Difference operator (Backshift)

we know that

$$\nabla y_t = y_t - y_{t-1}$$

where ∇ is the (backward) difference operator.

Another way to write it in terms of a backshift operator B, defined as

$$B y_t = y_{t-1}, \text{ so}$$

$$\nabla y_t = y_t - y_{t-1}$$

$$\nabla y_t = y_t - B y_t$$

$$\boxed{\nabla y_t = (1-B) y_t}$$

with $\nabla = 1-B$.

Differencing can be performed successively if necessary, ~~until the~~ the second difference is

$$\nabla^2 y_t = \nabla(\nabla y_t) = (1-B)^2 y_t$$

$$= (1 - 2B + B^2) y_t$$

$$\nabla^2 y_t = y_t - 2B y_t + B^2 y_t$$

$$\nabla^2 y_t = y_t - 2 y_{t-1} + y_{t-2}$$

$$\boxed{\nabla^2 y_t = y_t - 2y_{t-1} + y_{t-2}}$$

In general, powers of the backshift operator and the backward difference operator are defined as

$$\boxed{B^i y_t = y_{t-i}}$$

$$\nabla^i y_t = (1-B)^i y_t$$

with $\nabla^i = (1-B)^i$

Finite Order Moving Average Processes MA(q)

we know that

$$y_t = \mu + \sum_{i=0}^{\infty} \psi_i \varepsilon_{t-i} \quad -(1)$$

$\boxed{5.2}$

ε_t = white noise
with $E(\varepsilon_t) = 0$

lag h

$$\gamma_\varepsilon(h) = \begin{cases} \sigma^2 & \text{if } h = 0 \\ 0 & \text{if } h \neq 0 \end{cases}$$

eqn (1) can be written as

$$y_t = \mu + \underbrace{\psi_0}_{1} \varepsilon_t + \underbrace{\psi_1}_{-\theta_1} \varepsilon_{t-1} + \underbrace{\psi_2}_{-\theta_2} \varepsilon_{t-2} + \cdots$$

In finite order moving average or conventionally MA models,

$\boxed{\psi_0 \text{ is set to } 1}$ and the weights that are not to '0' are represented by the Greek letter θ with a minus (−) sign in front.

Hence a moving average process of order q [MA(q)] is given as

$$\boxed{y_t = \mu + \varepsilon_t - \theta_1 \underbrace{\varepsilon_{t-1}}_{B\varepsilon_t} - \cdots - \theta_q \underbrace{\varepsilon_{t-q}}_{B^q \varepsilon_t}} \quad -(2)$$

where $\{\varepsilon_t\}$ is white noise.

Equation (2) is a special case of Equation (1) with only finite weights,

an MA(q) process is always stationary regardless of values of the weights.

In terms of the backward shift operator, the MA(q) process is

$$y_t = \mu + \varepsilon_t - \theta_1 B \varepsilon_t - \cdots - \theta_q B^q \varepsilon_t$$

$[\because \; B^i \varepsilon_t = \varepsilon_{t-i}]$

$$y_t = \mu + (1 - \theta_1 B - \cdots - \theta_q B^q)\varepsilon_t$$

$$\boxed{y_t = \mu + \left(1 - \sum_{i=1}^{q} \theta_i B^i\right) \varepsilon_t} \quad -(3)$$

~~$y_t = \mu + (1 -$~~

$$y_t = \mu + \Theta(B)\varepsilon_t$$

where $\Theta(B) = 1 - \sum_{i=1}^{q} \theta_i B^i$

$\because \{\varepsilon_t\}$ is white noise, the expected value of the MA(q) process is simply

Mean

$E(ax) = aE(x)$

$$E(y_t) = E(\mu + \varepsilon_t - \theta_1 \varepsilon_{t-1} - \cdots - \theta_q \varepsilon_{t-q})$$

$$E(y_t) = E(\mu) + E(\varepsilon_t) - \theta_1 E(\varepsilon_{t-1}) - \cdots - \theta_q E(\varepsilon_{t-q})$$

$$\boxed{E(y_t) = \mu} \quad -(4)$$

$(\because E(\varepsilon_t) = 0$
$E(\varepsilon_{t-1}) = 0$ and so on$)$.

$\checkmark$ $Var(ax) = a^2 Var(x)$

and its $\boxed{\text{variance}}$ is from eq 2

$$Var(y_t) = \gamma_y(0) = Var(\mu + \varepsilon_t - \theta_1 \varepsilon_{t-1} - \cdots - \theta_q \varepsilon_{t-q})$$

lag

$$\gamma_y(0) = Var(\mu) + Var(\varepsilon_t) + \theta_1^2 Var(\varepsilon_{t-1}) + \cdots + \theta_q^2 Var(\varepsilon_{t-q})$$

$$\gamma_y(0) = 0 + \sigma^2 + \theta_1^2 \sigma^2 + \cdots + \theta_q^2 \sigma^2$$

$$\left[\because Var(ax) = a^2 Var(X)\right]$$

$$\boxed{\gamma_y(0) = \sigma^2 (1 + \theta_1^2 + \cdots + \theta_q^2)} \quad -(5)$$

Variance of MA(q) process $\checkmark$

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Similarly, the $\boxed{\text{auto covariance}}$ at $\boxed{\text{lag } k}$ can be calculated from

$$\boxed{\gamma_y(k) = \mathrm{cov}(y_t, y_{t+k})}$$

eq 2

$$\gamma_y(k) = E\left[\left(\varepsilon_t - \theta_1 \varepsilon_{t-1} - \dots - \theta_q \varepsilon_{t-q}\right) \times \left(\varepsilon_{t+k} - \theta_1 \varepsilon_{t+k-1} - \dots - \theta_q \varepsilon_{t+k-q}\right)\right]$$

Autocovariance function of y_t

$$\gamma_y(k) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \psi_i \psi_j \gamma_\varepsilon(i-j+k)$$

$$\gamma_y(k) = \sigma^2 \sum_{i=0}^{\infty} \psi_i \psi_{i+k}$$

$$\gamma_y(k) = \sigma^2 \left[\psi_0 \psi_k + \psi_1 \psi_{k+1} + \dots + \psi_q \psi_{k+q} + \dots\right]$$

$$= \sigma^2 \left[-\theta_k + \theta_1 \theta_{k+1} + \dots + \theta_q \theta_{k+q}\right]$$

Put $\psi_0 = 1$ and other weight in Greek letter θ with a minus $(-)$ sign in front.

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$$\gamma_y(k) = \sigma^2\left[-\theta_k + \theta_1\theta_{k+1} + \theta_2\theta_{k+2} + \cdots + \theta_q\theta_{q-k}\right]$$

$$\gamma_y(k) = \begin{cases} \sigma^2\left[-\theta_k + \theta_1\theta_{k+1} + \cdots + \theta_q\theta_{q-k}\right], & k = 1, 2, \ldots, q \\ 0, & k > q \end{cases} \quad -(6)$$

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From equation (5) and (6), the

{auto correlation function (ACF)} of

the MA(q) process is

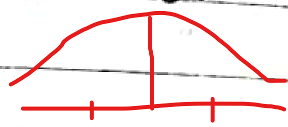
$$\rho_y(k) = \frac{\gamma_y(k)}{\gamma_y(0)}$$

$$\rho_y(k) = \begin{cases} \dfrac{-\theta_k + \theta_1\theta_{k+1} + \cdots + \theta_q\theta_{q-k}}{1 + \theta_1^2 + \cdots + \theta_q^2}, & k = 1, \ldots q \\ 0, & k > q \end{cases} \quad \text{—(7)}$$

ACF is very helpful in identifying the MA model and its appropriate order as it "cut off" after lag q.

In real life, the sample ACF, $r(k)$, will not necessarily be equal to zero after lag q.

It is expected to become very small absolute value after lag q.

For a data set of N observations, this is often tested against 

$\boxed{\pm \frac{2}{\sqrt{N}} \text{ limits,}}$ where $\frac{1}{\sqrt{N}}$ is the approximate value for standard deviation of the ACF for any lag under the assumption $\rho(k) = 0$ for all k's.