}$ $p$-optimal, $x_i = 0$ or $x_{i+k} = 0$). For example, if we have

$$[\Gamma, v] = \{\langle q, [0, 0.3] \rangle, \langle \neg q, 0.5 \rangle\}$$

then a possible 1-optimal repair would consist of replacing the interval [0,0.3] in the first statement by the interval [0,0.5]. However, from our repaired knowledge base we could infer that $\langle q, 0.5 \rangle$.

## 5. Examples and some remarks

In this section we consider some examples in order to illustrate some differences in $DC^p$ and the $p$-optimal repairs for distinct values of $p$. The values we will focus on are the most common when dealing with $p$-norms, i.e., 1, 2 and $\infty$.

Let us consider for our first example the set

$$[\Gamma, v] = \{\langle q_1 \& q_2, 0.2 \rangle, \langle q_1 \& q_3, 0.2 \rangle, \langle q_1, 0.1 \rangle\},$$

for $q_1, q_2, q_3 \in L$. We will have that $DC^1([\Gamma, v]) = 0.1$, $DC^2([\Gamma, v]) \simeq 0.082$ and $DC^\infty([\Gamma, v]) = 0.05$. The normalized values will be given by $\overline{DC}^1([\Gamma, v]) = \frac{1}{30}$, $\overline{DC}^2([\Gamma, v]) \simeq 0.047$ and $\overline{DC}^\infty([\Gamma, v]) = 0.05$. It is interesting to have a look at the optimal repairs induced by $DC^p$ for these three distinct values of $p$.

There is a unique 1-optimal repair in $R([\Gamma, v])$ and it yields a unique modification in the assignment of the sentence $q_1$ by a magnitude of 0.1, i.e., such optimal repair would consist of the replacement of $\langle q_1, 0.1 \rangle$ for $\langle q_1, [0.1, 0.2] \rangle$ (or its point-valued alternative $\langle q_1, 0, 2 \rangle$). Notice that, in such repair, the whole weight of the adjustment rests on a single statement (on $q_1$) and that, in the lack of knowledge of how reliable or accurate the assignment $v$ is on the distinct sentences in $\Gamma$, might not be the most reasonable repair strategy. Instead, the unique $\infty$-optimal repair in $R([\Gamma, v])$ offers in that sense a more balanced approach by inducing an adjustment of magnitude 0.05 in the assignment on each one of the sentences in $\Gamma$. The unique 2-optimal repair in $R([\Gamma, v])$ involves an adjustment of magnitude $\frac{1}{30}$ in the assignment on both $q_1 \& q_2$ and $q_1 \& q_3$ and an adjustment of magnitude $\frac{1}{15}$ on $q_1$.

Let us consider now

$$[\Gamma, v] = \{\langle q_1 \& q_2, 0.2 \rangle, \langle q_1, 0.1 \rangle\}.$$

The number of 1-optimal repairs in $R([\Gamma, v])$ is infinite. Any repair that yields a modification in the grades for $q_1 \& q_2$ and $q_1$ of the form $0.2 - x$ and $0.1 + y$ respectively, with $x, y \geq 0$ and $x + y = 0.1$, will be 1-optimal. As in our previous example, in the lack of any knowledge about how accurate or reliable our propositions in $[\Gamma, v]$ are and of any other rational criterion that would help us in discriminating among repairs, a balanced adjustment (for example, one which yields $x = y = 0.05$) seems a reasonable choice.

For our last example consider

$$[\Gamma, v]_k = \{\langle q_1, 1 \rangle, \ldots, \langle q_{k-1}, 1 \rangle, \langle \perp, 1 \rangle\},$$

with $q_1, \ldots, q_{k-1} \in L$ and $k \in \mathbb{N}$. We clearly have that

$$DC^1([\Gamma, v]_k) = DC^2([\Gamma, v]) = DC^\infty([\Gamma, v]) = 1$$

and that $\overline{DC}^1([\Gamma, v]) = 1/k$, $\overline{DC}^2([\Gamma, v]) = 1/\sqrt{k}$ and $\overline{DC}^\infty([\Gamma, v]) = 1$. $\overline{DC}^\infty$ assigns to $[\Gamma, v]$ the highest possible amount of inconsistency in a knowledge base in *FL*. Examples like this one in which the repair to restore consistency involves necessarily a severe adjustment in the assignment of a small number of them brings $\overline{DC}^p$, for big values of $p$, to assign high amounts of inconsistency regardless of the size of the knowledge base. Although arguable, it seems reasonable to consider $[\Gamma, v]_{k_1}$ more inconsistent than $[\Gamma, v]_{k_2}$ whenever $k_1 < k_2$ based on the simple fact that the former has larger cardinality. On that assumption, for knowledge bases like these in which the adjustment of possible optimal repairs concentrates on a small number of propositions it seems more reasonable to measure the $p$-distance to consistency by considering small values of $p$.

## 6. Properties of $DC^p$

In this section we will be showing some properties of the operator $DC^p$, most of which apply also trivially to its normalization $\overline{DC}^p$. To do so we will be considering throughout (unless otherwise stated) a knowledge base of the form $[\Gamma, v] \in FL$, with $\Delta \subset \Gamma$,

$$\Delta' = \Gamma - \Delta = \{\phi_1, \ldots, \phi_k\},$$

for some $k \in \mathbb{N}$, and $v(\phi_i) = [\underline{\eta}_i, \overline{\eta}_i]$ for all $i \in \{1, \ldots, k\}$. The knowledge base $[\Gamma, v_{\vec{x}}]$ will be defined from $[\Gamma, v]$ as in Section 4.

We first prove that if the set $S_\Delta^p([\Gamma, v])$ is not empty then it always has a *minimum* for $p \in [1, \infty]$. That is to say, that

$$\inf S_\Delta^p([\Gamma, v]) = \min S_\Delta^p([\Gamma, v])$$

or, in other words, that $\inf S_\Delta^p([\Gamma, v])$ is always attained (provided it is not empty) by some repair $\vec{x} \in R_\Delta([\Gamma, v])$.

**Proposition 14.** *Let $S_\Delta^p([\Gamma, v]) \neq \emptyset$. There exists a p-optimal repair $\vec{x} \in \mathbb{R}^{2k}$ (of $[\Gamma, v]$ with respect to $\Delta$) for all $p \in [1, \infty]$.*

**Proof.** The result follows from compactness of the space $[0, 1]^l$ and continuity of $\|\vec{x}\|_p$, as we next show in more detail.

Let $p \in [1, \infty]$ and assume that $S_\Delta^p([\Gamma, v]) \neq \emptyset$. We can define a decreasing sequence $\{a_n\}$ with

$$\lim_{n\to\infty} a_n = a = \inf S_\Delta^p([\Gamma, v])$$

and such that $a_n \in S_\Delta^p([\Gamma, v])$ for all $n \in \mathbb{N}$ (i.e., there exists a repair $\vec{x}_n \in R_\Delta([\Gamma, v])$ for which $\|\vec{x}_n\|_p = a_n$ for all $n \in \mathbb{N}$). We can assume, without loss of generality, that the sequences of the form $\{x_i^n\}$ are monotone and convergent, with limit $x_i^*$, for all $i \in \{1, \ldots, 2k\}$. Notice that we will have that $\|\vec{x}^*\| = a$ (where $\vec{x}^*$ is the vector given by the coordinates $x_i^*$).

Let us consider, for each $n \in \mathbb{N}$, an Ł-valuation $\omega_n$ such that

$$\omega_n(\phi_i) \in [\underline{\eta}_i - x_i^n, \overline{\eta}_i + x_{i+k}^n]$$

for all $i \in \{1, \ldots, k\}$ and $\omega_n(\theta) \in v(\theta)$ for all $\theta \in \Delta$. Let $\vec{\omega}_n \in \mathbb{D}_l$ be the vector representation of $\omega_n$, for all $n \in \mathbb{N}$. We can take a convergent subsequence $\{\vec{\omega}_{n_r}^1\}$ in the first coordinates of $\{\vec{\omega}_n\}$. We know such a convergent subsequence needs to exist and converge in the interval [0,1] by compactness. We can proceed in the same way for the other coordinates. The final subsequence, $\{\vec{\omega}_{n_r}^l\}$, will have as limit the Ł-valuation $\omega$ (with vector representation $\vec{\omega}$) such that

$$\omega(\phi_i) \in [\underline{\eta}_i - x_i^*, \overline{\eta}_i + x_{i+k}^*]$$

for all $i \in \{1, \ldots, k\}$ and $\omega(\theta) \in v(\theta)$ for all $\theta \in \Delta$. $\square$

Next we prove *monotonicity* of $DC^p$.

Let $[\Gamma_1, v_1], [\Gamma_2, v_2] \in FL$ be such that $v_1(\phi) = v_2(\phi)$ for all $\phi \in \Gamma_1 \cap \Gamma_2$ and $\Delta_1 \subset \Gamma_1$, $\Delta_2 \subset \Gamma_2$.

**Proposition 15.** *For all* $p \in [1, \infty]$ *we have that*

$$DC^p_{\Delta_1}([\Gamma_1, v_1]) \le DC^p_{\Delta_1 \cup \Delta_2}([\Gamma_1 \cup \Gamma_2, v_1 + v_2]).$$

**Proof.** It follows directly from the definition of $DC^p$. □

For the two propositions that follow consider $p_1, p_2 \in [1, \infty]$.

**Proposition 16.** *If* $p_1 < p_2$ *then* $DC^{p_1}_{\Delta}([\Gamma, v]) \ge DC^{p_2}_{\Delta}([\Gamma, v])$.

**Proof.** The result follows from the well known fact that, for general $\vec{x} \in \mathbb{R}^m$ and $m \in \mathbb{N}$, $\|\vec{x}\|_p$ is decreasing on $p$, for $p \in [1, \infty)$. □

For normalized operators we have what follows.

**Proposition 17.** *If* $p_1 < p_2$ *then* $\overline{DC}^{p_1}_{\Delta}([\Gamma, v]) \le \overline{DC}^{p_2}_{\Delta}([\Gamma, v])$.

**Proof.** The result follows from the fact that, for general $\vec{x} \in \mathbb{R}^m$ and $m \in \mathbb{N}$, $\|\vec{x}\|_p / k^{1/p}$ is increasing on $p$, for $p \in [1, \infty)$. □

**Proposition 18.** $DC^p_{\Delta}([\Gamma, v]) = 0$ *if and only if* $[\Gamma, v]$ *is satisfiable, for all* $p \in [1, \infty]$.

**Proof.** It follows directly from the definition of $DC^p$. □

For what follows let $p \in [1, \infty)$.

**Proposition 19.** *If* $S^p_{\Delta}([\Gamma, v]) \neq \emptyset$ *then* $DC^p_{\Delta}([\Gamma, v]) \le k^{1/p}$ *and the bound* $k^{1/p}$ *cannot be improved for any* $k \in \mathbb{N}$.

**Proof.** That $k^{1/p}$ is an upper bound for $DC^p_{\Delta}([\Gamma, v])$ is clear. In order to prove that such upper bound cannot be improved consider $K < k^{1/p}$ a real number. Notice that we can always find a collection of $k$ distinct rational numbers $\{r_1, \ldots, r_k\}$ such that

$$K < \left( \sum_{i=1}^{k} r_i^p \right)^{1/p} < k^{1/p}.$$

We can find a sentence $\phi_{1-r_i}$ that is maximally Ł$_{1-r_i}$-consistent for each $i \in \{1, \ldots, k\}$ (Proposition 8). We can assume that

$$L_{\phi_{1-r_i}} \cap L_{\phi_{1-r_j}} = \emptyset$$

for $i \neq j$. Consider now

$$[\Gamma, v] = \{\langle \phi_{1-r_i}, 1 \rangle | i \in \{1, \ldots, k\}\}.$$

We will have that $DC^p([\Gamma, v]) = (\sum_{i=1}^{k} r_i^p)^{1/p}$. □

**Corollary 20.** *If* $S^\infty_{\Delta}([\Gamma, v]) \neq \emptyset$ *then* $DC^\infty_{\Delta}([\Gamma, v]) \le 1$ *and the bound* 1 *cannot be improved*.

Next we want to prove that the operators $DC^{p_1}$ and $DC^{p_2}$, for $p_1 \neq p_2$, are *essentially* distinct, by which we mean that the difference between $DC^{p_1}$ and $DC^{p_2}$ does not reduce only to the (possible) difference in magnitude of the values they assign to a certain knowledge base in *FL* but also to the ordering they induce on *FL*. An identical result holds also for $\overline{DC}^{p_1}$ and $\overline{DC}^{p_2}$, as can be easily seen from the proof that follows.

**Proposition 21.** *$DC^{p_1}$ and $DC^{p_2}$ induce distinct orderings on FL.*

**Proof.** Let us assume that $p_1 < p_2$ and that $p_1, p_2 \in [1, \infty)$. Let us consider the set $[\Gamma_1, v_1] = \{\langle \top, 0\rangle\}$. We will have that

$$DC^{p_1}([\Gamma_1, v_1]) = DC^{p_2}([\Gamma_1, v_1]) = 1.$$

Consider now the set

$$[\Gamma_2, v_2] = \{\langle \phi_{1-\lambda}, 1\rangle, \langle \phi_{1-\mu}, 1, \rangle\},$$

with $\lambda, \mu \in \mathbb{Q} \cap [0, 1]$, where $\phi_{1-\lambda}$ is maximally Ł$_{1-\lambda}$-consistent and $\phi_{1-\mu}$ is maximally Ł$_{1-\mu}$-consistent (as defined in Proposition 8). We will also assume that $L_{\phi_{1-\mu}} \cap L_{\phi_{1-\lambda}} = \emptyset$.

We will have that

$$DC^{p_1}([\Gamma_2, v_2]) = (\lambda^{p_1} + \mu^{p_1})^{1/p_1}$$

and that

$$DC^{p_2}([\Gamma_2, v_2]) = (\lambda^{p_2} + \mu^{p_2})^{1/p_2}.$$

First notice that, for all $\lambda, \mu \in \mathbb{Q} \cap (0, 1]$, it is the case that

$$(\lambda^{p_1} + \mu^{p_1})^{1/p_1} > (\lambda^{p_2} + \mu^{p_2})^{1/p_2}.$$

Notice also that there need to exist values $\lambda, \mu \in \mathbb{Q} \cap [0, 1]$ such that

$$(\lambda^{p_1} + \mu^{p_1})^{1/p_1} > 1 > (\lambda^{p_2} + \mu^{p_2})^{1/p_2}$$

since $p_1 \in [1, \infty)$. This proves that $DC^{p_1}$ and $DC^{p_2}$ induce distinct orderings on *FL*. □

In simple words, what Proposition 21 tells us is that, for any two knowledge bases $[\Gamma_1, v_1]$ and $[\Gamma_2, v_2]$ in *FL* we can have that $[\Gamma_1, v_1]$ is *more* inconsistent than $[\Gamma_2, v_2]$ according to $DC^{p_1}$ but that $[\Gamma_1, v_1]$ is *less* inconsistent than $[\Gamma_2, v_2]$ according to $DC^{p_2}$, for $p_1 \neq p_2$.

## 7. The computation of $DC^p$ and $R$

In this section we deal with the computation of the $p$-distance of a general knowledge base $[\Gamma, v] \in FL$. In particular, we show the connection between its computation and the solution to certain constrained optimization problems.

Let us consider $[\Gamma, v]$, with $\Delta' = \Gamma - \Delta$, $\Delta \subset \Gamma$,

$$[\Delta', v_{|\Delta'}] = \{v(\phi_1) = [\underline{\eta}_1, \overline{\eta}_1], \ldots, v(\phi_k) = [\underline{\eta}_k, \overline{\eta}_k]\}$$

and

$$[\Delta, v_{|\Delta}] = \{v(\phi_{k+1}) = [\underline{\eta}_{k+1}, \overline{\eta}_{k+1}], \ldots, v(\phi_{k+t}) = [\underline{\eta}_{k+t}, \overline{\eta}_{k+t}]\},$$

for some $t \in \{0\} \cup \mathbb{N}$ and $k \in \mathbb{N}$.

We can assume, without loss of generality, that $L = \bigcup_{i=1}^{k+t} L_{\phi_i}$.

Consider the following constrained optimization problem with *optimization variable* the vector $\vec{x} \in \mathbb{R}^{2k}$ and $p \in [1, \infty]$:

$$\text{minimize } \|\vec{x}\|_p \tag{1}$$

subject to the following constraints:

- $\underline{\eta}_i - x_i \leq f_{\phi_i}(\vec{y}) \leq \overline{\eta}_i + x_{i+k}$ for each $i \in \{1, \ldots, k\}$,
- $\underline{\eta}_{k+i} \leq f_{\phi_{k+i}}(\vec{y}) \leq \overline{\eta}_{k+i}$ for each $i \in \{1, \ldots, t\}$,
- $\vec{y} \in \mathbb{D}_l$ (i.e., $0 \leq y_i \leq 1$ for each $i \in \{1, \ldots, l\}$),

- $x_i \geq 0$ for each $i \in \{1, \ldots, 2k\}$,
- $0 \leq \underline{\eta}_i - x_i \leq \overline{\eta}_i + x_{i+k} \leq 1$ for each $i \in \{1, \ldots, k\}$.

The value attained by the optimization variable $\vec{x}$ in (1) represents a possible repair of $[\Gamma, v]$ (an element in $R_\Delta([\Gamma, v])$) and $\vec{y}$ represents an Ł-valuation on $L$.

Let us denote the constrained optimization problem (1) by $\mathcal{C}^p_\Delta([\Gamma, v])$. We define $\mathcal{SC}^p_\Delta([\Gamma, v])$, the solution to $\mathcal{C}^p_\Delta([\Gamma, v])$, as follows:

$$\mathcal{SC}^p_\Delta([\Gamma, v]) = \inf_{\vec{y} \in \mathbb{D}_l} \{\|\vec{x}\|_p | (\vec{x}, \vec{y}) \in \mathbb{R}^{2k+l} \text{ is feasible}\}.$$

By $(\vec{x}, \vec{y})$ being feasible we mean that $(\vec{x}, \vec{y})$ satisfies the constraints in $\mathcal{C}^p_\Delta([\Gamma, v])$. The collection of all such vectors is called the *feasible set* (of $\mathcal{C}^p_\Delta([\Gamma, v])$)—see, for example, [6] for more on these concepts and, in general, on the terminology and basic definitions for constrained optimization problems. Notice that $\mathcal{SC}^p_\Delta([\Gamma, v])$ does not exist if the feasible set of $\mathcal{C}^p_\Delta([\Gamma, v])$ is empty.

It is clear from the definition of $DC^p_\Delta([\Gamma, v])$ and Proposition 14 that, if $S^p_\Delta([\Gamma, v]) \neq \emptyset$,

$$\mathcal{SC}^p_\Delta([\Gamma, v]) = DC^p_\Delta([\Gamma, v])$$

and that an *optimal point* $(\vec{x}, \vec{y}) \in \mathbb{R}^{2k+l}$ always exists (that is to say, a point $(\vec{x}, \vec{y})$ at which $\mathcal{SC}^p_\Delta([\Gamma, v])$ is attained). As is clear, $\vec{x}$ in an optimal point $(\vec{x}, \vec{y})$ yields a $p$-optimal repair of $[\Gamma, v]$.

Notice that the functions of the form $f_\phi$ in $\mathcal{C}^p_\Delta([\Gamma, v])$, McNaughton functions, need not be convex, let alone linear. That places our problem $\mathcal{C}^p_\Delta([\Gamma, v])$ within *non-convex* optimization grounds and, unfortunately, there are no *effective* methods to deal with non-convex optimization problems for the general case—see for example [5,6] for more on these issues. However, our case is peculiar in that the inequalities in $\mathcal{C}^p_\Delta([\Gamma, v])$ are either linear or *close* to linearity. In particular, McNaughton functions are *piecewise* linear, only the min operator breaks linearity (we assume that only basic logical symbols are used, i.e., $\rightarrow$ and $\bot$).

Our aim is to express the feasible set of $\mathcal{C}^p_\Delta([\Gamma, v])$ as the union of a collection of sets defined only by linear constraints and thus place our problem in the domain of convex optimization and, in particular for the 1-distance computation, linear optimization since, as is well known, there exist effective and efficient algorithms for convex optimization—see [6]. We can do so in a pretty trivial way by observing that, for general functions $g, h$ with domain in $\mathbb{R}^l$, the set

$$\{\vec{y} | \min\{g(\vec{y}), h(\vec{y})\} \in \mathcal{I}\},$$

with $\mathcal{I} \in \mathbb{I}$, is equivalent to the union of sets

$$\{\vec{y} | g(\vec{y}) \in \mathcal{I}, g(\vec{y}) \leq h(\vec{y})\} \cup \{\vec{y} | h(\vec{y}) \in \mathcal{I}, h(\vec{y}) \leq g(\vec{y})\}.$$

Notice that for $g, h$ linear the constraints in the sets $\{g(\vec{y}) \in \mathcal{I}, g(\vec{y}) \leq h(\vec{y})\}$ and $\{h(\vec{y}) \in \mathcal{I}, h(\vec{y}) \leq g(\vec{y})\}$ are linear too.

Let us consider a (singleton) constraint set of the form $\{f_\phi(\vec{y}) \in \mathcal{I}\}$, where $f_\phi$ is a McNaughton function, with $\mathcal{I} \in \mathbb{I}$ and $\vec{y} \in \mathbb{R}^l$ (for a constraint set of the form $\{f_\phi(\vec{y}) \in [\underline{\eta} - x_1, \overline{\eta} + x_2]\}$, with $\underline{\eta}, \overline{\eta} \in [0, 1]$ and $x_1, x_2$ variables, the process that we next describe would work exactly the same way). We pick one of the innermost min operators occurring in $f_\phi$ (that is to say, that there is no other min operator within the scope of the chosen one). Let us assume that the arguments of the min operator chosen are $g, h$. We eliminate such operator by generating two new sets of constraints: the first one,

$$C^1_\phi(\vec{y}) = \{f^1_\phi(\vec{y}) \in \mathcal{I}, g(\vec{y}) \leq h(\vec{y})\},$$

where $f^1_\phi(\vec{y})$ is obtained from $f_\phi(\vec{y})$ by replacing $\min\{g(\vec{y}), h(\vec{y})\}$ for $g(\vec{y})$, and the second one,

$$C^2_\phi(\vec{y}) = \{f^2_\phi(\vec{y}) \in \mathcal{I}, h(\vec{y}) \leq g(\vec{y})\},$$

where $f^2_\phi(\vec{y})$ is obtained from $f_\phi(\vec{y})$ by replacing $\min\{g(\vec{y}), h(\vec{y})\}$ for $h(\vec{y})$. We proceed in the same way for each new set of constraints obtained at each step in the iterative process until all the min operators are eliminated. The result will be a collection of sets of constraints, which we will denote by $\mathcal{G}_\phi$, that do not contain any min operator and the union of the sets defined by the distinct sets of constraints in $\mathcal{G}_\phi$ will define the same set as the initial constraint $f_\phi(\vec{y}) \in \mathcal{I}$. More formally,

$$\{\vec{y} | f_\phi(\vec{y}) \in \mathcal{I}\} = \bigcup_{C_\phi(\vec{y}) \in \mathcal{G}_\phi} \{\vec{y} | C_\phi(\vec{y})\}.$$

We can view this process as the generation of a tree where the nodes are given by the distinct constraint sets generated, with *root* $\{f_\phi(\vec{y}) \in \mathcal{I}\}$ and *leaves* the constraint sets in $\mathcal{G}_\phi$. An iteration in the process corresponds to the generation of exactly two new constraint sets that arise from the elimination of an innermost min operator in a previously obtained constraint set (in the tree, the *parent* of the two new constraint sets) in the way indicated above. The process terminates when the leaves of the generated tree have no min operators left.

Let us consider the collections of sets of constraints $\mathcal{G}_{\phi_1}, \ldots, \mathcal{G}_{\phi_{k+t}}$, one for each sentence in $\Gamma$. We define $\mathcal{G}$ to be the set given by the distinct collections of constraints obtained by the union of exactly one set of constraints in each collection $\mathcal{G}_{\phi_i}$, for $i \in \{1, \ldots, k+t\}$ (i.e., a constraint set in $\mathcal{G}$ consists of the union of exactly one constraint set in each $\mathcal{G}_{\phi_i}$).

We define, for each set of constraints $C^i(\vec{x}, \vec{y}) \in \mathcal{G}$, for $i \in \{1, \ldots, |\mathcal{G}|\}$, the following optimization problem with optimization variable the vector $\vec{x} \in \mathbb{R}^{2k}$ and $p \in [1, \infty]$:

$$\text{minimize} \quad \|\vec{x}\|_p \tag{2}$$

subject to the following constraints:

- $C^i(\vec{x}, \vec{y})$,
- $\vec{y} \in \mathbb{D}_l$ (i.e., $0 \le y_i \le 1$ for each $i \in \{1, \ldots, l\}$),
- $x_i \ge 0$ for each $i \in \{1, \ldots, 2k\}$,
- $0 \le \underline{\eta}_i - x_i \le \overline{\eta}_i + x_{i+k} \le 1$ for each $i \in \{1, \ldots, k\}$.

Let us denote each convex optimization problem of this form by $\mathcal{C}^{p,i}_\Delta([\Gamma, v])$, for $i \in \{1, \ldots, |\mathcal{G}|\}$, and define its solution as we did for (1) in the following way:

$$\mathcal{SC}^{p,i}_\Delta([\Gamma, v]) = \inf_{\vec{y} \in \mathbb{D}_l} \{\|\vec{x}\|_p | (\vec{x}, \vec{y}) \in \mathbb{R}^{2k+l} \text{ is feasible}\}.$$

Next we prove that $\mathcal{SC}^{p,i}_\Delta([\Gamma, v])$, if it exists, is always attained by a $p$-optimal value $\vec{x} \in \mathbb{R}^{2k}$.

**Proposition 22.** *If the feasible set of $\mathcal{C}^{p,i}_\Delta([\Gamma, v])$ is not empty then $\mathcal{C}^{p,i}_\Delta([\Gamma, v])$ has an optimal point $\vec{x} \in \mathbb{R}^{2k}$.*

**Proof.** The result follows by an argument similar to that in the proof of Proposition 14, i.e., from compactness of the feasible set of $\mathcal{C}^{p,i}_\Delta([\Gamma, v])$ and continuity of $\|\vec{x}\|_p$. □

The next result summarizes the relation between our collection of convex optimization problems of the form $\mathcal{C}^{p,i}_\Delta([\Gamma, v])$, for $i \in \{1, \ldots, |\mathcal{G}|\}$, our initial problem $\mathcal{C}^p_\Delta([\Gamma, v])$ and $DC^p_\Delta([\Gamma, v])$.

**Proposition 23.** *We have the following identity*:

$$DC^p_\Delta([\Gamma, v]) = \min\{\mathcal{SC}^{p,i}_\Delta([\Gamma, v]) | i \in \{1, \ldots, |\mathcal{G}|\}\}.$$

*If $\{\mathcal{SC}^{p,i}_\Delta([\Gamma, v]) | i \in \{1, \ldots, |\mathcal{G}|\}\} = \emptyset$ then $DC^p_\Delta([\Gamma, v]) = \infty$.*

**Proof.** The result follows from the fact that the union of the feasible sets of the optimization problems $\mathcal{C}^{p,i}_\Delta([\Gamma, v])$, for $i \in \{1, \ldots, |\mathcal{G}|\}$, is equal to the feasible set of $\mathcal{C}^p_\Delta([\Gamma, v])$. □

Summarizing, the computation of $DC^p_\Gamma([\Delta, v])$ reduces to finding the solution of a collection of convex optimization problems (for which there exist efficient algorithms). It is worth mentioning though that this collection of problems, at least in the worst case scenario, shows an exponential growth with respect to the min operators in the McNaughton functions that correspond to the sentences in $\Gamma$ and with respect to the number of sentences in $\Gamma$ itself. Notice that, for $\phi \in \Gamma$ and $k \in \mathbb{N}$ the number of min operators in $f_\phi$, the number of constraint sets that we need to generate, i.e., the cardinality of $\mathcal{G}_\phi$, is $2^k$ (that at least in principle, yet it will not be necessarily so in most cases. At each step in the generation of our tree structure for the constraint sets in $\mathcal{G}_\phi$ a *branch* corresponding to a certain constraint set can be *closed* whenever such set is known to be unfeasible. That could reduce the number of *leaves* in the final tree and thus the cardinality of $\mathcal{G}_\phi$ substantially). Notice also that $\mathcal{G}$ shows an exponential growth in the cardinality of $\Gamma$. $\mathcal{G}$ consists of all the unions of exactly one constraint set for each $\mathcal{G}_\phi$, for $\phi \in \Gamma$. However, in most cases many such unions will be unfeasible and thus the effective size of $\mathcal{G}$ would be reduced.

## 8. The medical expert system CADIAG2

The main motivation for the work presented in this paper is related to the medical expert system CADIAG2 and stems from the possibility of *inconsistency* in the (possibly) *vague* information processed by its inference mechanism. In this section we briefly introduce the system CADIAG2 and justify the potential usefulness of our theory in relation to CADIAG2 and, in general, CADIAG2-like systems.

CADIAG2 (Computer Assisted DIAGnosis) is a well-known rule-based expert system, presented in some literature as an example of a fuzzy expert system—see for example [12] or [24]—which is aimed at providing support in diagnostic decision making in the field of internal medicine. Its design and construction was initiated in the early 1980s at the Medical University of Vienna by K.P. Adlassnig—see [1–3,16] for more on the origins and design of CADIAG2.

CADIAG2 consists of two fundamental pieces: the knowledge base and the inference engine.

- *Knowledge base*: The knowledge base consists of a set of *IF–THEN* rules intended to represent certain and uncertain relationships between distinct medical entities: symptoms, findings, signs and test results (to which we will commonly refer as *symptoms*) on the one hand and diseases and therapies (to which we will commonly refer as *diseases*) on the other.

  The set of symptoms we will denote by $S$ and the set of diseases by $D$. We will identify $S \cup D$ with our propositional language $L$ in this section. Rules in CADIAG2 can be formalized as pairs of the form $\langle \theta \Rightarrow \phi, \eta \rangle$, where $\theta \in SL$ is the *antecedent*, $\phi \in SL$ the *consequent* and $\eta \in [0, 1]$ is intended to represent the degree to which $\theta$ confirms $\phi$.[3]

  It is convenient to distinguish among two main types of rules in CADIAG2:

  - *Binary rules*: Rules of the form $\langle q_1 \Rightarrow q_2, \eta \rangle$, with $q_1, q_2 \in L$ and $\eta \in [0, 1]$.
  - *Compound rules*: Rules of the form $\langle \theta \Rightarrow q, \eta \rangle$, with $\theta \in SL$, $q \in D$ and $\eta \in [0, 1]$.

  We stress the fact that the antecedent of some compound rules in CADIAG2 is represented by propositions in *SL* (formed by means of *conjunction*, *disjunction* and *negation* from *L*).

  For a more detailed description of CADIAG2's knowledge base one can look at, for example, [1,2,7].
- *Inference engine*: The inference engine is based on methodology from *fuzzy set theory*, in the sense of [22,23]. To be more precise, the inference mechanism of CADIAG2 rests mostly on a slight modification[4] of *fuzzy modus ponens*—see, for example, [9].

  For a detailed description of the inference process in CADIAG2 and alternative formalizations one can look at, for example, [1,7,19].

A run of the inference engine in CADIAG2 gets started from a set $\Gamma$ of medical entities in *SL* present in the patient along with some numbers in the interval [0,1] that represent their respective *degrees of presence* and are motivated by the possibly vague nature of the medical entities in $\Gamma$. Examples of vague entities that occur in CADIAG2 are '*reduced glucose in serum*' or '*normal fever*'.

We can thus represent our initial set of entities $\Gamma$ along with their respective degrees of presence in the patient by $[\Gamma, v] \in FL$, where $v$ is the assignment on $\Gamma$ given by the corresponding degrees of presence.

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[3] An interpretation of such degrees of confirmation in terms of a fuzzy semantics such as that suggested in [7] for the rules of CADIAG2 (thus not as certainty degrees but as degrees of truth), would also make the knowledge base of the system suitable for our approach.

[4] Such modification is due to the consideration of both 0 and 1 as maximal elements in the interval [0,1].

Degrees of presence of Boolean combinations of basic entities in $L$ in the patient are assumed to behave according to the following identities for conjunction, disjunction and negation, for general $\phi, \theta \in SL$:

- $v(\phi \wedge \theta) = \min\{v(\phi), v(\theta)\}$.
- $v(\phi \vee \theta) = \max\{v(\phi), v(\theta)\}$.
- $v(\neg\phi) = 1 - v(\phi)$.

It is of no central interest to us how the set $\Gamma$ of medical entities is initially identified and how their respective degrees of presence obtained in order for the system to start its inference engine—for more on this see [1,2]. What we want to stress is that, at least in principle, some of the medical entities in $\Gamma$ could be represented by combinations of propositions in $L$ that are not themselves (or at least not all) in $\Gamma$ (we could have, as a hypothetical example, a proposition of the form $\phi \vee \theta \in \Gamma$, with $\phi \notin \Gamma$, where $v(\phi \vee \theta)$ could be obtained not functionally from $v(\phi)$ and $v(\theta)$—which we might not know—but from the results of a medical test that yields a degree of presence of $\phi \vee \theta$ or by direct evaluation of the doctor). Such a *direct* evaluation of the degrees of presence of propositions in *SL* makes $[\Gamma, v]$ susceptible of *non-trivial* inconsistencies (i.e., not only of inconsistencies that originate from conflicting atomic graded statements of the form $(q, \eta_1), \dots, (q, \eta_k)$, for distinct values $\eta_1, \dots, \eta_k \in [0, 1]$, for $k \in \mathbb{N}$ and $q \in L$).

All that said, our approach proves suitable for evaluating and repairing inconsistency in knowledge bases with graded truth like the one with which the inference engine in CADIAG2 gets started. It is of course difficult to assess the usefulness of doing so (at least for inferential purposes) in fuzzy expert systems like CADIAG2 where the inference is based on the compositional rule and thus *ad hoc* in the sense that there is no known or clear semantics that motivates it. The inference engine of the system does not explode in the presence of inconsistency yet it is the case that a large amount of inconsistency and substantial changes induced by a possible repair for it can have a large influence in its inferential outcome. Such outcome is, given the *ad hoc* nature of the inference, certainly hard to interpret but, nevertheless, even in the context of CADIAG2 and its *unsound* inference, evaluating such possibly inconsistent information would help us understanding it better and would allow us to assess the impact it would have in the inferential outcome of the system. Starting with consistent information about the patient would allow us to be sure that no conflicting information is carried through the inference process to the inferential outcomes.

## 9. Conclusion and future work

We have presented and analyzed a family of measures (*p*-measures) aimed at helping the modeler in evaluating inconsistency in knowledge bases with graded truth, i.e., knowledge bases that consist of propositions that are not necessarily fully true and that occur in the knowledge base together with a degree of truth or set of degrees of truth. Our family of *p*-measures constitutes a graded approach to quantifying inconsistency in the sense that we take into account minimal adjustments in the degrees of truth that make the knowledge base consistent within the frame of Łukasiewicz semantics. We have also seen that the computation of the measures here presented yield suitable repairs to bring or restore consistency in such knowledge bases and that such computation reduces to finding the solution of a certain collection of convex optimization problems. Information processing in a well-known fuzzy medical expert system (CADIAG2) was described in the last section in order to motivate and justify most of the theory in this paper.

Much is left to be done about *p*-measures and related repairs. In particular, a deeper analysis of our *p*-measures and optimal repairs aimed at identifying reasonable criteria in order to facilitate the choice of distinct values of *p* would be desirable. Likewise, criteria in order to discriminate among alternative optimal repairs associated to a particular *p*-measure should be also identified. In this respect, some insight has already been provided in relation to our examples in Section 5 yet more clear, general criteria for the choice of alternative values of *p* and corresponding repairs would be desirable. Such an analysis will likely be taken as future work.

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## References


[1] K.P. Adlassnig, G. Kolarz, W. Scheithauer, H. Grabner, Approach to a hospital-based application of a medical expert system, Med. Inf. 11 (3) (1986) 205–223.
[2] K.P. Adlassnig, G. Kolarz, W. Scheithauer, H. Effenberger, H. Grabner, CADIAG: approaches to computer-assisted medical diagnosis, Comput. Biol. Med. 15 (1985) 315–335.
[3] K.P. Adlassnig, Fuzzy set theory in medical diagnosis, IEEE Trans. Syst. Man Cybern. 16 (2) (1986) 260–265.
[4] O. Ariely, M. Denecker, M. Bruynooghe, Distance semantics for database repairs, Ann. Math. Artif. Intell. 50 (3/4) (2007) 389–415.
[5] D. Bertsekas, Nonlinear Programming, Athena Scientific, 1999.
[6] S. Boyd, L. Vandenberghe, Convex Optimization, Cambridge University Press, 2004.
[7] A. Ciabattoni, T. Vetterlein, On the fuzzy (logical) content of CADIAG2, Fuzzy Sets Syst. 161 (14) (2009) 1941–1958.
[8] J. Grant, A. Hunter, Measuring inconsistency in knowledge bases, J. Intell. Inf. Syst. 27 (2006) 159–184.
[9] P. Hájek, Metamathematics of Fuzzy Logic, Kluwer Academic Publishers, 1998.
[10] A. Hunter, S. Konieczny, Measuring inconsistency through minimal inconsistent sets, in: KR Proceedings, AAAI Press, 2008, pp. 358–366.
[11] A. Hunter, S. Konieczny, Approaches to measuring inconsistent information, Lecture Notes in Computer Science, vol. 3300, Springer, Berlin, Heidelberg, 2004, pp. 189–234.
[12] G. Klir, T.A. Folger, Fuzzy Sets, Uncertainty and Information, Prentice-Hall International, 1988.
[13] P. Klinov, B. Parsia, D. Picado Muiño, The consistency of the medical expert system CADIAG2: a probabilistic approach, Journal of Information Technology Research 4 (1) (2011) 1–20 (Special issue on A. Rodríguez González, M.A. Mayer (Eds.), Medical Diagnosis Systems).
[14] K.M. Knight, Measuring inconsistency, J. Philos. Logic 31 (2002) 77–98.
[15] S. Konieczny, J. Lang, P. Marquis, Quantifying information and contradiction in propositional logic through epistemic actions, in: IJCAI Proceedings, 2003, pp. 106–111.
[16] H. Leitich, K. Adlassnig, G. Kolarz, Evaluation of two different models of semiautomatic knowledge acquisition for the medical consultant system CADIAG2/RHEUMA, Artif. Intell. Med. 25 (2002) 215–225.
[17] R. McNaughton, A theorem about infinite-valued sentential logic, J. Symb. Logic 16 (1) (1951) 1–13.
[18] D. Picado Muiño, Deriving Information from Inconsistent Knowledge Bases: A Probabilistic Approach, Ph.D. Thesis, School of Mathematics, The University of Manchester, 2008.
[19] D. Picado Muiño, A probabilistic interpretation of the medical expert system CADIAG2, Soft Comput. 15 (10) (2011) 2013–2020.
[20] D. Picado Muiño, Measuring and repairing inconsistency in probabilistic knowledge bases, Int. J. Approx. Reason. 52 (6) (2011) 828–840.
[21] M. Thimm, Measuring inconsistency in probabilistic knowledge bases, in: UAI Proceedings, 2009.
[22] L.A. Zadeh, Fuzzy sets, Inf. Control 8 (1965) 338–353.
[23] L.A. Zadeh, Fuzzy logic and approximate reasoning, Synthese 30 (1975) 407–428.
[24] H.J. Zimmermann, Fuzzy Set Theory and its Applications, Kluwer Academic Publisher, Boston, Dordrecht, London, 1991.