Lecture 8: Trees and Heap Sort

COMS10007 - Algorithms

Dr. Christian Konrad

19.02.2019

Sorting Algorithms seen so far

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Data Structures

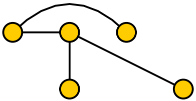
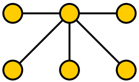
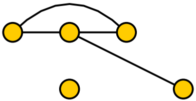
- *Data storage format that allows for efficient access and modification*
- Building block of many efficient algorithms
- For example, an array is a data structure

Trees

Definition: A *tree* $T = (V, E)$ of size n is a tuple consisting of

$$V = \{v_1, v_2, \dots, v_n\} \text{ and } E = \{e_1, e_2, \dots, e_{n-1}\}$$

with $|V| = n$ and $|E| = n - 1$ with $e_i = \{v_j, v_k\}$ for some $j \neq k$ such that for every node v_i there is at least one edge e_j such that $v_i \in e_j$. V are the nodes/vertices and E are the edges of T.

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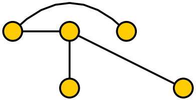

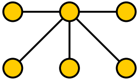

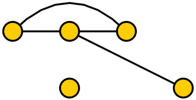

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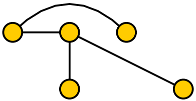

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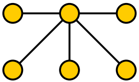

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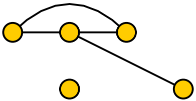

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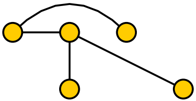

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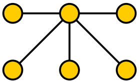

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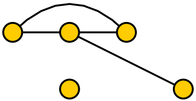

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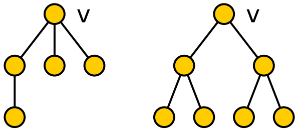
Rooted Trees

Definition: (rooted tree) A *rooted* tree is a triple $T = (v, V, E)$ such that $T = (V, E)$ is a tree and $v \in V$ is a designed node that we call the *root* of T.



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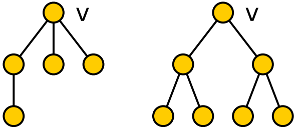

Definition: (leaf, internal node) A *leaf* in a tree is a node with exactly one incident edge. A node that is not a leaf is called an *internal node*.

Children, Parent, and Degree

Further Definitions:

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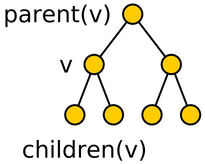

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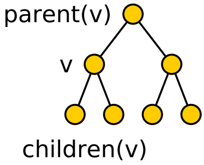

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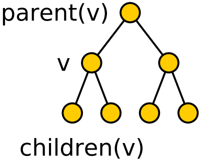

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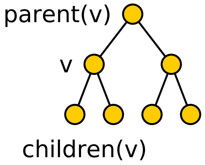

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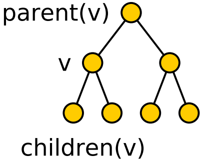

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- The *level* of a vertex v is the length of the unique path from the root to v plus 1.

Properties of Trees

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a contradiction to the fact that $\sum_{v \in V} \deg(v) = 2(n-1)$ in every tree. $\square$

Binary Trees

Definition: (k-ary tree) A (rooted) tree is *k-ary* if every node has at most k children. If $k = 2$ then the tree is called binary. A k ary tree is

- *full* if every internal node has exactly k children,
- *complete* if all levels except possibily the last is entirely filled (and last level is filled from left to right),
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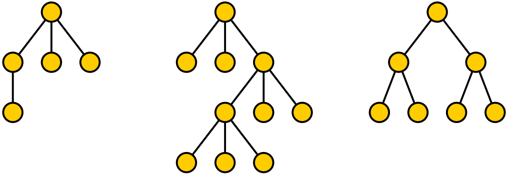

complete 3-ary tree   full 3-ary tree   perfect binary tree

Height of Perfect and Complete k-ary Trees

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Remark: The runtime of many algorithms that use tree data structures depends on the height of these trees. We are therefore interested in using complete/perfect trees.

Priority Queues

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Data structure that allows the following operations:

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- Extract-Max(.): Remove the maximum element from the data structure
- *others...*

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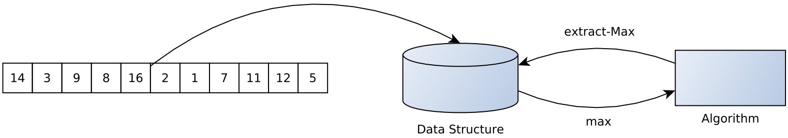
Sorting using a Priority Queue



From Array to Tree

Interpretation of an Array as a Complete Binary Tree

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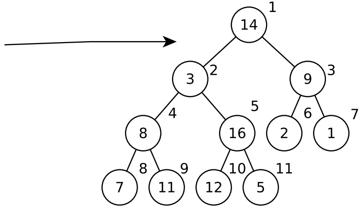

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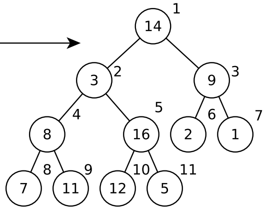

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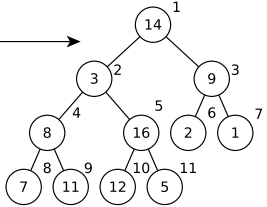

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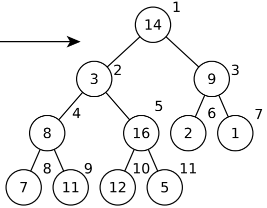

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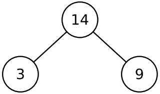
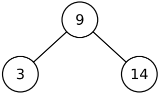
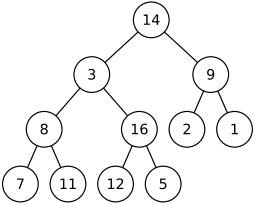
- Left/Right Child of i: $2i$ and $2i + 1$

The Heap Property

Key of nodes larger than keys of their children







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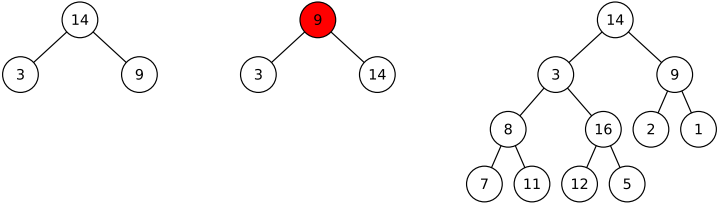

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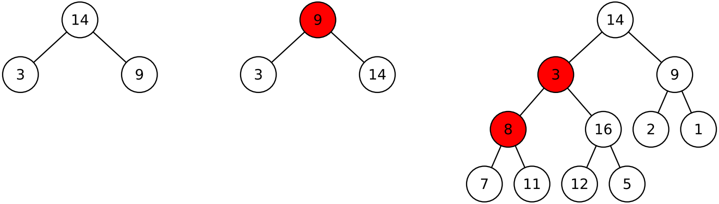

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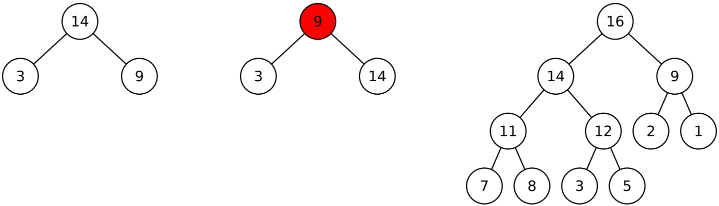

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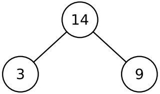

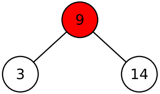

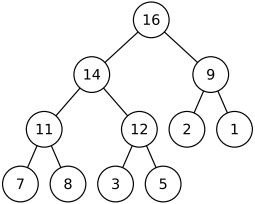

Heap Property → Maximum at root
Important for Extract-Max(.)

The Heapify Operation

Constructing a Heap

Given a binary tree, transform it into one that fulfills the Heap Property

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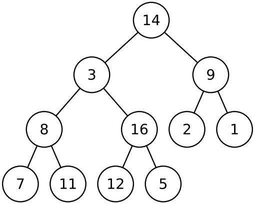

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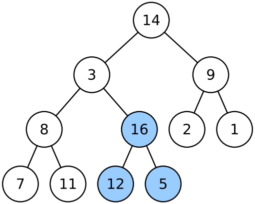

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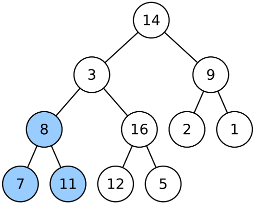

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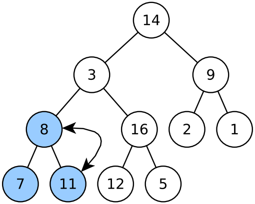

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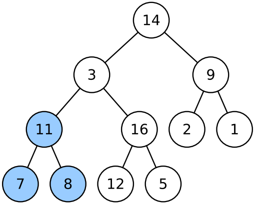

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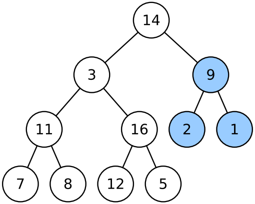

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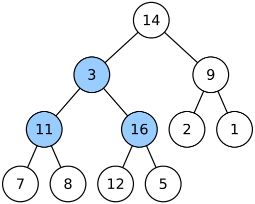

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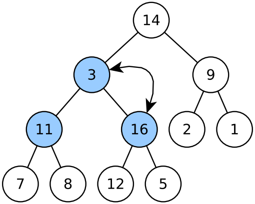

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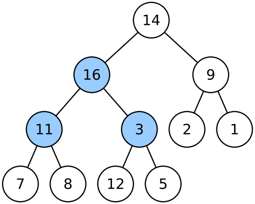

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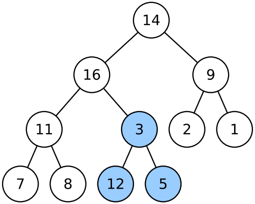

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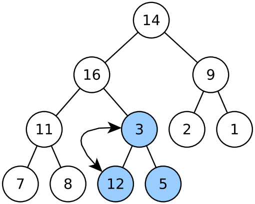

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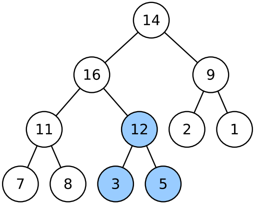

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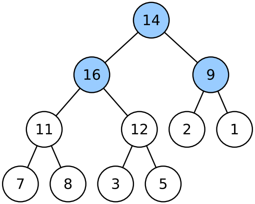

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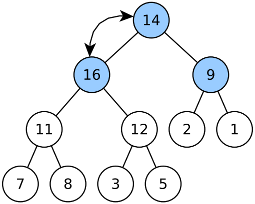

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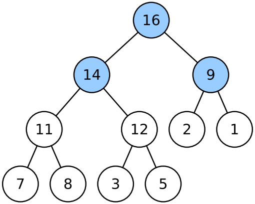

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Heapify()

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Constructing a Heap: Time $O(n \log n)$

Improved Analysis of Heap Construction

More Precise Analysis of the Heap Construction Step

- Heapify(x): $O(\text{depth of subtree rooted at } x) = O(\log n)$
- **Observe:** Most nodes close to the "bottom"

Analysis:

- Let i be the largest integer such that $n' := 2^i - 1$ and $n' < n$
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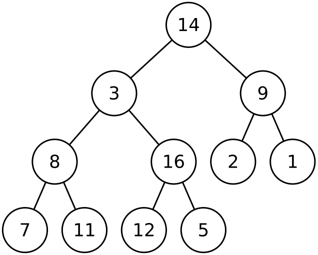
- These nodes are contained in a perfect binary tree
- This tree has height $i - 1$



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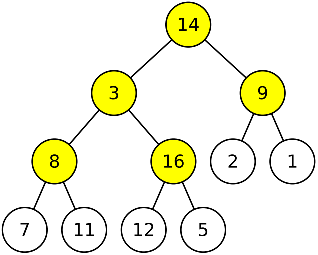

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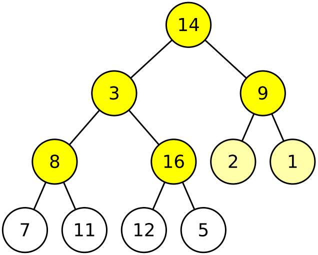


Analysis

We sum over all levels, count the number of nodes per level, and multiply with the depth of their subtrees:

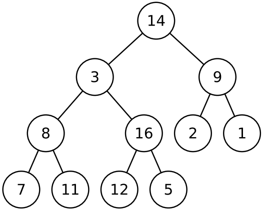
$$\sum_{j=1}^{i} \underbrace{2^{i-j}}_{\text{nodes in level } i-j} \cdot \underbrace{j}_{\text{depth of subtree}}$$

$$\sum_{j=1}^{i} 2^{i-j} \cdot j \;=\; 2^i \cdot \sum_{j=1}^{i} \frac{j}{2^j} = O(2^i) = O(n') = O(n) \; .$$

$\boxed{\text{We'll prove } \sum_{j=1}^{i} \frac{j}{2^j} = O(1) \text{ very soon...!}}$

Putting Everything Together

14	3	9	8	16	2	1	7	11	12	5



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The Complete Algorithm

Putting Everything Together

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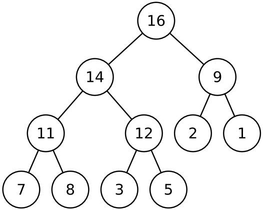
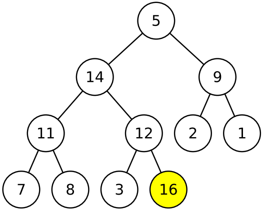


Putting Everything Together

5	14	9	11	12	2	1	7	8	3	16



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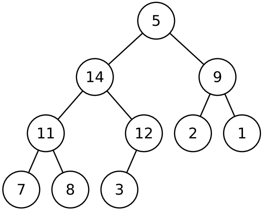

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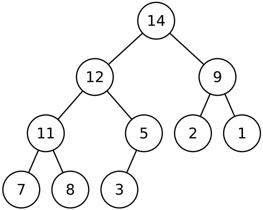

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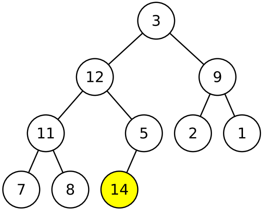

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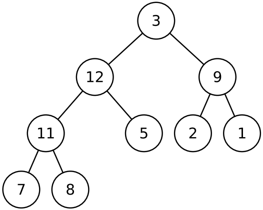

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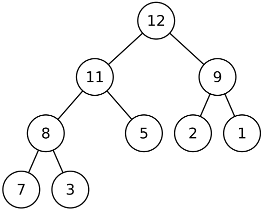

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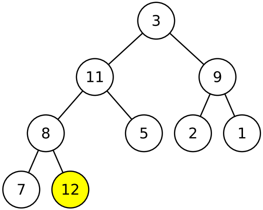

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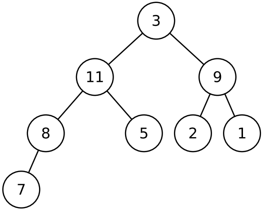

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The Complete Algorithm

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The Complete Algorithm

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1	2	3	5	7	8	9	11	12	14	16

1. Build-heap() $O(n)$
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Runtime: $O(n \log n)$

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How to bound $\sum_{i=0}^{n-1} \frac{i}{2^i}$:

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&= \sum_{i=0}^{n-1} \frac{1}{2^i} + \frac{n-1}{2^n} = \frac{\frac{1}{2^n} - \frac{1}{2}}{\frac{1}{2} - 1} + \frac{n-1}{2^n} = O(1) \ .
\end{aligned}$$