ECON0057: Advanced Macroeconomic Theory

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Tutorial 2

January 22, 2025

Introduction

- The McCall Search Model helps us understand decision-making under uncertainty in labour markets.
- Key concepts:
  - Wage offer distribution.
  - Reservation wage.
  - Optimal decision rules.
- Today's focus:
  - Solve exercises based on Bellman equations.
  - Analyse reservation wages under different scenarios.

Brief Review

- Basic premise:
  - An unemployed worker maximises discounted lifetime income
  - Receive (for now) one offer each period with wage drawn from $w \sim F(W)$
  - accept/reject - no recall!

Brief Review

- Formally, The objective function is

$$V(w) = \max \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t y_t, \quad \beta \in (0,1), \quad y = \begin{cases} b & \text{if unemployed} \\ w & \text{if employed} \end{cases}$$

- The value of accepting an offer with wage w is

$$V^A(w) = \sum_{t=0}^{\infty} \beta^t w = w \sum_{t=0}^{\infty} \beta^t = \frac{w}{1-\beta}$$

- The value of rejecting is

$$V^R(w) = b + \beta \mathbb{E} V\left(w'\right) = b + \beta \int V\left(w'\right) f\left(w'\right) dw' = b + \beta \int V\left(w'\right) dF\left(w'\right)$$

Exercise 6.1

Chance of an Offer

- Each period:
  - With probability ϕ, no wage offer is received.
  - With probability $1 - \phi$, a wage w is offered, drawn from a CDF $F(w)$.
- The worker's goal is to maximize:

$$\mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t y_t\right],$$

  where $y_t = w$ if employed and $y_t = c$ if unemployed.
- Write the Bellman equation for the worker's problem.

Exercise 6.1

- Let $V(w)$ be the expected value of $\sum_{t=0}^{\infty} \beta^t y_t$ for an unemployed worker who has offer w in hand and who behaves optimally.

- Let V_m be the expected value of an unemployed worker without an offer in hand. The value of not receiving an offer can be written as:

$$V_n = c + \beta\left(\phi V_n + (1-\phi)\mathbb{E}V(w)\right) = \frac{c}{(1-\beta\phi)} + \frac{\beta(1-\phi)}{(1-\beta\phi)}\mathbb{E}V(w)$$

  The value of receiving an offer w can be written as

$$\begin{aligned} V(w) &= \max\left\{\tfrac{w}{1-\beta}, V_n\right\} \\ &= \max\left\{\tfrac{w}{1-\beta}, \tfrac{c}{(1-\beta\phi)} + \tfrac{\beta(1-\phi)}{(1-\beta\phi)}\mathbb{E}V(w)\right\} \\ &= \max\left\{\tfrac{w}{1-\beta}, \tfrac{c}{(1-\beta\phi)} + \tfrac{\beta(1-\phi)}{(1-\beta\phi)}\int_{-\infty}^{\infty} V(w)dF(w)\right\} \end{aligned}$$

Exercise 6.2

Two Offers

- Consider an unemployed worker who each period can draw TWO iid wage offers from the cdf $F(w)$.
- The worker will work forever at the same wage after having accepted an offer. In the event of unemployment during a period, the worker receives unemployment compensation c.
- The worker derives a decision rule to maximize $\mathbb{E} \sum_{t=0}^{\infty} \beta^t y_t$ where $y_t = w$ or $y_t = c$, depending on whether she is employed or unemployed.
- (a) Formulate the Bellman equation for the worker's problem
- (b) Prove that the worker's reservation wage is higher than it would be had the worker faced the same c and been drawing only ONE offer from the same distribution $F(w)$ each period.

Exercise 6.2

- She would not accept a job if both wage offers are too low.

- Having two draws each period is equivalent to drawing from a "better" wage offer distribution If the two offered random wages are W_1 and W_2, we can define the equivalent problem of an agent receiving only one offer $W_m = \max\{W_1, W_2\}$.

$$\begin{aligned} G(w) &= \Pr(W_m < w) \\ &= \Pr(\max\{W_1, W_2\} < w) \\ &= \Pr(W_1 < w)\Pr(W_2 < w) \\ &= F(w)^2 = F^2(w) \end{aligned}$$

- The Bellman equation is therefore

$$V(w) = \max\left\{\frac{w}{1-\beta}, c + \beta \int_0^B V(w')\, dF^2(w')\right\}$$

- where w is the best offer in hand ($w = \max\{w_1, w_2\}$).

Exercise 6.2

- Bellman Equations:
  - two draws:

$$V_2(w) = \max\left\{\frac{w}{1-\beta}, c + \beta\int_0^B V_2\left(w'\right)dF^2\left(w'\right)\right\}$$

  - one draw:

$$V_1(w) = \max\left\{\frac{w}{1-\beta}, c + \beta\int_0^B V_1\left(w'\right)dF\left(w'\right)\right\}$$

Exercise 6.2

- Recall that the reservation wage w solves the following equation

$$w - c = \frac{\beta}{1-\beta} \int_{\bar{w}}^{B} \left(w' - w\right) dF\left(w'\right)$$

- We therefore need to show that

$$w_1 = c + \frac{\beta}{1-\beta} \int_{\bar{w}_1}^{B} \left(w' - w_1\right) dF\left(w'\right) <$$
$$w_2 = c + \frac{\beta}{1-\beta} \int_{\overline{w_1}}^{B} \left(w' - w_2\right) dF^2\left(w'\right)$$

- An equivalent question is asking what is the behaviour of the difference between the LHS and the RHS of this equation.

Exercise 6.2

- Define a function h_i :

$$h_i(w) = \frac{\beta}{1-\beta} \int_w^B (w' - w)\, d\left(F^i\left(w'\right)\right) + c - w$$

- where i denotes the exponent on the CDF and $h_i\left(\hat{w}_i\right) = 0$.
- The good thing about h_i is that we can easily check its properties.
- Compute the derivative using Leibniz's rule for differentiating under an integral

Leibniz's Rule

- If $f(x,z)$ is a function of x and z, and the limits of integration $a(z)$ and $b(z)$ depend on z, then the derivative of the integral with respect to z is given by:

$$\frac{\partial}{\partial z}\int_{a(z)}^{b(z)} f(x,z)\,dx = f(b(z),z)b'(z) - f(a(z),z)a'(z) + \int_{a(z)}^{b(z)} \frac{\partial f(x,z)}{\partial z}\,dx.$$

- where:
- $f(b(z),z)b'(z)$: Contribution from the upper limit of integration.
- $-f(a(z),z)a'(z)$: Contribution from the lower limit of integration.
- $\int_{a(z)}^{b(z)} \frac{\partial f(x,z)}{\partial z}\,dx$: Change due to the variation of $f(x,z)$ with z inside the integral.

Exercise 6.2

Function:

$$h_i(w) = \frac{\beta}{1-\beta}\int_w^B (w'-w)\, d(F^i(w')) + c - w.$$

- Differentiate with respect to w:

$$\frac{\partial h_i(w)}{\partial w} = \frac{\beta}{1-\beta}\frac{\partial}{\partial w}\int_w^B (w'-w)\, d(F^i(w')) - 1.$$

- Applying Leibniz's Rule:

$$\frac{\partial}{\partial w}\int_w^B (w'-w)\, d(F^i(w')) = (w'-w)\big|_{w'=w}\cdot(-1) = 0.$$

- Final derivative:

$$\frac{\partial h_i(w)}{\partial w} = \frac{\beta}{1-\beta}\left(F^i(w)-1\right) - 1.$$

- $h_i(w)$ is strictly decreasing because $\frac{\partial h_i(w)}{\partial w} < 0$.
- The difference reaches 0 at B - $F(B) = 1$

Exercise 6.2

- Thus it follows that, for the same w

$$h_2(w) > h_1(w)$$

- So in the end we can write

$$h_1\left(\hat{w}_1\right) = 0 = h_2\left(\hat{w}_2\right) > h_1\left(\hat{w}_2\right)$$

- so looking at the first and last element above

$$h_1\left(\hat{w}_1\right) > h_1\left(\hat{w}_2\right)$$

- and since h_1 is decreasing, that gives us that $\hat{w}_1 < \hat{w}_2$.

Exercise 6.2

Alternative

- Formula: $\int u\,dv = uv - \int v\,du$
- For our case:

$$u = w' - \bar{w} \qquad dv = f(w')dw'$$
$$du = dw' \qquad v = F(w')$$

- This transforms our integral from:

$$\int_{\bar{w}}^{B} (w' - \bar{w}) f(w') dw'$$

to:

$$\left[(w' - \bar{w})F(w')\right]_{\bar{w}}^{B} - \int_{\bar{w}}^{B} F(w')dw'$$
$$= (B - \bar{w})F(B) - (\bar{w} - \bar{w})F(\bar{w}) - \int_{\bar{w}}^{B} F(w')dw'$$
$$= (B - \bar{w}) \cdot 1 - 0 - \int^{B} F(w')dw'$$

Two Offers per Period

Alternative

- Consider an unemployed worker who each period can draw two iid wage offers from the cdf $F(w)$
- Having two draws is equivalent to drawing from $F^2(w)$
- We want to show that $\hat{w}_2 \geq \hat{w}_1$ (reservation wage is higher with two offers)

Proof Using Integration by Parts

Alternative

- We can rewrite the value difference using integration by parts:

$$
\begin{aligned}
h(\bar{w}) &= \int_{\bar{w}}^{B} (w' - \bar{w}) f(w') dw' \\
&= [(w' - \bar{w}) F(w')]_{\bar{w}}^{B} - \int_{\bar{w}}^{B} F(w') dw' \\
&= (B - \bar{w}) 1 - (0) - \int_{\bar{w}}^{B} F(w') dw' \\
&= B - \bar{w} - \int_{\bar{w}}^{B} F(w') dw'
\end{aligned}
$$

- Similarly for two offers:

$$
g(\bar{w}) = B - \bar{w} - \int_{\bar{w}}^{B} F^2(w') dw'
$$

Comparing One vs Two Offers

Alternative

- Since $F(w') \leq 1$ for all w':
- $F(w') \geq F^2(w')$ for all w'
- Therefore $-F(w') \leq -F^2(w')$
- This implies $\int_{\bar{w}}^{B} F(w')dw' \geq \int_{\bar{w}}^{B} F^2(w')dw'$
- Thus $g(\bar{w}) \geq h(\bar{w})$ for any $\bar{w}$

Since both functions are increasing in $\bar{w}$ and we're looking for where they equal zero:

$$g(\hat{w}_2) = 0 = h(\hat{w}_1) > h(\hat{w}_2)$$

Therefore $\hat{w}_2 \geq \hat{w}_1$

Exercise 6.9

Finite Horizon

- Consider a worker who lives two periods. In each period the worker, if unemployed, receives an offer of lifetime work at wage w, where w is drawn from a distribution F. Wage offers are iid over time. The worker's objective is to maximize $Ey_1 + \beta y_2$, where $y_t = w$ if employed and c otherwise.

- Analyse the worker's optimal decision rule. In particular establish that the optimal strategy is to choose a reservation wage. Show that the reservation wage decreases over time.

Exercise 6.9

Second Period

- Backwards induction from $t = T$
- The value is given by
$$V_2(w) = \max\{w, c\}$$
- so the optimal strategy is to accept all offers $w \geq c$ and reject all other offers. The second period reservation wage is c.

Exercise 6.9

First Period

- Accepting an offer w in the first period implies a lifetime utility of $w + \beta w = (1 + \beta)w$.

- The expected value of rejecting an offer is $c + \beta \int_0^B V_2(w')\, dF(w')$. The optimal value of the objective function for a worker in period one with offer w in hand is

$$V_1(w) = \max\left\{ w(1+\beta), c + \beta \int_0^B V_2(w')\, dF(w') \right\}$$

- The second term (reject) is constant while the first term (accept) is increasing in w.

- Therefore there will be a reservation wage below which the worker rejects all offers and above which she accepts all offers.

Exercise 6.9

- The reservation wage solves:

$$\hat{w}_1 (1+\beta) = c + \beta \int_0^B V_2(w')\, dF(w')$$

- Remembering that $V_2(w) = \max\{w, c\}$. Thus

$$\hat{w}_1 (1+\beta) = c + \beta c F(c) + \beta \int_c^B w' dF(w')$$

$$\hat{w}_1 (1+\beta) = c + \beta c + \beta c (F(c) - 1) + \beta \int_c^B w' dF(w')$$

$$\hat{w}_1 (1+\beta) = c + \beta c - \beta \int_c^B c dF(w') + \beta \int_c^B w' dF(w')$$

$$\hat{w}_1 = c + \frac{\beta}{1+\beta} \int_c^B (w' - c)\, dF(w') > c = \hat{w}_2$$

Therefore $\hat{w}_1 > \hat{w}_2$; the reservation wage is decreasing with time,