Preemptive Goal Linear Programming for Torque-Neutral Thruster Firing

1 Thruster Geometry and Model

For each thruster $i = 1, \dots, n$:

- Position in body frame: $\mathbf{r}_i \in \mathbb{R}^3$
- Fixed firing direction: $\mathbf{d}_i \in \mathbb{R}^3$
- Thrust magnitude (decision variable): $t_i \geq 0$

The resultant body-frame force and torque are:

$$\mathbf{F} = \sum_{i=1}^{n} t_i \mathbf{d}_i, \tag{1}$$

$$\boldsymbol{\tau} = \sum_{i=1}^{n} t_i \, (\mathbf{r}_i \times \mathbf{d}_i). \tag{2}$$

Define each thruster s torque arm vector:

$$\mathbf{h}_i := \mathbf{r}_i \times \mathbf{d}_i.$$

2 Linearized Constraints with Goal Variables

To maintain linearity, we introduce positive and negative deviation variables.

Primary goal: Zero torque

$$\sum_i h_{i,k} \, t_i + e_k^{\tau,-} - e_k^{\tau,+} = 0, \qquad k \in \{x, y, z\}. \tag{3}$$

Secondary goal: Desired force

$$\sum_i d_{i,k} \, t_i + e_k^{F,-} - e_k^{F,+} = F_k^{\text{des}}, \qquad k \in \{x, y, z\}. \tag{4}$$

Bounds

$$0 \leq t_i \leq \overline{T}_i, \qquad e_k^{\tau,\pm} \geq 0, \qquad e_k^{F,\pm} \geq 0. \tag{5}$$

3 Preemptive (Lexicographic) Optimization

We solve a sequence of linear programs where higher-priority goals are strictly preserved.

LP-1: Torque minimization

$$
\begin{aligned}
\min \quad & \sum_k \left(e_k^{\tau,+} + e_k^{\tau,-}\right) && (6)\\
\text{s.t.} \quad & \text{All constraints above.} && (7)
\end{aligned}
$$

Let the optimal value be $Z_\tau^\star$.

LP-2: Force minimization

$$
\begin{aligned}
\min \quad & \sum_k \left(e_k^{F,+} + e_k^{F,-}\right) && (8)\\
\text{s.t.} \quad & \sum_k \left(e_k^{\tau,+} + e_k^{\tau,-}\right) \le Z_\tau^\star, && (9)\\
& \text{All previous constraints.} && (10)
\end{aligned}
$$

Let the optimal value be $Z_F^\star$.

LP-3: Optional tertiary objective (fuel use)

$$
\begin{aligned}
\min \quad & \sum_i t_i && (11)\\
\text{s.t.} \quad & \sum_k \left(e_k^{\tau,+} + e_k^{\tau,-}\right) \le Z_\tau^\star, && (12)\\
& \sum_k \left(e_k^{F,+} + e_k^{F,-}\right) \le Z_F^\star. && (13)
\end{aligned}
$$

This *epsilon-constraint* sequence enforces strict preemptive priorities.

4 Single-Shot Weighted LP (Approximation)

For implementation simplicity, one can solve a single LP using large hierarchical weights:

$$
\min \quad W_\tau \sum_k (e_k^{\tau,+} + e_k^{\tau,-}) + W_F \sum_k (e_k^{F,+} + e_k^{F,-}) + \sum_i t_i, \tag{14}
$$

with $W_\tau \gg W_F \gg 1$. However, the sequential method is numerically more robust.

5 Matrix Formulation

Let $\mathbf{t} \in \mathbb{R}^n$ stack all thruster thrusts. Define:

$$
H = \begin{bmatrix} | & & | \\ \mathbf{h}_1 & \dots & \mathbf{h}_n \\ | & & | \end{bmatrix}, \qquad D = \begin{bmatrix} | & & | \\ \mathbf{d}_1 & \dots & \mathbf{d}_n \\ | & & | \end{bmatrix}.
$$

Then:

$$
\begin{aligned}
H\mathbf{t} + \mathbf{e}^{\tau,-} - \mathbf{e}^{\tau,+} &= \mathbf{0}, && (15)\\
D\mathbf{t} + \mathbf{e}^{F,-} - \mathbf{e}^{F,+} &= \mathbf{F}^{\text{des}}, && (16)\\
0 \le \mathbf{t} &\le \overline{\mathbf{T}}. && (17)
\end{aligned}
$$

6 Practical Notes

- **Deadbands:** Replace $= 0$ with inequalities $-\tau_{\max} \le (H\mathbf{t})_k \le \tau_{\max}$.
- **Thruster saturation:** Already handled by $0 \le t_i \le \overline{T}_i$.
- **Minimum impulse bit:** Add binaries $y_i \in \{0, 1\}$ and constraints $t_i \ge t_i^{\min} y_i,\ t_i \le \overline{T}_i y_i$.
- **Planar case:** Drop unused axes to reduce LP size.

7 Example: Planar Case

For planar motion (controlling F_x and τ_z):

$$F_x = d_{1x}t_1 + d_{2x}t_2 + d_{3x}t_3, \tag{18}$$
$$\tau_z = h_{1z}t_1 + h_{2z}t_2 + h_{3z}t_3. \tag{19}$$

LP-1:

$$\min \quad e_+^\tau + e_-^\tau \tag{20}$$
$$\text{s.t.} \quad h_{1z}t_1 + h_{2z}t_2 + h_{3z}t_3 + e_-^\tau - e_+^\tau = 0, \tag{21}$$
$$0 \le t_i \le \overline{T}_i. \tag{22}$$

LP-2:

$$\min \quad e_+^F + e_-^F \tag{23}$$
$$\text{s.t.} \quad d_{1x}t_1 + d_{2x}t_2 + d_{3x}t_3 + e_-^F - e_+^F = F_x^{\text{des}}, \tag{24}$$
$$e_+^\tau + e_-^\tau \le Z_\tau^\star. \tag{25}$$

Optionally, minimize $\sum_i t_i$ as a tertiary step.