Method of Moments for parameter estimation

Method of Moments Estimation

- One of the easiest methods of parameter estimation is the *method of moments* (MOM).
- The basic idea is to find expressions for the sample moments and for the population moments and equate them:

$$\frac{1}{n}\sum_{i=1}^{n} X_i^r = E(X^r)$$

- The $E(X^r)$ expression will be a function of one or more unknown parameters.
- If there are, say, 2 unknown parameters, we would set up MOM equations for $r = 1, 2$, and solve these 2 equations simultaneously for the two unknown parameters.
- In the simplest case, if there is only 1 unknown parameter to estimate, then we equate the sample mean to the true mean of the process and solve for the unknown parameter.

MOM with AR models

- First, we consider autoregressive models.
- In the simplest case, the $AR(1)$ model, given by $Y_t = \phi Y_{t-1} + e_t$, the true lag-1 autocorrelation $\boxed{\rho_1 = \phi}$.
- For this type of model, a method-of-moments estimator would simply equate the true lag-1 autocorrelation to the sample lag-1 autocorrelation r_1. or $\hat{\rho}_1$
- So our MOM estimator of the unknown parameter ϕ would be $\hat{\phi} = r_1$. or $\hat{\rho}_1$

MOM with an AR(2) model

- In the $AR(2)$ model, we have unknown parameters ϕ_1 and ϕ_2.
- From the Yule-Walker equations,

$$\rho_1 = \phi_1 + \rho_1\phi_2 \text{ and } \rho_2 = \rho_1\phi_1 + \phi_2$$

- In the method of moments, we will replace the true lag-1 and lag-2 autocorrelations, ρ_1 and ρ_2, by the sample autocorrelations r_1 and r_2, respectively.

or $\hat{\rho}_1$ or $\hat{\rho}_1$

MOM with an AR(2) model

- That gives the equations

$$r_1 = \phi_1 + r_1\phi_2 \text{ and } r_2 = r_1\phi_1 + \phi_2$$

which are then solved for ϕ_1 and ϕ_2 to obtain

$$\hat{\phi}_1 = \frac{r_1(1 - r_2)}{1 - r_1^2} \text{ and } \hat{\phi}_2 = \frac{r_2 - r_1^2}{1 - r_1^2}$$

- The general $AR(p)$ model is estimated in a similar way, with the Yule-Walker equations being used to obtain the *Yule-Walker estimates* $\hat{\phi}_1, \hat{\phi}_2, \ldots, \hat{\phi}_p$.

MOM with MA Models

- We run into problems when trying to using the method of moments to estimate the parameters of moving average models.

$y_t = \varepsilon_t - \theta \varepsilon_{t-1}$ or

- Consider the simple $MA(1)$ model, $Y_t = e_t - \theta e_{t-1}$.
- The true lag-1 autocorrelation in this model is $\rho_1 = -\theta/(1+\theta^2)$.

$\rho_1 = -\dfrac{\theta}{1+\theta^2}$

or $\hat{\rho}_1$

- If we equate ρ_1 to r_1, we get a quadratic equation in θ.

$r_1 + r_1\theta^2 + \theta = 0$

- If $|r_1| < 0.5$, then only one of the two real solutions satisfies the invertibility condition $|\theta| < 1$.

$r_1\theta^2 + \theta + r_1 = 0$

- That solution is $\hat{\theta} = \left(-1 + \sqrt{1 - 4r_1^2}\right)/(2r_1)$.

$\theta = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

- But if $|r_1| = 0.5$, no invertible solution exists, and if $|r_1| > 0.5$, then no real solution at all exists, and the method of moments fails to give any estimator of θ.

More MOM Problems with MA Models

- With higher-order $MA(q)$ models, the set of equations for estimating $\theta_1, \dots, \theta_q$ is highly nonlinear and could only be solved numerically.
- There would be many solutions, only one of which is invertible.
- In any case, for $MA(q)$ models, the method of moments usually produces poor estimates, so it is not recommended to use MOM to estimate MA models.

MOM Estimation of Mixed ARMA Models

- Consider only the simplest mixed model, the $ARMA(1,1)$ model.
- Since $\rho_2/\rho_1 = \phi$, a MOM estimator of ϕ is $\hat{\phi} = r_2/r_1$.
- Then the equation

$$r_1 = \frac{(1-\theta\hat{\phi})(\hat{\phi}-\theta)}{1-2\theta\hat{\phi}+\theta^2}$$

  can be used to solve for an estimate of θ.
- This is a quadratic equation is θ, and so we again keep only the invertible solution (if any exist) as our $\hat{\theta}$.

MOM Estimation of the Noise Variance

- We still must estimate the variance σ_e^2 of our error component.
- For any model, we first estimate the variance of the time series process itself, $\gamma_0 = var(Y_t)$, by the sample variance

$$\boxed{s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_t - \bar{Y})^2}$$

- Then we can take advantage of known relationships among the parameters in our specified model to obtain a formula for $\hat{\sigma}_e^2$.

Formulas for MOM Noise Variance Estimators in Common Models

- For $AR(p)$ models, $\hat{\sigma}_e^2 = (1 - \hat{\phi}_1 r_1 - \hat{\phi}_2 r_2 - \cdots - \hat{\phi}_p r_p)s^2$.
- For the $AR(1)$ model, this reduces to $\hat{\sigma}_e^2 = (1 - r_1^2)s^2$.
- For $MA(q)$ models,

$$\hat{\sigma}_e^2 = \frac{s^2}{1 + \hat{\theta}_1^2 + \hat{\theta}_2^2 + \cdots + \hat{\theta}_q^2}.$$

- For $ARMA(1,1)$ models,

$$\hat{\sigma}_e^2 = \frac{1 - \hat{\phi}^2}{1 - 2\hat{\phi}\hat{\theta} + \hat{\theta}^2} s^2.$$