Dissolution d'une particule d'alumine dans un bain électrolytique

On considère une particule sphérique de rayon r_0 dans un bain électrolytique. On suppose que son rayon r va varier au cours du temps selon l'éq^t

$$\begin{cases} \dfrac{dr}{dt} = -\dfrac{k}{r} & \text{si } t>0 \\ r(0) = r_0 \end{cases}$$

Ainsi, en notant $\mathring{r} = \dfrac{dr}{dt}$ on obtient successivement

$$r\mathring{r} = -k,$$

$$\frac{1}{2}\frac{d}{dt}r^2 = -k,$$

$$r^2 = -2kt + r_0^2,$$

$$r(t) = \sqrt{r_0^2 - 2kt}.$$

Si $T = \dfrac{r_0^2}{2k}$ on obtient $r(T) = 0$.

La particule de rayon r_0 sera dissoute après un temps $T = \dfrac{r_0^2}{2k}$.

Dissolution d'une population de particule $n_p(r)$.

On suppose que cette population se dissout selon l'eqt

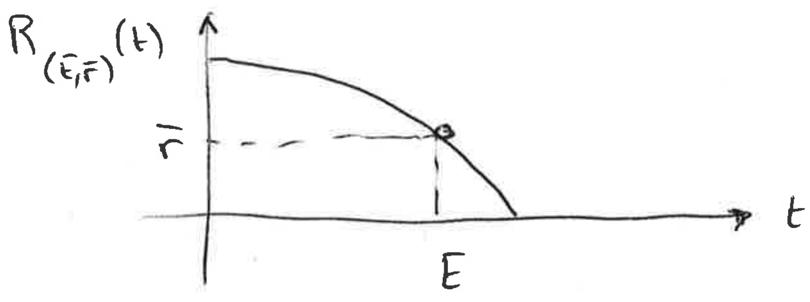
$$\begin{cases} \dfrac{\partial n_p}{\partial t} - k \dfrac{\partial}{\partial r}\left(\dfrac{1}{r} n_p\right) = 0 \\ n_p(0,r) = n_{p0}(r) \end{cases}$$

Ainsi on a $\dfrac{\partial n_p}{\partial t} - \dfrac{k}{r}\dfrac{\partial n_p}{\partial r} + \dfrac{k}{r^2} n_p = 0$. (1)

Si, dans le plan (t,r) on fixe $t=\bar{t}>0$, $r=\bar{r}>0$, l'équation de la courbe caractéristique qui passe par $(\bar{t},\bar{r})$ est donnée par

$$\frac{d}{dt} R_{(\bar{t},\bar{r})}(t) = -\frac{k}{R_{(\bar{t},\bar{r})}(t)}.$$

Ainsi $R_{(\bar{t},\bar{r})}(t) = \sqrt{\bar{r}^2 - 2k(t-\bar{t})}$



Notons $N_p(t) = n_p(t, R_{(\bar{t},\bar{r})}(t))$ la valeur de n_p sur cette ligne caractéristique

On a $\frac{d}{dt} N_p(t) = \frac{\partial n_p}{\partial t}(t, R_{(\bar{t},\bar{r})}(t)) + \frac{\partial}{\partial r} n_p (t, R_{(\bar{t},\bar{r})}(t)) \cdot \frac{d}{dt} R(t)$

$$= \frac{\partial n_p}{\partial t}(t, R_{(\bar{t},\bar{r})}(t)) - \frac{k}{R_{(\bar{t},\bar{r})}(t)} \frac{\partial}{\partial r} n_p (t, R(\bar{t},\bar{r})(t))$$

$$= - \frac{k}{R^2_{(\bar{t},\bar{r})}(t)} N_p(t).$$

Ainsi $\frac{1}{N_p(t)} \frac{d}{dt} N_p(t) = - \frac{k}{R^2_{(\bar{t},\bar{r})}(t)}$

qui implique

$$\frac{d}{dt} \ln N_p(t) = - \frac{k}{(\bar{r}^2 - 2k(t-\bar{t}))}.$$

En intégrant on aura

$$\ln N_p(t) = \frac{1}{2} \ln (\bar{r}^2 - 2k(t-\bar{t})) + D$$

et donc $N_p(t) = (\bar{r}^2 - 2k(t-\bar{t}))^{1/2} \cdot e^D$

Si $t=0$, on a $N_p(0) = e^D \sqrt{\bar{r}^2 + 2k\bar{t}} = n_{p_0}(R(0))$
$= n_{p_0}(\sqrt{\bar{r}^2+2k\bar{t}})$

Ainsi $e^D = \frac{n_{p_0}(\sqrt{\bar{r}^2+2k\bar{t}})}{\sqrt{\bar{r}^2+2k\bar{t}}}$ et

$$\boxed{N_p(t) = \frac{n_{p_0}(\sqrt{\bar{r}^2+2k\bar{t}})}{\sqrt{\bar{r}^2+2k\bar{t}}} \cdot (\bar{r}^2 - 2k(t-\bar{t}))^{1/2}}$$

Clairement $N_p(\bar{t}) = n_p(\bar{t},\bar{r})$ et donc

$$\boxed{n_p(\bar{t},\bar{r}) = \bar{r}\,\frac{n_{p0}\left(\sqrt{\bar{r}^2+2k\bar{t}}\right)}{\sqrt{\bar{r}^2+2k\bar{t}}}}$$ Solution exacte

Remarque

Supposons connu $n_p(\bar{t}-\Delta t, r)$. Nous avons ainsi

$$R_{(\bar{t},\bar{r})}(\bar{t}-\Delta t) = \sqrt{\bar{r}^2+2k\Delta t}$$

Puisque $N(t) = C_1\left(\bar{r}^2-2k(t-\bar{t})\right)^{1/2}$, alors

$$N(\bar{t}-\Delta t) = C_1\left(\bar{r}^2+2k\Delta t\right) = n_p\left(t-\Delta t, \sqrt{\bar{r}^2+2k\Delta t}\right)$$

Ainsi $C_1 = \dfrac{n_p\left(t-\Delta t, \sqrt{\bar{r}^2+2k\Delta t}\right)}{\bar{r}^2+2k\Delta t}$ et

donc ~~$N(\bar{t}-\Delta t) = \ldots$~~

$$N(t) = \frac{n_p\left(t-\Delta t, \sqrt{\bar{r}^2+2k\Delta t}\right)}{\sqrt{\bar{r}^2+2k\Delta t}}\left(\bar{r}^2-2k(t-\bar{t})\right)^{1/2}.$$

Puisque $N(\bar{t}) = n_p(\bar{t},\bar{r})$, on obtient

$$\boxed{n_p(\bar{t},\bar{r}) = \bar{r}^2\,\frac{n_p\left(t-\Delta t, \sqrt{\bar{r}^2+2k\Delta t}\right)}{\sqrt{\bar{r}^2+2k\Delta t}}}$$

Approximation numérique

~~Pas en temps, Δr~~ T_{max}, r_{max}

$\tau = \dfrac{T_{max}}{N}$, $\Delta r = \dfrac{r_{max}}{M}$, $t_i = i\tau$, $r_j = j\Delta r$

---- etc.