1 Introduction

Notation:

- F a totally real number field.
- $p > 3$ a prime, $L/\mathbb{Q}_p$ finite field extension, $\mathcal{O}$ the ring of integers of L with maximal ideal λ and residue field $\mathbb{F}$. L will be assumed to be "large enough". This means that we demand that L contains a primitive p-th root of unity, the images of all embeddings $F \hookrightarrow \overline{L}$ and $\mathbb{F}$ contains certain eigenvalues (of a mod λ Galois representation).
- G_K the absolute Galois group of a field K.

In talk 3 we have seen that we can always go from regular algebraic cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A}_F)$ to compatible systems of Galois representations. We would like to do the converse: Starting with a continuous Galois representation $\rho : G_F \to \mathrm{GL}_2(L)$ we would like to find a regular algebraic cuspidal automorphic representations π of $\mathrm{GL}_2(\mathbb{A}_F)$ s.t. ρ occurs in the compatible system associated with π. It is no loss of generality to assume $\mathrm{im}(\rho) \subset \mathrm{GL}_2(\mathcal{O})$ (one can always conjugate to achieve that).

The idea of modularity lifting is the following: Suppose we are given two continuous representations, $\rho_0, \rho : G_F \to \mathrm{GL}_2(\mathcal{O})$ s.t. we already know that ρ_0 is modular and s.t. the reductions of ρ and ρ_0 mod λ agree. Then, under certain additional conditions, we want to conclude that also ρ is modular. The precise statement is:

Theorem 1
Let $\rho_0, \rho : G_F \to \mathrm{GL}_2(\mathcal{O})$ be continuous representations s.t. $(\rho \bmod \lambda) = (\rho_0 \bmod \lambda) =: \overline{\rho}$. Assume that ρ_0 is modular, that ρ is geometric and that L is sufficiently large.[1] Assume further:

1) p is unramified in F.
2) $\mathrm{im}(\overline{\rho}) \supset \mathrm{SL}_2(\mathbb{F}_p)$.
3) For all embeddings $\sigma : F \hookrightarrow L$ and all places $v \mid p$ of F the following hold:
   a) $\mathrm{HT}_\sigma(\rho) = \mathrm{HT}_\sigma(\rho_0)$ and this set contains two distinct elements.
   b) The elements of $\mathrm{HT}_\sigma(\rho)$ differ by at most $p-2$.
   c) $\rho|_{G_v}$ and $\rho_0|_{G_v}$ are crystalline.

Then ρ is modular.

Recall that the condition that ρ is geometric means that ρ is unramified almost everywhere and that $\rho|G_{F_v}$ is de Rham for all places $v \mid p$ of F. Condition 1) is a condition on F and condition 2) is a condition on $\overline{\rho}$. Note that 2) implies that $\overline{\rho}$ is absolutely irreducible (i.e. irreducible after $\otimes\overline{\mathbb{F}}$). The conditions in 3) are Hodge-theoretic conditions. We won't focus on them in this talk.

2 Approach

The first part of the approach of proving the Theorem can be describes as follows. We want to define the following objects

- a "relevant" universal deformation ring $R_\varnothing{}^{\mathrm{univ}}$,

[1] Precisely, L should contain a primitive p-th root of unity and the images of all embeddings $L \hookrightarrow \overline{\mathbb{Q}_p}$ and additionally $\mathbb{F}$ should contain all eigenvalues of all $\overline{\rho}(g)$, $g \in G_F$.

- a "relevant" space of modular forms $S_\varnothing$,
- a "relevant" Hecke algebra $\mathbb{T}_\varnothing$
- a universal modular representation ${\rho_\varnothing}^{\mathrm{mod}} : G_F \to \mathrm{GL}_2(\mathbb{T}_\varnothing)$

s.t. the following conditions[2] hold:

D1) $\rho \in {R_\varnothing}^{\mathrm{univ}}(\mathcal{O})$

D2) $\mathbb{T}_\varnothing$ is reduced and acts faithfully on $S_\varnothing$.

D3) ${\rho_\varnothing}^{\mathrm{mod}}$ induces a surjection $\theta : R_\varnothing \twoheadrightarrow \mathbb{T}_\varnothing$.

D4) For $r \in {R_\varnothing}^{\mathrm{univ}}(\mathcal{O})$, the existence of a factorization

$$\begin{tikzcd}
{R_\varnothing}^{\mathrm{univ}} \arrow[rr, "\text{induced by } r"] \arrow[dr, two heads, "\theta"'] & & \mathcal{O} \\
& \mathbb{T}_\varnothing \arrow[ur, dotted] &
\end{tikzcd}$$

implies that r is modular.

Lemma 2 (Reduction to Commutative Algebra Statement)
In this situation ρ_0 is modular if

$$\mathrm{Supp}_{{R_\varnothing}^{\mathrm{univ}}}(S_\varnothing) = \mathrm{Spec}({R_\varnothing}^{\mathrm{univ}}).$$

Here, $S_\varnothing$ is a $\mathbb{T}_\varnothing$-module and becomes an ${R_\varnothing}^{\mathrm{univ}}$-modules via θ.

Proof.
Let $({R_\varnothing}^{\mathrm{univ}})_{\mathrm{red}}$ denote the quotient of ${R_\varnothing}^{\mathrm{univ}}$ by its nilradical and consider the following enlarged version of the diagram in D4):

$$\begin{tikzcd}
({R_\varnothing}^{\mathrm{univ}})_{\mathrm{red}} \arrow[drrr, bend left=50] & & & \\
& {R_\varnothing}^{\mathrm{univ}} \arrow[ul, two heads] \arrow[rr, "\text{induced by } r"] \arrow[dr, two heads, "\theta"'] & & \mathcal{O} \\
& & \mathbb{T}_\varnothing \arrow[ur, dotted] &
\end{tikzcd}$$

By D2), $\mathbb{T}_\varnothing$ is reduced so θ factors through ${R_\varnothing}^{\mathrm{univ}} \twoheadrightarrow ({R_\varnothing}^{\mathrm{univ}})_{\mathrm{red}}$ inducing $\overline{\theta} : ({R_\varnothing}^{\mathrm{univ}})_{\mathrm{red}} \twoheadrightarrow \mathbb{T}\varnothing$.
Now D1) and D4) clearly imply that it suffices to show that $\overline{\theta}$ is an isomorphism.
Take $x \in \ker(\theta)$, so x acts as 0 on $S_\varnothing$. Then x must be contained in any $\mathfrak{p} \in \mathrm{Spec}({R_\varnothing}^{\mathrm{univ}})$, since otherwise $S_{\varnothing,\mathfrak{p}} = 0$, contradicting $\mathrm{Supp}_{{R_\varnothing}^{\mathrm{univ}}}(S_\varnothing) = \mathrm{Spec}({R_\varnothing}^{\mathrm{univ}})$. Thus, x is nilpotent. This shows that $\ker(\theta)$ is contained in the nilradical of ${R_\varnothing}^{\mathrm{univ}}$, which means $\ker(\overline{\theta}) = 0$. □

Aim from heron: Define the objects listed above (and some more that will be used later) and check properties D1)-D4).

[2]Our notation is $C(B) := \mathrm{Hom}_A(C, B)$ for two A-algebras B, C.

3 Base Change Reduction Step

One can use base change to replace F by a solvable totally real extension. Doing this, one may assume that the following assumptions are satisfied (plus the assumptions of the Theorem of course):

1) $[F:\mathbb{Q}]$ is even.

2) For all places $v \nmid p$: $\overline{\rho}|_{G_{F_v}}$ is unramified.

3) For all places $v \nmid p$: $\rho(G_{F_v})$ and $\rho_0(G_{F_v})$ are unipotent.

4) For all places $v \nmid p$ s.t. ρ or ρ_0 is ramified: $\overline{\rho}|_{G_{F_v}} = 1$.

5) $\det(\rho) = \det(\rho_0)$.

4 Deformation Ring Machine

Recall the general machine: Given $T \subset S$ finite sets of finite places of F, $\overline{\rho} : G_F \to \mathrm{GL}_2(\mathbb{F})$ absolutely irreducible.
Input: $(\mathcal{D}_v)_{v \in S}$, $\chi : G_F \to \mathcal{O}^\times$, where

- $\mathcal{D}_v$ is a deformation problem, i.e. a certain subfunctor of $R^{\square_T}_{\overline{\rho}|_{G_{F_v}}}$.

- χ is a continuous group homomorphism.

Output:

- $R^{\square_T}_{S,\overline{\rho},\chi}$ represents T-framed deformations of $\overline{\rho}$ unramified outside S having determinant χ.

- $R^{\mathrm{univ}}_{S,\overline{\rho},\chi}$ represents deformations of $\overline{\rho}$ unramified outside S having determinant χ.

- $R^{\mathrm{loc}}_{S,\overline{\rho},\chi} := \hat{\bigotimes}_{v \in T} R^{\square}_{\overline{\rho}|_{G_{F_v}}, \chi|_{G_{F_v}}}$

We want to apply this to the situation of the Theorem. We take $\overline{\rho}$ as in the theorem and $T := T_p \sqcup T_r$, where $T_p := \{v \mid p\}$ and $T_r := \{v \mid \rho \text{ or } \rho_0 \text{ ramified at } v\}$. We take $\chi := \det\rho = \det\rho_0$.

Let Q be a finite set of finite places of F s.t. $Q \cap T = \varnothing$. Set $S := T \sqcup Q$. We also need to fix some choices:

- a primitive p-th root of unity $\zeta \in L$

- for each $v \in T_r$ some $\sigma_v \in I_v$ that maps to a topological generator in I_v/P_v

Then we define two families $(D_v)_{v \in S}$ and $(D'_v)_{v \in S}$ of local deformation problems (we refer to $(D_v)_{v \in S}$ as the "standard situation" and to $(D'_v)_{v \in S}$ as the "prime situation"; the standard situation is the situation we are interested in; the prime situation is nicely behaved; in the end of the proof we compare the standard situation with the prime situation to transfer information available in the prime situation to the standard situation; the relationship is that both situations agree mod λ):

	D_v	D'_v
$v \in T_p$	"crystalline lifts of correct HT weights"	"crystalline lifts of correct HT weights"
$v \in T_r$	lifts $\tilde{\rho}$ with $\mathrm{charpo}_{\tilde{\rho}(\sigma_v)} = (X-1)^2$	lifts $\tilde{\rho}$ with $\mathrm{charpo}_{\tilde{\rho}(\sigma_v)} = (X-\zeta)(X-\zeta^{-1})$
$v \in Q$	**all** lifts	**all** lifts

Output: ${R_Q}^{\mathrm{univ}}$, $R_Q^{\square}$, R_Q^{loc}

Check: D1) holds for $Q = \varnothing$. The only thing to note is that by the reduction step $\rho(I_v)$ is unipotent for $v \in T_r$, so the characteristic polynomial has the correct form.

5 Quaternionic Automorphic Representations

Idea: By the reduction step, $[F:\mathbb{Q}]$ is even. Since F is totally real, this means $\#\{v \mid \infty\}$ is even so we find a unique quaternion algebra D over F that is ramified precisely at $\{v \mid \infty\}$. By Jacquet-Langlands, every regular algebraic cuspidal automorphic representation of $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_2$ comes from a unique infinite-dimensional regular algebraic automorphic representation of $\mathrm{Res}_{F/\mathbb{Q}}D$ (of the same weight; note that such a representation of $\mathrm{Res}_{F/\mathbb{Q}}D$ is automatic cuspidal, since D is anisotropic modulo center). This leads to the

Aim:

- Construct an explicit space of modular forms (for D) that serves as a model for automorphic representations.
- Go further and define an integral version of this space (plus corresponding Hecke algebra) that is defined over $\mathcal{O}$. This then allows us to compare with $R_\varnothing{}^{\mathrm{univ}}$, which is also defined over $\mathcal{O}$.

The general construction goes as follows: Fix $\iota : \overline{L} \cong \mathbb{C}$.

Input:

- weights $(K_\tau, \eta_\tau)_{\tau\mid\infty}$ with $k_\tau \in \mathbb{Z}_{\geq 2}$ and $\eta_v \in \mathbb{Z}$ (think of k_τ as indexing discrete series representation and η_v as indexing (twists with) algebraic characters of the center of the universal enveloping algebra of the complexification of the Lie algebra of GL_2),
- S a finite set of finite places disjoint from $\{v \mid p\}$,
- $U = \;()v$ finite$U_v = U_S \prod_{v\notin S,\ v \text{ finite}} \mathrm{GL}_2(\mathcal{O}_v)$ an open compact subgroup.
- $\psi : U_S \to \mathcal{O}^\times$ a continuous homomorphism,
- $\chi_0 : \mathbb{A}_F^\times / \overline{(F_\infty^\times)^\circ F^\times} \to \overline{L}$,

plus some compatibility conditions.

Output: $S_U \coloneqq S_{k,\eta,\psi,\chi_0}(U,\mathcal{O})$

The presise definition is:

$$S_U \coloneqq \{\Phi : D^\times \backslash \mathrm{GL}_2(\mathbb{A}_F) \to \Lambda \mid \Phi(zgu) = \chi_0(z)\tilde{\psi}^{-1}(u)\Phi(g)\},$$

where

- $\Lambda \coloneqq \otimes_{\tau:F\hookrightarrow\mathbb{C}} \mathrm{Sym}^{k_\tau-1}(\mathcal{O}^2) \otimes (\bigwedge^2 \mathcal{O}^2)^{\eta_\tau}$
- $\tilde{\psi} : U \to \mathrm{GL}(\Lambda)$ the representation induced by $U \ni u_S u_p u_{\mathrm{rest}} \mapsto \psi(u_S)u_p \in \mathrm{GL}(\mathcal{O}^2)$

Fact: S_U is a finite free $\mathcal{O}$-module (since $D^\times \mathrm{GL}_2(\mathbb{A}_F^\infty)/U(\mathbb{A}_F^\infty)^\times$ is finite).

Now we want to choose the input data in a way that S_U is related to ρ and ρ_0. We take

- the HT weights of ρ (which are also the HT weights for ρ_0) as weights
- $S \coloneqq T_r \sqcup Q$ (note that we don't need T_p now, the information at p is contained in Λ)
- $U_S = \prod_{v\in S} U_v$ with

$$U_v \coloneqq \begin{cases} \{A \mid A \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod v\}, & v \in T_r \\ \{A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid A \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod v,\ k(v)^\times \ni (a/d \bmod v) \mapsto 1 \in \Delta_v\}, & v \in Q \end{cases}$$

  where Δ_v is the maximal p-power quotient of $k(v)^\times$.

- $\psi = 1$
- $\chi_0 := \epsilon_p \det\rho$

In this way we obtain for each Q as above a space S^Q of modular forms. Again, there is a prime construction of spaces S'_Q. The only difference is that ψ gets replaced by ψ' which is trivial at $v \in Q$ and sends A to $(a/d \bmod v)$ and then σ_v to ζ.

Remark 3
The definition of U is not random, but corresponds to the conductor of ρ (which is also the conductor of ρ_0).

Next, we associate a Hecke algebra with the above data:

$$\mathbb{T}^Q := \mathrm{im}(\mathcal{O}[T_v, S_v, \tilde{U}_\omega \mid v, \omega] \to \mathrm{End}(S^Q)),$$

where

- v runs over all finite places of F not contained in $T \sqcup Q$
- ω runs over all places in Q
- the map is given by

$$\begin{aligned} S_v &\mapsto [U_v \mathrm{diag}(\phi_v, 1) U_v] \\ T_v &\mapsto [U_v \mathrm{diag}(\phi_v, \phi_v) U_v] \\ \tilde{U}_\omega &\mapsto [U_\omega \mathrm{diag}(\phi_v, 1) U_\omega]. \end{aligned}$$

Here the brackets on the right hand side denote the usual double coset operator.

Similar one obtains $\mathbb{T}^{Q,'}$. Clearly, $\mathbb{T}^Q$ is a commutative $\mathcal{O}$-algebra under which S^Q is a module (same for prime situation).

Facts:

- S^Q is a finite free $\mathbb{T}^Q$-module (same for prime)
- $\mathbb{T}^Q$ acts faithfully on S^Q (same for prime)
- ($Q = \varnothing$) Let Π be the set of cuspidal automorphic representations of $D^\times(\mathbb{A}_F)$ unramified outside $S = T_r \sqcup T_p$ s.t. $\mathrm{Hom}_{U_S}(\mathbb{C}, \pi_S) \neq 0$ (trivial U_S-action on $\mathbb{C}$; this means $\pi^U \neq 0$; think of this as a condition that comes from a condition on the conductor $\mathcal{C}$ of ρ (of the form "something $\subset \mathcal{C}$"); it is satisfied by the reduction step!).

In particular:

- $\mathbb{T}^\varnothing$ is reduced
- $\mathrm{Hom}_{\mathcal{O}\text{-alg}}(\mathbb{T}^\varnothing, \mathbb{C}) \leftrightarrow \Pi$, where $\mathbb{C}$ becomes an $\mathcal{O}$-algebra via ι.
- $\mathbb{T}^\varnothing \otimes_{\mathcal{O},\iota} \mathbb{C} \cong \prod_{\pi\in\Pi} \mathbb{C}$

Desideratum: Universal Galois deformation $\rho : G_F \to \mathrm{GL}_2(\mathbb{T}^\varnothing)$ so that we get ${R_\varnothing}^{\mathrm{univ}} \to \mathbb{T}^\varnothing$ by universal property.
First guess: With every $\pi \in \Pi$ we can associate a Galois representation (talk 4) $\rho_\pi : G_F \to \mathrm{GL}_2(\mathbb{C})$. Take the product

$$G_F \to \prod_{\pi\in\Pi} \mathrm{GL}_2(\mathbb{C}) \cong \mathrm{GL}_2(\mathbb{T}^\varnothing \otimes_{\mathcal{O},\iota} \mathbb{C}).$$

Problems:

1) We want an integral version, i.e. we dont like $\otimes\mathbb{C}$.

2) $\mathbb{T}^{\varnothing}$ is not local (so the universal property of $R_{\varnothing}{}^{\mathrm{univ}}$, which is defined on the category of local complete Noetherian algebras, does not apply).

Solution: Make it local!
$\mathbb{T}^{\varnothing}$ is semilocal and and $\mathcal{O}$ henselian, so we can write

$$\mathbb{T}^{\varnothing} \cong \prod_{\mathfrak{m}} \mathbb{T}_{\mathfrak{m}},$$

where $\mathfrak{m}$ runs through the finitely many maximal ideals of $\mathbb{T}^{\varnothing}$.

Fact: One can associate with every maximal ideal a representation $\overline{\rho_{\mathfrak{m}}{}^{\mathrm{mod}}} : G_F \to \mathrm{GL}_2(\mathbb{T}^{\varnothing}/\mathfrak{m})$. If this representation is absolutely irreducible, one can associate further a representation

$$\rho_{\mathfrak{m}}{}^{\mathrm{mod}} : G_F \to \mathrm{GL}_2(\mathbb{T}^{\varnothing}_{\mathfrak{m}})$$

in a way s.t.

$$\mathrm{Hom}_{\mathcal{O}\text{-alg}}(T^{\varnothing}_{\mathfrak{m}}, \overline{L}) \leftrightarrow \{\pi \in \Pi \mid \mathrm{GaloisRep}(\pi) \text{ reduces to } \overline{\rho_{\mathfrak{m}}{}^{\mathrm{mod}}}\}$$

Remark 4
$SL_2(\mathbb{F}_p) \subset \mathrm{im}\overline{\rho}$ implies that ρ is absolutely irreducible.

Now we need to pick the correct maximal ideal $\mathfrak{m}$ of $\mathbb{T}^{\varnothing}$ s.t. $\overline{\rho_{\mathfrak{m}}{}^{\mathrm{mod}}} = \overline{\rho}$ (because precisely then the automorphic representation associated with ρ_0 occurs in $T^{\varnothing}_{\mathfrak{m}} \otimes \overline{L}$). **Idea:** Modularity of ρ_0 gives a homomorphism $\mathbb{T}^{\varnothing} \to \mathbb{C}$ that actually has values in $\mathcal{O}$. Take

$$\mathfrak{m} := \ker(\mathbb{T}^{\varnothing} \to \mathcal{O} \to \mathbb{F}).$$

Definition 5
$\mathbb{T}_{\varnothing} := (\mathbb{T}^{\varnothing})_{\mathfrak{m}}$, $S_{\varnothing} := (S^{\varnothing})_{\mathfrak{m}}$, $\rho^{\mathrm{mod}}_{\varnothing} := \rho^{\varnothing}_{\mathfrak{m}}$.

It remains to check D2)-D4):

- D2) is clear.
- The existence of the map in D3) follows, since $\mathbb{T}_{\varnothing}$ is local, Noetherian and complete. An explicit description of $\rho_{\mathfrak{m}}{}^{\mathrm{mod}}$ shows that all Hecke operators are in the image of $R_{\varnothing}{}^{\mathrm{univ}} \to \mathbb{T}_{\varnothing}$, so that this map is surjective.
- D4) follows from the definition of $\rho_{\varnothing}{}^{\mathrm{mod}}$, the definition of modularity and (matching Hecke eigenvalues and Frobenius eigenvalues in the unramified situation) and Chebotarev density.

Similarly we define $\mathbb{T}'_Q$ etc.

6 The second part of the proof

Aim: Prove $\mathrm{Supp}_{R_{\varnothing}{}^{\mathrm{univ}}}(S_{\varnothing}) = \mathrm{Spec}(R_{\varnothing}{}^{\mathrm{univ}})$.
Idea: Use the "Q-machine" to construct nice "resolution" $R_\infty \curvearrowright S_\infty$ of $R_{\varnothing}{}^{\mathrm{univ}} \curvearrowright S_{\varnothing}$ which give enough information to reach the Aim.

7 Back to Deformation Rings

For $\nu_0 \in T$ fixed set

$$\mathcal{J} := \mathcal{O}[X_{\nu,i,j} \mid \nu \in T,\ i,j \in \{1,2\}]/(X_{\nu_0,1,1}).$$

This is a complete local finite type $\mathcal{O}$-algebra with $\dim(\mathcal{J}) = 4\#T$.
Facts:

- ${\rho_Q}^{\mathrm{univ}}$ induces $R_Q^{\square} \cong {R_Q}^{\mathrm{univ}} \hat{\otimes} \mathcal{J}$.
- There is an ideal $\mathfrak{a}_Q \subset R_Q^{\square}$ s.t. $R_Q^{\square}/\mathfrak{a}_Q \cong {R_Q}^{\mathrm{univ}}$.
- All situations agree modulo λ.

8 Precise Idea of Proof

We want to construct the following:

- $\mathcal{O}$-algebra morphisms $\mathcal{J}_\infty \to R_\infty \twoheadrightarrow {R_\varnothing}^{\mathrm{univ}}$, where $\mathcal{J}_\infty$ and R_∞ are local complete Noetherian, together with an R_∞-module S_∞ and a surjection $S_\infty \twoheadrightarrow S_\varnothing$ of R_∞-modules,
- a local complete Noetherian $\mathcal{O}$-algebra R_∞ together with a diagram $\mathcal{J}_\infty \to R'_\infty \twoheadrightarrow R_\varnothing^{\mathrm{univ},\prime}$ of $\mathcal{O}$-algebra morphisms[3], a module S'_∞ and a surjection $S'_\infty \twoheadrightarrow S'_\varnothing$ of R'_∞-modules,
- some $r \in \mathbb{Z}_{\geq 0}$

s.t.

P1) $\dim(R_\infty) = \dim(R'_\infty) = \dim(J_\infty) = 4\#T + r$,

P2) $\mathcal{J}_\infty$ is regular and S_∞ and S'_∞ are finite free over $\mathcal{J}_\infty$,

P3) both situations (standard and prime) agree mod λ,

P4) reduction mod λ yields a bijection between the irreducible components of $\mathrm{Spec}(R_\infty)$ and $\mathrm{Spec}(R_\infty/\lambda)$,

P5) $\mathrm{Spec}(R'_\infty)$ is irreducible,

P6) there is an ideal $\mathfrak{a}_\infty \subset R_\infty$ s.t. $R_\infty/\mathfrak{a}_\infty \cong {R_\varnothing}^{\mathrm{univ}}$ and $S_\infty/\mathfrak{a}_\infty \cong S_\varnothing$.

Remark 6
We will construct R_∞ and R'_∞ as certain inverse limits of quotients of certain deformation rings $R^{\square}_{Q_N}$ for some sequence of finite sets of finite places $(Q_N)_N$. Then from the properties of these rings discussed before, one can see that achieving P3)-P5) seems feasible. The hard part is to achieve P1) and P2) simultaneously.

Lemma 7
In the situation described above,

$$\mathrm{Supp}_{{R_\varnothing}^{\mathrm{univ}}}(S_\varnothing) = \mathrm{Spec}({R_\varnothing}^{\mathrm{univ}})$$

holds true.

[3]Note that J_∞ does not have a prime: This is not a typo!

Proof.
First recall from commutative algebra: For a local Noetherian ring $(A, \mathfrak{m})$ and a finite A-module M one has:

- $\mathrm{Depth}_A(M) \le \dim(A/\mathfrak{p})$ for all minimal prime ideals $\mathfrak{p}$ of A that are minimal in $\mathrm{Supp}_A(M)$.
- If A is regular, then $\mathrm{Depth}_A(A) = \dim(A)$.

Now, since $\mathcal{J}_\infty$ is regular and S_∞ is finite free over $\mathcal{J}_\infty$, we have

$$\mathrm{Depth}_{\mathcal{J}_\infty}(S_\infty) = \mathrm{Depth}_{\mathcal{J}_\infty}(\mathcal{J}_\infty) = \dim(\mathcal{J}_\infty) = 4\#T + r.$$

Let $\mathfrak{p} \in \mathrm{Spec}(R_\infty)$ be a prime ideal which contained and minimal in $\mathrm{Supp}_{R_\infty}(S_\infty)$. Then we have:

$$4\#T + r = \dim(R_\infty) \ge \dim(R_\infty/\mathfrak{p}) \ge \mathrm{Depth}_{R_\infty}(S_\infty) \ge \mathrm{Depth}_{J_\infty}(S_\infty) = 4\#T + r$$

Hence, all these inequalities are equalities and as $\dim(R_\infty) = 4\#T + r$, we see that $\mathfrak{p}$ is a minimal prime ideal of R_∞. Thus, $\mathrm{Supp}_{R_\infty}(S_\infty)$ is a union of irreducible components of $\mathrm{Spec}(R_\infty)$. The same statement is true in the prime world.
Now $\mathrm{Spec}(R'_\infty)$ is irreducible, so $\mathrm{Supp}_{R'_\infty}(S'_\infty) = \mathrm{Spec}(R'_\infty)$. By P3) we get $\mathrm{Supp}_{R_\infty/\lambda}(S_\infty/\lambda)$. Using P4), we see $\mathrm{Supp}_{R_\infty}(S_\infty) = \mathrm{Spec}(R_\infty)$ and finally, by P6), $\mathrm{Supp}_{R_\varnothing{}^{\mathrm{univ}}}(S_\varnothing) = \mathrm{Spec}(R_\varnothing{}^{\mathrm{univ}})$. $\square$

Now how does one find r s.t. P1) and P2) both hold? We have the following facts:

- S_Q is finite free over $\mathcal{O}[\Delta_Q]$ (here using quaternion algebra makes life easier, since one only has to analyze functions on a finite set).
- One can find a sequence $(Q_N)_N$ of finite sets of finite places and a number r s.t.
  - $\#Q_N = r$ for all N,
  - $\forall v \in Q_N \;:\; \#k(v) \equiv 1 \mod p^N$,
  - $R^{\square}_{Q_N}$ is topologically generated over R^{loc} by $\#T - 1 - [F : \mathbb{Q}] + r$ elements,
  - $\overline{\rho}(\mathrm{Frob}_v)$ has two distinct eigenvalues for all $v \in Q_N$.

Comments:

- For $N \to \infty$, S_{Q_N} is free over $\mathcal{O}[\Delta_{Q_N}]$ with Δ_{Q_N} a larger and larger p-power group. So, the first guess is to put

  $$S_\infty = \varprojlim_N S_{Q_N}$$

  (or maybe replace the S_{Q_N} by some quotients). This does not make sense! We don't even have the maps between the different S_{Q_N} to build a projective system.
- The number of elements needed to present $R^{\square}_Q$ topologically over R^{loc} is measured by some cohomology group (depending on Q). There is a natural "global version" of this group, whose rank gives a candidate for the number r.
- Trick to build a projective system: Take quotients of S_{Q_N} that have finite cardinality and construct artificially a projective system which maps that exist for cardinality reasons.
- $\mathcal{J}_\infty$ is just a power series ring over $\mathcal{J}$ in the correct number of variables (namely r, so that $\dim(\mathcal{J}_\infty) = \dim(R_\infty)$ works out).