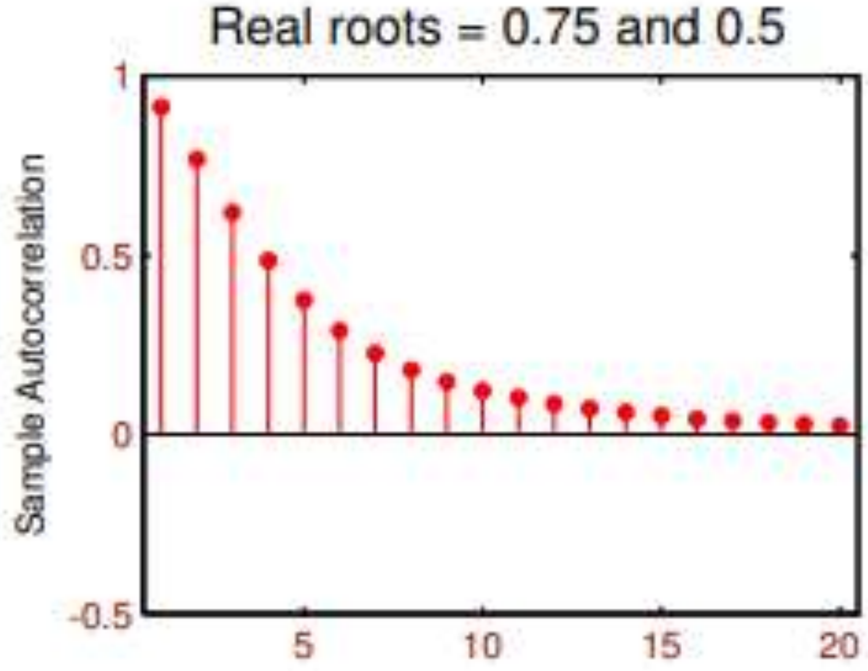
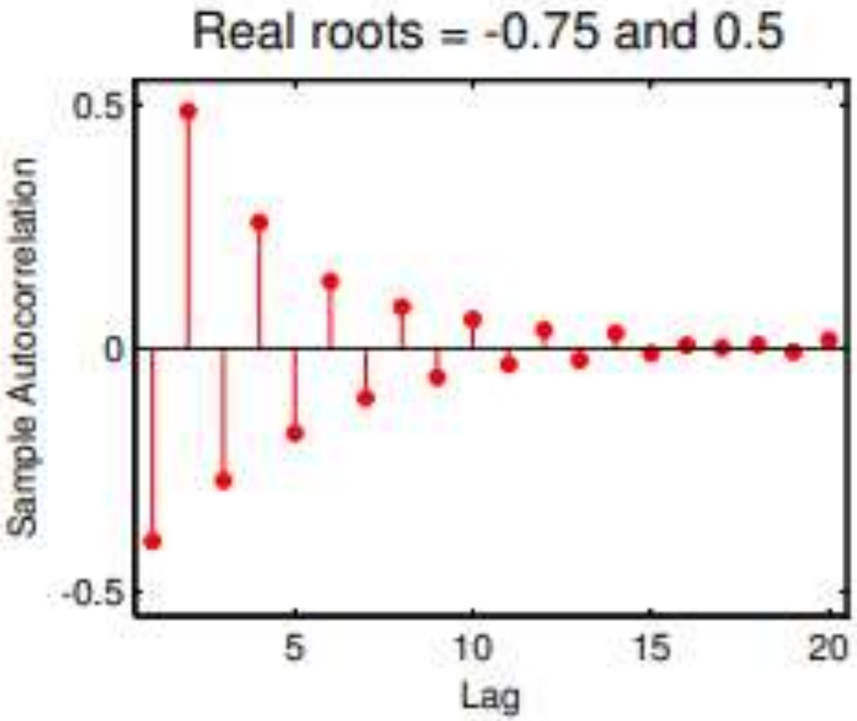
1.AutoCorrelation for AR process Diagram
2.Partial Autocorrelation Function

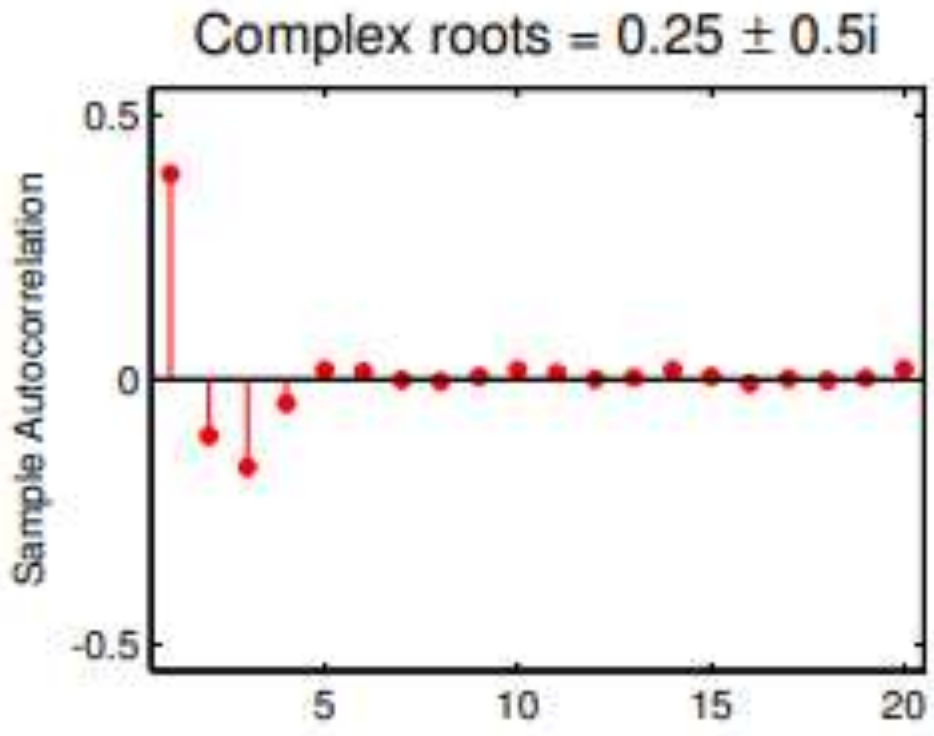
PACF

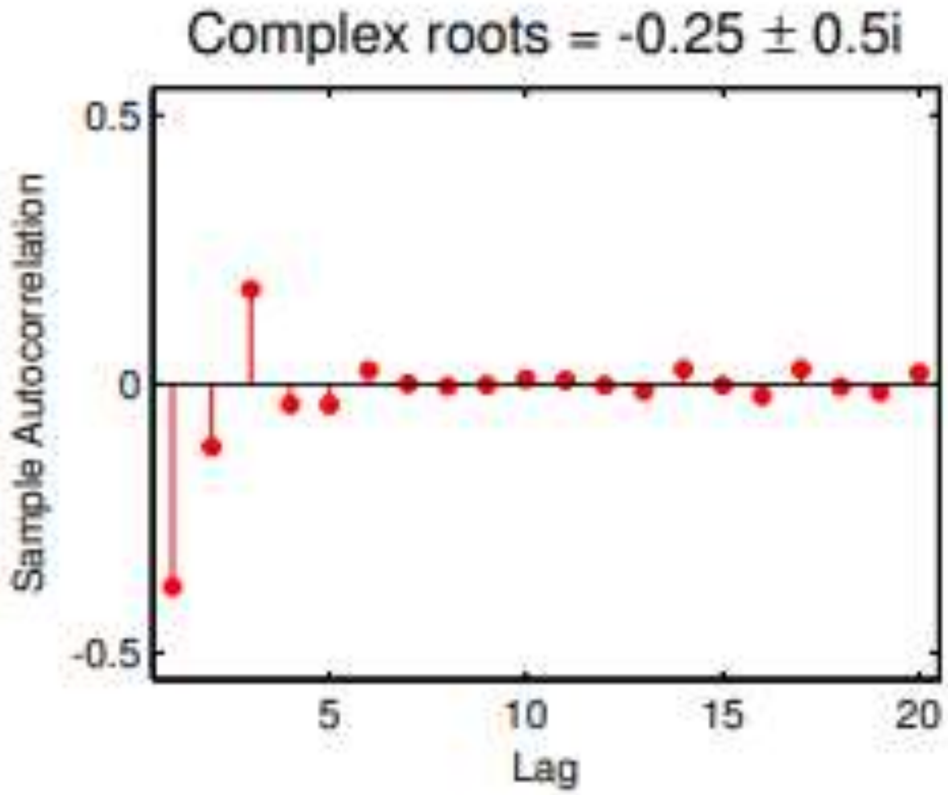
ACF for Real and Different roots (sign may be equal or opposite) in AR process.





ACF for Complex roots (sign may be equal or opposite) in AR process.





Partial Autocorrelation Function

we saw that the ACF is an excellent tool in identifying the order of an MA(q) process, because it is expected to “cut off” after lag q. However, the ACF is not as useful in the identification of the order of an AR(p) process for which it will most likely have a mixture of exponential decay and damped sinusoid expressions. Hence such behavior, while indicating that the process might have an AR structure, fails to provide further information about the order of such structure. For that, we will define and employ the partial autocorrelation function (PACF) of the time series. But before that, we discuss the concept of partial correlation to make the interpretation of the PACF easier.

Partial Autocorrelation Function

Determining the order of an autoregressive process from its autocorrelation function is difficult. To resolve this problem the partial autocorrelation function is introduced.

Partial Correlation Consider three random variables X, Y, and Z. Then consider simple linear regression of X on Z and Y on Z as

$$\hat{X} = a_1 + b_1 Z \quad \text{where } b_1 = \frac{\mathrm{Cov}(Z, X)}{\mathrm{Var}(Z)}$$

and

$$\hat{Y} = a_2 + b_2 Z \quad \text{where } b_2 = \frac{\mathrm{Cov}(Z, Y)}{\mathrm{Var}(Z)}$$

Then the errors can be obtained from

$$X^* = X - \hat{X} = X - (a_1 + b_1 Z)$$

and

$$Y^* = Y - \hat{Y} = Y - (a_2 + b_2 Z)$$

Partial Autocorrelation Function

Then the partial correlation between X and Y after adjusting for Z is defined as the correlation between $X^*$and Y^* ; corr(X^*, Y^*) = corr(X − $\hat{X}$, Y − $\hat{Y}$). That is, partial correlation can be seen as the correlation between two variables after being adjusted for a common factor that may be affecting them. The generalization is of course possible by allowing for adjustment for more than just one factor.

Partial Autocorrelation Function

Partial Autocorrelation Function Following the above definition, the **PACF** between y_t and y_{t-k} is the autocorrelation between y_t and y_{t-k} after adjusting for $y_{t-1}, y_{t-2}, \ldots, y_{t-k+1}$. Hence for an AR($p$) model the PACF between y_t and y_{t-k} for $k > p$ should be equal to zero. A more formal definition can be found below.

Consider a stationary time series model { y_t } that is not necessarily an AR process. Further consider, for any fixed value of k, the Yule–Walker equations for the ACF of an AR(p) process given in equation 1 as

$$\rho(j) = \sum_{i=1}^{k} \phi_{ik}\rho(j-i), \quad j = 1, 2, \ldots, k$$ **…….equation 1**

or

$$\begin{aligned}
\rho(1) &= \phi_{1k} + \phi_{2k}\rho(1) + \cdots + \phi_{kk}\rho(k-1) \\
\rho(2) &= \phi_{1k}\rho(1) + \phi_{2k} + \cdots + \phi_{kk}\rho(k-2) \\
&\vdots \\
\rho(k) &= \phi_{1k}\rho(k-) + \phi_{2k}\rho(k-2) + \cdots + \phi_{kk}
\end{aligned}$$

Partial Autocorrelation Function

Hence we can write **equation 1** in matrix form as

$$\begin{bmatrix} 1 & \rho(1) & \rho(2) & \dots & \rho(k-1) \\ \rho(1) & 1 & \rho(3) & \dots & \rho(k-2) \\ \rho(2) & \rho(1) & 1 & \dots & \rho(k-3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho(k-1) & \rho(k-2) & \rho(k-3) & \dots & 1 \end{bmatrix} \begin{bmatrix} \phi_{1k} \\ \phi_{2k} \\ \phi_{3k} \\ \vdots \\ \phi_{kk} \end{bmatrix} = \begin{bmatrix} \rho(1) \\ \rho(2) \\ \rho(3) \\ \vdots \\ \rho(k) \end{bmatrix}$$

….equation 2

or

$$\mathbf{P}_k \phi_k = \rho_k$$

….equation 3

Partial Autocorrelation Function

where

$$\mathbf{P}_k = \begin{bmatrix} 1 & \rho(1) & \rho(2) & \dots & \rho(k-1) \\ \rho(1) & 1 & \rho(3) & \dots & \rho(k-2) \\ \rho(2) & \rho(1) & 1 & \dots & \rho(k-3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho(k-1) & \rho(k-2) & \rho(k-3) & \dots & 1 \end{bmatrix},$$

$$\phi_k = \begin{bmatrix} \phi_{1k} \\ \phi_{2k} \\ \phi_{3k} \\ \vdots \\ \phi_{kk} \end{bmatrix}, \quad \text{and} \quad \rho_k = \begin{bmatrix} \rho(1) \\ \rho(2) \\ \rho(3) \\ \vdots \\ \rho(k) \end{bmatrix}.$$

Thus to solve for ϕ_k, we have

$$\phi_k = \mathbf{P}_k^{-1} \rho_k$$

Partial Autocorrelation Function

Thus to solve for ϕ_k, we have

$$\phi_k = \mathbf{P}_k^{-1} \rho_k$$

For any given k, k = 1, 2,… , the last coefficient ϕ_{kk} is called the partial autocorrelation of the process at lag k. Note that for an AR(p) process $\phi_{kk} = 0$ for k > p. Hence we say that the PACF cuts off after lag p for an AR(p). This suggests that the PACF can be used in identifying the order of an AR process similar to how the ACF can be used for an MA process.

From this definition it is clear that an AR(p) process will have the first p nonzero partial autocorrelation coefficients and, therefore, in the ***partial autocorrelation function (PACF) the number of nonzero coefficients indicates the order of the AR process.***

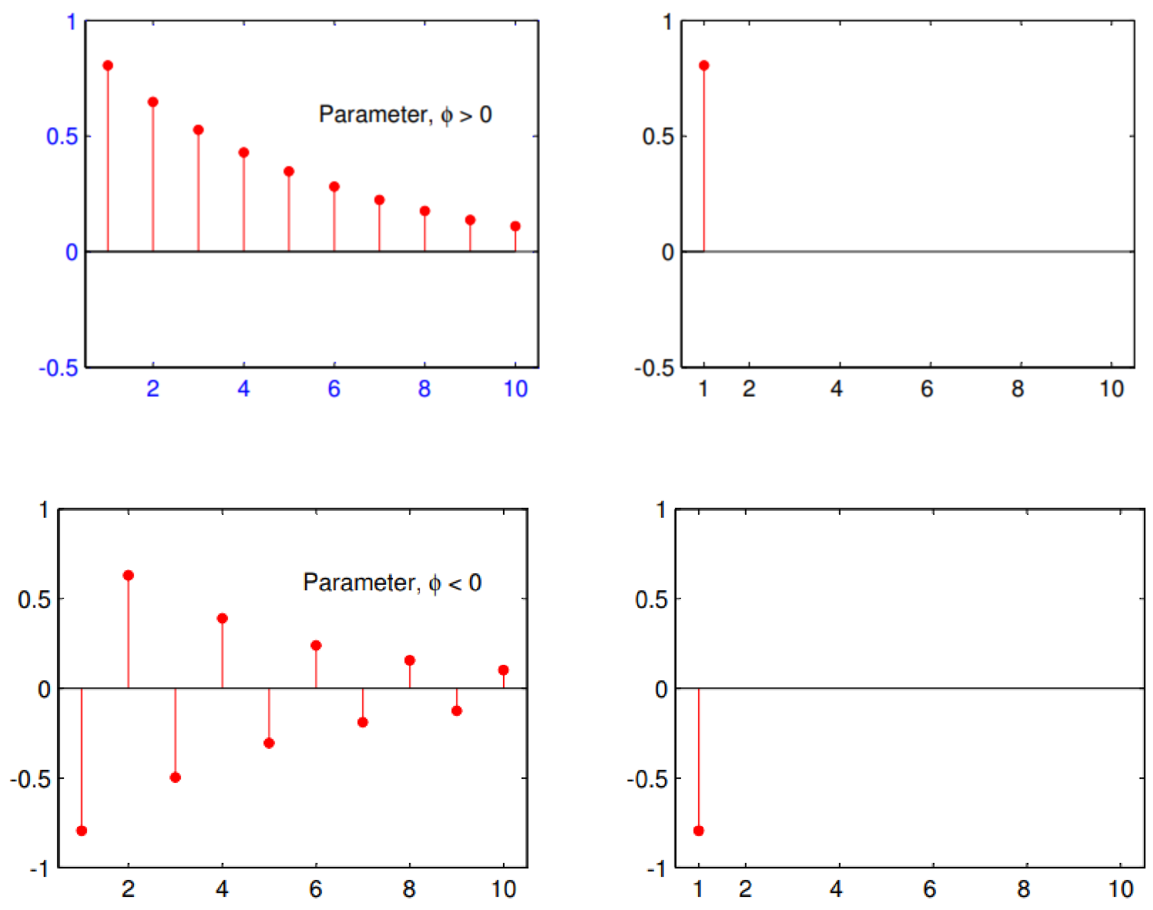
Partial Autocorrelation Function

For sample calculations, $\hat{\phi}_{kk}$, the sample estimate of ϕ_{kk}, is obtained by using the sample ACF, $r(k)$. Furthermore, in a sample of N observations from an AR(p) process, $\hat{\phi}_{kk}$ for $k > p$ is approximately normally distributed with

$$E(\hat{\phi}_{kk}) \approx 0 \quad \text{and} \quad \mathrm{Var}(\hat{\phi}_{kk}) \approx \frac{1}{N}$$

Hence the 95% limits to judge whether any $\hat{\phi}_{kk}$ is statistically significantly different from zero are given by ±2⁄ √ N.

The partial autocorrelation function - AR(1) models



The partial autocorrelation function - AR(2) models