ECON0057 Lecture 4

Heterogeneous Agents Model

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2025

Introduction

- Previous Lectures: Income Fluctuations Problem, exogenous prices (interest rate, wage)
- To determine prices we need a General Equilibrium model with market clearing conditions
- Goal of the lecture: incorporate Heterogeneous Agents in macro models
- How risk/heterogeneity impacts capital accumulation, business cycle fluctuations, inequality

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Model: Households

- Mass 1 of households
- Households **inelastically** supply 1 unit of labor:
  - Labor income is given by $w_t s_t$ with w_t wage at t, s_t household productivity
  - s_t is stochastic and follows a Markov process with kernel $Q(s, s')$
- Households save in a risk free asset a_t which pays $1 + r_t$ at the beginning of the period
  - Different timing compared to Lecture 3
  - In Aiyagari, the risk-free asset a_t is interpreted as capital k_t, rented to a representative firm
  - The return r_t is therefore $r_t^k - \delta$ where r_t^k is the rental rate of capital and δ the depreciation rate
- Households are subject to a borrowing constraint $a_t \geq -\phi$ at all t

Households' Problem

Households solve:

$$V\left(a_0, s_0, \{r_t, w_t\}_{t\geq 0}\right) \sup_{\{c_t\}_{t\geq 0}} \mathbb{E}_0 \left(\sum \beta^t u(c_t)\right)$$

$$s.t \quad a_{t+1} = (1+r_t)a_t + w_t s_t - c_t \quad a_{t+1} \geq -\phi, \quad a_0,\ s_0 \ given$$

The consumption and savings of the agent now depend **on the full sequence of prices** $\{r_t, w_t\}_{t\geq 0}$

The solution of the sequential problem can be written:

$$c_t^*\left(a_0, s^t, \{r_s, w_s\}_{s\geq 0}\right)$$
$$a_{t+1}^*\left(a_0, s^t, \{r_s, w_s\}_{s\geq 0}\right)$$

Or in the recursive formulation (Bellman):

$$c_t\left(a, s, \{r_{t+s}, w_{t+s}\}_{s\geq 0}\right)$$
$$a_{t+1}\left(a, s, \{r_{t+s}, w_{t+s}\}_{s\geq 0}\right)$$

If the Principle of Optimality applies:

$$c_t^*\left(a_0, s^t, \{r_t, w_t\}_{t\geq 0}\right) = c_t\left(a_t^*\left(a_0, s^{t-1}, \{r_t, w_t\}_{t\geq 0}\right), s_t, \{r_{t+s}, w_{t+s}\}_{s\geq 0}\right)$$
$$a_{t+1}^*\left(a_0, s^t, \{r_t, w_t\}_{t\geq 0}\right) = a_{t+1}\left(a_t^*\left(a_0, s^{t-1}, \{r_t, w_t\}_{t\geq 0}\right), s_t, \{r_{t+s}, w_{t+s}\}_{s\geq 0}\right)$$

Distribution of Households' Productivity

Denote by $\pi_0(s)$ the initial distribution of productivity, and $\pi_t(s)$ the distribution at t

Recall that productivity at t, s_t, is exogenous and Markov with kernel Q:

$$\begin{aligned}\pi_{t+1}(s_{t+1}) &= \int Q(s_t, s_{t+1})\pi_t(s_t)ds_t \\ &= \int Q(s_t, s_{t+1})Q(s_{t-1}, s_t)...Q(s_0, s_1)\pi_0(s_0)ds_t ds_{t-1}...ds_0\end{aligned}$$

The distribution of productivity could change from period to period (but only depends on $\pi_0(s)$).

To simplify, we are assuming that $\pi_0(s) = \pi^*(s)$ where π^* is a **stationary distribution** for Q:

$$\pi^*(s') = \int Q(s, s')\pi^*(s)ds$$

We therefore have:

$$\pi_{t+1}(s_{t+1}) = \pi^*(s_{t+1}) = \int Q(s_t, s_{t+1})Q(s_{t-1}, s_t)...Q(s_0, s_1)\pi^*(s_0)ds_t ds_{t-1}...ds_0$$

Individual productivity fluctuates but *the distribution of productivity stays the same.*

Firm Side

A representative firm rents capital and labor from households to produce a single good y_t:

$$y_t = F(K_t, L_t)$$

F is concave, satisfies Inada and has constant returns to scale $(F(aK_t, aL_t) = aF(K_t, L_t))$.

- Since $\pi_t = \pi^*$ (stationary distribution of s), effective labor supply, N_t, is constant:

$$N_t = N^* = \int s\pi^*(s)ds$$

- We normalize average productivity $\int s\pi^*(s)ds = 1$, so labor market clearing implies:

$$L_t = N^* = 1$$

- The representative firm pays their marginal productivity to labor and capital:

$$\begin{aligned} r_t &= r_t^k - \delta = F_K(K_t, 1) - \delta \\ w_t &= F_L(K_t, 1) \end{aligned}$$

First equation defines demand for capital $K(r_t)$, second defines w_t as a function of r_t, $w(r_t)$.

From Inada and concavity, $K(r)$ **decreasing**, with $\lim_{-\delta} K(r) = \infty$, $\lim_{\infty} K(r) = 0$ and $w(r)$ **decreasing**, with $\lim_{-\delta} w(r) = \infty$,$\lim_{\infty} w(r) = 0$

Supply of Capital

At t, the supply of capital is determined by the *distribution* of households' asset holdings a_t

Denote by $\lambda_t(a, s)$ the joint distribution of assets and productivity:

$$A_t = \mathbb{E}(a_t) = \int a\lambda_t(a, s)dads$$

Asset market clearing implies that r_t is determined by $K(r_t) = \mathbb{E}(a_t)$

r_t therefore depends on the distribution of agents λ_t: $r_t = r(\lambda_t)$, similarly $w_t = w(\lambda_t)$.

Agents' choices at t depend on the full sequence $\{\lambda_t, \lambda_{t+1}...\}$!

Note that λ_{t+1} is determined by the agents' choice at t, indeed $a_{t+1} = a(a_t, s_t, \{\lambda_{t+s}\}_{s\geq 0})$, so:

$$\lambda_{t+1}(a' \in A, s') = \int \lambda_t(a_t^{-1}(a' \in A, s, \{\lambda_{t+s}\}_{s\geq 0}), s)Q(s, s')ds$$

Where a_t^{-1} is wealth a_t such that $a' = a(a_t, s_t, \{\lambda_{t+s}\}_{s\geq 0})$

Supply of Capital

Since λ_{t+1} is determined by the agents' choice at t, it depends implicitly on λ_t and future distribution $\{\lambda_{t+1+s}\}_{s>0}$)

We can derive similar recursions at all dates, implicitly all λ_{t+s} are functions of λ_t only, and $a_{t+1} = a(a_t, s_t, \lambda_t)$.

This means that the Bellman equation of the agent has an infinite dimensional state variable, λ_t!

To avoid this issue we will focus first on a **stationary steady state**:

- We will consider an equilibrium where $r_t = r$
- Such that $\lambda_{t+1} = \lambda_t$ at all dates
- We can drop λ_t from Bellman (the agent's solution only depends the steady state r)
- We still need to make sure that for r^*, a stationary λ_t exists
- That is, given r and the agents' choice, can we find λ_0 s.t. the distribution does not move?

Stationary Equilibrium

Define $\Pi_r(a', s', a, s)$ the transition kernel for the process (a, s) which depends on 1.the kernel Q; 2.the agent policy function $a' = a(a, s, r)$ given the constant interest rate r.

This (endogenous) transition kernel $\Pi_r(a', s', a, s)$ characterizes the evolution of $\lambda_t(a, s)$:

$$\lambda_{t+1}(a', s') = \int \lambda_t(a, s)\Pi_r(a', s', a, s)da\, ds$$

Stationary Equilibrium

A Stationary Equilibrium is 1.a policy function $a^*(a, s, r)$; 2.a distribution $\lambda_r^*(a, s)$; and 3. a triple $\{K, r, w\}$ such that:

1. $a^*(a, s, r)$ solves the household problem given r and w
2. $\lambda_r^*(a, s)$ is stationary for the transition kernel $\Pi_r(a', s', a, s)$ induced by Q and $a(a, s, r)$
$$\lambda_r^*(a', s') = \int \lambda_r^*(a, s)\Pi_r(a', s', a, s)da\, ds$$
3. r and w satisfies $r = F_K(K, 1) - \delta$, $w = F_L(K, 1)$
4. Capital market clears, $K = \int a\, \lambda_r^*(a, s)da\, ds$.

And the good market ?

Remember that by Walras law demand for good will automatically be verified. We can verify this using the agent's problem:

$$\begin{aligned}
0 &= \int (a_{t+1} - (1+r)a_t - ws_t + c_t)\lambda_r^*(a,s) da\, ds \\
&= K_{t+1} - (1+r)K_t - wL_t + C_t \\
&= K_{t+1} - r^k K_t - wL_t - (1-\delta)K_t + C_t \\
\Rightarrow Y_t + (1-\delta)K_t &= K_{t+1} + C_t
\end{aligned}$$

The second line uses the stationarity of λ^*, the third the definition of r^k, the fourth the fact that F has constant returns (so $Y = r^k K + wL$).

Aggregate resources is equal to aggregate investment plus aggregate consumption

$\rightarrow$ the good market clears.

Computing the Equilibrium

Idea to compute the equilibrium (details in tutorials):

1. Start from a guess for aggregate capital K_0, then given K_i
2. Compute the prices r_i, w_i using the firm FOC
3. Solve the agents value function and policy function $V_i(a, s, r_i)$ and $a_i(a, s, r_i)$
4. Find the stationary distribution implied by $Q(s, s')$ and $a_i(a, s, r_i)$, $\lambda_i^*(a, s, r_i)$
5. Compute aggregate supply of capital:

$$K_i^s = \int a\lambda_i^*(a, s, r_i)\, da\, ds$$

6. Look at excess supply $\epsilon = K_i^s - K_i$
7. If $|\epsilon| < h$ where h is a prefixed tolerance level stop

   Else $K_{i+1} = \rho K_i + (1-\rho)K_i^s$ with $0 < \rho < 1$ and restart from 1

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Simplified Problem

To study our economy, (and prove existence of an equilibrium) we make some simplifying assumptions:

- s_t is iid, discrete, bounded with $0 < \underline{s} \leq s_t \leq \bar{s}$, with distribution π
- u is CRRA, $u(c) = c^{1-\gamma}/(1-\gamma)$

Since s is iid, we know that we can restate the agent's problem in terms of cash on hand:

$$z_t = (1+r)a_t + ws_t$$
$$V(z) = \sup_{a' > -\phi} u(z - a') + \beta \mathbb{E}\left(V((1+r)a' + ws')\right)$$

From Lecture 3 $a(z)$ is non decreasing and $c(z)$ is increasing and if ϕ is stricter than the NBR then we have $-(1+r)\phi + w\underline{s} < z^*$ such that the agent is constrained for $z \leq z^*$.

Note that the NBR depends on w and is given by $w\underline{s}/r$:

- We need the borrowing constraint ϕ to be weakly stricter than the NBR
- We redefine the borrowing constraint as $\tilde{\phi} = min(\phi, w\underline{s}/r)$ (and $\tilde{\phi} = \phi$ when $r \leq 0$)

Plan for Proving existence

To show that a stationary equilibrium exists, we simply need to study asset market clearing.

Demand for capital $K(r)$ is simple: it is a decreasing function that goes to 0 at ∞ and ∞ at $-\delta$

Capital supply $Ea(r) = \int a\lambda_r^*(a,s)da\,ds$, is more complex as it depends on the endogenous stationary distribution $\lambda_r^*(a,s)$ but we know a few things:

- If $(1+r)\beta \geq 1$, capital supply goes to ∞

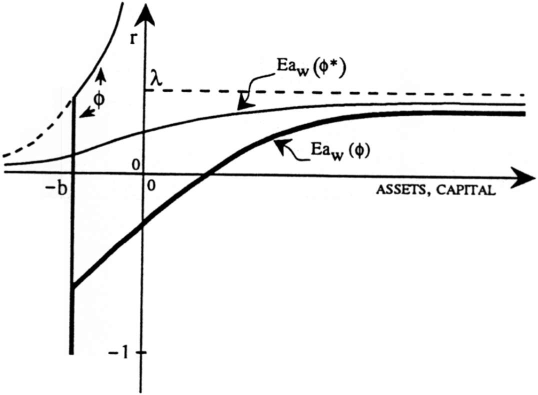
  $\rightarrow$ From Lecture 3, almost surely agents will accumulate infinite wealth

- If $r = -1$, agents consume everything: everyone is constrained and supply of capital is $-\phi$

Therefore there should be a point r at which supply and demand are equal if:

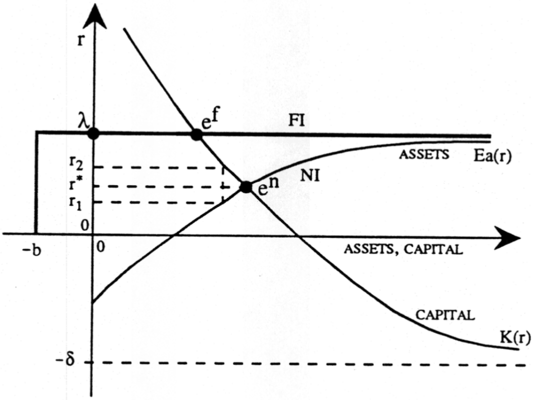
1. For any r there exist a unique stationary $\lambda_r^*(a,s)$
2. $\lambda_r^*(a,s)$ varies continuously with r

Capital Supply



Note: As ϕ increases (laxer borrowing constraint), $Ea(\phi)$ shifts up (here ϕ^* NBC)

Equilibrium



Discussion

- FI (Full Insurance) capital supply with no uncertainty, equilibrium e_f

  capital supply is a horizontal line at $1 + r = 1/\beta$ (why?)
- With risk (NI), at all r, demand for assets higher because of precautionary savings
- Higher demand for assets, $Ea(r)$ shifts down:

  $\rightarrow$ at equilibrium lower interest rate r higher level of capital (look at e_n vs e_f)
- First graph gives the intuition for comparative statics with respect to ϕ
  - larger ϕ, $Ea(r)$ shifts up: laxer borrowing limit reduces the need for precautionary saving
  - therefore with higher ϕ, higher interest rate and lower capital

Existence and Uniqueness of $\lambda_r^*(a, s)$

Our first step is to show that stationary capital supply is well defined for all r

This is given by the following result from Stokey, Lucas & Prescott

Theorem 12.12

Suppose that s is a Markov Process on an interval $[a, b]$ and that its transition kernel Π satisfies:

1. Π has the Feller Property: for any continuous f, $g(s) = \int f(s')\Pi(s, s')ds'$ is continuous
2. Π is monotone: for any non-decreasing f, $g(s) = \int f(s')\Pi(s, s')ds'$ is non-decreasing
3. Π satisfies a mixing condition: there exists $a < c < b$, $\epsilon > 0$ and $N \geq 1$ such that

$$\Pi^N(a, [c, b]) \geq \epsilon, \quad \Pi^N(b, [a, c]) \geq \epsilon$$

Then Π has a unique stationary distribution

Sketch of the Proof

Note that we have:

$$g(z) = \int f(z')\Pi_r(z,z')dz' = \sum_i f((1+r)a(z) + ws_i)\pi(s_i)$$

Feller is direct to verify: if f is continuous, since $a(z)$ is continuous then g is.

Monotonicity is also direct since $a(z)$ is non-decreasing

For the mixing property, the idea is to start from an arbitrary z_0 and defining

$$\underline{z}_{n+1} = (1+r)a(\underline{z}_n) + w\underline{s} \quad \textit{and} \quad \bar{z}_{n+1} = (1+r)a(\bar{z}_n) + z\bar{s}$$

show that the sequences converge to $\underline{z}$ and $\bar{z}$ respectively.

The only hard part is actually to prove that there exist $\bar{z}$ such that if $z_t \in [\underline{z}, \bar{z}]$, so is z_{t+1}

This is an interesting result, if agents only receive good shocks, they will accumulate resources at a slower and slower rate until they have a "satisfying buffer" $\bar{z}$.

Result. Suppose that $\beta(1+r) < 1$ then there exists z^s such that $z_t \geq z^s$ implies $z_{t+1} \leq z_t$

Proof. If $a(z)$ is bounded this is obvious, take $z^s = (1+r)\, sup(a(z)) + w\bar{s}$.

Assume $a(z)$ goes to ∞, note that for z_t large enough:

$$1 \leq \frac{\mathbb{E}(V'(z_{t+1}))}{V'((1+r)a(z_t) + w\bar{s})} \leq \frac{V'((1+r)a(z_t) + w\underline{s})}{V'((1+r)a(z_t) + w\bar{s})} = \left(\frac{c((1+r)a(z_t) + w\bar{s})}{c((1+r)a(z_t) + w\underline{s})}\right)^{\gamma}$$

since both $a(z)$ and $c(z)$ are increasing, we have:

$$c((1+r)a(z_t) + w\bar{s}) = c((1+r)a(z_t) + w\underline{s}) + w(\bar{s} - \underline{s})h(z)$$

with $0 \leq h(z) \leq 1$.

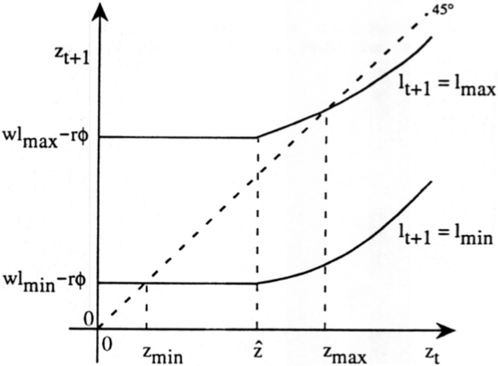
In addition $c(z)$ goes to ∞ at ∞, so $\mathbb{E}(V'(z_{t+1}))/V'((1+r)a(z_t) + w\bar{s})$ goes to 1.

We can therefore choose z^s such that for $z_t \geq z^s$:

$$V'(z_t) = \beta(1+r)\mathbb{E}(V'(z_{t+1})) < V'((1+r)a(z_t) + w\bar{s})$$

so $z_t > (1+r)a(z_t) + w\bar{s} \geq z_{t+1}$.

Illustration



To show that a stationary equilibrium exists, we only need to know that $\lambda^*(z, r)$ is continuous in r. We use the following results from SLP.

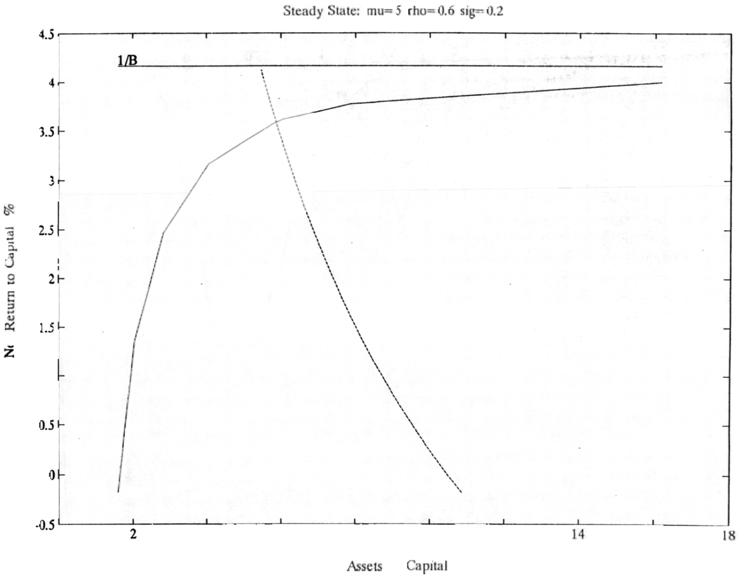
Theorem SLP 12.13

Suppose 1. that $s \in [a, b]$ 2. for θ_n, s_n converging to θ_0, s_0, $\Pi_{\theta_n}(s_n, s')$ converges towards $\Pi_{\theta_0}(s_0, s')$ for all s', and 3. that for all θ, Π_θ has a unique stationary distribution λ^*_θ. Then $\lambda^*_{\theta_n}$ converges towards $\lambda^*_{\theta_0}$

We can almost directly apply the theorem: we've already shown the first and third assumption.

To show the second we only need to show that $a(z, r, w(r))$ depends continuously on r and w. This is done by directly applying the tools developed in lecture 1 to $V(z, r)$. First showing that it is continuous, then using the ToM to show that $a(z, r, w(r))$ is as well.

Results


Steady State: mu=5 rho=0.6 sig=0.2
1/B
4.5
4
3.5
3
2.5
2
1.5
0.5
0
-0.5
2
14
18
Return to Capital %
Assets
Capital

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To understand how heterogeneity/risk impacts business cycle fluctuations, Krusell-Smith consider the Aiyagari model with aggregate risk.

The production function is now given by:

$$Z_t F(K, L)$$

Where Z_t follows a Markov process.

Because of aggregate risk there is no equilibrium with constant r

$\rightarrow$ A stationary equilibrium is a distribution over the aggregate state space (Z, λ)

We cannot remove the distribution from the agent problem; the aggregate state space is (Z_t, λ_t) and the Bellman equation is:

$$V(a, s; Z, \lambda) = \sup_{a' \geq -\phi} u((1 + r(Z, \lambda))a + w(Z, \lambda)s - a') + \beta \mathbb{E}\left(V\left(a', s'; Z', \lambda'\right) \mid s, Z\right)$$

The evolution of the exogenous state (s, Z) is given by a transition kernel $Q(Z, s; Z', s')$, but the law of motion of λ is endogenous: how do we specify it?

Recursive Equilibrium

KS propose the notion of **recursive equilibrium**. Define the perceived law of motion of λ as the operator $\mathcal{H}$ agents use to forecast λ':

$$\lambda' = \mathcal{H}(\lambda, Z, Z')$$

Recursive Equilibrium

A recursive equilibrium is 1.a policy function $a^*(a, s; Z, \lambda)$; 2.a couple of pricing function $w(Z, \lambda)$ and $r(Z, \lambda)$; and 3.a perceived law of motion $\mathcal{H}(\lambda, Z, Z')$ such that:

1. $a^*(a, s, Z, \lambda)$ solves the agent's problem given $\mathcal{H}$, prices $w(Z, \lambda)$ and $r(Z, \lambda)$
2. $w(Z, \lambda)$ and $r(Z, \lambda)$ are given by the firm FOC with $K = \int a\lambda dads$, $L = \int s\lambda dads$
3. The good market clears:

$$\int c(a, s, Z, \lambda)\lambda dads + \int a'(a, s, Z, \lambda)\lambda dads = ZF(K, L) + (1 - \delta)K$$

4. $\mathcal{H}(\lambda, Z, Z')$ is consistent with the realized law of motion defined by the agents' choices

$$\lambda'(A \times S; Z') = \int \mathcal{I}(a^*(a, s, Z, \lambda) \in A;\ s' \in S) \frac{Q(Z, s; Z', s')}{\int Q(Z, s; Z', s')ds'} \lambda \, dadsds'$$

Where $\mathcal{I}$ is an indicator function and Q is the kernel of the exogenous states

The definition of the recursive equilibrium simply states that the forecasting rule of the agent has to be consistent with the realized law of motion of the joint distribution of assets and productivity.

This does not help to solve the dimensionality issue, our Bellman equation still depends on λ.

Note that the distribution at t matters in the agent problem to forecast future prices $r(Z_{t+s}, \lambda_{t+s})$. But $r(Z_{t+s}, \lambda_{t+s})$ itself only depends on one moment $\mu^1_{t+s} = \int a\lambda_{t+s} dads$.

KS assume that there is a set of M moments of λ, $\mu = \mu^1, ..., \mu^M$ that is itself Markov:

$$\begin{aligned} \mu^i_{t+1} &= \mathcal{H}_{M,i}(Z, Z', \mu_t) \\ \text{and} \quad r(Z, \lambda) &= r(Z, \mu) \end{aligned}$$

Replacing λ and $\mathcal{H}$ by μ and $\mathcal{H}_M$ in the agent problem and the state space would be finite.

To verify that we have a recursive equilibrium, we then need to verify that the realized law of motion of μ is consistent with $\mathcal{H}_M$.

KS moments

The goal is to find a simple law of motion for μ^1. We can start by assuming that (μ^1, Z) is Markov.

If the realized law of (μ^1, Z) does not match $\mathcal{H}_1(Z, \mu^1)$ (for any $\mathcal{H}_1$), we will then start adding moments.

The main (numerical) result of KS is that $\mu^1 = \int a d\lambda = K$ is well approximated by a simple forecasting rule:

$$ln(K') = b^0(Z) + b^1(Z) ln(K)$$

Where the coefficients $b^0(Z)$ and $b^1(Z)$ depend on the aggregate state Z.

In KS Z only takes 2 values Z_G, Z_B

Computation

1. Guess the coefficients b_z^0, b_z^1, for $z = G, B$.

2. Solve the the agent problem given the law of motion for K:

$$ln(K') = b_z^0 + b_z^1 ln(K)$$

   and prices defined by $r(K,Z)$, $w(K,Z)$

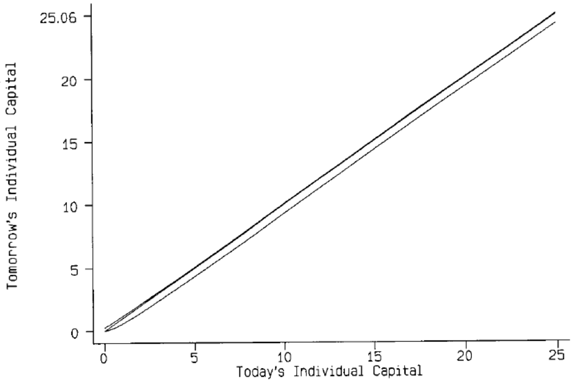
3. Set an arbitrary distribution λ_0 and simulate the economy for T periods given the solution of the agent problem $a^*(a,s,Z,K)$ and a randomly drawn path of Z.

4. Discard the first periods to avoid dependency on initial conditions and regress $ln(K_{t+1})$ on $ln(K_t)$ for each Z to estimate β_z^0, β_z^1

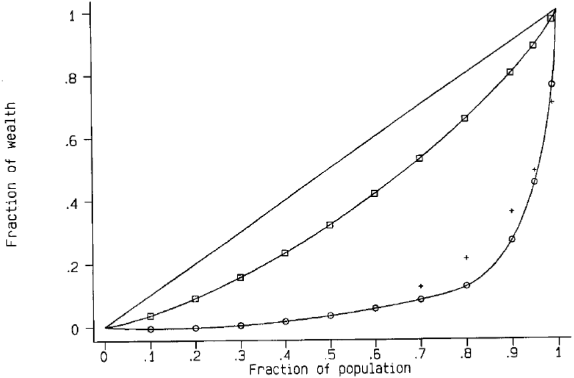
5. β_z^0, β_z^1 are not close to the guess update the guess and restart

6. If they are check the accuracy of the forecasting rule (the R^2 of the regression). If the R^2 is "high" we 're done otherwise add more moments or change the functional form of the forecasting rule.

In practice KS find very high R^2. Why? Agent savings are quasi linear (except at bottom)



Can reproduce wealth inequality but need a stochastic and persistent β (satisfactory?)



But in terms of aggregate variables, model identical to RBC with representative agent.