Alternate Derivation of Optimal k in "Optimal Currency Hedging for International Equity Portfolios"[1] Using Block Matrix Inverse

Inverse of block matrix[2]: $\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} + A^{-1}BS^{-1}CA^{-1} & -A^{-1}BS^{-1} \\ -S^{-1}CA^{-1} & S^{-1} \end{bmatrix}$

Where Schur complement of the block A is $S = D - CA^{-1}B$.

Covariance matrix of n currencies and 1 equity index: $\Sigma = \begin{bmatrix} \Sigma_{FX} & \Sigma_{EQFX} \\ \Sigma'_{EQFX} & \sigma^2_{EQ} \end{bmatrix} = \begin{bmatrix} A_{n\times n} & B_{n\times 1} \\ B'_{1\times n} & D_{1\times 1} \end{bmatrix}$.

Expected returns vector: $\alpha = \begin{bmatrix} \alpha_{FX} \\ \alpha_{EQ} \end{bmatrix} = \begin{bmatrix} v \\ a \end{bmatrix}$.

The Markowitz mean-variance optimal (maximal Sharpe ratio) portfolio is $\Sigma^{-1}\alpha$.

$$\begin{aligned} \begin{bmatrix} A & B \\ B' & D \end{bmatrix}^{-1} \begin{bmatrix} \alpha_{FX} \\ a \end{bmatrix} &= \begin{bmatrix} A^{-1} + A^{-1}BS^{-1}B'A^{-1} & -A^{-1}BS^{-1} \\ -S^{-1}B'A^{-1} & S^{-1} \end{bmatrix} \begin{bmatrix} v \\ a \end{bmatrix} \\ &= \begin{bmatrix} A^{-1}v + A^{-1}BS^{-1}B'A^{-1}v - A^{-1}BS^{-1}a \\ -S^{-1}B'A^{-1}v + S^{-1}a \end{bmatrix} \\ &= \begin{bmatrix} A^{-1}v - \left(\frac{a - B'A^{-1}v}{S}\right)A^{-1}B \\ \left(\frac{a - B'A^{-1}v}{S}\right) \end{bmatrix} \end{aligned}$$

The third equality above uses the fact that $CA^{-1}B = B'A^{-1}B$ and $B'A^{-1}v$ both have dimension $(1 \times n)(n \times n)(n \times 1)$, so S and $B'A^{-1}v$ are scalars. Scaling preserves Sharpe ratio; multiply by $\left(\frac{S}{a - B'A^{-1}v}\right)$ to attain a vector with 1 as the equity weight.

$$w_{opt,w_{EQ}=1} = \begin{bmatrix} -A^{-1}B + \left(\frac{S}{a - B'A^{-1}v}\right)A^{-1}v \\ 1 \end{bmatrix} = \begin{bmatrix} -{\Sigma_{FX}}^{-1}\Sigma_{EQFX} + k_{opt}{\Sigma_{FX}}^{-1}\alpha_{FX} \\ 1 \end{bmatrix}$$

$$k_{opt} = \frac{\sigma^2_{EQ} - \Sigma'_{EQFX}{\Sigma_{FX}}^{-1}\Sigma_{EQFX}}{\alpha_{EQ} - \Sigma'_{EQFX}{\Sigma_{FX}}^{-1}\alpha_{FX}} = \frac{\sigma^2_{MINVAR}}{\alpha_{MINVAR}}$$

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[1] Jacob Boudoukh, Matthew Richardson, Ashwin Thapar & Franklin Wang (2019) Optimal Currency Hedging for International Equity Portfolios, Financial Analysts Journal, DOI: 10.1080/0015198X.2019.1628556

[2] https://www.cis.upenn.edu/~jean/schur-comp.pdf

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September 16, 2019