22nd March, 2021

MAT2250 — Discrete Math

Mandatory assignment 1 of 1

Submission deadline

Thursday 8th April 2021, 14:30 in Canvas (canvas.uio.no).

Instructions

You can choose between scanning handwritten notes or typing the solution directly on a computer (for instance with LaTeX). The assignment must be submitted as a single PDF file. Scanned pages must be clearly legible. The submission must contain your name, course and assignment number.

It is expected that you give a clear presentation with all necessary explanations. Remember to include all relevant plots and figures. Students who fail the assignment, but have made a genuine effort at solving the exercises, are given a second attempt at revising their answers. All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you have understood the content you have handed in, we may request that you give an oral account.

In exercises where you are asked to write a computer program, you need to hand in the code along with the rest of the assignment. It is important that the submitted program contains a trial run, so that it is easy to see the result of the code.

Application for postponed delivery

If you need to apply for a postponement of the submission deadline due to illness or other reasons, you have to contact the Student Administration at the Department of Mathematics (e-mail: studieinfo@math.uio.no) well before the deadline.

All mandatory assignments in this course must be approved in the same semester, before you are allowed to take the final examination.

Complete guidelines about delivery of mandatory assignments:

uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

GOOD LUCK!

You are highly encouraged to collaborate with other students on the problem set. If you need help connecting with others in the course, let me know by email. You must include the names of any students who you collaborated with. Remember: the write up of all solutions must be your own!

Problem 1. Stirling numbers of the 2nd kind and generating functions

1. Use a counting argument to show that the Stirling numbers of the second kind satisfy $S_{n,2} = 2^{n-1} - 1$ for $n \geq 1$ (recall that $S_{0,k} = 0$). Express the generating function

$$\mathcal{S}_2(z) = \sum_{n=0}^{\infty} S_{n,2} z^n$$

as a rational function.

2. Use the recurrence relation $S_{n,k} = S_{n-1,k-1} + kS_{n-1,k}$ and the formula found above for $\mathcal{S}_2(z)$ to prove that

$$\mathcal{S}_3(z) := \sum_{n=0}^{\infty} S_{n,3} z^n = \frac{z^3}{(1-z)(1-2z)(1-3z)}.$$

3. Use a partial fraction decomposition of the generating function $\mathcal{S}_3(z)$ to obtain a closed formula for $S_{n,3}$.

Problem 2. Graphs from sets

1. Recall the hypercube graph is $Q_n = (V(Q_n), E(Q_n))$ where

$$V(Q_n) = \{u_1 \dots u_n \mid u_i = 0, 1\}$$

and

$$E(Q_n) = \{uv \mid u, v \text{ binary words differing in exactly one letter}\}.$$

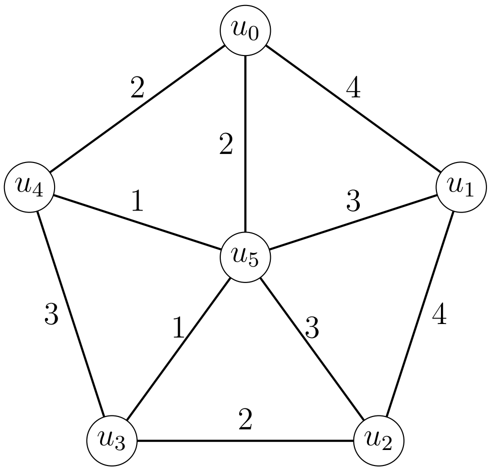
What are the sizes of $V(Q_n)$ and $E(Q_n)$?

2. Assuming all edges have weight 1, what is the distance in Q_n from $00\dots00$ to $u = u_1u_2\dots u_n$? What is the distance in Q_n between two arbitrary words u and v?

3. Orient the edge $k = uv$ of Q_n so that $u = k^-$ and $v = k^+$ if the word v contains more 1's than u. Prove that the orientation is acyclic. Which word corresponds to a sink and which word is a source?

Problem 3. Greedy algorithms

Consider the graph G with weight function prescribed along its edges:



1. Run Dijkstra's algorithm from starting vertex u_0 to find a tree containing all minimal weight paths from u_0 with the prescribed weight function. Write down which edge and vertex is selected at each step of the algorithm.

   What is the length of the shortest weighted path in G from u_0 to u_3?

2. Run Kruskal's algorithm to find a minimal spanning tree of G for the given weight function.

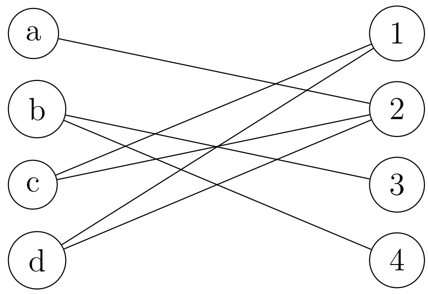
   What is the weight of a minimal spanning tree in G? Is there a unique minimal spanning tree in G with this weight function?

3. Show that if the greedy algorithm on matroids from Theorem 7.13 can select two distinct bases B_1 and B_2 of minimal weight, then there must exist $s_1 \in B_1$ and $s_2 \in B_2$ such that $w(s_1) = w(s_2)$. Conclude that if the weight function $w : S \to \mathbb{R}$ is an injective map then the greedy algorithm selects a unique minimal basis of a matroid.

   Find an example of a weighted graph which shows that the converse to the above statement is false. Namely that there is a graph G with weight function $w : E \to \mathbb{R}$ which is not injective but the greedy algorithm still selects a unique minimal spanning tree.

Problem 4. Matchings

1. Consider the bipartite graph:



   Determine the matching number of G and find a maximal matching.

2. The hypercube graph Q_n is bipartite, with the two disjoint vertex sets being determined by the binary words containing an even or odd number of 1's. The matching number of Q_n is 2^{n-1}. Describe a maximal matching of Q_n.

3. Let S be the set of binary words of length n with k number of $1's$ and let T be the set of binary words of length n with $k+1$ number of 1's. Let $E = \{st \mid s, t \text{ differ in exactly one letter}\}$ and set $H = (S \cup T, E)$ (notice H is an induced subgraph of Q_n).

   - What are the degrees in H of vertices in S and vertices in T?
   - Prove that for any subset A of S we have $(k+1)|N(A)| \geq (n-k)|A|$.

     Hint: consider the set of edges incident to A and the set of edges incident to $N(A)$.
   - Use Hall's matching condition to conclude that $m(H) = |S|$ for $k < n/2$.

   Conclude that there exists an injective map f from the subsets of size k of $\{1, \ldots, n\}$ to the subsets of size $k+1$ of $\{1, \ldots, n\}$ satisfying $I \subset f(I)$ for $k < n/2$.