ECON0057 Lecture 3

Income Fluctuation Problem II

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Introduction

- Last lecture: Consumption with Certainty Equivalence
  - Linear function of assets and (NPV) income (no interaction)
  - Risk does not matter
  - Numerous empirical problem (excess sensitivity /smoothness etc.)
- This lecture: Consumption with **precautionary motive** and **liquidity constraint**
  - Precautionary Motive: when risk increases (keeping NPV of income constant), agent saves more
  - Liquidity Constraint: cap on how much agent can borrow
  - How does that change consumption?
    - Suppose agents cannot borrow at all
    - With 0 wealth, cannot consume more than their current income
    - Consumption more sensitive to current income/transitory shocks

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A Two Period Model

Two periods, income at $t = 1$, y_1 is stochastic, u concave:

$$\sup_{c_0, c_1} u(c_0) + \beta \mathbb{E}(u(c_1))$$

$$s.t. \quad a_1 = R(y_0 - c_0), \quad a_2 \geq 0$$

At $t = 1$ agent consumes everything: $c_1 = a_1 + y_1$, so the Euler equation gives:

$$u'(y_0 - a_1/R) = \beta R \mathbb{E}(u'(a_1 + y_1))$$

- Do we still have certainty equivalence (consumption only depends on mean of y_1)?
- Is the consumption function still linear?

To answer the first question, consider a mean preserving spread of y_1:

- Income at $t = 1$ now given by $\tilde{y}_1 = y_1 + \epsilon$ with ϵ indep. of y_1, $\mathbb{E}(\epsilon) = 0$
- $\tilde{y}_1$ is a mean preserving spread of y_1, with certainty equivalence consumption should remain constant.

Result. If u' is strictly convex ($u''' > 0$), then $\tilde{a}_1 > a_1$ when $\tilde{y}_1$ is a mean preserving spread of y_1.

- When $u''' > 0$, we say u exhibits **prudence**
- Savings increase, consumption decreases when income is riskier: **precautionary savings**
- Consumption at $t = 0$ lower than in CE case where income at $t = 1$ is constant ($y_1 = \mathbb{E}(y_1)$)

Note on saving motives. When R increases, savings can increase through an **intertemporal motive**, higher R allows to consume more in the future through savings.

Here **precautionary motive**, agent saves more to self insure against future bad shocks.

Proof. Suppose $\tilde{a}_1 \leq a_1$, then:

$$\begin{aligned} & u'(\tilde{a}_1 + y_1 + \epsilon) \geq u'(a_1 + y_1 + \epsilon) \\ \Rightarrow & \mathbb{E}(u'(\tilde{a}_1 + y_1 + \epsilon)) \geq \mathbb{E}(u'(a_1 + y_1 + \epsilon)) \end{aligned}$$

Jensen Inequality: if f is strictly convex then $f(\mathbb{E}(X)) < \mathbb{E}(f(X))$

Applying Jensen inequality:

$$\mathbb{E}(u'(a_1 + y_1 + \epsilon) \mid y_1) > u'(\mathbb{E}(a_1 + y_1 + \epsilon \mid y_1)) = u'(a_1 + y_1)$$

$$\Rightarrow \mathbb{E}(u'(a_1 + y_1 + \epsilon)) > \mathbb{E}(u'(a_1 + y_1))$$

Therefore:

$$\begin{aligned} \beta R \mathbb{E}(u'(\tilde{a}_1 + y_1 + \epsilon)) &> \beta R \mathbb{E}(u'(a_1 + y_1)) \\ &= u'(y_0 - a_1/R) \\ &\geq u'(y_0 - \tilde{a}_1/R) \end{aligned}$$

A contradiction since we need $u'(y_0 - \tilde{a}_1/R) = \beta R \mathbb{E}(u'(\tilde{a}_1 + y_1 + \epsilon))$ to satisfy Euler

Extensions

In the T period problem with *iid* income, defining $w_t = a_t + y_t$, the FOC of Bellman is:

$$u'(w_t + (y_{t+1} - w_{t+1})/R) = \beta R \mathbb{E}(V'_{t+1}(w_{t+1}))$$

If V' is convex, can apply the same reasoning to show that agents dissave when risk increase.
$\rightarrow$ Sibley (1975): at T, $V_T = u(w_T)$ so V'_T convex if $u''' \geq 0$, then by induction V'_{T-s} convex.

Carroll & Kimball (1996):

- Generalize this to any income process (finite T in their paper)
- If u has prudence and is HARA ($u'''u'/u''^2 = k > 0$)
  - c is a **concave** function of assets
  - In particular when y_t is iid c is a function of w_t: $c(a_t + y_t)$, c reacts less to y_t when a_t is large.

$u''' > 0$ "common", satisfied by $u(c) = c^{(1-\gamma)}/(1-\gamma)$ (CRRA) with $\gamma \geq 0$

3 periods, u quadratic (no prudence) and $\beta R = 1$, agents cannot borrow ($a_t \geq 0$).

- At $t = 2$ agent consumes everything, $c_2 = a_2 + y_2$
- At $t = 1$ given a_1, if $a_2 > 0$:

$$c_1 = \mathbb{E}_1(c_2) = a_2 + \mathbb{E}_1(y_2)$$
$$\Rightarrow c_1 = \frac{R}{1+R}(a_1 + y_1 + \frac{1}{R}\mathbb{E}(y_2))$$

  If $\mathbb{E}(y_2) > a_1 + y_1$, then $c_1 > a_1 + y_1$: agent would need to borrow to smooth consumption

  Therefore:

$$c_1 = \frac{R}{1+R}(a_1 + y_1) + \frac{1}{1+R}\min(\mathbb{E}_1(y_2), a_1 + y_1)$$

  c_1 is a concave function of assets, c_1 reacts more to y_1 when a_1 is small
- At $t = 0$ assume that y_0 large enough (so the agent is not constrained in $t = 0$):

$$y_0 - \frac{a_1}{R} = \frac{R}{1+R}(a_1 + \mathbb{E}_0(y_1)) + \frac{1}{1+R}\mathbb{E}_0(\min(\mathbb{E}_1(y_2), a_1 + y_1))$$

Precautionary Savings without Prudence

Assume $a_1 + min(y_1) < \mathbb{E}(y_2)$, $a_1 + max(y_1) > \mathbb{E}(y_2)$

Consider a spread in y_1, $\tilde{y}_1 = y_1 + \epsilon$ with $\mathbb{E}(\epsilon) = 0$, suppose that $\tilde{a}_1 \leq a_1$:

$$\begin{aligned}
&\frac{R}{1+R}(\tilde{a}_1 + \mathbb{E}_0(y_1)) + \frac{1}{1+R}\mathbb{E}_0(\min(\mathbb{E}_1(y_2), \tilde{a}_1 + \tilde{y}_1)) \\
\leq &\frac{R}{1+R}(a_1 + \mathbb{E}_0(y_1)) + \frac{1}{1+R}\mathbb{E}_0(\min(\mathbb{E}_1(y_2), a_1 + \tilde{y}_1)) \\
< &\frac{R}{1+R}(a_1 + \mathbb{E}_0(y_1)) + \frac{1}{1+R}\mathbb{E}_0(\min(\mathbb{E}_1(y_2), a_1 + y_1)) \\
&= y_0 - \frac{a_1}{R} \leq y_0 - \frac{\tilde{a}_1}{R}
\end{aligned}$$

(third line Jensen with concave function)

Borrowing constraint generates precautionary savings without prudence!

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Model

Borrowing constraints and risk can generate precautionary savings and non-linear c

$\rightarrow$ More realistic consumption functions.

Deaton model:

- Infinite horizon
- y_t stochastic and Markov (with $y_{min} > 0$)
- Agent cannot borrow more than b.
- Finally Deaton assumes $\beta R < 1$

$$V(a_0, y_0) = \sup_{c_t} \mathbb{E}\left(\beta^t u(c_t)\right)$$

$$s.t. \quad a_{t+1} = R(a_t + y_t - c_t)\; s.t.\; a_t \geq -b$$

Note we do not need a "No Ponzi" condition, this is taken care of by the borrowing constraint

Suppose that we impose the No Ponzi condition $\lim_{t\to\infty} a_t/R^t \geq 0$ almost surely.

Expanding the budget constraint forward we get:

$$\lim_{s\to\infty} \frac{a_{t+s}}{R^s} = a_t + \sum \frac{y_{t+s}}{R^s} - \sum \frac{c_{t+s}}{R^s}$$

Therefore since $c_t \geq 0$ we have:

$$\begin{aligned} \lim_{s\to\infty} \frac{a_{t+s}}{R^s} &\geq 0 \\ \Rightarrow a_t &\geq -\sum \frac{y_{t+s}}{R^s} \quad \text{after all history } \{y_t, ..., y_{t+s}, ...\} \\ \Leftrightarrow a_t &\geq -\sum \frac{y_{min}}{R^s} = -\frac{R}{R-1} y_{min} \end{aligned}$$

Conversely, if we impose any constraint $a_t \geq -b \geq -\frac{R}{R-1} y_{min}$, we have $\lim_{t\to\infty} a_t/R^t \geq 0$

Borrowing Constraint and No Ponzi

- Imposing the No Ponzi is equivalent to imposing the borrowing constraint

$$b^* = \frac{R}{R-1} y_{min}$$

- b^* is called the **natural borrowing constraint** (NBR)
- NBR is the laxest constraint ensuring that agents can repay their debts after all histories
  - At t there's a positive risk that y_{min} is drawn for T periods (with arbitrary T)
  - So if the agent has to repay his debts after all histories b cannot be larger
- With Inada ($u'(0) = \infty$) and NBR, agent will never be constrained (except on a measure 0 set)
- From now on, $b < b^*$
  - Stricter than no Ponzi (and NBR)
  - can generate binding borrowing constraints (Deaton takes $b = 0$)

Euler

We have two state variables the endogenous a_t and the exogenous y_t, Bellman Equation:

$$V(a_t, y_t) = \max_{a_{t+1} \geq -b} u(a_t + y_t - a_{t+1}/R) + \beta \mathbb{E}(V(a_{t+1}, y_{t+1}) \mid y_t)$$

From Lecture 1, if u concave, increasing, differentiable, so is V

The envelope condition gives:

$$\partial_a V(a_t, y_t) = u'(c_t)$$

Noting λ_t the Lagrangian on the constraint, the FOC is:

$$u'(c_t) = \beta R \mathbb{E}(\partial_a V(a_{t+1}, y_{t+1} \mid y_t) + \lambda_t R$$

The Euler equation therefore becomes:

$$u'(c_t) \geq \beta R \mathbb{E}_t(u'(c_{t+1})) \quad \text{with equality if } a_{t+1} > -b$$

BC prevents agents to smooth consumption. When it binds agents want to borrow against future income to increase consumption today but cannot: the marginal utility is strictly larger today

IID Shocks

c is a function of a and y. We can show that $c(a, y)$ is increasing in a for all y.

Indeed, from the envelope condition:

$$\partial_{aa} V(a, y) = u''(c)\partial_a c$$

If u concave, V concave so $\partial_a c \geq 0$. Difficult to have a more precise characterization.

Consider the case where **income is IID**. Define cash on hand as total resources available to the agent: $w_t = a_t + y_t$, we can rewrite the budget constraint as:

$$w_{t+1} = R(w_t - c_t) + y_{t+1}$$

If y_{t+1} is a random shock independent of time and y_t, w_{t+1} and y_{t+1} are independent of y_t conditional on w_t, the problem is summarized by w_t.

More formally, consider two agents with a_t, y_t and a'_t, y'_t such that $a_t + y_t = a'_t + y'_t = w_t$:

- At t both agent can choose any $c \leq w_t + b/R$
- Conditional on c chosen at t, $a_{t+1} = a'_{t+1}$ and the distribution of y_{t+1} is the same as y'_{t+1}
- Since the problem is Markov, the continuation values of both agents are identical
- Therefore both agents choose the same c and the solution only depends on w_t

IID Shocks

When shocks are IID, the Bellman equation is:

$$V(w) = \sup_{0 \leq c \leq w + b/R} u(c) + \beta R \mathbb{E}(V(R(w-c) + y'))$$

And c is a function of w only (we have $c(w)$ instead of $c(a, y)$).

Result. Define $w_{min} = -b + y_{min}$. If $b < b^*$, there exists $w^* > w_{min}$ such that for all $w_t \leq w^*$:

$$c(w_t) = w_t + b/R \quad \text{and} \quad a_{t+1} = -b$$

Proof: $c > 0$ is feasible at w_{min}, therefore $c(w_{min}) > 0$ and $\partial_w V(w_{min}) = u'(c(w_{min})) < \infty$.

Assume the constraint never binds, in that case, for all w_t:

$$\partial_w V(w_t) = \beta R \mathbb{E}(\partial_w V(w_{t+1})) \leq \beta R \mathbb{E}(\partial_w V(w_{min})) < \partial_w V(w_{min})$$

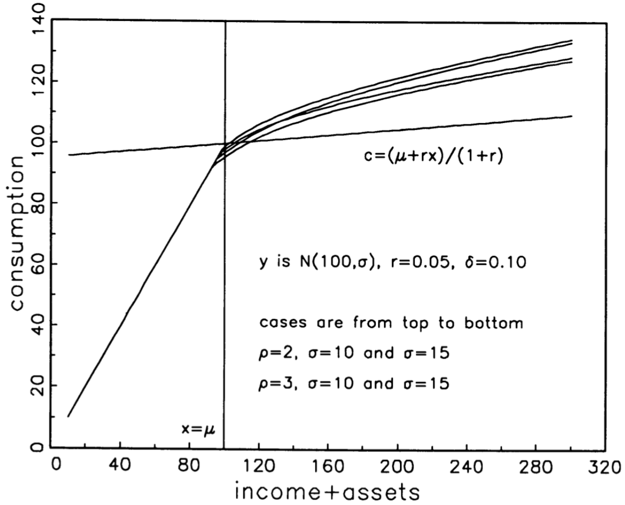
Taking the limit as $w_t \to w_{min}$ since $\partial_w V$ is continuous we have a contradiction.

Since $\partial_w V$ is decreasing , if constraint binds at w_0 for $w < w_0$ the constraint also binds

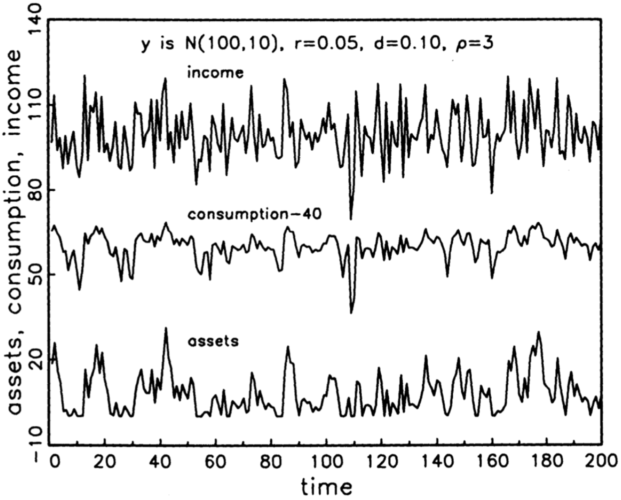
$$\partial_w V(w) \geq \partial_w V(w_0) > \beta R \mathbb{E}(\partial_w V(w_0')) \geq \beta R \mathbb{E}(\partial_w V(w'))$$

Exercise: show that $a_{t+1}(w_t)$ is non decreasing

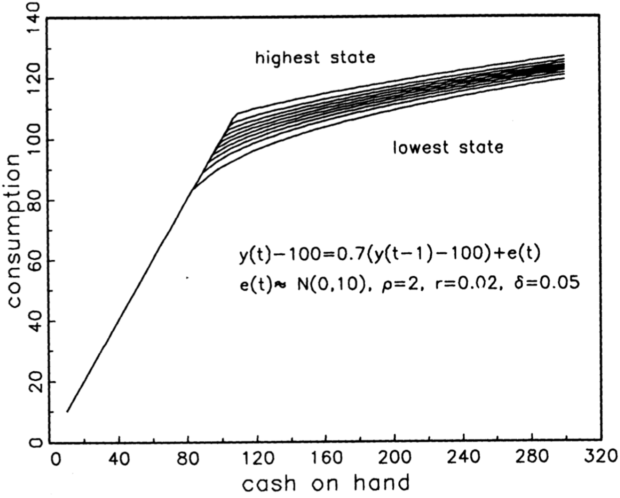
Deaton Results IID case



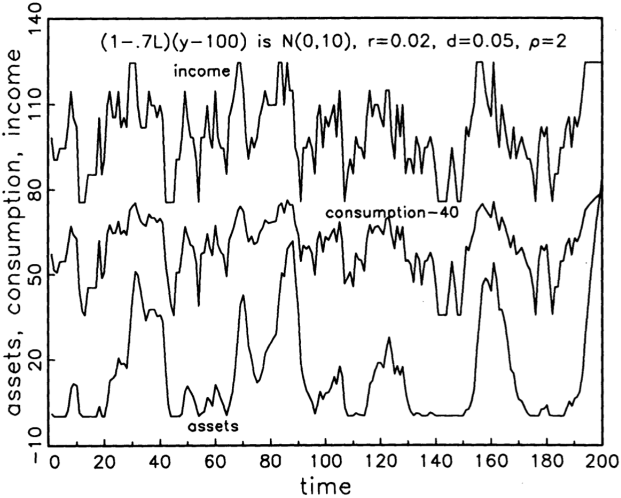
Deaton Results IID case



Deaton Results persistent case



Deaton Results persistent case



Complement: Why do we need $\beta R < 1$

Assume that u satisfies Inada ($\lim_{c \to 0} u'(c) = \infty$, $\lim_{c \to \infty} u'(c) = 0$), and that y is bounded.

A useful Theorem:

Doob Martingale Convergence Theorem (Special Case)

Let Z_t be a non negative supermartingale, that is Z_t satisfies $Z_t \geq \mathbb{E}_t(Z_{t+1}) \geq 0$ almost surely.

Then Z_t converges almost surely to random variable Z with $\mathbb{E}(Z) < \infty$

A supermartingale is simply a random sequence decreasing "on average"

From the Euler equation, we have $u'(c_t) \geq \beta R \mathbb{E}_t(u'(c_{t+1}))$

$(\beta R)^t u'(c_t)$ is a non negative supermartingale, so it converges a.e. to a finite random variable

Case $\beta R > 1$

- First consider the case $\beta R > 1$
- Then $(\beta R)^t \longrightarrow \infty$
- $(\beta R)^t u'(c_t)$ converges a.s. to a finite random variable, so $u'(c_t)$ converges to 0 a.s.
- Given Inada conditions, c_t converges to infinity almost surely
- In every period we have $c_t \leq a_t + y_t + b/R$
- Therefore a_t converges to infinity almost surely

  $\rightarrow$ if we assume $\beta R > 1$ then all agents will have infinite wealth in the long run!

Case $\beta R = 1$

Assume $\beta R = 1$ and y is IID and takes a finite number of values

(see Chamberlain and Wilson (2000) for a general proof)

- $u'(c_t)$ is a supermartingale it converges almost surely to a finite random variable
- Starting from any w_t and for any W, we can find s, s.t. $w_{t+s} > W$ with positive probability
- Suppose not and consider the sequence $w_{t+s+1} = a(w_{t+s}) + y_{max}$
- The sequence is bounded and increasing so it converges towards $\bar{w} = a(\bar{w}) + y_{max}$
- Euler gives:

$$\partial_w V(\bar{w}) \geq \mathbb{E}(\partial_w V(a(\bar{w}) + y)) > \partial_w V(a(\bar{w}) + y_{max}) = \partial_w V(\bar{w})$$

a contradiction

Case $\beta R = 1$

- $a(w)$ is non decreasing, so if T realizations of y_{max} occur, for any w_t, $w_{t+T} > W$
- T realizations of y_{max} happen with probability $P(y_{max})^T > 0$
- By Borel-Cantelli ("infinite monkey") this happens infinitely many times with probability 1 starting from any w_t
- Therefore $\lim w_t < W$ with probability 0
- Since it's true for any W, $\lim w_t = \infty$

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JPS and the tax rebate of 2001

- Economic Growth and Tax Relief Reconciliation Act of 2001 (EGTRRA)

  Cut lowest tax rate ($\leq$ 12000\$) from 15% to 10%

- Checks (typically \$300 or \$600) corresponding to "advance refund" for 2001

  Sent to 92 million taxpayers between Jul-Sep

- Features: anticipated (more or less), EGTRRA enacted in May, lump-sum (fixed amount per adult), **randomized timing**

- CEX added question in 2001 asking if rebate was received, when, and how much

- JPS estimate:

$$\Delta C_{i,t+1} = \beta Rebate_{i,t+1} + \alpha_0 X_{i,t+1} + \sum \alpha_{1,s} month_{s,i}$$

  Instrument $Rebate_{i,t+1}$ with $I(Rebate_{i,t+1} > 0)$ since timing is random

JPS Results

Table 2: The contemporaneous response of expenditures to the tax rebate

Dependent Variable:	ΔC Dollar change in			ΔC Dollar change in			ΔlnC Percent change in			ΔC Dollar change in		
	Food	Non-durable goods (strict)	Non-durable goods	Food	Non-durable goods (strict)	Non-durable goods	Food	Non-durable goods (strict)	Non-durable goods	Food	Non-durable goods (strict)	Non-durable goods
Estimation method:	OLS	OLS	OLS	OLS	OLS	OLS	OLS	OLS	OLS	2SLS	2SLS	2SLS
Rebate	0.109 (0.056)	0.239 (0.115)	0.373 (0.135)							0.108 (0.058)	0.202 (0.112)	0.375 (0.136)
I(Rebate>0)				51.5 (27.6)	96.2 (53.6)	178.8 (65.0)	2.72 (1.36)	1.76 (1.05)	3.16 (1.02)			
Age	0.570 (0.320)	0.449 (0.550)	1.165 (0.673)	0.552 (0.318)	0.391 (0.548)	1.106 (0.670)	0.035 (0.020)	0.005 (0.016)	0.023 (0.015)	0.569 (0.320)	0.424 (0.549)	1.166 (0.671)
Change in adults	130.3 (57.8)	285.8 (90.0)	415.8 (102.8)	131.1 (57.8)	287.7 (90.2)	418.6 (102.9)	6.16 (2.08)	6.22 (1.58)	7.55 (1.50)	130.3 (57.7)	286.2 (90.0)	415.7 (102.7)
Change in children	73.7 (45.3)	98.3 (82.4)	178.4 (98.3)	74.0 (45.3)	98.7 (82.5)	179.2 (98.3)	3.99 (2.36)	3.73 (1.66)	4.59 (1.66)	73.7 (45.3)	98.3 (82.5)	178.4 (98.3)
N	13,066	13,066	13,066	13,066	13,066	13,066	13,007	13,066	13,066	13,066	13,066	13,066

Notes: All regressions include a full set of month dummies. Reported standard errors are adjusted for arbitrary within-household correlations and heteroskedasticity. The third triplet of three columns is multiplied by 100 so as to report a percent change. The last three columns report results from two-stage least squares regressions where *I(Rebate>0)* with the other regressors are used as instruments for *Rebate*.

Kaplan & Violante (2014)

JPS: very large responses to tax rebates. Difficult to replicate in Deaton model. Indeed the MPC is large only when agents are near the constraint: would need an unreasonably large fraction of households with no wealth to replicate JPS.

Kaplan & Violante (2014) propose a model with 2 assets: a standard liquid asset m (cash, deposit etc.) with return R, and an illiquid asset a (housing, pension plan etc.) with return $R^a > R$. To use illiquid account need to pay a fix cost κ. Constraint on both assets

Idea: it is costly to use illiquid assets, agents draw/deposit on illiquid account infrequently. When they don't, they behave like Deaton agents with wealth m.

When they receive a small rebate/income shock, it is not to be worth it to draw on illiquid asset. Most agents have low liquid wealth m, so the response of consumption is large.

Simple version of the main equations to give an idea (lots of bells and whistles in the paper)

Kaplan & Violante (2014)

Bellman when agent doesn't change illiquid position

$$
\begin{aligned}
V_t^m(m,a,y) &= \sup_{m'} \ u(c) + \beta \mathbb{E}_t(V_t(m',a',y')) \\
s.t. \quad & m' = R(m+y-c) \quad m' \geq -b \\
& a' = R^a a
\end{aligned}
$$

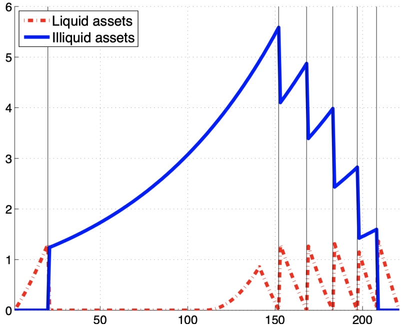
Bellman when agent does change illiquid position

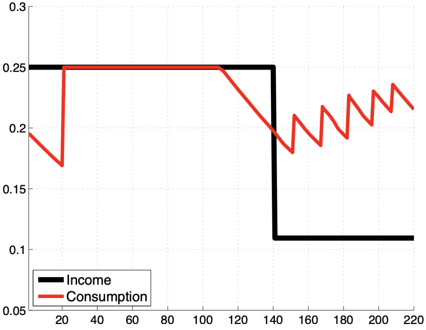
$$
\begin{aligned}
V_t^a(m,a,y) &= \sup_{m',a'} \ u(c) + \beta \mathbb{E}_t(V_t(m',a',y')) \\
s.t. \quad & m' = R(m+y+a-c-a'/R^a-\kappa) \quad m' \geq -b \quad a' \geq 0
\end{aligned}
$$

Full Bellman:

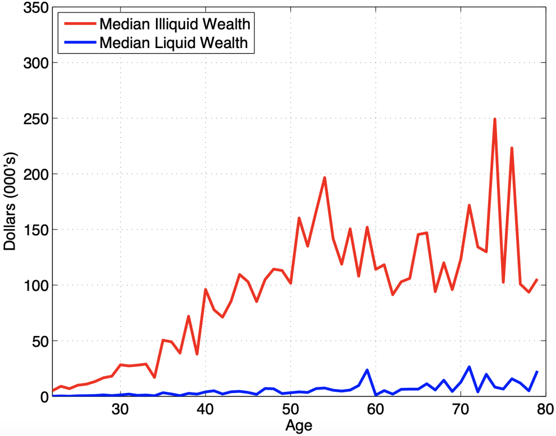
$$
V_t(a,m,y) = \max(V_t^m(m,a,y),\ V_t^a(m,a,y))
$$

Results: Lifecycle Consumption



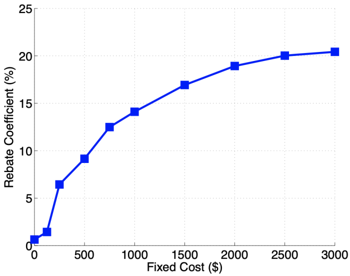


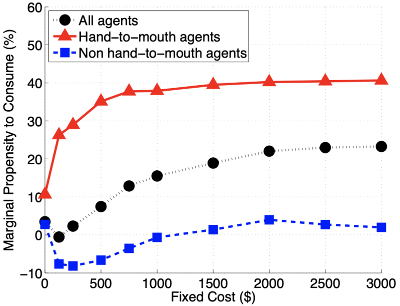
Results: Liquid/Illiquid Wealth



Results: Fraction of Hand-to-Mouth households

Results: Rebate Experiments





Results: Rebate Experiments, anticipation