Finite Order AutoRegressive Process

In autoregressive model, the current value of the process is expressed as a finite, linear aggregate of previous values of the process and a random disturbance term ε_t.

First-order Autoregressive Process, AR(1)

we know that

$$y_t = \mu + \sum_{i=0}^{\infty} \psi_i \varepsilon_{t-i} \qquad \text{—(1)}$$

$$y_t = \mu + \sum_{i=0}^{\infty} \psi_i B^i \varepsilon_t$$

$[B^i \varepsilon_t = \varepsilon_{t-i}]$
Backshift operator

$$y_t = \mu + \psi(B) \varepsilon_t$$

where $\psi(B) = \sum_{i=0}^{\infty} \psi_i B.$

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where $\{\varepsilon_t\}$ is white noise

$E(\varepsilon_t) = \text{mean} = 0$

$$\gamma_\varepsilon(h) \begin{cases} V(\varepsilon_t) = \sigma^2 & \text{for } h=0 \\ = 0 & \text{for } h \neq 0 \end{cases} \quad [\text{lag } h]$$

we create exponential decay pattern of weights.

For that we will set

$$\psi_i = \phi^i \quad ,$$

where $|\phi| < 1$ to guarantee the exponential "decay".

So, in this notation, the weights on the disturbances term starting from the current disturbance and going back in past will be $1, \phi, \phi^2, \phi^3, \ldots$.

Hence we can write eqn ① as

$$\left.\begin{aligned} y_t &= \mu + \varepsilon_t + \phi\,\varepsilon_{t-1} + \phi^2 \varepsilon_{t-2} + \cdots \\ y_t &= \mu + \sum_{i=0}^{\infty} \phi^i \varepsilon_{t-i} \end{aligned}\right\} \text{—②}$$

from eqn ②, we also have

$$y_{t-1} = \mu + \varepsilon_{t-1} + \phi\,\varepsilon_{t-2} + \phi^2 \varepsilon_{t-3} + \cdots \quad — ③$$

we can then combine eqn ② and ③ as

$$y_t = \mu + \varepsilon_t + \phi\,\varepsilon_{t-1} + \phi^2 \varepsilon_{t-2} + \cdots$$

$$y_t = \mu - \phi\mu + \phi\, y_{t-1} + \varepsilon_t$$

$$\boxed{y_t = \delta + \phi\, y_{t-1} + \varepsilon_t} \quad — ④ \Leftarrow$$

where $\delta = (1-\phi)\mu = \mu - \phi\mu$

The process in eqn ④ is called a first-order autoregressive process. AR(1) because it is a regression of y_t on y_{t-1}.

→ The assumption of $|\phi| < 1$ results in the weights that decay exponentially in time and also guaranteed that $\sum_{i=0}^{\infty} |\psi_i| < \infty$.

→ This means that an AR(1) process is stationary if $|\phi| < 1$.

The mean of a stationary AR(1) process

from eqn (4)

$$y_t = \delta + \phi y_{t-1} + \varepsilon_t$$

Taking expectation on both sides

$$E(y_t) = E(\delta) + \phi E(y_{t-1}) + E(\varepsilon_t)$$

$$\mu = \delta + \phi \mu + 0$$

$$\mu = \delta + \phi \mu$$

$$\mu - \phi \mu = \delta$$

$$(1 - \phi) \mu = \delta$$

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$$\boxed{\mu = \frac{\delta}{1-\phi}}$$ mean of AR(1) process — (5)

Variance of AR(1) process

constant
$Var(2) = 0$

$Var(ax) = a^2 Var(x)$

from eqn (4)

$$y_t = \delta + \phi y_{t-1} + \varepsilon_t$$

Taking variance on both sides

WN

$$Var(y_t) = Var(\delta) + Var(\phi y_{t-1}) + Var(\varepsilon_t)$$

$$Var(y_t) = \cancel{Var}\ 0 + \phi^2 Var(y_{t-1}) + \sigma^2$$

Assume stationary

$$Var(y_t) = Var(y_{t-1})$$

$$\therefore Var(y_t) = \phi^2 var(y_t) + \sigma^2$$

$$Var(y_t) - \phi^2 Var(y_t) = \sigma^2$$

$$Var(y_t)\left[1 - \phi^2\right] = \sigma^2$$

$$Var(y_t) = \frac{\sigma^2}{1-\phi^2}$$

or

$$Var(y_t) = \underline{\underline{\gamma(0)}} = \frac{\sigma^2}{1-\phi^2} \quad —(6)$$

Variance of $\underline{AR(1)}$ process.

{why it is $|\phi| < 1$} ?

$\phi = 1 \quad \gamma(0) = \infty$

$\phi > 1 \quad \gamma(0) = -ive$

$1 - \phi^2 < 1 \Rightarrow |\phi| < 1$

Stationary Cond.

Autocovariance function of a Stationary AR(1) process

From eqn (2) and eqn (3) we know that

$$y_t = \mu + \varepsilon_t + \phi \varepsilon_{t-1} + \phi^2 \varepsilon_{t-2} + \cdots$$

$$y_{t-1} = \mu + \varepsilon_{t-1} + \phi \varepsilon_{t-2} + \phi^2 \varepsilon_{t-3} + \cdots$$

$$\gamma_y(1) = \mathrm{cov}(y_t, y_{t-1})$$

$$\gamma_y(1) = E[(y_t - \mu)(y_{t-1} - \mu)]$$

$E(\varepsilon_t) = 0$

$$\gamma_y(1) = E\Big[(\varepsilon_t + \phi \varepsilon_{t-1} + \phi^2 \varepsilon_{t-2} + \cdots) \times (\varepsilon_{t-1} + \phi \varepsilon_{t-2} + \phi^2 \varepsilon_{t-3} + \cdots)\Big]$$

$$\gamma_y(1) = \phi E(\varepsilon_{t-1} \times \varepsilon_{t-1}) + \phi^3 E(\varepsilon_{t-2} \cdot \varepsilon_{t-2}) + \phi^5 E(\varepsilon_{t-3} \times \varepsilon_{t-3}) + \cdots$$

$$\gamma_y(1) = \phi \sigma^2 + \phi^3 \sigma^2 + \phi^5 \sigma^2 + \cdots$$

$$\gamma_y(1) = \phi \sigma^2 [1 + \phi^2 + \phi^4 + \cdots]$$

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$$\boxed{\gamma_y(1) = \frac{\phi^1 \sigma^2}{(1-\phi^2)}}$$

$$\left[\because (1-x^2)^{-1} = 1 + x^2 + (x^2)^2 + \cdots\right]$$

if we continue in the same way

$$\gamma_y(k) = \operatorname{cov}(y_t, y_{t-k})$$

$$\boxed{\gamma_y(k) = \phi^k \frac{\sigma^2}{1-\phi^2}} \quad \text{--- (7)}$$

Autocovariance of AR(1) process.

ACF

Auto correlation function for stationary time series AR(1) process

$$\rho(k) = \frac{\gamma(k)}{\gamma(0)}$$

from eqⁿ (6) & eqⁿ (7)

$$\rho(k) = \frac{\dfrac{\phi^k \cancel{\sigma^2}}{\cancel{1-\phi^2}}}{\dfrac{\cancel{\sigma^2}}{\cancel{1-\phi^2}}}$$

$$\boxed{\rho(k) = \phi^k} \quad \text{for } k = 0, 1, 2, \ldots$$

Hence ACF for an AR(1) process has an exponential decay form.

⇒ ACF of an AR(1) process is equal to the powers of the AR parameter of the process and decreases geometrically to zero.