Second order Autoregressive Process, AR(2)

The second order Autoregressive Process AR(2) is given by

$$\boxed{y_t = \delta + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \varepsilon_t} \quad -(1)$$

It is regression of y_t on y_{t-1} and y_{t-2}.

Now, we write eqn (1) in terms of backshift operator as

$$y_t = \delta + \phi_1 B y_t + \phi_2 B^2 y_t + \varepsilon_t$$

$$\left[\because B^i y_t = y_{t-i}\right]$$

$$y_t - \phi_1 B y_t - \phi_2 B^2 y_t = \delta + \varepsilon_t$$

$$(1 - \phi_1 B - \phi_2 B^2) y_t = \delta + \varepsilon_t \quad -(2)$$

or

$$\Phi(B) y_t = \delta + \varepsilon_t \quad -(3)$$

apply $\Phi(B)^{-1}$ to both sides, we obtain

$$y_t = \Phi(B)^{-1}\delta + \Phi(B)^{-1}\varepsilon_t$$

or

$$y_t = \mu + \Psi(B)\varepsilon_t \qquad -(4)$$

where $\mu = \Phi(B)^{-1}\delta$

$$\Psi(B) = \Phi(B)^{-1}$$

$$y_t = \mu + \sum_{i=0}^{\infty}\psi_i\varepsilon_{t-i}$$

$$y_t = \mu + \sum_{i=0}^{\infty}\psi_i B^i\varepsilon_t$$

where

$$\mu = \Phi(B)^{-1}\delta \qquad -(5)$$

and

$$\Phi(B)^{-1} = \sum_{i=0}^{\infty}\psi_i B^i = \Psi(B) \qquad -(6)$$

Now, we use eqⁿ (6) to obtain the weights in Eqⁿ (4) in terms of ϕ_1 and ϕ_2.

For that, we use

$$\Phi(B)\,\Psi(B) = 1 \qquad -(7)$$

i.e.

$$(1-\phi_1 B-\phi_2 B^2)(\psi_0+\psi_1 B+\psi_2 B^2+\cdots) = 1$$

or

$$\psi_0+(\psi_1-\phi_1\psi_0)B+(\psi_2-\phi_1\psi_1-\phi_2\psi_0)B^2$$
$$+\cdots+(\psi_j-\phi_1\psi_{j-1}-\phi_2\psi_{j-2})B^j+\cdots = 1 \qquad -(8)$$

On RHS of eqⁿ ⑧ there are no backshift operators, for $\Phi(B)\psi(B) = 1$,

we need

$$\left.\begin{aligned} \psi_0 &= 1 \\ (\psi_1 - \phi_1 \psi_0) &= 0 \\ (\psi_2 - \phi_1 \psi_1 - \phi_2 \psi_0) &= 0 \\ &\vdots \\ (\psi_j - \phi_1 \psi_{j-1} - \phi_2 \psi_{j-2}) &= 0 \end{aligned}\right\} \text{ for all } j = 2, 3, \ldots \quad (9)$$

we can solve above eqⁿ ⑨ to estimate infinitely many parameters

But ~~eqⁿ ⑨~~ ψ_j in eqⁿ ⑨ satisfy the second-order linear difference equation.

Now, we can expressed it in terms of two roots m_1 and m_2 of the associated polynomial

$$(\psi_2 - \phi_1 \psi_1 - \phi_2 \psi_0) = 0$$

$$\downarrow$$

$$\boxed{m^2 - \phi_1 m - \phi_2 = 0} \quad \text{--- (10)}$$

If the roots obtained by

$$m_1, m_2 = \frac{\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{2}$$

satisfy $|m_1|, |m_2| < 1$ then

we have $\sum_{i=0}^{\infty} |\psi_i| < \infty$.

Hence if the roots m_1 and m_2 both less than 1 in absolute value, then the AR(2) model is causal and stationary.

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If roots of eqn (10) are complex of the form $a \pm ib$, the condition for {stationary} is that $\sqrt{a^2+b^2} < 1$.

Under the condition that $|m_1|, |m_2| < 1$, the AR(2) time series, $\{y_t\}$, has an infinite MA representation.

$\therefore$ for the second order autoregressive process to be {stationary}, the parameters ϕ_1 and ϕ_2 must satisfy.

$$\phi_1 + \phi_2 < 1$$

$$\phi_2 - \phi_1 < 1$$

$$|\phi_2| < 1$$

Conditions for the stationary of an AR(2) process.

⇒ The mean of a stationary AR(2) ⇐

from eqⁿ (1)

$$y_t = \delta + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \varepsilon_t$$

Taking expectations on both sides of above eqⁿ

$$E(y_t) = E(\delta) + \phi_1 E(y_{t-1}) + \phi_2 E(y_{t-2}) + E(\varepsilon_t)$$

$$\mu = \delta + \phi_1 \mu + \phi_2 \mu + 0$$

$$\mu - \phi_1 \mu - \phi_2 \mu = \delta$$

$$\mu(1 - \phi_1 - \phi_2) = \delta$$

$$\mu = \frac{\delta}{1 - \phi_1 - \phi_2} \quad \text{--- (11)}$$

mean of AR(2) process.