General Autoregressive Process, AR(p)

The General p^{th} order AR model is given as

$$\boxed{y_t = \delta + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \varepsilon_t} \quad -(1)$$

$\phi_1 y_{t-1} \to By_t$, $\phi_2 y_{t-2} \to B^2 y_t$, $\phi_p y_{t-p} \to B^p y_t$

It is regression of y_t on $y_{t-1}, y_{t-2} \cdots y_{t-p}$.

Now, we can write eq^n (1) in terms of backshift operator as

$$y_t = \delta + \phi_1 B y_t + \phi_2 B^2 y_t + \cdots + \phi_p B^p y_t + \varepsilon_t$$

$$[\because B^i y_t = y_{t-i}]$$

$$y_t - \phi_1 B y_t - \phi_2 B^2 y_t - \cdots - \phi_p B^p y_t = \delta + \varepsilon_t$$

$$(1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p) y_t = \delta + \varepsilon_t \quad -(2)$$

or

$$\Phi(B) y_t = \delta + \varepsilon_t \quad -(3)$$

$A^{-1}A = 1$

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apply $\Phi(B)^{-1}$ to both sides, we obtain

$$y_t = \Phi(B)^{-1}\delta + \Phi(B)^{-1}\varepsilon_t$$

or

$$y_t = \mu + \psi(B)\varepsilon_t \qquad -(4)$$

where $\mu = \Phi(B)^{-1}\delta$

$$\psi(B) = \Phi(B)^{-1}$$

$$y_t = \mu + \sum_{i=0}^{\infty} \psi_i \varepsilon_{t-i}$$

$$y_t = \mu + \sum_{i=0}^{\infty} \psi_i B^i \varepsilon_t$$

where

$$\mu = \Phi(B)^{-1}\delta \qquad -(5)$$

and

$$\Phi(B)^{-1} = \sum_{i=0}^{\infty} \psi_i B^i = \psi(B) \qquad -(6)$$

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Now, we use eqⁿ 6 to obtain the weights in Eqⁿ 4 in terms of ϕ_1 and ϕ_2, --- ϕ_p.

For that, we use

$$\Phi(B)\,\Psi(B) = 1 \qquad — (7)$$

$\phi_r\ \phi_1(B)$

$$(1-\phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p) \times$$
$$(\psi_0 + \psi_1 B + \psi_2 B^2 + \cdots + \psi_p B^p + \cdots) = 1 \quad -(8)$$

on RHS of eq^n (8) there are no ~~parameters~~ backshift operator for

$$\Phi(B)\,\psi(B) = 1.$$

we need,

$$\left.\begin{array}{ll} \psi_j = 0, & j < 0 \\ \psi_0 = 1 & \\ \psi_j - \phi_1 \psi_{j-1} - \phi_2 \psi_{j-2} - \cdots - \phi_p \psi_{j-p} = 0 & \\ \quad \text{for } j = 1, 2, \ldots & \end{array}\right\} -(9)$$

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The AR(P) time series $\{y_t\}$ in eqn (1) is causal and stationary if the roots of the associated polynomial

$$\boxed{m^p - \phi_1 m^{p-1} - \phi_2 m^{p-2} - \cdots - \phi_p = 0} \quad (10)$$

AR(2) $\longrightarrow$ $m^2 - \phi_1 m - \phi_2 = 0$

are less than one in absolute value.

i.e.

$$|m_1|, |m_2|, \ldots, |m_p| < 1.$$

then we have $\sum_{i=0}^{\infty} |\psi_i| < \infty$.

mean of AR(p) process

we can easily show that, for the stationary AR(p) process

$$E(y_t) = \mu = \frac{\delta}{1 - \phi_1 - \phi_2 - \cdots - \phi_p} \quad \text{--- (II)}$$

Autocovariance and Autocorrelation of stationary AR(p) process

from eqn (1)

$$y_t = \delta + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \varepsilon_t \quad \text{--- (1)}$$

The autocovariance function is

$$\gamma(k) = cov(y_t, y_{t-k})$$

$$\gamma(k) = cov(\delta + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \dots + \phi_p y_{t-p} + \varepsilon_t, y_{t-k})$$

$$\boxed{\text{See AR(2) process Autocovariance}}$$

$$\gamma(k) = \sum_{i=1}^{p} \phi_i \, cov(y_{t-i}, y_{t-k}) + cov(\varepsilon_t, y_{t-k}) \quad \text{--- (2)}$$

$$\gamma(k) = \sum_{i=1}^{p} \phi_i \gamma(k-i) + \sigma^2 \quad \text{if } k=0$$

$$\gamma(k) = \sum_{i=1}^{p} \phi_i \gamma(k-i) + 0 \quad \text{if } k>0$$

Thus for $k=0$ we have

$$\gamma(0) = \sum_{i=1}^{p} \phi_i \gamma(i) + \sigma^2 \quad (3)$$

$\gamma(-i) = \gamma(i)$

$\Rightarrow \gamma(0) -$

$$\Rightarrow \gamma(0) - \sum_{i=1}^{p} \phi_i \gamma(i) = \sigma^2$$

$$\Rightarrow \gamma(0) \left[1 - \sum_{i=1}^{p} \phi_i \frac{\gamma(i)}{\gamma(0)} \right] = \sigma^2$$

$$\Rightarrow \gamma(0) \left[1 - \sum_{i=1}^{p} \phi_i \rho(i) \right] = \sigma^2 \quad (4)$$

$$\left[\because \rho(i) = \frac{\gamma(i)}{\gamma(0)} \right]$$

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$Var(y_t)$

Now divide eqn (2) by $\gamma(0)$ for $k>0$, it can be observed that the ACF of an AR(p) process satisfies the Yule-walker equations

$$\rho(k) = \sum_{i=1}^{p} \phi_i \, \rho(k-i) \quad , k=1,2,\ldots \qquad (5)$$

The equation (5) are p^{th} order linear difference equations, implying that the ACF for an AR(p) model can be found through the p roots of the associated polynomial in eqn (10)

i.e. $m^p - \phi_1 m^{p-1} - \phi_2 m^{p-2} - \cdots - \phi_p = 0.$

For ex. if roots are real and distinct we have

$$\rho(k) = c_1 m_1^k + c_2 m_2^k + \cdots + c_p m_p^k \quad , k=1,2,\ldots \qquad (6)$$

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where $c_1, c_2, \ldots, c_p$ are particular constants. However, the roots may not all be distinct or real.

Thus the ACF of an AR(p) process can be a mixture of exponential decay and damped sinusoid expressions depending on the roots of eqn (10)