Fluid

Professor Sean Peterson's Lecture transcribed by Thomas Gao.

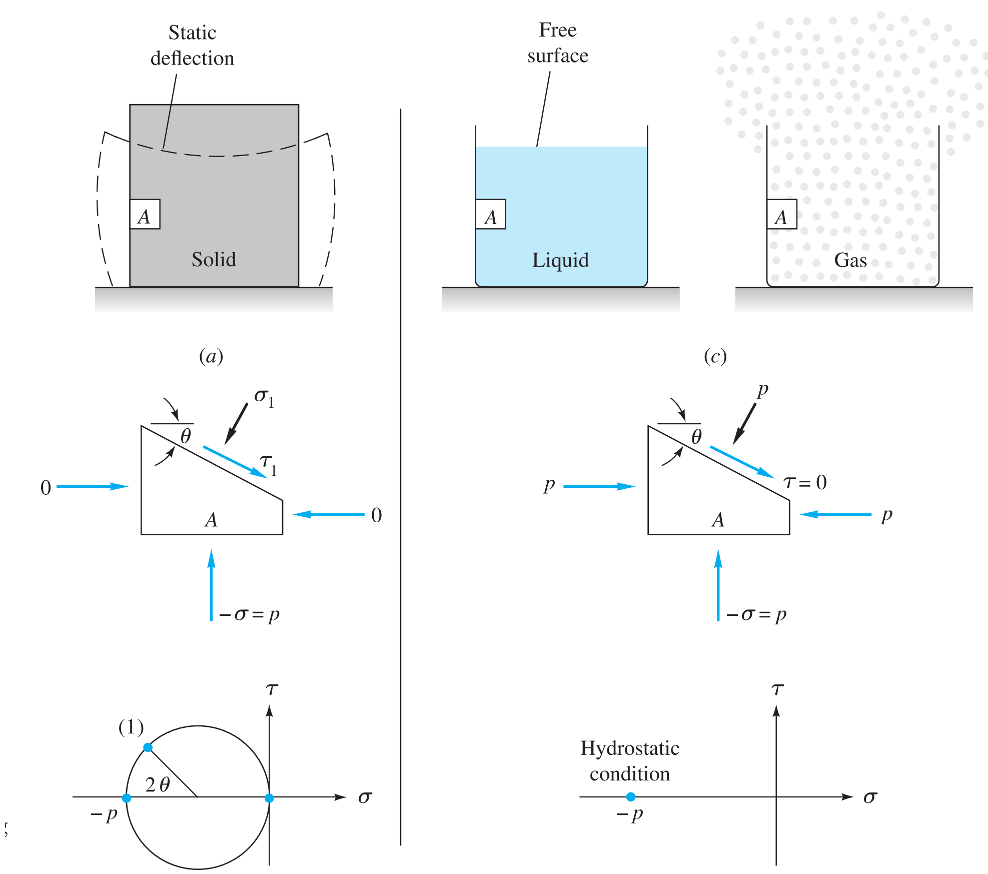
Introduction

A solid can resist a shear stress by a static deflection; a fluid cannot.

We can say that a fluid at rest must be in a state of zero shear stress, a state often called the **hydrostatic stress condition** in structural analysis. In this condition, Mohr's circle for stress reduces to a point, and there is no shear stress on any plane cut through the element under stress.

There are two classes of fluids, **liquids** and **gases**.

- A liquid, being composed of relatively close-packed molecules with strong cohesive forces, tends to retain its volume and will form a free surface in a gravitational field if unconfined from above. Free-surface flows are dominated by gravitational effects.
- Since gas molecules are widely spaced with negligible cohesive forces, a gas is free to expand until it encounters confining walls. A gas has no definite volume, and when left to itself without confinement, a gas forms an atmosphere that is essentially hydrostatic. Gases cannot form a free surface, and thus gas flows are rarely concerned with gravitational effects other than buoyancy.



> Some apparently "solid" substances such as asphalt and lead resist shear stress for short periods but actually deform slowly and exhibit definite fluid behavior over long periods. Other substances, notably colloid and slurry mixtures, resist small shear stresses but "yield" at large stress and begin to flow as fluids do.
>
> The study of more general deformation and flow is called **rheology**.
>
> Liquids and gases can coexist in two-phase mixtures, this is called **multiphase flows**.
>
> In some situations the distinction between liquid and gas blurs. This is the case at temperatures and pressures above the **critical point** of a substance, where only a single phase exists, primarily resembling a gas.

Velocity Field

From dynamics, the velocity of a particle is (units SI `[m/s]`)

$$\begin{aligned}\vec{V} &= \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k} \\ &= u\hat{i} + v\hat{j} + w\hat{k}\end{aligned}$$

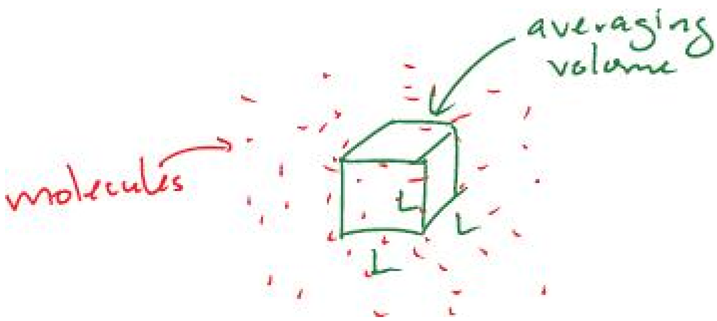
Fluid Parcel: a very small amount of fluid that moves together in one piece

Fluid Velocity: rate of change (ROC) of position with respect to (w.r.t.) time of a fluid parcel.

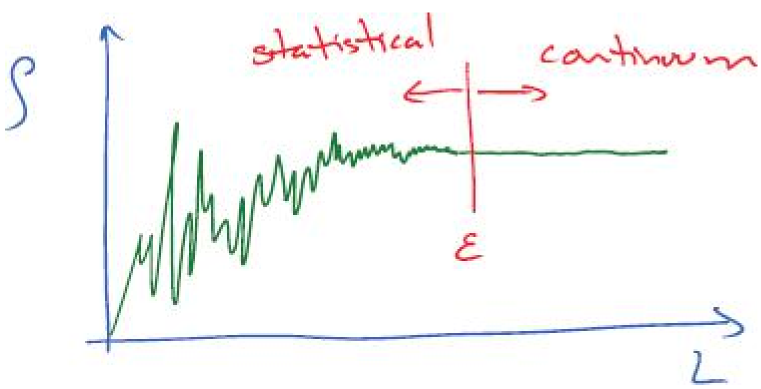
Note that this perspective (changes seen by parcel) is known as the **Lagrangian Perspective**. It implies the continuum assumption (sufficient molecules in parcel that properties are meaningful bulk averages of molecular motion).

Continuum Assumption: Properties at a point. Continuum approximation assumes that matter is infinitely divisible. It assumes that length scales of interest are large in comparison with molecular scales. And most importantly, each property is defined at a point, e.g. (x,y,z), (r,θ,z).

Density: $\rho = \lim_{L \to \epsilon} \frac{\sum m_i}{L^3}$



We can start box small and gradually increase L:



Cases where continuum assumption doesn't apply:

- In very small channels (capillary tubes?)
- Upper atmosphere (molecules are so far apart)
- Shock waves

Lagrangian Perspective vs **Eulerian Perspective**:

- Lagrangian Perspective focus on a single parcel the whole time. However, it is difficult to keep track of parcels without special techniques.

- Eulerian Perspective focus on position and measure the velocities of parcels as they move through a region of space. We will use this because its easier in practice.

Velocity Field: record of $\vec{V}$ at every point in the flow. $\vec{V}$ at a point (x, y, z) is the $\hat{V}$ of the parcel that fills that point at time instant t.

$$\vec{V}(x, y, z, t) = u(x, y, z, t)\hat{\imath} + v(x, y, z, t)\hat{\jmath} + w(x, y, z, t)\hat{k}$$

Similarly, we have scalar fields,

- $\rho = \rho(x, y, z, t)$

We also have tensor (e.g. stress tensor) $\overrightarrow{\overrightarrow{T}} = \overrightarrow{\overrightarrow{T}}(x, y, z, t)$

$$\overrightarrow{\overrightarrow{T}} = \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix}, \qquad \begin{cases} \text{row} \rightarrow \text{direction of stress} \\ \text{col} \rightarrow \text{stresses on plane} \end{cases}$$

Flow Visualization

Questions (dependent):

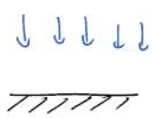
- How to mark the fluid
- How to interpolate what you see

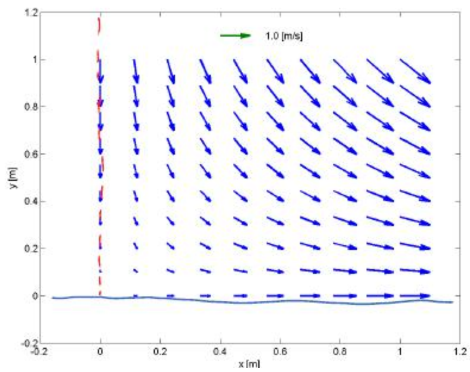
Vector Plot: at various points throughout the flow field, draw an arrow

- Base of arrow at the point
- Direction of arrow = direction of $\vec{V}$ at that point
- Length of arrow is proportional to $|\vec{V}|$ (speed)
- Color of arrow related to speed or other quantity (HW1 for example)

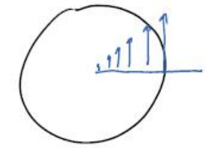
3 Examples: **Stagnation Flow** (flow come to a halt at a wall); Spinning bottle flow; Buth tub vortex.

stagnation flow

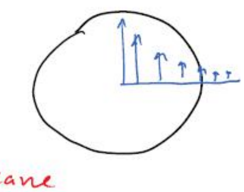




② Spinning bottle flow



③ Bath tub vortex



→ can combine to model a hurricane

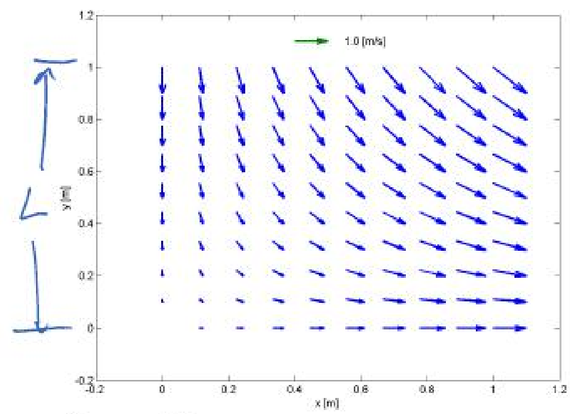
Example Question:

- Take a look at the top row (y=1m), then the vertical component of the vectors are the same.
- Similarly, take a look at bottom row,

Example: "reverse engineer" velocity field of stagnation flow

(at fixed z and time t_0)

$\vec{V}(x,y,z,t) = ?$



@ $y = 1\,m$; $v(x, 1m) \approx -1\,m/s$ for all x

$y = 0.5\,m$; $v(x, 0.5m) \approx -0.5\,m/s$ for all x

$y = 0\,m$; $v(x, 0m) \approx 0.0\,m/s$ for all x

$\Longrightarrow v(x,y,z,t_0) = -V_0 \frac{y}{L}$ where $V_0 = 1\,m/s$ $L = 1m$

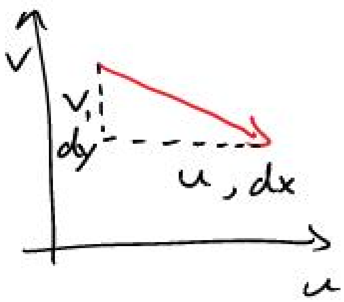
Similarly can show $u(x,y,z,t_0) = V_0 \frac{x}{L}$

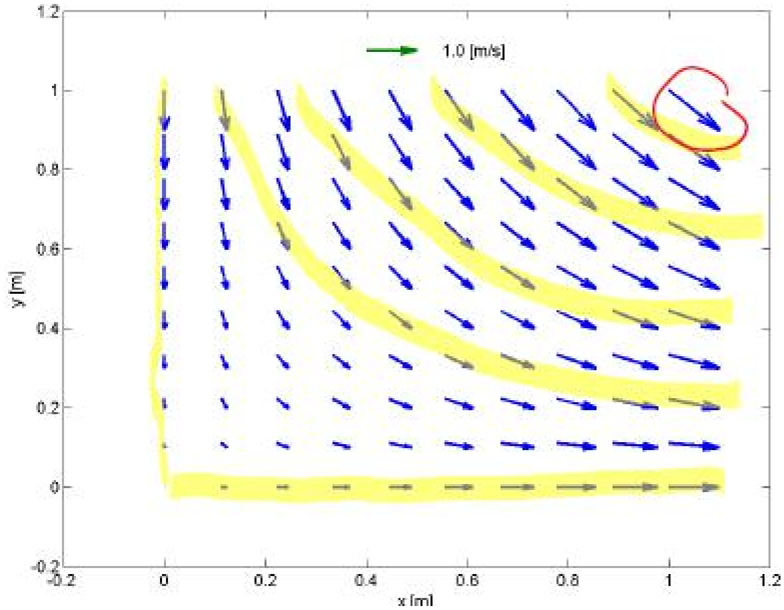
$$\boxed{\vec{V}(x,y,z,t_0) = \frac{V_0 x}{L}\hat{i} - \frac{V_0 y}{L}\hat{j} + 0\hat{k}}$$

Streamline: a line that is everywhere tangent to the local velocity vectors. At any point on a stream line

$$\frac{dy}{dx} = \frac{v}{u} \Longrightarrow \int_{x_0}^{x} \frac{dx}{u} = \int_{y_0}^{y} \frac{dy}{v}$$

where (x_0, y_0) is a reference point where you know the values, and (x, y) is any point on the streamline. As for u, v, they are the direction of this vector. Consider the example of stagnation flow, we have $u = \frac{V_0 x}{L}, v = -\frac{V_0 y}{L}$.

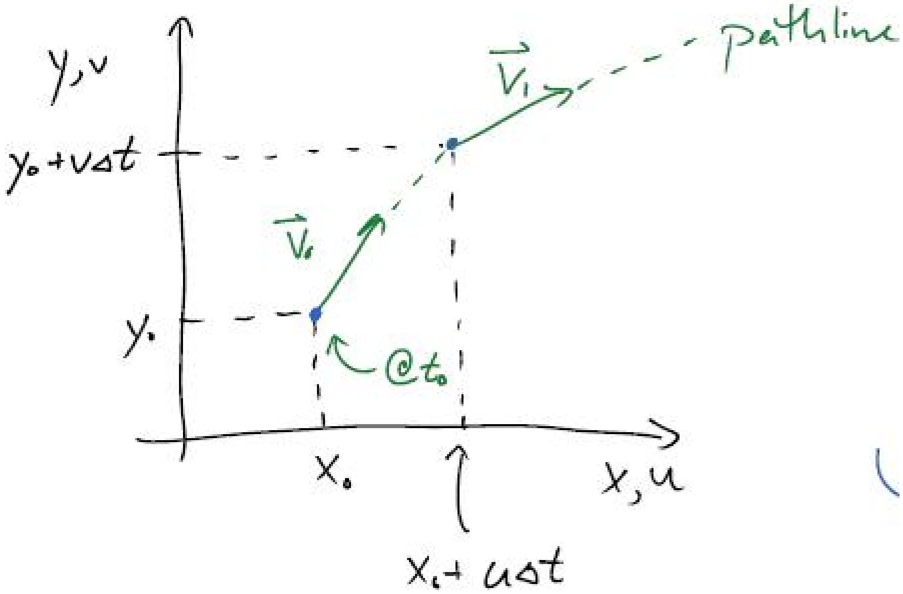




Pathline: Path followed by an individual fluid parcel.

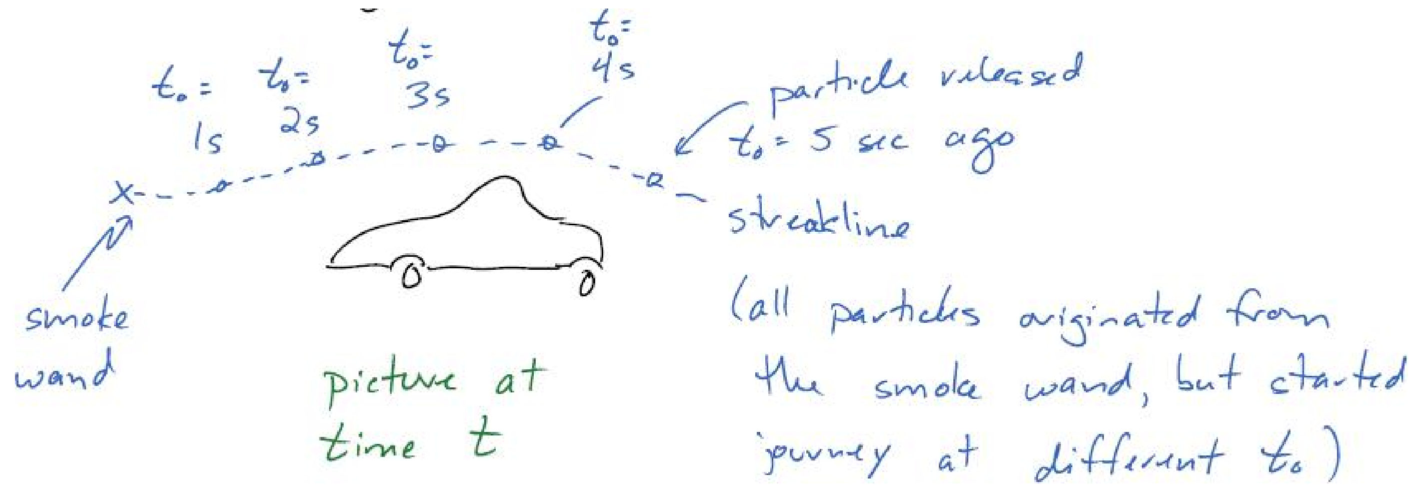
$$\begin{cases} \frac{dx}{dt} = u(x, y, z, t), & x(t_0) = x_0 \\ \frac{dy}{dt} = v(x, y, z, t), & y(t_0) = y_0 \end{cases}$$

After an initial value ODE, evaluate analytical or numerically.



Streakline: line joining the set of fluid parcels that have traveled through a fixed reference point. Use same ODE's as for pathline, but instead of being fixed and t evolving, now t is fixed, and t_0 is the variable, i.e. Time is present, t_0 are past frames.

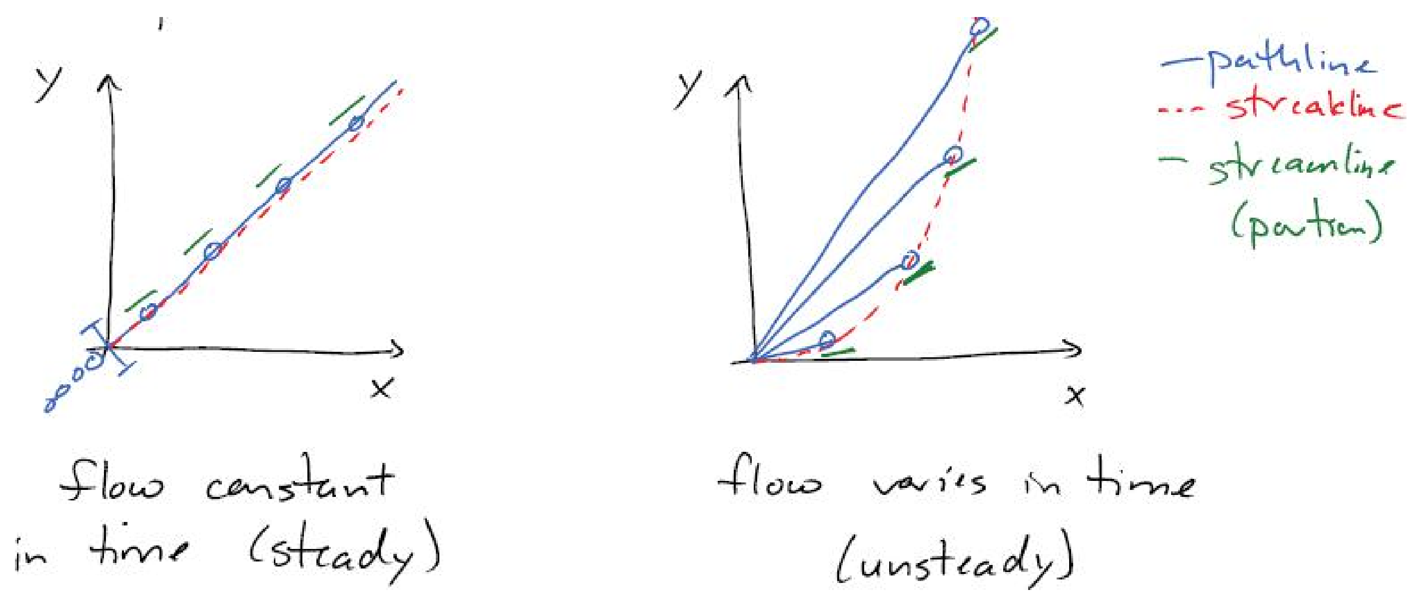
> This concept is hardest to picture. But imagine a line of smoke coming out of a single point A. At diffent point of this smoke line, it was a particle travelling through A t_0 time ago.



Comments:

1. Pathlines and streaklines evolve with time
2. Streamline pattern is a "snapshot" of complete flowfield at an instant in time.
3. Important: for steady flow, pathlines, streamlines, and streaklines coincide.
4. Flow does not cross a streamline by definition. Flow is contained within a streamline bundle (StreamTube).

Important Example:



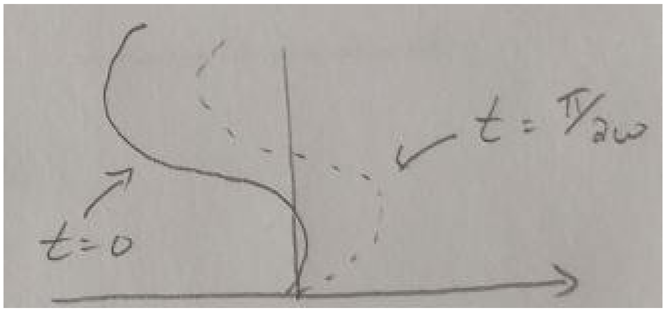
Example 2: Oscillating Sprinkler (garden sprinkler that's oscillating side to side). The velocity of the sprinkler is

$$\overrightarrow{V} = u_0 \sin(\omega(t - y/v_0)\hat{i} + v_0\hat{j}$$

Streamline: $\dfrac{dy}{dx} = \dfrac{v}{u} = \dfrac{V_0}{u_0 \sin(\omega(t - y/v_0)} \Longrightarrow u\left(\dfrac{v_0}{\omega}\right)\cos[\omega(t - y/v_0)] = v_0 x + c.$

Streamline @t=0: passing through $x = y = 0$, solve for initial condition and we get $c = \dfrac{u_0 v_0}{\omega}$. And we can solve for x

Streamline @t=$\pi/2\omega$ through origin, solve for initial condition and we get $c = 0$. As a result,

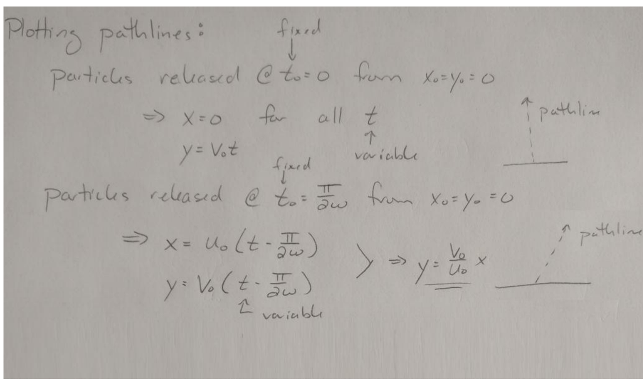


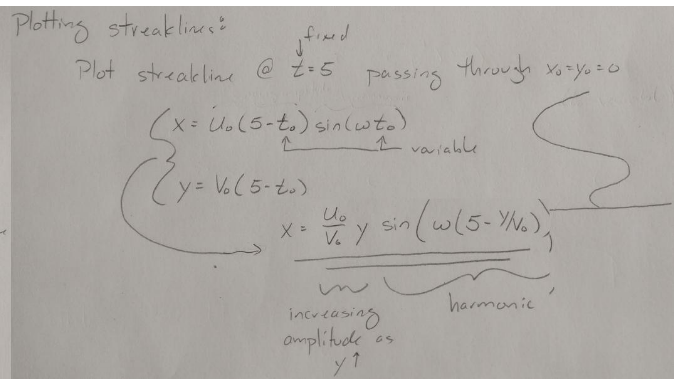
Pathline and Streakline:

$$\frac{dx}{dt} = u_0 \sin[\omega(t - y/V_0)] \Longrightarrow \int_{x_0}^{x} dx = \int_{t_0}^{t} u_0 \sin\left[\omega\left(\frac{y - y_0 + V_0 t_0}{V_0} - \frac{y}{V_0}\right)\right] dt$$

$$\Longrightarrow x - x_0 = -u_0 \sin\left[\omega\left(\frac{y_0 - V_0 t_0}{V_0}\right)\right](t - t_0)$$

$$\frac{dy}{dt} = V_0 \Longrightarrow \int_{y_0}^{y} dy = \int_{t_0}^{t} V_0 \, dt \Longrightarrow y - y_0 = V_0(t - t_0)$$





Example 3: Water in Glass on Turntable. Let $\overrightarrow{V}(x, y, z, t) = -\Omega y\hat{i} + \Omega x\hat{j} + 0\hat{k}$. Plot streamline through $(1m, 0m)$

$$\frac{dy}{dx} = \frac{v}{u} \Longrightarrow \Omega x dx + \Omega y dy = 0$$

$$\Longrightarrow \int_{x_0=1m}^{x_1} \Omega x \, dx + \int_{y_0=0m}^{y_1} \Omega y \, dy = 0$$

$$\Longrightarrow x_1^2 + y_1^2 = 1m \qquad \text{circle of radius 1m}$$

Terminology in Fluid Mechanics

In general, $\overrightarrow{V}(x, y, z, t)$ has 3 components (u,v,w), and 4 independent variables (x,y,z,t).

Dimensionality is the number of spatial dimensions that predominantly influence $\overrightarrow{V}$.

Example 1:

> $\vec{V} = xy\hat{\imath} + z\hat{\jmath}$ 2 component, 3D
>
> $\vec{V} = xyz\hat{\imath}$ 1 component, 3D
>
> $\vec{V} = t\hat{\imath} + 2t\hat{\jmath}$ 2 component, 0D
>
> Example 2: Consider clouds moving over a region. What are the number of components and dimensionality?
>
> Answer: since clouds move horizontally, the number of components=2 (u,v). Since wind at ground level is not the same as wind at higher altitude, dimensionality=1 (z).

We now move to types of flow:

Steady Flow: flow where $\dfrac{\partial \vec{V}}{\partial t} \approx 0$. The opposite of this is unsteady flow, e.g. stirred coffee cup coming rest.

> Example:
>
> Example: Car going 120 km/hr on straight highway with no wind. Is flow steady or unsteady
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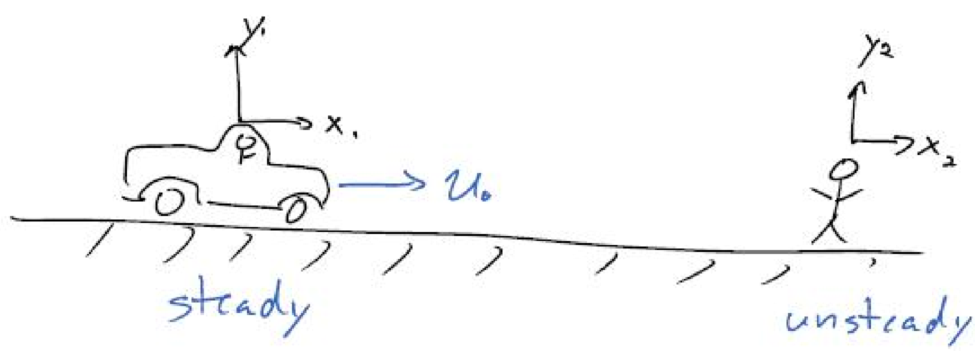
> 
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>
> depends on reference frame/coordinate system/point of view.

Laminar Flow:

- Flow where parcels move smoothly beside on another
- Generally requires low flow rates, small dimensions, and/or viscous (thick) fluids.
- Relatively uncommon in natural and industrial settings (shows up in micro-air vehicles, flow in small tubes, some biological flow.)

Turbulent Flow:

- Chaotic flow with a mean (average) flow and small scale 3D unsteady motion "superposed" on it.

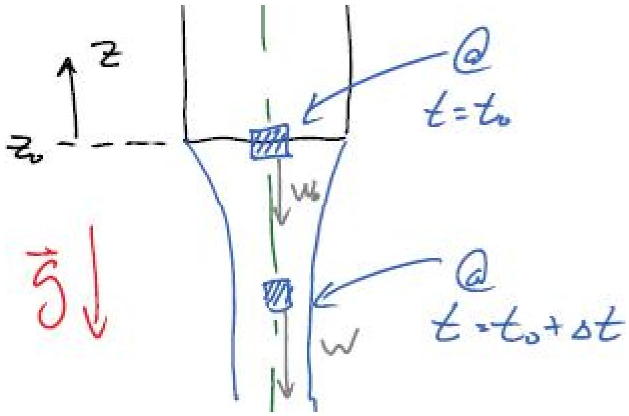
Factors for determining whether flow is lamina or turbulent:

- Flow speed
- Primary dimension (a length scale, e.g. tube diameter)
- Fluid propoerties (density and viscosity)

Acceleration Field

How do we calculate acceleration of parcels if we know the velocity field?

Example: Specific case of water jet issuing from a tap (small opening case). Consider a fluid parcel moving along the center-line: $(u = 0 = v, w \neq 0)$.



Firstly, from kinematics for a parcel:

$$\begin{cases} w(z,t) = -w_0 - gt \\ z(t) = z_0 - w_0 t - 0.5gt^2 \end{cases} \implies w(z) = -\sqrt{w_0^2 + 2g(z_0 - z)}$$

Note that w doesn't depend on t at a given z (velocity field is steady), but w changes with z. Question, is acceleration 0?

Secondly, using formal approach: $\vec{a}$ is R.O.C. of $\overrightarrow{V}$

$$a_z = \lim_{\Delta t \to 0} \frac{w(z_0 + \Delta z, t_0 + \Delta t) - w(z_0, t_0)}{\Delta t}$$

Recall from first-order approximation using Taylor series, we use

$$f(x + \Delta x, y + \Delta y) = f(x,y) + \left[\frac{\partial f(x,y)}{\partial x}\Delta x + \frac{\partial f(x,y)}{\partial y}\Delta y\right]$$

We then have

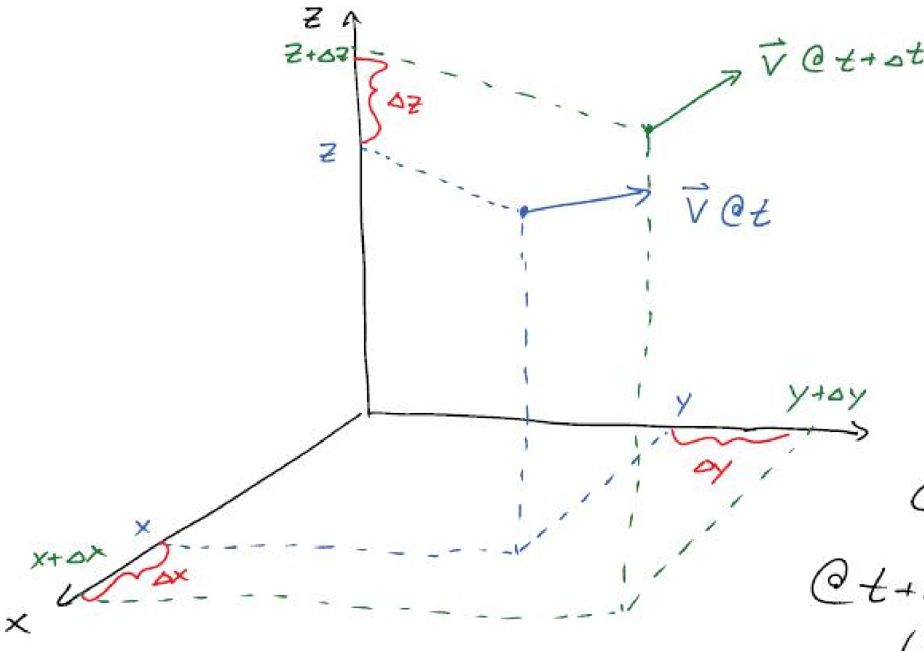
$$\begin{aligned} a_z &\approx \lim_{\Delta t \to 0} \left\{ \frac{[w(z_0,t_0) + \frac{\partial w}{\partial z}|_{z,t}\Delta z + \frac{\partial w}{\partial t}|_{z,t}\Delta t + \cdots] - w(z_0,t_0)}{\Delta t} \right\} \\ &= \lim_{\Delta t \to 0} \left\{ \frac{\partial w}{\partial z} w + \frac{\partial w}{\partial t} \right\} \qquad \text{since } \Delta z = w\Delta t \\ &= \underbrace{\frac{\partial w}{\partial t}}_{\substack{\textbf{local accel.} \\ \text{(change w. time)}}} + \underbrace{w\frac{\partial w}{\partial z}}_{\substack{\textbf{connection accel.} \\ \text{(change w. position)}}} \end{aligned}$$

In our example, $a_z = 0 + w\frac{\partial w}{\partial z} = -\sqrt{w_0^2 + 2g(z_0 - z)}\left[\frac{-2g}{-2\sqrt{w_0^2+2g(z_0-z)}}\right] = -g.$ □

Thirdly, using blind math approach (used in textbooks):

$$a_z = \frac{dw(z,t)}{dt} = \frac{\partial w}{\partial t}\frac{\partial t}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t} = \frac{\partial w}{\partial t} + w\frac{\partial w}{\partial z} \qquad \text{(chain rule)}$$

Fluid Acceleration General Case $\vec{a}(x, y, z, t)$



follow a fluid parcel in the flow given $\vec{V}(x, y, z, t)$

@ t parcel is @ (x, y, z)

@ $t+\Delta t$ parcel is @ $(x+\Delta x, y+\Delta y, z+\Delta z)$

as $\Delta t \to 0$ $\quad$ $\Delta x \to u\Delta t$

$\Delta y \to v\Delta t$

$\Delta z \to w\Delta t$

For fluid parcel,

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\vec{V}(x+\Delta x, y+\Delta y, z+\Delta z, t+\Delta t) - \vec{V}(x, y, z, t)}{\Delta t}$$

Analogous to $\vec{V}$, we can represent $\vec{a} = a_x \hat{i} + a_y \hat{j} + a_z \hat{k}$. Following the same approach as used in the last example, for each a_x, a_y, a_z component we get

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$
$$a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$
$$a_z = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

which is equivalent to

$$\vec{a} = \frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z} \quad \text{or} \quad \vec{a} = \frac{D}{Dt}\vec{V}$$

where (D/Dt) is the **material Lagrangian derivative substantial**.

> Example for $\partial \vec{V}/\partial t$:
>
> - Wind gusts: $\partial \vec{V}/\partial t = 0$
> - Flow from tap at constant opening: $\partial \vec{V}/\partial t = 0$.

Consider Lagrangian perspective $\vec{V}_{\text{parcel}} = \vec{V}(t)$, and we try to relate acceleration between Lagrangian and Eulerian ($\vec{V}(x, y, z, t)$ velocity field):

$$\vec{a}_{\text{parcel}} = \frac{d\vec{V}_{parcel}}{dt} = \frac{D}{Dt}\vec{V}$$

Since it's difficult to follow parcels and measure their properties, we do measurements in a fixed field.

> Example: Fluid converging in a nozzle: using square electrode to generate hydrogen bubbles in a pulse.
>
> Example #1: Flow in a Converging Nozzle
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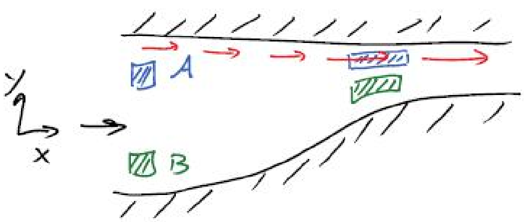
> 
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>
> (i) Velocity field
>
> - does not vary in time: steady
> - varies in space
>
> (ii) Acceleration
>
> - consider point A
>
> $\vec{a}_A = ?$ $\qquad \vec{a}_A > 0$
>
> Example 2: Glass of water on a turntable. Recall that $\vec{V} = -\Omega y\hat{i} + \Omega x\hat{j}$. Find $\vec{a} at $x = y = 1/\sqrt{2}$.
>
> $u = -\Omega y$
>
> $v = \Omega x$
>
> $w = 0 \qquad (\therefore a_z = 0)$
>
> | (x,y) | u | v |
> |---|---|---|
> | $(1,0)$ | 0 | Ω |
> | $(0,1)$ | $-\Omega$ | 0 |
> | $(1/\sqrt{2}, 1/\sqrt{2})$ | $-\Omega/\sqrt{2}$ | $\Omega/\sqrt{2}$ |
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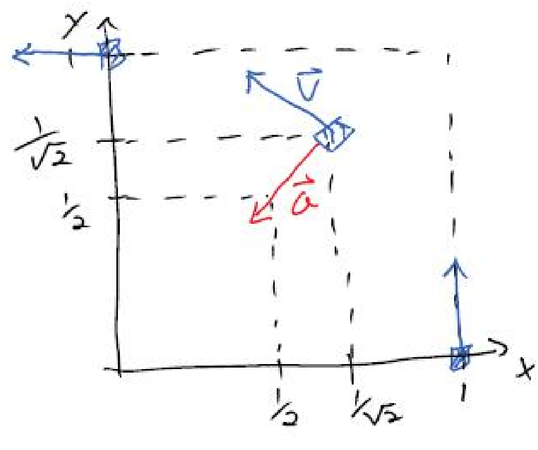
> 
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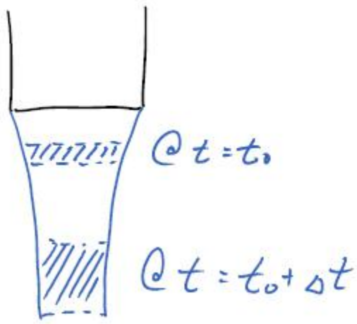
>
> Using the formula, we have $a_x = -\Omega^2 x, a_y = -\Omega^2 y$. And we get
>
> $$\vec{a} = -\Omega^2 x\hat{i} - \Omega^2 y\hat{j}, \quad \vec{a}(1/\sqrt{2}, 1/\sqrt{2}) = -\Omega^2/\sqrt{2}\hat{i} - \Omega^2/\sqrt{2}\hat{j}$$
>
> Note that speed doesn't change during streamline, with $|\vec{V}| = \Omega r$. The acceleration also doens't change and is $|\vec{a}| = |\vec{V}|^2/r$ which should be familiar from circular motion in kinematics. $\vec{a}$ acts towards the axis of rotation.

Fluid Strain Rates

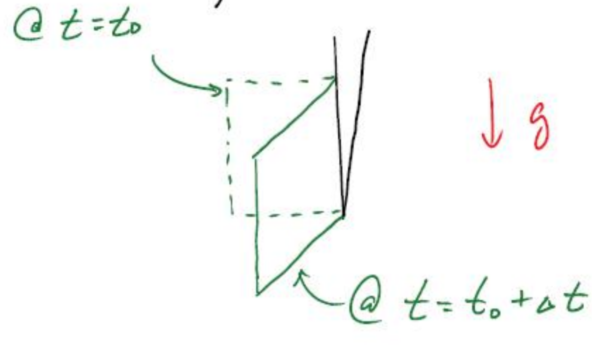
Recall the fluid converging in a nozzle example. We tracked the fluid parcel, and saw that it started square in shape, but shape changes continuously in two ways depending on relative motion of parcel faces.

More examples:

Water tap jet



Honey on knife



Parallel (opposite) faces
- volume/aspect ratio changes

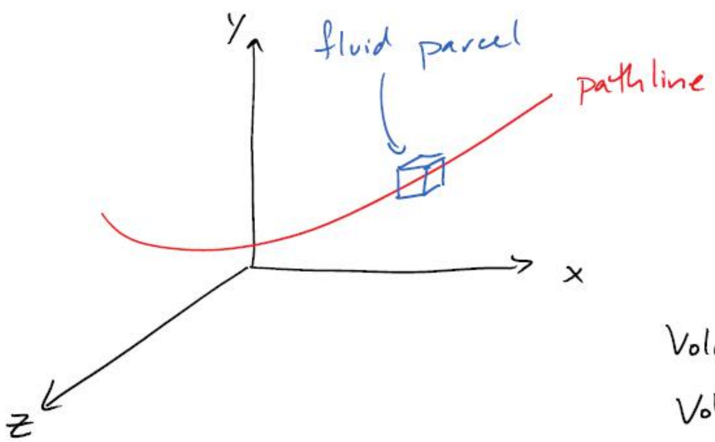
Volumetric Strain Rate
$\equiv$ R.O.C. of parcel volume w.r.t. time per unit volume

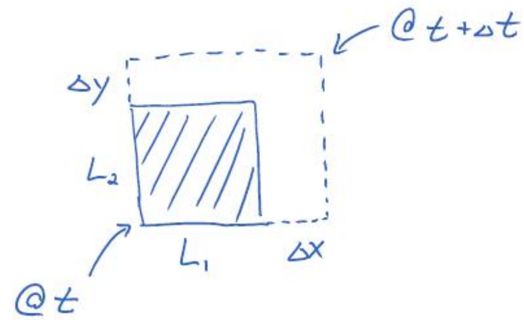
Perpendicular/adjacent faces
- angle changes

Shear Strain Rate
$\equiv$ R.O.C. of angle between adjacent faces

Volumetric Strain Rate: the rate of change of parcel volume with respect to time per unit volume.

Volumetric Strain Rate



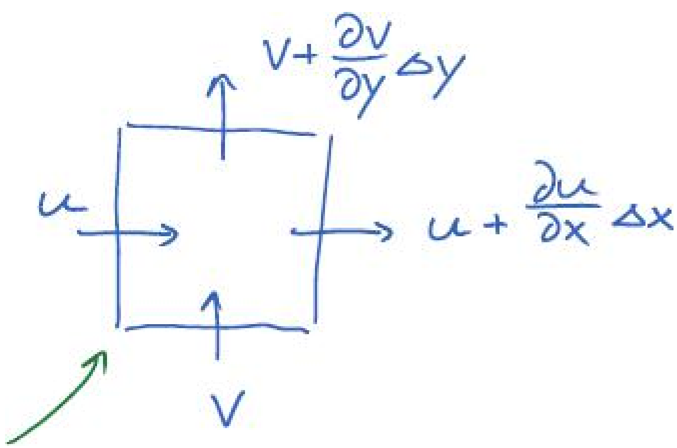


$$\text{Volume @ } t = L_1 L_2 L_3$$
$$\text{Volume @ } t + \Delta t = (L_1 + \Delta x)(L_2 + \Delta y)(L_3 + \Delta z)$$

> Note that for simplicity, this transcription uses $\mathbf{V}$ to represent volume, and $\vec{V}$ to represent velocity.

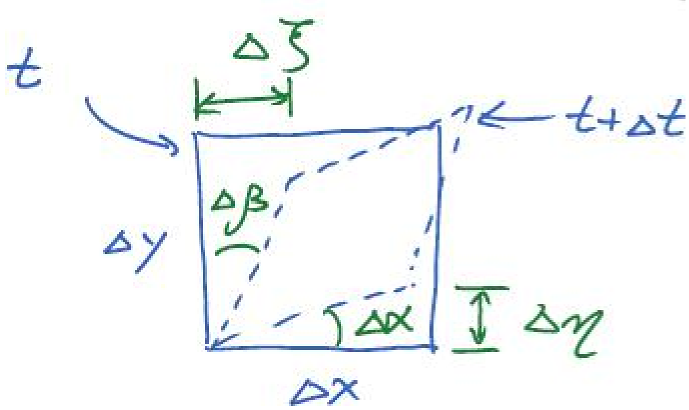
We have

$$
\begin{aligned}
\Delta \mathbf{V} &= \mathbf{V}(t+\Delta t) - \mathbf{V}(t) \\
&\approx L_2 L_3 \Delta x + L_1 L_3 \Delta y + L_1 L_2 \Delta z + \text{(H.O.T.)} \\
\Delta x &= \left(u + \frac{\partial u}{\partial x} L_1 - u \right) \Delta t \qquad \text{(see diagram below.)} \\
&= \frac{\partial u}{\partial x} L_1 \Delta t \\
\Delta y &= \frac{\partial v}{\partial y} L_2 \Delta t \\
\Delta z &= \frac{\partial w}{\partial z} L_3 \Delta t \\
\Delta V &= \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) L_1 L_2 L_3 \Delta t \\
\text{Vol. Strain} &= \frac{\Delta \mathbf{V}}{\mathbf{V}} \\
\text{Vol. Strain Rate} &= \frac{d}{dt} \frac{\Delta \mathbf{V}}{\mathbf{V}} \\
&= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \nabla \cdot \vec{V} \quad \text{(divergence operator)}
\end{aligned}
$$

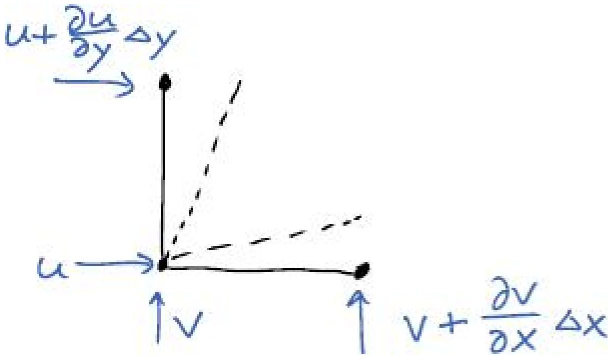


If volume is constant, fluid is said to be **incompressible**, i.e. $\nabla \cdot \vec{V} = 0$.

Shear Strain Rate: $\dot{\epsilon}_{xy} = \frac{d\beta}{dt} + \frac{d\alpha}{dt}$ (see diagram)



$$
\tan \Delta \alpha = \frac{\Delta \eta}{\Delta x} \approx \Delta \alpha \qquad \tan \Delta \beta = \frac{\Delta \zeta}{\Delta y} \approx \Delta \beta
$$



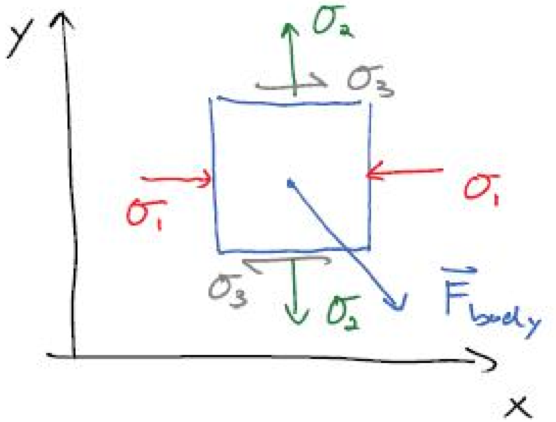
$$\Delta\eta = \left(v + \frac{\partial v}{\partial x}\Delta x - v\right)\Delta t = \frac{\partial v}{\partial x}\Delta x \Delta t \Longrightarrow \Delta\alpha = \frac{\partial v}{\partial x}\Delta t$$

$$\frac{d\alpha}{dt} \approx \frac{\Delta\alpha}{\Delta t} = \frac{\partial v}{\partial x}$$

$$\frac{d\beta}{dt} \approx \frac{\partial u}{\partial x} \qquad \text{similarly}$$

$$\dot{\epsilon}_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \qquad \dot{\epsilon}_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \qquad \dot{\epsilon}_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$

Forces on Fluids



(i) Stresses (act on surfaces)

σ_1 - compression (pressure)

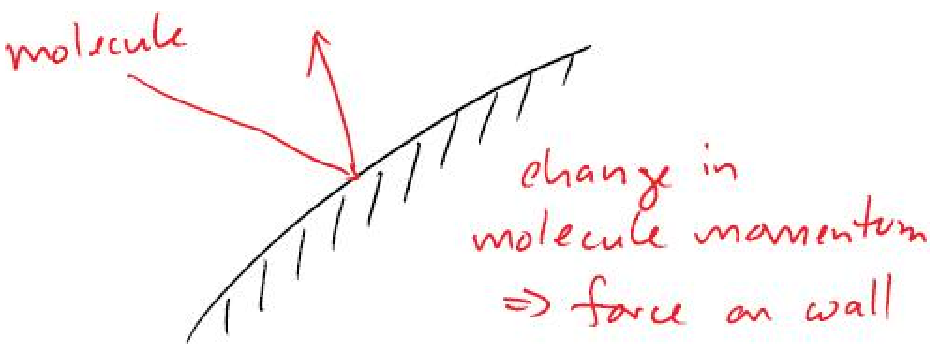
σ_2 - tension (rare)

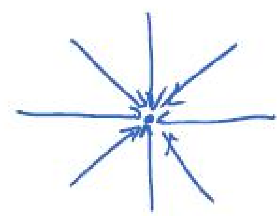
σ_3 - shear: friction, surface tension

- Compression: liquids are incompressible but gases can be compressed.
- Shear Stress: fluids cannot withstand (by definition)
- Body forces: act on every fluid parcel in the flow, e.g. gravity, electromagnetism, coriolis, etc.

Pressure

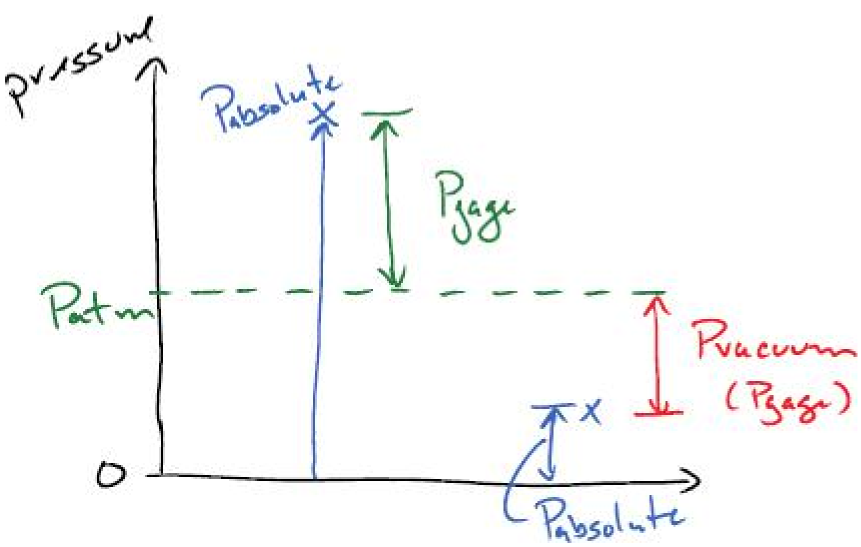
Pressure is the normal stress acting on a fluid element in compression. It is a continuum result of molecular momentom exchange; It acts in compression and normal to any plane in fluid; It acts uniformly in all directions at a point.



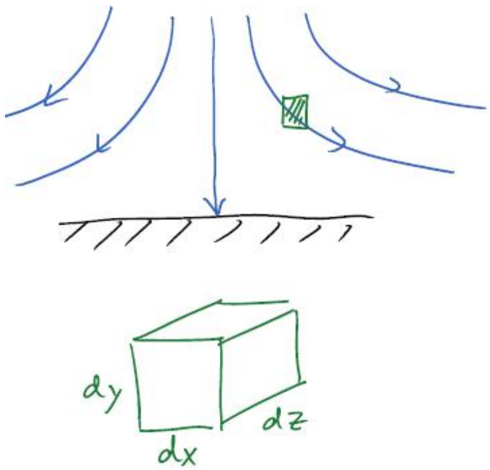


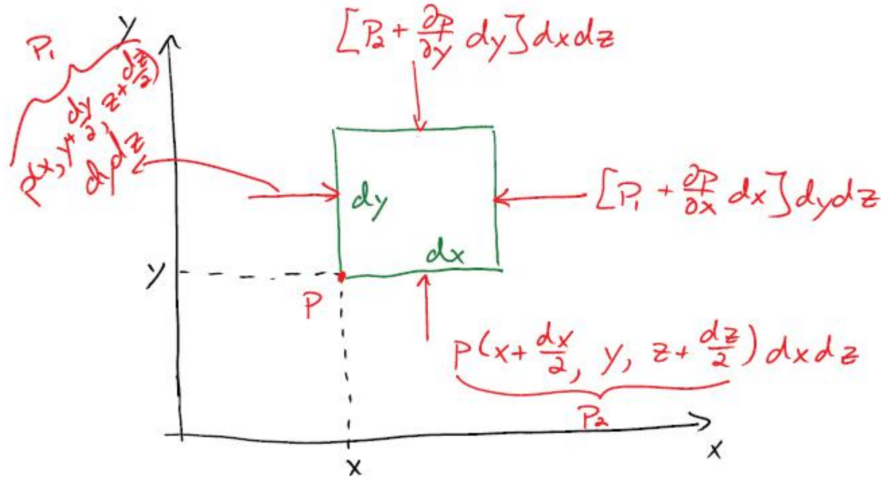
Atmospheric Pressure, $P_{atm} \approx 10^5$ Pa @ sea level.

- Absolute Pressure (w.r.t. vacuum)
- Relative Pressure (w.r.t. local atmosphere)
  - Gage Pressure $(P - P_{\text{atm}})$ where $P \geq P_{\text{atm}}$.
  - Vacuum Pressure $(P_{\text{atm}} - P)$ where $P \leq P_{\text{atm}}$.



Net force due to pressure $d\vec{F}_{\text{pressure}}$: consider a fluid element in a flow where $P(x, y, z, t)$





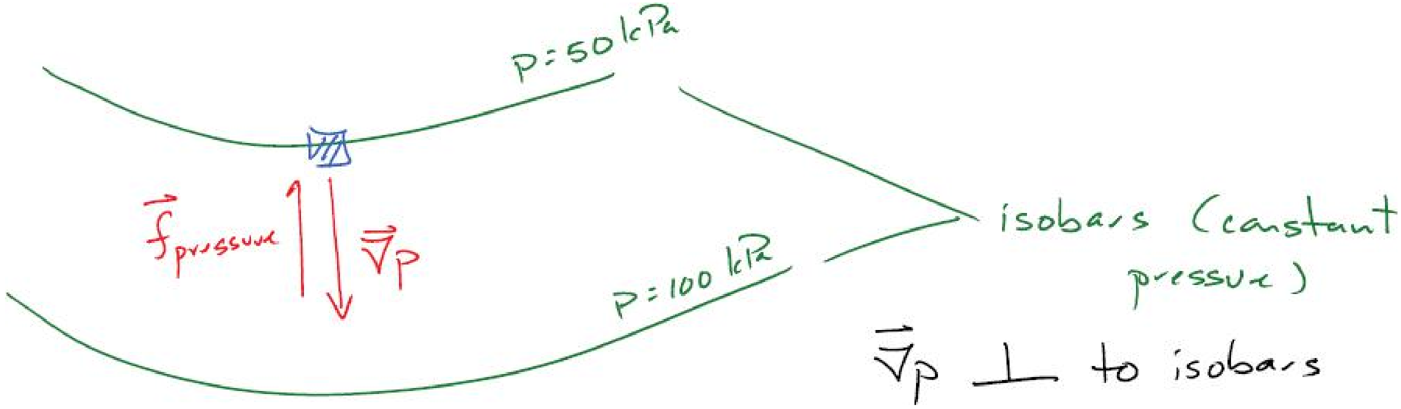
Deriving the components of $d\vec{F}_{\text{pressure}}$:

$$
\begin{aligned}
dF_x &= P_1 dy\,dz - \left[P_1 + \frac{\partial P}{\partial x}dx\right] dy\,dz = -\frac{\partial P}{\partial x}dx\,dy\,dz \\
dF_y &= P_2 dx\,dz - \left[P_2 + \frac{\partial P}{\partial y}dy\right] dx\,dz = -\frac{\partial P}{\partial y}dx\,dy\,dz \\
dF_z &= P_3 dx\,dy - \left[P_3 + \frac{\partial P}{\partial z}dz\right] dx\,dy = -\frac{\partial P}{\partial z}dx\,dy\,dz \\
d\vec{F}_{\text{pressure}} &= dF_x\,\hat{\imath} + dF_y\,\hat{\jmath} + dF_z\,\hat{k} \\
&= \left[-\frac{\partial P}{\partial x}\hat{\imath} - \frac{\partial P}{\partial y}\hat{\jmath} - \frac{\partial P}{\partial z}\hat{k}\right] \underbrace{dx\,dy\,dz}_{d\mathbf{V}} \\
&= -\vec{\nabla} P\, d\mathbf{V} \qquad \text{gradient operator}
\end{aligned}
$$

Define f to be the **net force per unit volume**.

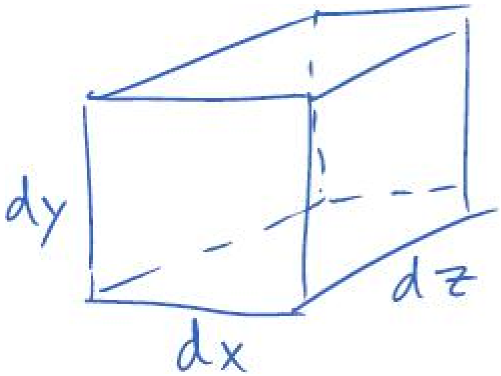
$$\vec{f}_{\text{pressure}} = -\vec{\nabla} P$$

where $\vec{\nabla} P$ is the vector that gives the magnitude and direction of maximum spatial increase in P.



Equilibrium of a Fluid Element

Consider a fluid element in the flow:



Apply Newton's 2nd Law, $\sum \vec{F}_{\text{applied}} = \vec{a}_{\text{mass}}$

Applied force typically come from gravity, pressure and friction.

$$\underbrace{\vec{F}_{gravity}}_{\rho\,dx\,dy\,dz\,\vec{g}} + \underbrace{\vec{F}_{pressure}}_{-\vec{\nabla}p\,dx\,dy\,dz} + \underbrace{\vec{F}_{friction}}_{\vec{f}_{friction}\,dx\,dy\,dz} = m\vec{a} = \rho\,\underbrace{dx\,dy\,dz}_{d\forall}\,\vec{a}$$

Per unit volume (divided by $d\mathbf{V}$ = $dx\ dy\ dz$)

$$\rho\vec{g} - \vec{\nabla}P + \vec{f}_{\text{friction}} = \rho\vec{a} = \rho\frac{D}{Dt}\vec{V}$$

And we arrive at the **governing equation**:

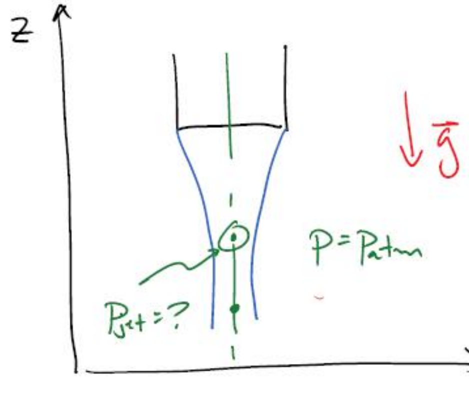
$$\rho\frac{D\vec{V}}{Dt} = \rho\vec{g} - \vec{\nabla}P + \vec{f}_{\text{friction}}$$

Note that $\sum \vec{F} = 0 \iff \vec{a} = 0$. Friction is present in most applications (opposes motion).

Need pressure or gravity to drive the flow.

Example: Pressure inside the water jet from tap (or air jet from compressed gas):

Example Pressure inside the water jet from tap (or air jet from compressed gas)



$$\rho \vec{a} = -\vec{\nabla} p + \rho \vec{g} + \vec{f}_{friction}$$

on the centerline:

$$a_z = -g$$

$u = 0 \implies a_x = 0$

$\implies$ no motion in x

$\rightarrow$ no friction in x

Project the governing equation on x

$\rightarrow$ x: $\rho \cancel{a_x}^{\,0} = -\frac{\partial p}{\partial x} \implies \frac{\partial p}{\partial x} = 0$

$\therefore \boxed{P_{jet} = P_{atm}}$

$$\rho a_z = -\cancel{\frac{\partial p}{\partial z}}^{\,0} - \rho g + \cancel{f_{fric}}^{\,0}$$

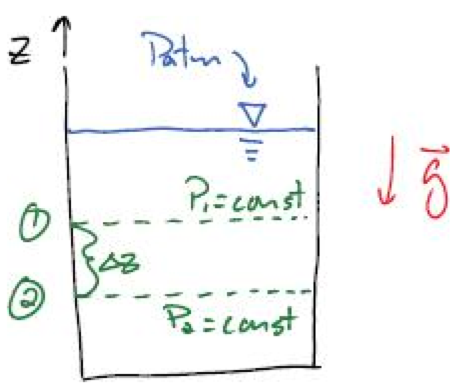
$$\cancel{\rho} a_z = -\cancel{\rho} g \qquad a_z = -g$$

Pressure Variation in Fluids

Hydrostatic Pressure: hydrostatic means fluid is at rest $(\vec{V} = 0 \implies \vec{a} = 0)$

$$\rho \vec{a} = \rho \frac{D\vec{V}}{Dt} = 0 = -\vec{\nabla} p + \rho \vec{g} + \vec{f}_{\text{friction}} = -\vec{\nabla} p + \rho \vec{g} \quad \text{(no motion implies no friction)}$$

Therefore $\vec{\nabla} p = \rho \vec{g}$.

> Example: Liquid in a container
>
> 
>
>
> Describing the pressure fluid: we can do so qualitatively, $\vec{\nabla} = \rho \vec{g}$. By definition $\vec{\nabla} p \perp$ to isobars. Isobars are also $\perp \vec{g}$ Also, sign of $\vec{\nabla} p$ is the same as a sign of $\rho \vec{g}$. This implies pressure increases in direction of $\vec{g}$
> .
>
> Quantitatively,

$x:\ \frac{\partial p}{\partial x} = 0 \quad \Rightarrow \quad p$ is independent of x

$y:\ \frac{\partial p}{\partial y} = 0 \quad \Rightarrow \quad$ " " " " y

$z:\ \frac{\partial p}{\partial z} = -\underbrace{\rho g}_{\gamma}$

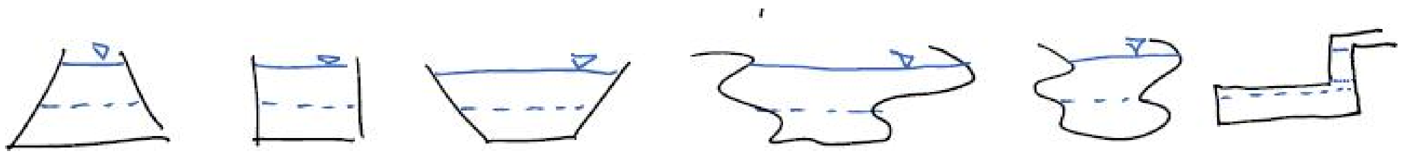
$\gamma \rightarrow$ "specific weight" $\equiv \frac{\text{weight of fluid}}{\text{unit volume}} = \frac{mg}{V} = \rho g \ [\frac{N}{m^3}]$

$\Rightarrow p = p(z)$

$$\Rightarrow \frac{dp}{dz} = -\rho g \Rightarrow \int_{1}^{2} dp = \int_{1}^{2} -\rho g\, dz$$

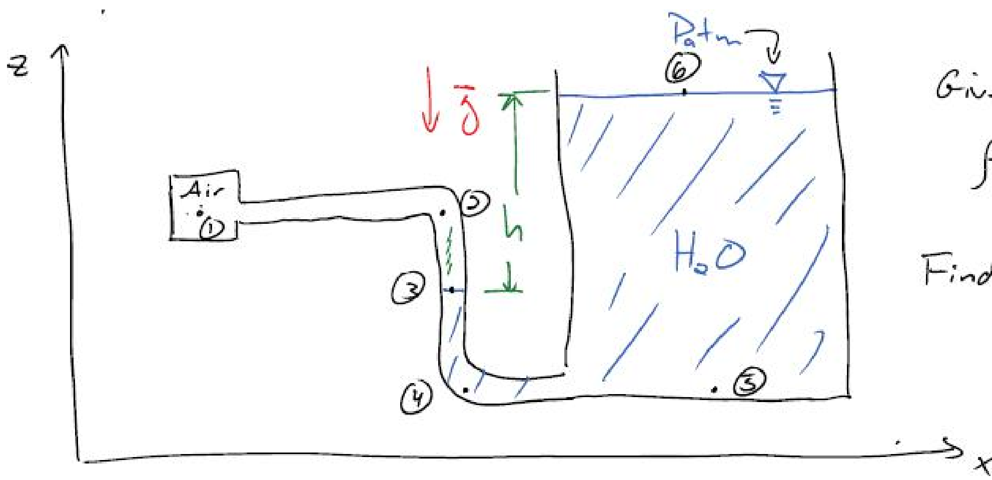
Which implies $P_2 - P_1 = \rho g(z_1 - z_2) = \rho g \Delta z$

Note that in continuous and uniform static fluid, $p = p(z)$ and does not depend on the shape of the container:



P increases with depth and decreases with altitude: $P_{\text{atm}} \approx 10^5 N/m^2 =$ weight of fluid column (atmosphere) above a 1 m^2 area.

Example 2: Hydrostatic Pressure in a Fluid System:



Given: fluids @ rest

ρ_{air}, ρ_{H_2O}, h

Find: P_{air} (gage)

$$\boxed{\begin{aligned}\rho_{air} &= 1.2\ kg/m^3 \\ \rho_{H_2O} &= 1000\ kg/m^3\end{aligned}}$$

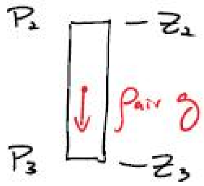
Method: begin at one end of the system and move systematically in z and x towards the other, and move 1 to 6.

① → ② $\vec{\nabla} p = \rho_{air} \vec{g} \implies x: \frac{\partial p}{\partial x} = 0 \implies \underline{P_1 = P_2}$

∴ same elevation ⇒ same pressure

② → ③

$$z: \quad \frac{\partial p}{\partial z} = -\rho_{air} g \implies \int_{P_2 = P_1}^{P_3} dp = -\int_{z_2}^{z_3} \rho_{air} g \, dz$$

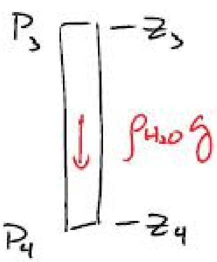


$$\underbrace{P_3 - P_2}_{\text{net force per unit area}} = \underbrace{-\rho_{air} g (z_3 - z_2)}_{\text{force due to gravity per unit area}}$$

$$\underline{P_3 = P_2 + \rho_{air} g (z_2 - z_3)} \qquad (P_3 > P_2 \text{ as expected})$$

Note: $\rho_{\rm air} = 1.2$ kg/m^3. If $\Delta z = 10m \implies \Delta p = (1.2)(9.8)(10) \approx 120$ Pa $\ll P_{\rm atm}$. For static gases, can often neglect Δp due to small elevation, for example $P_3 \approx P_2$.

③ → ④



$$\frac{dp}{dz} = -\rho_{H_2O} g$$

$$P_4 - P_3 = -\rho_{H_2O} g (z_4 - z_3)$$

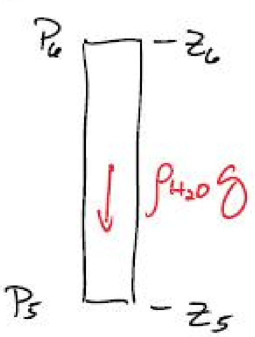
$$\underline{P_4 = P_3 + \rho_{H_2O} g (z_3 - z_4)} \qquad P_4 > P_3$$

Note: $\rho_{H_2O} = 1000$ kg/m^3 ⇒ weight is significant even for small Δz

④ → ⑤

$z = \text{const} \qquad \underline{P_5 = P_4}$

⑤ → ⑥



$$\frac{dp}{dz} = -\rho_{H_2O} g$$

$$P_6 - P_5 = -\rho_{H_2O} g (z_6 - z_5)$$

$$(*) \quad \underline{P_6 = P_5 - \rho_{H_2O} g (z_6 - z_5)} \qquad (P_6 < P_5)$$

Finally,

Combine:

$$P_5 = P_4$$
$$\left.\begin{array}{l} P_4 = P_3 + \rho_{H_2O}\, g\,(z_3 - z_4) \\ P_3 = P_2 = P_1 \end{array}\right\} \text{sub into } (*)$$

$$P_6 = P_{atm} = P_1 + \rho_{H_2O}\, g\,(z_3 - \cancel{z_4}) - \rho_{H_2O}\, g\,(z_6 - \cancel{z_5}) \qquad z_4 = z_5$$

$$P_{atm} = P_1 + \rho_{H_2O}\, g\,\underbrace{(z_3 - z_4)}_{-h}$$

$$\Rightarrow \boxed{\underbrace{P_1 - P_{atm}}_{P_{1,gage}} = \rho_{H_2O}\, g h}$$

Note that in practice, we can do this all in one line:

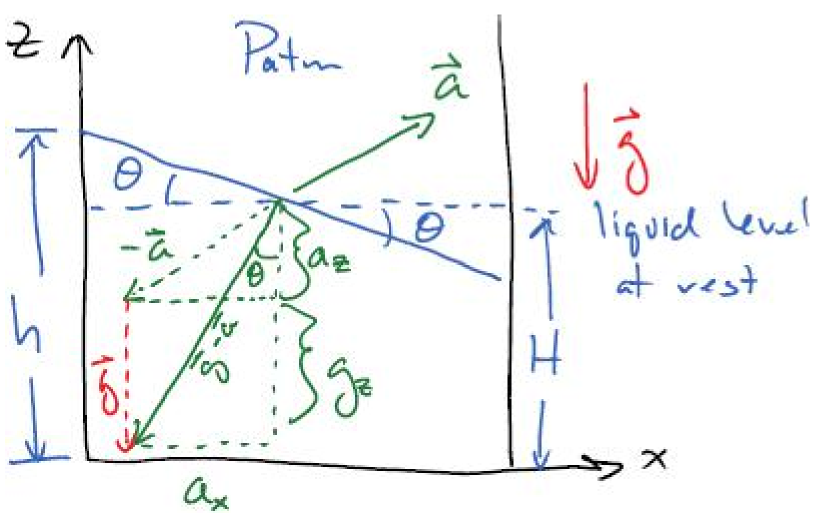
- 1 to 6: $P_1 + \rho_{\mathrm{H_2O}} g(z_3 - z_6) = P_{\mathrm{atm}}$
- $P_1 = P_2 \approx P_3$ and $P_3 - P_6 = \rho_{\mathrm{H_2O}} gh \Longrightarrow P_1 - P_{\mathrm{atm}} = \rho_{\mathrm{H_2O}} gh$

Fluid Pressure in Rigid Body Motion

Rigid body motion is all fluid parcels are in combined translation and rotation, with no relative motion (hence no friction)

$$\rho\vec{a} = -\vec{\nabla}p + \rho\vec{g} + \vec{f}_{\mathrm{friction}} \qquad \therefore \vec{\nabla}p = \rho(\vec{g} - \vec{a})$$

Important Example 1: **Uniform linear acceleration** (e.g. fuel tank on an acceleration car; or a glass on a turn table). Given $\vec{a}, \vec{g}, \rho$ geometry, find (a) shape of the free surface (b) expression for $p(x, z)$.



Solution: Given equation: $\vec{\nabla}p = \rho(\vec{g} - \vec{a})$

(a) **Free surface**: can solve based on vector calculus and geometry, $\vec{\nabla}P \perp$ to isobars, and therefore $(\vec{g} - \vec{a}) \perp$ to isobars. Free surface is an isobar $(P = P_{\mathrm{atm}} = \mathrm{const.})$. Therefore, free surface is a line

$$\tan\theta = \frac{a_x}{g+a_z}$$

Using Calculus,

$$\vec{\nabla}P = \frac{\partial P}{\partial x}\hat{\imath} + \frac{\partial P}{\partial y}\hat{\jmath} + \frac{\partial P}{\partial z}\hat{k} = \rho(\vec{g}-\vec{a})$$

In the x-axis, $\frac{\partial P}{\partial x} = -\rho a_x \neq 0$, note that $\frac{\partial P}{\partial x} \neq 0$ because acceleration is not just due to gravity. In the z-axis, $\frac{\partial P}{\partial z} = \rho(-g-a_z) = -\rho(g+a_z)$. Free surface being isobar means $dP = 0$ among free surface. For $p = p(x,z)$ where dP is the total differential:

$$\begin{aligned}
dP &= \frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial z}dz = -\rho a_x dx - \rho(g+a_z)dz = 0\\
\therefore \frac{dz}{dx} &= \frac{-a_x}{g+a_z} \quad \text{Note that } \frac{dz}{dx} = \tan(\pi-\theta) = -\tan\\
\therefore z &= -\frac{a_x}{g+a_z}x + h \quad \text{where } h \text{ is constant}
\end{aligned}$$

To find h, it must satisfy conservation of volume, i.e. $\mathbf{V}_{\text{tank}} = \text{const.}$

(b) To find genereal $p(x,z)$, we note

$$\begin{aligned}
\int dP &= \int -\rho a_x dx - \int \rho(g+a_z)dz\\
P &= -\rho a_x x - \rho(g+a_z)z + \text{const.}
\end{aligned}$$

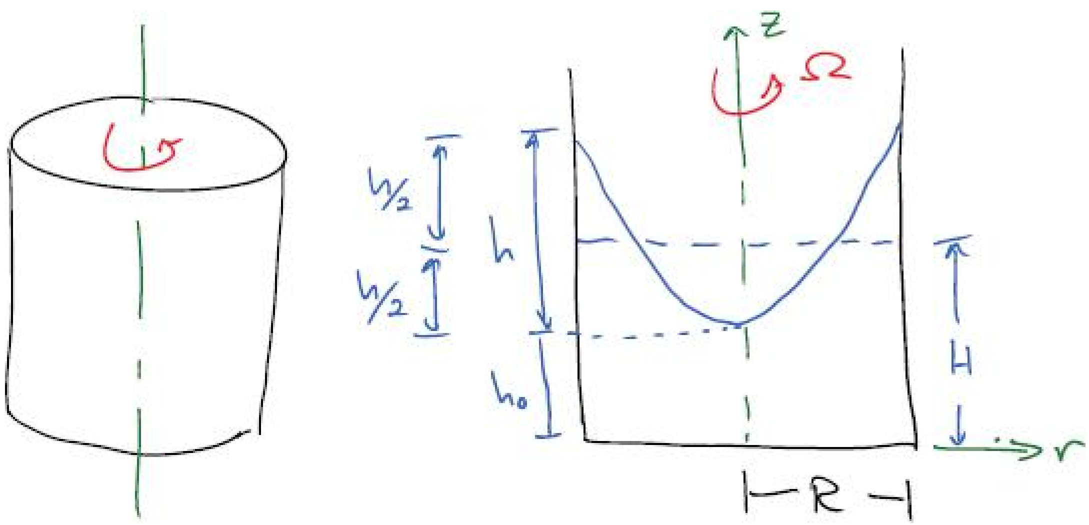
Again, the constant is based upon a known pressure in the fluid at a given point, e.g. $P(x=0, z=h) = P_{\text{atm}}$. From which we get $\text{const} = P_{\text{atm}} + \rho(g+a_z)h$. Hence, we arrive at

$$P = P_{\text{atm}} - \rho a_x x - \rho(g+a_z)(z-h)$$

A couple observations:

- For z=const, x increase $\implies$ p decreases
- For x=const, z increase $\implies$ p decreases
- Max P? from the diagram, it's the bottom left corner.

Important Example 2: **Glass on the middle of turntable**. Given H $[\text{m}]$, R $[\text{m}]$, Ω $[\text{rad/s}]$. Find (a) free surface shape, (b) equation for free surface (c) $P(r,z)$



Solution: given the following equation (note that $\frac{\partial P}{\partial \theta} = 0$)

$$\vec{\nabla} P = \frac{\partial P}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial P}{\partial \theta}\hat{e}_\theta + \frac{\partial P}{\partial z}\hat{e}_z = \rho(\vec{g} - \vec{a})$$

(a) Need to know acceleration field. Recall that acceleration is $|\vec{a}| = r\Omega^2$. Note: in our example $|\vec{a}|$ is a function of r and is difficult to use geometric approach. But we can say that since free surface should be locally perpendicular to $(\vec{g} - \vec{a})$, we expect curved isobars.

Project the governing equations:

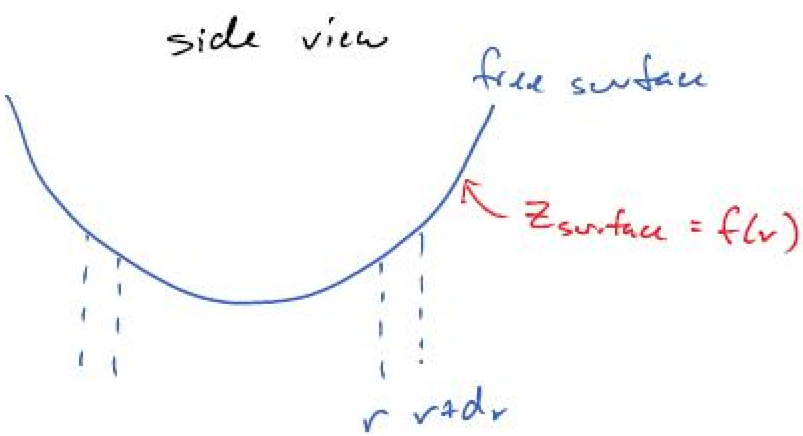
$$\begin{aligned}
\frac{\partial P}{\partial r} &= \rho[0 - (-r\Omega^2)] = \rho r \Omega^2 \\
\frac{\partial P}{\partial z} &= -\rho g \\
dP &= \frac{\partial P}{\partial r}dr + \frac{\partial P}{\partial z}dz = \rho r \Omega^2\, dr - \rho g\, dz
\end{aligned}$$

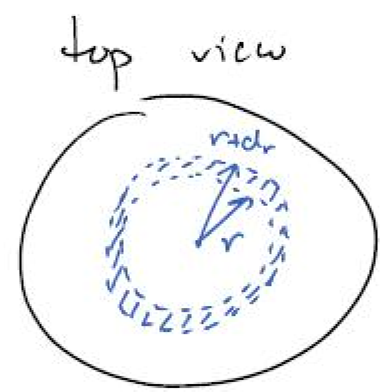
For free surface: $P = P_{\text{atm}} = \text{const} \implies dP = 0$. And then $\rho r \Omega^2\, dr - \rho g\, dz = 0 \implies dz = \frac{r\Omega^2}{g} dr$. Therefore free surface is a **paraboloid**

$$z = \frac{\Omega^2 r^2}{2g} + \text{const.}$$

(b) Need to determine the constant of the above result to quantitatively evaluate the change in free surface position. Again, we can obtain the constant by satisfying $\mathbf{V}$ = volume of fluid is constant. Note that

- At rest: $\mathbf{V} = \pi R^2 H$
- In motion: $\mathbf{V} = \text{vol. under free surface}$





$$\begin{aligned}
d\mathbf{V} &= \pi(r + dr)^2 z_s - \pi r^2 z_s \\
&= \pi(r^2 + 2r\, dr + dr^2) z_s - \pi r^2 z_s \\
&\approx 2\pi r\, dr\, z_{\text{surface}}(r)
\end{aligned}$$

For convenience, let $z_{\text{surf}}(r = 0) = h_0$ (the unknown constant) and that $z_{\text{surf}} = \frac{\Omega^2 r^2}{2g} + h_0$, we have

$$
\begin{aligned}
\mathbf{V} &= \int_0^R 2\pi r z_{\text{surf}}(r)\, dr \\
&= 2\pi \int_0^R \left(\frac{\Omega^2 r^3}{2g} + h_0 r \right)\, dr \\
&= \frac{\pi \Omega^2 R^4}{4g} + \pi R^2 h_0 = \pi R^2 H \qquad \text{(constant)} \\
\implies H &= \frac{\Omega^2 R^2}{4g} + h_0 \\
\implies h_0 &= H - \frac{\Omega^2 R^2}{4g}
\end{aligned}
$$

Therefore, we arrive at the **free-surface shape formula**:

$$
z_{\text{surf}} = \frac{\Omega^2}{2g}\left(r^2 - \frac{R^2}{2} \right) + H
$$

Position of top edge $(v = R)$:

$$
z(v = R) = h + h_0 \qquad h = \frac{\Omega^2 R^2}{2g}
$$

(c) Integrate $dP = \rho r \Omega^2\, dr - \rho g\, dz$,

$$
P = \frac{1}{2}\rho \Omega^2 r^2 - \rho g z + \text{const}
$$

Again, get constant from $P(r = 0, z = h_0) = P_{\text{atm}}$ (could use any other point on free surface), and we get $\text{const} = P_{\text{atm}} + \rho g h_0$. Therefore,

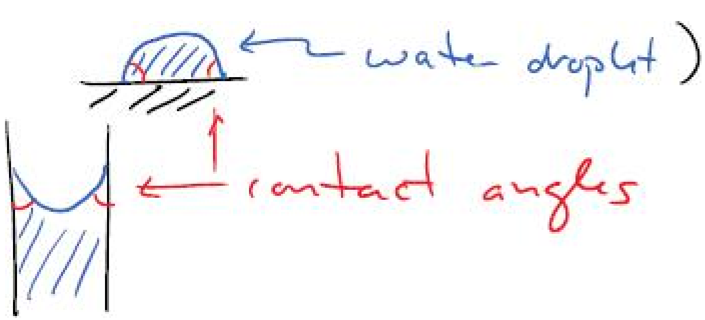
$$
P = P_{\text{atm}} + \rho \frac{\Omega^2 r^2}{2} + \rho g (H - \frac{\Omega^2 R^2}{4g} - z)
$$

Again, we note that

- z=const: as r increases, p increases
- r=const: as z increases, p decreases
- Location of max p: leftmost and rightmost bottom corner.

Free Surfaces and Surface Tension

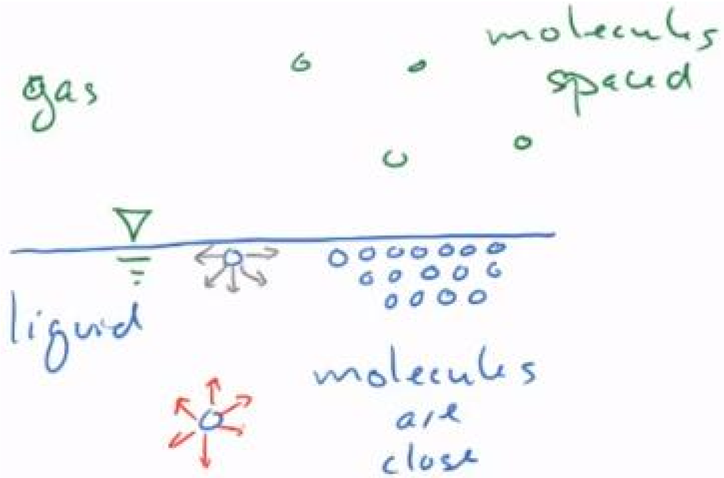
Interface between liquids and gases often curved (e.g. water droplet). E.g. meniscus in a tube. Keyword: **Contact Angles**.



Mechanism:

- Electric field forces between molecules.

- Imbalance at surface.
- Strong attraction between surface molecules.

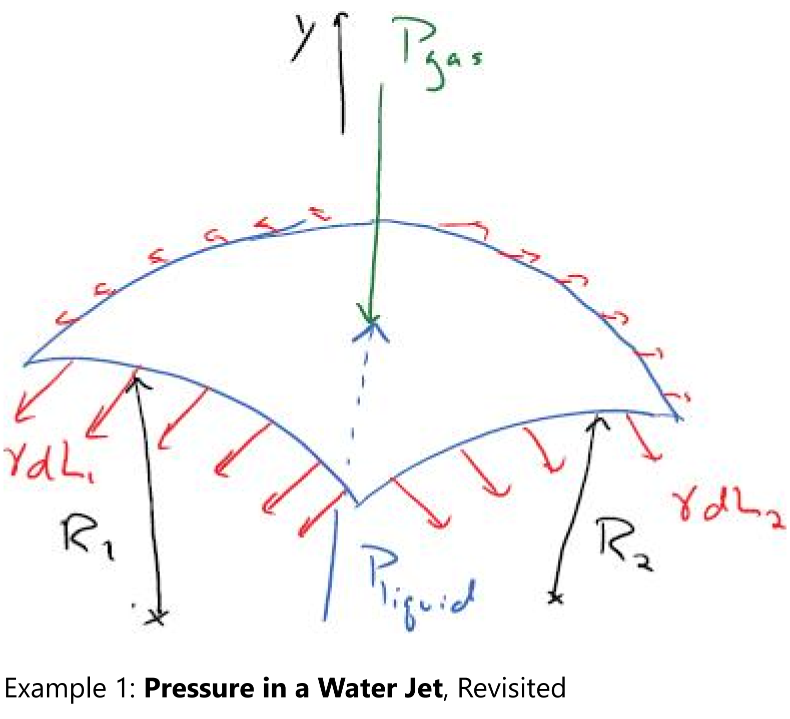


Tension force F is the product of coefficient of surface tension γ $[\mathrm{N/m}]$ and the length of the edge L (into the page). Note that γ depends on the specific liquid/gas combination. It is also sensitive to temperature (γ decreases as temperature increases) and impurities (suspended molecules).

$$F = \gamma L$$

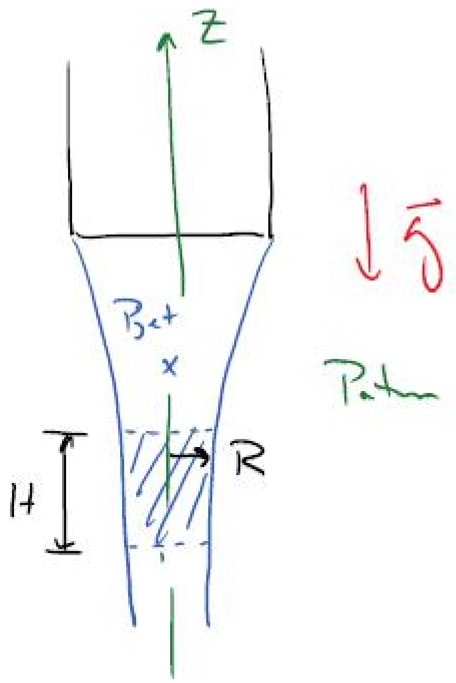
Relationship to Pressure:

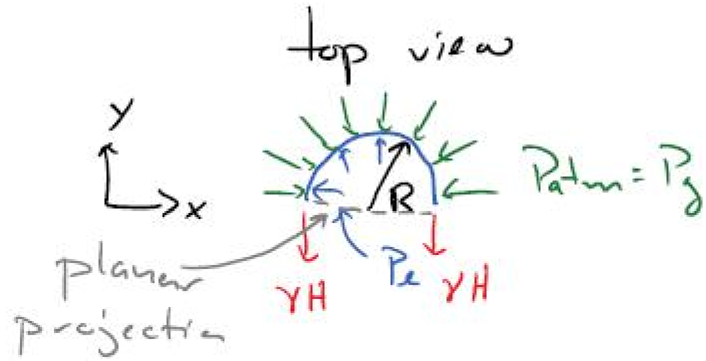
- Consider a small element of curved surface (curvature radii $R_1 \& R_2$)
- Consider forces in y direction (perpendicular to surface)
- Due to curvature γdL_1 and γdL_2 are in negative y direction.



Example 1: **Pressure in a Water Jet**, Revisited

Previously we argued that $P_{\text{jet}} = P_{\text{atm}}$, but we neglected surface tension. Take a short column of fluid, "sliced half". Note that P_l is pressure of liquid, and P_g is pressure of gas, and that $2RH$ is **planar projection**.





$\sum F_y = 0$ on surface, and so $-2\gamma H + P_l(2RH) - P_g(2RH) = 0$, we get

$$P_l = P_g + \gamma / R$$

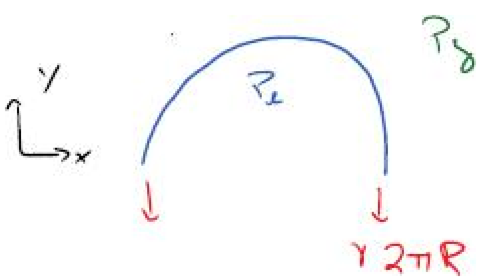
Finally, plugging in numbers:

- $\gamma = 0.073$ [N/m] for water/air at 20 degree celsius (Table A-5).
- Moderate flow: $R = 1\ \mathrm{cm} = 0.01\ \mathrm{m}$. And

$$P_l = P_{\mathrm{atm}} + 7.25\ [\mathrm{Pa}]$$

Example 2: Pressure in Water Drop: Consider a suspended spherical drop (e.g. rain drop):

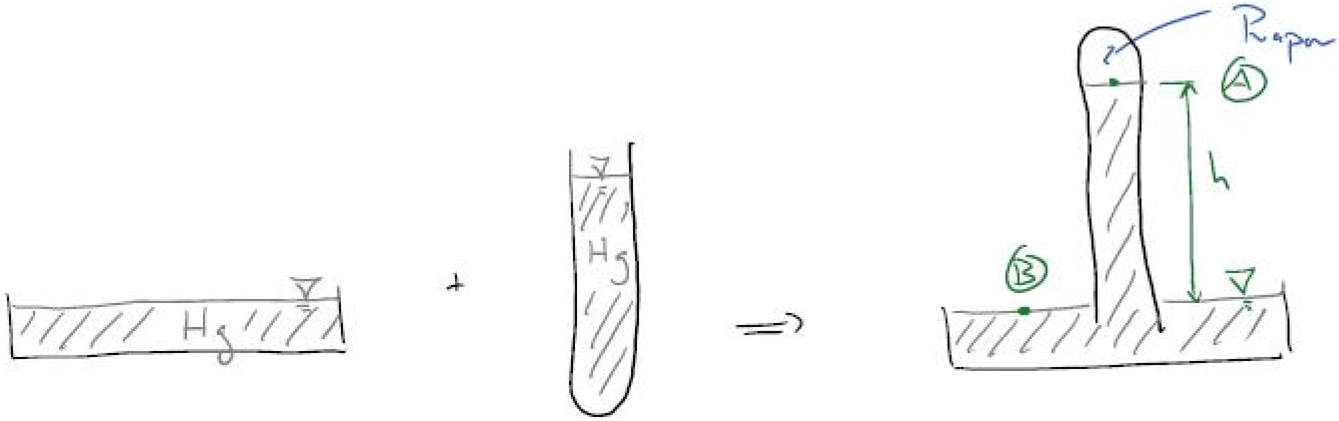
$$\sum F_y = 0 \qquad -2\pi R\gamma + P_l \pi R^2 - P_g \pi R^2 = 0$$



$$P_l = P_g + \frac{2\gamma}{R}$$

Pressure Measurement

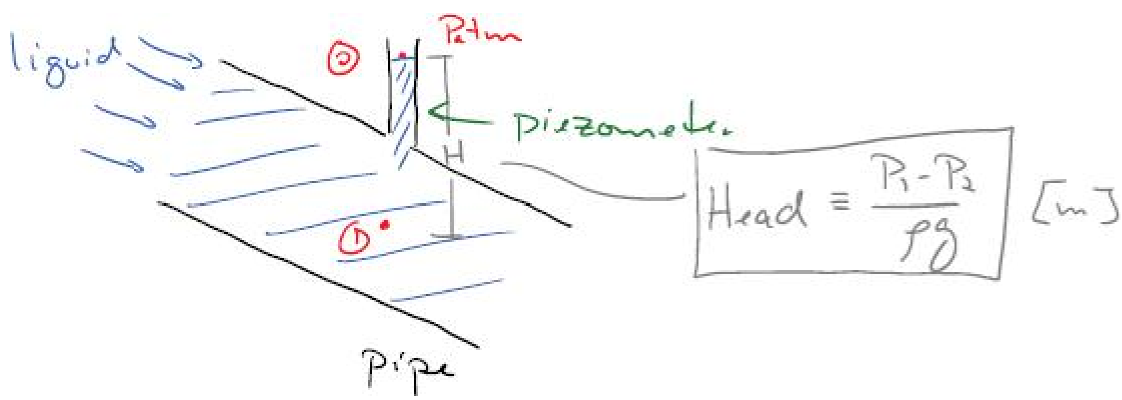
Barometer: used to measure atmospheric pressure. (e.g. Mercury Barometer)



$$P_A + \rho_{Hg} g h = P_B \qquad P_{\text{atm}} = P_{\text{vapor}} + \rho_{Hg} g h$$

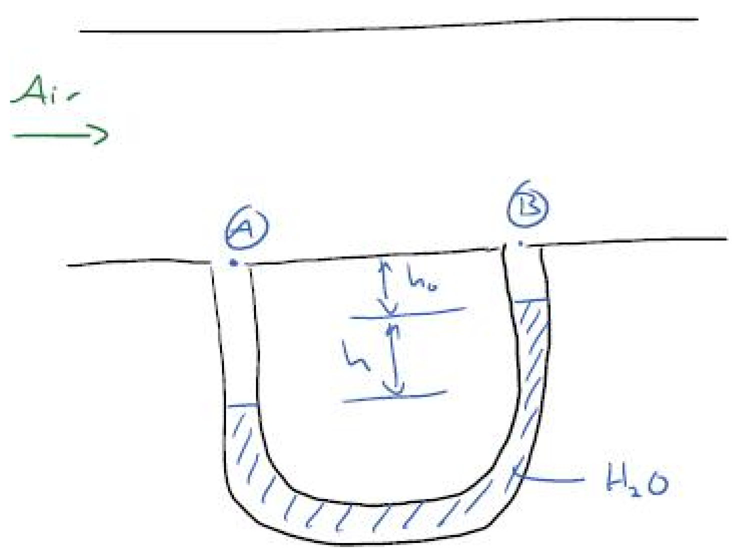
For 20 degree celsius, $P_{\text{vapor}} = 0.16 \text{ Pa}$ (can neglect), therefore $P_{\text{atm}} = \rho_{Hg} g h$. $h \approx 760 \text{ mm Hg}$ at sea level.

Piezometer Tube and **Pressure Head**: usually $P_2 = P_{\text{atm}}$.



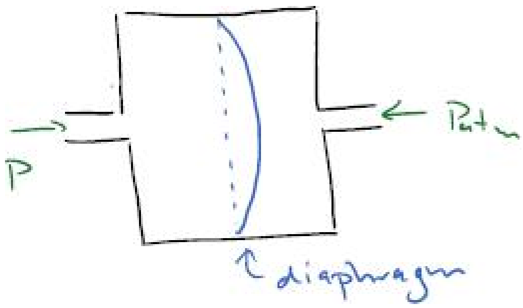
$$\text{Head} = \frac{P - P_{\text{atm}}}{\rho g}$$

U-Tube Manometer: Note that $P_A + \rho_{\text{air}} g h_0 + \rho_{\text{air}} g h - \rho_{H_2O} g h - \rho_{\text{air}} g h_0 = P_B$



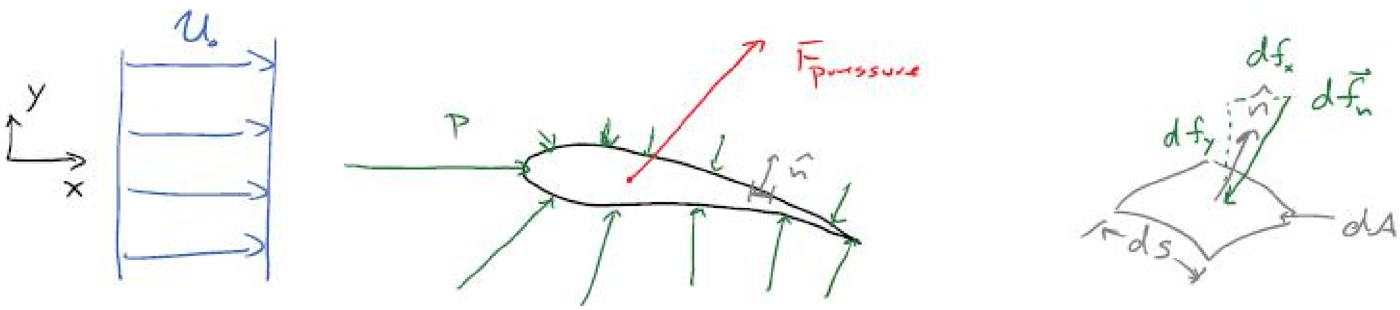
$$P_A - P_B = (\rho_{\text{H2O}} - \rho_{\text{air}}) g h \approx \rho_{\text{H2O}} g h$$

Pressure Gages: Used to measure gage pressure, as P increases the diaphragm deflects, read with high temperal resolution, and a bunch of other designs too.



Pressure Force on Submerged Bodies

General case: force f due to pressure on an airfoil; Note that $d\vec{f}_n = -p\,\hat{n}\,dA$, ($df_n$ positive in $\hat{n}$ direction.)



We have the general formula for computing net force due to pressure acting on a submerged surface:

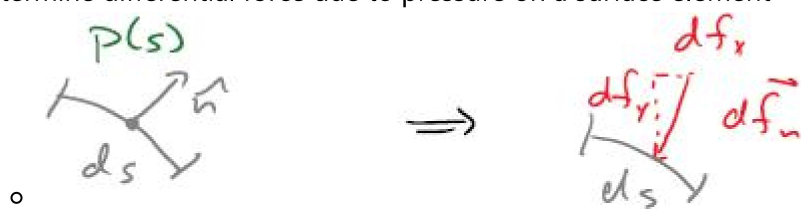
$$\vec{F}_{\text{pressure}} = \iint_A -p\,\hat{n}\,dA$$

Breaking down to components:

$$F_x = \iint_A df_x \quad \text{pressure drag}$$
$$F_y = \iint_A df_y \quad \text{lift}$$

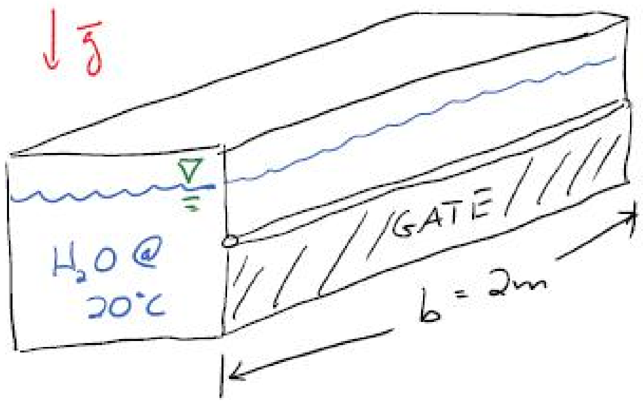
Practical Approach: (different than book; more general)

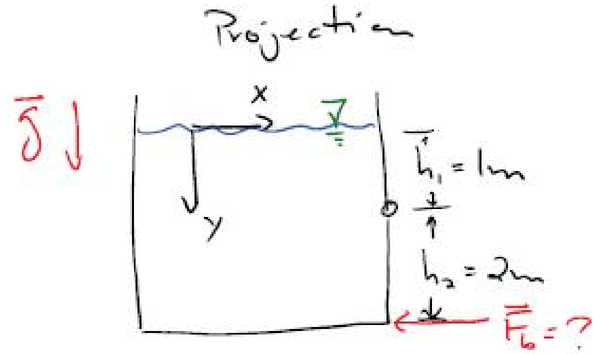
1. Isolate the body $\rightarrow$ free body diagram
2. Set up surface $(s, \hat{n})$ and global (x, y) coordinates and sign conventions for moments.
3. Determine differential force due to pressure on a surface element



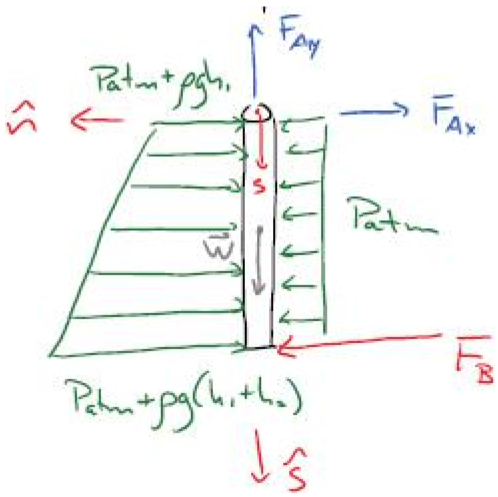
4. Add up (integrate) forces and moments as appropriate.

Example 1: **Hinged Gate**, find min $|\vec{F}_b|$



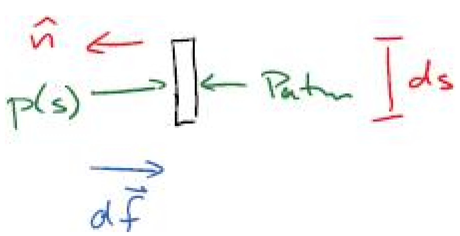


Step 1: Freebody Diagram:



Step 2: Coordinate system: Global (x, y); Surface (s, n); Moment sign (for this example we choose clockwise to be positive)

Step 3: Element Analysis:



where df_n is the projection of net force due to pressure on element (positive in $\hat{n}$), ρ is density of water at 20 degree celsius. We have

$$\begin{aligned} df_n &= P_{\text{atm}}\, b\, ds - P(s)\, b\, ds \qquad P(s) = P_{\text{atm}} + \rho g h_1 + \rho g s \\ &= [P_{\text{atm}} - P(s)] b\, ds \\ &= [P_{\text{atm}} - P_{\text{atm}} - \rho g(h_1 + s)] b\, ds \\ &= -\rho g(h_1 + s) b\, ds \end{aligned}$$

Note that $df_x = -df_n$; $df_y = 0$.

To find F_B, can consider moment balance about A. dM_A is the moment of df_n about A, and is $df_n \cdot s$:

$$dM_a = df_n \cdot s = -\rho g(h_1 + s) s\, b\, ds$$

Step 4: sum the forces and moments. Since gate is at rest, $\sum F_x = 0 = \sum F_y$, and $\sum M_a = 0$.

$$\sum M_A = F_{Bx} h_2 + \int dM_A = F_{Bx} h_2 + \int_{s=0}^{s=h_2} -\rho g (h_1 + s)\, s b\, ds = 0$$

$$= F_B h_2 - \rho g b \int_0^{h_2} (h_1 s + s^2)\, ds = F_B h_2 - \rho g b \left[\frac{h_1 s^2}{2} + \frac{s^3}{3} \right]\Bigg|_0^{h_2}$$

$$= F_B h_2 - \rho g b \left[\frac{h_1 h^2}{2} + \frac{h_2^3}{3} \right] = 0$$

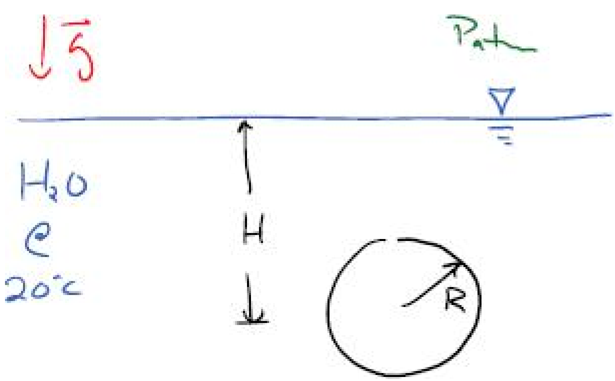
$$F_B = \rho g b \left[\frac{h_1 h_2}{2} + \frac{h_2^2}{3} \right] = (998\ \mathrm{kg/m^3})(9.81\ \mathrm{m/s^2})(2\,\mathrm{m}) \left[\frac{(1\,\mathrm{m})(2\,\mathrm{m})}{2} + \frac{(2\,\mathrm{m})^2}{3} \right]$$

$$= 45.7\ \mathrm{kN}$$

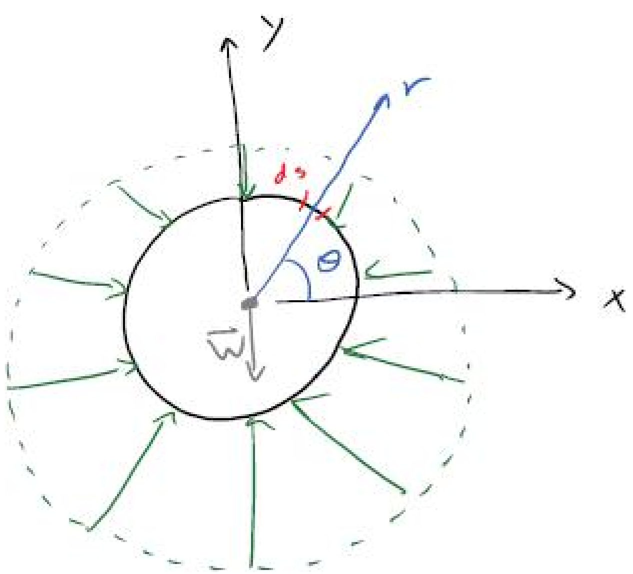
Concluding Notes:

- It is convenient to select $\hat{n}$ in the direction of a net force, and pick clock wise direction as positive direction of $\hat{n}$
- Do not need global condition for this example (simple linear geometry), but will for problems with curved surfaces.
- Didn't need reactions at A to solve in this case.

Example 2: Submerged log: Given H, R, L (length). Find the net force due to pressure acting on the log.



Step 1, Draw FBD:



Step 2, Coordinates: global (x, y) and local (r, θ).