Bounded Preemptive Goal Programming for a 6–Thruster System

We consider six thrusters with positions

$$\begin{aligned}
r_1 &= (-1,-1,-1), & a_1 &= (0,0,1),\\
r_2 &= (1,-1,-1), & a_2 &= (0,0,1),\\
r_3 &= (0,1,-1), & a_3 &= (0,0,1),\\
r_4 &= (-1,-1,1), & a_4 &= (0,0,-1),\\
r_5 &= (1,-1,1), & a_5 &= (0,0,-1),\\
r_6 &= (0,1,1), & a_6 &= (0,0,-1),
\end{aligned}$$

and thrust magnitudes

$$0 \le T_i \le 1, \quad i = 1, \ldots, 6.$$

Net Force

Since all thrusters act along $\pm z$,

$$F_x = 0, \qquad F_y = 0,$$

and

$$F_z = T_1 + T_2 + T_3 - T_4 - T_5 - T_6.$$

Net Torque

The torque from thruster i is

$$\tau_i = r_i \times (T_i a_i).$$

Using

$$r \times (0,0,1) = (y,-x,0), \qquad r \times (0,0,-1) = (-y,x,0),$$

we obtain the total torque

$$\tau = (\tau_x, \tau_y, \tau_z)$$

with

$$\begin{aligned}
\tau_x &= -T_1 - T_2 + T_3 + T_4 + T_5 - T_6,\\
\tau_y &= \quad T_1 - T_2 - T_4 + T_5,\\
\tau_z &= 0.
\end{aligned}$$

Goals and Priorities

We use bounded preemptive goal programming with the following priorities:

1. **Priority 1:** Eliminate torque:
$$\tau_x = 0, \qquad \tau_y = 0.$$
2. **Priority 2:** Maximize the vertical force F_z.
3. **Priority 3:** Minimize F_x and F_y (already identically zero).

Under the torque constraints and bounds $0 \le T_i \le 1$, we can solve for

$$T_1 = T_2 + T_4 - T_5, \qquad T_3 = 2T_2 - 2T_5 + T_6,$$

and one finds that

$$F_z = 4\,(T_2 - T_5), \qquad T_2 - T_5 \le \frac{1}{2},$$

so the maximum achievable vertical force with zero torque is

$$F_z^{\max} = 2.$$

One Optimal Solution

One convenient optimal solution is

$$(T_1, T_2, T_3, T_4, T_5, T_6) = (0.5,\ 0.5,\ 1,\ 0,\ 0,\ 0),$$

which satisfies

$$F = (F_x, F_y, F_z) = (0, 0, 2),$$
$$\tau = (\tau_x, \tau_y, \tau_z) = (0, 0, 0),$$

and respects the bounds $0 \le T_i \le 1$.