Numerical Methods (Math 385/685)

Homework 6

May 6, 2019

Due: Sunday, May 19th

Problems:

1. Prove *Theorem 3.4.2* by following the given outline:

   (a) Determine the $\max_{t\in[0,1]} |p_i(t)|$ for the four b-spline basis functions.

   (b) Verify that $\sum_{i=1}^{4} p_i(t) = 1$

   (c) Let $\sigma_i = (\sigma_{i,1}(t), \sigma_{i,2}(t))$ be a B-spline segment for σ_p and take $t_0 \in [0,1]$. Prove that if $x_0 = \sigma_{i,1}(t_0)$, then

   $$|f(x_0) - \sigma_{i,2}(t_0)| \leq \sum_{k=0}^{3} |p_k(t_0)||f(x_0) - f(x_{i+k})|.$$

   (d) Complete the proof by identifying a constant C depending on f that satisfies

   $$|f(x_0) - \sigma_{i,2}(t_0)| \leq C\Delta$$

   where $\Delta = \max_i(x_{i+1} - x_i)$

2. Let $f(x) = xe^{-x} - 1$ and set the interval to $[1,4]$. Compute the integral of f using the two point Gaussian quadrature. Compare these results with those obtained by Trapezoidal method in Homework 5.

3. The growth of a tumor can be modeled by the following ordinary differential equation

   $$\frac{dU}{dt} = aU^{\alpha} - bU^{\beta}$$

where α and β are chosen with respect to the tumor geometry,i.e U^α is the subpopulation of dividing cells and U^β is the aging subpopulation. The mitosis rate and death rate are give by a and b. A tumor initially consisting of 10 cells has the following parameters, $a = 5, b = 0.01, \alpha = 1$ and $\beta = 2$. Estimate the number of cells present after 10 iterations with a time step of 0.1 and

(a) Forward Euler

(b) A single corrector

(c) Midpoint scheme.

(d) Calculate the absolute error at the 10^{th} iteration of your prediction in each case (a-c) using the exact solution $U(t) = \frac{5000e^{5t}}{500+10(e^{5t}-1)}$. Discuss the accuracy of each scheme.

4. Consider the first order wave equation

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial u}{\partial x}$$

(a) Using forward difference in time and forward difference in space, render this PDE in finite difference format. In this case, it is usual to write $C = \frac{\alpha \Delta t}{\Delta x}$. C is called the *Courant number*.

(b) Execute FDM for the following setting.

$$\alpha = -1/2 \text{ and } \alpha = 1/2, \text{ Region} : [0, 300], \text{ Time interval} : [0, 5].$$

Initial conditions:

$$u(x, 0) = \begin{cases} 0 & \text{if } 0 \leq x \leq 50 \text{ or } 110 \leq x \leq 300 \\ 100 \sin[\pi \frac{(x-50)}{60}] & \text{if } 50 \leq x \leq 110 \end{cases}$$

and boundary condition:

$$u(0, t) = 0 \text{ and } u(300, t) = 0.$$

Set $\Delta x = 5, \Delta t = 0.00015$

(c) Use B-splines to display the output at times 1,3 and 5.

(d) Discuss the quality of your solutions for the two different values of α at $t = 5$

5. Consider the wave equation in problem 3,

(a) Execute Neumann stability analysis for $\alpha < 0$ and $\alpha > 0$

(b) What is the difference between the two cases and how does it compare with the computed solutions from problem 4.

Graduate Students

(a) Prove that the Crank Nicolson FDM applied to the 1D heat equation is unconditionally stable.

(b) use the discrete Fourier interpolation to get expressions for

$$\frac{|c_k^{n+1}|}{|c_k^{n+1/2}|}, \frac{|c_k^{n+1/2}|}{|c_k^{n}|}$$

(c) Multiply the two expressions to get an expression for $\frac{|c_k^{n+1}|}{|c_k^{n}|}$