Bruhat-Tits Theory

A quick recount of Prof. Fintzen's course
and a description of the Bruhat-Tits building

Definition 1. *Let V be a finite-dimensional real evctor space and A an affine space over V. An* ***affine root system*** *in A is a subset $\Psi \subset A^*$ of affine linear functionals on A satisfying*

1. *$A^* = \operatorname{Span}\Psi$ and Ψ contains no constant functionals,*
2. *given $\psi \in \Psi$ there exists $\psi^\vee \in V$ such that $\langle \psi, \psi^\vee \rangle := \nabla\psi(\psi^\vee) = 2$ and the reflection*
$$\begin{aligned} r_\psi : A &\to A \\ x &\mapsto x - \langle \psi, x \rangle \psi^\vee \end{aligned}$$
induces a map (by pre-composition) $r_\psi : A^* \to A^*$ which preserves Ψ,
3. *the pairings $\langle \psi, \eta^\vee \rangle$ are integers for all $\psi, \nabla \in \Psi$,*
4. *the* ***affine Weyl group*** *$W(\Psi) = \langle r_\psi \mid \psi \in \Psi \rangle \subset \operatorname{Aut}(A)$ acts properly on A (where A is given the Euclidean topology induced from that of $\mathbb{R}$).*

Lemma 1. *Let $\Phi \subset V^*$ be a root system on the real vector space V and let A be V's corresponding affine space. Then the set*

$$\Psi_\phi := \{\alpha + n \mid \alpha \in \Phi, n \in I_\alpha\} \subset A^*$$
$$I_\alpha := \begin{cases} \mathbb{Z} & \frac{\alpha}{2} \notin \Phi \\ 2\mathbb{Z}+1 & \frac{\alpha}{2} \in \Phi \end{cases}$$

is an affine root system in A and $\nabla\Psi_\Phi = \Phi$. Furthermore, we have an explicit description of the Weyl group

$$W(\Psi_\Phi) \cong \operatorname{Span}_{\mathbb{Z}}(\alpha^\vee \mid \alpha \in \Phi) \rtimes W(\Phi).$$

To do. *Figure out the example of a pair of reductive groups with isomorphic root spaces and root latices but non-isomorphic root datum.*

Definition 2. *Let Ψ be an affine root system on the affine space A. Connected components of the space*

$$A \setminus \Big(\bigcup_{\psi \in \Psi} H_\psi\Big)$$

are called ***chambers****, where H_ψ is the hyperplane defined by the affine functional $\psi \in A^*$. These are the "facets of dimension* $\dim A$*". Similarly, we define facets of dimension* $\dim A - 1$ *as connected components in*

$$\Big(\bigcup_{\psi} H_\psi\Big) \setminus \Big(\bigcup_{H_{\psi_1} \neq H_{\psi_2}} H_{\psi_1} \cap H_{\psi_2}\Big)$$

and so forth. Facets of smallest dimension are called ***vertices****.*

The closure $\overline{C}$ of a chamber C in A is a fundamental domain for the action of the Weyl group on A; in particular, $W(\Psi)$ acts simply transitively on $\overline{C}$ - this makes it particularly easy to determine when an element in $\operatorname{Aut}(A)$ doesn't lie in $W(\Psi)$.

Definition 3. *The* ***extended Weyl group*** *$W(\Psi)^{\text{ext}}$ of an affine root system $\Psi \subset A^*$ on A is the subgroup of* $\operatorname{Aut}(A)$ *which preserve Ψ and whose derivatives lie in $\nabla\Psi$.*

Definition 4. *Let $\mathscr{C} \subseteq A$ be a chamber. Define the set of roots*

$$\begin{aligned} \Psi(\mathscr{C})^+ &:= \{\psi \in \Psi \mid \psi(\mathscr{C}) \subset \mathbb{R}_{>0}\}, \\ \Psi(\mathscr{C})^o &:= \left\{\psi \in \Psi(\mathscr{C})^+ \mid H_\psi \cap \overline{\mathscr{C}} \text{ has dimension } \dim A - 1 \text{ and } \frac{1}{2}\psi \notin \Psi\right\}. \end{aligned}$$

Sets of roots of the form $\Psi(\mathscr{C})^o$ play the same role in the theory of affine root systems as bases do in regular ones, the following proposition illustrates.

Proposition 1. *1. Every $\psi \in \Psi(\mathscr{C})^+$ can be expressed as a non-negative integral linear combination of roots in $\Psi(\mathscr{C})^o$,*

2. the Weyl group $W(\Psi)$ is generated by reflections relative to roots in $\Psi(\mathscr{C})^o$.

3. If Ψ is an irreducible root system then there exists a root $\psi_0 \in \Psi(\mathscr{C})^o$ such that $\{\nabla\psi \mid \psi \in \Psi(\mathscr{C})^o \setminus \{\psi_0\}\}$ is a basis for the root system $\nabla\Psi$.

Definition 5. *Let $\Psi \subset A^*$ be an affine root system, $x \in A$ an arbitrary point. The **local** root system is the subset of roots*

$$\Psi_x := \{\psi \in \Psi \mid x \in H_\psi\}.$$

The subgroup $\mathrm{Stab}_{W(\Psi)}(x) \leq W(\Psi)$ acts on the set of roots in Ψ_x and we have an isomorphism between $\mathrm{Stab}_{W(\Psi)}(x)$ and the Weyl group of the root system $W(\nabla\Psi_x)$

Definition 6. *Let $x \in A$ be a point. x is **special** if for every root $\psi \in \Psi$ the hyperplane H_ψ is parallel to some hyperplane of the form H_η where $\eta \in \Psi_x$. x is said to be **extra-special** if the map $\nabla : \Psi \to \nabla\Psi$ is surjective when restricted to Ψ_x.*

Proposition 2. *The map $\nabla : W(\Psi) \to W(\nabla\Psi)$ is injective when restricted to $\mathrm{Stab}_{W(\Psi)}(x) \leq W(\Psi)$ and an isomorphism if x is special. In this case, the map identifies $W(\Psi)$ with $T(\Psi) \rtimes \mathrm{Stab}_{W(\Psi)}(x)$ where $T(\Psi) \leq W(\Psi)$ is the group of translations (and of course the kernel of ∇).*

Definition 7. *A **Tits system** is a quadruple (G, N, B, R) where G is a group, B, N subgroups such that $B \cap N$ is a normal subgroup in N and $R \subset N/B \cap N$ is a set of generators; these are required to satisfy the axioms:*

1. *G is generated by B and N,*
2. *elements in R have order two,*
3. *$r \in R, w \in W := N/B \cap N \implies BwB \cdot BrB \subset BwB \cup BwrB$,*
4. *$rBr^{-1} \neq B$.*

Lemma 2. *For a Tits system (G, B, N, R), we have $G = BNB$.*

Definition 8. *Let (G, N, B, R) be a Tits system. A subgroup $P \leq G$ is called a **standard parabolic** if it contains B; $P \leq G$ is called **parabolic** if it contains a conjugate of B.*

A very important fact we took without proof is that every standard parabolic P is of the form

$$P = BW_XB$$

where $X \subset R$ is some subset and W_X is the subgroup they generate in $N/B \cap N$.

Proposition 3. *1. (Bruhat decomposition) The map*

$$\begin{aligned} W &\longrightarrow B\backslash G/B \\ w &\longmapsto BwB \end{aligned}$$

is a bijection,

*2. any parabolic is conjugate to a **unique** standard parabolic,*

3. each parabolic subgroup is self-normalised in G.

Proposition 4. *Let (G, B, N, R) be a Tits system, $|R| < \infty$. Let $\mathscr{P}$ be the collection of maximal proper parabolic subgroups (note that these exist by the previous fact without proof - simply take X as being R minus an element), and B the collection of finite subsets of $\mathscr{P}$*

$$\mathscr{B} := \big\{\{P_1, \dots, P_n\} \subset \mathscr{P} \mid P_1 \cap \dots \cap P_n \text{ is a parabolic}\big\}.$$

Then the pair $(\mathscr{P}, \mathscr{B})$ defines an abstract symplicial complex structure on the set of maximal proper parabolic subgroups of G.

Definition 9. *Let $(\mathscr{V}, \mathscr{B})$ be a polysimplicial complex.*

1. *An element in $\mathscr{B}$ is called a **facet**, and for facets $A, B \in \mathscr{B}$, A is said to be a **face of B**'s if $A \subseteq B$. Maximal facets are called **chambers***

2. *A **subcomplex in** $(\mathscr{V}, \mathscr{B})$ is just a subset $\mathscr{B}' \subset \mathscr{B}$ such that faces of elements in $\mathscr{B}'$ also lie in $\mathscr{B}'$.*

3. *A **chamber complex** is a polysimplicial complex $(\mathscr{V}, \mathscr{B})$ in which every facet is contained in a maximal facet, and such that for every pair of chambers $C, C' \in \mathscr{B}$ there exists a "gallery" connecting them - i.e. a sequence of chambers*

$$C = C_1, C_2, \dots, C_{n-1}, C_n = C'$$

such that each of the consecutive intersections $C_i \cap C_{i+1}$ are maximal faces in C_i and C_{i+1} for all i. Such intersections are called facets of codimension one.

4. *A chamber complex is called **thick** if each facet of codimension one is a face of at least three chambers, and **thin** if of exactly two.*

Proposition 5. *1. Let A be an affine space and $\Psi \subset A^*$ an affine root system. The polysimplicial structure on A induced by Ψ is a thin chamber complex.*

2. *If (G, B, N, R) is a Tits system, the associated simplicial complex structure on the set $\mathscr{P}$ of maximal proper parabolic subgroups in G is a thick chamber complex.*

Proof. 1. That Ψ induces a chamber complex structure on A is left as an exercise and is just a matter or reminding oneself the axioms; it evidently forms a thin chamber complex since all hyperplanes split A into just two subsets.

2. Facets of codimension one $\mathscr{P}$ correspond to parabolic subgroups P of type $\{r\}$ where $r \in R$ (being minimal w.r.t. properly containing a Borel); the simplex corresponding to P is the face of at least three chambers because P contains at least three conjugates of B: if $P = B\{1, r\}B = B \cup BrB$ then

$$P \supseteq \begin{cases} B \\ rBr^{-1} \\ brBrb^{-1} \end{cases}$$

where b is any element in $B \setminus rBr^{-1}$.

■

In the above setting, any facet $\{P_1, \dots, P_n\} \in \mathscr{B}$ is completely determined by the parabolic subgroup $P = P_1 \cap \dots \cap P_n$.

Definition 10. *A building $\mathscr{B}$ is a chamber complex equipped with distinguished chamber sub-complexes $\mathscr{A} = \{A_\alpha\}_\alpha$ called **apartments**, satisfying:*

1. *$\mathscr{B}$ is a thick chamber complex,*
2. *apartments are thin chamber complexes,*
3. *any two chambers in $\mathscr{B}$ lie in an apartment,*
4. *for every pair of apartments A_1, A_2 in $\mathscr{B}$ and facets $\mathscr{F}_1, \mathscr{F}_2 \in A_1 \cap A_2$ there exists an isomorphism of chamber complexes*

$$A_1 \xrightarrow{\cong} A_2$$

fixing $\mathscr{F}_1$ and $\mathscr{F}_2$.

Tits-systems yield buildings to which we dedicate our interest (note that all that's left to do is give the simplicial structure on the collection of maximal proper parabolics $\mathscr{P}$ the "apartments" data):

Proposition 6. *Let (G, B, N, R) be a Tits-system such that R is finite. Set*

1. *$C := \{P \in \mathscr{P} \mid B \subset P\}$ the collection of standard parabolics,*
2. *$\overline{C} := \{$non-empty subsets $A \subset C\}$ the set of C's faces,*
3. *$A := \{n \cdot \mathscr{F} \mid \mathscr{F} = \{P_1, \dots, P_n\} \in \mathscr{B}, n \in N\}$ where $n \cdot \{P_1, \dots, P_n\} := \{nP_1n^{-1}, \dots, nP_nn^{-1}\}$ for any facet $\{P_1, \dots, P_n\} \in \mathscr{B}$*

*(C is called the **standard chamber** and A the **standard apartment**). Setting $\mathscr{A} = \{g \cdot A\}_{g \in G}$ yields a collection of apartments for $(\mathscr{P}, \mathscr{B})$.*
Furthermore, for every pair of facets $\mathscr{F}_1, \mathscr{F}_2 \in \mathscr{B}$ contained in $gA \cap A$ there exists $g' \in G$ such that $gA = g'A$ and conjugation by g fixes every vertex in $\mathscr{F}_1$ and $\mathscr{F}_2$ - in other words, the isomorphism in point 4 of definition 9 can be chosen as conjugation by some element in G.

The above building is called the Tits building.

Proposition 7. *Let (G, B, N, R) be a Tits-system and $\mathscr{B}$ the associated building. Then:*

1. *for every facet $\mathscr{F}$ in $\mathscr{B}$ we have*
$$P_{\mathscr{F}} := \bigcap_{P \in \mathscr{F}} P = \{g \in G \mid g \cdot \mathscr{F} = \mathscr{F}\},$$
2. *P_F acts transitively on the set of apartments containing it,*
3. *if $g \in G$ stabilises a facet $\mathscr{F}$ then it fixes all of $\mathscr{F}$'s vertices.*
4. *if $\mathscr{F}_1$ and $\mathscr{F}_2$ lie in the same chamber and $g\mathscr{F}_1 = \mathscr{F}_2$ then $\mathscr{F}_1 = \mathscr{F}_2$ and $g \in P_{\mathscr{F}_1} (= P_{\mathscr{F}_2})$,*
5. *for any pair of facets $\mathscr{F}_1, \mathscr{F}_2 \in A$ in a fixed apartment such that $\mathscr{F}_1 = g\mathscr{F}_2$ for some $g \in G$ we have $\mathscr{F}_1 = n\mathscr{F}_2$ for some $n \in N$.*

Definition 11. *Let (G, B, N, R) be a Tits system and express R as a dijoint union*
$$R = R_1 \dot\cup \ldots \dot\cup R_n$$
*where each R_i commutes with R_j in the Weyl group $N/B \cap N$. A subset $X \subset R$ is called **admissible** if it doesn't contain R_i for any i; a parabolic subgroup in G is called admissible if its type is.*

Proposition 8. *Let $\mathscr{B}$ be the Tits building associated to the Tits-system (G, B, N, R) and set $\mathscr{B}'$ to be the subcomplex defined by the admissible parabolics. Then $\mathscr{B}'$ is also a building, and its apartments are induced by those in $\mathscr{B}$.*

At this point in the course our aim was to discuss the axiomatic approach to the Bruhat-Tits building, mimicking the notion of symmetric spaces for Lie groups over the real numbers and bearing in mind the example we saw in the beginning of the course for $\mathrm{SL}_2(\mathbb{R}), \mathrm{SL}_2(\mathbb{Q}_p)$. For this we introduce the following canonical map: for any connected reductive group $\mathbb{G}$ over F and $A_{\mathbb{G}} \subset \mathbb{G}$ a maximal split torus lying in the centre $Z(\mathbb{G})$ we define
$$\omega_{\mathbb{G}} : \mathbb{G}(F) \to X_*(A_{\mathbb{G}}) \otimes_{\mathbb{Z}} \mathbb{Q}$$
$$g \mapsto \omega_{\mathbb{G}}(g)$$
where $\langle \omega_{\mathbb{G}}(g), \chi \rangle = \mathrm{val}(\chi(g))$ for every $\mathrm{Gal}(\overline{F}/F)$-invariant character $\chi \in X^*_F(\mathbb{G}) \xrightarrow{\mathrm{res}} X^*(A_{\mathbb{G}})$.

Question. *In the lectures we said that the map on characters* res *restricted to these Galois-invariant characters is injective and has finite-index image in $X^*(A_{\mathbb{G}})$; why should this determine the element $\omega_{\mathbb{G}}(g)$?*

We set $G(F)^1 := \ker(\omega_{\mathbb{G}})$. A nice fact is that the composition
$$S' \to S \to S/A_{\mathbb{G}}$$
is an isogeny, which implies $X_*(S') \cong X_*(S/A_{\mathbb{G}})$ whence the composition
$$\begin{tikzcd} X_*(S') \arrow[r] \arrow[rr, bend left, "\cong"] & X_*(S) \arrow[r] & X_*(S/A_{\mathbb{G}}) \end{tikzcd}$$
provides a section of the canonical map $X_*(S') \to X_*(S)$.

Question. *Why should cocharacters only depend on the isogeny class of a torus?*

Axioms of Bruhat-Tits theory. Let G be the F-rational points of a connected reductive group $\mathbb{G}$ over F.

1. (Existence of the BT building) There exists a building $\mathscr{B}(G, F) = \mathscr{B}(G)$ equipped with an action of G via polysimplicial automorphisms. The apartments are in G-equivariant bijection with G's maximal split tori.
A surjective homomorphism $\mathbb{G} \to \mathbb{G}'$ with central kernel induces an equivariant isomorphism of buildings $\mathscr{B}(G) \cong \mathscr{B}(G')$.
2. (Apartments are affine spaces) Each apartment $A(S)$ ($S \leq G$ being a maximal split torus) has the structure of an affine space over the real vector space $V(S') := X_*(S') \otimes_{\mathbb{Z}} \mathbb{R}$ where $S' := S \cap G_{\mathrm{der}} = S \cap [G, G]$.
For any $g \in G$ the isomorphism $g : A(S) \to g \cdot A(S) = A(gSg^{-1})$ is affine and its derivative is the natural isomorphism $V(S') \to V(gSg^{-1})$ induced by conjugation by g on $X_*(S') \to X_*(gS'g^{-1})$; in particular, $Z_G(S)(F)$ acts on $A(S)$ by translations, and the vector $v \in V(S)$ which determines the translation-action of $z \in Z_S(\mathbb{G})(F)$ on $A(S)$ is the image of $-\omega_{Z_{\mathbb{G}}(S)}(z)$ under the map $V(S) \twoheadrightarrow V(S')$ induced by the section described in the remarks before this paragraph.

3. (The affine tiling structure on apartments) The polysimplicial structure on each apartment $A(S)$ is given by the simplicial structure induced from some affine root system $\Psi(S) \subset A(S)^*$ whose derivative satisfies $\nabla\Psi(S) \cong \phi(G,S)$.

4. (The Tits system structure) Let $C \subset A(S)$ be a chamber in one of the apartments in $\mathscr{B}(G)$. There exists an (explicitly defined) finite index normal subgroup $\mathbb{G}(F)^o \subset \mathbb{G}(F)^1 \subset \mathbb{G}(F)$ such that

$$(\mathbb{G}(F)^o, \mathrm{Stab}_{\mathbb{G}(F)^o}(C), N_{\mathbb{G}(F)}(S) \cap \mathbb{G}(F)^o, R)$$

is a saturated Tits system, whose restricted building is $B(G)$ (an important fact that I forgot to mention is that R is uniquely determined by the first three pieces of data in a Tits system, provided such a set of generators exists).

Question. *Is $\mathbb{G}(F)^o$ not supposed to depend on the maximal split torus $S \subseteq G$?*

Definition 12. *Let $\mathscr{B}(G)$ be the Bruhat-Tits building attached to $\mathbb{G}/F$, and let $\mathscr{F} \subset \mathscr{B}(G)$ be a facet. Then $\mathbb{G}(F)^o_{\mathscr{F}} := \mathrm{Stab}_{\mathbb{G}(F)^o}$ is called the **parahoric subgroup** of $\mathbb{G}(F)$ associated to $\mathscr{F}$. If $\mathscr{F}$ is a chamber, then the associated parahoric $G(F)^o_{\mathscr{F}}$ is called **Iwahori**.*

Question. *Rephrasing my previous question, does a reductive group have just one Iwahori?*

To do. *Figure out what the Iwahori's are for $\mathrm{SL}_2(\mathbb{Q}_p)$ in terms of the Bruhat-Tits building constructed in the beginning of the course.*

The remaining portion of the course was devoted almost entirely to constructing the Bruhat Tits building, bearing the stated four axioms in mind.
Until stated otherwise, $\mathbb{G}$ will denote a split connected reductive group over F which is semi-simple and simply-connected, $T \subset G$ a split maximal torus.
For every root $\alpha \in \Phi(G,T) \subset X^*(T)$ we have a coroot $\alpha^\vee \in X_*(T)$ which is by definition a morphism

$$\alpha^\vee : \mathbb{G}_m \to T.$$

$\alpha^\vee$ has a derivative

$$d\alpha^\vee : \mathbb{G}_a \to \mathrm{Lie}\, T$$

which is completely determined by the image of $1 \in \mathbb{G}_a(F)$, which we'll denote by $H_\alpha \in \mathrm{Lie}\, T$.
For every non-zero element $X_\alpha \in \mathrm{Lie}\, U_\alpha(F) = \mathfrak{g}_\alpha(F)$ where $U_\alpha \subset G$ is the root group corresponding to α, there exists a unique 'opposite' element $X_{-\alpha} \in \mathfrak{g}_{-\alpha}(F)$ such that

$$[X_\alpha, X_{-\alpha}] = H_\alpha.$$

These three elements satisfy the classic $\mathfrak{sl}_2$-relations and yield an algebraic-group homomorphism

$$\mathrm{SL}_{2,F} \to \mathbb{G}.$$

whose derivative satisfies

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto X_\alpha,$$
$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mapsto X_{-\alpha},$$
$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto H_\alpha.$$

We denote by $w(X_\alpha) \in N_G(T)$ the image in G of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{SL}_2(F)$ under this group homomorphism, and by r_α the image of $w(X_\alpha)$ in the Weyl group $N_G(T)/T$.

Definition 13. *1. A **weak Chevalley system** is a set of the form*

$$X = \{\pm X_\alpha\}_{\alpha\in\Phi(G,T)}$$

where $X_\alpha \in \mathfrak{g}_\alpha(F) \setminus \{0\}$, such that the adjoint action of $w(X_\alpha) \in N_G(T)$ on $\mathfrak{g}(F)$ preserves X.

*2. A **Chevalley system** is a set $X = \{X_\alpha\}_{\alpha\in\Phi(G,T)}$ where $X_\alpha \in \mathfrak{g}_\alpha(F) \setminus \{0\}$ such that $\{\pm X_\alpha\}_\alpha$ is a weak Chevalley system and $[X_\alpha, X_{-\alpha}] = H_\alpha$ for all α.*

3. *In any of the above two situations, we call the group*

$$W \leq N_G(T)(F)\,,\;\; W = \langle w(X_\alpha) \mid X_\alpha \in X \rangle$$

the associated ***Tits group****.*

4. *A* ***pinning*** *of G is a tuple $(B, T, \{X_\alpha\}_{\alpha \in \Delta(B)})$ where $B \subset G$ is a Borel, $T \subset B$ a split maximal torus and $X_\alpha \in \mathfrak{g}_\alpha(F) \setminus \{0\}$ for every $\alpha \in \Delta(B) \subset \Phi(G, T)$.*

Proposition 9. 1. *For every pinning $(B, T, \{X_\alpha\}_{\alpha \in \Delta(B)})$ there's a unique Chevalley system extending it,*

2. *(in the general setting) for every split semi-simple connected reductive group $\mathbb{G}$ over F there exists a simply-connected reductive group $\mathbb{G}_{sc}$ over F together with an* ***isogeny*** *$\mathbb{G}_{sc} \to \mathbb{G}$.*

The previous proposition allows us to define (weak) Chevalley systems for reductive groups with aren't necessarily simply connected: these are simply images of weak Chevalley systems of $\mathbb{G}_{sc}$ via the derivative of the above isogeny.

Let $\{X_\alpha\}_\alpha$ be a Chevalley system; each $X_\alpha \in \mathfrak{g}_\alpha(F) \setminus \{0\}$ provides us with an isomorphism

$$u_\alpha : \mathbb{G}_a \to U_\alpha$$

which is completely determined by the requirement

$$du_\alpha(1) = X_\alpha \in \mathfrak{g}_\alpha(F).$$

The importance of Chevalley systems lies in their way of providing us with *valuations* that'll aid and are central to our construction of the Bruhat-Tits building for $\mathbb{G}$.

Definition 14. *Let $V(T) = X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ be the vector space aassociated to the maximal split torus T as defined in the Bruhat-Tits axioms. For every vector $v \in V(T)$ define the subgroup of G*

$$P_v := \langle T(F)^1\,,\, u_\alpha(\mathfrak{p}_F^{-\lfloor \alpha(v) \rfloor}) \mid \alpha \in \Phi(G, T) \rangle$$

where $\mathfrak{p}_F \subset \mathcal{O}_F$ is the unique maximal ideal.

P_v is an open, bounded and compact subgroup in $\mathbb{G}(F)$.

Definition 15. *A* ***Chevalley valuation*** *of the root datum $(X^*, \Delta(T), X_*, \Delta(T)^\vee)$ is the collection of valuations*

$$\varphi = \{\varphi_\alpha = \mathrm{val} \circ u_\alpha^{-1} : U_\alpha(F) \to \mathbb{Q} \cup \{\infty\}\}_{\alpha \in \Phi(G,T)}.$$

For any $v \in V(T) = X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ we define the shifted valuations as well:

$$\begin{aligned} \varphi_v = \{\varphi_{v,\alpha} : U_\alpha(F) &\to \mathbb{R} \cup \{\infty\}\}_\alpha \\ u &\mapsto \phi_\alpha(u) + \langle \alpha, v \rangle \end{aligned}$$

To do. *Figure out these and how they change depending on the Chevalley system for $\mathrm{SL}_{2,\mathbb{Q}_p}$.*

These permit us to introduce the apartment associated to the maximal split torus $T \subset G$ (note what we're now defining the apartments without having yet introduced the building these lie in).

Definition 16. *The* ***apartment associated to*** *T is the set*

$$\mathcal{A}(T) := \{\text{valuations } \phi_v = \{\phi_{\alpha,v}\}_\alpha \mid v \in V(T)\}$$

equipped with the structure of an affine space over $V(T)$ by means of the action

$$\phi_v + w := \phi_{v+w}$$

for $v, w \in V(T)$.

The choice of a Chevalley system above yields the root group maps

$$u_\alpha : \mathbb{G}_a \xrightarrow{\cong} U_\alpha$$

and thus the valuations $\{\phi_\alpha\}_\alpha$ whose shifts by (real linear combinations of) cocharacters give the whole apartment. The important fact to bare in mind is that the choice of a Chevalley system is equivalent to the choice of an *origin* in the apartment:

Proposition 10. *The set $\mathscr{A}(T)$ is independent on the choice of Chevalley system.*

Proof. Let $Y = \{Y_\alpha \in \mathfrak{g}_\alpha(F) \setminus \{0\}\}_\alpha$ be another Chevalley system and $\varphi_Y = \{\varphi_{Y,\alpha}\}_\alpha$ the corresponding valuations. If $B \supset T$ is any Borel then restricting each Chevalley system to the relative basis yields two distinct pinnings of G

$$(B, T, \{X_\alpha\}_{\alpha \in \Delta(B)})$$
$$(B, T, \{Y_\alpha\}_{\alpha \in \Delta(B)}).$$

Note that since each root space $\mathfrak{g}_\alpha(F)$ is one dimensional, there exist constants $\{c_\alpha\}_\alpha \in F \setminus \{0\}$ such that $c_\alpha X_\alpha = Y_\alpha$; as the roots in $\Delta(B)$ for a basis for all characters on T, there must exist a unique $t \in T$ such that $\alpha(t) = c_\alpha$ for each $\alpha \in \Delta(B) \implies \mathrm{Ad}(t) \cdot X_\alpha = Y_\alpha$. This implies $u_{Y,\alpha} = (u_{X,\alpha})^t$ hence

$$d(tu_\alpha(-)t^{-1}) = Y_\alpha$$

and comparing valuations yields

$$\begin{aligned}\varphi_{\alpha,Y}(x) &= \mathrm{val}(u_{Y,\alpha}^{-1}(x)) \\ &= \mathrm{val}(u_{X,\alpha}^{-1}(txt^{-1})) = \mathrm{val}(\alpha(t^{-1})u_{X,\alpha}^{-1}(x)) = \varphi_{X,\alpha}(x) + \langle \alpha, -w_T(t)\rangle,\end{aligned}$$

which shows $\varphi_Y = \varphi_X - w_T(t)$. ■

As desired in the axioms we discussed, we get an action of $N_G(T)(F)$ on the apartments via

$$n \in N_G(T)(F),\ x \in \mathscr{A}(T),\ \ n \cdot \varphi_x = n \cdot \{\varphi_{x,\alpha}(-)\}_\alpha := \left\{\varphi_{x,\alpha}({-}^{n^{-1}})\right\}.$$

To do. *Figure out the details here (?) What's the action of $t \in T(F)$, for instance?*

The next important step is to introduce the polysimplicial-complex structure on each apartment which coincides with that coming from an affine root system $\Psi \subset \mathscr{A}(T)^*$.

Definition 17. 1. *For every point in an apartment $x \in \mathscr{A}(T)$ and real number $r \in \mathbb{R}$ define the (open, compact) subgroup*

$$U_{\alpha,x,r} = \{u \in U_\alpha(F) \mid \varphi_{x,\alpha}(u) \geq r\}.$$

2. *For every affine functional $\psi \in \mathscr{A}(T)^*$ such that $\nabla\psi = \alpha \in \Phi(G,T)$, define*

$$U_\psi = U_{\alpha,x,\psi(x)}$$

for any $x \in \mathscr{A}(T)$ (note that this doesn't depend on $x \in \mathscr{A}(T)$).

3. *Lastly, define for any $\psi \in \mathscr{A}(T)^*$*

$$U_{\psi+} := \bigcup_{\epsilon>0} U_{\psi+\epsilon}.$$

Theorem 1. *The set*

$$\Psi := \{\psi \in \mathscr{A}(T)^* \mid \nabla\psi = \alpha \in \Phi(G,T), U_\psi \neq U_{\psi+}\}$$

is a reduced affine root system and (!!!!!) its derivative root system is $\Phi(G,T)$. More precisely $\Psi = \Psi_{\Phi(G,T)}$.

To do. *Figure this theorem out properly: the proof shouldn't be hard but try to motivate why you should get back precisely the root system you started off with by taking derivatives :)*

We now want to remove the assumption on the existence of a split maximal torus, so until stated otherwise $\mathbb{G}$ will be a reductive quasi-split group over F and $S \subset G$ will be a maximal split torus (a relevant result is that the subgroup $T = Z_{\mathbb{G}}(S)$ is a maximal torus, which is split only if it equals S). A good example to bare in mind which will turn out to be the *canonical* one so-to-speak is that of $\mathrm{SU}_{3,F}$ which is defined by

$$\mathrm{SU}_3(F') := \left\{g \in \mathrm{SL}_3(F' \otimes_F E) \mid g^\sigma = x(g^t)^{-1}x^{-1}\right\}$$

where E/F is a degree-two separable field extension, $\sigma \in \mathrm{Gal}(E/F)$ is the non-trivial element and x is the matrix

$$x = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The important thing to note is that the base change of SU_3 to E yields precisely $\mathrm{SL}_{3,F}$ which is split (because the base change to E of $Z_{\mathbb{G}}(S)$ is split) and thus the previous developed theory applies.

Definition 18. *The root system $\Phi = \Phi(G,S) \subset X^*(S)$ is called the* ***relative root system*** *of $\mathbb{G}$ corresponding to the maximal split torus S, and the root system $\Phi(G,T) \subset X^*(T)$ will be denoted by $\widetilde{\Phi}$ and is called the* ***absolute*** *root system.*
We have a restriction map

$$\widetilde{\Phi} \xrightarrow{\text{res}} \Phi$$

which is $\Gamma = \mathrm{Gal}(F^{\text{sep}}/F)$-invariant (in particular, its fibres are Galois orbits, because since S is split the Galois-action it's naturally endowed with is trivial).

For every character $\chi \in X^*(T)$ we denote by $\Gamma_x = \mathrm{Stab}_\Gamma(\chi) \leq \Gamma$ the stabiliser subgroup and by F_χ the corresponding intermediate extension $F^{\text{sep}} \supseteq F_\chi \supseteq F$.
When working with restricted root systems we have a slightly different notion of root subgroups.

Definition 19. *For every relative root $a \in \Phi(G,S)$ the corresponding root group $U_a \subset \mathbb{G}$ is the unique smooth closed subgroup of $\mathbb{G}$ normalised by S and whose Lie algebra satisfies*

$$\mathrm{Lie}\, U_a = \bigoplus_{n \in \mathbb{Z}_{\geq 1}} \mathfrak{g}_{n \cdot a} \subset \mathfrak{g}.$$

We can give explicit descriptions of these root subgroups:

1. if $a \in \Phi$ is neither "divisible" nor "multipliable" (i.e. $\frac{a}{2}, 2a \notin \Phi$) then given any root in its fibre

$$\widetilde{a} \in \widetilde{\Phi}\,, \ \ \mathrm{res}(\widetilde{a}) = a$$

the root subgroup is given by the Weil restriction of $\widetilde{a}$'s

$$U_a \cong \mathrm{Res}_{F_{\tilde{a}}/F}\, U_{\tilde{a}}.$$

This also gives us an explicit description of the F-rational points

$$U_a(F) \cong U_{\tilde{a}}(F_{\tilde{a}}) \cong F_{\tilde{a}}.$$

The first isomorphism is given by taking the norm map

$$F_{\tilde{a}} \otimes_F F_{\tilde{a}} \xrightarrow{\mathrm{Nm}_{F_{\tilde{a}}/F}} F_{\tilde{a}}.$$

2. if $a, 2a \in \Phi$ then matters are slightly more complex, and really leverage on the example described above for SU_3. Once again we fix a preimage $\tilde{a} \in \widetilde{\Phi}, \mathrm{res}(\widetilde{a}) = a$. In this case, there exists (**this is an exercise**) a unique $\sigma \in \Gamma$ such that $\widetilde{a} + \sigma(\widetilde{a}) \in \widetilde{\Phi}$ is also a root, which we'll denote by $\widetilde{b}$. Because res is Γ-equivariant, we get $\mathrm{res}(\widetilde{b}) = 2a$.

To do. *Solve the exercise :p*

Definition 20. *For any degree 2 separable field extension L/K, whose Galois group is generated by $\sigma \in \mathrm{Gal}(L/K) \cong \mathbb{Z}/(2)$, define the algebraic group over K*

$$U_{L/K}(A) := \left\{(u,v) \in (A \otimes_K L)^2 \mid v + v^\sigma = u^\sigma \cdot u\right\}$$

for all K-algebras A, with multiplication given by

$$(u,v) \cdot (u',v') := (u + u', v + v' + u^\sigma \cdot u).$$

Denote by $U^o_{L/K} \subset U_{L/K}$ the subgroup of pairs (u,v) where $u = 0$.

We have a chain of field extensions

$$F_{\tilde{a}} \supseteq F_{\tilde{b}} \supseteq F$$

where $F_{\tilde{a}}/F_{\tilde{b}}$ is separable of degree two. Over $F_{\tilde{b}}$ we have an isomorphism, giving us a description of the root group

$$\begin{tikzcd} \langle U_{\tilde{a}}, U_{\sigma(\tilde{a})}, U_{\tilde{b}} \rangle \arrow[r, "\cong"] & U_{F_{\tilde{a}}/F_{\tilde{b}}} \\ U_{\tilde{b}} \arrow[u, hook] \arrow[r, "\cong"] & U^o_{F_{\tilde{a}}/F_{\tilde{b}}} \arrow[u, hook] \end{tikzcd}$$

Taking Weil-restrictions gives us our desired root groups over F:

$$U_a \cong \mathrm{Res}_{F_{\tilde{b}}/F}(U_{\tilde{a}} U_{\sigma(\tilde{a})} U_{\tilde{b}}) \cong \mathrm{Res}_{F_{\tilde{b}}/F}\, U_{F_{\tilde{a}}/F_{\tilde{b}}},$$
$$U_{2a} \cong \mathrm{Res}_{F_{\tilde{b}}/F}\, U_{\tilde{b}} \cong \mathrm{Res}_{F_{\tilde{b}}/F}\, U^o_{F_{\tilde{a}}/F_{\tilde{b}}}.$$

Now that we've established what the root groups should be, all that's left to do for defining $\mathscr{B}(\mathbb{G})$'s apartments is carry over our construction of the Chevalley systems and their induced valuations.

Definition 21. *1. A **weak Chevalley-Steinberg system** for G is a Γ-invariant weak-Chevalley system for $G_{F^{\text{sep}}}$,*

$$\{\pm X_\alpha \in \mathfrak{g}_\alpha(F^{\text{sep}})\}_{\alpha\in\widetilde{\Phi}}.$$

*2. An F-**pinning** of $\mathbb{G}$ is a triple*

$$(B, T, \{X_\alpha\}_{\alpha\in\widetilde{\Delta(B)}})$$

where $B \subset \mathbb{G}$ is a Borel, $T \subset B$ is a maximal torus and the elements $X_alpha \in \mathfrak{g}_\alpha(F^{\text{sep}}) \setminus \{0\}$ are such that the set $\{X_\alpha\}_{\alpha\in\Delta(\tilde{B})}$ is Γ-invariant.

*3. A **Chevalley-Steinberg system** for $\mathbb{G}$ is a set $\{X_\alpha\}_{\alpha\in\widetilde{\Phi}}$ such that $\{\pm X_\alpha\}_{\alpha\in\widetilde{\Phi}}$ is a weak Chevalley-Steinberg system and the following conditions on compatibility with the restriction map* res *hold:*

- *$a \in \Phi$ not divisible or multipliable $\implies \tau(X_{\tilde{a}}) = X_{\tau(\tilde{a})}$ for all $\tau \in \Gamma$*
- *$a, 2a \in \Phi \implies \tilde{b} = \tilde{a} + \sigma(\tilde{a}), \tau(X_{\tilde{b}}) = \epsilon(\tau)X_{\tilde{b}}$ for all $\tau \in \Gamma_{\tilde{b}}$ where $\epsilon : \Gamma_{\tilde{b}}/\Gamma_{\tilde{a}} \to \{\pm 1\}$ is the only non-trivial group-homomorphism.*

In lectures we stated that, much like in the split case, every weak Chevalley-Steinberg system can be refined to a Chevalley-Steinberg system.

Definition 22. *Suppose $\{X_\alpha\}_{\alpha\in\Phi}$ is a Chevalley-Steinberg system; the corresponding **Chevalley valuation maps** $\{\varphi_a\}_{a\in\Phi}$ are defined by the following:*

1. if a is neither divisible nor multipliable, then $\varphi_a : U_a(F) \to \mathbb{R} \cup \{\infty\}$ is defined by the following composition

$$\varphi_a : U_a(F) \xleftarrow[\cong]{\text{Nm}} U_{\tilde{a}}(F_{\tilde{a}}) \xleftarrow[\cong]{du_{\tilde{a}}(1)=X_{\tilde{a}}} F_{\tilde{a}} \xrightarrow{\text{val}} \mathbb{R} \cup \{\infty\}$$

2. if both a and $2a$ are relative roots in Φ, then as previously fix $\widetilde{a}$ and $\widetilde{a}'$ such that $\text{res}(\widetilde{a}) = \text{res}(\widetilde{a}') = a$ *and $\widetilde{b} = \widetilde{a} + \widetilde{a}' \in \widetilde{\Phi}$ is also an absolute root. The Chevalley-Steinberg system yields isomorphisms*

$$\begin{aligned} u_{\tilde{a}} &: \mathbb{G}_a \xrightarrow{\cong} U_{\tilde{a}} \subset \mathbb{G}_{F_{\tilde{a}}} \\ u_{\tilde{a}'} &: \mathbb{G}_a \xrightarrow{\cong} U_{\tilde{a}'} \subset \mathbb{G}_{F_{\tilde{a}'}} = \mathbb{G}_{F_{\tilde{a}}} \\ u_{\tilde{b}} &: \mathbb{G}_a \xrightarrow{\cong} U_{\tilde{b}} \subset \mathbb{G}_{F_{\tilde{b}}} \end{aligned}$$

which glue to an isomorphism

$$U_{F_{\tilde{a}}/F_{\tilde{b}}} \xrightarrow{\cong} U_{\tilde{a}}U_{\tilde{a}'}U_{\tilde{b}} \subseteq \mathbb{G}_{F_{\tilde{b}}}$$

which restricts to another

$$U^o_{F_{\tilde{a}}/F_{\tilde{b}}} \xrightarrow{\cong} U_{\tilde{b}}.$$

From these can define the valuations φ_a and φ_{2a}:

$$\begin{aligned} \varphi_a : U_a(F) \cong U_{F_{\tilde{a}}/F_{\tilde{b}}}(F_{\tilde{b}}) &\longrightarrow \mathbb{R} \cup \{\infty\} \\ (u, v) &\longmapsto \frac{1}{2}\,\text{val}(v) \end{aligned}$$

$$\begin{aligned} \varphi_{2a} : U_{2a}(F) \cong U_{\tilde{b}}(F_{\tilde{b}}) &\longrightarrow \mathbb{R} \cup \{\infty\} \\ (0, v) &\longmapsto \text{val}(v). \end{aligned}$$

For every vector $v \in X_*(S) \otimes_{\mathbb{Z}} \mathbb{R} (= V(S))$ define $\phi_v := \{\phi_{a,v}\}_{a\in\Phi}$ by

$$\begin{aligned} \phi_{a,v} : U_a(F) &\to \mathbb{R} \cup \{\infty\} \\ u &\mapsto \phi_a(u) + \langle a, v \rangle \end{aligned}$$

We're now in the right spot for defining the apartment attached to $S \subseteq \mathbb{G}$.

Definition 23. *For every maximal split torus $S \subset \mathbb{G}$ define*

$$\mathscr{A}(S) := \{\phi_v \mid v \in V(S)\}\,.$$

To do. *Evidently in this quasi-split case the requirement that $\mathbb{G}$ is simply-connected isn't important since we can argue by taking the 'universal cover' which is only an isogeny away from $\mathbb{G}$. Fill in details for what does on in the general case: note that to assert existence results for Chevalley(-Steinberg) systems it was important to have G semisimple.*

Definition 24.

- *For every relative root $a \in \Phi(G,S), x \in \mathscr{A}(S), r \in \mathbb{R}$ set*

$$U_{a,x,r} := \{u \in U_a(F) \mid \varphi_{x,a}(u) \geq r\} \subset U_a(F).$$

- *For every affine linear functional $\psi \in \mathscr{A}(S)^*$ such that $\nabla\psi = a \in \Phi$ we set*

$$\begin{aligned} U_\psi &= U_{a,x,\psi(x)}, \\ U_{\psi+} &= \bigcup_{\epsilon>0} U_{\psi+\epsilon}. \end{aligned}$$

- *Define*

$$\begin{aligned} \Psi' &:= \{\psi \in \mathscr{A}(S)^* \mid \nabla\psi \in \Phi, U_\psi \neq U_{\psi+}\}\,, \\ \Psi &:= \{\psi \in \mathscr{A}(S)^* \mid \nabla\psi \in \Phi, U_\psi(F) \not\subseteq U_{\psi+}(F)U_{2\nabla\psi}(F)\}\,. \end{aligned}$$

Theorem 2. *Ψ and Ψ' are both affine root systems with derivate root systems equal Φ.*

To do. *Verify this theorem for $\mathrm{SU}_{3,\mathbb{Q}_p} = U_{\mathbb{Q}_p[\sqrt{p}]/\mathbb{Q}_p}$.*

Just as in the split case, each apartment $\mathscr{A}(S)$ inherits a canonical action of the normaliser $N_G(S)(F) \subseteq G$.
This theorem yields the polysimplicial-complex structure on each apartment (regardless of whether we use Ψ or Ψ', since both affine root systems yield the same hyperplanes in $\mathscr{A}(S)$).

We now want to introduce the filtration subgroups just as we did in Definition 14.

Definition 25. *Suppose $T \subseteq \mathbb{G}$ is a torus, E/F a finite Galois extension such that T_E splits and F^{un}/F the maximal unramified extension. We set*

$$T(F^{un})^o := \mathrm{im}(T(E^{un})^1 \xrightarrow{\mathrm{Nm}_{E^{un}/F^{un}}} T(F^{un}))$$

and $T(F)^o := T(F) \cap T(F^{un})^o$.
$T(F)^o$ is called the Iwahori subgroup of the torus T (to be thought of as the ***connected component of*** *$T(F)^1$).*

To do. *Take the torus over $F = \mathbb{Q}_p$ given by*

$$T(F') := \left\{ \begin{pmatrix} a & b \\ pb & a \end{pmatrix} \in \mathrm{SL}_2(F') \right\}$$

for all field extensions F'. Figure out what the Iwahori is here and why $T(\mathbb{Q}_p)^1 \supsetneq T(\mathbb{Q}_p)^o$.

Definition 26. *Let $x \in \mathscr{A}(S)$ be a point in the apartment corresponding to the maximal split torus $S \subseteq \mathbb{G}$, and let $T = Z_G(S)$ be the corresponding maximal torus. We define the* ***parahoric subgroup*** *associated to x as the subgroup*

$$\mathscr{P}_x^o := \langle T(F)^o, U_{a,x,0} \mid a \in \Phi = \Phi(G,S)\rangle.$$

(It turns out that -) $\mathscr{P}_x^o$ only depends on the facet $\mathscr{F}$ containing x, so often we'll denote $\mathscr{P}_x^o$ by $\mathscr{P}_{\mathscr{F}}^o$.
Whenever $\mathscr{F}$ is a *chamber*, $\mathscr{P}_{\mathscr{F}}^o$ is called an **Iwahori subgroup**.

Lemma 3. *Suppose $\mathscr{F}_1$ and $\mathscr{F}_2$ are two facets in $\mathscr{A}(S)$. Then:*

- $\mathscr{F}_2 \subset \overline{\mathscr{F}_1} \implies \mathscr{P}_{\mathscr{F}_1}^o \subseteq \mathscr{P}_{\mathscr{F}_2}^o$,
- $\mathscr{F}_1 \neq \mathscr{F}_2 \iff \mathscr{P}_{\mathscr{F}_1}^o \neq \mathscr{P}_{\mathscr{F}_2}^o$.

Theorem 3. *If $\mathscr{C} \subseteq \mathscr{A}(S)$ is any chamber and $x \in \mathscr{C}$ is any point, then we have a decomposition of the corresponding Iwahori*

$$\mathscr{P}_{\mathscr{C}}^o = \prod_{a<0} U_{a,x,r} \times T(F)^o \times \prod_{a>0} U_{a,x,r}.$$

where $r \in \{0, 0+\}$.

Definition 27. *For any quasi-split reductive group $\mathbb{G}$ over F define:*

- $\mathbb{G}(F)^\natural := \operatorname{im}(\mathbb{G}_{sc}(F) \to \mathbb{G}(F))$ *where* $\mathbb{G}_{sc} \to \mathbb{G}$ *is* $\mathbb{G}$*'s universal cover.*
- $\mathbb{G}(F)^o := \langle \mathbb{G}(F)^\natural, T(F)^o \rangle$.

$\mathbb{G}(F)^o$ is an open, normal subgroup and $G(F)^o \subset \mathbb{G}(F)^1$ is of finite index. Furthermore, if $\mathbb{G}$ is simply connected then $\mathbb{G}(F)^o = \mathbb{G}(F)^1 = \mathbb{G}(F)$.

A nice description of $\mathbb{G}(F)^\natural$ is the following: $\mathbb{G}(F)^\natural$ is the subgroup generated by *unipotent elements* (not to be confused with the unipotent radical, which is the largest unipotent *normal* subgroup).

With the definition of the subgroup $\mathbb{G}(F)^o$ at hand, we can now give a description of the Bruhat Tits building, linking all the constructed apartments $\mathscr{A}(S)$ into one huge chamber complex.

Theorem 4. *Let $\mathscr{C}$ be a chamber in $\mathscr{A}(S)$ and set $\mathscr{I} = \mathscr{P}^o_{\mathscr{C}}$ the corresponding Iwahori subgroup. Set*

$$N := N_G(S)(F) \cap \mathbb{G}(F)^o$$
$$R := \{r_\psi \mid \psi \in \Psi(\mathscr{C})^o\}\,.$$

Then $(\mathbb{G}(F)^o, \mathscr{I}, N, R)$ is a saturated (i.e. $\bigcap_{n\in N} n\mathscr{I}n^{-1} = N \cap \mathscr{I}$) Tits system, with Weyl group equal to $W(\Psi)$.

The Tits system in the above theorem is called the *Iwahori-Tits system* attached to $\mathbb{G}$.

To do. *Understand this proof from your notes.*

Remarkably, the above construction doesn't depend on the choice of a maximal split torus S.

Lemma 4. *Any two pairs $(S_1, \mathscr{C}_1), (S_2, \mathscr{C}_2)$ where S_i is a maximal split torus and $\mathscr{C}_i \subseteq \mathscr{A}(S_i)$ is a chamber for $i = 1, 2$ are conjugate under $\mathbb{G}(F)^\natural$.*

Proof. All maximal split tori are conjugate under $\mathbb{G}(F)^\natural$ (ok?) so we can assume $S_1 = S_2 = S$. Every pair of chambers in $\mathscr{A}(S)$ are conjugate under $W(\Psi)$ by definition of $\mathscr{A}(S)$'s polysimplicial structure, and we have a surjection

$$N_{\mathbb{G}(F)^\natural}(S) \twoheadrightarrow W(\Psi).$$

■

Definition 28 (The Bruhat-Tits building)**.** *The Bruhat-Tits building $\mathscr{B}(\mathbb{G})$ attached to the quasi-split reductive group $\mathbb{G}$ over F is the restricted building associated to the Iwahori Tits-system $(\mathbb{G}(F)^o, \mathscr{I}, N, R)$.*
For every pair $(S, \mathscr{C})$ where S is a maximal split torus and $\mathscr{C} \subset \mathscr{A}(S)$ is a chamber we get a distinguished apartment in $\mathscr{B}(G)$ and a distinguished chamber. The distinguished apartment identifies with the set of admissible parabolic subgroups of $\mathbb{G}(F)^o$ that contain an $N_{\mathbb{G}(F)^o}(S)$-conjugate of $\mathscr{I}$.

The following proposition illustrates how apartments in the Bruhat-Tits building indeed identify with the apartments $\mathscr{A}(S)$ thus constructed.

Proposition 11.
1. *The admissible parabolic subgroups of the Iwahori-Tits system are precisely the parabolic subgroups of $\mathbb{G}$.*
2. *The map $\mathscr{F} \mapsto \mathscr{P}^0_{\mathscr{F}}$ is an order-preserving bijection*

$$\{\textit{Facets in } \mathscr{A}(S)\} \xrightarrow{1:1} \{\textit{facets in a distinguished apt. in } \mathscr{B}(G)\}\,.$$

In particular, $\mathscr{A}(S)$ is the geometric realisation of the distinguished apartment.

Proof. Assume $\mathbb{G}$ is simply connected and Φ is irreducible. If $\mathscr{F}$ is a facet contained in the closure of $\mathscr{C}$, and

$$W(\Psi)_{\mathscr{F}} = \langle r_\psi \mid \psi \in \Psi(\mathscr{C})^o, \psi(\mathscr{F}) = 0 \rangle,$$

then we have a decomposition

$$\mathscr{P}^o_{\mathscr{F}} = \mathscr{I} W(\Psi)_{\mathscr{F}} \mathscr{I},$$

which we'll refrain from proving. This shows surjectivity since for every proper standard (admissible) parabolic subgroup P in the Iwahori-Tits system we have

$$P = \mathscr{I} W(\Psi)_X \mathscr{I}$$

for some proper subset $X \subset \Psi(\mathscr{C})^o$, hence

$$\bigcap_{\psi \in X} H_\psi$$

is non-empty and thus contains a facet $\mathscr{F}$ which must be such that $W(\Psi)_X = W(\Psi)_{\mathscr{F}}$. As for injectivity, simply take a look at Lemma 3. ■

The one thing we're missing from completing the construction is that since the Bruhat-Tits building is defined in terms of a Tits-system over $\mathbb{G}(F)^o$, it's only apparent that $\mathbb{G}(F)^o$ has an action on $\mathscr{B}(\mathbb{G})$. This follows from the following decomposition of $\mathbb{G}(F)$:

$$\mathbb{G}(F) = \mathbb{G}(F)^\natural \operatorname{Stab}_{N_G(\mathbb{G})(F)}(\mathscr{C})$$