Thruster Allocation:
Maximize F_x, Minimize Other Force, Minimize Torque

1. Thruster Geometry

We have four thrusters with directions a_i and positions r_i:

$$
\begin{aligned}
a_1 &= (2,\ 2,\ 0), & r_1 &= (100,\ 100,\ 100), \\
a_2 &= (2,\ -2,\ 0), & r_2 &= (100,\ -100,\ 100), \\
a_3 &= (-2,\ -2,\ 0), & r_3 &= (-100,\ -100,\ 100), \\
a_4 &= (-2,\ 2,\ 0), & r_4 &= (-100,\ 100,\ 100).
\end{aligned}
$$

Thruster commands T_i are bounded by

$$
0 \le T_i \le 1 \qquad (i = 1, 2, 3, 4).
$$

2. Force Model

The total force is

$$
\mathbf{F} = \sum_{i=1}^{4} T_i\, a_i = (F_x,\ F_y,\ F_z).
$$

For this specific geometry,

$$
\begin{aligned}
F_x &= 2(T_1 + T_2 - T_3 - T_4), \\
F_y &= 2(T_1 - T_2 - T_3 + T_4), \\
F_z &= 0.
\end{aligned}
$$

3. Torque Model

The total torque about the origin is

$$
\boldsymbol{\tau} = \sum_{i=1}^{4} T_i\, (r_i \times a_i) = (\tau_x,\ \tau_y,\ \tau_z).
$$

For the given r_i and a_i,

$$
\begin{aligned}
\tau_x &= -200T_1 + 200T_2 + 200T_3 - 200T_4, \\
\tau_y &= 200T_1 + 200T_2 - 200T_3 - 200T_4, \\
\tau_z &= 0.
\end{aligned}
$$

4. Auxiliary Variables for Magnitude Minimization

To represent absolute values in linear form, introduce nonnegative variables:

$$-u_y \leq F_y \leq u_y, \qquad u_y \geq 0,$$
$$-u_{\tau_x} \leq \tau_x \leq u_{\tau_x}, \qquad u_{\tau_x} \geq 0,$$
$$-u_{\tau_y} \leq \tau_y \leq u_{\tau_y}, \qquad u_{\tau_y} \geq 0.$$

Thus,

$$u_y \geq |F_y|, \qquad u_{\tau_x} \geq |\tau_x|, \qquad u_{\tau_y} \geq |\tau_y|.$$

5. Single Weighted Linear Program

A single LP encoding

$$\text{maximize } F_x, \quad \text{minimize other force } |F_y|, \quad \text{minimize torque}$$

is:

$$\max\left[F_x - \alpha\, u_y - \beta\, (u_{\tau_x} + u_{\tau_y}) \right],$$

with user-chosen weights $\alpha, \beta > 0$.
Subject to:

$$F_x = 2(T_1 + T_2 - T_3 - T_4),$$
$$F_y = 2(T_1 - T_2 - T_3 + T_4),$$
$$\tau_x = -200T_1 + 200T_2 + 200T_3 - 200T_4,$$
$$\tau_y = 200T_1 + 200T_2 - 200T_3 - 200T_4,$$

$$-u_y \leq F_y \leq u_y,$$
$$-u_{\tau_x} \leq \tau_x \leq u_{\tau_x},$$
$$-u_{\tau_y} \leq \tau_y \leq u_{\tau_y},$$
$$0 \leq T_i \leq 1 \quad (i = 1, \ldots, 4), \qquad u_y, u_{\tau_x}, u_{\tau_y} \geq 0.$$

6. Preemptive (Lexicographic) Priority Formulation

If we want strict priorities

$$\text{maximize } F_x \ \Rightarrow\ \text{minimize } |F_y| \ \Rightarrow\ \text{minimize torque},$$

we solve three LPs sequentially.

Priority P_1: Maximize F_x

Solve

$$\max F_x$$

subject to force/torque definitions and

$$0 \le T_i \le 1.$$

Let the optimum be F_x^*. To preserve this in later levels, add the achievement constraint

$$F_x \ge F_x^* - \varepsilon,$$

where $\varepsilon \ge 0$ is a small tolerance.

Priority P_2: Minimize Lateral Force $|F_y|$

Introduce u_y with

$$-u_y \le F_y \le u_y, \qquad u_y \ge 0,$$

and solve

$$\min u_y$$

subject to all previous constraints and

$$F_x \ge F_x^* - \varepsilon.$$

Let the resulting optimum be u_y^*. Then add an achievement constraint

$$u_y \le u_y^* + \varepsilon_y,$$

with small $\varepsilon_y \ge 0$.

Priority P_3: Minimize Torque

Introduce u_{τ_x}, u_{τ_y} with

$$-u_{\tau_x} \le \tau_x \le u_{\tau_x}, \quad -u_{\tau_y} \le \tau_y \le u_{\tau_y},$$

and solve

$$\min\left(u_{\tau_x} + u_{\tau_y}\right)$$

subject to

$$F_x \ge F_x^* - \varepsilon, \qquad u_y \le u_y^* + \varepsilon_y.$$

7. Summary

We have defined:

- A linear model of force and torque from 4 bounded thrusters.

- A single weighted LP that maximizes forward force F_x while penalizing side force and torque via auxiliary variables.

- A preemptive (goal-programming) formulation with the priorities:

  1. Maximize F_x,
  2. Minimize $|F_y|$,
  3. Minimize torque magnitude $|\tau_x| + |\tau_y|$.

  This XeLaTeX document can be compiled with `xelatex` and used as a basis for implementation or documentation.