Code verification for the manuscript: "Slate: extending Firedrake's domain-specific abstraction to hybridized solvers for geoscience and beyond"

Thomas H. Gibson[†,1]

[1]*Department of Mathematics, Imperial College, South Kensington Campus, London SW7 2AZ*
[†]*Email:* `t.gibson15@imperial.ac.uk`

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1 Code verification

To verify our computed results, we now perform a simple convergence study for a model Dirichlet problem. We seek a solution to the Poisson equation as a first-order system:

$$\boldsymbol{u} + \nabla p = 0 \text{ in } \Omega = [0,1]^2, \tag{1}$$

$$\nabla \cdot \boldsymbol{u} = f \text{ in } \Omega, \tag{2}$$

$$p = p_0 \text{ on } \partial\Omega_D, \tag{3}$$

where f and p_0 are chosen so that the analytic solution is the sinusoid $p(x,y) = \sin(\pi x)\sin(\pi y)$ and its negative gradient. We solve this problem by hybridizing the mixed formulation of (1), and employ our static condensation preconditioner described in Section 4.1.1 of the manuscript. All results were obtained in serial, with MUMPS providing the LU factorization algorithms for the condensed trace system.

Each mesh in our convergence study is obtained by generating a quadrilateral mesh with 2^r cells in each spatial direction, and dividing each quadrilateral cell into two equal simplicial elements. Once the solutions are obtained, we compute a post-processed scalar solution using the method described in Section 3.3.1 of the manuscript via Slate-generated kernels. Figure 1 displays the results for the hybridized Raviart-Thomas (RT) method. Our computations are in full agreement with the theory.

We repeat this experiment for the LDG-H method with varying choices of τ in order to verify how τ influences the convergence rates, comparing with the expected rates for the LDG-H method given a particular order of τ (see Table 1 for a summary). In all our experiments, we use the post-processing methods described in Sections 3.3.1 and 3.3.2 to produce approximations $p_h^\star$ and $\boldsymbol{u}_h^\star$. Error convergence plots from our tests are shown in Figure 2 that confirm the expected rates. This rather sensitive test verifies that our software framework is generating correct code.

Table 1: The expected convergence rates of the LDG-H method with a stability parameter τ of a particular order.

parameter	expected rates of convergence ($k \geq 1$)			
τ	$\lVert p - p_h \rVert_{L^2(\Omega)}$	$\lVert \boldsymbol{u} - \boldsymbol{u}_h \rVert_{\boldsymbol{L}^2(\Omega)}$	$\lVert p - p_h^\star \rVert_{L^2(\Omega)}$	$\lVert \boldsymbol{u} - \boldsymbol{u}_h^\star \rVert_{\boldsymbol{L}^2(\Omega)}$
$\mathcal{O}(1)$	$k+1$	$k+1$	$k+2$	$k+1$
$\mathcal{O}(h)$	k	$k+1$	$k+2$	$k+1$
$\mathcal{O}\left(h^{-1}\right)$	$k+1$	k	$k+1$	k

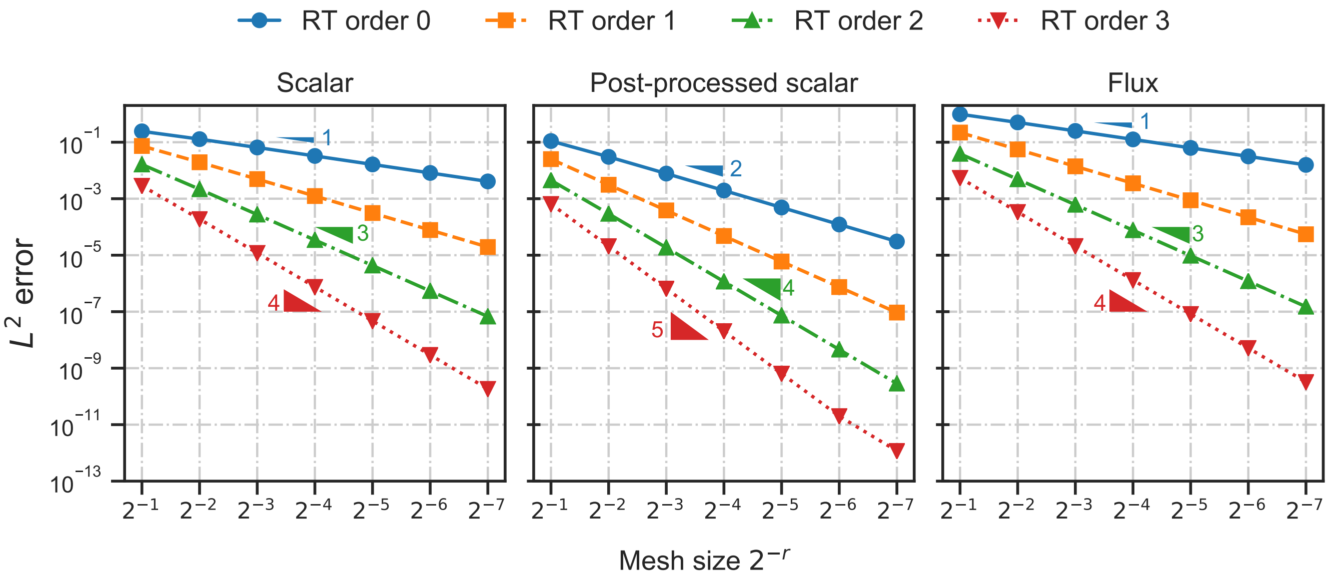

Figure 1: Error convergence rates for our implementation of the hybridized RT method of orders 0, 1, 2, and 3. We observe the expected rates for the scalar and flux solutions of the standard RT method: $k+1$ in the L^2-error for both the scalar and flux approximations. Additionally, we see the effects of post-processing the scalar solution, yielding superconvergent $k+2$ rates.

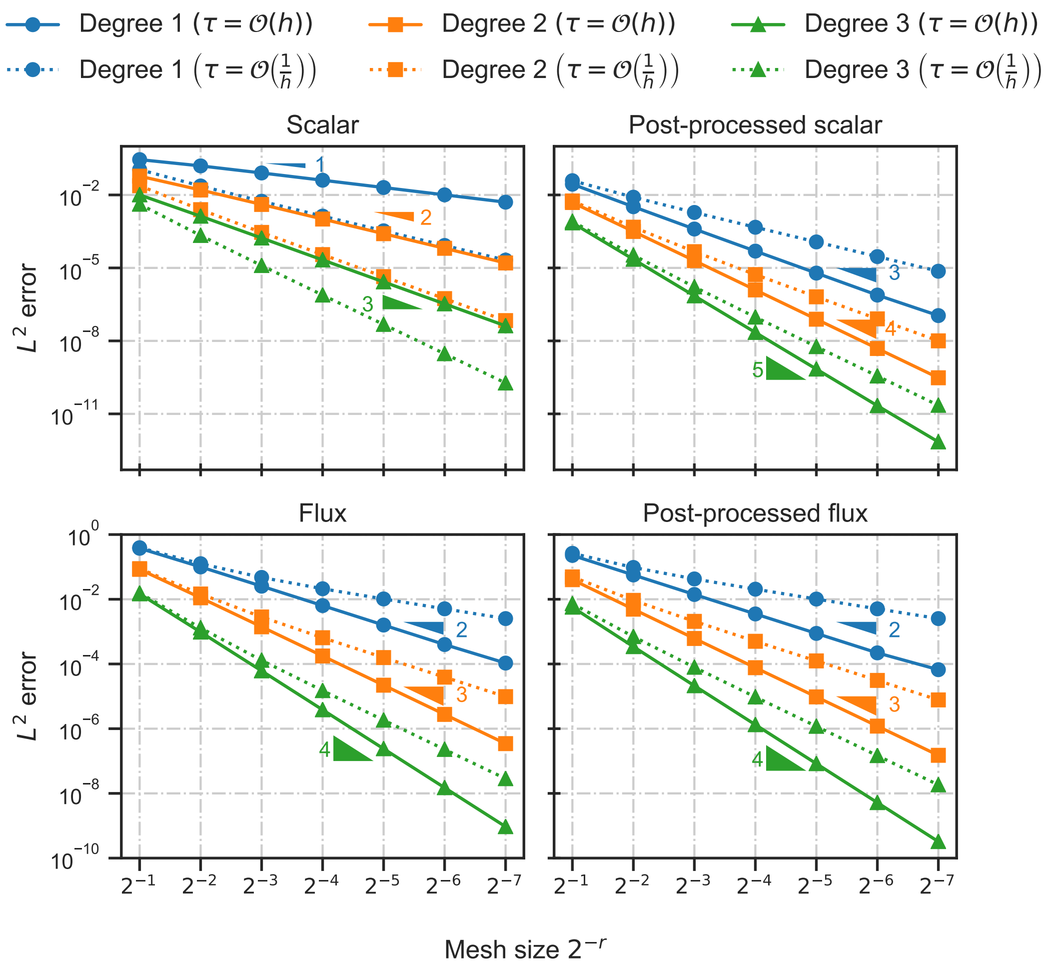

Figure 2: Error convergence rates for our implementation of the LDG-H method with $\tau = h$ and $\tau = \frac{1}{h}$. The expected sensitivity of this discretization subject to appropriate choices of stabilization parameter τ is verified. We see no change in the convergence rates between the scalar and post-processed scalar solutions when $\tau = \frac{1}{h}$. Superconvergence is achieved when taking $\tau = h$. The post-processed flux rates in both cases match the rates of the unprocessed flux.