Bounded Preemptive Goal Programming for a 12-Thruster Configuration

1. Thruster Configuration (CoM at Origin)

We consider 12 thrusters mounted on a rigid body with center of mass at the origin. Each thruster i has position $r_i \in \mathbb{R}^3$, direction $a_i \in \mathbb{R}^3$, and thrust magnitude $T_i \in [0, 1]$.

1.1 Radial Thrusters (T1–T6)

$$\begin{aligned}
\text{T1: } r_1 &= (L, 0, 0), & a_1 &= (1, 0, 0),\\
\text{T2: } r_2 &= (-L, 0, 0), & a_2 &= (-1, 0, 0),\\
\text{T3: } r_3 &= (0, L, 0), & a_3 &= (0, 1, 0),\\
\text{T4: } r_4 &= (0, -L, 0), & a_4 &= (0, -1, 0),\\
\text{T5: } r_5 &= (0, 0, L), & a_5 &= (0, 0, 1),\\
\text{T6: } r_6 &= (0, 0, -L), & a_6 &= (0, 0, -1).
\end{aligned}$$

These are radial: $r_i \parallel a_i$, so individually they produce zero torque.

1.2 Tangential Thrusters (T7–T12)

$$\begin{aligned}
\text{T7: } r_7 &= (0, L, 0), & a_7 &= (0, 0, 1),\\
\text{T8: } r_8 &= (0, -L, 0), & a_8 &= (0, 0, -1),\\
\text{T9: } r_9 &= (0, 0, L), & a_9 &= (1, 0, 0),\\
\text{T10: } r_{10} &= (0, 0, -L), & a_{10} &= (-1, 0, 0),\\
\text{T11: } r_{11} &= (L, 0, 0), & a_{11} &= (0, 1, 0),\\
\text{T12: } r_{12} &= (-L, 0, 0), & a_{12} &= (0, -1, 0).
\end{aligned}$$

2. Force and Torque Equations

Let $T_i \in [0, 1]$ be the thrust levels. The total force is

$$F = (F_x, F_y, F_z) = \sum_{i=1}^{12} T_i a_i.$$

By inspection,

$$F_x = T_1 - T_2 + T_9 - T_{10},$$
$$F_y = T_3 - T_4 + T_{11} - T_{12},$$
$$F_z = T_5 - T_6 + T_7 - T_8.$$

The torque from thruster i is $\tau_i = r_i \times (T_i a_i)$, and total torque $\tau = (\tau_x, \tau_y, \tau_z)$ is

$$\tau = \sum_{i=1}^{12} r_i \times (T_i a_i).$$

Radial thrusters T1–T6 generate no torque (since $r_i \times a_i = 0$), while:

$$\begin{aligned} r_7 \times a_7 &= (L, 0, 0), & r_8 \times a_8 &= (L, 0, 0), \\ r_9 \times a_9 &= (0, L, 0), & r_{10} \times a_{10} &= (0, L, 0), \\ r_{11} \times a_{11} &= (0, 0, L), & r_{12} \times a_{12} &= (0, 0, L). \end{aligned}$$

Hence

$$\tau_x = L(T_7 + T_8),$$
$$\tau_y = L(T_9 + T_{10}),$$
$$\tau_z = L(T_{11} + T_{12}).$$

3. Bounded Preemptive Goal Programming

We impose bounds

$$0 \le T_i \le 1, \quad i = 1, \dots, 12,$$

and use a preemptive (lexicographic) priority structure:

P1: Maximize τ_x (x-axis torque),

P2: Minimize other torque components $|\tau_y|, |\tau_z|$,

P3: Treat force as last priority (e.g. minimize net force magnitude).

Small nonnegative tolerances $\varepsilon, \varepsilon_y, \varepsilon_z$ are used in achievement constraints to allow numerical slack.

3.1 Priority P1: Maximize τ_x

Solve the LP:

$$\max\ \tau_x = L(T_7 + T_8)$$
$$\text{s.t.} \begin{cases} \tau_y = L(T_9 + T_{10}), \\ \tau_z = L(T_{11} + T_{12}), \\ F_x = T_1 - T_2 + T_9 - T_{10}, \\ F_y = T_3 - T_4 + T_{11} - T_{12}, \\ F_z = T_5 - T_6 + T_7 - T_8, \\ 0 \le T_i \le 1, \quad i = 1, \dots, 12. \end{cases}$$

Let the optimal value be τ_x^*. We then add the *achievement constraint* for all lower-priority levels:

$$\tau_x \ \geq \ \tau_x^* - \varepsilon.$$

3.2 Priority P2: Minimize Other Torque

Introduce nonnegative magnitude variables

$$\begin{aligned} -u_{\tau_y} \leq \tau_y \leq u_{\tau_y}, \quad u_{\tau_y} \geq 0, \\ -u_{\tau_z} \leq \tau_z \leq u_{\tau_z}, \quad u_{\tau_z} \geq 0. \end{aligned}$$

Solve:

$$\min u_{\tau_y} + u_{\tau_z}$$
$$\text{s.t.} \begin{cases} \tau_x = L(T_7 + T_8), \\ \tau_y = L(T_9 + T_{10}), \\ \tau_z = L(T_{11} + T_{12}), \\ F_x = T_1 - T_2 + T_9 - T_{10}, \\ F_y = T_3 - T_4 + T_{11} - T_{12}, \\ F_z = T_5 - T_6 + T_7 - T_8, \\ -u_{\tau_y} \leq \tau_y \leq u_{\tau_y}, \\ -u_{\tau_z} \leq \tau_z \leq u_{\tau_z}, \\ 0 \leq T_i \leq 1, \\ \text{Achievement from P1: } \tau_x \geq \tau_x^* - \varepsilon. \end{cases}$$

Let the optimal values be $u_{\tau_y}^*, u_{\tau_z}^*$. We then add *achievement constraints* for P2:

$$u_{\tau_y} \leq u_{\tau_y}^* + \varepsilon_y, \qquad u_{\tau_z} \leq u_{\tau_z}^* + \varepsilon_z.$$

3.3 Priority P3: Force as Last Priority

Now we treat the net force as the lowest priority quantity. Introduce magnitude variables:

$$\begin{aligned} -u_{Fx} \leq F_x \leq u_{Fx}, \quad u_{Fx} \geq 0, \\ -u_{Fy} \leq F_y \leq u_{Fy}, \quad u_{Fy} \geq 0, \\ -u_{Fz} \leq F_z \leq u_{Fz}, \quad u_{Fz} \geq 0. \end{aligned}$$

Solve:

$$
\min u_{Fx} + u_{Fy} + u_{Fz}
$$

$$
\text{s.t.} \begin{cases}
F_x = T_1 - T_2 + T_9 - T_{10}, \\
F_y = T_3 - T_4 + T_{11} - T_{12}, \\
F_z = T_5 - T_6 + T_7 - T_8, \\
\tau_x = L(T_7 + T_8), \\
\tau_y = L(T_9 + T_{10}), \\
\tau_z = L(T_{11} + T_{12}), \\
-u_{Fx} \leq F_x \leq u_{Fx}, \\
-u_{Fy} \leq F_y \leq u_{Fy}, \\
-u_{Fz} \leq F_z \leq u_{Fz}, \\
0 \leq T_i \leq 1, \\
\text{Achievement P1: } \tau_x \geq \tau_x^* - \varepsilon, \\
\text{Achievement P2: } u_{\tau_y} \leq u_{\tau_y}^* + \varepsilon_y,\ u_{\tau_z} \leq u_{\tau_z}^* + \varepsilon_z.
\end{cases}
$$

This finds thrust levels that:

- Preserve the **maximum** x-torque from P1,
- Preserve the **minimum** achievable other torques from P2,
- And, among those, **minimize net force** as the final priority.

4. Priority Summary

In words, the bounded preemptive goal programming priorities are:

P1: Maximize τ_x (primary torque objective),

P2: Minimize $|\tau_y|$ and $|\tau_z|$ (secondary torque cleanup),

P3: Minimize $|F_x|, |F_y|, |F_z|$ (force disturbance as last priority).