Preemptive (Priority) Goal Programming with Bounded Preemptive Simplex

We have four thrusters with bounds

$$0 \le t_i \le 1 \quad (i = 1, 2, 3, 4).$$

Geometry and directions:

$$\begin{aligned} r_1 &= (2,2,0),\ a_1 = (100,100,100), & r_2 &= (-2,-2,0),\ a_2 = (100,100,100), \\ r_3 &= (0,2,2),\ a_3 = (100,100,100), & r_4 &= (0,-2,-2),\ a_4 = (100,100,100). \end{aligned}$$

Force/torque maps

Each force column is a_i; each torque column is $r_i \times a_i$. With $a = (100, 100, 100)$ and $r = (x, y, z)$,

$$r \times a = \big(y \cdot 100 - z \cdot 100,\quad z \cdot 100 - x \cdot 100,\quad x \cdot 100 - y \cdot 100\big).$$

Thus, assembling $A = [a_1\, a_2\, a_3\, a_4]$ and $B = [r_1 \times a_1 \ \cdots \ r_4 \times a_4]$:

$$A = \begin{bmatrix} 100 & 100 & 100 & 100 \\ 100 & 100 & 100 & 100 \\ 100 & 100 & 100 & 100 \end{bmatrix}, \quad B = \begin{bmatrix} 200 & -200 & 0 & 0 \\ -200 & 200 & 200 & -200 \\ 0 & 0 & -200 & 200 \end{bmatrix}.$$

Hence for $t = (t_1, t_2, t_3, t_4)^\top$,

$$F = A\,t = 100\Big(\sum_{i=1}^4 t_i\Big) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad T = B\,t = \begin{bmatrix} 200(t_1 - t_2) \\ 200(-t_1 + t_2 + t_3 - t_4) \\ 200(-t_3 + t_4) \end{bmatrix}.$$

Stage P_1 (highest priority): Minimize torque $\|T\|_1$

Introduce deviational variables $\tau_x^\pm, \tau_y^\pm, \tau_z^\pm \ge 0$ and write

$$\begin{aligned} &200(t_1 - t_2) + \tau_x^- - \tau_x^+ = 0, \\ &200(-t_1 + t_2 + t_3 - t_4) + \tau_y^- - \tau_y^+ = 0, \qquad \min z_1 = (\tau_x^+ + \tau_x^-) + (\tau_y^+ + \tau_y^-) + (\tau_z^+ + \tau_z^-). \\ &200(-t_3 + t_4) + \tau_z^- - \tau_z^+ = 0, \end{aligned}$$

We keep bounds *inside* the simplex via LB/UB/B statuses.

P_1 Tableau 0 (start; make τ^+ basic in each row)

Basic	RHS	t_1	t_2	t_3	t_4	τ^-	τ^+	Status
τ_x^+	0	+200	−200	0	0	+1	1	B
τ_y^+	0	−200	+200	+200	−200	+1	1	B
τ_z^+	0	0	0	−200	+200	+1	1	B
z_1	0	**0**	**0**	**0**	**0**	**+3**	0	

Nonbasic statuses at start:

$$t_1, t_2, t_3, t_4 : \ \mathsf{LB}\ ([0,1]), \qquad \tau_x^-, \tau_y^-, \tau_z^- : \ \mathsf{LB}.$$

All four t_i have reduced cost 0 (we can move along directions that keep z_1 unchanged).

Pivot 1 (degenerate): bring t_1 into the x-torque row. Entering t_1 from LB (allowed with $r = 0$); pivot on $(\tau_x^+,\ t_1)$.

Solve τ_x^+-row for t_1:

$$t_1 = t_2 - \tfrac{1}{200}\tau_x^- + \tfrac{1}{200}\tau_x^+.$$

Basic	RHS	t_1	t_2	t_3	t_4	τ^-	τ^+	Status
t_1	0	1	-1	0	0	$-\frac{1}{200}$	$+\frac{1}{200}$	B
τ_y^+	0	0	0	$+200$	-200	$+1$	1	B
τ_z^+	0	0	0	-200	$+200$	$+1$	1	B
z_1	0	0	0	**0**	**0**	$+\mathbf{3}$	0	

Pivot 2 (degenerate): bring t_3 into the z-torque row. Pivot on $(\tau_z^+,\ t_3)$ and solve for t_3:

$$t_3 = t_4 + \tfrac{1}{200}\tau_z^- - \tfrac{1}{200}\tau_z^+.$$

Basic	RHS	t_1	t_2	t_3	t_4	τ^-	τ^+	Status
t_1	0	1	-1	0	0	$-\frac{1}{200}$	$+\frac{1}{200}$	B
τ_y^+	0	0	0	0	0	$+1$	1	B
t_3	0	0	0	1	-1	$+\frac{1}{200}$	$-\frac{1}{200}$	B
z_1	0	0	0	0	0	$+\mathbf{3}$	0	

Reading off the *zero-torque manifold* by setting all deviations $\tau^\pm = 0$:

$$\boxed{t_1 - t_2 = 0, \qquad t_3 - t_4 = 0}$$

(i.e. $t_1 = t_2$ and $t_3 = t_4$), and the y-torque row becomes redundant once these hold. Thus the best P_1 value is $\boxed{Z_1^\star = 0}$. We *lock* $t_1 = t_2$ and $t_3 = t_4$ for Stage P_2.

Stage P_2: Maximize force along $(1,1,1)$

Since $F = 100(\sum t_i)\,(1,1,1)$, maximizing along $(1,1,1)$ is equivalent to maximizing

$$\max\ z_2 = t_1 + t_2 + t_3 + t_4 \quad \text{s.t.} \quad t_1 = t_2, \quad t_3 = t_4, \quad 0 \le t_i \le 1.$$

Let $v_1 := t_1 = t_2$ and $v_2 := t_3 = t_4$ with $0 \le v_1, v_2 \le 1$. Then

$$z_2 = 2v_1 + 2v_2, \qquad F = 100(2v_1 + 2v_2)\,(1,1,1).$$

$\mathbf{P_2}$ Tableau A (start)

We keep the two equalities as definition rows; v_1, v_2 are nonbasic at LB.

Var	Value	Status	Bounds
v_1	0.0	LB	$[0,1]$
v_2	0.0	LB	$[0,1]$
t_1	0.0	B	$(t_1 = v_1)$
t_2	0.0	B	$(t_2 = v_1)$
t_3	0.0	B	$(t_3 = v_2)$
t_4	0.0	B	$(t_4 = v_2)$
z_2	0		$r_{v_1} = +2, \quad r_{v_2} = +2$

Bounded-preemptive step rule (maximization): - Nonbasic at LB with $r > 0$ can *increase*. No row blocks; only opposite bounds matter. - Both v_1 and v_2 hit their UBs first (pure *preemptive bound hits*, no pivot).

$\mathbf{P_2}$ Tableau B (after preemptive UB hits)

Var	Value	Status	Bounds
v_1	1.0	UB	$[0,1]$
v_2	1.0	UB	$[0,1]$
t_1	1.0	B	$(t_1 = v_1)$
t_2	1.0	B	$(t_2 = v_1)$
t_3	1.0	B	$(t_3 = v_2)$
t_4	1.0	B	$(t_4 = v_2)$
z_2	4		$r_{v_1} = +2, \; r_{v_2} = +2$ (cannot improve by decreasing)

Stage P_2 optimum and outputs

$\boxed{t_1 = t_2 = t_3 = t_4 = 1}, \qquad \boxed{z_2^\star = 4}, \qquad \boxed{F = 100 \cdot 4\,(1,1,1) = (400,400,400)}, \qquad \boxed{T = (0,0,0)}.$

Bounded-preemptive optimality check. At the end, v_1 and v_2 are nonbasic at UB. For maximization, a nonbasic at UB could only help if its reduced cost were *negative* (so we would decrease it). Here $r_{v_1} = r_{v_2} = +2 > 0$, so decreasing either worsens z_2. No other candidates $\Rightarrow$ optimal.