Notes on Galois Representations

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In this pdf I gather some notes written while studying for a course on the Arithmetic of the Langlands programme taught by Ana Caraiani in the winter term 2022/2023. I mainly follow Toby Gee's lecture notes [Gee] from the Arizona Winter School and Fargues' writing [Fa], but most of the content is from Professor Caraiani's lectures which I haven't quite matched up with any particular reference. These notes are quite lacking in proofs and proper arguments, and their purpose (while mostly being my personal learning and to create a sort-of birds eye view of the topic) is purely expository. I'm extremely happy if you can make any use of them and please contant me[1] if you're interested in sharing your remarks, typos, corrections or any suggestions to improve my writing.

1 Generalities

Definition 1.1. Let K'/K be a normal separable extension of fields. The Galois group $\mathrm{Gal}(K'/K)$ is endowed with a profinite topology by endowing each its finite quotients

$$\mathrm{Gal}(K'/K) \twoheadrightarrow \mathrm{Gal}(K''/K),\ K' \supseteq K''/K < \infty$$

with the discrete topology:

$$\mathrm{Gal}(K'/K) = \varprojlim_{K''/K<\infty} \mathrm{Gal}(K''/K).$$

If $\overline{K}$ is a fixed separable closure of K, we denote by G_K the Galois group $\mathrm{Gal}(\overline{K}/K)$.

Definition 1.2. A *Galois representation* is a continuous group homomorphism

$$\rho : G_K \to \mathrm{GL}_n(L)$$

where L is a topological field and $\mathrm{GL}_n(K) \subseteq L^{n^2}$ is given the subspace topology. If $L = \mathbf{C}$ then ρ is called an *Artin representation.*

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Example 1.3. Let $K/\mathbf{Q}$ be the splitting field of the polynomial $x^3 - x - 1$. We wonder under what assumptions on the prime p the polynomial $x^3 - x - 1$ splits completely in $\mathbf{F}_p[x]$. The Galois group $\mathrm{Gal}(K/\mathbf{Q})$ is isomorphic to the symmetric group S_3, since its order is 6 and is non-abelian. By considering the action of S_3 on the three roots of $x^3 - x - 1$ we get a continuous representation

$$G_{\mathbf{Q}} \twoheadrightarrow \mathrm{Gal}(K/\mathbf{Q}) \hookrightarrow \mathrm{GL}_3(\mathbf{C})$$

with finite image and, being a permutation action, we get a decomposition of $\mathbf{Q}^3$ into irreducible subrepresentations

$$\mathbf{C}^3 = \{(x_1, x_2, x_3) \mid x_1 + x_2 + x_3 = 0\} \oplus \{(x_1, x_2, x_3) \mid x_1 = x_2 = x_3\}$$

where $G_{\mathbf{Q}}$ acts trivially on the second. Restricting to the first we get a 2-dimensional representation

$$\rho : G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{C}).$$

If we fix an isomorphism between $\overline{\mathbf{Q}}_p$ and the complex numbers $\mathbf{C}$ we obtain a map

$$G_{\mathbf{Q}_p} \to G_{\mathbf{Q}}$$

and thus the representation ρ can be pulled back to a Galois representation of $G_{\mathbf{Q}_p}$. We have a short exact sequence

$$0 \to I_{\mathbf{Q}_p} \to G_{\mathbf{Q}_p} \to G_{\mathbf{F}_p} \cong \widehat{\mathbf{Z}} \to 0$$

where the right-most map simply restricts a $\mathbf{Q}_p$-automorphism $\sigma : \overline{\mathbf{Q}}_p \xrightarrow{\cong} \overline{\mathbf{Q}}_p$ to the integers $\overline{\mathbf{Z}}_p \subseteq \overline{\mathbf{Q}}_p$ and then reduces mod p to get an automorphism $\overline{\mathbf{F}}_p \to \overline{\mathbf{F}}_p$. We denote by $\mathrm{Frob}_p \in G_{\mathbf{F}_p}$ the Frobenius element

$$\mathrm{Frob}_p : x \mapsto x^p$$

which is a topological generator for $G_{\mathbf{F}_p}$. If we let F be any lift of Frob_p along $G_{\mathbf{Q}_p} \to G_{\mathbf{F}_p}$ then we have the equivalence

$$x^3 - x - 1 \text{ splits completely mod } p \iff \rho(F) = \mathrm{id}$$

because all roots of $x^3 - x - 1$ in $\overline{\mathbf{Q}}_p$ are integers and the image of $\mathcal{O}_K \subseteq \overline{\mathbf{Q}}_p$ in $\mathbf{F}_p$ is the splitting field of $x^3 - x - 1 \in \mathbf{F}_p[x]$. Since ρ's image is finite, F acting trivially on $\mathbf{C}^2$ via ρ is equivalent to requiring $\mathrm{tr}(\rho(F)) = \dim \rho = 2$ (because the eigenvalues of the operator $\rho(\mathbf{F})$ are necessarily roots of unity by the existence of an equivariant non-degenerate form on $\mathbf{C}^2$, from Maschke's theorem [RepSerre]). Note that $\mathrm{tr}(\rho(F))$ doesn't depend on the lift F, since all such lifts differ by elements in $I_{\mathbf{Q}_p}$, which acts trivially on $\mathbf{C}^2$ via ρ, and all isomorphisms $\overline{\mathbf{Q}}_p \cong \mathbf{C}$ yield conjugate images of F in $G_{\mathbf{Q}}$. We've thus translated the number-theoretic problem on determining properties on the roots of a polynomial into representation-theoretic ones concerning traces of Frobenius elements.

Remark 1.4. The preimage $W_{\mathbf{Q}_p}$ of the subgroup $\mathbf{Z} \subseteq \widehat{\mathbf{Z}} \cong G_{\mathbf{F}_p}$ generated by the Frobenius element is called the Weil group, and is endowed with the topological-group structure induced by requiring $I_{\mathbf{Q}_p} \subseteq W_{\mathbf{Q}_p}$ to be an open subgroup - in particular, it does not carry the subspace topology when viewed as a subgroup of $G_{\mathbf{Q}_p}$; it will turn out to be particularly important when studying Galois representations of $G_{\mathbf{Q}_p}$. The Weil group W_K of an arbitrary non-archimedean field K fits into the picture painted by class field theory which describes an isomorphism of topological groups

$$\mathrm{Art}_K : W_k^{\mathrm{ab}} \xrightarrow{\cong} K^\times$$

such that $\mathrm{Art}_{K'} = \mathrm{Art}_K \circ \mathrm{Nm}_{K'/K}$ for all finite extensions K'/K and the projection

$$W_K^{\mathrm{ab}} \to \mathbf{Z} \cdot \mathrm{Frob}_K \subseteq G_k$$

identifies with the valuation $\mathrm{val}_K : K^\times \to \mathbf{Z}$ under this isomorphism.

Artin representations, such as the one in Example 1.3, are particularly well-behaved, because of the huge differences between the Euclidean (locally connected) topology on $\mathbf{C}$ and the profinite (totally disconnected) one on Galois groups.

Proposition 1.5. *An Artin representation $\rho : G_K \to \mathrm{GL}_n(\mathbf{C})$ always has finite image.*

Proof. Using the exponential map $\exp : U \subseteq \mathfrak{gl}_n(\mathbf{C}) \to \mathrm{GL}_n(\mathbf{C})$ defined on a neighbourhood $U \subseteq \mathfrak{gl}_n(\mathbf{C})$ of the identity, which is a diffeomorphism onto an open subset of the identity $\mathrm{id} \in \mathrm{GL}_n(\mathbf{C})$, one can produce an open neighbourhood $V \subseteq \mathrm{GL}_n(\mathbf{C})$ of the identity which contains no subgroups of $\mathrm{GL}_n(\mathbf{C})$ aside from $\{\mathrm{id}\}$. Thus $\rho^{-1}(V)$ contains no subgroups of G_K and is open; hence $\rho^{-1}(V) = \ker \rho$ is open and $\implies$ $\mathrm{im}\,\rho$ is finite since ρ factors through one of the projections $G_K \twoheadrightarrow \mathrm{Gal}(K'/K)$ where $K'/K < \infty$. ■

Example 1.6. In contrast, a Galois representation with values in $\mathrm{GL}_n(\mathbf{Q}_p)$ might well have infinite image. For instance, suppose $\{\zeta_n\}_{n\geq 1}$ is a set of compatible p^n-th roots of unity in $\overline{\mathbf{Q}}$ (i.e. $\zeta_n^p = \zeta_{n-1}$). Then for every $\sigma \in G_{\mathbf{Q}}$ we have

$$\sigma(\zeta_n) = \zeta_n^{a_{\sigma,n}}, \quad (p, a_{\sigma,n}) = 1$$

and the elements $(\overline{a}_{\sigma,n} \in \mathbf{Z}/p^n\mathbf{Z})_n$ assemble to a p-adic unit $a_\sigma \in \mathbf{Z}_p^\times$ by compatibility of the family $(\zeta_n)_n$ of roots of unity. We thus get a surjective continuous group homomorphism

$$\epsilon_p : G_{\mathbf{Q}} \twoheadrightarrow \mathbf{Z}_p^\times \subseteq \mathrm{GL}_1(\mathbf{Q}_p)$$

- the continuity follows from the fact that ϵ_p is a limit of the maps

$$G_{\mathbf{Q}} \twoheadrightarrow \mathrm{Gal}(\mathbf{Q}(\zeta_n)/\mathbf{Q}) \xrightarrow{\cong} (\mathbf{Z}/p^n\mathbf{Z})^\times$$

which are all continuous.

Remark 1.7. Possibly the most fruitful and interesting source of Galois representations is from the ℓ-adic étale cohomology of an algebraic variety: if X/K is a variety where $K/\mathbf{Q}$ is a number field, one can define the ℓ-adic étale cohomology groups $H^i_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_\ell)$ which naturally carry an action of the Galois group G_K, since these groups are defined on the base change of X to K's separable closure $\overline{K}$. These $\overline{\mathbf{Q}}_p$ vector spaces will turn out to be our source of Galois representations when we'll be interested in constructing them from modular forms: the Deligne-Serre theorem addresses the problem on whether there exist Galois representations ρ as in Example 1.3 such that the traces of Frobenius elements are the prescribed elements $a_p \in \mathbf{Z}$ for varying p; above we had $a_p = 2$ if and only if $x^3 - x - 1$ splits completely modulo p.

2 Local representations for $\ell \neq p$

In this section we discuss representations of Galois groups G_K for $K/\mathbf{Q}_\ell$ a finite extension with values in $\mathrm{GL}_n(L)$ for a fixed algebraic extension $L/\mathbf{Q}_p$ with $p \neq \ell$.
A central tool in this setting is the wild ramification subgroup $P \subseteq \mathrm{Gal}(\overline{K}/K)$.

Definition 2.1. For every finite extension K'/K we define the inertia subgroup

$$I_{K'/K} = \left\{\sigma \in \mathrm{Gal}(K'/K) \mid \sigma_{|\mathcal{O}_{K'}/\mathfrak{m}_{K'}} = \mathrm{id}\right\}$$

and its wild ramification subgroup

$$P_{K'/K} = \left\{\sigma \in \mathrm{Gal}(K'/K) \mid \sigma_{|\mathcal{O}_{K'}/\mathfrak{m}^2_{K'}} = \mathrm{id}\right\} \subseteq I_{K'/K}.$$

The following lemma provides a description of $P_{K'/K}$; the real take-away is that it's a *normal* ℓ-Sylow of the inertia group $I_{K'/K}$ - evidently this will play a role when considering *p-adic* representations of $\mathrm{Gal}(K'/K)$, because of the natural contrast between of pro-p and pro-ℓ topologies.

Lemma 2.2. *Let K'/K be a finite Galois extension of degree e which is totally ramified (i.e. K' and K have equal residue fields).*

1. *$P_{K'/K} \subseteq I_{K'/K}$ is a normal subgroup,*
2. *$P_{K'/K}$ is an ℓ-group,*
3. *$I^t_{K'/K} = I_{K'/K}/P_{K'/K}$ has order equal to the prime-to-ℓ part of e (called the* tame inertia*).*
4. *There exists a canonical embedding*

$$\theta : I^t_{K'/K} \hookrightarrow \mu_e(k)$$

where $\mu_\ell(k)$ is the group of e-th power roots of unity in $\overline{k}$ and k is K's residue field, whose image is the subgroup of elements of order less than or equal to the prime-to-ℓ part of e.

Proof. We first define $\theta : I_{K'/K} \to \mu_e(k)$ and then check all the due properties by generalities on the group $\mu_e(k)$ and its prime-part-to-ℓ torsion subgroup. If $\varpi_{K'} \in K', \varpi_K \in K$ are uniformisers and by hypothesis on the extension K'/K we have the relation

$$\varpi_K = u\varpi^e_{K'}$$

for some unit $u \in \mathcal{O}^\times_{K'}$. For every element $\sigma \in I_{K'/K}$ we can apply σ to the above equation and thus

$$\varpi_K = \sigma(u)\sigma(\varpi_{K'})^e;$$

expressing $\sigma(\varpi_{K'}) = \theta_\sigma \varpi_{K'}$ it follows that

$$\frac{\sigma(u)}{u}\theta^e_\sigma = 1 \implies \overline{\theta}_\sigma \in \mu_e(\overline{k})$$

since $\sigma(u) \equiv u$ because $\sigma \in I_{K'/K}$. A simple check shows that

$$\begin{aligned} \theta : I_{K'/K} &\to \mu_e(\overline{k}) \\ \theta &\mapsto \theta_\sigma \end{aligned}$$

is a group homomorphism independent on our choices of uniformisers. The kernel

$$\ker\theta = \left\{\sigma \in I_{K'/K} \mid \sigma\varpi_{K'} = \varpi_{K'}\right\}$$

evidently just equals $P_{K'/K}$.
As for the second point, suppose $\sigma \in P_{K'/K}$ is an element of order m where m and ℓ are coprime and assume for the sake of contradiction that $\sigma(x) \neq x$ for some $x \in \mathcal{O}_{K'}$; this implies there exists $i \geq 0$ such that

$$\sigma(x) - x \in \mathfrak{m}^i_{K'}, \text{ and } \sigma(x) - x \notin \mathfrak{m}^{i+1}_{K'}.$$

Then $\sigma^{j+1}(x) - \sigma^j(x) \equiv \sigma(x) - x \mod \mathfrak{m}^{i+1}_{K'}$ since $\sigma \in I_{K'/K}$ and thus

$$\sigma^m(x) - x = \sigma(x)^m - \sigma^{m-1}(x) + \ldots + (-1)^m(\sigma(x) - x) \equiv m(\sigma(x) - x) \mod \mathfrak{m}^{i+1}_{K'}.$$

But since $\sigma^m(x) = x$ and m is invertible in k we get a contradiction by our hypothesis on i. ■

Remark 2.3. The groups $I_{K'/K}, \mu_e(\overline{k})$ both naturally carry an action of the full Galois group $\mathrm{Gal}(K'/K)$ since $I_{K'/K} \subseteq \mathrm{Gal}(K'/K)$ is normal. It thus makes sense to ask if θ respects these actions.

Lemma 2.4. *The map $\theta : I_{K'/K} \xrightarrow{\cong} \mu_e(k)$ is a morphism of $\mathrm{Gal}(K'/K)$-groups, in the sense that*

$$\theta(\tau\sigma\tau^{-1}) = \tau(\theta(\sigma))$$

for all $\tau \in \mathrm{Gal}(K'/K)$ and $\sigma \in I_{K'/K}$.

Proof. This is a pretty straightforward computation by the explicit description in the proof of Lemma 2.2. ∎

Remark 2.5. The previous lemma shows that in fact the subfield K^{tame}/K is not an abelian extension, if it's non-trivial.

Remark 2.6. Having constructed and described $P_{K'/K}$ for all finite totally ramified extensions K'/K we can set $P_K \subseteq I_K$ as the projective limit of all the finite wild ramification subgroups.

Remark 2.7. If we choose a compatible system of roots of unity $\zeta = (\zeta_m)_{(m,\ell)=1}$ then we have an isomorphism

$$t_\zeta : I_K/P_K \xrightarrow{\cong} \prod_{\ell \neq p} \mathbf{Z}_p = \varprojlim_{(m,\ell)=1} \mathbf{Z}/(m)$$

defined by

$$\frac{\sigma(\varpi_K^{\frac{1}{m}})}{\varpi_K^{\frac{1}{m}}} = \zeta_m^{t_\zeta(\sigma)}.$$

We denote by $t_{\zeta,p}$ the composition of t_ζ with the projection onto $\mathbf{Z}_p$. These *characters* will turn out to be useful in describing p-adic Galois representations.

Definition 2.8. Let L be a field of characteristic 0. A *Weil-Deligne representation* of W_K on a finite dimensional L-vector space V is a pair (r, N) where r is an open-kernel representation of W_K on V and $N \in \mathrm{End}(V)$ satisfies the following equality

$$r(\sigma)Nr(\sigma)^{-1} = (\#k)^{-\mathrm{val}_K(\sigma)}N$$

Definition 2.9.

1. An element $A \in \mathrm{GL}_n(L)$ is *bounded* if $\det A \in \mathcal{O}_L^\times$ and A's characteristic polynomial has coefficients in $\mathcal{O}_L$.
2. A Weil-Deligne representation is *bounded* if $r(\sigma)$ is bounded for all $\sigma \in W_K$.

Remark 2.10. The operator $A \in \mathrm{GL}_n(L)$ is bounded if and only if A stabilises an $\mathcal{O}_L$-lattice in L^n - note that this does not imply that any Weil-Deligne representation stabilises a lattice - this is true however if we know r's image is *finite*.

Theorem 2.11 (Grothendieck's Monodromy Theorem)**.** *Suppose $\ell \neq p$ are two primes, $K/\mathbf{Q}_\ell$ is a finite field extension and $L/\mathbf{Q}_p$ is an algebraic extension.*

1. *Given a Galois representation $\rho : G_K \to \mathrm{GL}(V)$ where V is a finite-dimensional L-vector space, there exists a finite extension K'/K and a unique endomorphism $N \in \mathrm{End}(V)$ such that the equation*

   $$\rho(\sigma) = \exp(N \cdot t_{\zeta,p}(\sigma))$$

   for all $\sigma \in I_{K'}$.

2. *There is an equivalence of categories between finite-dimensional continuous representations of G_K with values in L and the category of* bounded *Weil-Deligne representations with values in L; the equivalence maps the representation $\rho : G_K \to \mathrm{GL}_n(L)$ to the Weil-Deligne representation (r, N) where N is as above and r is defined by*

   $$r(\sigma) := \rho(\sigma)\exp(-t_{\zeta,p}(\phi^{-\mathrm{val}_K(\sigma)})\dot{N})$$

   where $\phi \in W_K$ is a fixed lift of $\mathrm{Frob}_K \in G_k$

Remark 2.12. The important consequence of the above theorem is that, up to passing to a finite field extension of K, the restriction to the Weil group W_K of a Galois representation $\rho : G_K \to \mathrm{GL}_n(L)$ is completely determined by the nilpotent operator $N \in \mathrm{Mat}_{n\times n}(L)$ and the image of a Frobenius lift $F \in W_K$ - furthermore, the restriction of ρ to the inertia subgroup $I_K \subseteq W_K$ *only depends on the image under the p-adic character* $t_{\zeta,p} : I_K \to I_K^t \to \mathbf{Z}_p$ described in Remark 2.7.

Proof. Since G_K is a compact subgroup, its image in $\mathrm{GL}_n(L)$ can be conjugated to a subgroup of $\mathrm{GL}_n(\mathcal{O}_L)$ and thus there exists some $\mathcal{O}_L$-lattice $\Lambda \subset V$ which is G_K-stable; composing with the projection map to $\Lambda/p\Lambda$ we get a group homomorphism

$$G_K \to \mathrm{GL}_n(\mathcal{O}_L/\varpi_L),$$

into a *finite group* of matrices; denote by $G_{K'}$ the kernel, which is the absolute Galois group of a finite extension K'/K.
Note that the kernel

$$\ker(\mathrm{GL}_n(\mathcal{O}_L) \to \mathrm{GL}_n(\mathcal{O}_L/\varpi_L)) = \{g \in \mathrm{GL}_n(\mathcal{O}_L) \mid g - \mathrm{id}_n \equiv 0 \mod \varpi_L\}$$

is pro-p, since we have an explicit system of neighbourhoods of the identity given by the congruence subgroups

$$U_m = \{g \in \mathrm{GL}_n(\mathcal{O}_L) \mid g - \mathrm{id}_n \equiv 0 \mod \varpi_L^m\}$$

which are such that U_m/U_{m+1} has order a prime power of p (I'm just too lazy to figure out which :p). Since the wild inertia subgroup $P_{K'} \subseteq I_{K'}$ is pro-ℓ and $\ell \neq p$, we get that $\rho\mid_{I_{K'}}$ factors through tame inertia $I_{K'}/P_{K'} \cong \prod_{l\neq q} \mathbf{Z}_q$ and since $\mathbf{Z}_q$ is pro-q for every q, we also get that it factors through the projection $t_{\zeta,p}$.
Fix $\sigma \in I_{K'}$ a lift of $1 \in \mathbf{Z}_p$ via $t_{\zeta,p}$ and consider the action of $\rho(\sigma)$ on V; we study $\rho(\sigma)$'s eigenvalues. Suppose $v \in V \otimes_L \overline{L}$ is such that $(\rho \otimes_L \overline{L})(\sigma)(v) = \lambda v$ for some $\lambda \in \overline{L}$. By the equation discussed in the previous lemma

$$\theta(\tau\sigma\tau^{-1}) = \tau(\theta(\sigma))$$

where τ acts on $\theta(\sigma)$ by G_K's action on the roots of unity in the residue field $\overline{k}$ (and then taking the *projective limit*[2] of all of these) we can specialise τ to be the Frobenius, which thus yields

$$\theta(\mathrm{Frob}_K\, \sigma\, \mathrm{Frob}_K^{-1}) = \sigma^p$$

and thus λ^p is also an eigenvalue of $\rho(\sigma)$, because $\rho(\sigma)$ has the same eigenvalues as any endomorphism conjugate to it, and λ^p is an eigenvalue of $\rho(\sigma)$ conjugated by $\rho(\mathrm{Frob}_K)$.
Since the set of eigenvalues of $\rho(\sigma)$ is finite, we get that these must be p-power roots of unity. Suppose they're all p^m-th roots of unity for $m \geq 1$. If we thus substitute K' with K'' so that $t_{\zeta,p}(I_{K''}^t)$ is given by the subgroup

$$p^m \cdot \mathbf{Z}_p$$

we get that $\rho(\sigma^{p^m})$ (i.e. $\rho(\sigma)$ for our newly replaced finite extension K'') only has 1 as its eigenvalues $\implies \rho(\sigma)$ is a unipotent matrix $\implies \rho(I_K)$ is unipotent (since ρ is continuous and unipotent matrices are a closed subgroup of $\mathrm{GL}_n(L)$). Since, as discussed $\rho\mid_{I_{K''}}$ factors through $t_{\zeta,p}$, we get that $\rho\mid_{I_{K''}}$ is in fact a 'one-parameter subgroup' of $\mathrm{GL}_n(L)$ whose image lies in the subgroup of unipotent matrices. If we call $\overline{\rho} : \mathbf{Z}_p \to \mathrm{GL}_n(L)$ the map induced by $\rho\mid_{I_{K''}}$. This means we have a logarithm map

$$\log(\overline{\rho}(-) - \mathrm{id}_n) : \mathbf{Z}_p \to \mathrm{Mat}_{n\times n}(L)$$

which is well-defined since $\overline{\rho}(-) - \mathrm{id}_n$ is a nilpotent endomorphism by our construction. Setting $N = \log(\overline{\rho}(1) - \mathrm{id}_n) = \log(\sigma - \mathrm{id}_n)$ yields the desired matrix. ■

[2]Note the colimit... as I struggled to realise for a while :p

3 Local representations with $p = \ell$ and p-adic Hodge theory

The theory of p-adic representations of $G_{\mathbf{Q}_p}$ is far more rich and involved that that of Artin and ℓ-adic representations: the freedom which comes from a compatility of the topologies on $G_{\mathbf{Q}_p}$ and $\mathrm{GL}_n(\mathbf{Q}_p)$ gives rise to an abundance of very different looking representations - in the past century many efforts were put into trying to tune the conditions on these representations to force them to arise *from geometry*, which is the notion encompassing the links between modular forms (the automorphic side), Galois representations (coming from geometry) and the ℓ-adic cohomology of smooth varieties; we'll discuss more details on this in the next section.

As motivation, we recall the classic Hodge decomposition for smooth proper varieties over $\mathbf{Q}$:

Theorem 3.1 (Hodge Decomposition). *Let $X/\mathbf{Q}$ be a smooth proper variety. Then there exists a natural isomorphism*

$$H^n(X(\mathbf{C}), \underline{\mathbf{C}}) \cong \bigoplus_{i+j=n} H^i(X, \Omega^j_{X_{\mathbf{C}}/\mathbf{C}})$$

Remark 3.2. An important remark to make is that, although both left and right hand sides both descend to vector spaces which are well-defined over $\mathbf{Q}$ - one could consider *rational* singular cohomology of the complex manifold $X(\mathbf{C})$ and the sheaves of *rational* differential forms $\Omega^j_{X_{\mathbf{Q}}/\mathbf{Q}}$ - the isomorphism in Theorem 3.1 is of *transcendental nature*, and requires an extension of scalars to what is called a *period ring*, chosen to contain all integrals of differential forms along paths in $X(\mathbf{C})$ (in this case our period ring is $\mathbf{C}$). This observation forces to question whether, in a p-adic setting, one should expect an analogue of the Hodge-Tate decomposition to carry a statement about the action of the absolute Galois group $G_{\mathbf{Q}_p}$, since the base change to $\mathbf{C}$ from $\mathbf{Q}$ is unavoidable in this setting. The Tate twists are important to present this analogue.

Definition 3.3. Let V be a G_K-representation over $\mathbf{Q}_p$. The *j-th Tate twist* $V(j)$ of V is the tensor product of V with the the j-power of the cyclotomic character

$$\epsilon_p^{\otimes j} : G_k \to \mathbf{Z}_p^\times.$$

We can now state the mentioned p-adic analogue of Theorem 3.1.

Theorem 3.4. *Let X/K be a smooth proper variety, where $K/\mathbf{Q}_p$ is a finite field extension. There exists a G_K-equivariant isomorphism*

$$H^n_{ét}(X_{\overline{K}}, \mathbf{Q}_p)_{\mathbf{C}_p} \cong \bigoplus_{i+j=n} H^i(X, \Omega^j_{X/K}) \otimes_K \mathbf{C}_p(-j).$$

An analysis of the Galois cohomology of $\mathbf{C}_p$ will allow us to understand what restrictions Theorem 3.4 forces on the associated Galois representation.

Theorem 3.5 (Tate). *For $i \neq 0$ we have*

$$H^0(G_K, \mathbf{C}_p(i)) = H^1(G_K, \mathbf{C}_p(i)) = 0$$

and for the zero-th Tate twist

$$H^0(G_K, \mathbf{C}_p) = H^1(G_K, \mathbf{C}_p) = K.$$

In particular, we see that there are no non-zero continuous G_K-homomorphisms $\mathbf{C}_p(i) \to \mathbf{C}_p(j)$ for distinct integers i and j.

Definition 3.6. The integers j appearing in the decomposition of Theorem 3.4 are called the Hodge-Tate weights.

Remark 3.7. By Theorem 3.5, we see that the Hodge Tate weights can be directly computed by taking invariants:

$$H^i(X, \Omega^j_{X/K}) \cong (H^{i+j}_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} \mathbf{C}_p(j))^{G_K}.$$

Define the *Hodge-Tate period ring* as the $\mathbf{C}_p$-algebra

$$B_{\text{HT}} = \mathbf{C}_p[t, t^{-1}]$$

which carries the G_K action by acting on the one-dimensional subspace $\mathbf{C}_p \cdot t^j$ as the j-th Tate twist of $\mathbf{C}_p$; then we have the equality

$$(H^n_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} B_{\text{HT}})^{G_K} = \bigoplus_{i+j=n} H^i(X, \Omega^j_{X/K}).$$

and the right-hand side is *essentially* equal to the étale cohomology group $H^n_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p)$ - the only difference being the missing Tate-twists (i.e. the Galois action doesn't match); more precisely, we see that it *is* isomorphic to $H^n_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p)$ as a $\mathbf{Q}_p$-vector space. These observations yield the equality

$$\dim_K (H^n_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p) \otimes B_{\text{HT}})^{G_k} = \dim_K H^n_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p).$$

Definition 3.8. A Galois representation V of G_K over $\mathbf{Q}_p$ is called *Hodge-Tate* or B_{HT}*-admissible* if the equality

$$\dim_K (V \otimes_K B_{\text{HT}})^{G_K} = \dim_{\mathbf{Q}_p} V$$

holds.

We have another stringent condition on Galois representations arising from the étale cohomology of varieties, under additional regularity assumptions. If X/K is a proper smooth variety with good reduction - i.e. X admits a proper *smooth* integral model $\mathfrak{X}/\mathcal{O}_K$ - then we have an isomorphism of Galois representations

$$H^i_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p) \cong H^i_{\text{ét}}(X_{\overline{k}}, \mathbf{Q}_p)$$

which follows from the smooth and proper base change theorems, cfr. Part 1 section 20 in Milne's notes [Mil]. This shows that the action of G_K on $H^i_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p)$ factors through the quotient $G_K \twoheadrightarrow G_k \cong \widehat{Z}$ by the inertia subgroup $I_K \subseteq G_K$.

Definition 3.9. A Galois representation ρ of G_K for K a local non-archimedean field is called *unramified* if it factors through the quotient $G_K \to G_k$ by its inertia subgroup. ρ is called *potentially unramified* if there exists a finite field extension K'/K such that the restriction $\rho_{|G_{K'}}$ is unramified.

Exercise 3.10 (Hodge-Tate characters)**.** We have a description of one-dimensional Hodge-Tate representations, following Exercise 6.4.3 in [ConBr].

1. A continuous character $\eta : G_K \to \mathbf{Z}_p^\times$ is Hodge-Tate if and only if $\eta(n) := \eta \otimes \epsilon_p(n)$ is potentially unramified for some $n \in \mathbf{Z}$.

2. The characters $(\epsilon_p^{p-1})^a$ are never Hodge-Tate if $a \in \mathbf{Z}_p \setminus \mathbf{Z}$.

3. A Galois character $\eta : G_K \to \overline{\mathbf{Q}}_p^\times$ is Hodge-Tate if and only if there exists an open subset $U \subseteq K^\times$ such that for all $a \in U$ we have

$$\eta(a) = \prod_{\tau \in I} \tau(a)^{n_\tau}$$

where I is some collection of embeddings $K \hookrightarrow \overline{\mathbf{Q}}_p$, n_τ are integers and we identify W_K^{ab} with $K^\times$ via the Artin map described in Remark 1.4 - note that $\eta_{|W_K}$ factors through $W_K \twoheadrightarrow W_K^{\text{ab}} \cong K^\times$ since $\overline{\mathbf{Q}}_p^\times$ is abelian.

Remark 3.11. While the previous exercise shows that for one-dimensional p-adic representations of G_K, Hodge-Tate admissibility completely characterises *geometric* representations. For higher dimensions this is no longer true and one has to replace the ring B_{HT} with a different period ring, refining this condition. We refrain from describing any further theory on this matter since the construction of B_{dR} is rather invoved and of similar spirit to B_{HT}; we mention however that B_{dR} is a filtered G_K-ring so that its associated graded ring is B_{HT}, and that there exists a G_K-equivariant isomorphism mimicking Theorem 3.4

$$H^n_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} B_{\mathrm{dR}} \cong H^n_{\mathrm{dR}}(X/K) \otimes_K B_{\mathrm{dR}}.$$

This yields an equation analogous to the one described in Remark 3.7. The condition

$$\dim_K (V \otimes_K B_{\mathrm{dR}})^{G_K} = \dim_{\mathbf{Q}_p} V$$

is analogously called *de Rham admissibility.*

4 Global Galois Representations and The Fontaine-Mazur Conjecture

We now focus on Galois representations of finite field extensions $F/\mathbf{Q}$ of the rational numbers, in an attempt to globalise results discussed in the previous two sections.

For each finite place v of F we denote by F_v the completion at v. If F'/F is a finite Galois extension, the Galois group $\mathrm{Gal}(F'/F)$ permutes transitives places of F' above v; we denote by $\mathrm{Gal}(F'/F)_w$ the stabiliser of the place w above v, often called the *decomposition group*. There is a natural isomorphism of Galois groups

$$\mathrm{Gal}(F'/F)_w \xrightarrow{\cong} \mathrm{Gal}(F'_w/F_v)$$

and thus an embedding $\mathrm{Gal}(F'_w/F_v) \hookrightarrow \mathrm{Gal}(F'/F)$ which is well-defined up to $\mathrm{Gal}(F'/F)$-conjugacy. If the extension F'/F is unramified at v, then $\mathrm{Frob}_{F_v} \in \mathrm{Gal}(F'_w/F_v)$ is a well-defined element and we obtain a conjugacy class $[\mathrm{Frob}_v]$ of Frobenius elements in $\mathrm{Gal}(F'/F)$ via the distinct conjugate embeddings $\mathrm{Gal}(F'_w/F_v) \hookrightarrow \mathrm{Gal}(F'/F)$. The following theorem illustrates why these many conjugacy classes are useful tools in our study.

Theorem 4.1 (Chebotarev Density Theorem)**.** *If F'/F is a Galois extension of number fields, unramified outside a finite set S of places of F, then the union of Frobenius-conjugacy classes*

$$\{[\mathrm{Frob}_v]\}_{v \notin S} \subseteq \mathrm{Gal}(F'/F)$$

is dense in $\mathrm{Gal}(F'/F)$.

A proof can be found in the online notes [Tri].

Remark 4.2. As described in [Tri], Theorem 4.1 is a generalisation of Dirichlet's theorem on primes in arithmetic progression, and is proven using similar analytic tools.

Remark 4.3. Paired with the Brauer-Nesbitt theorem, Theorem 4.1 can be used to show that completely reducible (continuous) Galois representations

$$\rho : G_F \to \mathrm{GL}_n(L)$$

are determined by the set of characteristic polynomials

$$\{\mathrm{char}(\rho(\mathrm{Frob}_v))\}_{v \notin S}$$

and, if L's characteristic is 0, even just their traces

$$\{\mathrm{tr}(\rho(\mathrm{Frob}_v))\}_{v \notin S}.$$

Definition 4.4. A Galois representation $\rho : G_F \to \mathrm{GL}_n(L)$ with $L/\mathbf{Q}_p$ a finite extension is called *geometric* if it unramified outside a finite set S of places of F and if for all places v over p the representation $\rho_{|G_{F_v}}$ is de Rham.

We're now at the point where we can state the crux of these notes:

Conjecture 4.5 (Fontaine-Mazur)**.** *Any irreducible geometric representation*

$$\rho : G_F \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$$

is a subquotient of a representation of the form

$$H^i_{\acute{e}t}(X_{\overline{F}}, \mathbf{Q}_p)(j)$$

for some proper smooth variety X over F.

The case where $n = 1$ can be boiled down to a statement about global class field theory, and requires an interpretation of one-dimensional Galois representations as automorphic forms; this is at the heart of the Langlands Correspondence.

5 The Global Langlands Correspondence for characters

For the rest of these notes, we reference Fargues' article [Fa] and discuss the Global Langlands Correspondence for characters.

We fix a number field $F/\mathbf{Q}$ and consider the induced torus

$$T = \mathrm{Res}_{F/\mathbf{Q}}\, \mathbf{G}_{m,F}$$

so that the base change to $\overline{\mathbf{Q}}$ splits

$$T_{\overline{\mathbf{Q}}} = \prod_{\tau : F \hookrightarrow \overline{\mathbf{Q}}} \mathbf{G}_{m,\overline{\mathbf{Q}}}.$$

Thus the groups of characters $X^*(T) = \mathrm{Hom}_{\overline{\mathbf{Q}}}(T_{\overline{\mathbf{Q}}}, \mathbf{G}_m)$ can be described as the lattice

$$X^*(T) = \left\{ \sum_{\tau : F \hookrightarrow \overline{\mathbf{Q}}} a_\tau[\tau] \mid a_\tau \in \mathbf{Z} \right\}.$$

Since T splits over F, we see that by considering the group of F-rational points and precomposing with the unit of the adjunction $- \times_{\mathbf{Q}} F \dashv \mathrm{Res}_{F/\mathbf{Q}}(-)$, any character gives rise to a group homomorphism

$$\begin{aligned} F^\times &\longrightarrow \overline{\mathbf{Q}}^\times \\ x &\longmapsto \prod_\tau \tau(x)^{a_\tau}. \end{aligned}$$

Thus $G_{\mathbf{Q}}$ acts on $X^*(T)$ via post-composing each of the embeddings τ with a field automorphism $\sigma : \overline{\mathbf{Q}} \to \overline{\mathbf{Q}}$. The use of T becomes apparent once we consider its $\mathbf{A}_{\mathbf{Q}}$-rational points, where $\mathbf{A}_{\mathbf{Q}}$ is the ring of $\mathbf{Q}$-adeles:

$$T(\mathbf{A}_{\mathbf{Q}}) = \mathbf{A}_F^\times.$$

Definition 5.1. A Hecke character of F is a continuous character

$$\chi : \mathbf{A}_F^\times / F^\times \to \mathbf{C}^\times$$

from the idéle class group of F to the complex numers.

Hecke characters form the *automorphic side* of the Global Langlands Correspondence for one-dimensional representations.

Remark 5.2. Any χ as in Definition 5.1 decomposes as a restricted product

$$\chi = \bigotimes_v{}' \chi_v = \chi_f \otimes \chi_\infty$$

where $\chi_\infty : F_\infty^\times / F^\times \to \mathbf{C}$ is continuous and $\chi_f : \mathbf{A}_{F,f}^\times / F^\times \to \mathbf{C}$ is trivial on an open compact subgroup of the finite adeles $\mathbf{A}_{F,f}^\times$. Thus χ_v is unramified for all but finitely many finite places v - i.e. is trivial on the inertia subgroup $\mathcal{O}_{F_v}^\times \subseteq F_v^\times \cong G_{K_v}^{\mathrm{ab}}$ (by Artin Reciprocity 1.4).

Definition 5.3. Suppose $\rho \in X^\times(T)$ is a character of T. A Hecke character χ of F if *algebraic of weight* ρ is

$$\chi_{|(F_\infty^\times)^\circ} = \rho^{-1} : T(\mathbf{R})^\circ \hookrightarrow T(\mathbf{R}) \hookrightarrow T(\mathbf{C}) \xrightarrow{\rho^{-1}} \mathbf{C}^\times .$$

Remark 5.4. Since all infinite places $v \mid \infty$ are so that F_v is either the real of complex numbers, we can characterise algebraic Hecke characters as follows:

- if v is real, then
$$\chi_v(x) = \mathrm{sign}(x)^{\pm 1} \cdot |x|^n$$
for some $n \in \mathbf{Z}$.
- if v is complex, then
$$\chi_v(z) = z^p \overline{z}^q$$
for some integers $p, q \in \mathbf{Z}$.

The first of the following examples should motivate this notion: algebraic Hecke characters are a generalisation of Artin representations for the abelianised (global) Galois group G_F.

Example 5.5. 1. If χ is algebraic of weight $\rho = 0$, then by definition
$$\chi : \mathbf{A}_F^\times / \overline{(F_\infty^\times)^\circ F^\times} \to \mathbf{C}^\times$$
and by global Artin reciprocity the isomorphism
$$\mathrm{Art}_F : \mathbf{A}_F^\times / \overline{(F_\infty^\times)^\circ F^\times} \xrightarrow{\cong} G_F^{\mathrm{ab}}$$
identifies χ as a one-dimensional Artin representation for F.

2. The idélic norm
$$|| - || : \mathbf{A}_F^\times / F^\times \to \mathbf{R}_+^\times$$
is algebraic of weight
$$\rho = \mathrm{Nm}_{F/\mathbf{Q}}^{-1} = - \sum_{\tau : F \hookrightarrow \overline{\mathbf{Q}}} [\tau]$$
since we have
$$T(\mathbf{R})^\circ = \prod_{\tau : F \hookrightarrow \mathbf{R}} \mathbf{R}_+ \times \prod_{\tau, \overline{\tau} : F \hookrightarrow \mathbf{C}} \mathbf{C}$$
and the adelic norm computed on $(x_\tau)_\tau \in T(\mathbf{R})^\circ$ equals
$$||(x_\tau)|| := \prod_\tau |x_\tau| = \prod_\tau x_\tau = \mathrm{Nm}_{F/\mathbf{Q}}((x_\tau)_\tau)$$
since all real numbers appearing in the above product are already positive, and for each complex number the conjugate also appears.

Lemma 5.6. *Let $\chi : \mathbf{A}_F^\times/F^\times \to \mathbf{C}^\times$ be an algebraic Hecke character of weight $\rho \in X^*(T)$. Then there exists a number field $E/\mathbf{Q}$ such that the image of χ_f is contained in $E^\times$.*

Proof. Suppose $U_f \subset \mathbf{A}_{F,f}^\times$ is a compact open subgroup contained in the kernel of χ_f. By finiteness of the class group, which is isomorphic to the double quotient group

$$F^\times \backslash \mathbf{A}_{F,f}^\times \big/ \prod_v \mathcal{O}_{F,f}^\times$$

we see that the *group*

$$F^\times \backslash \mathbf{A}_{F,f}^\times \big/ U_f$$

is finite. Since $\chi_f(x) = \pm\rho(x)$ for all $x \in F^\times$ (becasuse $\chi_f(x^{(\infty)})\chi_\infty(x_\infty) = 1$ whenever $x \in F^\times$ and $F^\times$) it follows that χ_f takes values in the number field generated by $\rho(u)$ where u varies among the finitely many elements in the group $F^\times \backslash \mathbf{A}_{F,f}^\times \big/ U_f$. ■

Now that we've introduced the automorphic side, starting from an algebraic Hecke character we may construct a Galois representation.

Definition 5.7. Let $\chi = \chi_f \otimes \chi_\infty$ be an algebraic Hecke character of weight ρ and $E/\mathbf{Q}$ a number field which contains the image of χ_f. We then we have a map

$$\rho^E : T = \mathrm{Res}_{F/\mathbf{Q}} \to \mathrm{Res}_{E/\mathbf{Q}}\, \mathbf{G}_m$$

which descends $\rho : T_{\overline{\mathbf{Q}}} \to \mathbf{G}_{m,\overline{\mathbf{Q}}}$ since for all $x \in F^\times = T(\mathbf{Q})$ we have

$$\rho(x) = \chi_f(x) \in E^\times = (\mathrm{Res}_{E/\mathbf{Q}}\, \mathbf{G}_{m,E})(\mathbf{Q})$$

and the $\mathbf{Q}$-rational points are Zariski-dense in both induced tori.
For a fixed prime p and a prime $\lambda \mid p$ lying over p in E, we define the continuous map

$$\begin{aligned} \psi_{\chi,\lambda} : \mathbf{A}_F^\times = F_\infty^\times \times (\mathbf{A}_{F,f}^p)^\times \times (F \otimes_{\mathbf{Q}} \mathbf{Q}_p)^\times &\longrightarrow E_\lambda^\times \\ (x_\infty,\, x_f^p,\, x_p) &\longmapsto \underbrace{\frac{\chi_\infty}{\rho^{-1}}(x_\infty)}_{=\pm 1} \cdot \underbrace{\chi_f(x_p x_f^p)}_{\in E^\times} \cdot \underbrace{\rho_\lambda^{-1}(x_p)}_{\in E_\lambda^\times} \end{aligned}$$

It now follows that $\rho_{\chi,\lambda}$ factors through $F^\times$ and $(F_\infty^\times)^\circ$, is continuous for the ℓ-adic topology and thus defines a continuous Galois character

$$\psi_{\chi,\lambda} : G_F^{\mathrm{ab}} \xrightarrow{\cong} \mathbf{A}_F^\times / \overline{F^\times (F_\infty^\times)^\circ} \to E_\lambda^\times.$$

We have that $\psi_{\chi,\lambda}$ satisfies the following properties.

Proposition 5.8. *Let S be a finite set of finite place of F such that χ is unramifies away from S. Then*

1. *the Galois reprepresentation $\psi_{\chi,\lambda}$ is unramified outside $S \cup \{v \mid p\}$ and the element*

   $$\psi_{\chi,\lambda}(\mathrm{Frob}_v) \in E^\times$$

   is independent of $\lambda \mid p$ and of p.

2. *for all places $v \mid p$ dividing p, the representation*

   $$\psi_{\chi,\lambda}\, |_{G_{F_v}}$$

   is potentially crystalline and is crystalline if and only if χ is unramified at v.

This defines for us one of the maps in the following result.

Theorem 5.9 (Global Langlands for $n = 1$)**.** *Let E be the normal closure of the number field $F/\mathbf{Q}$ and fix embeddings $\iota_\infty : \overline{E} \hookrightarrow \mathbf{C}, \iota_p : \overline{E} \hookrightarrow \overline{\mathbf{Q}}_p$. Then there exists a natural bijection between algebraic Hecke characters $\chi : \mathbf{A}_F^\times/F^\times \to \mathbf{C}^\times$ and continuous characters $G_F^{ab} \to \overline{\mathbf{Q}}_p^\times$ which are de Rham at all places lying over p.*

We end these notes by computing the Galois representation associated to the adelic norm

$$\chi := || - || : \mathbf{A}_F^\times \to \mathbf{R}_{>0} \subseteq \mathbf{C}$$

which, as mentioned, is an algebraic Hecke character of weight $\rho = \mathrm{Nm}_{F/\mathbf{Q}}^{-1} : T = \mathrm{Res}_{F/\mathbf{Q}} \mathbf{G}_m \to \mathbf{G}_m$. The ℓ-adic norm of any ℓ-adic number is rational by construction, so we may choose $E = \mathbf{Q}$, thus ρ^E coincides with ρ. If we fix a prime p, then the formula in Definition 5.7 yields

$$\begin{aligned} \psi_{\chi,p} : \mathbf{A}_{\mathbf{Q}}^\times &\longrightarrow \mathbf{Q}_p^\times \\ (x_\infty, x_f^p, x_p) &\longmapsto \frac{|x_\infty|}{x_\infty} \cdot ||x_f||_f \cdot x_p = \mathrm{sign}(x_\infty)||x_f||_f \cdot x_p. \end{aligned}$$

where $|| - ||_f$ denotes the adélic norm on the finite places. By the Chebotarev Density Theorem 4.1 we see that identifying the induced representation

$$\psi_{||-||,p} : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})^{\mathrm{ab}} \to \mathbf{Q}_p^\times$$

amounts to understanding what the values of $\psi_{||-||,p}$ on the conjugacy classes of Frobenius elements are. Under the global Artin reciprocity map, any lift of the Frobenius element $\mathrm{Frob}_q \in G_{\mathbf{F}_q}$ will map to the element $\mathrm{Art}(\mathrm{Frob}_q) \in \mathbf{A}_{\mathbf{Q}}^\times$ where

$$\mathrm{Art}(\mathrm{Frob}_q)_f^{(q)} = \mathrm{Art}(\mathrm{Frob}_q)_\infty = 1, \;\; \mathrm{Art}(\mathrm{Frob}_q)_q = q.$$

So on one hand we see that

$$\psi_{||-||,p}(\mathrm{Frob}_q) = q \in \mathbf{Q}_p^\times.$$

On the other, if $(\zeta_{p^n})_{n\geq 1} \subset \overline{\mathbf{Q}}^\times$ is a compatible system of primitive p^n-th roots of unity and

$$\phi \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$$

is a lift of $\mathrm{Frob}_q \in G_{\mathbf{F}_q}$ (where we implicitly fix an isomorphism $\overline{\mathbf{Q}}_q \cong \mathbf{C}$), we see that

$$\phi(\zeta_{p^n}) = \zeta_{p^n}^q$$

- indeed, by definition we have that $\phi(\zeta_{p^n}) = \zeta_{p^n}^q + qy$ for some integer $y \in \mathbf{Z}_q$, but

$$|\zeta_{p^n}^q - \zeta_{p^n}^k|_q = \begin{cases} 0 & \text{if } q \equiv k \mod p^n \\ 1 & \text{otherwise} \end{cases}$$

and thus can't be strictly less than 1 unless $k = q$. This shows us that

$$\epsilon_p(\phi) = \epsilon_p(\mathrm{Frob}_q) = q$$

where $\epsilon_p : G_{\mathbf{Q}} \twoheadrightarrow \mathbf{Z}_p^\times$ is the cyclotomic character. Since ϵ_p and $\psi_{||-||,p}$ agree on a dense subset of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ by Theorem 4.1, we can conclude that indeed the p-adic Galois representation attached to the adélic norm on the rationals corresponds to the p-adic cyclotomic character, which indeed is unramified outside p and de Rham at p.

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