ECON0057 Lecture 1

Stochastic Dynamic Programming

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Introduction

- Models studied in this course: **dynamic** and **stochastic**
  - capital accumulation with productivity shocks
  - consumption/savings with uncertain income
- Goal of the lecture:
  - Start from a complex sequential problem (SP):

    "How much to invest in each period given past history of shocks"
  - Transform it into a simpler quasi 2-period problem:

    "How much to invest today given "value" of capital tomorrow"
- Equivalence between the 2 problems: **Principle of Optimality** (PO)
  - Bellman's insight: "Bridge between complex and simple problems"
- Reference Stokey-Lucas-Prescott (SLP) chapters 3-4 and 9

Intro: Stochastic Growth

Neoclassical Growth Model with productivity shocks, output given by:

$$y_t = z_t f(k_t)$$

TFP shocks z_t is **stochastic**. We denote by $z^t = z_0, z_1, ..., z_t$ an **history** of shocks.

Sequential Problem **(SP)**:

Given initial capital $k_0 \geq 0$ and productivity z_0 find an optimal investment plan:

$$V^*(k_0, z_0) = \sup_{\{k_t(z^{t-1})\}_{t>0}} \mathbb{E}\left(\sum_{t=0}^{\infty} \beta^t u(c_t(z^t))\right)$$

$$\begin{aligned} s.t. \quad & c_t(z^t) + k_{t+1}(z^t) = z_t f(k_t(z^{t-1})) + (1-\delta)k_t(z^{t-1}) \\ & c_t(z^t) \geq 0, \quad k_{t+1}(z^t) \geq 0 \end{aligned}$$

Optimal investment at t, $k_t(z^{t-1})$ is a function of the full history of past shocks z^{t-1}

Unknown of the problem is the full sequence $\{k_1(z^0), ..., k_t(z^{t-1}), ...\}_{0<t}$

Value function $V^*(k_0, z_0)$: maximal welfare attainable given k_0 and z_0

Intro: Markov Processes

- z_t drawn from $Z = \{z_G, z_B\}$, with $z_G > z_B$
- transition probability $P(z_{t+1}|z_t)$ stacked in **transition kernel**

$$\Pi = \begin{pmatrix} P(z_G|z_G) & P(z_B|z_G) \\ P(z_G|z_B) & P(z_B|z_B) \end{pmatrix}$$

  with $P(z_G|z_i) + P(z_B|z_i) = 1$

- transitory shocks $P(z_i|z_j) = P(z_i|z_i)$, persistent shocks $P(z_i|z_j) < P(z_i|z_i)$
- probability of an history $z^t = \{z_0, ..., z_t\}$ is given by $P(z^t|z_0) = P(z_1|z_0)P(z_2|z_1)...P(z_t|z_{t-1})$

Intuitively, V^* should solve (but it's a bit technical to show):

$$\begin{aligned} V^*(k_t, z_t) &= \sup_{k_{t+1}} \left\{ u(c_t) + \beta \sup_{\{k_{t+2}, k_{t+3}, \ldots\}} \mathbb{E} \left(\sum_{s=0}^{\infty} \beta^s u(c_{t+1+s}) | k_{t+1}, z_t \right) \right\} \\ &= \sup_{k_{t+1}} \left\{ u(c_t) + \beta \mathbb{E} \left(V^*(k_{t+1}, z_{t+1}) | z_t \right) \right\} \end{aligned}$$

Idea of dynamic programming, solve the **functional equation** (FE):

$$\begin{aligned} V(k, z) &= \sup_{k'} u(c) + \beta \mathbb{E} \left(V(k', z') | z \right) \\ s.t. \quad c + k' &= zf(k) + (1 - \delta)k \\ c &\geq 0, \quad k' \geq 0 \end{aligned}$$

The unknown is a function $V(k, z)$ and hopefully $V^*(k, z) = V(k, z)$

Usefulness of Dynamic Programming

- (PO) says (FE) and (SP) are equivalent (under some conditions)
  - optimal plan of (SP) $k_{t+1}(z^t)$ is then defined recursively as $k_{t+1}(z^t) = k'(k_t(z^{t-1}), z_t)$ with k' defined in (FE)
- (FE) simpler and useful to:
  - solve the problem numerically (via iteration)
  - derive properties of V^*: continuity, concavity, differentiability
  - easier to derive the Euler equation (direct link between c_t and c_{t+1}, crucial for empirics)

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1. Principle of optimality

2. The Bellman Equation

3. Euler Equation

- Exogenous random shocks z drawn every period in $Z \subset \mathbb{R}^d$ (think TFP shocks)
  - Formally should define a measurable space $(Z, \mathcal{Z})$, $\mathcal{Z}$ a σ-algebra of Z (optional, read chapter 7-8)
- z is a Markov chain: the distribution of z_{t+1} only depends on z_t
- Evolution of z defined by a transition kernel $Q : Z \times Z \to [0, 1]$, where $Q(z, \cdot)$ is a probability distribution over Z
  - informally, the probability of $z_{t+1} = z'$ given $z_t = z$ is given by $P(z'|z) = Q(z, z')$.
  - Q defines probability distribution over any sequence $\{z_0, ..., z_t\}$:

$$P(z^t|z_0) = Q(z_0, z_1)Q(z_1, z_2)...Q(z_{t-1}, z_t)$$

- Example: $z_{t+1} = \rho z_t + \epsilon_{t+1}$ with $0 \leq \rho < 1$ and ϵ_t iid and $\epsilon_t \sim \mathcal{N}(0, \sigma)$
  - $Z = \mathbb{R}$
  - $Q(z, z') = 1/\sigma\sqrt{2\pi} exp(-0.5(z' - \rho z)^2/\sigma^2)$

Reward Function and Feasible Plan

- Endogenous state at period beginning is x in $X \subset \mathbb{R}^d$ (think capital k_t)
- Agent chooses next period state y in $X \subset \mathbb{R}^d$ under constraint $y \in \Gamma(x, z)$
  - in NGM $x = k_t$, $y = k_{t+1}$, $\Gamma(x, z) = \{y, s.t.\ 0 \leq y \leq zf(x) + (1-\delta)x\}$
- Choosing y given x, z, agent receives $F(x, y, z) \geq 0$ in current period
  - This is the reward function (in general utility or profit function)
  - in NGM $F(x, y, z) = u(zf(x) + (1-\delta)x - y)$
- Define a feasible plan π as a sequence of functions $\{\pi_t(z^t)\}_{t\geq 0}$ with:

$$\pi_0 \in \Gamma(x_0, z_0)$$
$$\pi_t(z^t, x_0) \in \Gamma(\pi_{t-1}(z^{t-1}), z_t)$$

Denote by $\Pi(x_0, z_0)$ the set of all feasible plans starting from x_0, z_0

  - in NGM, a feasible plan is a sequence of investment choices $\{k_{t+1}(z^t, k_0)\}_{t\geq 0}$ with $k_{t+1} \geq 0$ and $c_t \geq 0$

Sequential Problem and Functional Equation

Sequential Problem:

$$V^*(x_0, z_0) = \sup_{\pi \in \Pi(x_0, z_0)} \mathbb{E}\left(\sum_{t \geq 0} \beta^t F\left(\pi_{t-1}(z^{t-1}), \pi_t(z^t), z_t\right) \Big| z_0 \right)$$

(slight abuse of notation, $\pi_{-1} = x_0$)

Bellman Equation:

$$V(x, z) = \sup_{y \in \Gamma(x,z)} \left\{ F(x, y, z) + \beta \int V(y, z') Q(z, z') dz' \right\}$$

- Principle of Optimality: conditions under which the 2 pbs are equivalent

Principle of Optimality

High level assumptions (will be verified with more structure on F Q and Γ)

A1 Solution of (FE), V, satisfies for all $\pi \in \Pi(x_0, z_0)$

$$\lim_{t\to\infty} \mathbb{E}\left(\beta^t V(\pi_{t-1}(z^{t-1}), z_t)|z_0\right) = 0$$

A2 For any x and z, there exists $y^*(x,z) \in \Gamma(x,z)$ such that

$$V(x,z) = F(x,z,y^*(x,z)) + \beta \int V(y^*(x,z), z')Q(z,z')dz'$$

Theorem

If V solves (FE) and satisfies **A1** and **A2** then:

- $V(x_0, z_0) = V^*(x_0, z_0)$
- π^* defined by $\pi_0^* = y^*(x_0, z_0)$, $\pi_t^*(z^t) = y^*(\pi_{t-1}^*(z^{t-1}), z_t)$ reaches the optimum of (SP)

Proof

Take any $\pi \in \Pi$, by definition of V,

$$V(x_0, z_0) \geq F(x_0, \pi_0, z_0) + \beta\mathbb{E}(V(\pi_0, z_1)|z_0)$$

Applying the same to $V(\pi_0, z_1)$, ..., $V(\pi_t, z_{t+1})$:

$$V(x_0, z_0) \geq \mathbb{E}\left(\sum_{t=0}^{t} \beta^s F(\pi_{s-1}, \pi_s, z_s)\middle| z_0\right) + \beta^{t+1}\mathbb{E}(V(\pi_{t+1}, z_{t+2})|z_0)$$

Using **A1**: $V(x_0, z_0) \geq \mathbb{E}\left(\sum_{t=0}^{\infty} \beta^s F(\pi_{s-1}, \pi_s, z_s)\middle| z_0\right) \Rightarrow V(x_0, z_0) \geq V^*(x_0, z_0)$

Using **A2** and applying the same expansion with π^*:

$$V(x_0, z_0) = \mathbb{E}\left(\sum_{t=0}^{\infty} \beta^s F(\pi^*_{s-1}, \pi^*_s, z_s)\middle| z_0\right)$$

$\Rightarrow V(x_0, z_0) = V^*(x_0, z_0)$ and π^* solves (SP)

Discussion

- A partial converse also exists:
  - Under additional assumptions, if V^*, π^* solve (SP) then they solve (FE)
  - Slightly more technical (and less useful) see SLP theorem 9.4 for details
- Theorem only says "if a solution of (FE) exists then it satisfies (SP)"
- Does NOT say that a solution of (FE) exists
- Next step: show that it does and can have nice properties (continuity, concavity, differentiability)

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A Functional Operator

Bellman Equation:

$$V(x,z) = \sup_{y \in \Gamma(x,z)} \left\{ F(x,y,z) + \beta \int V(y,z')Q(z,z')dz' \right\}$$

defines V as a fixed point of an operator T^* that takes a function $f : X \times Z \to \mathbb{R}$ and gives a new function $T^* f : X \times Z \to \mathbb{R}$:

$$(T^* f)(x,z) = \sup_{y \in \Gamma(x,z)} \left\{ F(x,y,z) + \beta \int f(y,z')Q(z,z')dz' \right\}$$

The Bellman equation can be rewritten $T^* V = V$: fixed point of a functional operator

Simple but powerful theorem when T^* is a **contraction** on a **Banach space** will allow us to prove that (FE) has a unique solution

Contraction Mapping Theorem

You don't really need to know what a Banach space is (but it's useful! read SLP chapter 3 for a simple intro to functional analysis)

For our purposes, the space $\mathcal{C}(Y)$ of continuous bounded functions from $Y \subset \mathbb{R}^d \to \mathbb{R}$ with the norm $||f|| = \sup_{y \in Y} |f(y)|$ is a Banach space

Contraction Mapping Theorem (SLP Chapter 3)

Consider a Banach space S with norm $|| \cdot ||$, a mapping $T : S \to S$ and $0 < \beta < 1$, such that for any $f, g \in S$:

$$||Tf - Tg|| \leq \beta ||f - g||$$

Then T is a contraction: it has a unique fixed point $TV = V$ and for any $v_0 \in S$ $||T^n v_0 - V|| \leq \beta^n ||v_0 - V||$.

Idea of the proof: take any v and define $v^n = T^n v$, for n large enough v_m and v_n are close:

$$||v^m - v^n|| \leq ||v^m - v^{m-1}|| + \ldots + ||v^{n+1} - v^n|| \leq (\beta^{m-1} + \ldots + \beta^n)||v^0 - v^1|| \leq \frac{\beta^n}{1 - \beta}||v^0 - v^1||$$

v^n converges and the limit v satisfies $Tv = v$

Discussion

- If we can show that T^* is a contraction then we're done!
- Very useful in practice: just start from any guess $V_0(x, z)$, apply T^* "enough" times and you get V that solves (FE)
- Showing that an operator T is a contraction can be a bit hard, but we have useful sufficient condition

Blackwell Sufficient Conditions (SLP Chapter 3)

Consider set of bounded functions from Y to $\mathbb{R}$, $\mathcal{B}(Y)$ and an operator $T : \mathcal{B}(Y) \to \mathcal{B}(Y)$. If T satisfies:

- For any $f, g \in \mathcal{B}(Y)$, with $f \leq g$, $Tf \leq Tg$
- There exists some β such that $T(f + a) \leq Tf + \beta a$ for all $f \in \mathcal{B}(Y)$, $a > 0$

Then T is a contraction

Theorem of the Maximum

Slight difficulty T^* has a *sup* in it, we need to know if taking the sup conserves property of functions, in particular continuity.
Consider the problem:

$$h(x,z) = \sup_{y \in \Gamma(x,z)} g(x,y,z)$$

Under some technical assumptions (see SLP 3.3) on Γ and g, h will be a continuous function of x, z:

Theorem of the Maximum (SLP Chapter 3)

If g is continuous and Γ is compact valued and continuous then h is continuous

The technicality comes from the fact that Γ is not a function but a "correspondence" (its value is a set). In most applications, $\Gamma(x,z)$ will be an interval where the bounds depend on x, z: $\Gamma(x,z) = [l(x,z), u(x,z)]$. Just think of continuity of Γ as continuity of $l(x,z)$ and $u(x,z)$.

Technical Assumptions

Goal now, simply to impose assumptions such that CMT and ToM apply:

B1. X is convex (used to show concavity of V)
B2. $F(x, y, z)$ is bounded continuous and $0 < \beta < 1$
B3. Γ is compact valued and continuous
B4. Z compact and Q satisfies the Feller property: for any continuous $f(x, \cdot)$,

$$g(x, z) = \int f(x, z')Q(z, z')dz'$$

is continuous in z

Unlike **A1** and **A2**, **B1**-**B4** can be directly checked. Moreover **B1**-**B4** imply **A1** and **A2** (you'll see!): our previous results are valid!

Z compact can be a bit constraining in Economic applications, see Chapter 12 for a generalization

A Technical Lemma

Lemma

Suppose that **B1** and **B4** hold. For any $f \in \mathcal{C}(X \times Z)$ (continuous and bounded),

$$g(y,z) = \int f(y,z')Q(z,z')dz'$$

is bounded and continuous. If $f(\cdot, z)$ is increasing (concave), then $g(\cdot, z)$ is increasing (concave)

Proof. Boundedness: $|g(y,z)| \leq \int ||f||Q(z,z')dz' = ||f||$.
Continuity: Take a sequence $(y_n, z_n) \to (y,z)$

$$|g(y,z) - g(y_n,z_n)| \leq \left| \int f(y,z')Q(z,z')dz' - \int f(y,z')Q(z_n,z')dz' \right|$$
$$+ \int |f(y,z') - f(y_n,z')|Q(z_n,z')dz'$$

First term vanishes by **B4**. Second vanishes since for N large enough (y_n, z') belongs to a compact $D \times Z$: f continuous $\Rightarrow f$ uniformly continuous on $D \times Z \Rightarrow |f(y,z') - f(y_n,z')|$ goes to 0 for all z'
See SLP for monotonicity/concavity

Existence and Uniqueness of V

Theorem

Suppose that **B1**-**B4** hold. Then:

- for any $f \in \mathcal{C}(X \times Z)$, $T^* f \in \mathcal{C}(X \times Z)$
- there exists a unique $V \in \mathcal{C}(X \times Z)$ such that $V = T^* V$ and for any $V_0 \in \mathcal{C}(X \times Z)$, $||V - (T^*)^n V_0|| \leq \beta^n ||V - V_0||$

Proof. Lemma shows that $\int f(y, z') Q(z, z') dz'$ is continuous. Since F is continuous and Γ compact valued and continuous, the ToM applies and $T^* f$ is continuous.

Direct to show that if $f \leq g$ then $T^* f \leq T^* g$ (do it!) and $T^*(f + a) = T^* f + \beta a$ hence blackwell sufficiency conditions apply: T^* is a contraction and the CMT applies.

Remark. **A1** is satisfied since V is bounded. **A2** is satisfied since V is continuous and $\Gamma(x, z)$ is compact.

Theorem

1. Suppose in addition to **B1**-**B4** that $F(\cdot, y, z)$ is increasing and that for $x \leq x'$, $\Gamma(x, z) \subset \Gamma(x', z)$ then $V(\cdot, z)$ is increasing
2. Suppose in addition to **B1**-**B4** that $F(\cdot, \cdot, z)$ is concave and that for $0 < \theta < 1$ $\theta\Gamma(x, z) + (1-\theta)\Gamma(x', z) \subset \Gamma(\theta x + (1-\theta)x', z)$ then $V(\cdot, z)$ is concave

The space of weakly-increasing (or concave), bounded and continuous functions are closed subsets of Banach spaces (subspaces of $\mathcal{C}(X \times Z)$). With our Lemma, we can easily show that T^* maps increasing (concave) functions to increasing (concave) functions and use the CMT to prove the result.

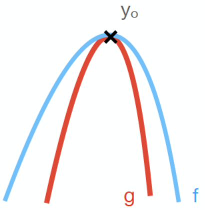
When 2 is satisfied, we have a useful additional property. Take an arbitrary V_0, bounded continuous and concave in its first 2 arguments and define $V_{n+1} = T^* V_n$, then the approximate policy function $\pi^n(x, z)$ defined as:

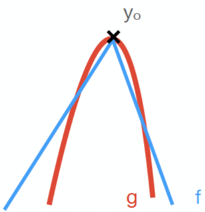
$$\pi^n(x, z) = \text{argmax}_{y \in \Gamma(x,z)} F(x, y, z) + \beta \mathbb{E}\left(V_n(y, z')|z\right)$$

converges towards the optimal policy function π^*

Benveniste & Scheinkman

Consider a convex set $Y \subset \mathbb{R}^d$ and $f : Y \to \mathbb{R}$ a concave function. Let $y_0 \in intY$ and $N(y_0)$ be a neighborhood of of y_0.
If there exists a concave, differentiable function $g : N(y_0) \to \mathbb{R}$ such that $f(y_0) = g(y_0)$ and $g(y_0) \leq f(y_0)$ on $N(y_0)$, then f is differentiable at y_0 and its derivatives are the same as g.





Idea of the proof: g constrains the *sub-gradients* of f

Differentiability of V

Theorem

Suppose that in addition to **B1**-**B4** that $F(\cdot, \cdot, z)$ is differentiable and concave and that for $0 < \theta < 1$ $\theta\Gamma(x, z) + (1-\theta)\Gamma(x', z) \subset \Gamma(\theta x + (1-\theta)x', z)$. Then V is differentiable and:

$$\nabla_x V(x, z) = \nabla_x F(x, y^*(x, z), z)$$

fix x_0 z_0 and consider the function:

$$g(x, z_0) = F(x, y^*(x_0, z_0), z_0) + \beta\mathbb{E}\left(V(y^*(x_0, z_0), z_0)|z_0\right)$$

Clearly $g(x, z_0) \leq V(x, z_0)$, $g(x_0, z_0) = V(x_0, z_0)$ and g is concave and differentiable in $x \Rightarrow$ Benveniste & Scheinkman applies

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Euler Equation

- The Bellman Equation is a relationship between the value function V "today" and "tomorrow"
- But V is not "observable" (measurable in datasets), for empirical purposes it would be better to have a direct relationship on the policy function (eg. consumption, investment, etc.)
- Also useful for computation
- Present here an informal derivation, see SLP for technical details
- We assume for simplicity that $X \subset \mathbb{R}$ (but as easy to do in higher dimensions)

Recall the Bellman Equation:

$$V(x,z) = \sup_{y \in \Gamma(x,z)} F(x,y,z) + \beta \mathbb{E}\left(V(y,z')|z\right)$$

If F strictly concave and differentiable, V concave and the optimal y (if interior) is defined by the first order condition (FOC):

$$\partial_y F(x,y,z) = -\beta \mathbb{E}\left(\partial_x V(y,z')|z\right)$$

From our theorem, we know $\partial_x V(y,z')$, here I give an informal but useful derivation of the "envelope theorem"

$$\begin{aligned} \partial_x V(x,z) &= \partial_x F(x, y^*(x,z), z) \\ &+ \partial_y F(x, y^*(x,z), z) \partial_x y^*(x,z) + \beta \mathbb{E}\left(\partial_x V(y^*(x,z), z') \partial_x y^*(x,z)|z\right) \end{aligned}$$

Using the (FOC), the second line is 0 and we have $\partial_x V(x,z) = \partial_x F(x, y^*(x,z), z)$

Euler Equation: Informal Derivation 2

Putting the FOC and the envelope condition together, we get the Euler equation:

$$\partial_y F(x, y^*(x,z)z) = -\beta \mathbb{E}\left(\partial_x F(y^*(x,z), y^*(y^*(x,z),z')), z')|z\right)$$

In the NGM, recall that we have $F(x,y,z) = u(zf(x) + (1-\delta)x - y)$, with $x = k_t$, $y = k_{t+1}$, $c_t = zf(x) + (1-\delta)x - y$, so Euler reads:

$$u'(c_t) = \beta \mathbb{E}(R_t u'(c_{t+1}))$$

with $R_t = z_{t+1} f'(k_{t+1}) + 1 - \delta$. Constrains the growth rate of consumption

Transversality Condition

The Euler equation by itself does not pin down the solution of (SP),intuitively it only constrains the growth not the level of the solution. An additional condition is needed to select the optimal plan:

$$\lim_{t\to\infty} \beta^t \partial_x F(x_t, y_t, z_t) x_t = 0 \qquad \textit{almost surely}$$

This is a "terminal" condition. Think of the NGM in finite horizon: in the last period, if the marginal utility of consumption is positive, everything is consumed ($k_{t+1} = 0$). Since the horizon is infinite, the transversality condition says that at the limit, this is satisfied (Note: the transversality condition is a "complementary slackness" condition: either the costate or the state variable is 0 at the limit)

Theorem

Suppose that **B1**-**B4** hold, that $F(\cdot, \cdot, z)$ is differentiable and concave and that $\Gamma(\cdot, z)$ is concave. The sequence $x_0, .., .x_t(z^{t-1}), ...$ that satisfies the Euler Equation, the transversality condition and $x_{t+1}(z^t) \in int\Gamma(x_t(z^{t-1}), z_t)$ is optimal for the (SP) problem