LINEAR MODELS FOR STATIONARY TIME SERIES

In statistical modeling, we are often engaged in an endless pursuit of finding the ever elusive true relationship between certain inputs and the output. These efforts usually result in models that are nothing but approximations of the “true” relationship. This is generally due to the choices the analyst makes along the way to ease the modeling efforts. A major assumption that often provides relief in modeling efforts is the linearity assumption. A linear filter, for example, is a linear operation from one time series x_t to another time series y_t.

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$$y_t = L(x_t) = \sum_{i=-\infty}^{+\infty} \psi_i x_{t-i} \tag{5.1}$$

with $t = \ldots, -1, 0, 1, \ldots$. In that regard the linear filter can be seen as a "process" that converts the input, x_t, into an output, y_t, and that conversion is not instantaneous but involves all (present, past, and future) values of the input in the form of a summation with different "weights", $\{\psi_i\}$, on each x_t. Furthermore, the linear filter in Eq. (5.1) is said to have the following properties:

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1. **Time-invariant** as the coefficients $\{\psi_i\}$ do not depend on time.
2. **Physically realizable** if $\psi_i = 0$ for $i < 0$; that is, the output y_t is a linear function of the current and past values of the input: $y_t = \psi_0 x_t + \psi_1 x_{t-1} + \cdots$.
3. **Stable** if $\sum_{i=-\infty}^{+\infty} |\psi_i| < \infty$.

In linear filters, under certain conditions, some properties such as stationarity of the input time series are also reflected in the output.

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For a time-invariant and stable linear filter and a stationary input time series x_t with $\mu_x = E(x_t)$ and $\gamma_x(k) = \mathrm{Cov}(x_t, x_{t+k})$, the output time series y_t given in Eq. (5.1) is also a stationary time series with

$$E(y_t) = \mu_y = \sum_{-\infty}^{\infty} \psi_i \mu_x$$

and

$$\mathrm{Cov}(y_t, y_{t+k}) = \gamma_y(k) = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \psi_i \psi_j \gamma_x(i - j + k)$$

It is then easy to show that the following stable linear process with white noise time series, ε_t, is also stationary:

$$y_t = \mu + \sum_{i=0}^{\infty} \psi_i \varepsilon_{t-i} \tag{5.2}$$

with $E(\varepsilon_t) = 0$, and

$$\gamma_\varepsilon(h) = \begin{cases} \sigma^2 & \text{if } h = 0 \\ 0 & \text{if } h \neq 0 \end{cases}$$

So for the autocovariance function of y_t, we have

$$\begin{aligned} \gamma_y(k) &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \psi_i \psi_j \gamma_\varepsilon(i - j + k) \\ &= \sigma^2 \sum_{i=0}^{\infty} \psi_i \psi_{i+k} \end{aligned} \tag{5.3}$$

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We can rewrite the linear process in Eq. (5.2) in terms of the **backshift operator**, B, as

$$\begin{aligned} y_t &= \mu + \psi_0 \varepsilon_t + \psi_1 \varepsilon_{t-1} + \psi_2 \varepsilon_{t-2} + \cdots \\ &= \mu + \sum_{i=0}^{\infty} \psi_i B^i \varepsilon_t \\ &= \mu + \underbrace{\left(\sum_{i=0}^{\infty} \psi_i B^i \right)}_{=\Psi(B)} \varepsilon_t \\ &= \mu + \Psi(B)\, \varepsilon_t \end{aligned} \tag{5.4}$$

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This is called the **infinite moving average** and serves as a general class of models for any stationary time series. This is due to a theorem by Wold (1938) and basically states that **any** nondeterministic weakly stationary time series y_t can be represented as in Eq. (5.2), where $\{\psi_i\}$ satisfy $\sum_{i=0}^{\infty} \psi_i^2 < \infty$.

White noise

The concept of white noise is essential for time series analysis and forecasting. In the most simple words, white noise tells you if you should further optimize the model or not.

White noise is a series that's not predictable, as it's a sequence of random numbers. If you build a model and its residuals (the difference between predicted and actual) values look like white noise, then you know you did everything to make the model as good as possible. On the opposite side, there's a better model for your dataset if there are visible patterns in the residuals.

The following conditions must be satisfied for a time series to be classified as white noise:

- The average value (mean) is zero
- Standard deviation is constant — it doesn't change over time
- The correlation between time series and its lagged version is not significant.

White noise

There are three (easy) ways to test if time series resembles white noise:

- By plotting the time series
- By comparing mean and standard deviation over time
- By examining autocorrelation plots