ECON0057: Advanced Macroeconomic Theory

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Tutorial 3

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Introduction

- Focus today on the Neoclassical Growth Model
  - Bellman Equation
  - Closed-form solution
  - Steady state
  - Consumption policy function
- Numerical Solution for the Neoclassical Growth Model
- Ljunqvist and Sargent exercise

Forming the Bellman

- Asked to solve the following Bellman equation:

$$V(k) = \max_{k'} \quad U(c) + \beta V(k')$$

- such that

$$k' + c = f(k)$$

- substitute in for c using the resource constraint so that

$$V(k) = \max_{k'} \quad U(f(k) - k') + \beta V(k')$$

Closed form solution

- Sometimes, we can characterise the problem analytically
- Define $U(c) = \log(c)$ and $f(k) = Ak^{\alpha}$

$$V(k) = \max_{k'} \quad \log(Ak^{\alpha} - k') + \beta V(k')$$

- FOC wrt k'

$$\frac{-1}{Ak^{\alpha} - k'} + \beta V'(k') = 0 \implies \dots \implies k' = \alpha\beta Ak^{\alpha}$$

- the solution to this problem is the function $k'(k)$, which gives an optimal choice of k' for every value of k

Closed form solution

- You can find notes on how to solve for this function in the file "DSGE_by _hand.pdf"

- Aim for problem set 2 is to compare the **analytical solution** to the **numerical approximation**

Steady State

- Defining $A = (\alpha\beta)^-1$, gives us steady state capital of 1

$$k' = \alpha\beta A k^{\alpha} = \alpha\beta\frac{1}{\alpha\beta}k^{\alpha} \implies k' = k^{\alpha}$$

- in steady state, $k' = k = k^{\alpha} \implies k^{ss} = 1$

Consumption Policy Function

- In addition to the capital policy function $k' = g(k)$, we can also recover the implied optimal level of consumption as a function of k

- From the constraint:

$$c + k' = f(k) \implies c = f(k) - k' \implies c = f(k) - k'(k)$$

- subbing in our capital policy function yields

$$c(k) = f(k) - \alpha\beta A k^{\alpha} \implies$$

$$c(k) = A k^{\alpha} - \alpha\beta A k^{\alpha} \implies$$

$$c(k) = A k^{\alpha}(1 - \alpha\beta)$$

- Once we find the optimal capital function, we recover the optimal level of consumption at each level of capital $(c(k))$

Numerical implementation of utility maximisation

Toy example

- Define $u = \log(f(k) - k')$ with $f(k) = (\beta\alpha)^{-1}$ and $\beta = 0.96$, $\alpha = 0.25$
- $k, k' \in \{1, 2\}$. Only two values for capital, gives us 4 possible values of utility
  - if $(k, k') = (1, 1)$, $u_1 = \log(f(1) - 1) = 1.15$
  - if $(k, k') = (1, 2)$, $u_2 = 0.77$
  - if $(k, k') = (2, 2)$, $u_3 = 1.20$
  - if $(k, k') = (2, 1)$, $u_4 = 1.46$
- Utility highest in the 4th case

Numerical implementation of utility maximisation

- We construct a utility matrix with 4 entries.
- In each entry, we need to compute utility for the pairs

$$\begin{bmatrix} (1,1) & (1,2) \\ (2,1) & (2,2) \end{bmatrix}$$

- To compute this, we start with two auxiliary matrices

$$k_M = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \quad \text{and} \quad k'_M = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$$

- Then we can compute $U_M = \log\left((\beta\alpha)^{-1} k_M^{\alpha} - k'_M\right) = \begin{bmatrix} 1.15 & 0.77 \\ 1.20 & 1.46 \end{bmatrix}$

Numerical implementation of utility maximisation

- Finally, we can maximise and find the highest value of utility for every value of $\boldsymbol{k}$. In this case
  - If $k = 1$ (first row), $u_{\max} = 1.15$
  - If $k = 2$ (second row), $u_{\max} = 1.46$
- In addition, we can find the best choice of $\boldsymbol{k}'$ for every value of $\boldsymbol{k}$.
- This is called the capital policy function. In this case:
  - If $k = 1$, $k' = 1$
  - If $k = 2$, $k' = 2$

Ljunqvist and Sargent

Exercise 6.8 - Wage Growth

- Relative to previous questions, the wage now grows at rate ϕ

- So wage $w_t = w\phi^t, \quad \phi > 1, \qquad \beta\phi < 1$

- If an offer is accepted, get:

$$w + \beta\phi w + \beta^2\phi^2 w + \ldots = \sum_{t=0}^{\infty} w(\beta\phi)^t = \frac{w}{1-\beta\phi}$$

- We can then write the Bellman equation as:

$$V(w) = \max_{a,r}\left\{\frac{w}{1-\beta\phi}, c + \beta\int_0^B V(w')dF(w')\right\}$$

Exercise 6.8

- The reservation is the value of w at which the agent is indifferent between accepting and rejecting

$$\frac{\bar{w}}{1-\beta\phi} = c + \beta \int_0^B V(w')dF(w')$$

$$= c + \beta \int_0^{\bar{w}} \frac{\bar{w}}{1-\beta\phi} dF(w') + \beta \int_{\bar{w}}^B \frac{w'}{1-\beta\phi} dF(w')$$

$$+ \beta \int_{\bar{w}}^B \frac{\bar{w}}{1-\beta\phi} dF(w') - \beta \int_{\bar{w}}^B \frac{\bar{w}}{1-\beta\phi} dF(w')$$

$$(1-\beta)\bar{w} - \beta \int_w^B (w' - \bar{w}) dF(w') = (1-\beta\phi)c$$

- define the LHS as a function $h(w)$ and show that it is increasing in w

Exercise 6.8

- The function $h(w)$ represents the net benefit of accepting a job at wage w:

$$h(w) = \int_{w}^{B} (w' - w) dF(w')$$

- We want to show $h'(w) > 0$ for the case where wages grow at rate $\phi > 1$:

- Using Leibniz rule with $a = w$ and $b = B$:

$$\begin{aligned} h'(w) &= (1-\beta) - \beta \left\{ \int_{w}^{B} -1 \, dF(w') + (B-w)0 - (w-w) \cdot 1 \right\} \\ &= (1-\beta) - \beta[-F(w')]_{w}^{B} \\ &= (1-\beta) - \beta[-1 + F(w)] \\ &= 1 - \beta F(w) > 0 \end{aligned}$$

Exercise 6.8

- Since $h'(w) > 0$, function is strictly increasing because:
  - $\beta < 1$ (discount factor)
  - $F(w) \leq 1$ (CDF property)
  - Therefore $1 - \beta F(w) > 0$ always holds
- The '1' term represents the direct benefit of higher wages
- The '$-\beta F(w)$' term captures the option value of waiting
- Higher growth rate ϕ makes current wages more valuable:
  - Growth compounds over time through ϕ^t
  - Higher base wage increases the growth trajectory
  - Option value of waiting decreases with higher growth rates