Numerical Methods (Math 385/685)

Homework 3

February 27, 2019

Due: Sunday, March 10th

Problems:

1. Find the LU decomposition of the following matrices using Mathematica's inbuilt function. Write out the matrices, L, U and P. Verify your decomposition by showing that $PA = LU$.

   (a) $\begin{pmatrix} 1. & 1. & 0 \\ 1 & 1. & 3 \\ 0 & 1. & -1 \end{pmatrix}$

   (b) $\begin{pmatrix} 1. & 2. & 1 & 7 \\ 2 & 0. & 1 & 4 \\ 1 & 0 & 2 & 5 \\ 1 & 2 & 3 & 11 \end{pmatrix}$

   (c) $\begin{pmatrix} 1. & 2. & 3 \\ 1 & 1 & 1 \\ 9 & 7 & 9 \end{pmatrix}$

2. The *Mathematica* function *Eigensystem* returns n+1 vectors when an nxn matrix is supplied. The entries of the first vector are the eigenvalues of A. The remaining vectors are the corresponding eigenvectors. Apply *Eigensystem* to the matrices listed in problem 1. Recall that a matrix is singular if it has zero as an eigenvalue. Also when looking at computer output a number close to zero should be considered as zero.

(a) Which of the matrices in problem 1 are singular?

(b) Which of the matrices in problem 1 are ill conditioned?

3. Let $A = [a_{i,j}]$ be an $n \times n$ matrix with real entries. Suppose that there is an m with $a_{i,j} = 0$ for $i \geq m, j \leq m$ and $a_{i,i} \neq 0$ for $1 \leq i < m$. Prove that A is singular.

4. Prove that the inverse of a triangular matrix is a triangular matrix of the same type. *(Hint: The product of a matrix and its inverse should produce an identity matrix)*

5. Suppose that an object can be at anyone of $n+1$ equally spaced points $x_0, x_1, x_2, \cdots, x_n$. When an object is at location x_j it is equally likely to move to either x_{i-1} or x_{i+1} and cannot directly move to any other location. Consider the probability P_i that an object starting at location x_i will reach the left end point x_0 before reaching the right end point x_n.Clearly $P_0 = 1$ and $P_n = 0$. Since the object can move to x_i only from x_{i-1} or x_{i+1} and does so with probability $1/2$, then the equation

$$P_i = \frac{1}{2}P_{i-1} + \frac{1}{2}P_{i+1}$$

holds for each location $i = 1, 2, \cdots, n-1$

(a) Set up a linear system $Ax = b$ for the problem.

(b) Execute *LinearSolve* for the linear system to calculate the respective probabilities for $n = 10$.

(c) From your results from part (b), what is the probability that from x_5 an object will reach x_0 before x_{10}

(d) Repeat part (b) for $n = 100, 1000, 10000, 100000$. Use the *Timing* function to determine the required CPU time. Plot CPU time against the matrix size.

(e) Comment on your results from part(d).

Graduate Students

1. Prove that given a normed linear space, V,

$$d(u, v) = \| u - v \|$$

defines a metric for $u, v \in V$.

2. Prove that for real or complex matrices

$$\| A \| \| B \| \geq \| AB \| \ .$$

3. Complete the proof. If A is an $n \times n$ real symmetric matrix with orthonormal eigenvectors $v_1, v_2, \cdots v_n$ then

$$\| A^k \| = \| A \|^k \ .$$