Unit 2

STATIONARY TIME SERIES MODELS

Wold representation of linear stationary processes, Study of linear time series models: Autoregressive, Moving Average and Autoregressive Moving average models and their statistical properties like ACF and PACF function.

Stationary Time series

The stationarity of a time series is related to its statistical properties in time. That is, in the more strict sense, a stationary time series exhibits similar "statistical behavior" in time and this is often characterized as a constant probability distribution in time. However, it is usually satisfactory to consider the first two moments of the time series and define stationarity (or weak stationarity) as follows:

(1) the expected value (mean) of the time series does not depend on time and

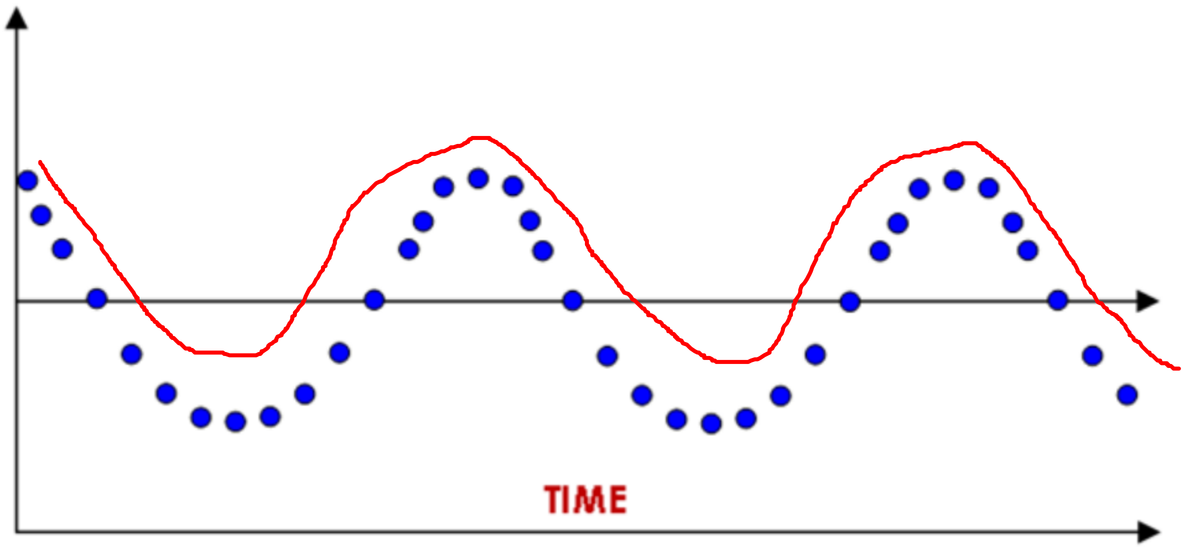
(2) the autocovariance function defined as Cov(y_t, y_{t+k}) for any lag k is only a function of k and not time; that is, $\gamma_y(k)$ = Cov(y_t, y_{t+k})

Stationary Time series

In a crude way, the stationarity of a time series can be determined by taking arbitrary “snapshots” of the process at different points in time and observing the general behavior of the time series. If it exhibits “similar” behavior, one can then proceed with the modeling efforts under the assumption of stationarity.

Further preliminary tests also involve observing the behavior of the autocorrelation function. A strong and slowly dying ACF will also suggest deviations from stationarity

Stationary Time series



A stochastic process is said to be stationary if its mean and variance are constant over time and the value of the covariance be-tween the two time periods depends only on the distance or gap or lag between the two time periods and not the actual time at which the covariance is computed. Such a time series will tend to return to its mean (called mean reversion) and fluctuations around this mean (measured by its variance) will have a broadly constant amplitude.

Stationary time series

A very important type of time series is a stationary time series. A time series is said to be strictly stationary if its properties are not affected by a change in the time origin. That is, if the joint probability distribution of the observations y_t, $y_{t+1} \dots . y_{t+n}$ is exactly the same as the joint probability distribution of the observations y_{t+k}, y_{t+k+1}…….. y_{t+k+n} then the time series is strictly stationary. When n = 0 the stationarity assumption means that the probability distribution of y_t is the same for all time periods and can be written as f(y).

Stationary Time series

Stationary implies a type of statistical equilibrium or stability in the data. Consequently, the time series has a constant mean defined in the usual way as

$E(x) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$

mean

$$\mu_y = E(y) = \int_{-\infty}^{\infty} yf(y)dy$$

$V(x) = \int_{-\infty}^{\infty} \left[x - E(x)\right]^2 f(y)\,dx$

and constant variance defined as

$$\sigma_y^2 = \mathrm{Var}(y) = \int_{-\infty}^{\infty} (y - \mu_y)^2 f(y)dy.$$

Stationary Time series

The sample mean and sample variance are used to estimate these parameters. If the observations in the time series are $y_1, \ldots, y_T$, then the sample mean is

$$\bar{y} = \hat{\mu}_y = \frac{1}{T} \sum_{t=1}^{T} y_t$$

and the sample variance is

$$s^2 = \hat{\sigma}_y^2 = \frac{1}{T} \sum_{t=1}^{T} (y_t - \bar{y})^2.$$

Autocovariance and Autocorrelation Functions

$\mathrm{cov}(X,Y)$ $\quad$ X $\quad$ Y $\quad$ kg $\quad$ gm

$\mathrm{cov}(x,y) = E[(x-\bar{x})(y-\bar{y})]$

$\mathrm{Var}(X) = E(x-\bar{x})^2 = E(X^2) - [E(x)]^2$

If a time series is stationary this means that the joint probability distribution of any two observations, say, y_t and y_{t+k}, is the same for any two time periods t and t + k that are separated by the same interval k. Useful information about this joint distribution, and hence about the nature of the time series, can be obtained by plotting a scatter diagram of all of the data pairs y_t , y_{t+k} that are separated by the same interval k. The interval k is called the lag.

Autocovariance and Autocorrelation Functions

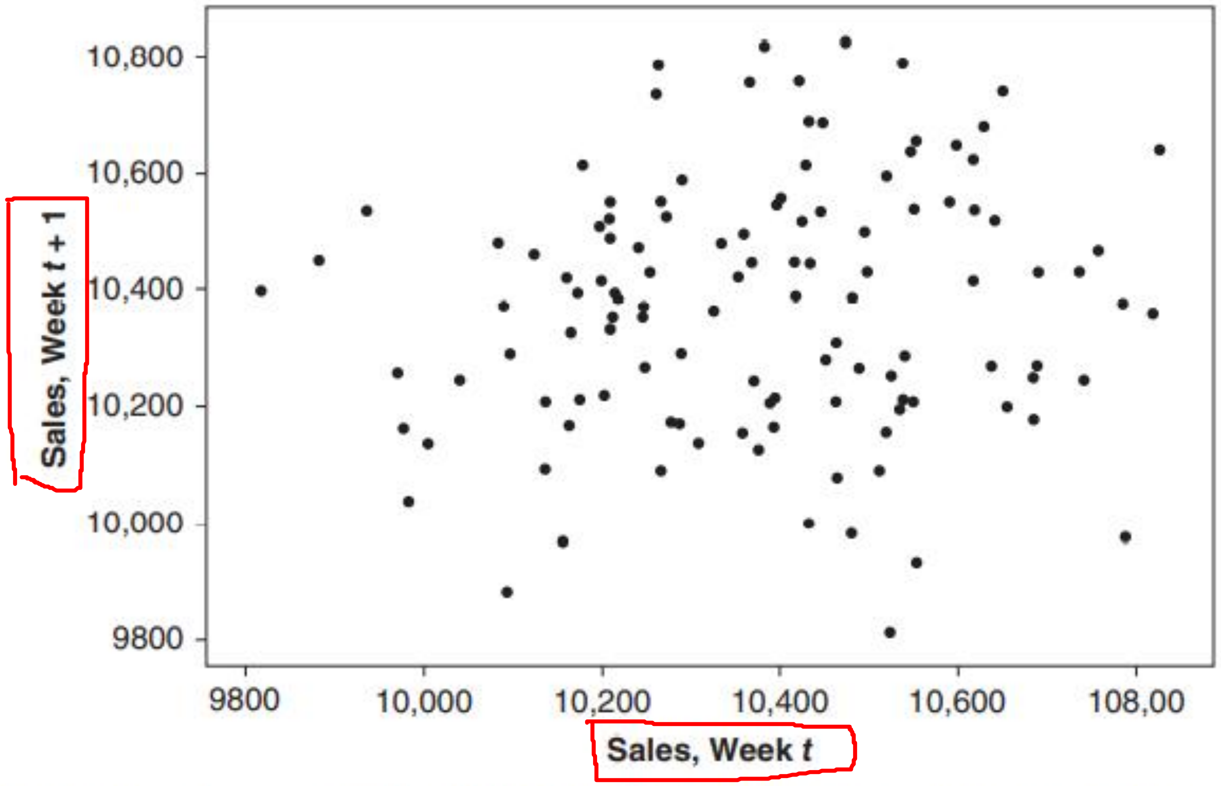
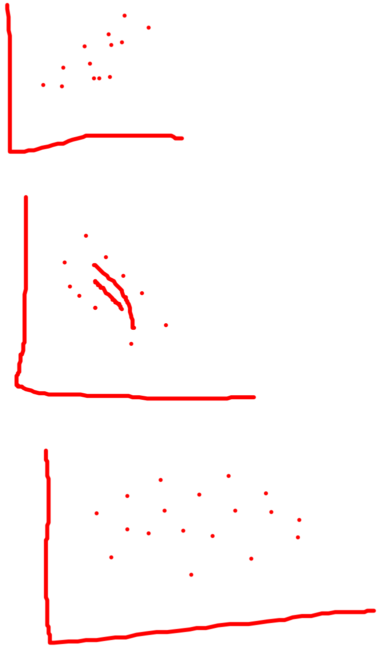


FIGURE 2.10 Scatter diagram of pharmaceutical product sales at lag $k = 1$.



Autocovariance and Autocorrelation Functions

Figure 2.10 is a scatter diagram for the pharmaceutical product sales for lag k = 1 and Figure 2.11 is a scatter diagram for the chemical viscosity readings for lag k = 1. Both scatter diagrams were constructed by plotting y_{t+1} *versus* y_t. Figure 2.10 exhibits little structure; the plotted pairs of adjacent observations y_t, y_{t+1} seem to be uncorrelated. That is, the value of y in the current period does not provide any useful information about the value of y that will be observed in the next period. A different story is revealed in Figure 2.11, where we observe that the

Autocovariance and Autocorrelation Functions

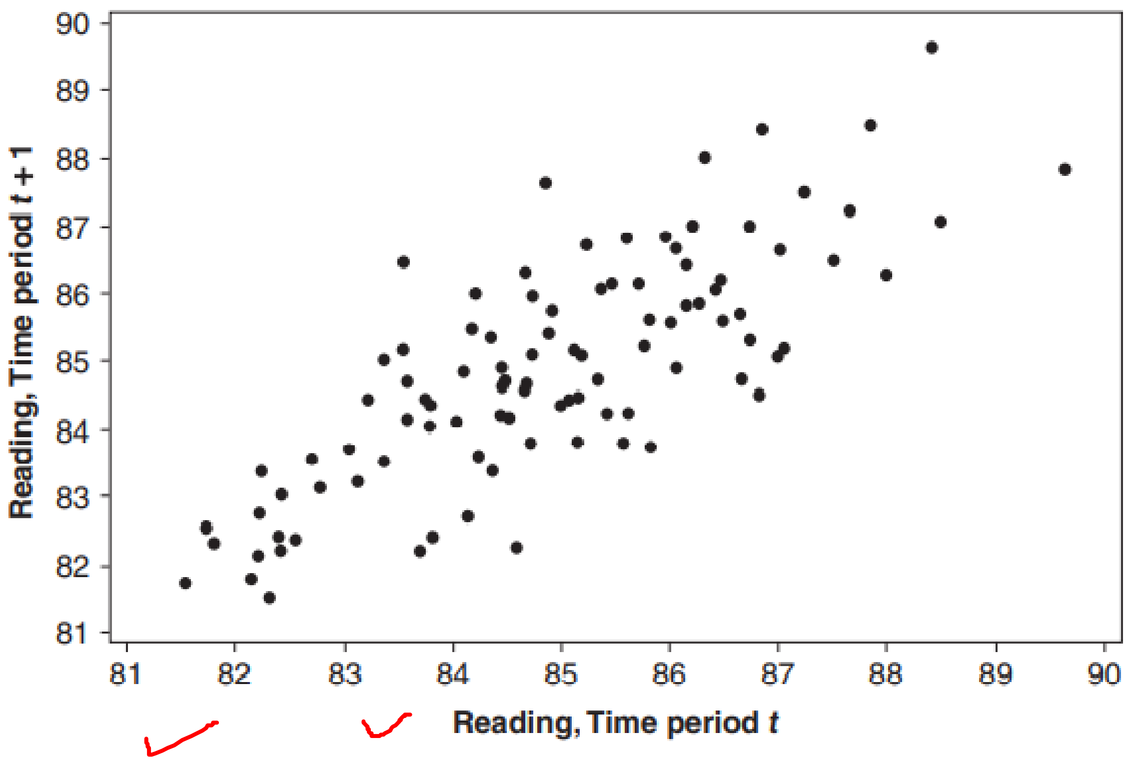


FIGURE 2.11 Scatter diagram of chemical viscosity readings at lag $k = 1$.

pairs of adjacent observations y_{t+1}, y_t are positively correlated. That is, a small value of y tends to be followed in the next time period by another small value of y, and a large value of y tends to be followed immediately by another large value of y. Note from inspection of Figures 2.10 and 2.11 that the behavior inferred from inspection of the scatter diagrams is reflected in the observed time series.

Autocovariance and Autocorrelation Functions

$k=0$

$V(y_t)=E(y_t-\mu)^2$

The covariance between y_t and its value at another time period, say, y_{t+k} is called the **autocovariance** at lag k, defined by

$$\gamma_k = \mathrm{Cov}(y_t, y_{t+k}) = E[(y_t - \mu)(y_{t+k} - \mu)]. \tag{2.10}$$

The collection of the values of γ_k, $k = 0, 1, 2, \ldots$ is called the **autocovariance function**. Note that the autocovariance at lag $k = 0$ is just the variance of the time series; that is, $\gamma_0 = \sigma_y^2$,which is constant for a stationary time series. The **autocorrelation coefficient** at lag k for a stationary time series is

Var is constant

r

$$\rho_k = \frac{E[(y_t - \mu)(y_{t+k} - \mu)]}{\sqrt{E[(y_t - \mu)^2]E[(y_{t+k} - \mu)^2]}} = \frac{\mathrm{Cov}(y_t, y_{t+k})}{\mathrm{Var}(y_t)} = \frac{\gamma_k}{\gamma_0}. \tag{2.11}$$

$$r = \frac{\mathrm{cov}(x,y)}{\sqrt{\mathrm{Va}(x)\,\mathrm{Va}(y)}}$$

$$\rho_k = \frac{\gamma_k}{\gamma_0}$$

Autocovariance and Autocorrelation Functions

$$\rho_k = \frac{E[(y_t - \mu)(y_{t+k} - \mu)]}{\sqrt{E[(y_t - \mu)^2]E[(y_{t+k} - \mu)^2]}} = \frac{\mathrm{Cov}(y_t, y_{t+k})}{\mathrm{Var}(y_t)} = \frac{\gamma_k}{\gamma_0}. \quad (2.11)$$

The collection of the values of ρ_k, $k = 0, 1, 2, \ldots$ is called the **autocorrelation function (ACF)**. Note that by definition $\boxed{\rho_0 = 1}$. Also, the ACF is independent of the scale of measurement of the time series, so it is a dimensionless quantity. Furthermore, $\boxed{\rho_k = \rho_{-k}}$; that is, the ACF is **symmetric** around zero, so it is only necessary to compute the positive (or negative) half.

Autocovariance and Autocorrelation Functions

It is necessary to estimate the autocovariance and ACFs from a time series of finite length, say, $y_1, y_2, \ldots, y_T$. The usual estimate of the autocovariance function is

$$c_k = \hat{\gamma}_k = \frac{1}{T}\sum_{t=1}^{T-k}(y_t - \bar{y})(y_{t+k} - \bar{y}), \quad k = 0, 1, 2, \ldots, K \tag{2.12}$$

and the ACF is estimated by the **sample autocorrelation function** (or **sample ACF**)

$$r_k = \hat{\rho}_k = \frac{c_k}{c_0}, \quad k = 0, 1, \ldots, K \tag{2.13}$$

Autocovariance and Autocorrelation Functions

A good general rule of thumb is that at least 50 observations are required to give a reliable estimate of the ACF, and the individual sample autocorrelations should be calculated up to lag K, where K is about $T/4$.

Often we will need to determine if the autocorrelation coefficient at a particular lag is zero. This can be done by comparing the sample autocorrelation coefficient at lag k, r_k, to its standard error. If we make the assumption that the observations are uncorrelated, that is, $\rho_k = 0$ for all k, then the variance of the sample autocorrelation coefficient is

$\hat{\rho}_k$

$$\mathrm{Var}(r_k) \cong \frac{1}{T} \tag{2.14}$$

and the standard error is

$$se(r_k) \cong \frac{1}{\sqrt{T}} \tag{2.15}$$

LINEAR MODELS FOR STATIONARY TIME SERIES

In statistical modeling, we are often engaged in an endless pursuit of finding the ever elusive true relationship between certain inputs and the output. These efforts usually result in models that are nothing but approximations of the “true” relationship. This is generally due to the choices the analyst makes along the way to ease the modeling efforts. A major assumption that often provides relief in modeling efforts is the linearity assumption. A linear filter, for example, is a linear operation from one time series x_t to another time series y_t.

LINEAR MODELS FOR STATIONARY TIME SERIES

$$y_t = L(x_t) = \sum_{i=-\infty}^{+\infty} \psi_i x_{t-i} \qquad (5.1)$$

with $t = \ldots, -1, 0, 1, \ldots$. In that regard the linear filter can be seen as a "process" that converts the input, x_t, into an output, y_t, and that conversion is not instantaneous but involves all (present, past, and future) values of the input in the form of a summation with different "weights", $\{\psi_i\}$, on each x_t. Furthermore, the linear filter in Eq. (5.1) is said to have the following properties:

LINEAR MODELS FOR STATIONARY TIME SERIES

1. **Time-invariant** as the coefficients $\{\psi_i\}$ do not depend on time.
2. **Physically realizable** if $\psi_i = 0$ for $i < 0$; that is, the output y_t is a linear function of the current and past values of the input: $y_t = \psi_0 x_t + \psi_1 x_{t-1} + \cdots$.
3. **Stable** if $\sum_{i=-\infty}^{+\infty} |\psi_i| < \infty$.

In linear filters, under certain conditions, some properties such as stationarity of the input time series are also reflected in the output.