Laplace Transform Examples

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Example 1 Determine the Laplace transform of the constant function $f(t) = 1, t \geq 0$.

Solution Using the definition of the transform, we compute

$$F(s) = \int_0^\infty e^{-st} \cdot 1 \, dt = \lim_{N\to\infty} \int_0^N e^{-st} dt$$

$$= \lim_{N\to\infty} \frac{-e^{-st}}{s} \bigg|_{t=0}^{t=N} = \lim_{N\to\infty} \left[\frac{1}{s} - \frac{e^{-sN}}{s} \right] .$$

Since $e^{-sN} \to 0$ when $s > 0$ is fixed and $N \to \infty$, we get

$$F(s) = \frac{1}{s} \quad \text{for} \quad s > 0 \; .$$

When $s \le 0$, the integral $\int_0^\infty e^{-st} dt$ diverges. (Why?) Hence $F(s) = 1/s$, with the domain of $F(s)$ being all $s > 0$. ◆

Example 2 Determine the Laplace transform of $f(t) = e^{at}$, where a is a constant.

Solution Using the definition of the transform,

$$
\begin{aligned}
F(s) &= \int_0^\infty e^{-st} e^{at}\,dt = \int_0^\infty e^{-(s-a)t}\,dt \\
&= \lim_{N\to\infty} \int_0^N e^{-(s-a)t}\,dt = \lim_{N\to\infty} \frac{-e^{-(s-a)t}}{s-a}\bigg|_0^N \\
&= \lim_{N\to\infty} \left[\frac{1}{s-a} - \frac{e^{-(s-a)N}}{s-a} \right] \\
&= \frac{1}{s-a} \quad \text{for} \quad s > a\ .
\end{aligned}
$$

Again, if $s \leq a$ the integral diverges, and hence the domain of $F(s)$ is all $s > a$. ◆

It is comforting to note from Example 2 that the transform of the constant function $f(t) = 1 = e^{0t}$ is $1/(s-0) = 1/s$, which agrees with the solution in Example 1.

Example 3 Find $\mathscr{L}\{\sin bt\}$, where b is a nonzero constant.

Solution We need to compute

$$\mathscr{L}\{\sin bt\}(s) = \int_0^\infty e^{-st} \sin bt \, dt = \lim_{N\to\infty} \int_0^N e^{-st} \sin bt \, dt \ .$$

Referring to the table of integrals on the inside front cover, we see that

$$\begin{aligned}
\mathscr{L}\{\sin bt\}(s) &= \lim_{N\to\infty} \left[\frac{e^{-st}}{s^2 + b^2}(-s \sin bt - b \cos bt) \Big|_0^N \right] \\
&= \lim_{N\to\infty} \left[\frac{b}{s^2 + b^2} - \frac{e^{-sN}}{s^2 + b^2}(s \sin bN + b \cos bN) \right] \\
&= \frac{b}{s^2 + b^2} \quad \text{for} \quad s > 0
\end{aligned}$$

(since for such s we have $\lim_{N\to\infty} e^{-sN}(s \sin bN + b \cos bN) = 0$; see Problem 32). ◆

Example 4 Determine the Laplace transform of

$$f(t) = \begin{cases} 2\ , & 0 < t < 5\ , \\ 0\ , & 5 < t < 10\ , \\ e^{4t}\ , & 10 < t\ . \end{cases}$$

Solution Since $f(t)$ is defined by a different formula on different intervals, we begin by breaking up the integral in (1) into three separate parts.[†] Thus,

$$\begin{aligned}
F(s) &= \int_0^\infty e^{-st} f(t)\,dt \\
&= \int_0^5 e^{-st}\cdot 2\,dt + \int_5^{10} e^{-st}\cdot 0\,dt + \int_{10}^\infty e^{-st} e^{4t}\,dt \\
&= 2\int_0^5 e^{-st}\,dt + \lim_{N\to\infty} \int_{10}^N e^{-(s-4)t}\,dt \\
&= \frac{2}{s} - \frac{2e^{-5s}}{s} + \lim_{N\to\infty}\left[\frac{e^{-10(s-4)}}{s-4} - \frac{e^{-(s-4)N}}{s-4}\right] \\
&= \frac{2}{s} - \frac{2e^{-5s}}{s} + \frac{e^{-10(s-4)}}{s-4} \quad \text{for} \quad s > 4\,. \quad \blacklozenge
\end{aligned}$$

Notice that the function $f(t)$ of Example 4 has jump discontinuities at $t = 5$ and $t = 10$. These values are reflected in the exponential terms e^{-5s} and e^{-10s} that appear in the formula for $F(s)$. We'll make this connection more precise when we discuss the unit step function in Section 7.6.

An important property of the Laplace transform is its **linearity.** That is, the Laplace transform $\mathcal{L}$ is a linear operator.

Example 5 Determine $\mathscr{L}\{11 + 5e^{4t} - 6 \sin 2t\}$.

Solution From the linearity property, we know that the Laplace transform of the sum of any finite number of functions is the sum of their Laplace transforms. Thus,

$$\begin{aligned}\mathscr{L}\{11 + 5e^{4t} - 6\sin 2t\} &= \mathscr{L}\{11\} + \mathscr{L}\{5e^{4t}\} + \mathscr{L}\{-6\sin 2t\} \\ &= 11\mathscr{L}\{1\} + 5\mathscr{L}\{e^{4t}\} - 6\mathscr{L}\{\sin 2t\}\ .\end{aligned}$$

In Examples 1, 2, and 3, we determined that

$$\mathscr{L}\{1\}(s) = \frac{1}{s}\ , \qquad \mathscr{L}\{e^{4t}\}(s) = \frac{1}{s-4}\ , \qquad \mathscr{L}\{\sin 2t\}(s) = \frac{2}{s^2+2^2}\ .$$

Using these results, we find

$$\begin{aligned}\mathscr{L}\{11 + 5e^{4t} - 6\sin 2t\}(s) &= 11\left(\frac{1}{s}\right) + 5\left(\frac{1}{s-4}\right) - 6\left(\frac{2}{s^2+4}\right) \\ &= \frac{11}{s} + \frac{5}{s-4} - \frac{12}{s^2+4}\ .\end{aligned}$$

Since $\mathscr{L}\{1\}$, $\mathscr{L}\{e^{4t}\}$, and $\mathscr{L}\{\sin 2t\}$ are all defined for $s > 4$, so is the transform $\mathscr{L}\{11 + 5e^{4t} - 6\sin 2t\}$. ◆

Example 6 Show that

$$f(t) = \begin{cases} t \ , & 0 < t < 1 \ , \\ 2 \ , & 1 < t < 2 \ , \\ (t-2)^2 \ , & 2 \le t \le 3 \ , \end{cases}$$

whose graph is sketched in Figure 7.4 is piecewise continuous on $[0, 3]$.

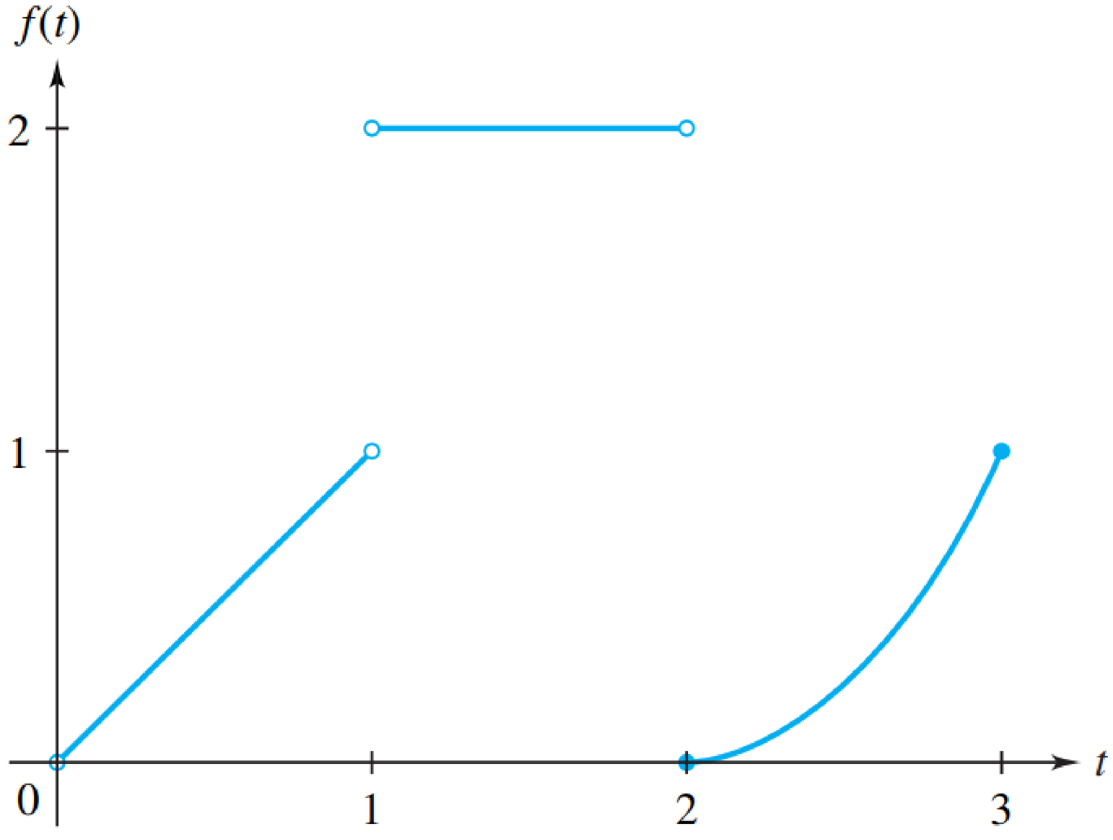


Figure 7.4 Graph of $f(t)$ in Example 6

Solution From the graph of $f(t)$ we see that $f(t)$ is continuous on the intervals (0, 1), (1, 2), and (2, 3]. Moreover, at the points of discontinuity, $t = 0, 1,$ and 2, the function has jump discontinuities, since the one-sided limits exist as finite numbers. In particular, at $t = 1$, the left-hand limit is 1 and the right-hand limit is 2. Therefore $f(t)$ is piecewise continuous on $[0, 3]$. ◆

Observe that the function $f(t)$ of Example 4 is piecewise continuous on $[0, \infty)$ because it is piecewise continuous on every finite interval of the form $[0, N]$, with $N > 0$. In contrast, the function $f(t) = 1/t$ is not piecewise continuous on any interval containing the origin, since it has an "infinite jump" at the origin (see Figure 7.5).

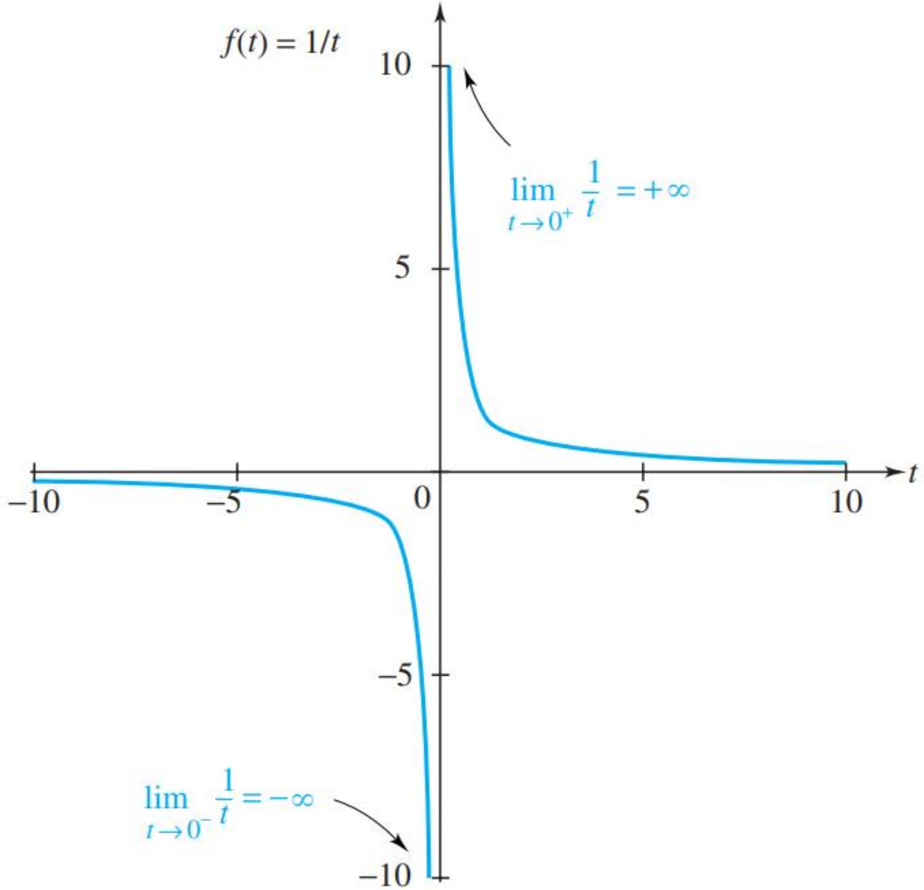


Figure 7.5 Infinite jump at origin

Example 1 Determine the Laplace transform of $e^{at} \sin bt$.

Solution In Example 3 in Section 7.2, we found that

$$\mathcal{L}\{\sin bt\}(s) = F(s) = \frac{b}{s^2 + b^2} .$$

Thus, by the translation property of $F(s)$, we have

$$\mathcal{L}\{e^{at}\sin bt\}(s) = F(s - a) = \frac{b}{(s - a)^2 + b^2} . \quad \blacklozenge$$

Example 2 Using Theorem 4 and the fact that

$$\mathscr{L}\{\sin bt\}(s) = \frac{b}{s^2 + b^2} ,$$

determine $\mathscr{L}\{\cos bt\}$.

Laplace Transform of the Derivative

Theorem 4. Let $f(t)$ be continuous on $[0, \infty)$ and $f'(t)$ be piecewise continuous on $[0, \infty)$, with both of exponential order α. Then, for $s > \alpha$,

$$\mathscr{L}\{f'\}(s) = s\mathscr{L}\{f\}(s) - f(0) . \tag{2}$$

Solution Let $f(t) := \sin bt$. Then $f(0) = 0$ and $f'(t) = b \cos bt$. Substituting into equation (2), we have

$$\begin{aligned}
\mathcal{L}\{f'\}(s) &= s\mathcal{L}\{f\}(s) - f(0) \ , \\
\mathcal{L}\{b \cos bt\}(s) &= s\mathcal{L}\{\sin bt\}(s) - 0 \ , \\
b\mathcal{L}\{\cos bt\}(s) &= \frac{sb}{s^2 + b^2} \ .
\end{aligned}$$

Dividing by b gives

$$\mathcal{L}\{\cos bt\}(s) = \frac{s}{s^2 + b^2} \ . \quad ◆$$

Example 3 Prove the following identity for continuous functions $f(t)$ (assuming the transforms exist):

(5)
$$\mathscr{L}\left\{\int_0^t f(\tau)d\tau\right\}(s) = \frac{1}{s}\mathscr{L}\{f(t)\}(s) \ .$$

Solution Define the function $g(t)$ by the integral

$$g(t) := \int_0^t f(\tau) d\tau \ .$$

Observe that $g(0) = 0$ and $g'(t) = f(t)$. Thus, if we apply Theorem 4 to $g(t)$ $\left[\text{instead of } f(t)\right]$, equation (2) on page 361 reads

$$\mathscr{L}\{f(t)\}(s) = s\mathscr{L}\left\{\int_0^t f(\tau) d\tau\right\}(s) - 0 \ ,$$

which is equivalent to equation (5).

Example 4 Determine $\mathscr{L}\{t \sin bt\}$.

We already know that

$$\mathscr{L}\{\sin bt\}(s) = F(s) = \frac{b}{s^2 + b^2} .$$

Solution Differentiating $\mathcal{L}\{\sin bt\}(s) = F(s)$

$$\frac{dF}{ds}(s) = \frac{-2bs}{(s^2 + b^2)^2} .$$

Hence, using formula (6), we have $\mathcal{L}\{tf(t)\}(s) = (-1)\dfrac{dF}{ds}(s)$

$$\mathcal{L}\{t \sin bt\}(s) = -\frac{dF}{ds}(s) = \frac{2bs}{(s^2 + b^2)^2} . \quad \blacklozenge$$

Example 1 Determine $\mathscr{L}^{-1}\{F\}$, where

(a) $F(s) = \dfrac{2}{s^3}$. **(b)** $F(s) = \dfrac{3}{s^2 + 9}$. **(c)** $F(s) = \dfrac{s - 1}{s^2 - 2s + 5}$.

Solution To compute $\mathscr{L}^{-1}\{F\}$, we refer to the Laplace transform table on page 359.

$$\textbf{(a)}\quad \mathscr{L}^{-1}\left\{\frac{2}{s^3}\right\}(t) = \mathscr{L}^{-1}\left\{\frac{2!}{s^3}\right\}(t) = t^2$$

$$\textbf{(b)}\quad \mathscr{L}^{-1}\left\{\frac{3}{s^2+9}\right\}(t) = \mathscr{L}^{-1}\left\{\frac{3}{s^2+3^2}\right\}(t) = \sin 3t$$

$$\textbf{(c)}\quad \mathscr{L}^{-1}\left\{\frac{s-1}{s^2-2s+5}\right\}(t) = \mathscr{L}^{-1}\left\{\frac{s-1}{(s-1)^2+2^2}\right\}(t) = e^t\cos 2t$$

In part (c) we used the technique of completing the square to rewrite the denominator in a form that we could find in the table. ◆

Example 2 Determine $\mathscr{L}^{-1}\left\{\dfrac{5}{s-6} - \dfrac{6s}{s^2+9} + \dfrac{3}{2s^2+8s+10}\right\}$.

Solution We begin by using the linearity property. Thus,

$$\mathcal{L}^{-1}\left\{\frac{5}{s-6}-\frac{6s}{s^2+9}+\frac{3}{2(s^2+4s+5)}\right\}$$

$$=5\mathcal{L}^{-1}\left\{\frac{1}{s-6}\right\}-6\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\}+\frac{3}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2+4s+5}\right\} .$$

Referring to the Laplace transform tables, we see that

$$\mathcal{L}^{-1}\left\{\frac{1}{s-6}\right\}(t)=e^{6t} \quad \text{and} \quad \mathcal{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\}(t)=\cos 3t .$$

This gives us the first two terms. To determine $\mathcal{L}^{-1}\{1/(s^2+4s+5)\}$, we complete the square of the denominator to obtain $s^2+4s+5=(s+2)^2+1$. We now recognize from the tables that

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+1^2}\right\}(t)=e^{-2t}\sin t .$$

Hence,

$$\mathcal{L}^{-1}\left\{\frac{5}{s-6}-\frac{6s}{s^2+9}+\frac{3}{2s^2+8s+10}\right\}(t)=5e^{6t}-6\cos 3t+\frac{3e^{-2t}}{2}\sin t . \quad ◆$$

Example 3 Determine $\mathscr{L}^{-1}\left\{\dfrac{5}{(s+2)^4}\right\}$.

Solution The $(s+2)^4$ in the denominator suggests that we work with the formula

$$\mathscr{L}^{-1}\left\{\frac{n!}{(s-a)^{n+1}}\right\}(t) = e^{at}t^n \ .$$

Here we have $a = -2$ and $n = 3$, so $\mathscr{L}^{-1}\{6/(s+2)^4\}(t) = e^{-2t}t^3$. Using the linearity property, we find

$$\mathscr{L}^{-1}\left\{\frac{5}{(s+2)^4}\right\}(t) = \frac{5}{6}\mathscr{L}^{-1}\left\{\frac{3!}{(s+2)^4}\right\}(t) = \frac{5}{6}e^{-2t}t^3 \ . \ \blacklozenge$$

Example 4 Determine $\mathscr{L}^{-1}\left\{\dfrac{3s+2}{s^2+2s+10}\right\}$.

Solution By completing the square, the quadratic in the denominator can be written as

$$s^2 + 2s + 10 = s^2 + 2s + 1 + 9 = (s + 1)^2 + 3^2 \ .$$

The form of $F(s)$ now suggests that we use one or both of the formulas

$$\mathcal{L}^{-1}\left\{\frac{s - a}{(s - a)^2 + b^2}\right\}(t) = e^{at}\cos bt \ ,$$

$$\mathcal{L}^{-1}\left\{\frac{b}{(s - a)^2 + b^2}\right\}(t) = e^{at}\sin bt \ .$$

In this case, $a = -1$ and $b = 3$. The next step is to express

$$\frac{3s + 2}{s^2 + 2s + 10} = A\frac{s + 1}{(s + 1)^2 + 3^2} + B\frac{3}{(s + 1)^2 + 3^2} \ , \tag{5}$$

where A, B are constants to be determined. Multiplying both sides of (5) by $s^2 + 2s + 10$ leaves

$$3s + 2 = A(s + 1) + 3B = As + (A + 3B) \ ,$$

which is an identity between two polynomials in s. Equating the coefficients of like terms gives

$$A = 3 \ , \qquad A + 3B = 2 \ ,$$

so $A = 3$ and $B = -1/3$. Finally, from (5) and the linearity property, we find

$$\mathcal{L}^{-1}\left\{\frac{3s + 2}{s^2 + 2s + 10}\right\}(t) = 3\mathcal{L}^{-1}\left\{\frac{s + 1}{(s + 1)^2 + 3^2}\right\}(t) - \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{3}{(s + 1)^2 + 3^2}\right\}(t)$$

$$= 3e^{-t}\cos 3t - \frac{1}{3}e^{-t}\sin 3t \ . \ \blacklozenge$$

method of partial fractions
1. Nonrepeated Linear Factors

Example 5 Determine $\mathscr{L}^{-1}\{F\}$, where

$$F(s) = \frac{7s - 1}{(s + 1)(s + 2)(s - 3)} .$$

Solution

We begin by finding the partial fraction expansion for $F(s)$. The denominator consists of three distinct linear factors, so the expansion has the form

$$\textbf{(6)} \qquad \frac{7s-1}{(s+1)(s+2)(s-3)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s-3} \, ,$$

where A, B, and C are real numbers to be determined.

One procedure that works for all partial fraction expansions is first to multiply the expansion equation by the denominator of the given rational function. This leaves us with two identical polynomials. Equating the coefficients of s^k leads to a system of linear equations that we can solve to determine the unknown constants. In this example, we multiply (6) by $(s+1)(s+2)(s-3)$ and find

$$\textbf{(7)} \qquad 7s - 1 = A(s+2)(s-3) + B(s+1)(s-3) + C(s+1)(s+2) \, ,^{\dagger}$$

which reduces to

$$7s - 1 = (A + B + C)s^2 + (-A - 2B + 3C)s + (-6A - 3B + 2C) \, .$$

Equating the coefficients of s^2, s, and 1 gives the system of linear equations

$$A + B + C = 0 \, , \quad -A - 2B + 3C = 7 \, , \quad -6A - 3B + 2C = -1 \, .$$

Solving this system yields $A = 2$, $B = -3$, and $C = 1$. Hence,

$$\textbf{(8)} \qquad \frac{7s-1}{(s+1)(s+2)(s-3)} = \frac{2}{s+1} - \frac{3}{s+2} + \frac{1}{s-3} \, .$$

An alternative method for finding the constants A, B, and C from (7) is to choose three values for s and substitute them into (7) to obtain three linear equations in the three unknowns. If we are careful in our choice of the values for s, the system is easy to solve. In this case, equation (7) obviously simplifies if $s = -1$, -2, or 3. Putting $s = -1$ gives

$$\begin{aligned} -7 - 1 &= A(1)(-4) + B(0) + C(0) \, , \\ -8 &= -4A \, . \end{aligned}$$

Hence $A = 2$. Next, setting $s = -2$ gives

$$\begin{aligned} -14 - 1 &= A(0) + B(-1)(-5) + C(0) \, , \\ -15 &= 5B \, , \end{aligned}$$

and so $B = -3$. Finally, letting $s = 3$, we similarly find that $C = 1$. In the case of nonrepeated linear factors, the alternative method is easier to use.

Now that we have obtained the partial fraction expansion (8), we use linearity to compute

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{7s-1}{(s+1)(s+2)(s-3)}\right\}(t) &= \mathcal{L}^{-1}\left\{\frac{2}{s+1} - \frac{3}{s+2} + \frac{1}{s-3}\right\}(t) \\ &= 2\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}(t) - 3\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}(t) \\ &\quad + \mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\}(t) \\ &= 2e^{-t} - 3e^{-2t} + e^{3t} \, . \quad ◆ \end{aligned}$$

method of partial fractions
2. Repeated Linear Factors

Example 6 Determine $\mathscr{L}^{-1}\left\{\dfrac{s^2+9s+2}{(s-1)^2(s+3)}\right\}$.

Solution Since $s - 1$ is a repeated linear factor with multiplicity two and $s + 3$ is a nonrepeated linear factor, the partial fraction expansion has the form

$$\frac{s^2 + 9s + 2}{(s-1)^2(s+3)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+3} .$$

We begin by multiplying both sides by $(s-1)^2(s+3)$ to obtain

$$s^2 + 9s + 2 = A(s-1)(s+3) + B(s+3) + C(s-1)^2 . \tag{9}$$

Now observe that when we set $s = 1$ (or $s = -3$), two terms on the right-hand side of (9) vanish, leaving a linear equation that we can solve for B (or C). Setting $s = 1$ in (9) gives

$$1 + 9 + 2 = A(0) + 4B + C(0) , \qquad 12 = 4B ,$$

and, hence, $B = 3$. Similarly, setting $s = -3$ in (9) gives

$$9 - 27 + 2 = A(0) + B(0) + 16C \qquad -16 = 16C .$$

Thus, $C = -1$. Finally, to find A, we pick a different value for s, say $s = 0$. Then, since $B = 3$ and $C = -1$, plugging $s = 0$ into (9) yields

$$2 = -3A + 3B + C = -3A + 9 - 1$$

so that $A = 2$. Hence,

$$\frac{s^2 + 9s + 2}{(s-1)^2(s+3)} = \frac{2}{s-1} + \frac{3}{(s-1)^2} - \frac{1}{s+3} . \tag{10}$$

We could also have determined the constants A, B, and C by first rewriting equation (9) in the form

$$s^2 + 9s + 2 = (A + C)s^2 + (2A + B - 2C)s + (-3A + 3B + C) .$$

Then, equating the corresponding coefficients of s^2, s, and 1 and solving the resulting system, we again find $A = 2$, $B = 3$, and $C = -1$.

Now that we have derived the partial fraction expansion (10) for the given rational function, we can determine its inverse Laplace transform:

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{s^2 + 9s + 2}{(s-1)^2(s+3)}\right\}(t) &= \mathscr{L}^{-1}\left\{\frac{2}{s-1} + \frac{3}{(s-1)^2} - \frac{1}{s+3}\right\}(t) \\ &= 2\mathscr{L}^{-1}\left\{\frac{1}{s-1}\right\}(t) + 3\mathscr{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\}(t) - \mathscr{L}^{-1}\left\{\frac{1}{s+3}\right\}(t) \\ &= 2e^t + 3te^t - e^{-3t} . \quad ◆\end{aligned}$$

We could also have determined the constants A, B, and C by first rewriting equation (9) in the form $s^2 + 9s + 2 = (A + C)s^2 + (2A + B - 2C)s + (-3A + 3B + C)$.

Then, equating the corresponding coefficients of s^2, s, and 1 and solving the resulting system, we again find $A = 2$, $B = 3$, and $C = -1$.

method of partial fractions

3. Quadratic Factors

Example 7 Determine $\mathscr{L}^{-1}\left\{\dfrac{2s^2 + 10s}{(s^2 - 2s + 5)(s + 1)}\right\}$.

Solution We first observe that the quadratic factor $s^2 - 2s + 5$ is irreducible (check the sign of the discriminant in the quadratic formula). Next we write the quadratic in the form $(s - \alpha)^2 + \beta^2$ by completing the square: $s^2 - 2s + 5 = (s - 1)^2 + 2^2$.

Since $s^2 - 2s + 5$ and $s + 1$ are nonrepeated factors, the partial fraction expansion has the form

$$\frac{2s^2 + 10s}{(s^2 - 2s + 5)(s + 1)} = \frac{A(s - 1) + 2B}{(s - 1)^2 + 2^2} + \frac{C}{s + 1} .$$

When we multiply both sides by the common denominator, we obtain

(11) $$2s^2 + 10s = [A(s - 1) + 2B](s + 1) + C(s^2 - 2s + 5) .$$

In equation (11), let's put $s = -1$, 1, and 0. With $s = -1$, we find

$$2 - 10 = [A(-2) + 2B](0) + C(8) , \qquad -8 = 8C ,$$

and, hence, $C = -1$. With $s = 1$ in (11), we obtain

$$2 + 10 = [A(0) + 2B](2) + C(4) ,$$

and since $C = -1$, the last equation becomes $12 = 4B - 4$. Thus $B = 4$. Finally, setting $s = 0$ in (11) and using $C = -1$ and $B = 4$ gives

$$0 = [A(-1) + 2B](1) + C(5) , \quad 0 = -A + 8 - 5 , \quad A = 3 .$$

Hence, $A = 3$, $B = 4$, and $C = -1$ so that

$$\frac{2s^2 + 10s}{(s^2 - 2s + 5)(s + 1)} = \frac{3(s - 1) + 2(4)}{(s - 1)^2 + 2^2} - \frac{1}{s + 1} .$$

With this partial fraction expansion in hand, we can immediately determine the inverse Laplace transform:

$$\mathcal{L}^{-1}\left\{\frac{2s^2 + 10s}{(s^2 - 2s + 5)(s + 1)}\right\}(t) = \mathcal{L}^{-1}\left\{\frac{3(s - 1) + 2(4)}{(s - 1)^2 + 2^2} - \frac{1}{s + 1}\right\}(t)$$

$$= 3\mathcal{L}^{-1}\left\{\frac{s - 1}{(s - 1)^2 + 2^2}\right\}(t) + 4\mathcal{L}^{-1}\left\{\frac{2}{(s - 1)^2 + 2^2}\right\}(t) - \mathcal{L}^{-1}\left\{\frac{1}{s + 1}\right\}(t)$$

$$= 3e^t \cos 2t + 4e^t \sin 2t - e^{-t} . \quad ◆$$

Initial Value Problems

Example 1 Solve the initial value problem

$$y'' - 2y' + 5y = -8e^{-t} ; \qquad y(0) = 2 , \qquad y'(0) = 12 . \tag{1}$$

Solution The differential equation in (1) is an identity between two functions of t. Hence equality holds for the Laplace transforms of these functions:

$$\mathscr{L}\{y'' - 2y' + 5y\} = \mathscr{L}\{-8e^{-t}\} .$$

Using the linearity property of $\mathscr{L}$ and the previously computed transform of the exponential function, we can write

$$\mathscr{L}\{y''\}(s) - 2\mathscr{L}\{y'\}(s) + 5\mathscr{L}\{y\}(s) = \frac{-8}{s+1} . \tag{2}$$

Now let $Y(s) := \mathscr{L}\{y\}(s)$. From the formulas for the Laplace transform of higher-order derivatives (see Section 7.3) and the initial conditions in (1), we find

$$\mathscr{L}\{y'\}(s) = sY(s) - y(0) = sY(s) - 2 ,$$
$$\mathscr{L}\{y''\}(s) = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 2s - 12 .$$

Substituting these expressions into (2) and solving for $Y(s)$ yields

$$\begin{aligned}
[s^2Y(s) - 2s - 12] - 2[sY(s) - 2] + 5Y(s) &= \frac{-8}{s+1} \\
(s^2 - 2s + 5)Y(s) &= 2s + 8 - \frac{8}{s+1} \\
(s^2 - 2s + 5)Y(s) &= \frac{2s^2 + 10s}{s+1} \\
Y(s) &= \frac{2s^2 + 10s}{(s^2 - 2s + 5)(s+1)} .
\end{aligned}$$

Our remaining task is to compute the inverse transform of the rational function $Y(s)$. This was done in Example 7 of Section 7.4, where, using a partial fraction expansion, we found

$$y(t) = 3e^t \cos 2t + 4e^t \sin 2t - e^{-t} , \tag{3}$$

which is the solution to the initial value problem (1). ◆

Example 2 Solve the initial value problem

$$y'' + 4y' - 5y = te^t \; ; \qquad y(0) = 1 \; , \qquad y'(0) = 0 \; . \tag{4}$$

Solution

Let $Y(s) := \mathscr{L}\{y\}(s)$. Taking the Laplace transform of both sides of the differential equation in (4) gives

$$\mathscr{L}\{y''\}(s) + 4\mathscr{L}\{y'\}(s) - 5Y(s) = \frac{1}{(s-1)^2} \quad (5)$$

Using the initial conditions, we can express $\mathscr{L}\{y'\}(s)$ and $\mathscr{L}\{y''\}(s)$ in terms of $Y(s)$. That is,

$$\mathscr{L}\{y'\}(s) = sY(s) - y(0) = sY(s) - 1 \ ,$$
$$\mathscr{L}\{y''\}(s) = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - s \ .$$

Substituting back into (5) and solving for $Y(s)$ gives

$$[s^2Y(s) - s] + 4[sY(s) - 1] - 5Y(s) = \frac{1}{(s-1)^2}$$
$$(s^2 + 4s - 5)Y(s) = s + 4 + \frac{1}{(s-1)^2}$$
$$(s+5)(s-1)Y(s) = \frac{s^3 + 2s^2 - 7s + 5}{(s-1)^2}$$
$$Y(s) = \frac{s^3 + 2s^2 - 7s + 5}{(s+5)(s-1)^3} \ .$$

The partial fraction expansion for $Y(s)$ has the form

$$\frac{s^3 + 2s^2 - 7s + 5}{(s+5)(s-1)^3} = \frac{A}{s+5} + \frac{B}{s-1} + \frac{C}{(s-1)^2} + \frac{D}{(s-1)^3} \ . \quad (6)$$

Solving for the numerators, we ultimately obtain $A = 35/216$, $B = 181/216$, $C = -1/36$, and $D = 1/6$. Substituting these values into (6) gives

$$Y(s) = \frac{35}{216}\left(\frac{1}{s+5}\right) + \frac{181}{216}\left(\frac{1}{s-1}\right) - \frac{1}{36}\left(\frac{1}{(s-1)^2}\right) + \frac{1}{12}\left(\frac{2}{(s-1)^3}\right) ,$$

where we have written $D = 1/6 = (1/12)2$ to facilitate the final step of taking the inverse transform. From the tables, we now obtain

$$y(t) = \frac{35}{216}e^{-5t} + \frac{181}{216}e^{t} - \frac{1}{36}te^{t} + \frac{1}{12}t^2e^{t} \quad (7)$$

as the solution to the initial value problem (4). ◆

Example 3 Solve the initial value problem

$$w''(t) - 2w'(t) + 5w(t) = -8e^{\pi - t}\,; \qquad w(\pi) = 2\,, \qquad w'(\pi) = 12\,. \tag{8}$$

Solution To use the method of Laplace transforms, we first move the initial conditions to $t = 0$. This can be done by setting $y(t) := w(t + \pi)$. Then

$$y'(t) = w'(t + \pi) \ , \qquad y''(t) = w''(t + \pi) \ .$$

Replacing t by $t + \pi$ in the differential equation in (8), we have

$$w''(t + \pi) - 2w'(t + \pi) + 5w(t + \pi) = -8e^{\pi-(t+\pi)} = -8e^{-t} \ . \tag{9}$$

Substituting $y(t) = w(t + \pi)$ in (9), the initial value problem in (8) becomes

$$y''(t) - 2y'(t) + 5y(t) = -8e^{-t} \ ; \qquad y(0) = 2 \ , \qquad y'(0) = 12 \ .$$

Because the initial conditions are now given at the origin, the Laplace transform method is applicable. In fact, we carried out the procedure in Example 1, where we found

$$y(t) = 3e^{t}\cos 2t + 4e^{t}\sin 2t - e^{-t} \ . \tag{10}$$

Since $w(t + \pi) = y(t)$, then $w(t) = y(t - \pi)$. Hence, replacing t by $t - \pi$ in (10) gives

$$\begin{aligned} w(t) = y(t - \pi) &= 3e^{t-\pi}\cos\left[2(t - \pi)\right] + 4e^{t-\pi}\sin\left[2(t - \pi)\right] - e^{-(t-\pi)} \\ &= 3e^{t-\pi}\cos 2t + 4e^{t-\pi}\sin 2t - e^{\pi-t} \ . \end{aligned}$$ ◆

Example 4 Solve the initial value problem

$$y'' + 2ty' - 4y = 1 \ , \qquad y(0) = y'(0) = 0 \ . \tag{13}$$

Solution Let $Y(s) = \mathcal{L}\{y\}(s)$ and take the Laplace transform of both sides of the equation in (13):

$$\mathcal{L}\{y''\}(s) + 2\mathcal{L}\{ty'(t)\}(s) - 4Y(s) = \frac{1}{s} \,. \tag{14}$$

Using the initial conditions, we find

$$\mathcal{L}\{y''\}(s) = s^2Y(s) - sy(0) - y'(0) = s^2Y(s)$$

and

$$\begin{aligned} \mathcal{L}\{ty'(t)\}(s) &= -\frac{d}{ds}\mathcal{L}\{y'\}(s) \\ &= -\frac{d}{ds}[sY(s) - y(0)] = -sY'(s) - Y(s) \,. \end{aligned}$$

Substituting these expressions into (14) gives

$$s^2Y(s) + 2[-sY'(s) - Y(s)] - 4Y(s) = \frac{1}{s}$$

$$-2sY'(s) + (s^2 - 6)Y(s) = \frac{1}{s}$$

$$Y'(s) + \left(\frac{3}{s} - \frac{s}{2}\right)Y(s) = \frac{-1}{2s^2} \,. \tag{15}$$

Equation (15) is a linear first-order equation and has the integrating factor

$$\mu(s) = e^{\int(3/s - s/2)ds} = e^{\ln s^3 - s^2/4} = s^3e^{-s^2/4}$$

(see Section 2.3). Multiplying (15) by $\mu(s)$, we obtain

$$\frac{d}{ds}\{\mu(s)Y(s)\} = \frac{d}{ds}\left\{s^3e^{-s^2/4}Y(s)\right\} = -\frac{s}{2}e^{-s^2/4} \,.$$

Integrating and solving for $Y(s)$ yields

$$s^3e^{-s^2/4}Y(s) = -\int \frac{s}{2}e^{-s^2/4}\,ds = e^{-s^2/4} + C$$

$$Y(s) = \frac{1}{s^3} + C\frac{e^{s^2/4}}{s^3} \,. \tag{16}$$

Now if $Y(s)$ is the Laplace transform of a piecewise continuous function of exponential order, then it follows from equation (12) that

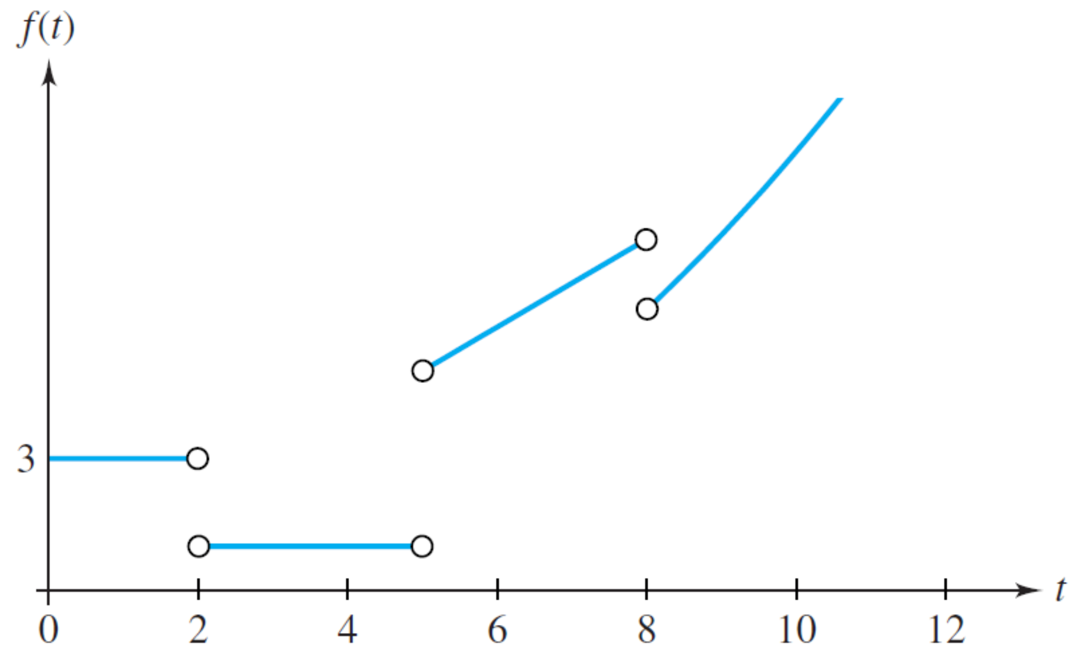
$$\lim_{s\to\infty} Y(s) = 0 \,.$$

For this to occur, the constant C in equation (16) must be zero. Hence, $Y(s) = 1/s^3$, and taking the inverse transform gives $y(t) = t^2/2$. We can easily verify that $y(t) = t^2/2$ is the solution to the given initial value problem by substituting it into (13). ◆

Transformation of Discontinuous and Periodic Functions

Example 1 Write the function

$$\textbf{(4)} \qquad f(t) = \begin{cases} 3\ , & t < 2\ , \\ 1\ , & 2 < t < 5\ , \\ t\ , & 5 < t < 8\ , \\ t^2/10\ , & 8 < t \end{cases}$$



Solution Clearly, from the figure we want to window the function in the intervals (0, 2), (2, 5), and (5, 8), and to introduce a step for $t > 8$. From (5) we read off the desired representation as

$$f(t) = 3\Pi_{0,2}(t) + 1\Pi_{2,5}(t) + t\Pi_{5,8}(t) + (t^2/10)u(t-8). \tag{5}$$ ◆

Example 2 Determine the Laplace transform of $t^2u(t-1)$.

Solution To apply equation (11), we take $g(t) = t^2$ and $a = 1$. Then

$$g(t + a) = g(t + 1) = (t + 1)^2 = t^2 + 2t + 1 \ .$$

Now the Laplace transform of $g(t + a)$ is

$$\mathscr{L}\{g(t + a)\}(s) = \mathscr{L}\{t^2 + 2t + 1\}(s) = \frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s} \ .$$

So, by formula (11), we have

$$\mathscr{L}\{t^2 u(t - 1)\}(s) = e^{-s}\left\{\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right\} \ . \ ◆$$

Example 3 Determine $\mathscr{L}\{(\cos t)u(t-\pi)\}$.

Solution Here $g(t) = \cos t$ and $a = \pi$. Hence,

$$g(t + a) = g(t + \pi) = \cos(t + \pi) = -\cos t \ ,$$

and so the Laplace transform of $g(t + a)$ is

$$\mathcal{L}\{g(t + a)\}(s) = -\mathcal{L}\{\cos t\}(s) = -\frac{s}{s^2 + 1} \ .$$

Thus, from formula (11), we get

$$\mathcal{L}\{(\cos t)u(t - \pi)\}(s) = -e^{-\pi s}\frac{s}{s^2 + 1} \ . \ \blacklozenge$$

Example 4 Determine $\mathscr{L}^{-1}\left\{\dfrac{e^{-2s}}{s^2}\right\}$ and sketch its graph.

Solution To use the translation property (9), we first express e^{-2s}/s^2 as the product $e^{-as}F(s)$. For this purpose, we put $e^{-as} = e^{-2s}$ and $F(s) = 1/s^2$. Thus, $a = 2$ and

$$f(t) = \mathscr{L}^{-1}\left\{\frac{1}{s^2}\right\}(t) = t \ .$$

It now follows from the translation property that

$$\mathscr{L}^{-1}\left\{\frac{e^{-2s}}{s^2}\right\}(t) = f(t-2)u(t-2) = (t-2)u(t-2) \ .$$

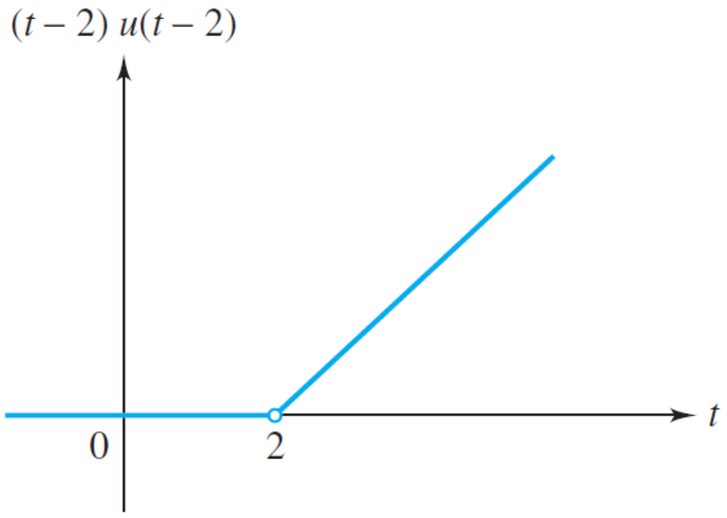


Figure 7.12 Graph of solution to Example 4

Example 5 The current I in an LC series circuit is governed by the initial value problem

$$I''(t) + 4I(t) = g(t) \; ; \qquad I(0) = 0 \; , \qquad I'(0) = 0 \; , \tag{12}$$

where

$$g(t) := \begin{cases} 1 \; , & 0 < t < 1 \; , \\ -1 \; , & 1 < t < 2 \; , \\ 0 \; , & 2 < t \; . \end{cases}$$

Determine the current as a function of time t.

Solution Let $J(s) := \mathscr{L}\{I\}(s)$. Then we have $\mathscr{L}\{I''\}(s) = s^2 J(s)$.

Writing $g(t)$ in terms of the rectangular window function $\Pi_{a,b}(t) = u(t-a) - u(t-b)$, we get

$$g(t) = \Pi_{0,1}(t) + (-1)\Pi_{1,2}(t) = u(t) - u(t-1) - [u(t-1) - u(t-2)]$$
$$= 1 - 2u(t-1) + u(t-2),$$

and so $\quad \mathscr{L}\{g\}(s) = \dfrac{1}{s} - \dfrac{2e^{-s}}{s} + \dfrac{e^{-2s}}{s}$.

Thus, when we take the Laplace transform of both sides of (12), we obtain

$$\mathscr{L}\{I''\}(s) + 4\mathscr{L}\{I\}(s) = \mathscr{L}\{g\}(s)$$
$$s^2 J(s) + 4J(s) = \frac{1}{s} - \frac{2e^{-s}}{s} + \frac{e^{-2s}}{s}$$
$$J(s) = \frac{1}{s(s^2+4)} - \frac{2e^{-s}}{s(s^2+4)} + \frac{e^{-2s}}{s(s^2+4)} .$$

To find $I = \mathscr{L}^{-1}\{J\}$, we first observe that

$$J(s) = F(s) - 2e^{-s}F(s) + e^{-2s}F(s) , \quad \text{where}$$
$$F(s) := \frac{1}{s(s^2+4)} = \frac{1}{4}\left(\frac{1}{s}\right) - \frac{1}{4}\left(\frac{s}{s^2+4}\right) .$$

Computing the inverse transform of $F(s)$ gives

$$f(t) := \mathscr{L}^{-1}\{F\}(t) = \frac{1}{4} - \frac{1}{4}\cos 2t .$$

Hence, via the translation property (9), we find

$$I(t) = \mathscr{L}^{-1}\{F(s) - 2e^{-s}F(s) + e^{-2s}F(s)\}(t)$$
$$= f(t) - 2f(t-1)u(t-1) + f(t-2)u(t-2)$$
$$= \left(\frac{1}{4} - \frac{1}{4}\cos 2t\right) - \left[\frac{1}{2} - \frac{1}{2}\cos 2(t-1)\right]u(t-1)$$
$$+ \left[\frac{1}{4} - \frac{1}{4}\cos 2(t-2)\right]u(t-2) .$$

The current is graphed in Figure 7.13. Note that $I(t)$ is smoother than $g(t)$; the former has discontinuities in its second derivative at the points where the latter has jumps. ◆

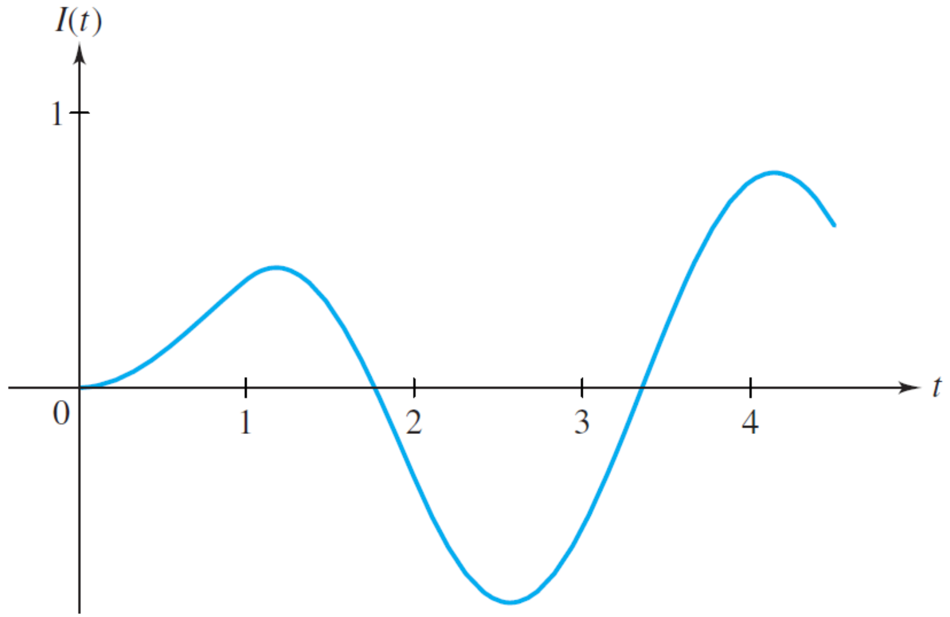


Figure 7.13 Solution to Example 5

Example 6 Determine $\mathscr{L}\{f\}$, where f is the square wave function in Figure 7.14.

(13) $$f(t) := \begin{cases} 1\ , & 0 < t < 1\ , \\ -1\ , & 1 < t < 2\ , \end{cases} \quad \text{and } f(t) \text{ has period 2.}$$

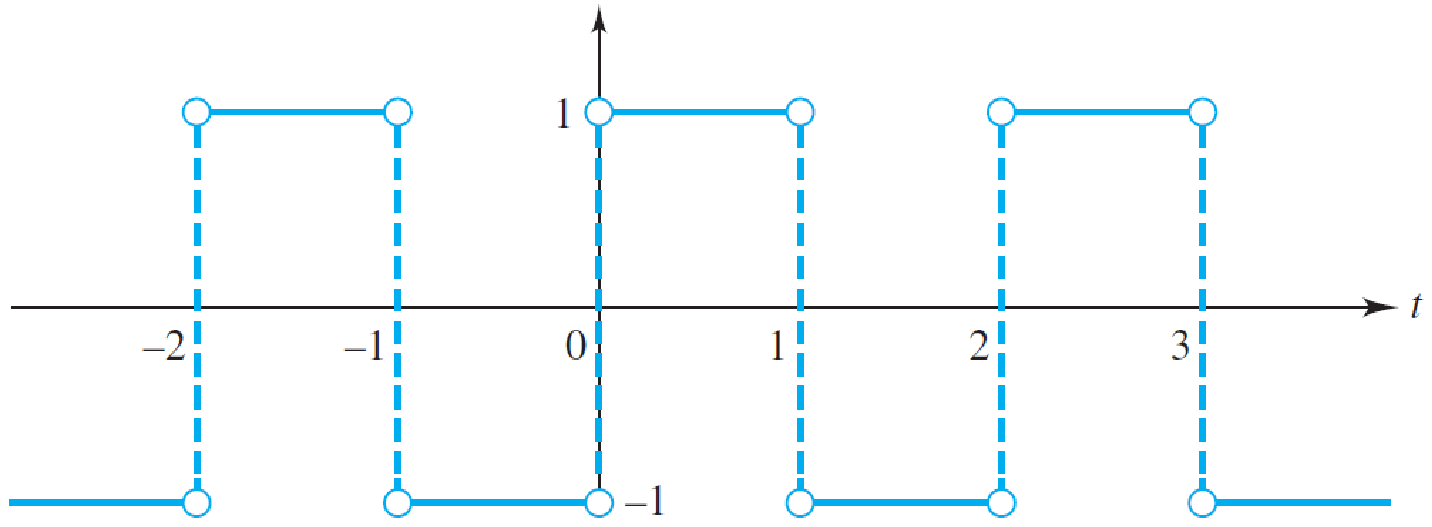


Figure 7.14 Graph of square wave function $f(t)$

Solution Here $T = 2$. Windowing the function results in $f_T(t) = \Pi_{0,1}(t) - \Pi_{1,2}(t)$, so by (7) we get $F_T(s) = (1 - e^{-s})/s - (e^{-s} - e^{-2s})/s = (1 - e^{-s})^2/s$. Therefore (15) implies

$$\mathscr{L}\{f\} = \frac{(1 - e^{-s})^2/s}{1 - e^{-2s}} = \frac{1 - e^{-s}}{(1 + e^{-s})s} \ . \quad \blacklozenge$$

Example 7 Determine $\mathscr{L}\{f\}$, where

$$f(t) := \begin{cases} \dfrac{\sin t}{t}, & t \neq 0, \\ 1, & t = 0. \end{cases}$$

Solution We begin by expressing $f(t)$ in a Taylor series[†] about $t = 0$. Since

$$\sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots ,$$

then dividing by t, we obtain

$$f(t) = \frac{\sin t}{t} = 1 - \frac{t^2}{3!} + \frac{t^4}{5!} - \frac{t^6}{7!} + \cdots$$

for $t > 0$. This representation also holds at $t = 0$ since

$$\lim_{t \to 0} f(t) = \lim_{t \to 0} \frac{\sin t}{t} = 1 .$$

Observe that $f(t)$ is continuous on $[0, \infty)$ and of exponential order. Hence, its Laplace transform exists for all s sufficiently large. Because of the linearity of the Laplace transform, we would expect that

$$\begin{aligned}
\mathscr{L}\{f\}(s) &= \mathscr{L}\{1\}(s) - \frac{1}{3!}\mathscr{L}\{t^2\}(s) + \frac{1}{5!}\mathscr{L}\{t^4\}(s) + \cdots \\
&= \frac{1}{s} - \frac{2!}{3!s^3} + \frac{4!}{5!s^5} - \frac{6!}{7!s^7} + \cdots \\
&= \frac{1}{s} - \frac{1}{3s^3} + \frac{1}{5s^5} - \frac{1}{7s^7} + \cdots .
\end{aligned}$$

Indeed, using tools from analysis, it can be verified that this series representation is valid for all $s > 1$. Moreover, one can show that the series converges to the function $\arctan(1/s)$ (see Problem 54). Thus,

$$\mathscr{L}\left\{\frac{\sin t}{t}\right\}(s) = \arctan \frac{1}{s} . \quad \blacklozenge \tag{17}$$

Example 1 Use the convolution theorem to solve the initial value problem

$$y'' - y = g(t) \; ; \qquad y(0) = 1 \; , \qquad y'(0) = 1 \; , \tag{11}$$

where $g(t)$ is piecewise continuous on $[0, \infty)$ and of exponential order.

Solution Let $Y(s) = \mathscr{L}\{y\}(s)$ and $G(s) = \mathscr{L}\{g\}(s)$. Taking the Laplace transform of both sides of the differential equation in (11) and using the initial conditions gives

$$s^2Y(s) - s - 1 - Y(s) = G(s) .$$

Solving for $Y(s)$, we have

$$Y(s) = \frac{s+1}{s^2-1} + \left(\frac{1}{s^2-1}\right)G(s) = \frac{1}{s-1} + \left(\frac{1}{s^2-1}\right)G(s) .$$

Hence,

$$\begin{aligned} y(t) &= \mathscr{L}^{-1}\left\{\frac{1}{s-1}\right\}(t) + \mathscr{L}^{-1}\left\{\frac{1}{s^2-1}\,G(s)\right\}(t) \\ &= e^t + \mathscr{L}^{-1}\left\{\frac{1}{s^2-1}\,G(s)\right\}(t) . \end{aligned}$$

Referring to the table of Laplace transforms on the inside back cover, we find

$$\mathscr{L}\{\sinh t\}(s) = \frac{1}{s^2-1} ,$$

so we can now express

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2-1}\,G(s)\right\}(t) = \sinh t * g(t) .$$

Thus

$$y(t) = e^t + \int_0^t \sinh(t-v)g(v)\,dv$$

is the solution to the initial value problem (11). ◆

Example 2 Use the convolution theorem to find $\mathscr{L}^{-1}\{1/(s^2 + 1)^2\}$.

Solution Write

$$\frac{1}{(s^2+1)^2} = \left(\frac{1}{s^2+1}\right)\left(\frac{1}{s^2+1}\right).$$

Since $\mathscr{L}\{\sin t\}(s) = 1/(s^2+1)$, it follows from the convolution theorem that

$$\begin{aligned}
\mathscr{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\}(t) &= \sin t * \sin t = \int_0^t \sin(t-v)\sin v\,dv \\
&= \frac{1}{2}\int_0^t [\cos(2v-t) - \cos t]\,dv^{\dagger} \\
&= \frac{1}{2}\left[\frac{\sin(2v-t)}{2}\right]_0^t - \frac{1}{2}t\cos t \\
&= \frac{1}{2}\left[\frac{\sin t}{2} - \frac{\sin(-t)}{2}\right] - \frac{1}{2}t\cos t \\
&= \frac{\sin t - t\cos t}{2}.
\end{aligned}$$

◆

Example 3 Solve the integro-differential equation

$$y'(t) = 1 - \int_0^t y(t - v)e^{-2v}\,dv \ , \qquad y(0) = 1 \ . \tag{12}$$

Solution Equation (12) can be written as

$$y'(t) = 1 - y(t) * e^{-2t} . \tag{13}$$

Let $Y(s) = \mathscr{L}\{y\}(s)$. Taking the Laplace transform of (13) (with the help of the convolution theorem) and solving for $Y(s)$, we obtain

$$sY(s) - 1 = \frac{1}{s} - Y(s)\left(\frac{1}{s+2}\right)$$

$$sY(s) + \left(\frac{1}{s+2}\right)Y(s) = 1 + \frac{1}{s}$$

$$\left(\frac{s^2 + 2s + 1}{s+2}\right)Y(s) = \frac{s+1}{s}$$

$$Y(s) = \frac{(s+1)(s+2)}{s(s+1)^2} = \frac{s+2}{s(s+1)}$$

$$Y(s) = \frac{2}{s} - \frac{1}{s+1} .$$

Hence, $y(t) = 2 - e^{-t}$. ◆