Laplace Transform: Definitions, Theorems and Proofs

Thomas Gao

DEFINITION

Laplace Transform

Definition 1. Let $f(t)$ be a function on $[0, \infty)$. The **Laplace transform** of f is the function F defined by the integral

$$F(s) := \int_0^\infty e^{-st} f(t)\,dt \; . \tag{1}$$

The domain of $F(s)$ is all the values of s for which the integral in (1) exists.† The Laplace transform of f is denoted by both F and $\mathcal{L}\{f\}$.

Notice that the integral in (1) is an **improper** integral. More precisely,

$$\int_0^\infty e^{-st} f(t)\,dt := \lim_{N\to\infty} \int_0^N e^{-st} f(t)\,dt$$

whenever the limit exists.

PROVE

Linearity of the Transform

Theorem 1. Let f, f_1, and f_2 be functions whose Laplace transforms exist for $s > \alpha$ and let c be a constant. Then, for $s > \alpha$,

$$\mathscr{L}\{f_1 + f_2\} = \mathscr{L}\{f_1\} + \mathscr{L}\{f_2\} , \tag{2}$$

$$\mathscr{L}\{cf\} = c\mathscr{L}\{f\} . \tag{3}$$

Proof. Using the linearity properties of integration, we have for $s > \alpha$

$$\begin{aligned}
\mathscr{L}\{f_1 + f_2\}(s) &= \int_0^\infty e^{-st}[f_1(t) + f_2(t)]dt \\
&= \int_0^\infty e^{-st} f_1(t)\,dt + \int_0^\infty e^{-st} f_2(t)\,dt \\
&= \mathscr{L}\{f_1\}(s) + \mathscr{L}\{f_2\}(s) \ .
\end{aligned}$$

Hence, equation (2) is satisfied. In a similar fashion, we see that

$$\begin{aligned}
\mathscr{L}\{cf\}(s) &= \int_0^\infty e^{-st}[cf(t)]dt = c\int_0^\infty e^{-st} f(t)\,dt \\
&= c\mathscr{L}\{f\}(s) \ . \quad \blacklozenge
\end{aligned}$$

DEFINITION

A function $f(t)$ on $[a, b]$ is said to have a **jump discontinuity** at $t_0 \in (a, b)$ if $f(t)$ is discontinuous at t_0, but the one-sided limits

$$\lim_{t \to t_0^-} f(t) \quad \text{and} \quad \lim_{t \to t_0^+} f(t)$$

exist as finite numbers. If the discontinuity occurs at an endpoint, $t_0 = a$ (or b), a jump discontinuity occurs if the one-sided limit of $f(t)$ as $t \to a^+$ $(t \to b^-)$ exists as a finite number. We can now define piecewise continuity.

Piecewise Continuity

Definition 2. A function $f(t)$ is said to be **piecewise continuous on a finite interval** $[a, b]$ if $f(t)$ is continuous at every point in $[a, b]$, except possibly for a finite number of points at which $f(t)$ has a jump discontinuity.

A function $f(t)$ is said to be **piecewise continuous on** $[\mathbf{0}, \infty)$ if $f(t)$ is piecewise continuous on $[0, N]$ for all $N > 0$.

Exponential Order α

Definition 3. A function $f(t)$ is said to be of **exponential order** $\boldsymbol{\alpha}$ if there exist positive constants T and M such that

(4) $$|f(t)| \leq Me^{\alpha t}, \qquad \text{for all } t \geq T.$$

PROVE

Conditions for Existence of the Transform

Theorem 2. If $f(t)$ is piecewise continuous on $[0, \infty)$ and of exponential order α, then $\mathscr{L}\{f\}(s)$ exists for $s > \alpha$.

Proof. We need to show that the integral

$$\int_0^\infty e^{-st} f(t)\, dt$$

converges for $s > \alpha$. We begin by breaking up this integral into two separate integrals:

$$\int_0^T e^{-st} f(t)\, dt + \int_T^\infty e^{-st} f(t)\, dt \ , \tag{5}$$

where T is chosen so that inequality (4) holds. The first integral in (5) exists because $f(t)$ and hence $e^{-st}f(t)$ are piecewise continuous on the interval $[0, T]$ for any fixed s. To see that the second integral in (5) converges, we use the **comparison test for improper integrals.**

Since $f(t)$ is of exponential order α, we have for $t \geq T$

$$|f(t)| \leq M e^{\alpha t} \ ,$$

and hence

$$|e^{-st} f(t)| = e^{-st} |f(t)| \leq M e^{-(s-\alpha)t} \ ,$$

for all $t \geq T$. Now for $s > \alpha$.

$$\int_T^\infty M e^{-(s-\alpha)t}\, dt = M \int_T^\infty e^{-(s-\alpha)t}\, dt = \frac{M e^{-(s-\alpha)T}}{s - \alpha} < \infty \ .$$

Since $|e^{-st} f(t)| \leq M e^{-(s-\alpha)t}$ for $t \geq T$ and the improper integral of the larger function converges for $s > \alpha$, then, by the comparison test, the integral

$$\int_T^\infty e^{-st} f(t)\, dt$$

converges for $s > \alpha$. Finally, because the two integrals in (5) exist, the Laplace transform $\mathcal{L}\{f\}(s)$ exists for $s > \alpha$. ◆

PROVE

Translation in s

Theorem 3. If the Laplace transform $\mathscr{L}\{f\}(s) = F(s)$ exists for $s > \alpha$, then

$$\mathscr{L}\{e^{at}f(t)\}(s) = F(s - a) \tag{1}$$

for $s > \alpha + a$.

Proof. We simply compute

$$
\begin{aligned}
\mathscr{L}\{e^{at}f(t)\}(s) &= \int_0^\infty e^{-st}e^{at}f(t)\,dt \\
&= \int_0^\infty e^{-(s-a)t}f(t)\,dt \\
&= F(s-a)\ . \quad \blacklozenge
\end{aligned}
$$

Theorem 3 illustrates the effect on the Laplace transform of multiplication of a function $f(t)$ by e^{at} .

PROVE

Laplace Transform of the Derivative

Theorem 4. Let $f(t)$ be continuous on $[0, \infty)$ and $f'(t)$ be piecewise continuous on $[0, \infty)$, with both of exponential order α. Then, for $s > \alpha$,

$$\mathscr{L}\{f'\}(s) = s\mathscr{L}\{f\}(s) - f(0) \quad \text{(2)}$$

Proof. Since $\mathscr{L}\{f'\}$ exists, we can use integration by parts $\left[\text{with } u = e^{-st} \text{ and } dv = f'(t)dt\right]$ to obtain

$$\text{(3)} \qquad \mathscr{L}\{f'\}(s) = \int_0^\infty e^{-st}f'(t)dt = \lim_{N\to\infty} \int_0^N e^{-st}f'(t)dt$$

$$= \lim_{N\to\infty} \left[e^{-st}f(t)\Big|_0^N + s\int_0^N e^{-st}f(t)dt \right]$$

$$= \lim_{N\to\infty} e^{-sN}f(N) - f(0) + s\lim_{N\to\infty}\int_0^N e^{-st}f(t)dt$$

$$= \lim_{N\to\infty} e^{-sN}f(N) - f(0) + s\mathscr{L}\{f\}(s) \ .$$

To evaluate $\lim_{N\to\infty} e^{-sN}f(N)$, we observe that since $f(t)$ is of exponential order α, there exists a constant M such that for N large,

$$|e^{-sN}f(N)| \le e^{-sN}Me^{\alpha N} = Me^{-(s-\alpha)N} \ .$$

Hence, for $s > \alpha$,

$$0 \le \lim_{N\to\infty} |e^{-sN}f(N)| \le \lim_{N\to\infty} Me^{-(s-\alpha)N} = 0 \ ,$$

so

$$\lim_{N\to\infty} e^{-sN}f(N) = 0$$

for $s > \alpha$. Equation (3) now reduces to

$$\mathscr{L}\{f'\}(s) = s\mathscr{L}\{f\}(s) - f(0) \ . \ \blacklozenge$$

PROVE

Laplace Transform of Higher-Order Derivatives

Theorem 5. Let $f(t), f'(t), \ldots, f^{(n-1)}(t)$ be continuous on $[0, \infty)$ and let $f^{(n)}(t)$ be piecewise continuous on $[0, \infty)$, with all these functions of exponential order α. Then, for $s > \alpha$,

$$\mathscr{L}\left\{f^{(n)}\right\}(s) = s^n \mathscr{L}\{f\}(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0) . \tag{4}$$

Using induction, we can extend the last theorem to higher-order derivatives of $f(t)$. For example,

$$\begin{aligned}\mathscr{L}\{f''\}(s) &= s\mathscr{L}\{f'\}(s) - f'(0)\\ &= s\left[s\mathscr{L}\{f\}(s) - f(0)\right] - f'(0)\ ,\end{aligned}$$

which simplifies to

$$\boldsymbol{\mathscr{L}\{f''\}(s) = s^2\mathscr{L}\{f\}(s) - sf(0) - f'(0)}\ .$$

PROVE

Derivatives of the Laplace Transform

Theorem 6. Let $F(s) = \mathscr{L}\{f\}(s)$ and assume $f(t)$ is piecewise continuous on $[0, \infty)$ and of exponential order α. Then, for $s > \alpha$,

$$\mathscr{L}\{t^n f(t)\}(s) = (-1)^n \frac{d^n F}{ds^n}(s) \ . \tag{6}$$

$$F'(s) = \mathscr{L}\{-tf(t)\}(s) \ .$$

Proof. Consider the identity

$$\frac{dF}{ds}(s) = \frac{d}{ds}\int_0^\infty e^{-st}f(t)\,dt \ .$$

Because of the assumptions on $f(t)$, we can apply a theorem from advanced calculus (sometimes called **Leibniz's rule**) to interchange the order of integration and differentiation:

$$\begin{aligned}\frac{dF}{ds}(s) &= \int_0^\infty \frac{d}{ds}(e^{-st})\,f(t)\,dt \\ &= -\int_0^\infty e^{-st}tf(t)\,dt = -\mathcal{L}\{tf(t)\}(s) \ .\end{aligned}$$

Thus,

$$\mathcal{L}\{tf(t)\}(s) = (-1)\frac{dF}{ds}(s) \ .$$

The general result (6) now follows by induction on n. ◆

DEFINITION

Inverse Laplace Transform

Definition 4. Given a function $F(s)$, if there is a function $f(t)$ that is continuous on $[0, \infty)$ and satisfies

$$\mathcal{L}\{f\} = F \,, \tag{2}$$

then we say that $f(t)$ is the **inverse Laplace transform** of $F(s)$ and employ the notation $f = \mathcal{L}^{-1}\{F\}$.

PROVE

Linearity of the Inverse Transform

Theorem 7. Assume that $\mathscr{L}^{-1}\{F\}$, $\mathscr{L}^{-1}\{F_1\}$, and $\mathscr{L}^{-1}\{F_2\}$ exist and are continuous on $[0, \infty)$ and let c be any constant. Then

$$\mathscr{L}^{-1}\{F_1 + F_2\} = \mathscr{L}^{-1}\{F_1\} + \mathscr{L}^{-1}\{F_2\} , \tag{3}$$

$$\mathscr{L}^{-1}\{cF\} = c\mathscr{L}^{-1}\{F\} . \tag{4}$$

Proof: TODO

DEFINITION

Unit Step Function

Definition 5. The **unit step function** $u(t)$ is defined by

$$u(t) := \begin{cases} 0\,, & t < 0\,, \\ 1\,, & 0 < t\,. \end{cases} \tag{1}$$

Rectangular Window Function

Definition 6. The **rectangular window function** $\Pi_{a,b}(t)$ is defined by†

$$\Pi_{a,b}(t) := u(t-a) - u(t-b) = \begin{cases} 0, & t < a\,, \\ 1, & a < t < b\,, \\ 0, & b < t\,. \end{cases} \tag{3}$$

PROVE

Translation in t

Theorem 8. Let $F(s) = \mathscr{L}\{f\}(s)$ exist for $s > \alpha \geq 0$. If a is a positive constant, then

(8) $$\mathscr{L}\{f(t-a)u(t-a)\}(s) = e^{-as}F(s) \ ,$$

and, conversely, an inverse Laplace transform†† of $e^{-as}F(s)$ is given by

(9) $$\mathscr{L}^{-1}\{e^{-as}F(s)\}(t) = f(t-a)u(t-a) \ .$$

Proof. By the definition of the Laplace transform, we have

$$\mathcal{L}\{f(t-a)u(t-a)\}(s) = \int_0^\infty e^{-st} f(t-a)u(t-a)\,dt \tag{10}$$
$$= \int_a^\infty e^{-st} f(t-a)\,dt\ ,$$

where, in the last equation, we used the fact that $u(t-a)$ is zero for $t < a$ and equals 1 for $t > a$. Now let $v = t - a$. Then we have $dv = dt$, and equation (10) becomes

$$\mathcal{L}\{f(t-a)u(t-a)\}(s) = \int_0^\infty e^{-as} e^{-sv} f(v)\,dv$$
$$= e^{-as} \int_0^\infty e^{-sv} f(v)\,dv = e^{-as} F(s)\ . \quad \blacklozenge$$

DEFINITION

Periodic Function

Definition 7. A function $f(t)$ is said to be **periodic of period T** ($\neq 0$) if

$$f(t+T) = f(t)$$

for all t in the domain of f.

It is convenient to introduce a notation for the "windowed" version of a periodic function $f(t)$ (using a rectangular window whose width is the period):

$$f_T(t) := f(t)\Pi_{0,T}(t) = f(t)[u(t) - u(t-T)] = \begin{cases} f(t)\ , & 0 < t < T\ , \\ 0\ , & \text{otherwise}\ . \end{cases} \tag{14}$$

The Laplace transform of $f_T(t)$ is given by

$$\boxed{F_T(s) = \int_0^{\infty} e^{-st} f_T(t)\,dt = \int_0^{T} e^{-st} f(t)\,dt\ .}$$

PROVE

Transform of Periodic Function

Theorem 9. If f has period T and is piecewise continuous on $[0, T]$, then the Laplace transforms $F(s) = \int_0^{\infty} e^{-st} f(t)\,dt$ and $F_T(s) = \int_0^{T} e^{-st} f(t)\,dt$ are related by

(15) $$F_T(s) = F(s)\left[1 - e^{-sT}\right] \quad \text{or} \quad F(s) = \frac{F_T(s)}{1 - e^{-sT}}.$$

Proof. From (14) and the periodicity of f, we have

$$f_T(t) = f(t)u(t) - f(t)u(t-T) = f(t)u(t) - f(t-T)u(t-T)\,, \tag{16}$$

so taking transforms and applying (8) yields $F_T(s) = F(s) - e^{-sT}F(s)$, which is equivalent to (15). ◆

DEFINITION

Gamma Function

Definition 8. The **gamma function** $\Gamma(t)$ is defined by

$$\Gamma(t) := \int_0^\infty e^{-u} u^{t-1} du \,, \qquad t > 0 \,. \tag{18}$$

PROVE

Prove the recursive identity

It can be shown that the integral in (18) converges for $t > 0$. A useful property of the gamma function is the recursive relation

$$\Gamma(t+1) = t\Gamma(t) \; . \tag{19}$$

This identity follows from the definition (18) after performing an integration by parts:

$$\begin{aligned}
\Gamma(t+1) &= \int_0^\infty e^{-u}u^t\,du = \lim_{N\to\infty}\int_0^N e^{-u}u^t\,du \\
&= \lim_{N\to\infty}\left\{-e^{-u}u^t\Big|_0^N + \int_0^N te^{-u}u^{t-1}\,du\right\} \\
&= \lim_{N\to\infty}(-e^{-N}N^t) + t\lim_{N\to\infty}\int_0^N e^{-u}u^{t-1}\,du \\
&= 0 + t\Gamma(t) = t\Gamma(t)\ .
\end{aligned}$$

When t is a positive integer, say $t = n$, then the recursive relation (19) can be repeatedly applied to obtain

$$\begin{aligned}
\Gamma(n+1) &= n\Gamma(n) = n(n-1)\Gamma(n-1) = \cdots \\
&= n(n-1)(n-2)\cdots 2\Gamma(1)\ .
\end{aligned}$$

It follows from the definition (18) that $\Gamma(1) = 1$, so we find

$$\boldsymbol{\Gamma(n+1) = n!}\ .$$

Thus, the gamma function extends the notion of factorial!

PROVE

Verify the following formula for Laplace Transform

As an application of the gamma function, let's return to the problem of determining the Laplace transform of an arbitrary power of t. We will verify that the formula

$$\mathscr{L}\{t^r\}(s) = \frac{\Gamma(r+1)}{s^{r+1}} \tag{20}$$

holds for every constant $r > -1$. (Since factorial only covers integer r, this covers all cases)

By definition,

$$\mathscr{L}\{t^r\}(s) = \int_0^\infty e^{-st} t^r \, dt \ .$$

Let's make the substitution $u = st$. Then $du = s\,dt$, and we find

$$\mathscr{L}\{t^r\}(s) = \int_0^\infty e^{-u} \left(\frac{u}{s}\right)^r \left(\frac{1}{s}\right) du$$

$$= \frac{1}{s^{r+1}} \int_0^\infty e^{-u} u^r \, du = \frac{\Gamma(r+1)}{s^{r+1}} \ .$$

Notice that when $r = n$ is a nonnegative integer, then $\Gamma(n+1) = n!$, and so formula (20) reduces to the familiar formula for $\mathscr{L}\{t^n\}$.

DEFINITION

PROVE

Convolution

Definition 9. Let $f(t)$ and $g(t)$ be piecewise continuous on $[0, \infty)$. The **convolution** of $f(t)$ and $g(t)$, denoted $f * g$, is defined by

$$(3)\qquad (f * g)(t) := \int_0^t f(t - v)g(v)\,dv \ .$$

Properties of Convolution

Theorem 10. Let $f(t)$, $g(t)$, and $h(t)$ be piecewise continuous on $[0, \infty)$. Then

$$(4)\qquad f * g = g * f \ ,$$

$$(5)\qquad f * (g + h) = (f * g) + (f * h) \ ,$$

$$(6)\qquad (f * g) * h = f * (g * h) \ ,$$

$$(7)\qquad f * 0 = 0 \ .$$

TODO: (Partial)

Proof. To prove equation (4), we begin with the definition

$$(f * g)(t) := \int_0^t f(t - v)g(v)dv \ .$$

Using the change of variables $w = t - v$, we have

$$(f * g)(t) = \int_t^0 f(w)g(t - w)(-dw) = \int_0^t g(t - w)f(w)dw = (g * f)(t) \ ,$$

which proves (4). The proofs of equations (5) and (6) are left to the exercises

Equation (7) is obvious, since $f(t - v) \cdot 0 \equiv 0$. ◆

PROVE

Convolution Theorem

Theorem 11. Let $f(t)$ and $g(t)$ be piecewise continuous on $[0, \infty)$ and of exponential order α and set $F(s) = \mathcal{L}\{f\}(s)$ and $G(s) = \mathcal{L}\{g\}(s)$. Then

$$\mathcal{L}\{f * g\}(s) = F(s)G(s) \ , \tag{8}$$

or, equivalently,

$$\mathcal{L}^{-1}\{F(s)G(s)\}(t) = (f * g)(t) \ . \tag{9}$$

Proof. Starting with the left-hand side of (8), we use the definition of convolution to write for $s > \alpha$

$$\mathscr{L}\{f * g\}(s) = \int_0^{\infty} e^{-st}\left[\int_0^t f(t-v)g(v)\,dv\right]dt\ .$$

To simplify the evaluation of this iterated integral, we introduce the unit step function $u(t-v)$ and write

$$\mathscr{L}\{f * g\}(s) = \int_0^{\infty} e^{-st}\left[\int_0^{\infty} u(t-v)f(t-v)g(v)\,dv\right]dt\ ,$$

where we have used the fact that $u(t-v) = 0$ if $v > t$. Reversing the order of integration[†] gives

$$\mathscr{L}\{f * g\}(s) = \int_0^{\infty} g(v)\left[\int_0^{\infty} e^{-st}u(t-v)f(t-v)\,dt\right]dv\ . \tag{10}$$

Recall from the translation property in Section 7.6 that the integral in brackets in equation (10) equals $e^{-sv}F(s)$. Hence,

$$\mathscr{L}\{f * g\}(s) = \int_0^{\infty} g(v)e^{-sv}F(s)\,dv = F(s)\int_0^{\infty} e^{-sv}g(v)\,dv = F(s)G(s)\ .$$

This proves formula (8). ◆