Preemptive Goal Linear Programming for Force-Neutral Thruster Firing

1 Thruster Geometry

For thruster $i = 1, \dots, n$:

- Position (lever arm) in body frame: $\mathbf{r}_i \in \mathbb{R}^3$
- Unit direction vector: $\mathbf{d}_i \in \mathbb{R}^3$
- Thrust magnitude (decision variable): $t_i \geq 0$

Resultant body-frame force and torque:

$$\mathbf{F} = \sum_{i=1}^{n} t_i \mathbf{d}_i, \tag{1}$$

$$\boldsymbol{\tau} = \sum_{i=1}^{n} t_i (\mathbf{r}_i \times \mathbf{d}_i), \tag{2}$$

where

$$\mathbf{h}_i := \mathbf{r}_i \times \mathbf{d}_i$$

is the torque arm vector for thruster i.

2 Linear Constraints with Goal Variables

We introduce positive and negative deviation variables to maintain linearity.

Primary goal: Force neutralization

$$\sum_i d_{i,k} t_i + e_k^{F,-} - e_k^{F,+} = 0, \qquad k \in \{x, y, z\}. \tag{3}$$

Secondary goal: Desired torque

$$\sum_i h_{i,k} t_i + e_k^{\tau,-} - e_k^{\tau,+} = \tau_k^{\text{des}}, \qquad k \in \{x, y, z\}. \tag{4}$$

Bounds

$$0 \leq t_i \leq \overline{T}_i, \qquad e_k^{F,\pm} \geq 0, \qquad e_k^{\tau,\pm} \geq 0. \tag{5}$$

3 Preemptive (Lexicographic) Optimization

We use a lexicographic (preemptive) sequence of linear programs.

LP-1: Force minimization (highest priority)

$$\begin{aligned}
\min \quad & \sum_k \left(e_k^{F,+} + e_k^{F,-}\right) && (6)\\
\text{s.t.} \quad & \text{All constraints above.} && (7)
\end{aligned}$$

Let the optimal value be $Z_F^\star$.

LP-2: Torque tracking/minimization (lower priority)

$$\begin{aligned}
\min \quad & \sum_k \left(e_k^{\tau,+} + e_k^{\tau,-}\right) && (8)\\
\text{s.t.} \quad & \sum_k \left(e_k^{F,+} + e_k^{F,-}\right) \le Z_F^\star, && (9)\\
& \text{All previous constraints.} && (10)
\end{aligned}$$

Let the optimal value be $Z_\tau^\star$.

LP-3: Optional fuel minimization (tie-breaker)

$$\begin{aligned}
\min \quad & \sum_i t_i && (11)\\
\text{s.t.} \quad & \sum_k \left(e_k^{F,+} + e_k^{F,-}\right) \le Z_F^\star, && (12)\\
& \sum_k \left(e_k^{\tau,+} + e_k^{\tau,-}\right) \le Z_\tau^\star. && (13)
\end{aligned}$$

4 Matrix Formulation

Let $\mathbf{t} \in \mathbb{R}^n$ collect the thrust magnitudes. Define matrices:

$$D = \begin{bmatrix} | & & | \\ \mathbf{d}_1 & \dots & \mathbf{d}_n \\ | & & | \end{bmatrix}, \qquad H = \begin{bmatrix} | & & | \\ \mathbf{h}_1 & \dots & \mathbf{h}_n \\ | & & | \end{bmatrix}.$$

Then:

$$\begin{aligned}
D\mathbf{t} + \mathbf{e}^{F,-} - \mathbf{e}^{F,+} &= \mathbf{0}, && (14)\\
H\mathbf{t} + \mathbf{e}^{\tau,-} - \mathbf{e}^{\tau,+} &= \boldsymbol{\tau}^{\text{des}}, && (15)\\
0 \le \mathbf{t} &\le \overline{\mathbf{T}}. && (16)
\end{aligned}$$

5 Optional Modifications

- **Force deadband:** replace equality with inequalities
  $$-F_{\max} \le (D\mathbf{t})_k \le F_{\max}.$$
- **Torque saturation:** enforce $-\tau_{\max} \le (H\mathbf{t})_k \le \tau_{\max}$.
- **Mixed-integer extension:** include binary variables $y_i \in \{0,1\}$ and
  $$t_i^{\min} y_i \le t_i \le \overline{T}_i y_i$$
  to capture minimum impulse bits or on/off thrusters.
- **Planar case:** drop unused axes (e.g., y, z) for a 2D vehicle.

6 Planar Example (2D)

For planar motion, control F_x, F_y (to zero) and τ_z (to a desired value):

$$\sum_i d_{ix} t_i + e^{Fx,-} - e^{Fx,+} = 0, \tag{17}$$

$$\sum_i d_{iy} t_i + e^{Fy,-} - e^{Fy,+} = 0, \tag{18}$$

$$\sum_i h_{iz} t_i + e^{\tau,-} - e^{\tau,+} = \tau_z^{\mathrm{des}}. \tag{19}$$

Primary goal: minimize $e^{Fx,\pm}, e^{Fy,\pm}$. Secondary goal: minimize $e^{\tau,\pm}$.