

On the Analytic Continuation of Field Configurations in Classical Theories

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Outline

Introduction

Mathematical Framework

First tests

Classical field theory situations

Conclusions & Future Work

Introduction

Field Theories in Nature

- ▶ The Standard Model of elementary particles
- ▶ The theory of gravitation
- ▶ Hydrodynamics
- ▶ Neuron activity in the brain

Field theory reminds

A field theory is defined by its Lagrangian density, or action:

$$S = \int d^4x \mathcal{L}(x^\mu, \phi, \partial_\mu \phi).$$

The first order of variation of the action must vanish around the classical field configuration:

$$\delta S = \int d^4x \delta \mathcal{L}(x^\mu, \phi, \partial_\mu \phi),$$

leading to the field equations:

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} = 0.$$

A particular field configuration: Soliton

A soliton is a solution of the field equation which:

- ▶ is time independant
- ▶ has a finite energy

Many modern particle theories exhibit such solutions on the classical level.

They are very popular in cosmology:

- ▶ Cosmic strings, domain walls, and attempts to localize gravity, wrap dimensions
- ▶ Magnetic monopoles as WIMPs
- ▶ Monopoles are a feature of many GUTs

A particular field configuration: Instanton

An instanton is a solution of the field equations which is

- ▶ in euclidian time,
- ▶ localised in time and space.

Again, instanton is common in many nowadays theories.

Mathematical Framework

The problem of analytic continuation

The situation arise when we have a field configuration in Minkowski time we want to evaluate in Euclidian time:

$$r = \sqrt{x^2 + \tau^2} \xrightarrow{\tau \rightarrow -it} \sqrt{x^2 - t^2}. \quad (1)$$

If $t > x$:

$$r = \pm i \sqrt{|t^2 - x^2|} = \pm i s.$$

The problem of analytic continuation

Let $\vec{f}(r)$ be a function defined by:

$$\vec{f}'(x) = \vec{F}(x, \vec{f}(x)).$$

The analytic continuation of $f(r)$, noted $\tilde{f}(z)$ is obtained by extending the derivative to the complex part:

$$\begin{aligned} f'(x) \rightarrow \frac{\partial f}{\partial z} &\equiv \partial_x f_R(x + iy) + i \partial_x f_I(x + iy), \\ \Rightarrow \partial_x f_R(z) + i \partial_x f_I(z) &= F_R(z, \tilde{f}(z)) + F_I(z, \tilde{f}(z)). \end{aligned}$$

Cauchy-Riemann equations

The Cauchy-Riemann equations allow us to write the system in terms of the derivatives w.r.t the complex coordinate:

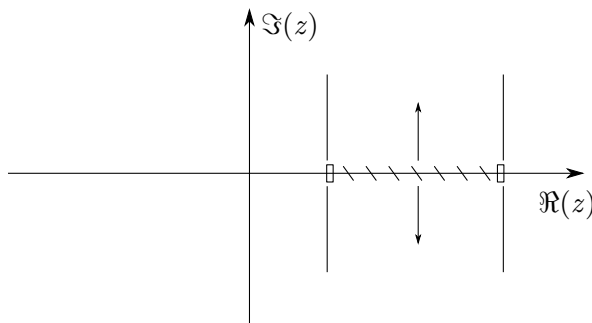
$$\begin{cases} \partial_x f_R = \partial_y f_I, \\ \partial_x f_I = -\partial_y f_R, \end{cases}$$

which lead to the Cauchy problem:

$$\Rightarrow \begin{cases} \partial_y f_I = F_R(z, \tilde{f}(z)), \\ \partial_y f_R = -F_I(z, \tilde{f}(z)). \end{cases}$$

Choice of coordinate system

This is not enough in the following situation:



but can be addressed with a change of coordinate.

Change to polar coordinate

Define the new polar coordinates:

$$\begin{cases} x = r \cos \theta, \\ y = r \sin \theta. \end{cases}$$

Using the Cauchy-Riemann equations, we can rewrite the previous Cauchy problem for the θ variable:

$$\begin{bmatrix} -\partial_{\theta} f_{\mathbf{R}}(r, \theta) \\ \partial_{\theta} f_{\mathbf{I}}(r, \theta) \end{bmatrix} = r \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} F_{\mathbf{I}}(z, \tilde{f}(z)) \\ F_{\mathbf{R}}(z, \tilde{f}(z)) \end{bmatrix}.$$

First Tests

We now want to test the framework on a set of known functions:

- ▶ The square root
- ▶ The natural logarithm
- ▶ An instanton of the quantum pendulum equation

The square root

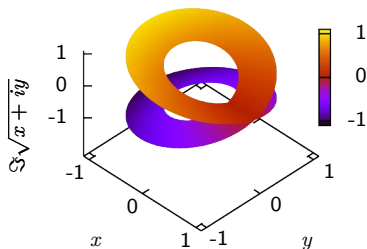
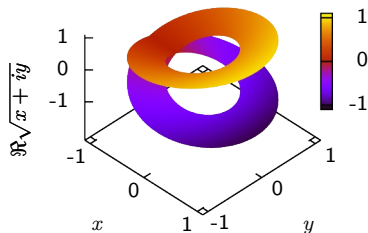
1. The square root is defined by the differential equation:

$$f(x) = \sqrt{x} \Rightarrow f'(x) = \frac{1}{2} \frac{1}{\sqrt{x}} = \frac{1}{2f(x)},$$

2. which leads to the Cauchy problem:

$$\left\{ \begin{array}{l} \partial_y f_I(z) = \frac{f_R(z)}{2(f_R^2(z) + f_I^2(z))} \\ -\partial_y f_R(z) = -\frac{f_I(z)}{2(f_R^2(z) + f_I^2(z))} \end{array} \right. \quad \left| \quad \begin{array}{l} f_R(x + i0) = \sqrt{x} \\ f_I(x + i0) = 0 \end{array} \right.$$

Numerical analytic continuation of the Square Root



The Natural Logarithm

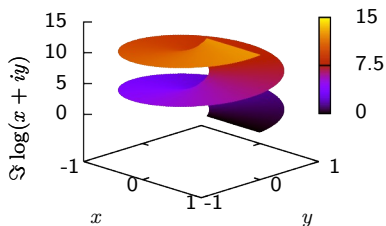
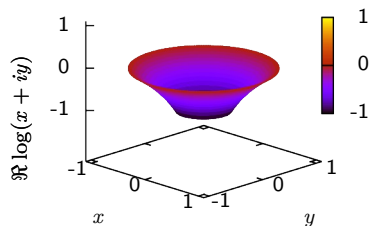
1. The natural logarithm is defined by the differential equation:

$$f(x) = \log(x) \Rightarrow f'(x) = \frac{1}{x} = \exp(-f(x)),$$

2. which leads to the Cauchy problem:

$$\left\{ \begin{array}{l} \partial_y f_I(z) = \exp(-f_R(z)) \cos(f_I) \\ -\partial_y f_R(z) = -\exp(-f_R(z)) \sin(f_I) \end{array} \right. \quad \left| \quad \begin{array}{l} f_R(x + i0) = \log(x) \\ f_I(x + i0) = 0 \end{array} \right.$$

Numerical analytic continuation of the Logarithm



The Quantum Pendulum

We consider the Lagrangian of a quantum particle in a cosine potential:

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - (1 - \cos(\phi))$$

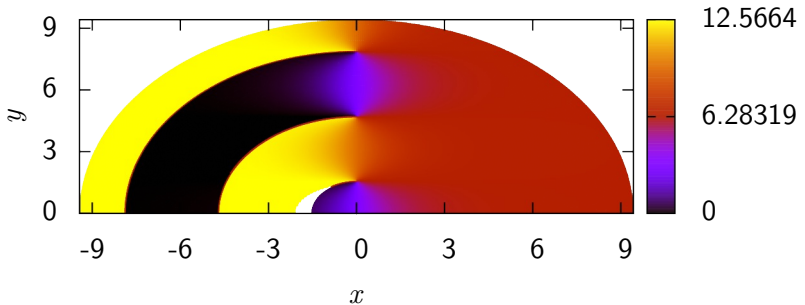
- ▶ Equation of motion: $\ddot{\phi} = \sin(\phi)$,
- ▶ Energy: $E = \frac{1}{2}\dot{\phi}^2 + (1 - \cos(\phi))$.

A solution of 0 energy reads:

$$\phi(t) = 4 \arctan(\exp(t)).$$

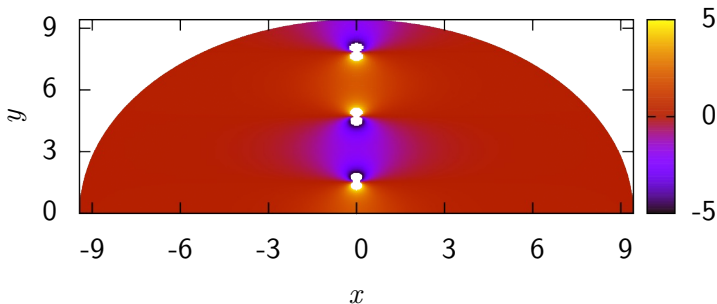
Numerical analytic continuation of the Quantum pentulum

Real part:



Numerical analytic continuation of the Quantum pentulum

Imaginary part:



Classical field theory situations

- ▶ Abelian Vortex of a complex scalar in $(2 + 1)$ dimensions, with $U(1)$ gauge symmetry
- ▶ Non-Abelian Monopole of a real scalar triplet in $(3 + 1)$ dimensions, with $SU(2)$ gauge symmetry. (Georgi-Glashow model)

Abelian Vortex

The Lagrangian of the theory reads:

$$\mathcal{L} = \frac{1}{2} D_\mu \phi^* D_\mu \phi - \frac{\lambda}{2} (\phi^* \phi - v^2)^2 - \frac{1}{4} F_{\mu\nu}^2.$$

- Field equations:

$$\begin{aligned} \partial^\nu F_{\nu\mu} &= ie(\phi^* D_\mu - \phi D_\mu \phi^*), \\ D^\mu D_\mu \phi + \lambda(\phi^* \phi - v^2)\phi &= 0 \text{ \& } c.c.. \end{aligned}$$

- Energy functional, static case:

$$E = 4\pi \int_0^\infty r^2 dr \left(\frac{1}{4} F_{ij} F_{ij} + \frac{1}{2} D_i \phi D_i \phi^* + \frac{\lambda}{2} (\phi^* \phi - v^2)^2 \right).$$

Static, Spherically symmetric ansatz

The simplest spherically symmetric static solution can be expressed with the ansatz:

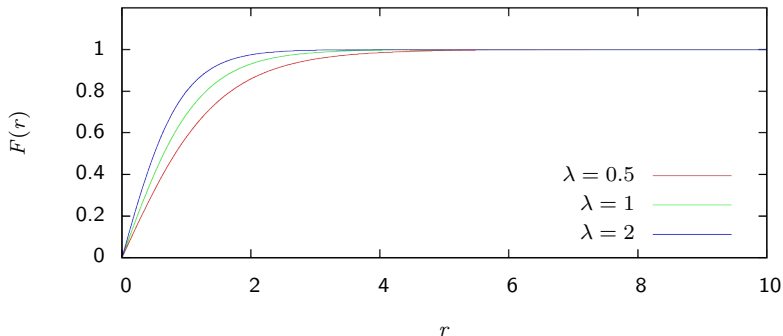
$$\begin{aligned}\phi(r, \theta) &= v e^{i\theta} F(r), \\ A_i(r) &= -\frac{1}{er} \varepsilon_{ij} n_j A(r).\end{aligned}$$

Leading to the equations for $f(a)$ and $a(r)$:

$$\begin{aligned}-\frac{d}{dr} \left[\frac{1}{r} \frac{dA}{dr} \right] - 2e^2 v^2 \frac{F^2(1-A)}{r} &= 0, \\ -\frac{d}{dr} \left[r \frac{dF}{dr} \right] + \lambda v^2 r (F^2 - 1) F + \frac{F}{r} (A - 1)^2 &= 0.\end{aligned}$$

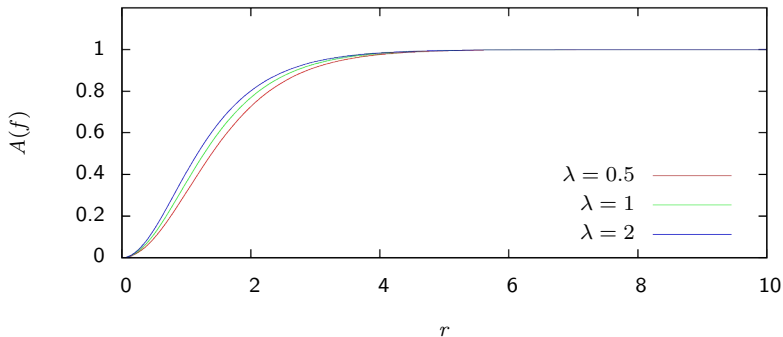
Vortex profile

Higgs field:



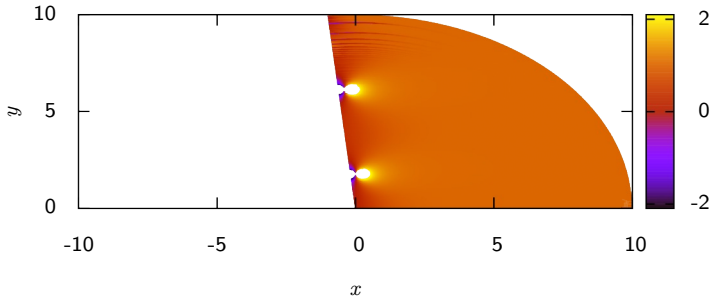
Vortex profile

Gauge Field:



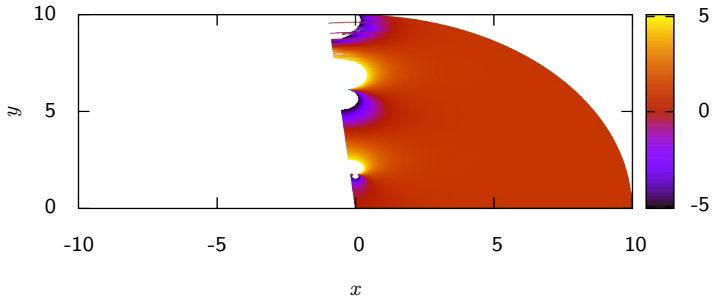
Analytic continuation of the Vortex

Real part of the Higgs field:



Analytic continuation of the Vortex

Real part of the gauge field:



Non-Abelian Monopole

The Lagrangian of the theory reads:

$$\mathcal{L} = \frac{1}{2} D_\mu \phi^a D_\mu \phi^a - \frac{\lambda}{2} (\phi^a \phi^a - v^2)^2 - \frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a.$$

- Field equations:

$$\begin{aligned} D_\mu D_\mu \phi^a + \lambda \phi^a (\phi^b \phi^b - v^2) &= 0, \\ \partial^\nu F_{\nu\mu}^a &= g \varepsilon^{abc} (A_\nu^b F_\mu^{c,\nu} + D_\mu \phi^b \phi^c). \end{aligned}$$

- Energy functional, static case:

$$E = 4\pi \int_0^\infty r^2 dr \left(\frac{1}{4} F_{ij}^a F_{ij}^a + \frac{1}{2} D_i \phi^a D_i \phi^a + \frac{\lambda}{4} (\phi^a \phi^a - v^2)^2 \right).$$

't Hooft–Polyakov ansatz

The first non-trivial static field configuration was proposed by 't Hooft & Polyakov:

$$\begin{cases} \phi^a(r, \theta) = v n^a h(r), \\ A_i^a(r) = \frac{1}{gr^2} \varepsilon^{aij} n_j (1 - f(r)), \quad A_0^a = 0. \end{cases}$$

Which lead to the field equations for $f(r)$ and $h(r)$:

$$\begin{aligned} r^2 h'' &= 2h f^2 + \lambda v^2 h (h^2 - r^2), \\ r^2 f'' &= f(f^2 - 1) - g v^2 h^2 f. \end{aligned}$$

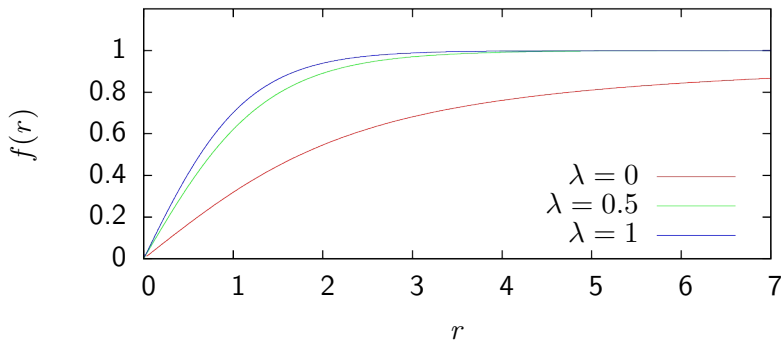
Bogomol'nyi–Prasad–Sommerfield limit

If we set the Higgs coupling constant to zero, the solution can be expressed in a simple form:

$$f(r) = \frac{r}{\sinh(r)} \quad ; \quad h(r) = \sqrt{gv^2}(r \coth(r) - 1).$$

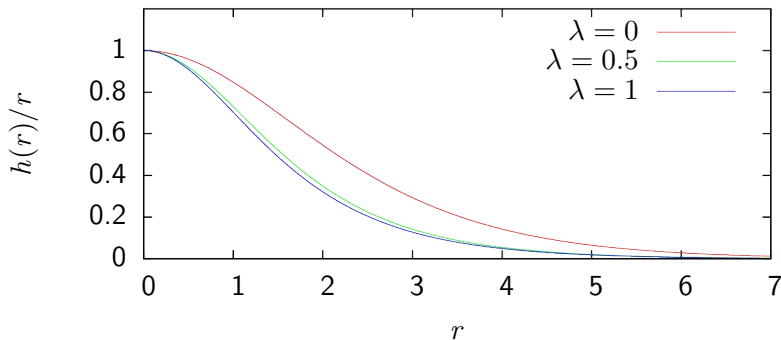
Monopole profile

Higgs field profiled:



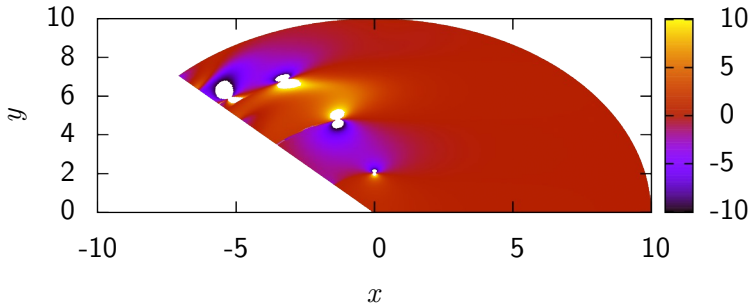
Monopole profile

Gauge field profile:



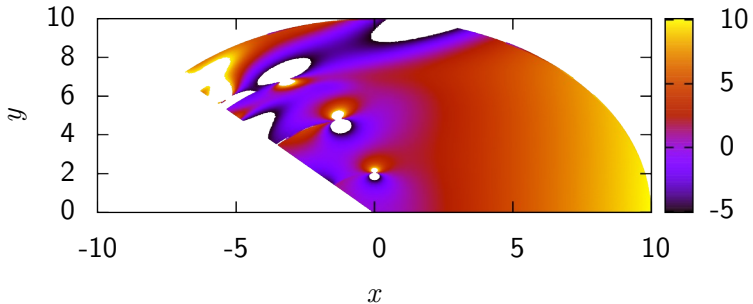
Analytic continuation of the Monopole

Real part of the Higgs field:



Analytic continuation of the Monopole

Real part of the gauge field:



Conclusions & Future Work

- ▶ Numerical framework to find the analytical continuation of a function,
- ▶ Successfully tested on a set of different situations,
- ▶ Proved on the numerical level that the Wick rotation of the abelian vortex and non-abelian vortex is valid.
- ▶ Additional arguments are required in order to corroborate numerical results.

Questions?