Unit -3

ESTIMATION OF ARMA MODELS

Estimation of ARMA models: Yule- Walker estimation of AR Processes, Maximum likelihood and least squares estimation for ARMA Processes, Residual analysis and diagnostic checking.

ARMA (p,q) process

In general, an ARMA(p, q) model is given as

$$y_t = \delta + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \varepsilon_t - \theta_1 \varepsilon_{t-1} - \theta_2 \varepsilon_{t-2} - \cdots - \theta_q \varepsilon_{t-q}$$

$$= \delta + \sum_{i=1}^{p} \phi_i y_{t-i} + \varepsilon_t - \sum_{i=1}^{q} \theta_i \varepsilon_{t-i}$$ **...equation 1**

or

$$\Phi(B)\, y_t = \delta + \Theta(B)\, \varepsilon_t$$ **...equation 2**

where ε_t is a white noise process.

Estimation of ARMA models

$$
\begin{aligned}
y_t &= \delta + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \varepsilon_t - \theta_1 \varepsilon_{t-1} \\
&\quad - \theta_2 \varepsilon_{t-2} - \cdots - \theta_q \varepsilon_{t-q} \\
&= \delta + \sum_{i=1}^{p} \phi_i y_{t-i} + \varepsilon_t - \sum_{i=1}^{q} \theta_i \varepsilon_{t-i}
\end{aligned}
$$

is characterized by $\boxed{p + q + 1}$ unknown parameters

- $\phi = (\phi_1, \ldots, \phi_p)'$ $\checkmark$ $=$ $\phi = \begin{bmatrix} \phi_1 \\ \vdots \\ \phi_p \end{bmatrix}$
- $\theta = (\theta_1, \ldots, \theta_q)'$
- σ^2 $\checkmark$

that need to be estimated.

Estimation of ARMA models

We considers three techniques for estimation of the parameters ϕ, θ and σ^2They are:

1. Yule-Walker Estimation
2. Maximum Likelihood Estimation
3. least squares estimation

The Yule-Walker Estimation

Consider an autoregressive stochastic process x_t of order p. It is well known that there is a link among the autoregressive coefficients and the autocovariances.In particular, we have

$$\Gamma\phi = \gamma$$

and

$$\sigma^2 = \gamma(0) - \phi'\gamma$$

where

$$\Gamma = \begin{bmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_{p-1} \\ \gamma_1 & \gamma_0 & \cdots & \gamma_{p-2} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{p-1} & \gamma_{p-2} & \cdots & \gamma_0 \end{bmatrix}$$

is the covariance matrix and

$$\gamma = (\gamma_1, \ldots, \gamma_p)'$$

The Sample Yule-Walker Estimation

If we replace the theoretical autocovariances by the corresponding sample autocovariances,we obtain

$$\hat{\Gamma}\phi = \hat{\gamma}$$

where

$$\hat{\Gamma} = \begin{bmatrix} \hat{\gamma}_0 & \hat{\gamma}_1 & \cdots & \hat{\gamma}_{p-1} \\ \hat{\gamma}_1 & \hat{\gamma}_0 & \cdots & \hat{\gamma}_{p-2} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{\gamma}_{p-1} & \hat{\gamma}_{p-2} & \cdots & \hat{\gamma}_0 \end{bmatrix}$$

is the sample autocovariance matrix and

$$\hat{\gamma} = (\hat{\gamma}_1, \ldots, \hat{\gamma}_p)'$$

The Yule-Walker Estimation

Var.

We assume $\hat{\gamma}(0) > 0$.To obtain the Yule-Walker estimators as a function of the autocorrelation function, we divide the two sides of equation

$$\boxed{\hat{\Gamma}\phi = \hat{\gamma}}$$

by $\hat{\gamma}(0) > 0$.
We have

$$\hat{R}\phi = \hat{\rho}$$

Sample AC

where

$$\hat{R} = \begin{bmatrix} \hat{\rho}_0 & \hat{\rho}_1 & \cdots & \hat{\rho}_{p-1} \\ \hat{\rho}_1 & \hat{\rho}_0 & \cdots & \hat{\rho}_{p-2} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{\rho}_{p-1} & \hat{\rho}_{p-2} & \cdots & \hat{\rho}_0 \end{bmatrix}$$

is the sample autocorrelation matrix and

$$\hat{\rho} = (\hat{\rho}_1, \dots, \hat{\rho}_p)'$$

$$\boxed{\rho_k = \frac{\gamma(k)}{\gamma(0)}}$$

$$\phi = \begin{bmatrix} \phi_1 \\ \vdots \\ \phi_p \end{bmatrix}_{1 \times p}$$

$$\phi' = [\phi_1 \dots \phi_p]'$$

The Yule-Walker Estimation

It is possible to show that

$$\underline{\underline{\hat{\gamma}(0) > 0}} \Rightarrow \boxed{\det \hat{R} \neq 0}$$

The Yule-Walker Estimation

AR(p)

Thus we can solve the system

$$\hat{R}\phi = \hat{\rho}$$

obtaining the so-called Yule-Walker estimators, namely

$$\hat{\phi} = \hat{R}^{-1}\hat{\rho}$$

and

$$\hat{\sigma}^2 = \hat{\gamma}(0)\left[1 - \hat{\rho}'\hat{R}^{-1}\hat{\rho}\right]$$

The Yule-Walker Estimation

Theorem. If y_t is a zero-mean stationary autoregressive process of order p with $\varepsilon_t \sim iid(0, \sigma^2)$, and $\hat{\phi}$ is the Yule-Walker estimator of ϕ, then

$$T^{1/2}(\hat{\phi} - \phi) = \sqrt{n}(\hat{\phi} - \phi) \sim N(0, \sigma^2 \Gamma^{-1})$$

has a limiting normal distribution with mean $\mathbf{0}$ and covariance matrix $\sigma^2\Gamma^{-1}$. Moreover

$$\hat{\sigma}^2 \xrightarrow{P} \sigma^2$$

Thus, under the assumption that the order p of the fitted model is the correct value, we can use the asymptotic distribution of ϕ to derive approximate large-sample confidence regions for ϕ and for each of its components.

$$y_t = \underset{\phi_1}{\underline{0.75}}\, y_{t-1} - \underset{\phi_2}{\underline{0.50}}\, y_{t-2} + \varepsilon_t$$

$$\hat{\rho} = \begin{bmatrix} \hat{\rho}_0 \\ \hat{\rho}_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$$

$$\hat{R} = \begin{bmatrix} \hat{\rho}_0 & \hat{\rho}_1 \\ \hat{\rho}_1 & \hat{\rho}_0 \end{bmatrix} = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$$

$$\rho_1 = \frac{\phi_1}{1-\phi_2} = \frac{0.75}{1+0.5} = 0.5$$

$$\boxed{\hat{\phi} = \hat{R}^{-1}\hat{\rho}}$$

$$\begin{bmatrix} \hat{\phi}_1 \\ \hat{\phi}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} \quad — (1)$$

$$\begin{bmatrix} \hat{\phi}_1 \\ \hat{\phi}_2 \end{bmatrix} = \frac{1}{0.75} \begin{bmatrix} 1 & -0.5 \\ -0.5 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} = \frac{1}{0.75} \begin{bmatrix} 0.75 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$1 \quad 0.5$

estimator

$$\hat{\phi}_1 = 1$$

$$\hat{\phi}_2 = 0$$

$$A^{-1} = \frac{\operatorname{Adj} A}{|A|}$$

$|A| \neq 0$ (inverse exist)

$$|\hat{R}| = \begin{vmatrix} 1 & 0.5 \\ 0.5 & 1 \end{vmatrix} = 1 - 0.25 = 0.75$$

$$\hat{R}^{-1} = \frac{\operatorname{Adj} \hat{R}}{|\hat{R}|} = \frac{1}{0.75} \begin{bmatrix} 1 & -0.5 \\ -0.5 & 1 \end{bmatrix} =$$