A multigrid method with non-nested space hierarchies: applications with hybridization

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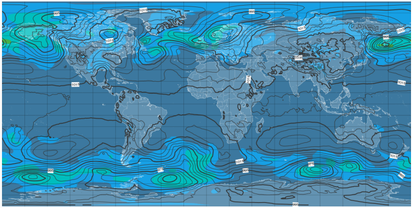
Motivation

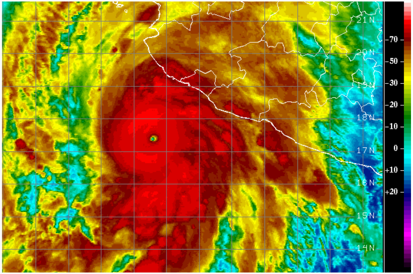
Challenges in numerical weather prediction



- Substantial increase in global resolution: $25\mathrm{km} \rightarrow 5-8\mathrm{km}$
- 5 day forecasts $\Longrightarrow$ model runtime needs to be under an hour!
- Semi-implicit time-integrators $\Longrightarrow$ Need scalable solvers!

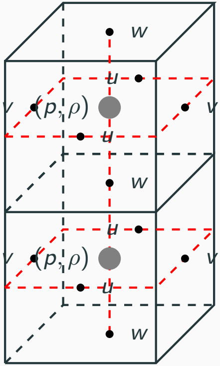
Requires repeatedly solving complex PDE systems:

$$\boxed{\begin{bmatrix} M_1 & B^T & Q^T \\ B & M_2 & 0 \\ Q & 0 & M_3 \end{bmatrix} \begin{Bmatrix} \mathbf{u} \\ p \\ \theta \end{Bmatrix} = \begin{Bmatrix} R_1 \\ R_2 \\ R_3 \end{Bmatrix}}$$

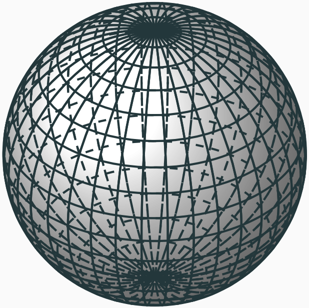


Numerics for weather and oceans

Traditional codes typically use C-grid finite differences on structured grids



Arakawa C-grid in 3D



"Lat-long" horizontal mesh

The mixed system is avoided by eliminating into an *elliptic* pressure equation:

$$\boxed{-\omega\left[\nabla^2_{\mathcal{S}}\Pi + \frac{\lambda^2}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \Pi}{\partial r}\right)\right] + \Pi = \ldots}$$

Towards finite elements

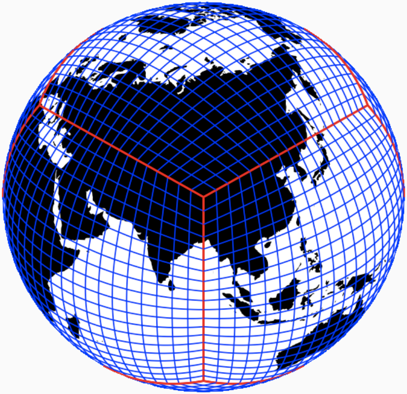
Finite element alternatives are popular:

- Discontinuous Galerkin methods
- Mixed ("compatible") methods (more recent)
    - Raviart-Thomas / Brezzi-Douglas-Marini mixed methods

because:

- methods are mesh-agnostic (avoids the pole problem)
- In the case of compatible finite element methods, preserved the mimetic properties of the C-grid.
    - Calculus identities are discretely preserved:

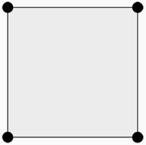
$$\nabla \cdot (a\mathbf{b}) = \mathbf{b} \cdot \nabla a + a \nabla \cdot \mathbf{b}$$
$$\nabla \times \nabla a = 0$$
$$\nabla \cdot \nabla \times \mathbf{b} = 0$$



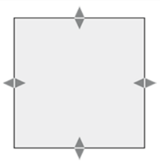
quasi-uniform cubed-sphere mesh

Continuous interval

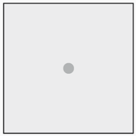
Discontinuous interval



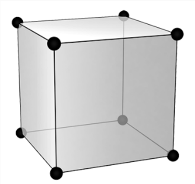
Cont. quad



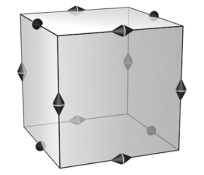
“Velocity” element



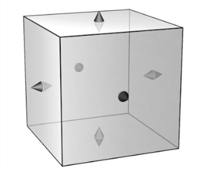
Discont. quad



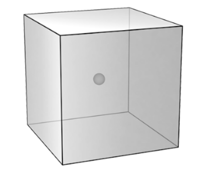
$\mathbb{W}_0$



$\mathbb{W}_1$



$\mathbb{W}_2$



$\mathbb{W}_3$

Illustrating the problem: a simple model

Linear shallow water equations (no rotation, mean depth $H = 1$):

$$\frac{\partial \mathbf{u}}{\partial t} + g\nabla h = 0,$$
$$\frac{\partial h}{\partial t} + \nabla \cdot \mathbf{u} = 0.$$

Compatible finite element discretization (lowest order): find $\mathbf{u} \in RT_0$, $h \in DG_0$:

$$\boxed{\begin{aligned} \int_\Omega \mathbf{w} \cdot \frac{\partial \mathbf{u}}{\partial t} \mathrm{d}x - g \int_\Omega h \nabla \cdot \mathbf{w} \mathrm{d}x = 0, \\ \int_\Omega \phi \frac{\partial h}{\partial t} \mathrm{d}x + \int_\Omega \phi \nabla \cdot \mathbf{u} \mathrm{d}x = 0, \end{aligned}}$$

$\forall \mathbf{w}, \phi \in RT_0 \times DG_0$.

Discretizing (implicitly) in time requires the solution of the saddle-point system:

$$\boxed{\begin{bmatrix} M_1 & -gB^T \\ B & M_2 \end{bmatrix} \begin{Bmatrix} \mathbf{u}^n \\ h^n \end{Bmatrix} = \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix}}$$

Elliptic equation

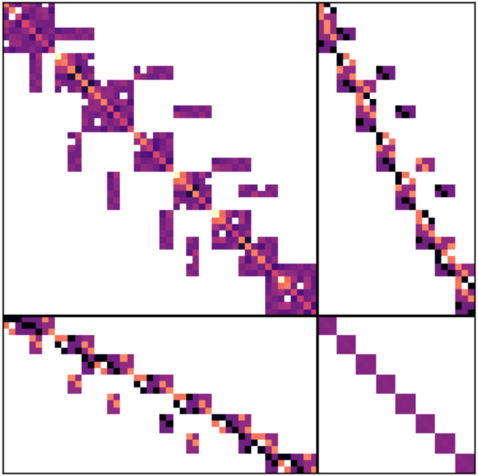
An elliptic problem can be obtained by eliminating $\mathbf{u}^n$:

$$\left(M_2 + gBM_1^{-1}B^T\right) h^n = \ldots$$

Problem!

Not practical in real applications because of the dense (globally) inverse M_1^{-1}.

Global sparsity pattern



Hybridization of the mixed method

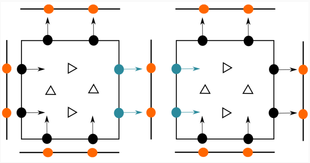
Discrete *hybridizable* problem: find $\mathbf{u}^n \in RT_0^d$, $h^n \in DG_0$, $\lambda \in T_0$:

$$\int_\Omega \mathbf{w} \cdot \mathbf{u}^n \mathrm{d}x - g\alpha\Delta t \int_\Omega h^n \nabla \cdot \mathbf{w} \mathrm{d}x + \sum_K g\alpha\Delta t \int_{\partial K} \lambda \mathbf{w} \cdot \mathbf{n} \mathrm{d}S = \dots,$$

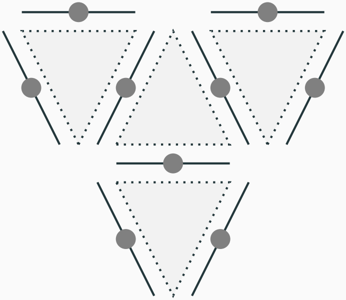
$$\int_\Omega \phi h^n \mathrm{d}x + \alpha\Delta t \int_\Omega \phi \nabla \cdot \mathbf{u}^n \mathrm{d}x = \dots,$$

$$\sum_K \int_{\partial K} \gamma \mathbf{u} \cdot \mathbf{n} \mathrm{d}S = 0$$

$\forall \mathbf{w}, \phi, \gamma \in RT_0 \times DG_0 \times T_0$.



RT_1^d and T_1 on quadrilaterals

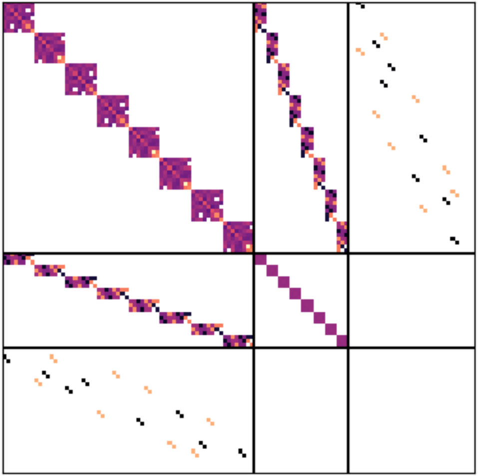


Global trace space T_0 on triangles

Matrix system

$$\begin{bmatrix} \tilde{M}_1 & -g\tilde{B}^T & K^T \\ \tilde{B} & M_2 & 0 \\ K & 0 & 0 \end{bmatrix} \begin{Bmatrix} \mathbf{u}^n \\ h^n \\ \lambda \end{Bmatrix} = \begin{Bmatrix} \tilde{R}_1 \\ R_2 \\ 0 \end{Bmatrix}$$

Global sparsity pattern (hybridizable)



Eliminating and recovering velocity / depth locally

A *sparse* elliptic equation for λ can be obtained

$$S\lambda = E,$$

where S and E are obtained via *local* static condensation:

$$S = \begin{bmatrix} K & 0 \end{bmatrix} \begin{bmatrix} \tilde{M}_1 & -g\tilde{B}^T \\ \tilde{B} & M_2 \end{bmatrix}^{-1} \begin{bmatrix} K^T \\ 0 \end{bmatrix},$$

$$E = \begin{bmatrix} K & 0 \end{bmatrix} \begin{bmatrix} \tilde{M}_1 & -g\tilde{B}^T \\ \tilde{B} & M_2 \end{bmatrix}^{-1} \begin{Bmatrix} \tilde{R}_1 \\ R_2 \end{Bmatrix}$$

$\mathbf{u}^n$ and h^n can be recovered locally:

$$\begin{Bmatrix} \mathbf{u}^n \\ h^n \end{Bmatrix} = \begin{bmatrix} \tilde{M}_1 & -g\tilde{B}^T \\ \tilde{B} & M_2 \end{bmatrix}^{-1} \left(\begin{Bmatrix} \tilde{R}_1 \\ R_2 \end{Bmatrix} - \begin{bmatrix} K^T \\ 0 \end{bmatrix} \lambda \right)$$

What about solvers for $S\lambda = E$?

$$S\lambda = E$$

What we know

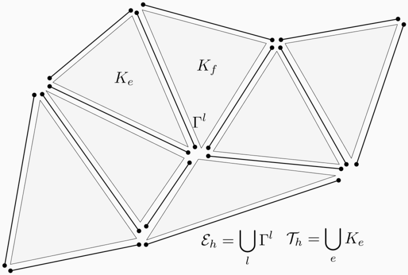
- λ is an approximation of h on the edges/faces of the mesh
- S is spectrally similar to the Schur-complement from the original (unhybridized) system (Gopolakrishnan, 2003)
- λ is the unique solution to an elliptic variational problem (Cockburn, 2004):

$$s(\lambda, \gamma) = e(\gamma), \forall \gamma \in T_k$$

- S is positive definite;
    - SPD if rotation is not included

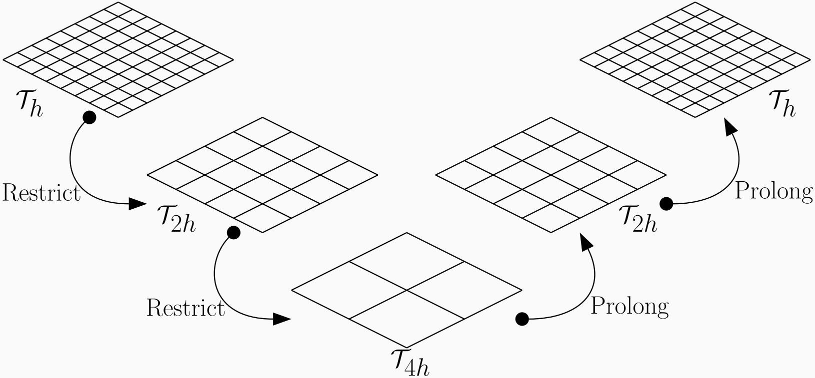
This *screams* multigrid. But:

- Geometric multigrid directly on the trace system?



Mesh hierarchy on $\mathcal{E}_h$ vs $\mathcal{T}_h$? Smoothers?

(Geometric) Multigrid: a (very) quick overview



- Restriction operator: R_{2h} moves data from fine to coarse ($h \to 2h$)
- Prolongation operator: P_h moves data from coarse to fine ($2h \to h$).
- Very important: smoothers S (Jacobi/Gauss-Seidel/Successive over-relaxation (SOR)/Richardson)
- Direct down-up traversal of the grid hierarchy is called a *V-cycle*.

Multigrid operators for H^1 finite elements

H^1 (elliptic) problem

Find $p \in V_h(\mathcal{T}_h) \subset H^1(\mathcal{T}_h)$ such that

$$a(p, q) = L(q), \forall p \in V_h.$$

Traditional MG methods construct a hierarchy: $\mathcal{T}_1, \cdots, \mathcal{T}_J \equiv \mathcal{T}_h$, with

$$V_1 \subset \cdots \subset V_J \equiv V_h,$$

With grid operators

$$(A_j\phi, \psi)_k \equiv a(\phi, \psi), \forall \phi, \psi \in V_j,$$

Prolongation

Typically taken to be the natural injection operator: $P_j : V_{j-1} \to V_j$

$$P_j\phi = \phi, \forall \phi \in V_{j-1}$$

Restriction

Restriction is the adjoint of prolongation w.r.t the inner products $(\cdot,\cdot)_j$ and $(\cdot,\cdot)_{j-1}$:

$$(P_j\phi, \psi)_j = (\phi, R_{j-1}\psi)_{j-1},$$

$\forall \phi \in V_{j-1}$, $\psi \in V_j$.

Back to hybridization

Mixed Poisson example (Cockburn and Gopalakrishnan 2004)

Find $\mathbf{u} \in U_h^d$, $p \in V_h$, $\lambda \in T_h$ such that

$$\int_{\mathcal{T}_h} \mathbf{w} \cdot \mathbf{u} \mathrm{d}x - \int_{\mathcal{T}_h} p \nabla \cdot \mathbf{w} \mathrm{d}x + \sum_K \int_{\partial K} \lambda \mathbf{w} \cdot \mathbf{n} \mathrm{d}S = 0,$$

$$\int_{\mathcal{T}_h} \phi \nabla \cdot \mathbf{u} \mathrm{d}x = \int_{\mathcal{T}_h} f \phi \mathrm{d}x,$$

$$\sum_K \int_{\partial K} \gamma \mathbf{u} \cdot \mathbf{n} \mathrm{d}S = 0,$$

for all $\mathbf{w}, \phi, \gamma$. The solutions have the form: $\mathbf{u} = \mathbf{Q}\lambda + \mathbf{Q}f$, $p = U\lambda + Uf$, where

$\mathbf{Q}\lambda$ and $U\lambda$ are contributions of the solutions determined by λ:

$$\begin{bmatrix} A & -B^T \\ B & 0 \end{bmatrix} \begin{Bmatrix} \mathbf{Q}\lambda \\ U\lambda \end{Bmatrix} = \begin{Bmatrix} -K^T\lambda \\ 0 \end{Bmatrix}$$

$\mathbf{Q}f$ and Uf are contributions determined by the source:

$$\begin{bmatrix} A & -B^T \\ B & 0 \end{bmatrix} \begin{Bmatrix} \mathbf{Q}f \\ Uf \end{Bmatrix} = \begin{Bmatrix} 0 \\ f \end{Bmatrix}$$

Variational problem for λ

Characterization of λ

$\lambda \in T_h$ is the unique solution to the variational problem:

$$s(\lambda, \gamma) = \int_{\mathcal{T}_h} \mathbf{Q}\lambda\mathbf{Q}\gamma \mathrm{d}x = \int_{\mathcal{T}_h} fU\gamma \mathrm{d}x = e(\gamma), \forall \gamma \in T_h$$

Relation to a primal problem (Gopalakrishnan 2003)

$$s(\lambda, \gamma) \sim a(p, q) = \int_{\mathcal{T}_h} \nabla p \cdot \nabla q \mathrm{d}x$$

We can use this relation to construct a new multigrid algorithm for solving the trace system. We solve for λ using a multigrid method for the primal problem!

A non-nest multigrid algorithm for λ (Gopalakrishnan and Tan, 2009)

$\mathcal{T}_1$ coarse mesh, refinements $\mathcal{T}_j$, $j = 2, \cdots, J$, $\mathcal{T}_J = \mathcal{T}_h$ (with skeleton $\mathcal{E}_h$).

Space hierarchy

Define the $J + 1$ spaces:

$$V_k = \begin{cases} P_1(\mathcal{T}_{k+1}) \subset H^1(\mathcal{T}_{k+1}), & k = 0, \cdots J - 1 \\ T_k(\mathcal{E}_h) \subset L^2(\mathcal{E}_h) & k = J. \end{cases}$$

So we have $V_0 \subset V_1 \subset \cdots \subset V_{J-1} \not\subset V_J$! Non-Nested!

Prologation

With Π_{V_J} denoting the L^2 orthogonal projection onto the trace space $T_k(\mathcal{E}_J)$:

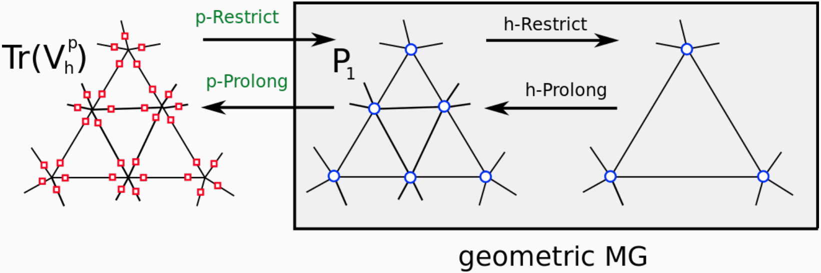
$$P_j\psi = \begin{cases} \psi & j < J \\ \Pi_{V_J}(\psi|_{\mathcal{E}_J}) & k = J. \end{cases}$$

Grid operators defined as

$$(A_j\psi, \phi)_j = \begin{cases} \int_{\mathcal{T}_j} \nabla\psi \cdot \nabla\phi \mathrm{d}x & j < J \\ \int_{\mathcal{T}_h} \mathbf{Q}\psi\mathbf{Q}\phi \mathrm{d}x & j = J. \end{cases}$$

Can be interpreted as a 2-level method with an H^1 multigrid method as a coarse solver.

Firedrake implementation



- Firedrake can already do hybridization and static condensation (`firedrake.HybridizationPC` and `firedrake.SCPC`)
- Trace multigrid method implemented via `firedrake.GTMGPC`
  - Problem specific!

```python
def get_p1_space():
    return FunctionSpace(mesh, "P", 1)

def p1_callback():
    P1 = get_p1_space()
    p = TrialFunction(P1)
    q = TestFunction(P1)
    return inner(grad(p), grad(q))*dx

def get_p1_prb_bcs():
    return DirichletBC(get_p1_space(), g, ids)

appctx = {'get_coarse_operator': p1_callback,
          'get_coarse_space': get_p1_space,
          'coarse_space_bcs': get_p1_prb_bcs()}
solve(a == L, w, solver_parameters={...},
      appctx=appctx)
```

Invoking GTMGPC

```
solver_parameters = {
    'mat_type': 'matfree',
    'ksp_type': 'preonly',
    'pc_type': 'python',
    'pc_python_type': 'firedrake.HybridizationPC',
    'hybridization': {
        'ksp_type': 'cg',
        'ksp_rtol': 1e-8,
        'pc_type': 'python',
        'pc_python_type': 'firedrake.GTMGPC',
        'gt': {'mg_levels': {'ksp_type': 'chebyshev',
                             'pc_type': 'jacobi',
                             'ksp_max_it': 2},
               'mg_coarse': {'ksp_type': 'preonly',
                             'pc_type': 'mg',
                             'mg_levels': {'ksp_type': 'chebyshev',
                                           'pc_type': 'jacobi',
                                           'ksp_max_it': 2}}}}}
```

GTMG convergence (mixed Poisson)

Table 1: CG iterations using GTMGPC on the traces for the hybridizable RT method ($k = 0, 1, 2, 3$). Residual reduction by a factor of 10^8. Coarse mesh: 8×8 unit square.

GMG	CG iterations (trace sys.)				
n_{levels}	n_{cells} (fine)	HRT_0	HRT_1	HRT_2	HRT_3
1	372	8	9	9	11
2	1,416	8	8	9	11
3	5,520	7	8	9	11
4	21,792	7	8	8	10
5	86,592	7	8	8	10
6	345,216	7	8	8	9

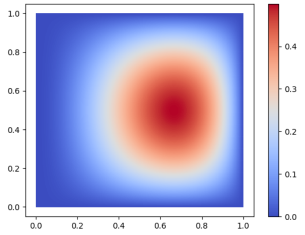
Poisson example

$$-\nabla^2 p = f, \text{ in } \Omega = [0,1]^2,$$

$$p = 0 \text{ on } \partial\Omega$$

Manufactured solution:

$$p = \sin(\pi x)\tan(\pi x/4)\sin(\pi y)$$



References

- **Hybridization**:
    - Cockburn, Bernardo, and Jayadeep Gopalakrishnan. "A characterization of hybridized mixed methods for second order elliptic problems." (2004)
    - Gopalakrishnan, Jayadeep. "A Schwarz preconditioner for a hybridized mixed method." (2003)
    - Cockburn, Bernardo, Jayadeep Gopalakrishnan, and Raytcho Lazarov. "Unified hybridization of discontinuous Galerkin, mixed, and continuous Galerkin methods for second order elliptic problems." (2009)
- **Fancy multigrid for HDG and hybrid-mixed**:
    - Gopalakrishnan, Jayadeep, and Shuguang Tan. "A convergent multigrid cycle for the hybridized mixed method." (2009)
    - Cockburn, Bernardo, et al. "Multigrid for an HDG method." (2014)
- **Hybridization and static condensation in Firedrake**:
    - Gibson, Thomas H., Mitchell, Lawrence, Ham, David A., Cotter, Colin J. "Slate: extending Firedrakes domain-specific abstraction to hybridized solvers for geoscience and beyond." (2019).
- **Compatible finite elements for GFD**:
    - Gibson, Thomas H., McRae, Andrew T.T., Cotter, Colin J., Mitchell, Lawrence, Ham, David A. "Compatible Finite Element Methods for Geophysical Flows: Implementation and Automation using Firedrake" (2019).

Conclusions

- MG directly on the trace space is headache-inducing.
    - Replacing the trace system with a P_1-problem is much nicer!!
    - Implementation: P_1 is never assumed to be the only coarse space.
- New Firedrake feature: `firedrake.GTMGPC`!
    - Potential use for geophysical models (Tom Gregory's poster)
    - Used within the context of HDG as well (Jack's talk)
- **Composability is amazing.**