Numerical Methods (Math 385/685)

Homework 5

April 5, 2019

Due: Sunday, April 21^{st}

Problems:

1. For $f(x) = xe^{-x} - 1$

   (a) Calculate the cubic Taylor polynomial of f about x= 2.5. Plot f and the Taylor polynomial on the same axis. Calculate the norm error and mean norm error for the approximation over the interval $[1, 4]$.

   (b) Compute the Lagrange polynomial interpolation,p of $f(x)$ using the nodes $x = 1, 2, 3, 4$. Plot f and p on the same axes for the interval $[1, 4]$. Calculate the norm error and mean norm error of the approximation.

   (c) Is p better or worse than the cubic Taylor interpolant?

   (d) Use Equation 3.1.2 in the text to estimate an upper bound for the maximal absolute error for the Lagrange interpolation.

2. Consider the function $f(x) = \frac{1}{1+x^2}$ on the interval $[-4, 4]$.

   (a) Determine the maximal value for $\mu = f^{(3)}(x)$ on the interval.

   (b) Divide the interval into 40 subintervals of length 0.2. In particular determine a partition $-4 = a_0 < a_1 < \cdots < a_{40} = 4$ with each $a_{k+1} = a_k + 0.2$

   (c) Compute the second degree polynomial interpolation of f on the subinterval $[a_k, a_{k+1}]$ using the three values a_k, $(a_k+a_{k+1})/2$, a_{k+1}, $k = 0, 1, 2, \cdots 39$.

(d) Plot the result of Part c and overlay the plot of f.

(e) Use Theorem 3.1.3 in the text to prove that the absolute error $|e(x)|$ is bounded by $\frac{\mu(0.2)^3}{6}$, where μ is the value computed in Part a.

3. (a) Verify that the Hermite cubics (page 73 of text) determine a Hermite interpolation on the interval $[\alpha, \beta]$

   (b) Repeat problem 4(c) but this time use the Hermite cubics to interpolate the function over each subinterval $[a_k, a_{k+1}]$.

4. The data below was generated from the function $f(x) = e^{2x} - \cos(2x)$. Use the forward, backward and central difference schemes to estimate the derivative at each point. Calculate the error bounds for each of the schemes you have used and comment on which scheme is most accurate.

x	$f(x)$	$f'(x)$
-0.3	-0.27652	
-0.2	-0.25074	
-0.1	-0.16134	
0	0	

5. Let $f(x) = xe^{-x} - 1$ and set the interval to $[1, 4]$. Consider the partition $1 < 1.5 < 2 < 2.5 < 3 < 3.5 < 4$ with 6 subintervals.

   (a) Write a *Mathematica* code to compute the integral of f using trapezoid method for the given partition. Your code should work for any number of subintervals.

   (b) Use the *Mathematica* inbuilt function *Integrate* to calculate the true value of the integral. Calculate the absolute error for your estimate in part (a).

   (c) Repeat the computation in part(a) but this time with $n = 10, 14, 18, 20$ subintervals. Calculate the absolute error in each case and comment on your results.

   (d) Estimate upper bounds for the numerical integration error in your estimates from part (a) using the bounds derived in class. Compare your bounds with your absolute errors calculated in part(c) and comment on the appropriateness of the bounds.

Graduate

1. Prove that for the piece-wise polynomial interpolation in problem 4, as the number of subintervals goes to ∞, then the error converges to zero.
2. Complete the proof of Theorem 3.5.2 in the text.