COSMOLOGY

A study of the interactions between black holes and magnetic monopoles

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Abstract

In this document I organize and present the work I do every day during my Master thesis.

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1 Georgi-Glashow model with $SU(2)$ gauge symmetry

The Lagrangian of the model is given by:

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a,\mu\nu} + \frac{1}{2}D_\mu\phi^a D^\mu\phi^a - \frac{\lambda}{4}\left(\phi^a\phi^a - v^2\right) \tag{1}$$

where

$$D_\mu\phi^a \equiv \phi^a{}_{;\mu} = \partial_\mu\phi^a + g\varepsilon^{abc}A^b_\mu\phi^c \tag{2}$$
$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\varepsilon^{abc}A^b_\mu A^c_\nu \tag{3}$$

By requiring that the first order of the variation of the action around the classical trajectory, we find the following equivalent of the Euler-Lagranges equations for the equation of motion:

$$\frac{\partial\mathcal{L}}{\partial\phi^a} - D_\mu\frac{\mathcal{L}}{\partial\phi^a{}_{;\mu}} = 0, \quad \forall a = 1,2,3 \tag{4}$$
$$\frac{\partial\mathcal{L}}{\partial A^a_\mu} - \partial_\nu\frac{\partial\mathcal{L}}{\partial A^a_{\mu,\nu}} = 0, \quad \forall\mu = 0,1,2,3,\ a = 1,2,3 \tag{5}$$

These lead to the following equations for motion:

$$D_\mu D^\mu\phi^a + \lambda\phi^a\left(\phi^b\phi^b - v^2\right) = 0 \tag{6}$$
$$g\varepsilon^{abc}A^b_\nu F^{c,\nu}_\mu + g\varepsilon^{abc}D_\mu\phi^b\phi^c = \partial^\nu F^a_{\nu\mu} \tag{7}$$

This last equation can be written in the following form:

$$D^\mu F^a_{\mu\nu} = gj^a_\nu \tag{8}$$

where

$$j^a_\mu = -\varepsilon^{abc}D_\mu\phi^b\phi^c \tag{9}$$

comes from the variation of the scalar part of the action with respect to A^a_μ.

We now assume a spherical symmetry for the solution of the fields ϕ^a and A^a_μ:

$$\begin{aligned}&\phi^a = vn^a\left(1 - H(r)\right), \quad \partial_0\phi^a = 0\\ &A^a_\mu = \frac{1}{gr}\varepsilon^{aij}n_j\left(1 - F(r)\right), \quad A^a_0 = 0, \quad \partial_0 A^a_i = 0\end{aligned} \tag{10}$$

Recalling that

$$\partial_i r = n_i, \quad \partial_i n_j = \frac{1}{r}\left(\delta_{ij} - n_i n_j\right) \tag{11}$$

we find by direct substitution into (6),(7) two second-order differential equation for the fonctions f and h:

$$r(rh'' + 2h') - 2h(f-1)^2 + \lambda v^2 r^2 h(1-h^2) = 0 \tag{12}$$
$$r^2 f'' - f(f-1)(f-2) = -g^2v^2r^2h^2(1-f) \tag{13}$$

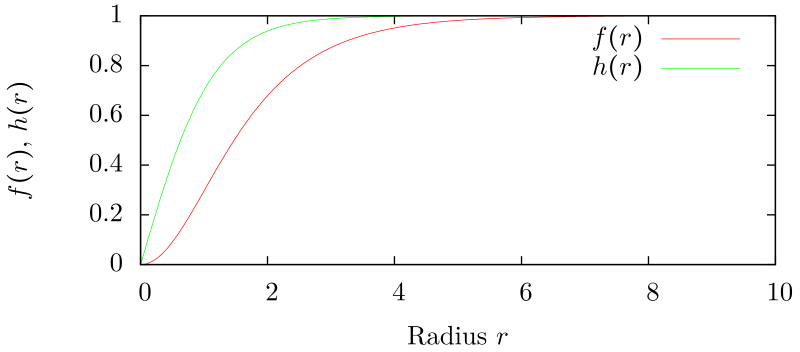

Figure 1: *Numerical integration of the non-linear set of equation for the 't Hooft–Polyakov monopole. We have chosen* $R = 10$*,* $N = 300$*,* $\lambda v^2 = g^2 v^2 = 1$*.*

More details on this computations are provided in Appendix B. For convenience, we can proceed to the following change of variable:

$$H(r) := rh(r) \quad \Rightarrow \quad H''(r) = rh''(r) + 2h'(r) \tag{14}$$

$$F(r) := f(r) - 1 \quad \Rightarrow \quad F''(r) = f''(r) \tag{15}$$

and by direct substitution we find the two equivalent equations:

$$r^2 H'' = 2HF^2 + \lambda v^2 H(H^2 - r^2) \tag{16}$$

$$r^2 F'' = F(F^2 - 1) - gv^2 H^2 F \tag{17}$$

which correspond to the original equations found by 't Hooft and Polyakov. Numerical integration of the system (12)-(13) is done in a similar way as in Sec. 2.1. The results are presented on Fig. 1.

1.1 Energy functionnal

The mass (or energy) of the monopole is one of the quantity which can be estimated analytically. We can show the the mass only depends upon the ratio of the two free parameters of the Lagrangian:

$$E \sim \frac{v}{g} \tag{18}$$

Introducing a non-Minkowskian metric $g_{\mu\nu}$, the symmetric energy-momentum tensor writes:

$$\delta_{g_{\mu\nu}} S = \frac{1}{2} \int \mathrm{d}^4x \sqrt{-g}\bar{T}^{\mu\nu} \delta g_{\mu\nu} \tag{19}$$

We easily find that for the Lagrangian (1), the corresponding $\bar{T}^{\mu\nu}$ is

$$\bar{T}_{\mu\alpha} = -F_{\mu\nu} F_{\alpha}{}^{\nu} + \mathrm{D}_\mu \phi^a \mathrm{D}_\alpha \phi^a - \eta_{\mu\alpha} \mathcal{L} \tag{20}$$

For a static configuration of the fields, the $\bar{T}_{00}$ component reduces to:

$$\bar{T}_{00} = \frac{1}{2} F_{0i} F_{0i} + \frac{1}{4} F_{ij} F_{ij} + \frac{1}{2} \mathrm{D}_0 \phi^a \mathrm{D}_0 \phi^a + \frac{1}{2} \mathrm{D}_i \phi^a \mathrm{D}_i \phi^a + \frac{\lambda}{4} \left(\phi^a \phi^a - v^2\right)^2 \tag{21}$$

Considering the 't Hooft Polyakov ansatz (10), the expression reduce even further, and the energy functionnal writes:

$$E = 4\pi \int_0^\infty r^2 \mathrm{d}r \left(\frac{1}{4} F_{ij} F_{ij} + \frac{1}{2} \mathrm{D}_i \phi^a \mathrm{D}_i \phi^a + \frac{\lambda}{4} \left(\phi^a \phi^a - v^2 \right)^2 \right) \tag{22}$$

In terms of f and h, it becomes:

$$E = \frac{4\pi}{g^2} \int_0^\infty \mathrm{d}r \Bigg[f'^2 + \frac{f^2 (f-2)^2}{2r^2} + \frac{v^2 g^2}{2} r^2 h'^2 + v^2 g^2 h^2 (1-f)^2 + \frac{\lambda}{4} v^4 g^2 r^2 (h^2 - 1)^2 \Bigg] \tag{23}$$

Proceeding to the following change of variable:

$$f(r) = 1 - F(r); \quad h(r) = \frac{H(r)}{r}; \quad \xi = gvr \tag{24}$$

The energy functional can be recast into

$$E = \frac{4\pi v}{g} C\left(\frac{\lambda}{g^2} \right) \tag{25}$$

We note that that the minimization of the energy functional 23 with respect to f and h lead to the same field equations as (12)-(13).

Given the numerical approximations for f and h found in previous section, we can estimate the energy in the following way:

$$\int_0^\infty \mathcal{E}(r, f, f', h, h') \mathrm{d}r \;\rightarrow\; h \sum_{i=1}^{N} \mathcal{E}\left(r_i, f_i, \frac{f_{i+1} - fi - 1}{h}, h_i, \frac{h_{i+1} - h_{i-1}}{h} \right) \tag{26}$$

where $\mathcal{E}$ is the energy density.

1.2 Effects of the parameters

It is interesting to find how the energy of the monopole depends upon the three parameters, v, g and λ. For all the simulation, we integrated in the interval $r \in [0, 30]$, with the number of points $N = 513$. Fig. 2-3 present the dependance of the fields for different values of the Higgs expectation value v, and Fig. 4-5 present the same data, but rather for different values of the Higgs coupling constant λ.

On Fig. 5 we see that the energy dependance is roughly a square root of the Higgs coupling constant λ, shifted from 4π. This is visible on Fig. 5, where $\lambda = 1$. It correspond to the case where

$$E = \frac{4\pi}{g^2} C(0) \tag{27}$$

We note that the energy dependance is clearly linear in the vacuum expectation. The typical size of the monopole vary greatly with v. It would be interesting to plot the dependance of the size with respect to v. (For example, the typical size is the radii r_f, r_h where $f(r_f) = 0.5$, $h(r_h) = 0.5$).

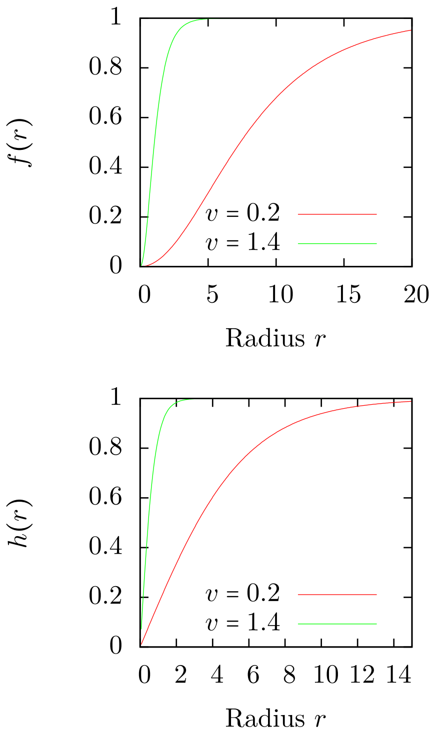

Figure 2: *Shape of the solution for different values of the Higgs field vacuum expectation value v, with $g = 1$, $\lambda = 1$*

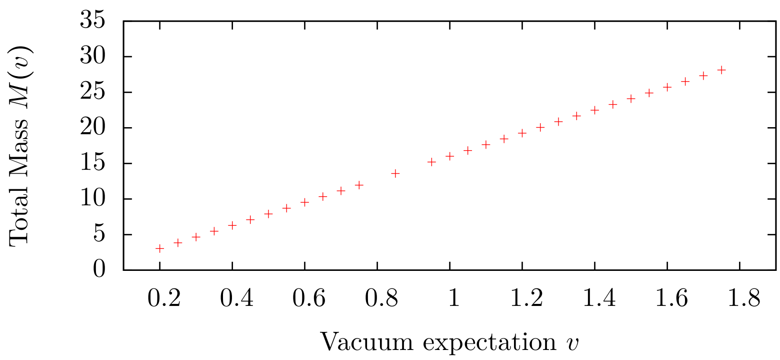

Figure 3: *Dependence of the energy with respect to the Higgs field vacuum expectation value v, with $g = 1$, $\lambda = 1$*

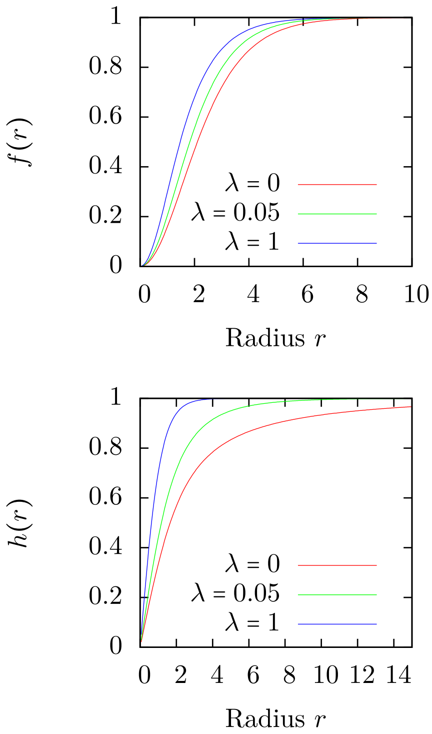

Figure 4: *Shape of the solution for different values of the Higgs field interaction constant* λ*, with* $g = 1$*,* $v = 1$

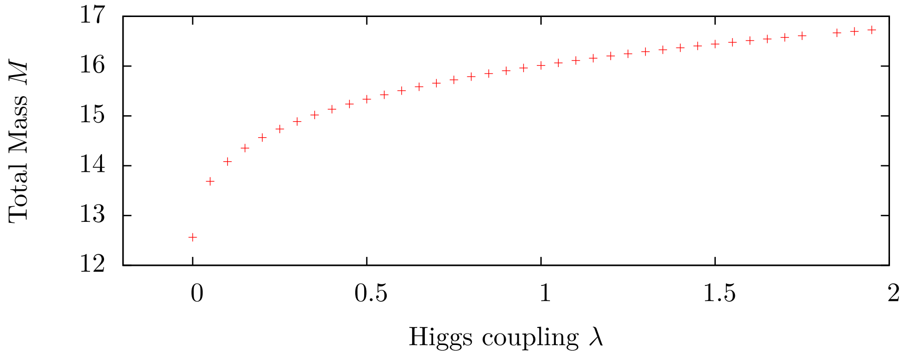

Figure 5: *Dependence of the energy with respect to the Higgs field interaction constant* λ*, with* $g = 1$*,* $v = 1$

2 The Vortex in $(2+1)$ dimensions with $U(1)$ gauge symmetry

The Lagrangian for the model is given by:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^* (D^\mu\phi) - \frac{\lambda}{2}\left(\phi^*\phi - v^2\right)^2 \tag{28}$$

where

$$F_{\mu\nu} = \partial_\mu\phi_\nu - \partial_\nu\phi_\mu \tag{29}$$
$$D_\mu\phi = \partial_\mu\phi - ieA_\mu\phi \tag{30}$$

The equations of motion obtained by varying the action with respect to ϕ, ϕ^* and A_μ are:

$$\partial^\nu F_{\nu\mu} = ie\left(\phi^* D_\mu - \phi\,(D_\mu\phi)^*\right) \tag{31}$$
$$(D^\mu D_\mu\phi)^* + \lambda\left(\phi^*\phi - v^2\right)\phi^* = 0 \tag{32}$$
$$D^\mu D_\mu\phi + \lambda\left(\phi^*\phi - v^2\right)\phi = 0 \tag{33}$$

We assume a static solution, with a polar symmetry. We choose the fields in the form:

$$A_i = -\frac{1}{er}\varepsilon_{ij}n_j A(r) \tag{34}$$
$$\phi = ve^{i\theta}F(r) \tag{35}$$

By substituing this ansatz in the equations of motion, we find two new equations for $A(r)$ and $F(r)$. For the left hand side of (31) we have

$$\partial_i A_j = \frac{1}{e}\left[\left(\frac{2A}{r^2} - \frac{A'}{r}\right)\varepsilon_{jk}n_i n_k + \frac{A}{r^2}\varepsilon_{ij}\right] \tag{36}$$
$$\Longrightarrow \quad \partial_i A_j - \partial_j A_i = \frac{1}{e}\left[\left(\frac{2A}{r^2} - \frac{A'}{r}\right)(\varepsilon_{jk}n_i - \varepsilon_{ik}n_j)\,n_k + \frac{2A}{r^2}\varepsilon_{ij}\right] \tag{37}$$
$$\Longrightarrow \quad \partial^i F_{ij} = -\frac{1}{e}\left[\frac{A'}{r}\right]'\varepsilon_{jk}n_k \tag{38}$$

and for the right hand side:

$$D_i\phi = v^2 F\left(i\partial_i\theta F + F'n_i + i\frac{AF}{r}\varepsilon_{ij}n_j\right) \tag{39}$$
$$\Longrightarrow \quad ie\left[\phi^* D_i\phi - \phi D_i\phi^*\right] = ie2i\Im(\phi^* D_i\phi) \tag{40}$$
$$= -2ev^2 F\left(\partial_i\theta F + \frac{AF}{r}\varepsilon_{ij}n_j\right) \tag{41}$$

Recalling that

$$\partial_i\theta = -\frac{1}{r}\varepsilon_{ij}n_j \tag{42}$$

we get:

$$ie\left[\phi^* D_j\phi - \phi D_j\phi^*\right] = 2ev^2\frac{F^2(1-A)}{r}\varepsilon_{jk}n_k \tag{43}$$

and the first equation of motion for A and F writes:

$$-\left[\frac{A'}{r}\right]' - 2e^2v^2\frac{F^2(1-A)}{r} = 0 \tag{44}$$

Let's now find the second equation:

$$D_i\phi = iv\frac{(A-1)F}{r}e^{i\theta}\varepsilon_{ij}n_j + vF'e^{i\theta}n_i \tag{45}$$

$$D_iD^i\phi = v\frac{(A-1)^2F}{r^2}e^{i\theta} - v\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left[r\frac{\mathrm{d}F}{\mathrm{d}r}\right]e^{i\theta} \tag{46}$$

The potential part of the equation gives:

$$\lambda\left(\phi^*\phi - v^2\right)\phi = \lambda v^3(F^2-1)Fe^{i\theta} \tag{47}$$

One finally obtain

$$-\frac{\mathrm{d}}{\mathrm{d}r}\left[r\frac{\mathrm{d}F}{\mathrm{d}r}\right] + \lambda v^2 r(F^2-1)F + \frac{F}{r}(A-1)^2 = 0 \tag{48}$$

2.1 Numerical integration of the equations of motion

In the last paragraph, we have obtained a differential system of two equations:

$$-\frac{\mathrm{d}}{\mathrm{d}r}\left[r\frac{\mathrm{d}F}{\mathrm{d}r}\right] + \lambda v^2 r(F^2-1)F + \frac{F}{r}(A-1)^2 = 0 \tag{49}$$

$$-\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{r}\frac{\mathrm{d}A}{\mathrm{d}r}\right] - 2e^2v^2\frac{F^2(1-A)}{r} = 0 \tag{50}$$

with the following asymptotic behaviour:

$$F(r) \to 1, \qquad A(r) \to 1, \qquad \text{as } r \to \infty \tag{51}$$

$$F(r) \to 0, \qquad A(r) \to 0, \qquad \text{as } r \to 0 \tag{52}$$

We are looking for a numerical scheme which converge to the analytic solution. Let's define for convenience the two auxiliary functions:

$$M(r,x,y) = \alpha r^2(y^2-1)y \tag{53}$$

$$N(r,x,y) = \beta r y^2(1-x) \tag{54}$$

such that the system to solve writes:

$$\begin{cases} r^2F'' + rF' - M(r,A,F) = 0 \\ rA'' - A' + N(r,A,F) = 0 \end{cases} \tag{55}$$

The boundary condition at infinity is hard to implement, so in a first approximation, we choose a maximal radius R where both function A and F eventually reach 1. We hope that the result of the simulation will be a good approximation of the solution if R is sufficiently large.

We discretize the intervalle $[0,R]$ into N points in the following way:

$$r_i = ih, \quad h = \frac{R}{N+1} \quad \Longrightarrow \quad r_0 = 0,\ r_{N+1} = R \tag{56}$$

and we write

$$a_i = A(r_i), \quad f_i = F(r_i) \tag{57}$$

The non-linear system (55) gives the following finite differences scheme:

$$\frac{r_i^2}{h^2}(f_{i+1} - 2f_i + f_{i-1}) + \frac{r_i}{2h}(f_{i+1} - f_{i-1}) - M(r_i, a_i, f_i) = 0 \tag{58}$$

$$\frac{r_i}{h^2}(a_{i+1} - 2a_i + a_{i-1}) - \frac{1}{2h}(a_{i+1} - a_{i-1}) + N(r_i, a_i, f_i) = 0 \tag{59}$$

$\forall\ i = 1, \ldots, N$ with the boundary conditions:

$$a_0 = f_0 = 0, \quad a_{N+1} = f_{N+1} = 1 \tag{60}$$

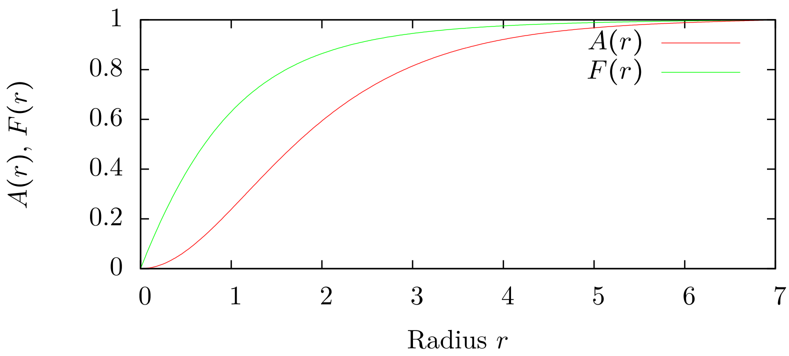

Figure 6: *Numerical integration of the non-linear system of equation for the vortex. We have chosen $R = 7$, $N = 400$, $\alpha = \beta = 1$. We note that obviously, for $r \to 0$, we have $f(r) \neq O(r^2)$.*

In order to solve for the unknown a_i, f_i, let's write

$$x_i = \begin{cases} a_i, & \forall i = 1, \ldots, N \\ f_i, & \forall i = N+1, \ldots, 2N \end{cases} \tag{61}$$

and the numerical scheme becomes

$$\vec{g}(x_1, \ldots, x_{2N}) = \begin{bmatrix} g_1(x_1, \ldots, x_{2N}) \\ \vdots \\ g_{2N}(x_1, \ldots, x_{2N}) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \tag{62}$$

where

$$g_1(x) = \frac{r_1}{h^2}(x_2 - 2x_1) - \frac{1}{2h}x_2 + N(r_1, x_1, x_{N+1}) \tag{63}$$

$$g_N(x) = \frac{r_N}{h^2}(1 - 2x_N + x_{N-1}) - \frac{1}{2h}(1 - x_{N-1}) + N(r_N, x_N, x_{2N}) \tag{64}$$

$$g_{N+1}(x) = \frac{r_1^2}{h^2}(x_{N+2} - 2x_{N+1}) + \frac{r_1}{2h}x_{N+2} - M(r_1, x_1, x_{N+1}) \tag{65}$$

$$g_{2N}(x) = \frac{r_N^2}{h^2}(1 - 2x_{2N} + x_{2N-1}) + \frac{r_N}{2h}(1 - x_{2N-1}) - M(r_N, x_N, x_{2N}) \tag{66}$$

$$g_i(x) = \frac{r_i}{h^2}\left(x_{i+1} - 2x_i + x_{i-1}\right) - \frac{1}{2h}\left(x_{i+1} - x_{i-1}\right) \\ + N(r_i, x_i, x_{N+i}), \quad \forall i = 2, \ldots, N-1 \tag{67}$$

$$g_i(x) = \frac{r_{i-N}^2}{h^2}\left(x_{i+1} - 2x_i + x_{i-1}\right) - \frac{r_{i-N}}{2h}\left(x_{i+1} - x_{i-1}\right) \\ + M(r_{i-N}, x_{i-N}, x_i), \quad \forall i = N+2, \ldots, 2N-1 \tag{68}$$

We then use Newton-Raphson method to find the solution for x_i's:

$$x^{i+1} = x^i - \frac{\vec{g}(x)}{J[\vec{g}]_x} \quad \Longleftrightarrow \quad J[\vec{g}]_x^{-1}\left(x^{i+1} - x^i\right) = \vec{g}(x) \tag{69}$$

where $J[\vec{g}]_x$ is the Jacobian matrix of $\vec{g}$ evaluated at x. The solution is given by $x = \lim_{i\to\infty} x^i$, and the convergence is faster if we choose cleverly the initial point x^0.

The results are shown on Fig. 6.

3 Analytical continuation of the String field configuration in $(1+1)$ dimensions

3.1 Introduction

3.2 Numerical Scheme

3.3 Integrating the Vortex and Monopole

3.4 Integrating the solution on the complex plane.

3.5 Validity tests

3.5.1 Riemann sheet: the square root

3.5.2 The logarithm

3.5.3 Sine-Gordon field equation

3.6 Application to the Vortex in $(2+1)$ dimensions

3.6.1 BPS limit

3.6.2 Non-BPS limit

3.7 Application to the Monopole in $(3+1)$ dimensions

3.7.1 BPS limit

3.7.2 Non-BPS limit

3.8 Conclusion

4 Monopoles in Gravity

The Lagrangian we are interested in for a first try is

$$\mathcal{L} = (\phi^* \phi + C)\, R + (D_\mu \phi)^* (D^\mu \phi) - \frac{\lambda}{2}(\phi^* \phi - v^2)^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{70}$$

where R is the Ricci scalar, $C,\ \lambda,\ v$ real constants, ϕ is a complex scalar field, and

$$D_\mu \phi = (\partial_\mu - ieA_\mu)\phi \tag{71}$$

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu \tag{72}$$

with $\nabla_\mu A_\nu = \partial_\mu A_\nu - \Gamma^\alpha_{\mu\nu} A_\alpha$ the usual covariant derivative under change of coordinate system. It is understood that the indices are moved up and down using the metric tensor $g^{\mu\nu}$ or its inverse: $A^\mu = g^{\mu\alpha} A_\alpha$.

The lagrangian (70) is invariant under $U(1)$ gauge transformation and under arbitrary change of coordinate.

4.1 Field equations

Let's split the Lagrangian into two parts, as usually done:

$$\mathcal{L} = \mathcal{L}_{\mathrm{G}} + \mathcal{L}_{\mathrm{M}} \tag{73}$$

where

$$\mathcal{L}_{\mathrm{G}} = (\phi^*\phi + C)\, R, \tag{74}$$

$$\mathcal{L}_{\mathrm{M}} = (D_\mu\phi)^*(D^\mu\phi) - \frac{\lambda}{2}(\phi^*\phi - v^2)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{75}$$

The variation of the action for the gravitational field with respect to the metric gives:

$$\delta S_{\mathrm{G}} = \int \mathrm{d}^4x\delta\left[\sqrt{-g}(\phi^*\phi + C)R\right] \tag{76}$$

$$= \int \mathrm{d}^4x\left[\delta\sqrt{-g}(\phi^*\phi + C)R + \sqrt{-g}(\phi^*\phi + C)\delta R\right] \tag{77}$$

By definition of the Ricci's scalar, we have:

$$\delta R = \delta\left(g^{\mu\nu}R_{\mu\nu}\right) = \delta g^{\mu\nu}R_{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu} \tag{78}$$

The first term lead to the usual Einstein tensor, and the second term can be written as a total covariant derivative:

$$\delta R_{\mu\nu} = \delta\left(\partial_\alpha\Gamma^\alpha_{\mu\nu} - \partial_\nu\Gamma^\alpha_{\alpha\mu} + \Gamma^\alpha_{\alpha\rho}\Gamma^\rho_{\mu\nu} - \Gamma^\alpha_{\nu\rho}\Gamma^\rho_{\alpha\mu}\right) \tag{79}$$

$$= \partial_\alpha\delta\Gamma^\alpha_{\mu\nu} - \Gamma^\rho_{\alpha\mu}\delta\Gamma^\alpha_{\nu\rho} + \Gamma^\alpha_{\alpha\rho}\delta\Gamma^\rho_{\mu\nu} - \left(\partial_\nu\delta\Gamma^\alpha_{\alpha\mu} - \Gamma^\rho_{\mu\nu}\delta\Gamma^\alpha_{\alpha\rho} + \Gamma^\alpha_{\nu\rho}\delta\Gamma^\rho_{\alpha\mu}\right) \tag{80}$$

$$= \nabla_\alpha\delta\Gamma^\alpha_{\mu\nu} - \nabla_\nu\delta\Gamma^\alpha_{\alpha\mu} \tag{81}$$

In the last equality, we add two terms which cancel each other, in order to rebuild a valid covariant derivative of the variation of the Christoffel symbols.

By definition of the covariant derivative we have:

$$g^{\mu\nu}\left(\nabla_\alpha\delta\Gamma^\alpha_{\mu\nu} - \nabla_\nu\delta\Gamma^\alpha_{\mu\nu}\right) = \nabla_\alpha\left(g^{\mu\nu}\delta\Gamma^\alpha_{\mu\nu}\right) - \nabla_\nu\left(g^{\mu\nu}\delta\Gamma^\alpha_{\alpha\mu}\right) \tag{82}$$

$$= \nabla_\alpha\left(g^{\mu\nu}\delta\Gamma^\alpha_{\mu\nu} - g^{\mu\alpha}\delta\Gamma^\beta_{\beta\mu}\right) \tag{83}$$

$$= \nabla_\alpha W^\alpha \tag{84}$$

and using the fact that, for an arbitrary vector:

$$\partial_\alpha\left(\sqrt{-g}A^\alpha\right) = \sqrt{-g}\nabla_\alpha A^\alpha \tag{85}$$

the variation of the action gives:

$$\delta S = \int \mathrm{d}^4\Bigg[\sqrt{-g}\left(\phi^*\phi + C\right)\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\delta g^{\mu\nu} \tag{86}$$

$$+ (\phi^*\phi + C)\partial_\alpha\left(\sqrt{-g}W^\alpha\right)\Bigg] \tag{87}$$

$$\tag{88}$$

At this stage we may integrate by part the last term, drop the boundary terms,and express W^α in terms of the variation of the metric. We eventually get:

$$\delta S = \int \mathrm{d}^4x\sqrt{-g}\left((\phi^*\phi + C)G_{\mu\nu} - \partial_\alpha\phi^2 W^\alpha\right) \tag{89}$$

$$\begin{aligned}
W^\alpha =& g^{\mu\nu}\Gamma_{\alpha\mu\nu}\delta g^{\sigma\alpha} - g^{\mu\sigma}\Gamma_{\alpha\mu\nu}\delta g^{\nu\alpha} && (90)\\
&+ \frac{1}{2}\left(g^{\mu\nu}g^{\sigma\rho} - g^{\mu\sigma}g^{\nu\rho}\right)\left(\delta g_{\rho\mu,\nu} + \delta g_{\rho\nu,\mu} - \delta g_{\mu\nu,\rho}\right) && (91)\\
=& g^{\mu\nu}\Gamma_{\alpha\mu\nu}\delta g^{\sigma\alpha} - g^{\mu\sigma}\Gamma_{\alpha\mu\nu}\delta g^{\nu\alpha} && (92)\\
&+ \left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma}\right)\delta g_{\rho\mu,\nu} && (93)
\end{aligned}$$

Substituting this result into the variation of the action and integrating by part the last term leads to:

$$\begin{aligned}
\delta S_{\mathrm{G}} =& \int \mathrm{d}^4x\sqrt{-g}G_{\mu\nu}\delta g^{\mu\nu} && (94)\\
&+ \int \mathrm{d}^4x\sqrt{-g}\partial_\sigma\phi^2\left(g^{\mu\sigma}\Gamma_{\alpha\mu\nu}\delta g^{\nu\alpha} - g^{\mu\nu}\Gamma_{\alpha\mu\nu}\delta g^{\sigma\alpha}\right) && (95)\\
&+ \int \mathrm{d}^4x\sqrt{-g}\partial_\nu\left[\sqrt{-g}\partial_\sigma\phi^2\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma}\right)\right]\delta g_{\rho\mu} && (96)
\end{aligned}$$

Using the fact that:

$$\begin{aligned}
\partial_\mu\left(\sqrt{-g}A_\nu\right) &= \sqrt{-g}\left(\nabla_\mu A_\nu + \Gamma^\lambda_{\mu\nu}A_\lambda + \Gamma^\lambda_{\mu\lambda}A_\nu\right) && (97)\\
\partial_\mu\left(\sqrt{-g}A^\nu\right) &= \sqrt{-g}\left(\nabla_\mu A^\nu - \Gamma^\nu_{\mu\lambda}A^\lambda + \Gamma^\lambda_{\mu\lambda}A_\nu\right) && (98)
\end{aligned}$$

the last term writes:

$$\begin{aligned}
&\sqrt{-g}\nabla_\nu\partial_\sigma\phi^2\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma}\right)\delta g_{\rho\mu} && (99)\\
&+\sqrt{-g}\left(\Gamma^\lambda_{\nu\sigma}\partial_\lambda\phi^2 + \Gamma^\lambda_{\nu\lambda}\partial_\sigma\phi^2\right)\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma}\right)\delta g_{\rho\mu} && (100)\\
&+\sqrt{-g}\partial_\sigma\phi^2\partial_\nu\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma}\right)\delta g_{\rho\mu} && (101)
\end{aligned}$$

Expressing these terms with the variation of the inverse metric $\delta g_{\rho\mu} = -g_{\rho\beta}g_{\mu\delta}\delta g^{\beta\delta}$ we get:

$$\begin{aligned}
&\sqrt{-g}\left(\nabla_\nu\partial^\nu\phi^2 g_{\beta\delta} - \nabla_\delta\partial_\beta\phi^2\right)\delta g^{\beta\delta} && (102)\\
&+ \sqrt{-g}\left(\partial_\lambda\phi^2 g^{\lambda\nu}\Gamma_{\delta\beta\nu} - \partial_\beta\phi^2 g^{\lambda\nu}\Gamma_{\delta\lambda\nu}\right)\delta g^{\beta\delta} && (103)
\end{aligned}$$

and the variation of the gravity part of the lagrangian reduce to:

$$\delta\mathcal{S}_{\mathrm{G}} = \int \mathrm{d}^4x\sqrt{-g}\left[G_{\mu\nu} + \nabla^2\phi^2 g_{\mu\nu} - \nabla_\mu\nabla_\nu\phi^2\right]\delta g^{\mu\nu} \tag{104}$$

Noting that, due to the symmetry of the lower indices of the Christoffel symbols, the energy-strength tensor do not depend on the metric, neither does the covariant derivative of the Higgs field ϕ, the variation of the matter's action with respect to the metric is easy to compute, and gives:

$$\delta\mathcal{S}_\phi = \int \mathrm{d}^4x\sqrt{-g}\left(\left(\mathrm{D}_\mu\phi\right)^*\mathrm{D}_\nu\phi + \frac{1}{2}F_{\mu\alpha}F^\alpha{}_\nu\right)\delta g^{\mu\nu} \tag{105}$$

The field equation for the gravity eventually writes:

$$(\phi^*\phi + C)G_{\mu\nu} + \nabla^2\phi^2 g_{\mu\nu} - \nabla_\mu\nabla_\nu\phi^2 + (\mathrm{D}_\mu\phi)^* \mathrm{D}_\nu\phi + \frac{1}{2}F_{\mu\alpha}F^\alpha{}_\nu = 0 \tag{106}$$

The equations for the Higgs field are:

$$-R\phi^* + \mathrm{D}^\mu\mathrm{D}_\mu\phi^* + \lambda\left(\phi^*\phi - v^2\right)\phi^* = 0 \tag{107}$$

$$\text{c. c.} = 0 \tag{108}$$

where

$$\mathrm{D}^\mu\mathrm{D}_\mu\phi^* = \nabla^\mu(\mathrm{D}_\mu\phi^*) + ieA^\mu(\mathrm{D}_\mu\phi^*) \tag{109}$$

Eventually, the field equation for the gauge field is:

$$\nabla_\nu F^{\mu\nu} = ie\left(\phi^*\mathrm{D}^\mu\phi - \phi\mathrm{D}^\mu\phi^*\right) \tag{110}$$

Hence, the system of field equation to solve is the following:

$$(\phi^*\phi + C)G_{\mu\nu} + \nabla^2\phi^2 g_{\mu\nu} - \nabla_\mu\nabla_\nu\phi^2 + (\mathrm{D}_\mu\phi)^* \mathrm{D}_\nu\phi + \frac{1}{2}F_{\mu\alpha}F^\alpha{}_\nu = 0 \tag{111}$$

$$\nabla_\nu F^{\mu\nu} + ie\left(\phi\mathrm{D}^\mu\phi^* - \phi^*\mathrm{D}^\mu\phi\right) = 0 \tag{112}$$

$$-R\phi^* + \mathrm{D}^\mu\mathrm{D}_\mu\phi^* + \lambda\left(\phi^*\phi - v^2\right)\phi^* = 0 \tag{113}$$

$$\text{c. c.} = 0 \tag{114}$$

where $G_{\mu\nu}$ is Einstein's tensor.

4.2 Spherically symmetric ansatz

The field equations are dimension-agnostic. We choose to work in a $(2+1)$ dimensions space, with a spherical symmetry about the coordinate origin. We assume that the most general form of a spherically symmetric metric tensor is the following:

$$\mathrm{d}s^2 = -\beta(r)\mathrm{d}t^2 + \alpha(r)\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 \tag{115}$$

The ansatz for the Higgs and gauge fields is:

$$\phi(r,\theta) = ve^{i\theta}F(r) \tag{116}$$

$$A_i(r,\theta) = -\frac{1}{er}\varepsilon_{ij}n_jK(r) \tag{117}$$

The only non zero componants of Einstein's tensor are:

$$G_{11} = -\frac{\alpha'\beta}{2r\alpha^2 r} \tag{118}$$

$$G_{22} = -1/2\frac{\beta'}{r\beta} \tag{119}$$

$$G_{33} = -\frac{r^2}{4}\frac{2\alpha\beta\beta'' - \alpha\beta'^2 - \alpha'\beta'\beta}{\alpha^2\beta^2} \tag{120}$$

$$\tag{121}$$

With this ansatz the field equation reduce to five non trivial relation. The diagonal componants of (111) give three equations:

$$\frac{2v^2\beta F'^2}{\alpha} - \frac{c\beta\alpha'}{2r\alpha^2} - \frac{v^2F^2\beta\alpha'}{2r\alpha^2} + v^2FF'\left(\frac{2\beta}{r\alpha} - \frac{\beta\alpha'}{\alpha^2}\right) + \frac{2v^2F\beta F''}{\alpha} = 0 \tag{122}$$

$$v^2F'^2 - \frac{K'^2}{2e^2r^2} - \frac{c\beta'}{2r\beta} - \frac{v^2F^2\beta'}{2r\beta} + v^2FF'\left(-\frac{2}{r} - \frac{\beta'}{\beta}\right) = 0 \tag{123}$$

$$\begin{aligned} &-\frac{2r^2v^2F'^2}{\alpha} - \frac{K'^2}{2e^2\alpha} + v^2FF'\left(\frac{r^2\alpha'}{\alpha^2} - \frac{r^2\beta'}{\alpha\beta}\right) - \frac{2r^2v^2FF''}{\alpha} \\ &\quad + (c + v^2F^2)\left(\frac{r^2\alpha'\beta'}{4\alpha^2\beta} + \frac{r^2\beta'^2}{4\alpha\beta^2} - \frac{r^2\beta''}{2\alpha\beta}\right) + v^2F^2(1-K)^2 = 0 \end{aligned} \tag{124}$$

Equation (112) reduce to the θ, θ componant:

$$-2ev^2F^2(K-1) + \frac{1}{e\alpha}\left(\frac{K'\alpha'}{2\alpha} + \frac{K'}{r} - \frac{K'\beta'}{2\beta} - K''\right) = 0 \tag{125}$$

And equation (113) gives:

$$\begin{aligned} &\frac{vF}{\alpha}\left(-\frac{\alpha'}{r\alpha} + \frac{\beta'}{r\beta} - \frac{\alpha'\beta'}{2\alpha\beta} - \frac{\beta'^2}{2\beta^2} + \frac{\beta''}{\beta}\right) - \frac{vF\,(K-1)^2}{r^2} \\ &\quad + vF'\left(\frac{1}{r\alpha} - \frac{\alpha'}{2\alpha^2} + \frac{\beta'}{2\alpha\beta}\right) + \frac{vF''}{\alpha} + v^3\lambda F\left(F^2 - 1\right) = 0 \end{aligned} \tag{126}$$

Since we have five equations and only four degrees of freedom, for the system to be consistant they must be dependant, and one equation should find how one equation can be expressed in terms of the others.

Appendices

A Sign conventions

The Riemann's tensor is defined up to a sign. We distinguish two conventions.

A.1 Wald conventions

The action is defined by:

$$S = \int \mathrm{d}^4 x \sqrt{-g} \left(\frac{1}{16\pi G} R + \mathcal{L}_{\mathrm{M}} \right) \tag{127}$$

where the Riemann tensor is defined by:

$$\begin{aligned} R_{abc}{}^{d} &= 2\left[-\partial_{[a} \Gamma^{d}{}_{b]c} + \Gamma^{e}{}_{c[a} \Gamma^{d}{}_{b]e} \right] && (128) \\ &= \partial_b \Gamma^{d}{}_{ac} - \partial_a \Gamma^{d}{}_{bc} + \Gamma^{d}{}_{be} \Gamma^{e}{}_{ca} - \Gamma^{d}{}_{ae} \Gamma^{e}{}_{cb} && (129) \end{aligned}$$

The Einstein field equation hence writes:

$$R_{ab} - \frac{1}{2} g_{ab} R - g_{ab} \lambda = 8\pi G T_{ab} \tag{130}$$

where

$$T_{ab} = -2 \frac{\delta \mathcal{L}_{\mathrm{M}}}{\delta g^{ab}} - g_{ab} \mathcal{L}_{\mathrm{M}} \tag{131}$$

A.2 Weinberg conventions

In Weinberg convention, the sign of the Riemann tensor is the opposite.

$$R^{\lambda}{}_{\mu\nu\kappa} = \partial_\kappa \Gamma^{\lambda}{}_{\mu\nu} - \partial_\nu \Gamma^{\lambda}{}_{\mu\kappa} + \Gamma^{\lambda}{}_{\kappa\eta} \Gamma^{\eta}{}_{\mu\nu} - \Gamma^{\lambda}{}_{\nu\eta} \Gamma^{\eta}{}_{\mu\kappa} \tag{132}$$

The Einstein field equation hence writes:

$$R_{ab} - \frac{1}{2} g_{ab} R - g_{ab} \lambda = -8\pi G T_{ab} \tag{133}$$

B Substitution of the 't Hooft-Polyakov ansatz

We only present the intermediate steps without details which are necessary to get to eq. (12)-(13).

$$\begin{aligned} \mathrm{D}_i \phi^a &= v h' n_i n_a + v \frac{h(1-f)}{r} (\delta_{ia} - n_i n_a) && (134) \\ \partial_i \left(\mathrm{D}_i \phi^a \right) &= v \frac{r^2 h'' + 2 r f' + 2 h (f-1)}{r^2} n_a && (135) \\ g \varepsilon^{abc} A_i^b \mathrm{D}_i \phi^c &= 2 v \frac{f h (1-f)}{r^2} n_a && (136) \\ \Rightarrow \mathrm{D}_i \mathrm{D}_i \phi^a &= v \frac{r^2 h'' + 2 r f' + 2 h (f-1)}{r^2} n_a + 2 v \frac{f h (1-f)}{r^2} n_a && (137) \\ \lambda \phi^a (\phi^b \phi^b - v^2) &= \lambda v^3 h (h^2 - 1) n_a && (138) \end{aligned}$$

$$\partial_i A^a_j = \frac{rf' - 2f}{gr^2}\varepsilon^{ajk}n_i n_k + \frac{f}{gr^2}\varepsilon^{aji} \tag{139}$$

$$\partial_i A^a_j - \partial_j A^a_i = \frac{rf' - 2f}{gr^2}(\varepsilon^{ajk}n_i n_k - \varepsilon^{aik}n_j n_k) + \frac{2f}{gr^2}\varepsilon^{aji} \tag{140}$$

$$g\varepsilon^{abc}A^b_i A^c_j = \frac{f^2}{gr^2}\varepsilon^{ijp}n_a n_p \tag{141}$$

$$\Rightarrow F^a_{ij} = \frac{rf' - 2f}{gr^2}(\varepsilon^{ajk}n_i n_k - \varepsilon^{aik}n_j n_k) + \frac{2f}{gr^2}\varepsilon^{aji} + \frac{f^2}{gr^2}\varepsilon^{ijp}n_a n_p \tag{142}$$

$$\partial_i F^a_{ij} = \frac{r^2 f'' - 2f + f^2}{gr^3}\varepsilon^{ajk}n_k \tag{143}$$

$$g\varepsilon^{abc}A^b_i F^c_{ij} = \frac{2f^2 - f^3}{gr^3}\varepsilon ajcn_c \tag{144}$$

$$\Rightarrow \mathrm{D}^i F^a_{ij} = -\frac{r^2 f'' - 2f + 3f^2 - f^3}{gr^3}\varepsilon^{ajk}n_k \tag{145}$$

$$gj^a_j = -gv^2\frac{h^2(1-f)}{r}\varepsilon^{ajc}n_c \tag{146}$$

C Energy functional for the monopole

The three terms in the energy functional (eq. (22)), in terms of f and h, writes:

$$F^a_{ij}F^a_{ij} = \frac{1}{g^2 r^4}\left[4(rf' - 2f)(rf' + 2f) + 2f^2(f^2 - 4f + 12)\right] \tag{147}$$

$$D_i\phi^a D_i\phi^a = v^2 h'^2 + 2v^2\frac{h^2(1-f^2)}{r^2} \tag{148}$$

$$\frac{\lambda}{4}(\phi^a\phi^a - v^2)^2 = \frac{\lambda v^4}{4}(h^2 - 1)^2 \tag{149}$$

D LU decomposition algorithm

Let A be a $N \times N$ regular matrix. One can show that there exist two $N \times N$ matrices L and U such that

$$A = LU \tag{150}$$

where L is lower triangular with unit diagonal, and U is upper triangular. We are looking for an algorithm to compute the components of L and U, given A.

We note:

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,N} \\ a_{2,1} & & & \\ \vdots & & & \\ a_{N,1} & \cdots & & a_{N,N} \end{bmatrix} \tag{151}$$

$$L = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ l_{2,1} & 1 & & \\ \vdots & & & \\ l_{N,1} & \cdots & l_{N,N-1} & 1 \end{bmatrix}, \; U = \begin{bmatrix} u_{1,1} & u_{1,2} & \cdots & u_{1,N-1} & u_{1,N} \\ 0 & u_{2,2} & & & \vdots \\ \vdots & & & & \\ 0 & 0 & \cdots & 0 & u_{N,N} \end{bmatrix} \tag{152}$$

Choosing the diagonal of L to be ones allow us to deduce the first row of U:

$$a_{1,i} = \sum_{k=1}^{N} l_{1,k} u_{k,i} = u_{1,i} \tag{153}$$

We can then deduce the first row of L:

$$a_{i,1} = \sum_{k=1}^{N} L_{i,k} U_{k,1} = l_{i,1} u_{1,1} \;\Rightarrow\; l_{i,1} = \frac{a_{i,1}}{u_{1,1}} \tag{154}$$

Following the similar procedure allow us to build U rows after rows, and L columns after columns. The induction step is the following:

Given $u_{k,i}\ \forall i, \forall k < K$, and $l_{i,k}\ \forall i,\ \forall k < K$ we can first compute $u_{K,i}$, and then $l_{i,K}$ as follow:

$$a_{K,i} = \sum_{m=1}^{N} l_{K,m} u_{m,i} = \sum_{m=1}^{K-1} l_{K,m} u_{m,i} + u_{K,i} \tag{155}$$

$$\Longrightarrow\; u_{K,i} = a_{K,i} - \sum_{m=1}^{K-1} l_{K,m} u_{m,i},\ \ \forall i = K, \dots, N \tag{156}$$

and similarly,

$$a_{i,K} = \sum_{m=1}^{N} l_{i,m} u_{m,K} = \sum_{m=1}^{K-1} l_{i,m} u_{m,K} + l_{j,K} u_{K,K} \tag{157}$$

$$\Longrightarrow l_{i,K} = \frac{1}{u_{K,K}} \left(a_{i,K} - \sum_{m=1}^{K-1} l_{i,m} u_{m,K} \right),\ \ \forall i = K+1, \dots, N \tag{158}$$

This induction step goes on until $K = N - 1$, and the last step of the algorithm serves to compute $u_{N,N}$ only, since $l_{N,N} = 1$:

$$a_{N,N} = \sum_{m=1}^{N-1} l_{N,m} u_{m,N} + u_{N,N} \tag{159}$$

$$\Longrightarrow u_{N,N} = a_{N,N} - \sum_{m=1}^{N-1} l_{N,m} u_{m,N} \tag{160}$$

We note that each component of the matrix A is used only once. As a consequence, generalizing the algorithm to make it *in-place* is straightforward.

E Noether and Symmetric Energy-Momentum tensor for the free Electrodynamics

The Lagrangian of free electrodynamics is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{161}$$

We easily find the field equations A^μ:

$$\partial_\mu F^{\mu\nu} = 0, \;\; \forall \nu = 0, 1, 2, 3 \tag{162}$$

We note that the Lagrangian and hence the field equations are gauge invariant under

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu \alpha(x^\sigma) \tag{163}$$

with α an arbitrary function of space-time.

Using Noether's theorem for translation invariance, we can compute the Energy-momentum tensor T^μ_ν:

$$^{\text{Noe}}T^\mu_\nu = \frac{\partial \mathcal{L}}{\partial A^\alpha_{,\mu}} \partial_\nu A^\alpha - \delta^\mu_\nu \mathcal{L} \tag{164}$$

$$= -F^{\mu\alpha} \partial_\nu A_\alpha + \frac{1}{4} \delta^\mu_\nu F_{\alpha\beta} F^{\alpha\beta} \tag{165}$$

We note that this tensor is neither symmetric, nor gauge invariant, but it is conserved:

$$\partial_\mu^{\text{Noe}} T^\mu_\nu = 0 \tag{166}$$

We shall now derive the symmetric Energy-momentum tensor defined by:

$$^{\text{Symm}}T_{\nu\mu} = \frac{1}{2} \frac{\delta\left(\sqrt{-g}\mathcal{L}\right)}{\delta g^{\mu\nu}} \tag{167}$$

$$= -F^\alpha_\mu F_{\nu\alpha} + \frac{1}{4} \eta_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \tag{168}$$

We note that this time the tensor is, as expected, symmetric, but it is also gauge invariant. The difference between the two is the single term

$$F^{\mu\alpha} \partial_\alpha A_\nu \tag{169}$$

F Some useful relation in Gravity and Gauge theories

The Christoffel symbols of the first kind write:

$$\Gamma_{\mu\alpha\beta} = \frac{1}{2} \left(\partial_\alpha g_{\mu\beta} + \partial_\beta g_{\mu\alpha} - \partial_\mu g_{\alpha\beta}\right) = \frac{1}{2} \left(g_{\mu\beta,\alpha} + g_{\mu\alpha,\beta} - g_{\alpha\beta,\mu}\right) \tag{170}$$

and the Christoffel symbols of the second kind:

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} \left(\partial_\alpha g_{\nu\beta} + \partial_\beta g_{\nu\alpha} - \partial_\nu g_{\alpha\beta}\right) = \frac{1}{2} g^{\mu\nu} \left(g_{\nu\beta,\alpha} + g_{\nu\alpha,\beta} - g_{\alpha\beta,\nu}\right) \tag{171}$$

The covariant derivative of a scalar reduce to the usual derivative:

$$\nabla_\mu \Phi = \partial_\mu \phi \tag{172}$$

while the covariant derivative of a covariant vector field is defined by:

$$\nabla_\mu A_\nu = \partial_\mu A_\nu \tag{173}$$