Variance and Autocovariance function of a stationary AR(2)

from eqn (1)

$$y_t = \delta + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \varepsilon_t \quad -(1)$$

The autocovariance function is

$$\gamma(k) = \operatorname{cov}(y_t, y_{t-k})$$

$$\gamma(k) = \operatorname{cov}(\delta + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \varepsilon_t, y_{t-k})$$

$$\gamma(k) = \phi_1 \operatorname{cov}(y_{t-1}, y_{t-k}) + \phi_2 \operatorname{cov}(y_{t-2}, y_{t-k}) + \operatorname{cov}(\varepsilon_t, y_{t-k}) \quad -(2)$$

$V(\varepsilon_t) = \sigma^2$ lag $k = 0$

$V(\varepsilon_t) = 0$ $k \neq 0$

$$\gamma(k) = \phi_1 \gamma(k-1) + \phi_2 \gamma(k-2) + \sigma^2 \quad \text{if } k=0$$

$$\gamma(k) = \phi_1 \gamma(k-1) + \phi_2 \gamma(k-2) + 0 \quad \text{if } k>0$$

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Auto covariance f^n

Thus for $k=0$ we have

$$\gamma(0) = \phi_1 \gamma(1) + \phi_2 \gamma(2) + \sigma^2$$

and

$$\gamma(k) = \phi_1 \gamma(k-1) + \phi_2 \gamma(k-2), \quad k = 1, 2, 3, \ldots \quad \text{--- (3)}$$

$\gamma(-1) = \gamma(1)$

$\gamma(-2) = \gamma(2)$

The eq^n in (3) are called the Yule-Walker equations for $\gamma(k)$.

$\operatorname{Cov}(y_t, y_{t-k})$

$\gamma(0)$

Var

Again from eq^n ① $V(\text{constant}) = 0$

$$y_t = \delta + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \varepsilon_t$$

Constant

Taking variance on both sides

$$Var(y_t) = Var(\underbrace{\phi_1 y_{t-1}}_{X} + \underbrace{\phi_2 y_{t-2}}_{Y} + \underbrace{\varepsilon_t}_{Z})$$

$\boxed{\text{NOTE}} \Rightarrow$ $\boxed{Var(X+Y+Z)} = Var(X) + Var(Y) + Var(Z) + 2\,cov(XY) + 2\,cov(YZ) + 2\,cov(XZ)$

$Var(ax) = a^2\, var(x)$

$$Var(y_t) = Var(\phi_1 y_{t-1}) + Var(\phi_2 y_{t-2}) + Var(\varepsilon_t) + 2\,cov(\phi_1 \phi_2 y_{t-1} y_{t-2}) + \underbrace{2\,cov(\phi_2 y_{t-2} \varepsilon_t)}_{=0} + \underbrace{2\,cov(\phi_1 y_{t-1} \varepsilon_t)}_{=0}$$

$$Var(y_t) = \phi_1^2 \, var(y_{t-1}) + \phi_2^2 \, var(y_{t-2})$$
$$+ var(\varepsilon_t)$$
$$+ 2\phi_1\phi_2 \, cov(y_{t-1}, y_{t-2})$$

Var(

or

$$\gamma(0) = \phi_1^2 \gamma(0) + \phi_2^2 \gamma(0) + \sigma^2 + 2\phi_1\phi_2 \gamma(1) \quad \text{--- (4)}$$

$$\left[\gamma(0) = var(y_t) = var(y_{t-1}) = var(y_{t-2}) \right.$$
$$\left. \gamma(1) = cov(y_{t-1}, y_{t-2}) \right]$$ Stationary

From eqn (3) $\gamma(k) = \phi_1 \gamma(k-1) + \phi_2 \gamma(k-2)$

$k = 1, 2, 3, ..$

for $k=1$

$$\gamma(1) = \phi_1 \gamma(0) + \phi_2 \gamma(-1)$$

$$\gamma(1) = \phi_1 \gamma(0) + \phi_2 \gamma(1)$$

$$[\because \gamma(-1) = \gamma(1)]$$

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which provides

$$\gamma(1) - \phi_2 \gamma(1) = \phi_1 \gamma(0)$$

$$(1-\phi_2)\gamma(1) = \phi_1 \gamma(0)$$

$$\boxed{\gamma(1) = \frac{\phi_1 \gamma(0)}{1-\phi_2}}$$

Put this value of $\gamma(1)$ in eq^n (4) we get

$$\gamma(0) = \frac{(1-\phi_2)\sigma^2}{(1+\phi_2)\underbrace{(1-\phi_1-\phi_2)}_{[1-(\phi_1+\phi_2)]}\underbrace{(1+\phi_1-\phi_2)}_{2 \quad 4}} \quad (5)$$

Variance for AR(2) process

$$\phi_1 + \phi_2 < 1$$

$$\phi_2 < \phi_1 \Leftarrow \phi_2 - \phi_1 < 1$$

$$|\phi_2| < 1$$

Auto correlation function of a stationary AR(2) process

$$\rho(k) = \frac{\gamma(k)}{\gamma(0)} = \frac{cov(y_t, y_{t-k})}{Var\, y_t}$$

From eqn (2) Yule-walker eqn

$$\gamma(k) = \phi_1 \gamma(k-1) + \phi_2 \gamma(k-2)$$

$k = 1, 2, 3, \ldots$

dividing above equation by $\gamma(0)$ (Variance of y_t)

$$\frac{\gamma(k)}{\gamma(0)} = \phi_1 \frac{\gamma(k-1)}{\gamma(0)} + \phi_2 \frac{\gamma(k-2)}{\gamma(0)}$$

$$\rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2} \quad \text{--- (1)}$$

$k = 1, 2, 3, \ldots$

For $k=1$

$$\rho_1 = \phi_1 \rho_0 + \phi_2 \rho_{-1}$$

$\rho_0 = 1$

$\rho_{-1} = \rho_1$

or $$\rho_1 = \phi_1 \rho_0 + \phi_2 \rho_1$$

$$(\because \rho_{-1} = \rho_1)$$

we get

$$\cancel{\rho_1 = \phi_2 \rho}$$

$$\rho_1 - \phi_2 \rho_1 = \phi_1 \rho_0$$

$$(1-\phi_2)\rho_1 = \phi_1 \rho_0$$

$$\rho_1 = \frac{\phi_1 \rho_0}{1-\phi_2} \qquad (\because \rho_0 = 1)$$

$$\rho_1 = \frac{\phi_1}{1-\phi_2} \quad \text{— } (2)$$

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<u>for $k=2$</u>

$$\rho_2 = \phi_1 \rho_1 + \phi_2 \rho_0$$

$$\rho_2 = \phi_1 \rho_1 + \phi_2 \qquad (\because \rho_0 = 1)$$

$$\rho_2 = \phi_1 \times \frac{\phi_1}{1-\phi_2} + \phi_2 \qquad [\text{from } (2)]$$

$$\rho_2 = \frac{\phi_1^2}{1-\phi_2} + \phi_2 \qquad (3)$$

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For $k \geq 3$

the autocorrelation coefficient can be obtained from recursively starting from the difference equation (1).

A general solution can be obtained through the roots m_1 and m_2 of the associated polynomial

C-I

$$m^2 - \phi_1 m - \phi_2 = 0.$$

as $$\rho(k) = c_1 m_1^k + c_2 m_2^k \quad \text{--- (4)}$$

where c_1 and c_2 are particular constant to be determined from the initial conditions $\rho_0 = 1$ $(c_1 + c_2 = 1)$ and

$$\rho_1 = \frac{\phi_1}{1-\phi_2}.$$

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According to eqn (4), the coefficients $\rho(k)$ will be ≤ 1 if

$|m_1| < 1$ and $|m_2| < 1$, which are the conditions of stationary of the process.

and, in this case the ACF is a mixture of two exponential decay terms.

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{CASE-II} If m_1 and m_2 are complex conjugates in the form $a \pm ib$, we then have

$$\sqrt{a^2+b^2} < 1$$

$$P(k) = R^k\left[C_1 \cos(\lambda k) + C_2 \sin(\lambda k)\right]$$

$$k \geq 3$$

where $R = |m_i| = \sqrt{a^2+b^2}$

and λ is determined by

$$\cos(\lambda) = \frac{a}{R}$$

$$\sin(\lambda) = \frac{b}{R}$$

Hence $a + ib = R\left[\cos(\lambda) + i\sin(\lambda)\right]$.

C_1 and C_2 are particular constant.

The ACF in this case has the form of a sinusoid with damping factor R and frequency λ. i.e., the period is $\frac{2\pi}{\lambda}$.

CASE-III If there is one real root m_0

$m_1 = m_2 = m_0$, we then have

$$P(k) = (C_1 + C_2 k)\, m_0^k \qquad k \geq 3$$

in this case ACF will exhibit an exponential decay pattern.