On the Analytic Continuation of Field Configurations in Classical Theories

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Outline

Introduction

Field Theories in Nature

- The Standard Model of elementary particles
- The theory of gravitation
- Hydrodynamics
- Neuron activity in the brain

Field Theory Recalls

A field theory is defined by its Lagrangian density, or action:

$$S = \int \mathrm{d}^4 x \mathcal{L}(x^\mu, \phi, \partial_\mu \phi).$$

The first order of variation of the action must vanish around the classical field configuration:

$$\delta S = \int \mathrm{d}^4 x \delta \mathcal{L}(x^\mu, \phi, \partial_\mu \phi),$$

leading to the field equations:

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} = 0.$$

A particular field configuration: Soliton

A soliton is a solution of the field equation which:

- is time independant
- has a finite energy

Many modern particle theories exibit such solutions on the classical level.

They are very popular in cosmology:

- Cosmic strings, domain walls, and attempts to localize gravity, wrap dimensions
- Magnetic monopoles as WIMPs
- Monopoles are a feature of many GUTs

A particular field configuration: Instanton

An instanton is a solution of the field equations which is

- in euclidian time,
- localised in time and space.

Again, instanton is common in many nowadays theories.

Mathematical Framework

The problem of analytic continuation

The situation arise when we have a field configuration in Minkowski time we want to evaluate in Euclidian time:

$$r = \sqrt{x^2 + \tau^2} \xrightarrow{\tau \to -it} \sqrt{x^2 - t^2}. \tag{1}$$

If $t > x$:

$$r = \pm i \sqrt{|t^2 - x^2|} = \pm i s.$$

The problem of analytic continuation

Let $\vec{f}(r)$ be a function defined by:

$$\vec{f}'(x) = \vec{F}(x, \vec{f}(x)).$$

The analytic continuation of $f(r)$, noted $\tilde{f}(z)$ is obtained by extending the derivative to the complex part:

$$f'(x) \rightarrow \frac{\partial f}{\partial z} \equiv \partial_x f_{\mathrm{R}}(x+iy) + i\partial_x f_{\mathrm{I}}(x+iy),$$
$$\Rightarrow \partial_x f_{\mathrm{R}}(z) + i\partial_x f_{\mathrm{I}}(z) = F_{\mathrm{R}}(z, \tilde{f}(z)) + F_{\mathrm{I}}(z, \tilde{f}(z)).$$

Cauchy-Riemann equations

The Cauchy-Riemann equations allow us to write the system in terms of the derivatives w.r.t the complex coordinate:

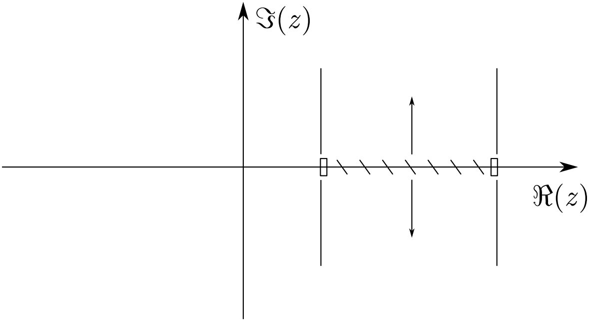
$$\begin{cases} \partial_x f_{\mathrm{R}} = \partial_y f_{\mathrm{I}}, \\ \partial_x f_{\mathrm{I}} = -\partial_y f_{\mathrm{R}}, \end{cases}$$

which lead to the Cauchy problem:

$$\Rightarrow \begin{cases} \partial_y f_{\mathrm{I}} = F_{\mathrm{R}}(z, \tilde{f}(z)), \\ \partial_y f_{\mathrm{R}} = -F_{\mathrm{I}}(z, \tilde{f}(z)). \end{cases}$$

Choice of coordinate system

This is not enough in the following situation:



but can be addressed with a change of coordinate.

Change to polar coordinate

Define the new polar coordinates:

$$\begin{cases} x = r\cos\theta, \\ y = r\sin\theta. \end{cases}$$

Using the Cauchy-Riemann equations, we can rewrite the previous Cauchy problem for the θ variable:

$$\begin{bmatrix} -\partial_\theta f_{\mathrm{R}}(r,\theta) \\ \partial_\theta f_{\mathrm{I}}(r,\theta) \end{bmatrix} = r \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} F_{\mathrm{I}}(z,\tilde{f}(z)) \\ F_{\mathrm{R}}(z,\tilde{f}(z)) \end{bmatrix}.$$

First Tests

We now want to test the framework on a set of known functions:

- The square root
- The natural logarithm
- An instanton of the quantum pendulum equation

The square root

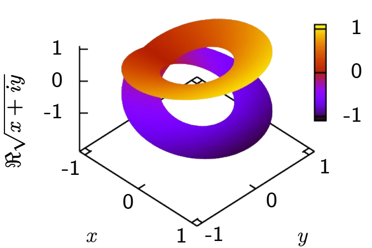
1. The square root is defined by the differential equation:

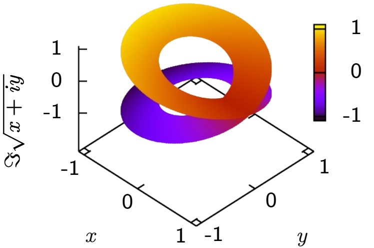
$$f(x) = \sqrt{x} \Rightarrow f'(x) = \frac{1}{2}\frac{1}{\sqrt{x}} = \frac{1}{2f(x)},$$

2. which leads to the Cauchy problem:

$$\left\{\begin{aligned} \partial_y f_I(z) &= \frac{f_R(z)}{2(f_R^2(z)+f_I^2(z))} \\ -\partial_y f_R(z) &= -\frac{f_I(z)}{2(f_R^2(z)+f_I^2(z))} \end{aligned}\right. \qquad \left| \qquad \begin{aligned} f_{\mathrm{R}}(x+i0) &= \sqrt{x} \\ f_{\mathrm{I}}(x+i0) &= 0 \end{aligned}\right.$$

Numerical analytic continuation of the Square Root





The Natural Logarithm

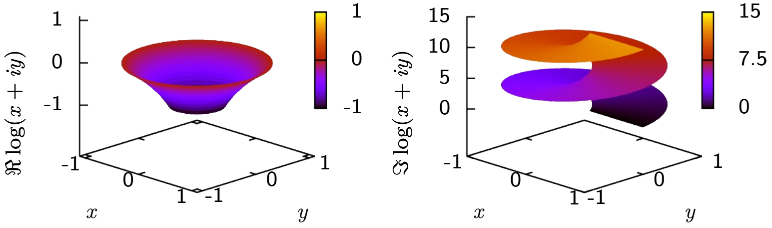
1. The natural logarithm is defined by the differential equation:

$$f(x) = \log(x) \Rightarrow f'(x) = \frac{1}{x} = \exp(-f(x)),$$

2. which leads to the Cauchy problem:

$$\begin{cases} \partial_y f_I(z) = \exp(-f_R(z)) \cos(f_I) \\ -\partial_y f_R(z) = -\exp(-f_R(z)) \sin(f_I) \end{cases} \quad \Bigg| \quad \begin{aligned} f_{\mathrm{R}}(x + i0) &= \log(x) \\ f_{\mathrm{I}}(x + i0) &= 0 \end{aligned}$$

Numerical analytic continuation of the Logarithm



The Quantum Pendulum

We consider the Lagrangian of a quantum particle in a cosine potential:

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - (1 - \cos(\phi))$$

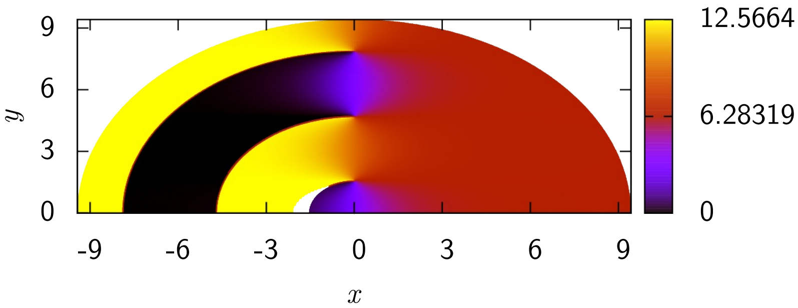
- Equation of motion: $\ddot{\phi} = \sin(\theta)$,
- Energy: $E = \frac{1}{2}\dot{\phi}^2 + (1 - \cos(\phi))$.

A solution of 0 energy reads:

$$\phi(t) = 4\arctan(\exp(t)).$$

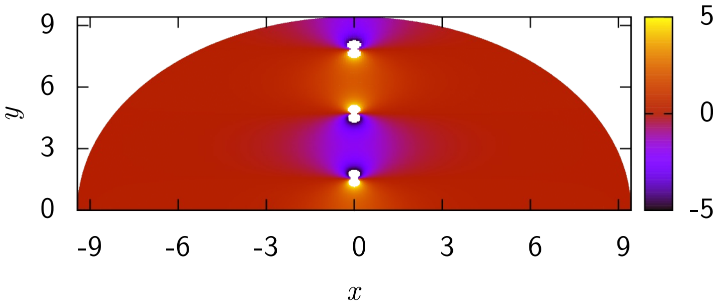
Numerical analytic continuation of the Quantum pentulum

Real part:



Numerical analytic continuation of the Quantum pentulum

Imaginary part:



Classical field theory situations

- Abelian Vortex of a complex scalar in $(2+1)$ dimensions, with $U(1)$ gauge symmetry
- Non-Abelian Monopole of a real scalar triplet in $(3+1)$ dimensions, with $SU(2)$ gauge symmetry. (Georgi-Glashow model)

Abelian Vortex

The Lagrangian of the theory reads:

$$\mathcal{L} = \frac{1}{2}\mathrm{D}_\mu\phi^*\mathrm{D}_\mu\phi - \frac{\lambda}{2}(\phi^*\phi - v^2)^2 - \frac{1}{4}F_{\mu\nu}^2.$$

- Field equations:

$$\partial^\nu F_{\nu\mu} = ie(\phi^*\mathrm{D}_\mu - \phi\mathrm{D}_\mu\phi^*),$$
$$\mathrm{D}^\mu\mathrm{D}_\mu\phi + \lambda(\phi^*\phi - v^2)\phi = 0 \ \& \ c.c..$$

- Energy functional, static case:

$$E = 4\pi\int_0^\infty r^2\mathrm{d}r\left(\frac{1}{4}F_{ij}F_{ij} + \frac{1}{2}\mathrm{D}_i\phi\mathrm{D}_i\phi^* + \frac{\lambda}{2}(\phi^*\phi - v^2)^2\right).$$

Static, Spherically symmetric ansatz

The simplest spherically symmetric static solution can be expressed with the ansatz:

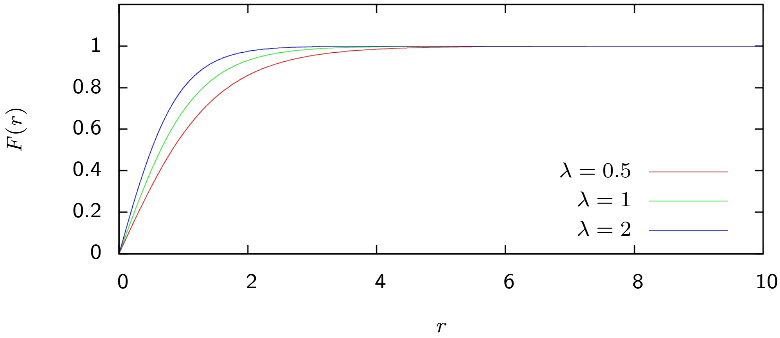
$$\phi(r,\theta) = ve^{i\theta}F(r),$$
$$A_i(r) = -\frac{1}{er}\varepsilon_{ij}n_j A(r).$$

Leading to the equations for $f(a)$ and $a(r)$:

$$-\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{r}\frac{\mathrm{d}A}{\mathrm{d}r}\right] - 2e^2v^2\frac{F^2(1-A)}{r} = 0,$$
$$-\frac{\mathrm{d}}{\mathrm{d}r}\left[r\frac{\mathrm{d}F}{\mathrm{d}r}\right] + \lambda v^2 r(F^2-1)F + \frac{F}{r}(A-1)^2 = 0.$$

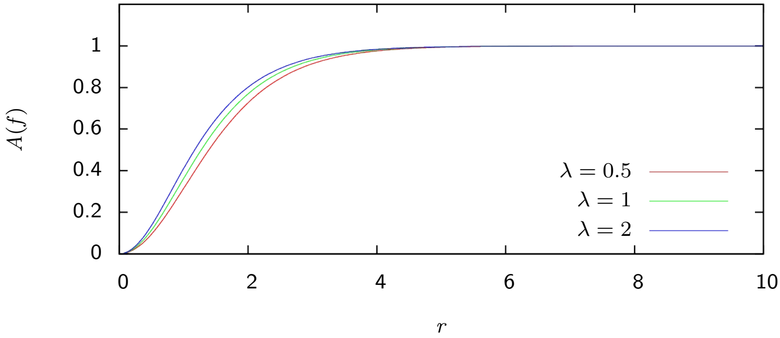
Vortex profile

Higgs field:



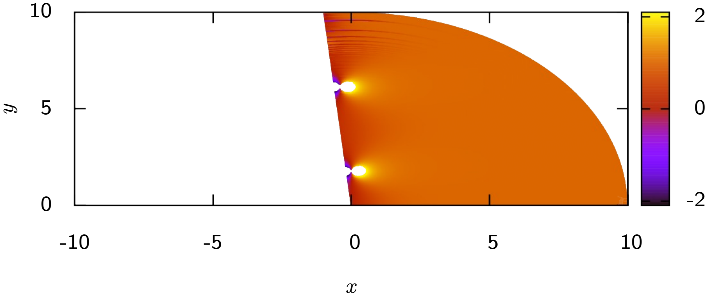
Vortex profile

Gauge Field:



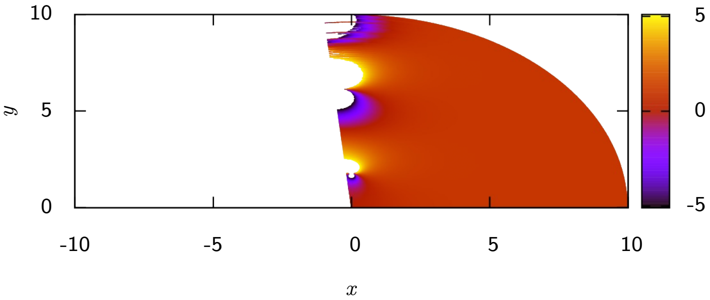
Analytic continuation of the Vortex

Real part of the Higgs field:



Analytic continuation of the Vortex

Real part of the gauge field:



Non-Abelian Monopole

The Lagrangian of the theory reads:

$$\mathcal{L} = \frac{1}{2}\mathrm{D}_\mu \phi^a \mathrm{D}_\mu \phi^a - \frac{\lambda}{2}(\phi^a\phi^a - v^2)^2 - \frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu}.$$

- Field equations:

$$\mathrm{D}_\mu \mathrm{D}_\mu \phi^a + \lambda\phi^a(\phi^b\phi^b - v^2) = 0,$$
$$\partial^\nu F^a_{\nu\mu} = g\varepsilon^{abc}(A^b_\nu F^{c,\nu}_\mu + \mathrm{D}_\mu\phi^b\phi^c).$$

- Energy functional, static case:

$$E = 4\pi\int_0^\infty r^2 \mathrm{d}r \left(\frac{1}{4}F^a_{ij}F^a_{ij} + \frac{1}{2}\mathrm{D}_i\phi^a\mathrm{D}_i\phi^a + \frac{\lambda}{4}(\phi^a\phi^a - v^2)^2\right).$$

't Hooft–Polyakov ansatz

The first non-trivial static field configuration was proposed by 't Hooft & Polyakof:

$$\begin{cases} \phi^a(r,\theta) = vn^a h(r), \\ A_i^a(r) = \dfrac{1}{gr^2}\varepsilon^{aij} n_j (1 - f(r)), \ A_0^a = 0. \end{cases}$$

Which lead to the field equations for $f(r)$ and $h(r)$:

$$r^2 h'' = 2hf^2 + \lambda v^2 h(h^2 - r^2),$$
$$r^2 f'' = f(f^2 - 1) - gv^2 h^2 f.$$

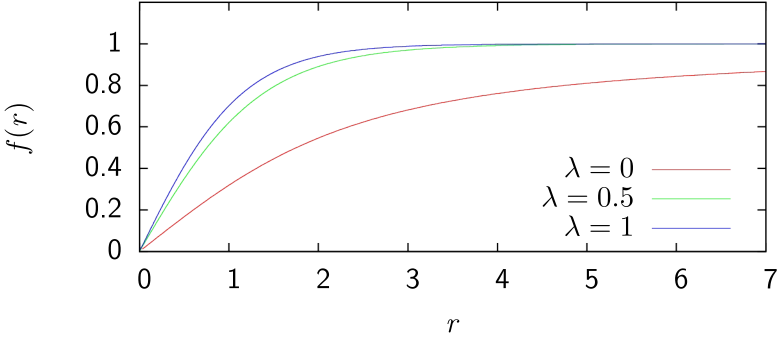
Bogomol'nyi–Prasad–Sommerfield limit

If we set the Higgs coupling constant to zero, the solution can be expressed in a simple form:

$$f(r) = \frac{r}{\sinh(r)} \quad ; \quad h(r) = \sqrt{gv^2}(r\coth(r) - 1).$$

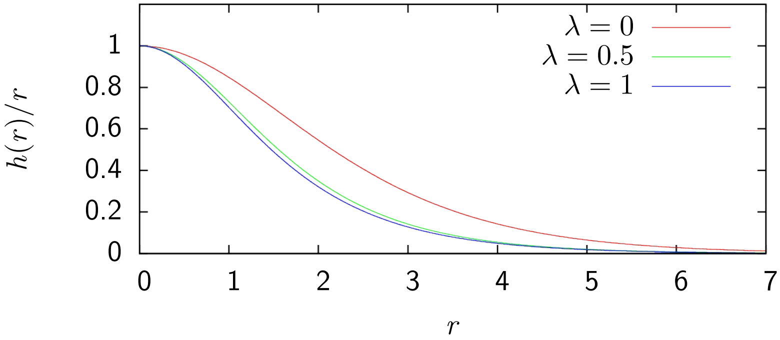
Monopole profile

Higgs field profiled:



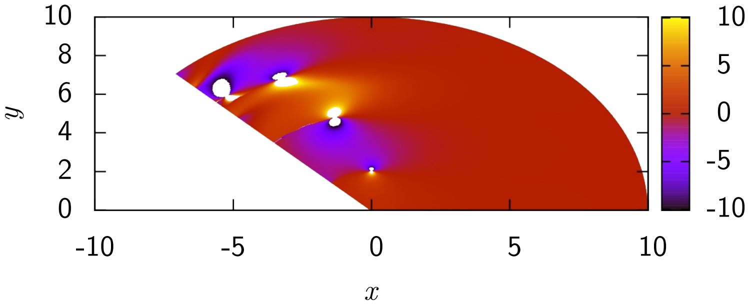
Monopole profile

Gauge field profile:



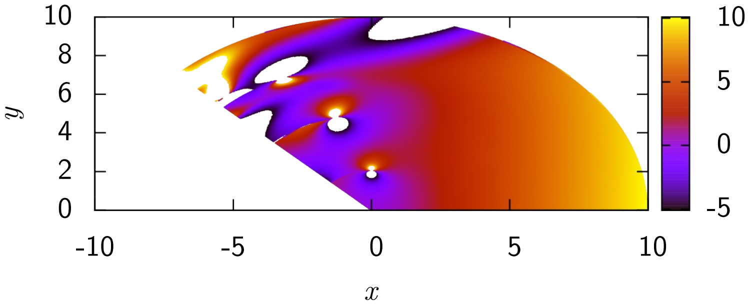
Analytic continuation of the Monopole

Real part of the Higgs field:



Analytic continuation of the Monopole

Real part of the gauge field:



Conclusions & Future Work

- Numerical framework to find the analytical continuation of a function,
- Successfully tested on a set of different situations,
- Proved on the numerical level that the Wick rotation of the abelian vortex and non-abelian vortex is valid.
- Additional arguments are required in order to corroborate numerical results.

Questions?