1) Problem Template (Geometry $\to$ Linear Maps)

Given thrusters $(\mathbf{r}_i, \mathbf{a}_i)$, throttles $\mathbf{t} \in \mathbb{R}^m$ with bounds

$$\boldsymbol{\ell} \leq \mathbf{t} \leq \mathbf{u}.$$

Force:

$$\mathbf{F} = A\,\mathbf{t} \in \mathbb{R}^3, \quad A = [\,\mathbf{a}_1 \ \cdots \ \mathbf{a}_m\,].$$

Torque:

$$\mathbf{T} = B\,\mathbf{t} \in \mathbb{R}^3, \quad B = [\,\mathbf{r}_1 \times \mathbf{a}_1 \ \cdots \ \mathbf{r}_m \times \mathbf{a}_m\,].$$

Optional hard linear constraints:

$$C\mathbf{t} + \mathbf{s} = \mathbf{b}, \qquad \mathbf{s} \geq 0.$$

You will solve several staged LPs

$$P_1 \succ P_2 \succ \cdots$$

using lexicographic goal programming.

2) Goals (All Linearizable)

Directional torque about unit axis $\mathbf{d}_T$:

$$\max\ \mathbf{d}_T^\top \mathbf{T} = (\mathbf{d}_T^\top B)\,\mathbf{t}.$$

Directional force along $\mathbf{d}_F$:

$$\max\ \mathbf{d}_F^\top \mathbf{F} = (\mathbf{d}_F^\top A)\,\mathbf{t}.$$

L1 magnitudes: introduce deviations $\mathbf{u} \geq 0$; e.g., minimize $\|\mathbf{T}\|_1$ using

$$-\mathbf{u} \leq B\mathbf{t} \leq \mathbf{u}, \qquad \min \mathbf{1}^\top \mathbf{u}.$$

Total thrust (fuel proxy): minimize $\mathbf{1}^\top \mathbf{t}$ or weighted sum $\mathbf{w}^\top \mathbf{t}$.

3) Variables' Status Inside the Simplex

Each original decision variable t_j has a status:

- LB if $t_j = \ell_j$ (nonbasic at lower bound),
- UB if $t_j = u_j$ (nonbasic at upper bound),
- B if basic (interior; except degeneracy).

Slack variables from $C\mathbf{t} + \mathbf{s} = \mathbf{b}$ are basic candidates with $\ell = 0$, $u = +\infty$.
This keeps real bounds inside the algorithm—no substitutions like $t = \ell + \tilde{t}$.

4) Reduced-Cost Eligibility Rules

Let r_j be the reduced cost of nonbasic t_j.

Maximization:

$$\text{LB variable enters by increasing if } r_j > 0; \qquad \text{UB variable enters by decreasing if } r_j < 0.$$

Minimization:

$$\text{LB variable enters by increasing if } r_j < 0; \qquad \text{UB variable enters by decreasing if } r_j > 0.$$

Use tolerance ε_{rc} (e.g. 10^{-10}) to ignore noise.

5) Augmented Ratio Test With Opposite Bound

Suppose t_e is eligible to enter.

Direction:

$$d_e = +1 \text{ if increasing from LB}, \qquad d_e = -1 \text{ if decreasing from UB}.$$

Let

$$\mathbf{d} := B^{-1} A_{:e},$$

so basics change as:

$$\Delta \mathbf{t}_B = -\,\mathbf{d}\,\Delta, \qquad \Delta t_e = d_e\,\Delta.$$

Row limits (a basic hits a bound):

If $d_e = +1$:

$$\theta_i = \begin{cases} \dfrac{t_{B_i} - \ell_{B_i}}{d_i}, & d_i > 0, \\ \dfrac{u_{B_i} - t_{B_i}}{-d_i}, & d_i < 0. \end{cases}$$

If $d_e = -1$: swap LB/UB and signs.

Opposite bound of entering:

$$\theta_{\text{bound}} = \begin{cases} u_e - t_e, & d_e = +1, \\ t_e - \ell_e, & d_e = -1. \end{cases}$$

Let

$$\theta = \min\{\theta_{\text{bound}},\ \min_i \theta_i\}.$$

If $\theta = \theta_{\text{bound}}$ strictly smaller (preemptive bound hit):

$$t_e \leftarrow \text{opposite bound}, \qquad \mathbf{t}_B \leftarrow \mathbf{t}_B - \mathbf{d}\,\theta\,d_e.$$

No pivot. Recompute reduced costs and continue.

If a row ties or wins: pivot that row/column. Entering becomes B, leaving becomes LB/UB.

If $\theta = 0$: degeneracy. If row-limited, pivot; if bound-limited, just flip t_e's status.

Anti-cycling: use Bland's rule.

6) Feasible Start (Phase I, if Needed)

If $\mathbf{t} = \boldsymbol{\ell}$ doesn't satisfy equalities:

Add artificial $\mathbf{a} \geq 0$ to $E\mathbf{t} = \mathbf{f}$:

$$E\mathbf{t} + \mathbf{a} = \mathbf{f}.$$

Phase I objective: minimize $\mathbf{1}^\top \mathbf{a}$ (or maximize $-\mathbf{1}^\top \mathbf{a}$).

Run bounded simplex. If optimum > 0: infeasible. Else remove artificials, keep resulting basis, proceed to P_1.

7) Lexicographic (Preemptive) Loop

For priorities $P_1 \succ P_2 \succ \cdots \succ P_K$:

Stage k:

1. Build stage-k LP.
2. Solve $\to$ value Z_k^*, solution $\mathbf{t}^{(k)}$.
3. Lock achievement for next stage:
    - Equality goal: add as equality row.
    - Max goal $g^\top \mathbf{t}$: lock $g^\top \mathbf{t} \geq Z_k^* - \delta$.
    - Min goal $h^\top \mathbf{t}$: lock $h^\top \mathbf{t} \leq Z_k^* + \delta$.

    Convert $\leq$ / $\geq$ to equalities with slacks. Use tiny δ.
4. Warm-start next stage with current basis/statuses.

8) What to Record in a Tableau

At each iteration, keep:

- Row equations for basics with RHS.
- For each nonbasic t_j: value (LB or UB), reduced cost r_j.
- Status tags LB/UB/B for every variable.
- Objective row with reduced costs.
- Step decision: entering variable, direction ($\uparrow$ from LB or $\downarrow$ from UB), step θ, bound hit or pivot (and which row left).

This is exactly what is needed to reproduce the full tableau with LB/UB/B tags.

9) Negative Lower Bounds & Other Details

Formulas use distances $(t - \ell)$ and $(u - t)$, so negative ℓ is fine.

Unbounded sides: treat as $+\infty$ distance.

Scaling: rescale rows/columns if magnitudes are large.

Neutral directions: reduced cost $= 0$ gives multiple optima; any representative works or use tiny tie-break weight.

10) Common Two-Stage Pattern

Stage P_1 (Min Thrust)

$$\min\ \mathbf{1}^\top \mathbf{t} \quad \text{s.t.} \quad \mathbf{d}_T^\top B\,\mathbf{t} \geq \tau_{\min},\ \boldsymbol{\ell} \leq \mathbf{t} \leq \mathbf{u},\ C\mathbf{t} + \mathbf{s} = \mathbf{b}.$$

Implement $\mathbf{d}_T^\top B\mathbf{t} \geq \tau_{\min}$ as

$$-\,\mathbf{d}_T^\top B\,\mathbf{t} + s_\tau = -\tau_{\min}, \qquad s_\tau \geq 0.$$

Solve with bounded simplex $\Rightarrow Z_1^*$.
Lock using equality $\mathbf{1}^\top \mathbf{t} = Z_1^*$ (or $\leq Z_1^* + \delta$).

Stage P_2 (Max Torque)

$$\max\ \mathbf{d}_T^\top B\,\mathbf{t} \quad \text{s.t.} \quad \mathbf{1}^\top \mathbf{t} = Z_1^*,\ \boldsymbol{\ell} \leq \mathbf{t} \leq \mathbf{u},\ C\mathbf{t} + \mathbf{s} = \mathbf{b}.$$

Run bounded simplex; you may see preemptive bound hits or row pivots as in examples.

11) Stopping & Optimality

For a stage:
If no nonbasic variable satisfies eligibility (Sec. 4) $\Rightarrow$ optimal.
If reduced costs ≈ 0: you're on a face of optimal solutions; any LB/UB reassignments that keep feasibility preserve optimality.