Method of Moments for parameter estimation

Method of Moments Estimation

- One of the easiest methods of parameter estimation is the method of moments (MOM).
- The basic idea is to find expressions for the sample moments and for the population moments and equate them:

$$\underbrace{\frac{1}{n}\sum_{i=1}^{n}X_{i}^{r}}_{=}=\underbrace{E(X^{r})}_{=}$$

- ▶ The $E(X^r)$ expression will be a function of one or more unknown parameters.
- If there are, say, 2 unknown parameters, we would set up MOM equations for r = 1, 2, and solve these 2 equations simultaneously for the two unknown parameters.
 - ▶ In the simplest case, if there is only 1 unknown parameter to estimate, then we equate the sample mean to the true mean of the process and solve for the unknown parameter.

MOM with AR models

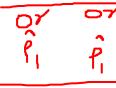
- First, we consider autoregressive models.
- In the simplest case, the AR(1) model, given by $Y_t = \phi Y_{t-1} + \epsilon_t$, the true lag-1 autocorrelation $\rho_1 = \phi$.
- For this type of model, a method-of-moments estimator would simply equate the true lag-1 autocorrelation to the sample lag-1 autocorrelation r_1 .
- So our MOM estimator of the unknown parameter ϕ would be $\hat{\phi} = r_1$.

MOM with an AR(2) model

- ▶ In the AR(2) model, we have unknown parameters ϕ_1 and ϕ_2 .
- From the Yule-Walker equations,

$$\rho_1 = \phi_1 + \rho_1 \phi_2$$
 and $\rho_2 = \rho_1 \phi_1 + \phi_2$

In the method of moments, we will replace the true lag-1 and lag-2 autocorrelations, ρ₁ and ρ₂, by the sample autocorrelations r₁ and r₂, respectively.



MOM with an AR(2) model

That gives the equations

$$r_1 = \phi_1 + r_1\phi_2$$
 and $r_2 = r_1\phi_1 + \phi_2$

which are then solved for ϕ_1 and ϕ_2 to obtain

$$\hat{\phi}_1 = \frac{r_1(1-r_2)}{1-r_1^2}$$
 and $\hat{\phi}_2 = \frac{r_2-r_1^2}{1-r_1^2}$

The general AR(p) model is estimated in a similar way, with the Yule-Walker equations being used to obtain the Yule-Walker estimates $\hat{\phi}_1, \hat{\phi}_2, \dots, \hat{\phi}_p$.

MOM with MA Models

- We run into problems when trying to using the method of moments to estimate the parameters of moving average models. Y = Ex - OE = -1
- ▶ Consider the simple MA(1) model, $Y_t = e_t \theta e_{t-1}$.
- The true lag-1 autocorrelation in this model is
- $\rho_1 = -\theta/(1+\theta^2).$ If we equate ρ_1 to r_1 , we get a quadratic equation in θ .
- ▶ If $|r_1|$ < 0.5, then only one of the two real solutions satisfies the invertibility condition $|\theta| < 1$.
- That solution is $\hat{\theta} = \left(-1 + \sqrt{1 4r_1^2}\right)/(2r_1)$.
 - ▶ But if $|r_1| = 0.5$, no invertible solution exists, and if $|r_1| > 0.5$, then no real solution at all exists, and the method of moments fails to give any estimator of θ .

$$\gamma_{1} + \gamma_{1} \otimes^{2} + \emptyset = 0$$

More MOM Problems with MA Models

- With higher-order MA(q) models, the set of equations for estimating $\underline{\theta_1, \ldots, \theta_q}$ is highly nonlinear and could only be solved numerically.
- There would be many solutions, only one of which is invertible.
- ► In any case, for MA(q) models, the method of moments usually produces poor estimates, so it is not recommended to use MOM to estimate MA models.

MOM Estimation of Mixed ARMA Models

- Consider only the simplest mixed model, the <u>ARMA(1,1)</u> model.
- ▶ Since $\rho_2/\rho_1 = \phi$, a MOM estimator of ϕ is $\hat{\phi} = r_2/r_1$.
- ► Then the equation

$$r_1 = rac{(1- heta\hat{\phi})(\hat{\phi}- heta)}{1-2 heta\hat{\phi}+ heta^2}$$

can be used to solve for an estimate of θ .

This is a quadratic equation is θ , and so we again keep only the invertible solution (if any exist) as our $\hat{\theta}$.

MOM Estimation of the Noise Variance

- We still must estimate the variance σ_e^2 of our error component.
- For any model, we first estimate the variance of the time series process itself, $\gamma_0 = var(Y_t)$, by the sample variance

$$s^2 = \frac{1}{n-1} \sum_{i=1}^n (Y_t - \bar{Y})^2$$

► Then we can take advantage of known relationships among the parameters in our specified model to obtain a formula for $\hat{\sigma}_{e}^{2}$.

Formulas for MOM Noise Variance Estimators in Common Models

- For AR(p) models, $\hat{\sigma}_e^2 = (1 \hat{\phi}_1 r_1 \hat{\phi}_2 r_2 \dots \hat{\phi}_p r_p) s^2$.
 - For the AR(1) model, this reduces to $\hat{\sigma}_e^2 = (1 r_1^2)s^2$.
 - ► For *MA*(*q*) models,

$$\hat{\sigma}_e^2 = \frac{s^2}{1 + \hat{\theta}_1^2 + \hat{\theta}_2^2 + \dots + \hat{\theta}_q^2}.$$

For ARMA(1,1) models,

$$\hat{\sigma}_e^2 = \frac{1 - \hat{\phi}^2}{1 - 2\hat{\phi}\hat{\theta} + \hat{\theta}^2}s^2.$$