24/11 ([2] chap. 4)

Quantum Groups, $U_q(\mathfrak{g}(A)), U_q(A)$

Definition: Given $A \in M(n \times n, \mathbb{Z})$ a symmetrisable Generalised Cartan Matrix, DA symmetric with $D \in M(n \times n, \mathbb{Z})$ diagonal with diagonal entries $d_1, \dots, d_n$, consider the associated Kac-Moody Lie algebra $\mathfrak{g}(A) = \widetilde{\mathfrak{g}}(A)/I$. Define the quantum universal enveloping algebra of $U_q(\mathfrak{g}(A))$ as the $k(q)$-algebra generated by elements

$$E_i, F_i, K_i^{\pm 1} \text{ for } 1 \le i \le n$$

with the *quantum analogues of the Serre relations* given by

(R1) $K_i K_i^{-1} = K_i^{-1} K_i = 1$, $K_i K_j = K_j K_i$ for all $i, j = 1, \dots, n$;

(R2) $K_i E_j = q^{d_i a_{i,j}} E_j K_i$, $K_i F_j = q^{d_i a_{i,j}} F_j K_i$;

(R3) $E_i F_j - F_j E_i = \delta_{i,j} \frac{K_i - K_i^{-1}}{q^{d_i} - q^{-d_i}}$;

(R4) (q-Serre relations) for all $i \neq j$

$$S^+_{i,j} := \sum_{s=0}^{1-a_{i,j}} (-1)^s \begin{bmatrix} 1 - a_{i,j} \\ s \end{bmatrix}_{q^{d_i}} E_i^{1-a_{i,j}-s} E_j E_i^s = 0,$$

$$S^-_{i,j} := \sum_{s=0}^{1-a_{i,j}} (-1)^s \begin{bmatrix} 1 - a_{i,j} \\ s \end{bmatrix}_{q^{d_i}} F_i^{1-a_{i,j}-s} F_j F_i^s = 0,$$

Similar to how we constructed the classic Kac-Moody Lie algebras, denote by $\widetilde{U}_q(A)$ the $k(q)$-algebra with the same generators as $U_q(A)$ and relations (R1),(R2) and (R3).

Remark: If A is the 1×1 matrix (2) then we recover the classic $U_q(\mathfrak{sl}_2)$ over the field $k(q)$.

Remark: Note that the relations given in (R4) are precisely the *quantised* Serre relations introduced when studying Kac-Moody Lie algebras, since in general we have the formula, valid in $U(\mathfrak{g})$,

$$(\mathrm{ad}(x))^n(y) = \sum_{s=0}^{n} (-1)^s \binom{n}{s} x^{n-s} y x^s.$$

This hints at somewhat of a general strategy which we shall adopt throughout most of the constructions: often times it's unlikely that we'll have a precise q-analogue of certain instruments for the classical theory developed over the standard universal enveloping algebra $U(\mathfrak{g})$, so what we'll do is turn our considered object into a combinatorial identity and *"quantise"* the obtained formula by swapping out the integral-coefficients appearing in it with their quantum-analogues.

Definition (notation): Let $I = (\alpha_{i_1}, \dots, \alpha_{i_r})$ be a finite sequence of simple roots (i.e. elements in $\mathfrak{h}^*$ given in the minimal realisation for A). Define

$$E_I := E_{i_1} \dots E_{i_r}, E_\emptyset := 1,$$
$$F_I := F_{i_1} \dots F_{i_r}, F_\emptyset := 1,$$
$$K_\mu := K_1^{m_1} \dots K_n^{m_n} \text{ for every } \mu \in Q := \bigoplus_\alpha \mathbb{Z}\alpha \subseteq \mathbb{C}^{\dim \mathfrak{h}} \text{ where } \mu = \textstyle\sum_i n_i \alpha_i.$$

Proposition: $U_q(A)$ is a Hopf algebra via the formulas

$$\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i,$$

$$\Delta(F_i) = 1 \otimes F_i + F_i \otimes K_i^{-1}$$

$$\Delta(K_i) = K_i \otimes K_i$$

and η, S defined just as with $U_q(\mathfrak{sl}_2)$.

Observations: We have naturally-defined subalgebras $\widetilde{U}_q^-, \widetilde{U}_q^0, \widetilde{U}_q^+$ generated by F_i's, $K_i^{\pm}$'s and E_i's respectively. If we set $\deg(E_i) = -\deg(F_i) = 1$ and $\deg(K^{\pm 1}) = 0$ then we have that the relations (R1),(R2),(R3) and (R4) are homogeneous with respect to this grading, thus $U_q(A)$ has a natural $\mathbb{Z}$-grading.

Proposition: The set $\{F_I, K_\mu, E_J \mid I, J \subseteq \{\alpha_1, \dots, \alpha_n\}, \mu \in Q\} \subseteq U_q(A)$ for a basis of $\widehat{U}_q(A)$.

Proof: Omitted. □

Corollary:

1. The multiplication map
$$\widetilde{U_q}^- \otimes \widetilde{U}_q^0 \otimes \widetilde{U}_q^+ \to \widetilde{U}_q$$
$$u_1 \otimes u_2 \otimes u_3 \mapsto u_1 u_2 u_3$$
is an isomorphism of vector spaces.
2. The E_I's, F_J's and K_μ's form a basis for $\widetilde{U_q}^-$, $\widetilde{U_q}^0$ and $\widetilde{U_q}^+$ respectively.
3. The multiplication map
$$\widetilde{U_q}^+ \otimes \widetilde{U_q}^0 \otimes \widetilde{U}_q^-$$
is an isomorphism of vector spaces.

Corollary: For every $i = 1, \dots, n$ the algebra homomorpism
$$\phi_i : U_{q^{d_i}}(\mathfrak{sl}_2) \to \widetilde{U}_q(A)$$
$$E \mapsto E_i$$
$$F \mapsto F_i$$
$$K \mapsto K_i$$
is an inclusion.

Theorem: The multiplication map
$$U_q^- \otimes U_q^0 \otimes U_q^+ \to U_q$$
$$u_1 \otimes u_2 \otimes u_3 \mapsto u_1 u_2 u_3$$
is an isomorphism of vector spaces.

Proof: Let $p : \widetilde{U}_q \twoheadrightarrow U_q$ be the canonical projection and $I \subseteq \widetilde{U}_q$ the ideal generated by the q-Serre relations elements $S_{i,j}^\pm \in \widetilde{U}_q$. Define $I^\pm := \widetilde{U}_q^\pm \cap I$.

Claim: *The ideal in $\widetilde{U}_q$ generated by the $S_{i,j}^+$'s is the image* **im** *via the multiplication map applied to the subalgebra $\widetilde{U}_q^- \otimes \widetilde{U}_q^0 \otimes I^+$.*

"$\supseteq$": clear.

"$\subseteq$": Evidently the image of $1 \otimes 1 \otimes S_{i,j}^+$ is $S_{i,j}^+$ and thus **im** contains all the generators of I^+ as an ideal. It follows that all we have to do it show **im** is a two-sided ideal; it's evidently a left ideal by construction and furthermore every element in **im** is a linear combination of terms of the form
$$u S_{i,j}^+ E_I,\ u \in \widetilde{U}_q.$$
multiplying any of these on the right by E_r, K_μ, F_r yields
$$u S_{i,j}^+ E_I E_r = u S_{i,j}^+ E_{I \cup r} \in \mathbf{im}$$
$$u S_{i,j}^+ E_I K_\mu = q^{\cdots} u S_{i,j}^+ K_\mu E_I = q^{\cdots} u K_\mu S_{i,j}^+ E_I \in \mathbf{im}$$
$$u S_{i,j}^+ E_I F_r = u F_r S_{i,j}^+ E_I - u[F_r, S_{i,j}^+] E_I - u S_{i,j}^+ [F_r, E_I] = u F_r S_{i,j}^+ E_I - u S_{i,j}^+ [F_r, E_I] \in \mathbf{im}$$
where in the last one we used that $[F_r, S_{i,j}^+] = 0$ and that $[F_r, E_I]$ lies in $\widetilde{U}_q^+$ and thus the term $u S_{i,j}^+ [F_r, E_I]$ lies in **im** by the first calculation.
Thus the claim follows.

Following a similar argument we also have that I^- is given precisely by the image under the multiplication map of the subalgebra $I^- \otimes \widetilde{U}_q^0 \otimes \widetilde{U}_q^+$. Since $I = I^+ \oplus I^-$ we have that I is the image under the multiplication map of
$$\widetilde{U}_q^- \otimes \widetilde{U}_q^0 \otimes I^+ + I^- \otimes \widetilde{U}_q^0 \otimes \widetilde{U}_q^+.$$

For the rest of the argument consult Jantzen Theorem 4.21 (haven't got time for this lol). □

Problem: We're missing an analogue of the PBW theorem for the quantum universal enveloping algebra $\widetilde{U}_q(A)$... The problem is that although we do have analogues of elements in $\widetilde{U}_q(A)$ corresponding to "simple" roots e_α for $\alpha \in \pi$, we're lacking some sort-of analogue of the remaining roots.
A useful tool in the classical theory of semisimple Lie algebras is the action of the Weyl group W; recall that W acts on the set of roots - W acts on $\mathfrak{g}$ and $w(\mathfrak{g}_\alpha) \subseteq \mathfrak{g}_{w(\alpha)}$ - and, in particular, we can construct *all* roots by taking the simple ones and analysing the orbit of W's action on them.

26/11 - 3/12 ([1] chap. 3)

Defining an analogue of the Weyl group action for $\mathfrak{g}(A)$ for A GCM symmetrisable

Definition: Let V be a (possibly infinite-dimensional) complex vector space, $x \in \mathrm{End}_{\mathbb{C}}(V)$ and $v \in V$ any vector.

1. x is *locally finite at* v if there exists an x-invariant subspace W containing v.
2. x is a *locally finite* endomorphism if it's locally finite at all vectors $v \in V$.

Definition: If $x \in \mathrm{End}_{\mathbb{C}}(V)$ is a locally finite endomorphism, define

$$\exp(x) := \sum_{n \geq 0} x^n/n! \in \mathrm{End}_{\mathbb{C}}(V).$$

Note that for every finite-dimensional x-invariant vector space $W \subseteq V$ we have that $\mathrm{End}_{\mathbb{C}}(W)$ is endowed with a norm $||-||$ satisfying $||A|| \cdot ||B|| \geq ||AB||$ (since $\mathrm{End}_{\mathbb{C}}(W)$ can be identified with euclidean space $\mathbb{C}^{\dim W^2}$) thus $\exp(x)|_W$ is well defined. Since x is locally finite by assumption, formally speaking we set

$$\exp(x) = \operatorname*{colim}_{W \subseteq V, x(W) \subseteq W} \exp(x)|_W \in \mathrm{End}_{\mathbb{C}}(V).$$

Remark: If x is *nilpotent* then $\exp(x)$ is *polynomial* (with globally bounded degree) in x.
If x is *locally nilpotent*, then it is a polynomial in x when restricted to each x-invariant finite-dimensional subspace.

Remark: A standard calculation shows that for every $a \in \mathbb{Z}$ we have $\exp(x)^a = \exp(ax)$; in particular, we have that $\exp(x)$ is *invertible* for every $x \in \mathrm{End}_{\mathbb{C}}(V)$ locally finite.

Lemma: Let A be an algebra, $\partial : A \to A$ a derivation; the following formulae hold $x, y \in A$.

1. $\partial^n([x,y]) = \sum_{i=0}^{n} \binom{n}{i} [\partial^i(x), \partial^{n-i}(y)]$,
2. $x^n(y) = \sum_{i=0}^{n} \binom{n}{i} (\mathrm{ad}(x))^i(y) x^{n-i}$,
3. $(\mathrm{ad}(x))^n(y) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} x^{n-i} y x^i$.

Proof: All of these follow quite directly from a simple induction. $\square$

Corollary: Let $x, y \in \mathrm{End}_{\mathbb{C}}(V)$, and assume y is locally finite and $\mathrm{ad}(y) \in \mathrm{End}_{\mathbb{C}}(\mathrm{End}_{\mathbb{C}}(V))$ locally finite at x. Then

$$\exp(y) \cdot x \cdot \exp(-y) = \sum_{n \geq 0} \frac{(\mathrm{ad}(y))^n}{n!}(x) = \exp(\mathrm{ad}(y))(x).$$

Proof: Since y is locally finite the left and right hand sides are well defined. We thus have

$$\exp(y) \cdot x \cdot \exp(-y) = (\sum_{n \geq 0} \frac{y^n}{n!}) \cdot x \cdot (\sum_{n \geq 0} \frac{(-y)^n}{n!}) = \sum_{n \geq 0} \sum_{i=0}^{n} (-1)^i \frac{y^{n-i} x y^i}{(n-i)! \cdot i!} \overset{\text{Lemma}}{=} \sum_{n \geq 0} \frac{\mathrm{ad}(y)^n}{n!} x = \exp(\mathrm{ad}(y))(x). \square$$

Remark:

- Let $x \in \mathrm{End}_{\mathbb{C}}(V)$ be locally finite. For every isomorphism $f \in \mathrm{GL}_{\mathbb{C}}(V)$ we have that conjugation $f \circ - \circ f^{-1} \in \mathrm{End}_{\mathbb{C}}(\mathrm{End}_{\mathbb{C}}(V))$ of course fixes the subspace of locally finite endomorphisms, i.e. $f \cdot x \cdot f^{-1} \in \mathrm{End}_{\mathbb{C}}(V)$ is again locally finite.

- Let $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of a complex Lie algebra $\mathfrak{g}$.
  If $\mathrm{ad}(x) : \mathfrak{g} \to \mathfrak{g}$ is locally finite, then $\mathrm{ad}(\pi(x)) \in \mathrm{End}_{\mathbb{C}}(\mathrm{im}(\pi)) \subseteq \mathfrak{gl}(V)$ is locally finite.

Proof: Let $z = \pi(u)\mathrm{im}(\pi)$ and let $U \subseteq \mathfrak{g}$ be a finite-dimensional vector subspace containing u which is $\mathrm{ad}(x)$-invariant. It follows that the image $\pi(U) \subseteq \mathrm{im}(\pi)$ is of course also finite-dimensional, contains z and is $\mathrm{ad}(\pi(x))$-invariant since

$$y = \pi(w) \in \pi(U) \implies \mathrm{ad}(\pi(x))(\pi(w)) = \pi([x,w]) \in \pi(U)$$

since U is $\mathrm{ad}(x)$-stable. □

Lemma: Assume $\mathfrak{g}$ is a Lie algebra with Lie-algebra generators $\{z_i\}_{i \in I}$.
Suppose $x \in \mathfrak{g}$ is such that $\mathrm{ad}(x)$ acts locally nilpotently on each z_i. Then $\mathrm{ad}(x) \in \mathrm{End}_{\mathbb{C}}(\mathfrak{g})$ is locally nilpotent.

Proof: For each $i \in I$, let $N_i \geq 0$ be such that $\mathrm{ad}(x)^{N_i}(z_i) = 0$. Thus for each $N \geq 0$ we have

$$(\mathrm{ad}(x))^N([z_i, z_j]) = \sum_{r=0}^{N} [(\mathrm{ad}(x))^r(z_i), \mathrm{ad}(x)^{N-r}(z_j)],$$

hence taking $N = N_i + N_j + 1$ yields that $\mathrm{ad}(x)$ acts locally nilpotently on $[z_i, z_j]$ as well. Inductively it follows that $\mathrm{ad}(x)$ acts locally nilpotently on a basis of $\mathfrak{g}$, since the collection of Lie-words in the z_i's form a set of generators; thus $\mathrm{ad}(x)$ is locally nilpotent. □

Corollary: Let A be a Generalised Cartan Matrix. Then $\mathrm{ad}(e_i), \mathrm{ad}(f_i) \in \mathrm{End}_{\mathbb{C}}(\mathfrak{g}(A))$ act locally nilpotently on $\mathfrak{g}(A)$.

Proof: The previous lemma shows that it is sufficient to test nilpotency of $\mathrm{ad}(e_i)$ and $\mathrm{ad}(f_i)$ on a set of generators for $\mathfrak{g}(A)$ as a Lie algebra. The Serre relations thus allow us to conclude. □

Definition: Let A be a Generalised Cartan Matrix and M a $\mathfrak{g}(A)$-module. M is said to be *integrable* if:

- M has a weight space decomposition: $M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda$,
- e_i, f_i act locally nilpotently on M.

We denote by $\mathscr{O}_{\mathrm{int}}(\mathfrak{g}(A)) \subseteq \mathrm{Mod}_{\mathfrak{g}(A)}$ the full subcategory of integrable $\mathfrak{g}(A)$-modules.
The previous corollary shows us that $\mathfrak{g}(A)$ itself lies in $\mathscr{O}_{\mathrm{int}}(\mathfrak{g}(A))$ via the adjoint representation.
The following proposition shows the importance of this definition. For every $1 \leq i \leq n$ let $\mathfrak{g}_{(i)}$ be the Lie subalgebra in $\mathfrak{g}(A)$ generated by the standard generators e_i, f_i - thus $\mathfrak{g}_{(i)}$ is spanned by e_i, f_i and $\alpha_i^\vee$ and is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$.

Proposition: Let V be an integrable $\mathfrak{g}(A)$-module.

1. V decomposes as a direct sum of finite dimensional irreducible $\mathfrak{g}_{(i)} \cong \mathfrak{sl}_2(\mathbb{C})$-modules which are also $\mathfrak{h}$-invariant.

2. Let $\lambda \in \mathfrak{h}^*$ be a weight of V and let α_i be a simple root of $\mathfrak{g}(A)$. Denote by M_{λ,α_i} the set of roots of V that lie in the α_i*-string through* λ - in other words $M_{\lambda,\alpha_i} := \{\lambda + t\alpha_i \in \mathfrak{h}^* \mid t \in \mathbb{Z}, M_{\lambda+t\alpha_i} \neq (0)\}$. Denote by m_t the dimension of the weight space $M_{\lambda+t\alpha_i}$ for each $\lambda + t\alpha_i \in M_{\lambda,\alpha_i}$. Then:

   (a) M_{λ,α_i} is a *string* (as its name suggests), i.e. it is of the form

   $$M_{\lambda,\alpha_i} = \{\lambda + (-p)\alpha_i, \lambda + (-p+1)\alpha_i, \ldots, \lambda + (q-1)\alpha_i, \lambda + q\alpha_i\}$$

   where p and q are both non-negative integers or infinite.

   (b) $p, q < \infty \implies p - q = \langle \lambda, \alpha_i^\vee \rangle$.

   (c) $\dim M_\lambda \leq \infty \implies p, q < \infty$.

   (d) $e_i : V_{\lambda+t\alpha_i} \to V_{\lambda+(t+1)\alpha_i}$ is injective if $-p \leq t < \frac{-\langle \lambda, \alpha_i^\vee \rangle}{2}$,

(e) $t \mapsto m_t$ is non-decreasing for $-p \leq t < \frac{-\langle \lambda, \alpha_i^\vee \rangle}{2}$,

(f) $t \mapsto m_t$ is symmetric with respect to the midpoint $\frac{-\langle \lambda, \alpha_i^\vee \rangle}{2}$,

(g) if both λ and $\lambda + \alpha_i$ lie in M_{λ, α_i}, then $e_i(V_\lambda) \neq (0)$.

Proof: In the universal enveloping algebra $U(\mathfrak{sl}_2(k))$ we have formulas

$$[h, f^k] = -2kf^k,$$

$$[h, e^k] = 2ke^k,$$

$$[e, f^k] = -k(k-1)f^{k-1} + kf^{k-1}h.$$

Proof of formulae: The first two follow quite simply from the fact that $f \in U(\mathfrak{sl}_2(k))_{-2}$ and $e \in U(\mathfrak{sl}_2(k))_2$. The last one follows by a simple induction, where for $k = 1$ we have the standard relation $[e, f] = h$:

$$[e, f^k] = ef^k - f^k e = f^{k-1}ef + [e, f^{k-1}]f - f^{k-1}ef + f^{k-1}h$$
$$= [e, f^{k-1}]f + f^{k-1}h.$$

With the third formula at hand, we have

$$e_i f_i^k(v) := f_i^k e_i(v) + k(1 - k + \langle \lambda, \alpha_i^\vee \rangle) f_i^{k-1}(v)$$

for each $v \in V_\lambda$.
It follows that the vector subspace

$$U = \sum_{k, m \geq 0} (f_i^k e_i^m(v))$$

is invariant under both the actions of $\mathfrak{g}_{(i)}$ and $\mathfrak{h}$. Since e_i and f_i are locally nilpotent on V, U must have finite dimension, and is thus a finite-dimensional $\mathfrak{sl}_2(\mathbb{C})$-module. By $\mathfrak{sl}_2(\mathbb{C})$'s simplicity, U must be a direct sum of simple $\mathfrak{sl}_2(\mathbb{C})$-modules.
Since $v \in V$ was chosen arbitrarily, it follows that V is a direct sum of irreducible finite-dimensional $\mathfrak{g}_{(i)}$-submodules. (a) thus follows.
The remaining results follow from the classical representation theory of $\mathfrak{sl}_2(\mathbb{C})$ and a few combinatorial arguments, applied to the $\mathfrak{g}_{(i)}$-submodule given by $\sum_{k \in \mathbb{Z}} V_{\lambda + k\alpha_i}$. □

Definition: For each $i = 1, \ldots, n$ define s_{α_i} the *reflection with respect to* α_i in $\mathrm{Aut}_{\mathbb{C}}(\mathfrak{h}^*)$ by

$$s_{\alpha_i}(\lambda) = \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i.$$

We define the Weyl group $\mathscr{W}$ as being the subgroup of $\mathrm{Aut}_{\mathbb{C}}(\mathfrak{h}^*)$ generated by the reflections s_{α_i}.
$\mathscr{W}$ also acts on $\mathfrak{h}$ via the pairing $\alpha \mapsto \alpha^\vee$, namely

$$s_{\alpha_i}(h) := h - \langle h, \alpha_i \rangle \alpha_i^\vee.$$

Definition: Assume $M \in \mathscr{O}_{\mathrm{int}}(\mathfrak{g}(A))$, A a symmetrisable GCM. Suppose M is given by $\pi : \mathfrak{g}(A) \to \mathfrak{gl}(M)$. Define

$$s_i^\pi = \exp(\pi(f_i)) \exp(-\pi(e_i)) \exp(\pi(f_i))$$

(also denoted by s_i^M when π is implicit).

Goal: Understand s_i^{ad}.

Lemma:

1. $s_i^{\mathrm{ad}}(f_i) = -e_i$,
2. $s_i^{\mathrm{ad}}(e_i) = -f_i$,
3. $s_i^{\mathrm{ad}}(\alpha_i^\vee) = -\alpha_i^\vee$.

Proof: Compute :) □

Lemma: Assume $M \in \mathcal{O}_{\text{int}}(\mathfrak{g}(A))$, A symmetrisable, $\pi : \mathfrak{g}(A) \to \mathfrak{gl}(M)$. Then

$$(s_i^M)^{-1}\pi(x)s_i^M = \pi(\exp(-\mathrm{ad}(f_i))\exp(\mathrm{ad}(e_i))\exp(-\mathrm{ad}(f_i))x)$$

Proof: We have, explicitly

$$(s_i^M)^{-1}\pi(x)s_i^M = \exp(-\pi(f_i))\exp(\pi(e_i))\exp(-\pi(f_i))\pi(x)\exp(\pi(f_i))\exp(-\pi(e_i))\exp(\pi(f_i)).$$

If we call $S = \exp(\pi(e_i))\exp(-\pi(f_i))\pi(x)\exp(\pi(f_i))\exp(-\pi(e_i))$ the middle part then

$$(s_i^M)^{-1}\pi(x)s_i^M = \exp(-\pi(f_i))S\exp(\pi(f_i)) = \exp(-\mathrm{ad}(\pi(f_i)))(S)$$

$$= \exp(-\mathrm{ad}(\pi(f_i)))(\exp(\mathrm{ad}(\pi(e_i))))\exp(-\mathrm{ad}(\pi(f_i)))\pi(x) = \pi(\exp(-\mathrm{ad}(f_i))\exp(\mathrm{ad}(e_i))\exp(-\mathrm{ad}(f_i))(x))$$

since π is a Lie-algebra homomorphism. □

We require a few simple results on working with *integrability*.

Lemma: Assume $\mathfrak{g}$ is a Lie algebra, $\pi : \mathfrak{g} \to \mathfrak{gl}(M)$ a representation. Assume $\mathfrak{g}$ is generated as a Lie algebra by the elements

$$\Phi = \{x \in \mathfrak{g} \mid \mathrm{ad}(x) \text{ is locally finite on } \mathfrak{g}, \pi(x) \text{ is locally finite on } M\}$$

Then

1. $\mathfrak{g}$ is generated *as a vector space* by the elements in Φ and
2. if furthermore $\mathfrak{g}$ is finite-dimensional (thus $\mathfrak{g}$ is generated as a Lie algebra by elements acting locally finitely on M) then any $m \in M$ is contained in a finite-dimensional $\mathfrak{g}$-submodule $W \subseteq M$.

Proof: Since $\mathfrak{g}$ is spanned by Lie words in the elements of Φ, it's enough to show that Φ is actually closed under $\mathfrak{g}$'s Lie bracket.
Let $x, y \in \Phi$; we thus have that $\exp(\mathrm{ad}(y))$ is well-defined. We actually have that $\exp(\mathrm{ad}(y))(x) \in \Phi$:

$$\pi(\exp(\mathrm{ad}(y))(x)) = \exp(\mathrm{ad}(\pi(y)))(\pi(x)) = \exp(\pi(y))\pi(x)\exp(-\pi(y))$$

which is locally finite. This in particular works if we set $\pi = \mathrm{ad}$ since $M = \mathfrak{g}, \pi = \mathrm{ad}$ also satisfies the hypothesis of the lemma. Thus $\mathrm{ad}(\exp(\mathrm{ad}(y)(x)))$ is also locally finite and hence $\exp(\mathrm{ad}(y)(x)) \in \Phi$.
Finally, simply observe that

$$\frac{\exp(t\,\mathrm{ad}(y))(x) - x}{t} \in \Phi$$

for every $t \neq 0$ since Φ is a vector subspace of $\mathfrak{g}$, and taking the limit as $t \to 0$ yields

$$[y, x] = \lim_{t \to 0} \frac{\exp(t\,\mathrm{ad}(y))(x) - x}{t} \in \Phi$$

As for our second claim, if we fix a PBW basis $\mathscr{B}$ for $U(\mathfrak{g})$ associated to the basis $x_1, \ldots, x_n \in \mathfrak{g}$ of $\mathfrak{g}$ as a vector space, we have that finding $m \in N \subseteq M$ such that $N = U(\mathfrak{g})N$ amounts to finding such an N for each of the one-dimensional lie algebras spanned by the x_i's. This follows straight from the first part, since $\mathfrak{g}$ is spanned by elements acting locally finitely on M. □

We may now discuss where the action of the braid group $B_{\mathscr{W}}$ associated to the Weyl group $\mathscr{W}$ arises.

Proposition: Let $M \in \mathcal{O}_{\text{int}}(\mathfrak{g}(A))$, A a symmetrisable GCM, $\pi : \mathfrak{g}(A) \to \mathfrak{gl}(M)$.

1. $s_i^\pi(M_\lambda) \subseteq M_{s_{\alpha_i}(\lambda)}$ for all $\lambda \in \mathfrak{h}^*$,
2. s_i^{ad} is a Lie-algebra homomorphism,
3. $(s_i^M)^2(v) = (-1)^{\langle \lambda, \alpha_i^\vee \rangle} v$ for all $v \in M_\lambda$,
4. let $m_{i,j} = \mathrm{ord}(s_{\alpha_i} s_{\alpha_j})$ for $i, j = 1, \ldots, n$ as elements in $\mathscr{W}$. If $m_{i,j} < \infty$ then

$$s_i^M s_j^M s_i^M \cdots = s_j^M s_i^M s_j^M$$

where on both sides we have $m_{i,j}$ factors. This implies we have an action of the Braid group $B_{\mathscr{W}}$ associated to $\mathscr{W}$ on M.

Proof:

1. Let $v \in M_\lambda$ be any weight vector, and $h \in \mathfrak{h}$. If $\langle h, \alpha_i \rangle = 0$ then h commutes with e_i and f_i, thus
$$hs_i^\pi(v) = s_i^\pi(hv) = \lambda(h)s_i^\pi(v) = s_{\alpha_i}(\lambda)(h)s_i^\pi(v)$$
thus $s_i^\pi(v) \in M_{s_{\alpha_i}(\lambda)}$ since by definition we have $s_{\alpha_i}(\lambda)(h) = \lambda(h) - \langle \lambda, \alpha_i^\vee \rangle \langle \alpha_i, h \rangle = \lambda(h)$.
On the other hand, if $h = \alpha_i^\vee$ then
$$s_{\alpha_i}(v)(h) = \langle \lambda, \alpha_i^\vee \rangle - \langle \lambda, \alpha_i^\vee \rangle \langle \alpha_i, \alpha_i^\vee \rangle = \langle \lambda, \alpha_i^\vee \rangle - 2\langle \lambda, \alpha_i^\vee \rangle = -\langle \lambda, \alpha_i^\vee \rangle.$$
It follows that
$$(s_i^\pi)^{-1}\pi(\alpha_i^\vee)s_i^\pi = \pi(\exp(-\mathrm{ad}(f_i))\exp(\mathrm{ad}(e_i))\exp(-\mathrm{ad}(f_i))) = \pi(-\alpha_i^\vee).$$
The reverse inclusion follows from the fact s_i^π is invertible and part 2.

2. We have
$$s_i^\pi(xv) = s_i^\pi(\pi(x)(s_i^\pi)^{-1}s_i^\pi(v))) = \pi(s_i^{\mathrm{ad}}(x))s_i^\pi(v).$$
If we take $M = \mathfrak{g}$ and $\pi = \mathrm{ad}$ then we have
$$s_i^{\mathrm{ad}}([x,v]) = [s_i^{\mathrm{ad}}(x), s_i^{\mathrm{ad}}(v)].$$

3. Since M is a direct sum of $\mathfrak{sl}_2(\mathbb{C})$-representations, we may assume without loss of generality that $\mathfrak{g}$ is equal to $\mathfrak{g}_{(i)}$ its Lie subalgebra generated by elements $e_i, f_i, \alpha^\vee$.
By our previous lemma there exists $v \in N \subseteq M$ a finite dimensional subrepresentation containing v and thus $\langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geq 0}$ by the classical representation theory of $\mathfrak{sl}_2(\mathbb{C})$. Furthermore, since $\mathfrak{sl}_2(\mathbb{C})$ is the Lie algebra associated to the Lie group $\mathrm{SL}_2(\mathbb{C})$ which is simply connected, by *fully faithfulness* we have that the Lie-algebra homomorphism $\pi : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{gl}(N)$ can be ***integrated*** to a Lie-group homomorphism $\widetilde{\pi} : \mathrm{SL}_2(\mathbb{C}) \to GL(N)$. Thus
$$(\mathrm{Exp}(f)\mathrm{Exp}(-e)\mathrm{Exp}(f))^2 = -\mathrm{id} = \mathrm{Exp}(i\pi h)$$
and $\mathrm{Exp}(i\pi h)v = e^{i\pi\langle \lambda, h \rangle} = (-1)^{\lambda, h}v$.

4. Since in this case we're just considering the action of s_i^M and s_j^M on M, we may assume A is a 2×2 matrix of the form
$$A = \begin{pmatrix} 2 & a_{i,j} \\ a_{j,i} & 2 \end{pmatrix}$$
By this it follows that $s_{\alpha_i} = \begin{pmatrix} -1 & -a_{i,j} \\ 0 & 1 \end{pmatrix}$ and $s_{\alpha_j} = \begin{pmatrix} 1 & 0 \\ -a_{j,i} & -1 \end{pmatrix}$ in the basis α_i, α_j.
A simple linear algebra argument shows the matrix
$$s_{\alpha_i}s_{\alpha_j} = \begin{pmatrix} -1 + a_{i,j}a_{j,i} & a_{i,j} \\ -a_{j,i} & -1 \end{pmatrix}$$
is of finite order if and only if $a_{i,j}a_{j,i} < 4$ which implies A is of finite type and thus $\mathfrak{g}(A)$ is finite dimensional.

Given any $v \in M$ by the previous lemma - since we've shown we may consider solely the case $\dim \mathfrak{g}(A) < \infty$ - we may fit v inside some finite-dimensional subrepresentation $N \subseteq M$ and, once again, ***integrate*** the Lie-algebra homomorpism $\pi : \mathfrak{g}(A) \to \mathfrak{gl}(N)$ to get a Lie group homomorphism $G(A) \to \mathrm{GL}(N)$ where $G(A)$ is the simply-connected two-dimensional Lie group whose Lie algebra is $\mathfrak{g}(A)$ (since we've reduced to the case $a_{i,j}a_{j,i} < 4$ we essentially just have to consider $G(A)$ equal to one of 4 possible groups). Once we've done this, it's enough to check $s_i = \mathrm{Exp}(f_i)\mathrm{Exp}(-e_i)\mathrm{Exp}(f_i)$ satisfy the stated braid group relations. This is done explicitly in all the relevant cases in Springer's book on algebrai groups.

Corollary:

1. $s_i^{\mathrm{ad}}|_{\mathfrak{h}} = s_{\alpha_i}$,
2. If $\alpha_j = w(\alpha_i)$, then $w(\alpha_i^\vee) = \alpha_j^\vee$ for every $w \in \mathscr{W}$.

Proof:

1. explicitly we have $s_i^{\mathrm{ad}}(h) = \exp(\mathrm{ad}(f_i))\exp(-\mathrm{ad}(e_i))\exp(\mathrm{ad}(f_i))(h)$ and
$$\begin{aligned} h &\mapsto h + \langle \alpha_i, h\rangle f_i \\ &\mapsto h + \langle \alpha_i, h\rangle f_i - \langle \alpha_i, h\rangle e_i - \langle \alpha_i, h\rangle \alpha_i^\vee + \frac{\langle \alpha_i, h\rangle \langle \alpha_i^\vee, \alpha_i\rangle}{2} e_i \\ &\mapsto h + \langle \alpha_i, h\rangle f_i - \langle \alpha_i, h\rangle \alpha_i^\vee + \langle \alpha_i, h\rangle f_i - \langle \alpha_i, h\rangle \langle \alpha_i, \alpha_i^\vee\rangle f_i \\ &= h - 2\langle \alpha_i, h\rangle f_i = s_{\alpha_i}(h) \end{aligned}$$
2. The previous part gives a Lie algebra homomorphism $\widehat{w} : \mathfrak{g}(A) \to \mathfrak{g}(A)$ defined by $\widehat{w}\mid_{\mathfrak{h}} = w$. Thus $[\widehat{w}(e_i), \widehat{w}(f_i)] = \widehat{w}(\alpha_i^\vee)$ and $w(\alpha_i) = \alpha_j$ implies $w(e_i) \in \mathfrak{g}_{\alpha_j}$ and $w(e_j) \in \mathfrak{g}_{\alpha_{-j}}$. Thus $\widehat{w}(\alpha_i^\vee) \in [\mathfrak{g}_{\alpha_i}, \mathfrak{g}_{-\alpha_j}]$ which is spanned by $\alpha_j^\vee$ (recall that the weight spaces relative to simple roots are one-dimensional - **not true for arbitrary roots**). It follows that we have $\widehat{w}(\alpha_i^\vee) = c\alpha_j^\vee$ for some scalar $c \in \mathbb{C}$. However applying α_j yields
$$2c = \alpha_j(\widehat{w}(\alpha_i^\vee)) = \widehat{w}(\alpha_i^\vee)(\alpha_j) = \alpha_i^\vee(\widehat{w}^{-1}(\alpha_j)) = \alpha_i^\vee(\alpha_i) = 2. \ \square$$

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Braid group action on $U_q(\mathfrak{g}(A))$

As usual, we start with $U = U_q(\mathfrak{sl}_2), q \neq \sqrt{-1}$.
Consider $\mathrm{Rep}_1(U) = \{$ f.d. re. of U of type 1$\}$ i.e. the collection of U-modules M such that $\dim M < \infty$, $M = \bigoplus_{j\in\mathbb{Z}} M_j$ where $M_j = \{m \in M \mid Km = q^j m\}$.

Definition: For $r \in \mathbb{N}_0$ define $E^{(r)} = \frac{E^r}{[r]!}, F^{(r)} = \frac{F^r}{[r]!}$ the *divided powers.*

Definition: For any $M \in \mathrm{Rep}_1(U)$ define linear endomorphisms T of M by:
$$\begin{aligned} v \in M_m &\mapsto T(v) = \sum_{a,b,c\in\mathbb{N}_0, -a+b-c=m} (-1)^b q^{b-ac} E^{(a)}F^{(b)}E^{(c)}v \\ v \in M_m &\mapsto T'(v) = \sum_{a,b,c\in\mathbb{N}_0, .a+b-c=m} (-1)^b q^{ac-b} E^{(a)}F^{(b)}E^{(c)}v \\ v \in M_m &\mapsto {}^{\omega}T(v) = \sum_{a,b,c\in\mathbb{N}_0, a-b+c=m} (-1)^b q^{b-ac} E^{(a)}F^{(b)}E^{(c)}v \\ v \in M_m &\mapsto {}^{\omega}T'(v) = \sum_{a,b,c\in\mathbb{N}_0, a-b+c=m} (-1)^b q^{ac-b} E^{(a)}F^{(b)}E^{(c)}v \end{aligned}$$

Remark: Since E, F act locally nilpotently on M, these sums are in fact finite and define linear endomorphisms of M.

Remark: Consider the cartan involution $\omega : U \to U$, and for $M \in \mathrm{Rep}_1(U)$ let ${}^{\omega}M$ be M with *the twisted* U-action
$$u \cdot m := \omega(u)m$$
It follows that, by how the previous endomorphisms are constructed,
$${}^{\omega}M \ni^{\omega} T(v) := T(v) \in {}^{\omega}T(v)$$
The above describes precisely the action of ${}^{\omega}T :^{\omega} M \to^{\omega} M$ in terms of $T : M \to M$.

Action on $L = L(n,+1) \in \mathbf{Rep}_1(U)$: recall that L is spanned by $m_0, m_1, m_2, \ldots, m_n$ and $Fm_j = m_{j+1}$. Since we want to consider the action of the constructed endomorphisms, it makes sense to rescale and define
$$v_i := \begin{cases} \frac{m_i}{[i]!} & 0 \le i \le n \\ 0 & \text{otherwise} \end{cases}$$
We thus have the identities
$$Fv_i = \begin{cases} [i+1]v_{i+1} & 0 \le i < n \\ 0 & i = n \end{cases}, \quad Ev_i = \begin{cases} [n-i+1]v_{i-1} & 0 < i \le n \\ 0 & i = 0 \end{cases}$$
Thus the divided powers act as
$$F^{(r)}v_i = \begin{bmatrix} r+i \\ r \end{bmatrix} v_{i+r}, E^{(r)}v_i = \begin{bmatrix} n+r-i \\ r \end{bmatrix} v_{i-r}.$$

Lemma: We have for $M = L(n,+1), v_i \in M$ the following formulas

$$T(v_i) = (-1)^{n-i} q^{(n-i)(i+1)} v_{n+i},\ T'(v_i) = (-1)^{n-i} q^{-(n-i)(i+1)} v_{n+i}$$

$$^{\omega}T(v_i) = (-1)^{n-i} q^{i(n-i+1)} v_{n+i},\ ^{\omega}T'(v_i) = (-1)^{n-i} q^{-i(n-i+1)} v_{n+i}$$

Proof: We have

$$T(v_i) = \sum_{-a+b-c=m} (-1)^b q^{b-ac} \begin{bmatrix} n+a-i+c-b \\ a \end{bmatrix} \begin{bmatrix} b+i-c \\ b \end{bmatrix} \begin{bmatrix} n+c-i \\ c \end{bmatrix} v_{n-i} = \text{[Jantzen appendix]}\square$$

Claim: $\Phi : L(n,+1) \xrightarrow{\cong} {}^{\omega}L(n,+1), v_i \mapsto v_{n-i}$ is an isomorphism of U-modules.

Proof: Simply look at symmetry of the Casimir's action.$\square$

Corollary:

1. $T, T', {}^{\omega}T, {}^{\omega}T'$ are invertible linear endomorphisms of any $M \in \mathrm{Rep}_1(U)$. Explicitly, $T^{-1} = {}^{\omega}T'$ and $(T')^{-1} = {}^{\omega}T$.
2. ${}^{\omega}T = (-q)^{-m}T$ on M_m and ${}^{\omega}T' = (-q)^{m}T'$ on M_m.

Proof: Follows from previous formulas. $\square$

Proposition: For all $M \in \mathrm{Rep}_1(U)$, $v \in M$ we have the following identities

$$T(Ev) = (-FK)T(v),$$

$$T(Fv) = (-K^{-1}E)T(v)$$

$$T(Kv) = K^{-1}T(v)$$

$$ET(v) = T(-K^{-1}Fv)$$

$$FT(v) = T(-EKv)$$

$$KT(v) = T(K^{-1}v).$$

Proof: It's enough to consider $M = L(n,+)$. We'll just show the first one... not really $\square$

Proposition: The assignment $T(E) = -FK, T(F) = -K^{-1}E, T(K) = K^{-1}$ defines an algebra automorphism of U, with inverse $T^{-1}(E) = -K^{-1}F, T^{-1}(F) = -EK, T^{-1}(K) = K^{-1}$.

Proof: Follows from the next proposition.

Proposition: For any $u \in U$ there exists a unique $u' \in U$ such that $T(uv) = u'T(v)$ for every $v \in M, M \in \mathrm{Rep}_1(U)$. Moreover, $\phi : u \mapsto u'$ is an algebra homomorphism of U.

Proof: Assuming the well-defined'ness and uniqueness it's clear that $(\lambda u_1 + \mu u_2)' = \lambda u_1' + \mu u_2'$ and $(u_1u_2)' = u_1'u_2'$. Thus the considered assignment is an algebra homomorphism.
Furthermore, by the previous calculation, u' exists on generators E, F, K, K^{-1}, and one can extend the definition to all elements via decreeing that $u \mapsto u'$ be an algebra homomorphism.
As for uniqueness, if $u'v = u''v$ for all $v \in M$ and $M \in \mathrm{Rep}_1(U)$. This implies $u' - u''$ acts by zero on all representations of U and thus $u' - u'' = 0$ in U since $\mathrm{Rep}_1(U)$ is faithful. $\square$

Remark: T is an isomorphism. In fact, the generators E, F, K lie in the image by our calculations, thus it is surjective. Furthermore if $u \in U$ is such that $T(u) = 0$ then $T(uv) = 0$ for all $v \in M, M \in \mathrm{Rep}_1(U)$, and thus implies $uv = 0$ since T acts invertibly on M. Just as we argued before it follows that u acts as zero on all representations of U and thus $u = 0$.

The general case $U = U_q(\mathfrak{g}(A))$ for A symmetrisable

Recall: We have an embedding $U_{q_i}(\mathfrak{sl}_2) \hookrightarrow U_q(\mathfrak{g}(A))$, $q_i := q^{d_i}$ mapping E to E_i and F to F_i.

Definition: For $M \in \mathrm{Rep}_1(U)$ define

$$M_{\lambda,\sigma} := \{m \in M \mid K_\mu m = \sigma(\mu) q^{(\lambda,\mu)} m \text{ for all } \mu \in \bigoplus_i \mathbb{Z}\alpha_i\}$$

where $\sigma : \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_n \to \{\pm 1\}$ is a group homomorphism and λ is an *integral weight* (definition to come).

Definition: λ is an *integral weight* if $\lambda = \sum_{i=1}^{n} m_i \omega_i$ for $\mathfrak{m}_i \in \mathbb{Z}$ and $\omega_1, \ldots, \omega_n \in \mathfrak{h}^*$ the *fundamental weights*. The fundamental weights ω_i are defined by

$$\langle \omega_i, \alpha_j^\vee \rangle = \delta_{i,j}.$$

Remark: For $\lambda = \sum_i m_i \omega_i$ an integral dominant weight, $\mu \in \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_n$ we have

$$(lambda, \mu) = \sum_i m_i \mu_i \in \mathbb{Z}$$

Proposition: Assume A is of finite-type (i.e. $\mathfrak{g}(A)$ is a simple finite-dimensional Lie algebra). If M if a finite-dimensional $U_q(A)$-module and $q \neq \sqrt{-1}$. Then

1. E_i, F_i act nilpotently on M,
2. $M = \bigoplus_{\lambda,\sigma} M_{\lambda,\sigma}$
3. $E_i M_{\lambda,\sigma} \subseteq M_{\lambda+\alpha_i,\sigma}$ and $F_i M_{\lambda,\sigma} \subseteq M_{\lambda-\sigma,\sigma}$.

Proof:

1. $\mathfrak{sl}_2(\mathbb{C})$-calculation.
2. Consider M as a $U_{q_i}(\mathfrak{sl}_2)$-module.

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Proposition: M a finite-dimensional representation of $U_q(\mathfrak{g}(A))$, where A is a **Cartan Matrix** (if A is not Cartan then by the exercise sheet $U_q(\mathfrak{g}(A))$ has no non trivial f.d. representations). Then

1. E_i, F_i act (locally) nilpotently,
2. $M = \bigoplus_{\lambda,\sigma} M_{\lambda,\sigma}$,
3. $E_i M_{\lambda,\sigma} \subseteq M_{\lambda+\alpha_i,\sigma}$ and $F_i M_{\lambda,\sigma} \subseteq M_{\lambda-\alpha_i,\sigma}$.

Proof:

1. Follows from the $\mathfrak{sl}_2$ case and the inclusion $U_q(\mathfrak{sl}_2(\mathbb{C})) \hookrightarrow U_q(\mathfrak{g}(A))$.
2. Again by restriction to $U_{q_i}(\mathfrak{sl}_2)$, K_i is diagonalisable with eigenvalues of the form $\pm q^m$ with $m \in \mathbb{Z}$.
   Since the K_i's commute, they are simultaneously diagonalisable over M, which yields an eigenspace decomposition with eigenvectors of the for m v such that

   $$K_i v = \sigma_i q^{m_i} v, \sigma_i \in \{\pm 1\}$$

   If we set $\sigma : \mathbb{Z}\alpha_i \oplus \cdots \oplus \mathbb{Z}\alpha_n \to \{\pm 1\}$ by setting $\sigma(\alpha_i) = \sigma_i$, then we have $K_\mu v = \sigma(\mu) q^{m_i \mu_i} v = \sigma(\mu) q^{(\lambda,\mu)} v$.
   If we now set λ such that $\langle \lambda, \alpha_i^\vee \rangle = m_i$ then v must lie in $M_{(\lambda,\sigma)}$. And thus the second point follows.
3. This one's clear by our previous remarks. □

Definition: Let M be as in the previous lemma, $M = \bigoplus_\sigma M^\sigma$ where $M^\sigma = \oplus_\lambda M_{(\lambda,\sigma)}$.
We call M *of type* σ if $M = M^\sigma$. And we say M is of type 1 if is of type $\sigma \equiv 1$.

Remark: We have an equivalence of categories (for **any** σ)

$$\{\text{finite-dimensional representations of type1}\} \cong \{\text{finite-dimensional representations of type } \sigma\}$$

$$M \mapsto^\phi M$$

where $\phi : U_q(A) \to U_q(A)$ is defined by $\phi(E_i) = \sigma(\alpha_i) E_i, \phi(F_i) = F_i, \phi(K_i) = \sigma(\alpha_i) K_i$ (note that $\phi^2 = 1$).

Definition: For A a Cartan Matrix let $\mathrm{Rep}_1(U_q(A))$ be the category of finite-dimensional representations of type 1.

Definition: For a general GCM A define

$$E_i^{(r)} := \frac{E_i^r}{[r]!_{q_i}}, F_i^{(r)} = \frac{F_i^r}{[r]_{q_i}!}$$

for all $r \in \mathbb{Z} \geq 0$ (as elements in $U_q(A)$).

Definition: Let $M \in \mathrm{Rep}_1(U_q(A))$. Define linear endomorphisms of M as follows:

$$1 \leq i \leq n,\ T_i(v) := \sum_{-a+b-c=m} (-1)^b q^{b-ac} E_i^{(a)} F_i^{(b)} E_i^{(c)} v$$

where $m := \langle \lambda, \alpha_i^\vee \rangle$.

Remark: ${}^{\omega}T_i, T_i', {}^{\omega}T_i'$ similarly (as last lecture).

Finite dimensional representations of $U_q(A)$, A a Cartan matrix

Let A be (provisionally) a GCM, and $U_q^{\geq 0}(A)$ be the subalgebra of $U_q(A)$ generated by $K_i^{\pm 1}$'s and E_i's.

Note that $U_q^{\geq 0}(A)$ is a Hopf subalgebra

We have a triangular decomposition: $U_q^{<0}(A) \otimes U_q^0(A) \otimes U_q^{>0}(A) \xrightarrow{\cong} U_q(A)$ (proof omitted).

Remark (*construction of the Verma's*): In particular $U_q(A)$ is free as a $U_q^{\geq 0}$-module with basis any one for $U_q^{<0}(A)$.
Let $\lambda \in \mathfrak{h}^*$ ($\mathfrak{h}$ = "Cartan for underlying Lie algebra associated with A"). Assume $\langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}$ for all i (i.e. λ is an integral weight). And let h_λ be the 1-dimensional $U_q^{\geq 0}(A)$-module generated by v defined by

$$K_i v = q^{(\lambda, \alpha_i)} v = q_i^{\langle \lambda, \alpha_i^\vee \rangle} v$$

$$E_i v = 0 \text{ for all } i.$$

We thus define the "*Verma*" $U_q(A)$-module

$$M(\lambda) := U_q(A) \otimes_{U_q^{\geq 0}} h_\lambda$$

of highest weight λ. For any $U_q(A)$-module M, $m \in M$ is said to be a weight vector of weight $\lambda \in \mathfrak{h}^*$ if $K_\mu m = q^{(\lambda, \mu)} m$ for all $\mu \in \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_n$.

Remark: $M(\lambda)$ has a unique irreducible quotient, denoted by $L(\lambda)$.

Theorem: A a Cartan Matrix, $\lambda \in \mathfrak{h}^*$ a dominant integral weight (i.e. $\langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geq 0}$ for all i).

1. For each i there is a homomorphism of $U_q(A)$-modules

   $$M(\lambda - (m+1)\alpha_i) \hookrightarrow M(\lambda) \text{ with } m = \langle \lambda, \alpha_i^\vee \rangle \text{ for all } i,$$

   defined by

   $$1 \otimes 1 \mapsto F_i^{m+1} \cdot 1 \otimes 1.$$

2. Define $\widetilde{L}(\lambda) = M(\lambda)/(\sum_i \mathrm{im}\phi_i)$. The $\widetilde{L}(\lambda)$ is *finite dimensional*.

3. We have a bijection

   $$\{\text{irreducible finite dimensional } U_q(A)-\text{modules of highest weight}\}/\cong \xrightarrow{\sim} X^+ := \{\text{dominant integral weights}\}$$

   $$L(\lambda) \mapsto \lambda$$

Proof:

1. The vectors $1 \otimes 1 \in M(\lambda-(n+1)\alpha_i)$ and $F_i^{m+1} 1 \otimes 1$ have the same weight. Moreover
$$E_j F_i^{m+1} \cdot 1 \otimes 1 = F_i^{m+1} E_j \cdot 1 \otimes 1 = 0 \text{ if } i \neq j$$
$$E_i F_i^{m+1} \cdot 1 \otimes 1 = 0 \text{ by a } U_q(\mathfrak{sl}_2)-\text{calculation.}$$
The universal property of $M(\lambda-(m+1)\alpha_i)$ yields the first part.

2. It's enough to show $\dim \widetilde{L}(\lambda)_\mu = \dim \widetilde{L}(\lambda)_{w(\mu)}$ for all $w = s_{\alpha_i}$ and $1 \le i \le n$ (note that $\dim \widetilde{L}(\lambda)_\mu < \infty$ by construction). Draw a picture of the weights and see why this is the case :)

   I want to go to mensa

   We have $\widetilde{L}(\lambda) = U_q(A)/\mathrm{Ann}_{U_q(A)}(1 \otimes 1) = U_q(A)/(\sum_i U_q(A)E_i + \sum_i U_q(A)(K_i - q^{(\lambda,\alpha_i)}) + \sum_i \sum_i U_q(A)F_i^{m+1}) \implies$ [Jantzen 5.7] E_i, F_i act locally nilpotently on $\widetilde{L}(\lambda)$. Consider for every i the module
$$M := \bigoplus_{r \in \mathbb{Z}} \widetilde{L}(\lambda)_{\mu - r\alpha_i}$$
   on which $U_{q_i}(\mathfrak{sl}_2) \cong \langle E_i, F_i \rangle$ acts.
   M has finite dimensional weight spaces, and E_i and F_i act locally nilpotently on M.
   Let $v \in M \setminus \{0\}$ be a non-zero vector such that $E_i v = 0$. The $U_{q_i}(\mathfrak{sl}_2)$-module generated by v thus yields a surjection of a Verma module
$$M(q^n) \twoheadrightarrow U_{q_i}(\mathfrak{sl}_2)v$$
   for some q^n ($M(q^n)$ is the $U_q(\mathfrak{sl}_2)$ Verma).
   However, $F_i^s v = 0$ for s large enough, which implies $L(q^n, +1) \overset{f}{\twoheadrightarrow} U_{q_i}(\mathfrak{sl}_2)v$. Consider
$$M' := M/\mathrm{im}(f).$$
   The dimension of the 0 or 1-weight space of M' is *smaller* than that of M - as $U_{q_i}(\mathfrak{sl}_2)$-modules.
   By repeating this argument we obtain a finite Jordan-Holder series of M as a quantum $\mathfrak{sl}_2$-module:
$$M \supseteq M^1 \supseteq M^2 \supseteq \cdots \supseteq M^r \supseteq (0)$$
   where the quotients of each of these are finite-dimensional.
   It follows that $\dim M_\mu = \dim M_{s_{\alpha_1}(\mu)}$ since it holds for any subquotient and thus for M.

3. $L(\lambda)$ is finite-dimensional for λ dominant integral, since $\widetilde{L}(\lambda)$ is and $M(\lambda)$ has unique irreducible quotient - $\widetilde{L}(\lambda) \twoheadrightarrow L(\lambda)$. Thus the map $\lambda \mapsto L(\lambda)$ is well-defined.
   Viceversa, given $L(\lambda)$ an irreducible finite dimensional $U_q(A)$-module, by the $U_q(\mathfrak{sl}_2)$ theory which we developed, $\langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{\ge 0}$ for all i; thus λ is dominant integral. □

Proposition: A a Cartan Matrix. For all $u \in U_q(A)$ there exists a direct sum of irreducible finite-dimensional highest-weight modules such that $uM \neq 0$.

Proof: For not-quantised version: exercise.
For quantised omitted. □

What about when A is not necessarily CM? What are the *integrable* representations?

Proposition (construction): Let A be a symmetrisable GCM. Construct $L(\lambda)$ as before over $U_q(A)$. Then $L(\lambda)$ is integrable if and only if $\lambda \in X^+ :=$ {dominant integral weights} $= \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{\ge 0}\}$.
Proof next time (:

Definition: A a symmetrisable GCM and M a direct sum of integrable irreducible representations or just integrable (this is one of those two definitions for the price of one sort-of-thing).
Define $T_i(v)$ as before for all $v \in M$. Similarly ${}^\omega T_i, T_i', {}^\omega T_i'$.

Proposition:

1. $T_i, T_i', {}^\omega T_i, {}^\omega T_i'$ are invertible endomorphisms.
2. ${}^\omega T_i(v) = (-q_i)^{-\langle \lambda, \alpha_i \rangle} T_i(v)$ for all $v \in M_\lambda$.
3. ${}^\omega T_i'(v) = (-q_i)^{\langle \lambda, \alpha_i^\vee \rangle} T_i'(v)$.

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Last time we discussed a classification result of the highest weight modules for $U_q(\mathfrak{g}(A))$ where A is a GCM. We prove this now:

Theorem:

1. $L(\lambda)$ is integrable $\iff \lambda \in \Lambda^+$,

2. {irred. integrable hw representations of $\mathfrak{g}(A)$}$/\cong \xrightarrow{\cong} \Lambda^+$.

Proof: Let $v_\lambda = 1 \otimes 1 \in L(\lambda)$. If $L(\lambda)$ is integrable then $f_i^N v_\lambda = 0$ for N large enough, and choose N as small as can be. It follows that

$$v_\lambda, f_i v_\lambda, \dots, f_i^{N-1} v_\lambda$$

spans a finite-dimensional $\mathfrak{sl}_2$-module, which implies $\langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geq 0}$.
For the converse, suppose $\langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geq 0}$ for all i. It's clear that $L(\lambda)$ has a weight decomposition, since $M(\lambda)$ does. What we have to show is that if $v \in L(\lambda)$ then $e_i^N v = 0$ and $f_i^N v = 0$ for some N large enough.
Without loss of generality, suppose $v \in L(\lambda)_\mu$ for some weight $\mu = \lambda - \sum_{i=0}^n n_i \alpha_i$. For N large enough we have that, by the structure of the weights in $L(\lambda)$, $\mu + N\alpha_i$ is not smaller than λ, and is thus not a weight in $M(\lambda)$ or $L(\lambda)$. Thus $e_i^N v = 0$.
As for the f_i-condition, let $v = f_1 \dots f_{i_n} v_\lambda$ for some $1 \leq i \leq n$, and *assume* for now that $f_i^N v_\lambda = 0$. If we set $y = f_1 \dots f_{i_n}$ then

$$f_i^s y v_\lambda = \sum_{k=0}^s \binom{s}{k} (\mathrm{ad} f_i)^k (y) f_i^{s-k} v_\lambda.$$

In general, if $D : U \to U$ is any derivation (U any algebra) then

$$D^n(u_1 \dots u_r) = \sum_{\sum_i m_i = m, m_i \geq 0} \binom{m}{m_1, \dots, m_r} D^{m_1}(u_1) \dots D_r^m(u_r)$$

- a simple induction shows.
If we now take $D = \mathrm{ad} f_i$ and $u_1 \dots u_r = f_1 \dots f_{i_n} = y$. By the Serre relations and our assumption on v_λ, we have that $f_i^s y v_\lambda = 0$ for s large enough.
Thus, all that's left to show is that f_i acts locally nilpotently at v_λ. For this, we can show $N = m_i + 1$ works, where $m_i = \langle \lambda, \alpha_i^\vee \rangle$.
Assume by contradiction that $v' := f_i^{m_i+1} v_\lambda \neq 0$. We ave $e_j v = e_j f_i^{m_i+1} v_\lambda = 0$ (by the relations in $\mathfrak{g}(A)$ for $i \neq j$ and the $\mathfrak{sl}_2$-calculations for $i = j$).
This implies v' is actually a highest weight vector, different to v_λ! This contradicts $L(\lambda)$'s irreducibility, because $U(\mathfrak{g}(A))v' = U(\mathfrak{n}_-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}_+) v' = U(\mathfrak{n}_-) v' \notin v_\lambda$ is a proper, non-zero, subrepresentation. This implies both 1 and 2. □

Remark:

1. The theorem holds for A a GCM.

2. If we go back to the case in which A is symmetrisable, $L(\lambda)$ has finite-dimensional weight spaces (because $M(\lambda)$ has as well). This implies we have a function $\mathrm{char}(L(\lambda))$ given by the character of $L(\lambda)$.
   There exists for $\lambda \in X^+$ a character formula for $L(\lambda)$ ("*Kac*-formula") which is totally analogous to the Weyl character formula for semisimple Lie algebras (take a look at the exercise sheet for the affine case).

3. A "*nice*" character formula exists for any integrable highest weight module. This can be tackled via BGG resolutions.

4. A nice consequence of this is ***any integrable highest weight module is irreducible*** (Cor 10.4 [1]).
   Gist: If L is an integrable highest weight module and L' is irreducible quotient, compare characters :)

Theorem: A a symmetrisable GCM, and $M \in \mathcal{O}_{\mathrm{int}}(\mathfrak{g}(A))$ with finite-dimensional weight spaces, and weights contained in a finite union of sets of the form $\{\mu \mid \mu \leq \lambda_i\}$, then we have a decomposition of M into highest weight submodules

$$M = \bigoplus_{\lambda \in X^+} L(\lambda)^{\oplus a_\lambda}$$

where $a_\lambda \in \mathbb{Z}_{\geq 0}$.

Remark: This applies in particular to $L(\lambda)\otimes L(\mu)$ with $\lambda,\mu\in X^+$; although unfortunately this does not hold for $L(\lambda)\otimes^{\omega} L(\mu)$ in general.

Proof: Let $N\subseteq M$ be the subspace annihilated by $\mathfrak{n}_+$, which is a non-zero subspace by our assumption on the weights of M. Furthermore, N is also $\mathfrak{h}$-invariant since it is spanned by weights.
Let $\{v_i\}_{i\in I}$ be a basis of N consisting of weight vectors, each v_i of weight μ_i.
We have homomorphisms

$$M(\mu_i)\twoheadrightarrow U(\mathfrak{g}(A))v_i\subseteq M$$
$$1\otimes 1\mapsto v_i$$

for each i. The image is an integrable hw module (as a submodule of M), and the previous remark shows that its image is irreducible. We thus have an induced homomorphism

$$\phi_i : L(\mu_i)\twoheadrightarrow U(\mathfrak{g}(A))v_i\subseteq M.$$

Adding these homomorphisms we have a homomorphism

$$\bigoplus_{i\in I} L(\mu_i)\xrightarrow{\phi:=\sum_i\phi_i} M$$

We actually have that ϕ is an isormorphism of $U(\mathfrak{g}(A))$-modules:

Injectivity: Let $v\in\ker\phi$ be such that $\mathfrak{n}_+v=0$, $v=\sum_{i\in I}c_iv_{\mu_i}$ where $v_{\mu_i}=1\otimes 1\in L(\mu_i)$ and $c_i\in\mathbb{C}$. Since each v_{μ_i} is sent to v_i and these form a set of linearly independent vecotors, it follows that $c_i=0$ for all i.

Surjectivity: Assume we have a ses

$$\bigoplus_{i\in I} L(\mu_i)\overset{\phi}{\hookrightarrow} M\overset{\pi}{\twoheadrightarrow} Q$$

with $Q\neq 0$. If we choose $L(\mu)$ a submodule of Q (as above), replacing M by $\pi^{-1}(L(\mu))$ allows us to assume wlog that $Q=L(\mu)$.
It follows that there exists a weight vector $v\in\pi^{-1}(v_\lambda)$. On v, the Casimir must acts as it does on v_μ, and thus in the same way it acts on $L(\mu)$. Moreover, $e_iv\in\ker\pi$ for all i and thus in the image of ϕ.
If we express $e_iv=\sum_{j\in I}z_j$ where $z_j\in L(\mu_j)$, then $z_j\neq 0$ implies

$$\mu+\alpha_i\leq\mu_j,$$

$$(\mu_j+\rho,\mu_j+\rho)=(\mu+\rho,\mu+\rho)$$

(these scalars are the ones as which the Casimir acts on $L(\mu_j)$ resp. $L(\mu)$) and the Casimir acts on $L(\mu)$ in the same way as on $L(\mu_j)$ if $z_j\neq 0$.
Recall that ρ is chosen such that $\langle\rho,\alpha_i^\vee\rangle=1$ for all i.
We have the following claim:

Given $\mu,\lambda\in\mathfrak{h}^*$ such that $\langle\lambda,\alpha_i^\vee\rangle\geq 0$ for all i, $\mu\leq\lambda$, and $\langle\mu,\alpha_i^\vee\rangle\geq 0$ for all i, then $v=\lambda+\rho$.

Applying the claim to $\mu:=\mu+\rho,\lambda=\mu_j$ yields $\mu+\rho=\mu_j+\rho$ hence $\mu=\mu_j$, which contradicts $\mu+\alpha_i\leq\mu_j$.
The claim is a simple calculation exercise...

□

Lemma: M is integrable if and only if ${}^{\omega}M$ is integrable.

Proof: ${}^{\omega}M$ has a weight space decomposition induced by M's, and the local nilpotency of the e_i's and f_i's follows directly.

Warning: If M an integrable highest weight module, then ${}^{\omega}M$ is integrable of highest weight.

Remark: If M is irreducible, then ${}^{\omega}M$ is irreducible.

Fact (without proof): The construction of (integrable) highest weight modules $L(\lambda)$ goes through in the quantised setting as well. We have a bijection

$$\{\text{irreducible integrable highest weight modules of } U_q(\mathfrak{g}(A))\}/\cong\xrightarrow{\cong} X^+\times\{\sigma:\mathbb{Z}\alpha_1\oplus\cdots\oplus\mathbb{Z}\alpha_n\to\{\pm 1\}\text{ group homomorphisms}\}.$$

Construction: We may thus define
$$T_i, {}^{\omega}T_i, T_i', {}^{\omega}T_i'$$
for any $M = L(\lambda,\sigma)$ or ${}^{\omega}L(\lambda,\sigma)$, with $\lambda \in X^+$. Just as last time, we restrict our attention to when $\sigma \equiv 1$ and thus the representations in the category $\mathrm{Rep}_1(U_q(\mathfrak{g}(A)))$ of "$U_q(\mathfrak{g}(A))$-modules of type 1" which are direct sums of $L(\lambda,1)$ or ${}^{\omega}L(\lambda,1)$ for $\lambda \in X^+$.

Proposition:

1. $T_i, {}^{\omega}T_i, T_i', {}^{\omega}T_i'$ is an invertible endomorphism for any $M \in \mathrm{Rep}_1^{int}(U_q(A))$.
2. ${}^{\omega}T_i(v) = (-q_i)^{-\langle\lambda,\alpha_i^vee\rangle} T_i(v)$ and ${}^{\omega}T_i'(v) = (-q_i)^{\langle\lambda,\alpha_i^\vee\rangle} T_i'(v)$.

Proof: Restrict to $U_{q_i}(\mathfrak{sl}_2)$ and use formulas we proved there.

Remark: By construction T_i maps M_λ to $M_{s_{\alpha_i}(\lambda)}$.

Lemma: If A is symmetrisable, $M \in \mathrm{Rep}_1^{\mathrm{int}}(U_q(A))$ then

1. $T_i(K_\mu v) = K_{s_{\alpha_i}(\mu)} T_i(v)$ for all $v \in M$,
2. $T_i(E_i(v)) = -F_i K_i T(v), T_i(F_i(v)))$... (for formulas, take a look at the $\mathfrak{sl}_2$ case),
3. (these are the *new* formulas) $T_i(E_j v) = E_j T_i(v)$ if $a_{i,j} = 0$ and $T_i(F_j(v)) = F_j(T_i(v))$.

Proof: Clear by the $\mathfrak{sl}_2$ case, and the last one follows from the Serre relations ($a_{i,j} = 0$ implies E_i and F_i commute with E_j and F_j).

Proposition: $M \in \mathrm{Rep}_1^{\mathrm{int}}(U_q(\mathfrak{g}(A)))$, A symmetrisable GCM. Then for every $1 \le i \ne j \le n$ we have the formula
$$T_i(E_j v) = (\mathrm{ad} E_i^{(r)}) E_j T_i(v)$$
where $r = -\langle \alpha_j, \alpha_i^\vee \rangle$.

Formula intuition: (We'll prove the formula later) We may *define*, for every Hopf algebra H we have the *adjoint representation* of H on H given by
$$\mathrm{ad}(h)(x) := \sum h_{(1)} x S(h_{(2)})$$
with $\Delta(h) = h_{(1)} \otimes h_{(2)}$ where S is the antipode.

Examples (of adjoint representation of a Hopf algebra): If $H = U(\mathfrak{g})$ where $\mathfrak{g}$ is a Lie algebra, then
$$\Delta(h) = 1 \otimes h + h \otimes 1, S(h) = -h$$
$$h \in \mathfrak{g}, x \in U(\mathfrak{g}) \implies \mathrm{ad}(h)(x) = hx - xh.$$
If we instead take $H = kG$ where G is a finite group, then
$$\mathrm{ad}(g)(x) = gxg^{-1}$$
for every $g \in G, x \in kG$.
Lastly, let $H = U_q(A)$ where A is a symmetrisable GCM, or if you like $H = \widetilde{U_q(A)}$. We have the following formulas
$$\mathrm{ad}(E_i)u = E_i u - K_i u K_i^{-1} E_i$$
$$\mathrm{ad}(F_i)u = -uF_i K_i + F_i u K = (F_i u - uF_i) K_i$$
$$\mathrm{ad}(K_i)u = K_i u K_i^{-1}$$
since
$$\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i$$
$$\Delta(F_i) = 1 \otimes F_i + F_i \otimes K_i^{-1}.$$

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Proposition: In the notations for last lecture, $T_i(E_j v) = \mathrm{ad}(E_i^{(r)})(E_j)T_i v$ where $r = -\langle \alpha_j, \alpha_i^\vee \rangle$ and the adjoint representation is given by $\mathrm{ad}(E_i)(u) = E_i u - K_i u K_i^{-1} E_i$.

Proof: Inductively, for all $r \geq 0$ we have

$$\mathrm{ad}(E_i^(r))u = \sum_{s=0}^{r} q_i^{s(r-s)} \begin{bmatrix} r \\ s \end{bmatrix}_{q_i} E_i^{r-s} K_i^s u K_i^{-s} (-1)^s q_i^{s(s-1)} E_i^s$$

$$= \sum_{s=0}^{r} (-1)^s q_i^{s(r-1)} \begin{bmatrix} r \\ s \end{bmatrix}_{q_i} E_i^{r-s} K_i^s u K_i^{-s} E_i^s.$$

Now just use divided powers $\mathrm{ad}(E_i^{(k)}) = \sum_{s=0}^{k} (-1)^s q_i^{s(k-1-r)} E_i^{k-s} E_j E_i^{(s)}$.

Lemma: If we set $r = 1 - a_{i,j}$ then we have the formulas in $\widetilde{U}_q(\mathfrak{g}(A))$

$$\mathrm{ad}(E_i^r)(E_j) = S_{i,j}^+$$

$$\mathrm{ad}(F_i^r)(E_j K_j) = S_{i,j}^- K_j K_i^r$$

Proof: $\mathrm{ad}(E_i^r)(E_j) = \sum_{s=0}^{r} (-1)^s q_i^{s(r-s)} \begin{bmatrix} r \\ s \end{bmatrix}_{q_i} E^{r-s} K_i^s E_j K_i^{-s} E_i^s$. □

Remarks: From this one can deduce $\mathrm{ad}(E_i^s)(E_j) = 0 = \mathrm{ad}(F_i^s)(F_j)$ for $s \geq r$ in $U_q(\mathfrak{g}(A))$. It follows (via some further calculations) that $[F_k, S_{i,j}^\pm] = 0 = [E_k, S_{i,j}^\pm]$ for all k.

Proposition: $T_i(F_j(v)) = \sum_{s=0}^{r} (-1)^s q_i^s F_i^{(s)} F_j F_i^{(r-s)} T_i(v)$.

Proof: This follows from the following lemma:

Lemma: Assume for $u, u' \in U_q(\mathfrak{g}(A))$ we have $T_i(uv) = u' T_i(v)$ for any $v \in M \in \mathrm{Rep}_1^{\mathrm{int}}(\mathfrak{g}(A))$. Then

1. ${}^\omega T_i(\omega(u)v) = \omega(u')^\omega T_i(v)$,
2. $T_i(\omega(u)v) = q^{-\langle \mu, \alpha_i^\vee \rangle} \omega(u') T_i(v)$ for $u \in U(\mathfrak{g}(A))$.

Proof:

1. For every $v \in^\omega$ we have ${}^\omega T_i(\omega(u)v) = T_i(uv) = u' T_i(v) = \omega(u') T_i(v) = \omega(u')^\omega T_i(v)$.
2. $T_i(\omega(u)v) = (-q_i)^{\langle \lambda - \mu, \alpha_i^\vee \rangle} \cdot^\omega T_i(\omega(u)v) = (-q_i)^{\langle \lambda - \mu, \alpha_i^\vee \rangle} (-q_i)^{\langle \lambda, \alpha_i^\vee \rangle} \omega(u')^\omega T_i(v)$. □

Proof of proposition: $T_i(F_j v) = (-q_i)^r \omega((\mathrm{ad}(E_i^{(r)})(E_j)) T_i(v)$:) □

Theorem: Let $1 \leq i \leq n$ for all $u \in U_q(\mathfrak{g}(A))$. Then there exists $u' \in U_q(\mathfrak{g}(A))$ such that $T_i(uv) = u' T_i(v)$ for all $v \in M \in \mathrm{Rep}_1^{\mathrm{int}}(\mathfrak{g}(A))$. Furthermore, $u \mapsto u'$ defines an algebra automorphism of $U_q(\mathfrak{g}(A))$.

Proof: As for $U_{q_i}(\mathfrak{sl}_2)$, using that for all $u \in U_q(\mathfrak{g}(A))$ there exists $v \in M \in \mathrm{Rep}_1^{\mathrm{int}}(U_q(\mathfrak{g}(A)))$ such that $uv \neq 0$. □

Theorem:

1. There are algebra homomorphisms
$$T_i : U_q(\mathfrak{g}(A)) \to U_q(\mathfrak{g}(A))$$
such that
$$T_i(E_i) = -F_i K_i$$
$$T_i(F_i) = -K_i^{-1} E_i$$
$$T_i(E_j) = \sum_s (-1)^s q_i^{-s} E_i^{(r-s)} E_j E_i^{(s)}$$
$$T_i(F_j) = \sum_s (-1)^s q_i^s F_i^{(s)} F_j F_j^{(r-s)}$$
$$T_i(K_\mu) = K_{s_{\alpha_i}(\mu)}$$
for all $\mu \in \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_n$ where $r = -\langle \alpha_j, \alpha_i^\vee \rangle$.

2. These satisfy the braid relations:
$$T_iT_jT_i\dots = T_jT_iT_j\dots$$
where on each side we have $m_{i,j} = \mathrm{ord}(s_{\alpha_i}s_{\alpha_i})$.

Proof:

1. :)
2. long direct calculation... □

Example: $\langle \alpha_j, \alpha_i^\vee \rangle = -1$.
We have $T_i(E_j) = E_iE_j - q^{-1}E_jE_i$ and $T_j(E_i) = E_jE_i - q^{-1}E_iE_j$ - these elements are called q^{-1}-commutators. Another check yields $T_jT_i(E_j) = E_i$ and $T_iT_j(E_i) = E_j$.
Nice calculation :) $T_iT_jT_i(E_j) = T_i(E_i) = -F_iK_i$ and $T_jT_iT_j(E_j) = T_jT_i(-F_jK_j) = -F_iK_{s_{\alpha_j}s_{\alpha_i}(\alpha_j)}$ and $s_{\alpha_j}s_{\alpha_i}(\alpha_j) = \alpha_i$.

We now compare this Braid group action on $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$. Consider $w_0 = s_{\alpha_1}s_{\alpha_2}s_{\alpha_1}$ the longest element in Weyl group and the roots
$$\{\alpha_1, s_{\alpha_1}(\alpha_2) = \alpha_1+\alpha_2, s_{\alpha_1}s_{\alpha_2}(\alpha_1) = \alpha_2\}$$
which by the exercise sheet are precisely the positive roots of the associated root system. Consider the root space
$$\mathfrak{g}_{\alpha_1} = \mathbb{C}e_1$$
Thus
$$s_1^{\mathrm{ad}}(e_2) = \text{long calculation} = [e_1, e_2]$$
$$s_1^{\mathrm{ad}}s_2^{\mathrm{ad}}s_1^{\mathrm{ad}}(e_2) = -[e_1, e_2]$$

Observations/facts:

- Let $w \in \langle s_{\alpha_i}, s_{\alpha_j}\rangle$ with $i \neq j$ and $w(\alpha_j) > 0$. Then $T_w(E_i) \in U_q^{>0}(\mathfrak{g}(A))$ (like the previous example).
- If $w(\alpha_i) = \alpha_j$ then $T_w(E_i) = E_j$.
- Consider $\mathfrak{g} = \mathfrak{sl}_3$:
$$T_1T_2(E_1^c), T_1(E_2^b), E_1^a,$$
and
$$T_2T_1(E_2^c), T_2(E_1^b), E_2^a$$
One can show these two triples span the same subspace in $U_q(\mathfrak{sl}_3)$. This holds in fact when we take $w = s_{\alpha_{i_1}}\dots s_{\alpha_{i_r}}$ reduced expression. The subspace spanned by these is called $U_q(A)[w_0]$ - w_0 being the longest element in the Weyl group for ss Lie algebra.
- The space $U_q(A)[w_0]$ is stable under right multiplication with E_i's.

Theorem: Let A be a GCM, $w_0 = s_{\alpha_1}\dots s_{\alpha_{i_r}}$ a chosen reduced expression. Then
$$T_{i_r}T_{i_{r-1}}\dots T_{i_2}(E_{i_1})$$
form a basis of $U_q^{>0}(\mathfrak{g}(A))$. This is called the PBW-basis.

Example, $\mathfrak{gsl}_3$: Let $E_1 = z_1, T_1(E_2) = z_2, T_1T_2(T_1) = E_2$. We have the formula
$$E_1E_2 = q^{-1}E_2E_1 + (E_1E_2 - q^{-1}E_2E_1)$$
$$z_1z_3 = qz_3z_1 + z_2$$

Theorem: A a CM, and let $w_0 = s_{\alpha_{i_1}}\dots s_{\alpha_{i_r}}$ be a fixed reduced expression of the longest element, with corresponding PBW basis
$$Z_1 = E_{i_r}, Z_2 = T_{i_r}(E_{i_{r-1}}), \dots, Z_r = T_{i_r}T_{i_{r-1}}\dots T_{i_2}(E_{i_1})$$
then for $i < j$ we have
$$Z_iZ_j = q^{\{\mathrm{wt}(Z_i), \mathrm{wt}(Z_j)\}}Z_jZ_i + \sum_{I(i,j)} c(b_{i+1}, \dots, b_{j-1})Z_{i+1}^{-b_{i+1}}\dots Z_{j-1}^{b_{j-1}}$$
where $I(i,j) = \{(b_{i+1}, \dots, b_{j-1}) \in \mathbb{Z}_{\geq 0}^{j-i+1} \mid \sum_{t=i+1}^{j-1} b_t\mathrm{wt}(Z_t) = \mathrm{wt}(Z_i) + \mathrm{wt}(Z_j)\}$.

Application: We have this nice result :)

Theorem: A a CM, $U_q^{>0}(A)$ is an iterated Ore extension.

Lemma: Let R be a ring, $\beta : R \to R$ a ring endomorphism. For $r \in R$ let $\partial_r(x) = rx - \beta(x)r$. Then ∂_r is a β-derivation.

Proof of theorem: Let Y_j be the subalgebra of $U_q^{>0}(A)$ generated by $Z_1, \ldots, Z_j$ with $Y_0 = k(q)$. Our claim is that $Y_j = Y_{j-1}[Z_j; \beta_j, \partial_j]$ where $\beta_j : Y_{j-1} \to Y_{j-1}$ is given by

$$\beta_j(Z_i) = q^{\{\mathrm{wt}(Z_i), \mathrm{wt}(Z_j)\}} Z_i$$

for $i < j$ and ... mensa time :((actually more like :))

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New topic: Crystals

Setup: Given a symmetrisable Kac-Moody Lie algebra $\mathfrak{g}(A) = \mathfrak{g}$, M an integrable/finite-dimensional representation of $\mathfrak{g}$ or $U_q(\mathfrak{g}(A))$. Question: is there a basis B of M such that

$$x \cdot v \in kB \cup \{0\}$$

for all $v \in M$ and $x = E_i, F_i, K_i^{\pm 1}$ (or $x = e_i, f_i, \alpha_i^\vee$ in the non-quantised version).

Observations: We start off by looking at $\mathfrak{sl}_2$ (as usual): let M be a finite-dimensional irreducible representation, then as we saw in the beginning of the course, such a basis exists (regardless of whether q is or isn't a root of unity). Since all representations for $\mathfrak{sl}_2$ are completely reducible, such a basis always exists.

Question: Can be construct such a *simultaneous* basis for all $\mathfrak{sl}_2$ triples in a KM Lie algebra $\{e_i, f_i, \alpha_i^\vee\}$ resp. $\{E_i, F_i, K_i^{\pm 1}\}$? In the classical situation, all representations of semisimple Lie algebras admit such a basis (check Humphrey's book) - these are called *Chevalley bases*.
In the quantum case, we can also construct such a basis for the very particular case of the natural representation of $U_q(\mathfrak{sl}_n)$.
Unfortunately this **does not work for a general KM Lie algebra, in the quantum case.**

Remark (*Kashiwara*): Such a basis does exist when "$q \to 0$" - this is called the *"crystal limit"*; these bases are called *crystal bases*.

Goals for the rest of course:

- Properly define "$q \to 0$".
- Construction of crystal bases for irreducible representations in the case for which $\mathfrak{g}$ is semisimple. This is based on a big induction called the *Grand loop argument*.
- Define the related *canonical basis (Lusztig)*.

Recall the theorem:

Theorem: Every finite dimensional $U_q(\mathfrak{g})$-module is completely reducible.

A quick lemma first:

Lemma: If λ is a dominant integrable weight, consider the module $L(\lambda)^*$. We have an isomorphism $L(\lambda)^* \cong L(-w_0(\lambda))$ where w_0 is the longest element in the Weyl group (for $U(\mathfrak{g})$ or $U_q(\mathfrak{g})$) (Recall that for any $U_q(\mathfrak{g})$-module M we have the dual $U_q(\mathfrak{g})$-module M^* since $U_q(\mathfrak{g})$ is a Hopf algebra).

Proof: $L(\lambda)$ is finite dimensional and irreducible, thus $L(\lambda)^*$ is finite-dimensional and irreducible; its highest weight (in the classical case at least) is the negative of the lowest weight in $L(\lambda)$. Let ν be this lowest weight: $\nu \leq \mu$ for all $\mu \in P(L(\lambda)) = \{\text{weights in } L(\lambda)\}$. Thus $w_0(\nu) \geq w_0(\mu)$ for all $\mu \in P(L(\lambda))$. By the braid group action we have that $P(L(\lambda))$ is stable under the action of w_0. It follows that $w_0(\nu)$ is the highest weight in $P(L(\lambda))$, hence $w_0(\nu) = \lambda$. $\square$

Proof (of theorem): It's enough to show that any short exact sequence (ses) $L(\lambda) \hookrightarrow M \twoheadrightarrow L(\mu)$ splits, where λ, μ are dominant integral weights.
(ses) induces for every ν integral weight a short exact sequence of vector spaces $L(\lambda)_\nu \hookrightarrow M_\nu \twoheadrightarrow L(\mu)_\nu$.
Let $v\mu \in L(\mu)_\mu$ and v a preimage under f (where $f : M \to L(\mu)$). Our claim is if $E_i v = 0$ for all i, then the sequence splits (we'll show this in a sec). Under this hypothesis we have a surjection $M(\mu) \twoheadrightarrow U_q(\mathfrak{g})v$. Since $U_q(\mathfrak{g})v$ is finite dimensional, this surjection factors through yielding an isomorphism $L(\mu) \xrightarrow{\cong} U_q(\mathfrak{g})v$ (recall that $L(\mu)$ is the quotient of $M(\mu)$ by all its submodules of the form $M(\lambda)$ where λ is not dominant integral).
We thus have $f(U_q(\mathfrak{g})v) = U_q(\mathfrak{g})f(v) = U_q(\mathfrak{g})v_\mu \cong L(\mu)$ which implies $U_q(\mathfrak{g})v \cap \ker f = \{0\}$ by exactness of the considered sequence. Thus $M = U_q(\mathfrak{g})v \oplus \ker(f)$ which yields our claim.
We now have to show that $E_i v = 0$ for all i.

Case 1: If $\mu \geq \lambda$ then the claim follows simple by the fact that μ must be a maximal weiht in M.

Case 2: If $\mu \leq \lambda$ then we may dualise (ses) to obtain the dual short exact sequence (ses*)

$$L(\mu)^* \cong L(-w_0(\mu)) \hookrightarrow M^* \twoheadrightarrow L(\lambda)^* \cong L(-w_0(\lambda))$$

which shows by case 1 $M^* \cong L(\mu)^* \oplus L(\lambda)^*$ hence $(M^*)^* \cong (L(\mu)^*)^* \oplus (L(\lambda)^*)^*$. Now we have an isomorphism (***not valid for all Hopf algebras***) $\Phi : N \to (N^*)^*$ for any finite-dimensional $\mathfrak{g}$-module where $\phi(n)(f) = f(K_{2\rho})^{-1}n$ and $2\rho = \sum_{\alpha \geq 0} \alpha$. □

Remark: One can consider irreducible finite dimensional representations.

Let M be a finite-dimensional $U_q(\mathfrak{g}) = U_q$-module, and we can consider it as a $U_{q_i}(\mathfrak{sl}_2)$-module by restriction of scalars.
We have a decomposition $M \cong L_1 \oplus \cdots \oplus L_r$ where $L_j \cong L(a_s)$ is irreducible as a $U_{q_i}(\mathfrak{sl}_2)$-modules, $a_s \in X^+ = \mathbb{N}_{\geq 0}$.
If we pick $m(s)$ a highest weight vector for $L(a_s)$, we have a basis for M

$$F_i^{(j)} m(s) \quad 1 \leq s \leq r \; 0 \leq j \leq a_s.$$

If $x \in M$ is a weight vector of weight l, we can express x as

$$x = \sum_{j \geq 0} \sum_{s, a_s - 2_j = l} b_s F_i^{(j)} m(s)$$

for some $b_s \in k$. We can also group terms as:

$$x = \sum_{j \geq 0 j \geq -l} F_i^{(j)} x_j$$

where $x_j := \sum_{s a_s - 2j = l} b_s m(s)$ (this way of expressing weight vectors is **crucial**).
Note, we need $j \geq -l$: if l is non-negative then in the above sums we have no extra constraint of course ($j \geq -l$ is automatically implies by $j \geq 0$), and if l is non-positive then... I didn't get this part (POST: check proof for following lemma)

We have a few easy facts about these summands:

Lemma:

1. $E_i x_j = 0$ for all j,
2. x_j has weight $l + 2j$,
3. if $x = \sum_{j \geq 0, j \geq -l} F_i^{(j)} x'_j$ with x'_j satisfying parts 1 and 2 of this lemma, then $x_j = x'_j$.

Proof: Pick $j \geq 0, -l$. Part 1 is pretty clear, and for part 2 note that $m(s)$ has weight $a_s = l + 2j$ by definition of x_j. As for part 3, define the $U_{q_i}(\mathfrak{sl}_2)$-module $N = U_{q_i}(\mathfrak{sl}_2)x'_j$; N is either irreducible or zero of highest weight $l + 2j$. If it's non-zero then $N \cong L(l + 2j)$ and $F^{(j)} x'_j$ is the projection onto the isotypic component for $L(l + 2j)$ of x.
Now since $j \geq -l$ we have $-j \geq l$ and thus $j \leq l + 2j \implies F_i^{(j)}$ acts injectively on $L(l + 2j)_{l+2j}$ so $F_i^{(j)} x'_j = F_i^{(j)} x_j \implies x'_j = x_j$. □

Definition (*Kashiwara operators*): Given M as above, define $\widetilde{E}_i, \widetilde{F}_i : M \to M$ as follows. Given $x \in M$ and its decomposition into x_j's as above, we define

$$\widetilde{F}_i x := \sum_{j \geq 0, j \geq -l} F_i^{(j+1)} x_j$$

$$\widetilde{E}_i x := \sum_{j \geq 0, j \geq -l} F_i^{(j-1)} x_j$$

A few easy properties:

Lemma: Let $x \in M_\lambda$ written as before; then

1. $\widetilde{F_i}\widetilde{E_i}x = x - x_0$,
2. $\widetilde{E_i}\widetilde{F_i}x = x - F_i^{(r)}x_r$,
3. $\widetilde{F_i}M = F_i M$,
4. if $M' \subseteq M$ is a submodule, then M' is $\widetilde{F_i}$ and $\widetilde{E_i}$-invariant,
5. if $\phi : M \to N$ is a U_q-homomorphism then ϕ commutes with $\widetilde{E_i}$ and $\widetilde{F_i}$ for all i.

Proof: Parts one and two are clear, and for part three simply note

$$\widetilde{F_i}x = F_i(\sum_j [j+1]_{q_i}^{-1} F_i^{(j)} x_j) \in F_i M$$

$$F_i x = \widetilde{F_i}(\sum_j [j+1]_{q_i} F_i^{(j)} x_j) \in \widetilde{F_i}M.$$

Part four follows from part 5 by taking the inclusion homomorphism, and for part 5 let x be a weight vector in M. We have

$$\phi(x) = \phi(\sum_j F_i^{(j)} x_j) = \sum_j F_i^{(j)} \phi(x_j)$$

$$\phi(\widetilde{F_i}x) = \sum_j F_i^{(j+1)} \phi(x_j)$$

so it's now enough to show

$$\phi(x_j) = \phi(x)_j$$

and for this we can use the previous lemma on the characterisation of the j-th components constructed:

- $E_i \phi(x_j) = \phi(E_i x_j) = 0$ for all j and
- $\phi(x_j)$ has the same weight as x_j.

Since $\phi(x) = \sum_j F_i^{(j)} \phi(x)_j$ and $\phi(x) = \sum_j F_i^{(j)} \phi(x_j)$ we have $\phi(x_j) = \phi(x)_j$. □

Proposition: Every finite-dimensional $U_q(\mathfrak{sl}_2)$-module of type 1 has a basis $\{v_s\}_{s \in I}$ such that $\widetilde{F}v_s = v_t$ for some t or $\widetilde{F}v_s = 0$. Similarly for $\widetilde{E}$.

Proof: Assuming M is an irreducible finite-dimensional $U_q(\mathfrak{sl}_2)$-module, such a basis exists because of the theory we developed at the beginning of the course. □

Idea for "$q \to 0$": Define the ground field $\mathbb{Q}(q) = k$. We have an intermediate ring: $\mathbb{Q}(q) \supseteq A \supseteq \mathbb{Q}[q]$ where $A = \{\frac{f}{g} \mid g(0) \neq 0\}$. Recall that A is a DVR with residue field $\mathbb{Q}$. If M is any U_q-module, then (essentially by definition) it's a k-vector space and can be restricted to an A-module. Next time we define what lattices in M are :)

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Definition: M a finite dimensional U_q-module; an *admissible lattice* $\mathscr{L}$ in M is an $A := \mathbb{Q}[q]_{(q)}$-submodule which is finitely generated over A such that

(L1) it generates M over $k = \mathbb{Q}(q)$ (i.e. $k \otimes_{\mathbb{Q}} \mathscr{L} \xrightarrow{\text{"mult"}} M$ is surjective)

(L2) $\mathscr{L} = \bigoplus_{\lambda \in X} \mathscr{L}_\lambda$ where X is the set of integral weights and $\mathscr{L}_\lambda := \mathscr{L} \cap M_\lambda$ and

(L3) is stable under the action of the Kashiwara operators $\widetilde{F}_i$ and $\widetilde{E}_i$ for all i.

Remarks: From the first property it follows that $k \otimes_A \mathscr{L} \xrightarrow{\text{"mult"}} M$ is actually an isomorphism; indeed, $\frac{f}{g} \otimes x = \frac{1}{g} \otimes fx$ is mapped to zero if and only if $g^{-1}fx = 0$ if and only if $fx = 0$ if and only if $x = 0$ since M is a torsion-free A-module. From the second property it also follows that $\mathscr{L}$ is a *free* A-module.
And from the last one since $\mathscr{L}$ is stable under $\widetilde{E}_i$ and $\widetilde{F}_i$ we have that the Kashiwara operators induce maps

$$\widetilde{E}_i, \widetilde{F}_i : \mathscr{L}/q\mathscr{L} \to \mathscr{L}/q\mathscr{L}.$$

Definition: Let M be a finite dimensional U_q-module. *A crystal basis* of M is a pair $(\mathscr{L}, B)$ where $\mathscr{L}$ is an admissible lattice in M and B is basis of $\mathscr{L}/q\mathscr{L}$ as a $\mathbb{Q}$-vector space satisfying:

(B1) $B = \bigsqcup_\lambda B_\lambda$ with $B_\lambda := B \cap \mathscr{L}_\lambda / q\mathscr{L}_\lambda$,

(B2) $\widetilde{F}_i B \subseteq B \cup \{0\}, \widetilde{E}_i B \subseteq B \cup \{0\}$,

(B3) for all basis elements $b, b' \in B$ we have $b' = \widetilde{F}_i b$ if and only if $\widetilde{E}_i b' = b$.

Remark: All this works exactly in the same way for integral representations of $U_q(\mathfrak{g}(A))$ where A is symmetrisable.

Example: Let M be a finite dimensional $U_q(\mathfrak{sl}_2)$-module (e.g. M the 4-dimensional representation of type 1). We take

$$v_0 \in M$$
$$v_1 = Fv_0 = F^{(1)}v_0 = \widetilde{F}v_0$$
$$v_2 = \widetilde{F}v_1 = F^{(2)}v_0$$
$$v_3 = \widetilde{F}v_2 = F^{(3)}v_0$$

If we define $\mathscr{L} = A$-submodule generated by v_0, v_1, v_2, v_3 and $B = (v_0, v_1, v_2, v_3) \subseteq \mathscr{L}/q\mathscr{L}$ we get a crystal basis.

Definition: Given a crystal basis $(\mathscr{L}, B)$ of M we may assign to it a graph, called the *crystal graph of* $(\mathscr{L}, B)$, defined as the oriented coloured graph, where the colours are $\{1, \ldots, n\}$, the vertices are elements in B and $b \xrightarrow{i} b'$ if and only if $b' = \widetilde{F}_i b$.

Problem: Uniqueness of crystal basis? Existence?

Proposition: Let M_1, M_2 be finite-dimensional U_q-modules with (*arbitrary*) A-submodules $\mathscr{L}_i \subseteq M_i$ then

1. $\mathscr{L}_1 \otimes \mathscr{L}_2$ is an admissible lattice for the U_q-module $M_1 \otimes M_2$ if and only if $\mathscr{L}_1$ and $\mathscr{L}_2$ are admissible latices in M_1 and M_2 respectively.

2. $(\mathscr{L}_1 \otimes \mathscr{L}_2, B_1 \times \{0\} \cup \{0\} \times B_2)$ is a crystal basis of $M_1 \oplus M_2$ if and only if $(\mathscr{L}_1, B_1)$ and $(\mathscr{L}_2, B_2)$ are crystal bases for M_1 and M_2 respectively.

Remark/construction: Let $M = L(\lambda)$ with $\lambda \in X^+$; we want to construct a good candidate for a crystal basis for M: pick a vector $0 \neq v_\lambda \in L(\lambda)_\lambda$ and define $\mathscr{L} := A$-submodule of M generated by all $\widetilde{F}_{i_r} \ldots \widetilde{F}_{i_2}\widetilde{F}_{i_1}v_\lambda$ for all tuples of indices $(i_{r_1}, \ldots, i_{r_n})$ where $1 \leq i_j \leq n$ and n is the size of the cartan matrix.
Is $\mathscr{L}$ an admissible lattice? It is surely finitely generated by definition, since only finitely many of the $\widetilde{F}_{i_r} \ldots \widetilde{F}_{i_1}\widetilde{F}_{i_1}v_\lambda$ are non-zero, and they generated $L(\lambda)$ over $\mathbb{Q}(q) = k$; indeed, we have a surjection $M(\lambda) \twoheadrightarrow L(\lambda)$, $M(\lambda)$ is spanned by the $F_{i_r} \ldots F_{i_2}\widetilde{F}_{i_1} 1 \otimes 1$ and we have $\widetilde{F}_i N = F_i N$ for all U_q-modules N.
We also have $v_\lambda \in \mathscr{L} \cap M_\lambda = \mathscr{L}_\lambda$ and $\widetilde{F}_i v_\lambda \in \mathscr{L} \cap M_{\lambda - \alpha_i}$ which implies $\mathscr{L} = \bigoplus_\lambda \mathscr{L}_\lambda$.
The first part of the third property is clear, since $\mathscr{L}$ is evidently stable under $\mathscr{F}_i$; **what is not clear (and this will require some work) is that $\widetilde{E}_i\mathscr{L} \subseteq \mathscr{L}$.** We'll accomplish this is a while.
Assuming this last part, is $(\mathscr{L}, B)$ a crystal basis?
B spans $\mathscr{L}/q\mathscr{L}$ over $\mathbb{Q}$ evidently, $B = \bigcup_\lambda$ where $B_\lambda = B \cap \mathscr{L}_\lambda / q\mathscr{L}_\lambda$ and lastly $\widetilde{F}_i B \subseteq B \cup \{0\}$ by definition. **However, it requires some effort to argue why $\widetilde{E}_i B \subseteq B \cup \{0\}$.**

Remark: The proposed crystal basis depends on the choise of v_λ, but the crystal graph doesn't!
We call the proposed crystal basis $(\mathscr{L}(\lambda), B(\lambda))$ - which we remark once more *depends on the choice of* v_λ.

Lemma:

1. If $\mathfrak{g} = \mathfrak{sl}_2$ then $(\mathscr{L}(\lambda), B(\lambda))$ is a crystal basis,

2. If $\lambda = 0$ then $(\mathscr{L}(\lambda), B(\lambda))$ is a crystal basis (for any semisimple lie algebra $\mathfrak{g}$),

3. The same is true in case λ is *a miniscule dominant weight* (namely $\langle \lambda, \alpha^\vee \rangle \in \{0, 1\}$ for all positive roots α).

4. The same is further true in case λ is a maximal short root in a component of $\mathfrak{g}$.

Proof:

1. In this case we can just appeal to the calculations in the previous example,
2. trivial :) ($\dim L(0) = 1$)
3. Exercise :(
4. Maybe exercise :/ $\square$

Theorem (uniqueness of crystals): Let $\lambda \in X^+$ be a dominant integral weight and $(\mathscr{L}(\lambda), B(\lambda))$ the crystal basis of $L(\lambda)$. Let $(\mathscr{L}, B)$ be another crystal basis for $L(\lambda)$. Then there exists a non-zero scalar $a \in k$ such that $\mathscr{L} = a\mathscr{L}(\lambda)$ and $\mathscr{L}(\lambda)/q\mathscr{L}(\lambda) \to \mathscr{L}/q\mathscr{L}$ sends $B(\lambda)$ bijectively to B.
First a few lemmas.

Corollary: The crystal graph for $L(\lambda)$ is uniquely determined.

Lemma: M a finite dimensional U_q-module, $x \in M_\lambda$ and $\alpha = \alpha_i$ a simply root.
We express like last time $x = \sum_{j \geq 0, j \geq \langle \lambda, \alpha^\vee \rangle} F_i^{(j)} x_j$. Let $\mathscr{L} \subseteq M$ be an admissible lattice. Then

1. $x \in \mathscr{L} \implies x_j \in \mathscr{L}$ for all j,
2. $\widetilde{E}_i x \in q\mathscr{L} \implies x_j \in q\mathscr{L}$ for all $j \geq 1$.

Proof: Note that we have, by last time's lemma, $\widetilde{E}_i x = \sum_{j \geq 1, j \geq \langle \lambda, \alpha_i^\vee \rangle} F_i^{(j-1)} x_j$ with $\widetilde{E}_i x_j = 0$ and $j \geq \langle \lambda, \alpha_i, \alpha_i^\vee \rangle = -\langle \lambda, \alpha_i^\vee \rangle - 2$ where $\langle \lambda + \alpha_i \rangle$ *is the weight of* $\widetilde{E}_i x$ so this expression is the x-decomposition for the vector $\widetilde{E}_i x$.

1. We have $x_0 = x - \widetilde{F}_i \widetilde{E}_i x \in \mathscr{L}$ and by induction of weights using the above expression we have $x_j \in \mathscr{L}$ for $j \geq 1$.
2. Since $\mathscr{L}$ is an admissible lattice we have that $q\mathscr{L}$ is also an admissible lattice, and then apply the previous part of this lemma to get the claim. $\square$

Lemma: Let M be a finite-dimensional U_q-module, $(\mathscr{L}, B)$ a crystal basis. For any subset $S \subseteq \mathscr{L}/q\mathscr{L}$ define $HW(S) := \{x \in S \mid \widetilde{E}_i x = 0 \text{ for all } i\}$ the *set of highest weight elements in* S.
Then

1. for all $b \in B$ there exists a $b' \in HW(B)$ such that $b = \widetilde{F}_{i_r} \dots \widetilde{F}_{i_1} b'$ for some $1 \leq i_j \leq n$.
2. $HW(\mathscr{L}_\lambda / q\mathscr{L}_\lambda)$ is the $\mathbb{Q}$-span of $HW(B_\lambda)$ for all λ.
3. $HW(B_\mu) = \emptyset$ if $\mu \notin X^+$.

Proof:

1. If the weight is maximal then $b \in HW(B)$, so define $b' = b$ and $r = 0$. Otherwise pick i such that $\widetilde{E}_i b \neq 0$, and let $b_1 = \widetilde{E}_i b$. By our induction hypothesis we have $b_1 = \widetilde{F}_{i_r} \dots \widetilde{F}_{i_1} b'$ for some $b' \in HW(B)$ thus $b = \widetilde{F}_i \widetilde{F}_{i_r} \dots \widetilde{F}_{i_1} b'$ since $b = \widetilde{F}_i \widetilde{E}_i b = \widetilde{F}_i b'$ (by (B2) and (B3)).
2. The relation "$\supseteq$" is clear. For the converse, let $x \in \mathscr{L}_\lambda / q\mathscr{L}_\lambda$ and write $x = \sum_{b \in B_\lambda} \gamma_b b$ for coefficients $\gamma_b \in \mathbb{Q}$ (by definition of crystal bases). Then $\widetilde{E}_i x = \sum_{b \in B_\lambda} \gamma_b \widetilde{E}_i b$.
If $\widetilde{E}_i x = 0$ then $\gamma_b = 0$ for all b such that $\widetilde{E}_i b \neq 0$ ($\widetilde{E}_i b \neq 0 \implies \widetilde{E}_i b \in B$). This implies $\gamma_b = 0$ for all $b \notin HW(B_\lambda)$. Thus x is a linear combination of highest weight vectors, i.e. $x \in \mathrm{span}_{\mathbb{Q}}(HW(B_\lambda))$.
3. Let $b \in \mathscr{L}/q\mathscr{L}$ be a non-zero vector and suppose $\widetilde{E}_i b = 0$; to show: $\langle \lambda, \alpha_i^\vee \rangle \geq 0$ (this implies our claim directly).
If $\langle \lambda, \alpha_i^\vee \rangle < 0$ then express $b = \sum_{j \geq 0, j \geq -\langle \lambda, \alpha_i^\vee \rangle} F^{(j)} b_j$ where by assumption $\langle \lambda, \alpha_i^\vee \rangle \geq 1$.
Take now $b \in HW(B_\mu)$. Since we have $\widetilde{E}_j b \in q\mathscr{L}$ for all j (by assumption) and no b_j occurs in the expression for $b \implies b_j \in q\mathscr{L}$ for all $j \implies b \in q\mathscr{L}$ which contradicts $b \neq 0$ hence $HW(B_\mu) \neq \emptyset \implies \mu \in X^+$.

Lemma: In the same setup as the statement of the above theorem, we have

1. $HW(\mathscr{L}(\lambda)_\lambda / q\mathscr{L}(\lambda)_\lambda) = \mathbb{Q} v_\lambda$,
2. if $\mu \neq \lambda$ then $\{x \in L(\lambda)_\mu \mid \widetilde{E}_i x \in \mathscr{L}(\lambda) \text{ for all } i\} = L(\lambda)_\mu$.

Lemma (lattice determined by highest weight): Let $\mathscr{L} \subseteq L(\lambda)$ be an A-submodule ($\lambda \in X^+$ dominant integral) and $\mathscr{L} = \bigoplus_\mu \mathscr{L}_\mu$ where $\mathscr{L}_\mu = \mathscr{L} \cap (L(\lambda))_\mu$; and *assume* $\mathscr{L}_\lambda = Av_\lambda$.

1. $\widetilde{F}_i \mathscr{L} \subseteq \mathscr{L}$ for all i implies $\mathscr{L}(\lambda) \subseteq \mathscr{L}$.
2. $\widetilde{E}_i \mathscr{L} \subseteq \mathscr{L}$ for all i implies $\mathscr{L} \subseteq \mathscr{L}(\lambda)$.

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I like chips. Last time we took a look at the lemmas:

Lemma: Let M be the finite-dimensional $U_q(\mathfrak{g})$-module $L(\lambda)$; assume $(\mathscr{L}(\lambda), B(\lambda))$ is a crystal basis for M. Then:

1. $HW(B(\lambda)) = \{v_\lambda\}$,
2. $\{x \in L(\lambda)_\mu \mid \widetilde{E}_i x \in \mathscr{L}(\lambda) \forall i\} = \mathscr{L}(\lambda)_\mu$ for all $\mu \neq \lambda$.

Proof:

1. The relation "$\supseteq$" is clear. For the converse, suppose $0 \neq b \in HW(B(\lambda))$ and we can express $b = \widetilde{F}_{i_r} \ldots \widetilde{F}_{i_1} v_\lambda$. If $r > 0$ then we can set $b' = \widetilde{F}_{i_{r-1}} \ldots \widetilde{F}_{i_1} v_\lambda$. We have $0 \neq b = \widetilde{F}_{i_r} b'$ and thus by the defining property of crystal bases we have $\widetilde{E}_{i_r} b = b' \neq 0$ which contradicts $b \in HW(B(\lambda)) \implies r = 0$.
2. As before, the relation "$\supseteq$" is clear; and for the converse suppose $0 \neq x \in L(\lambda)_\mu$ with $\widetilde{E}_i x \in \mathscr{L}(\lambda)_\mu$. Since $\mathscr{L}(\lambda)_\mu$ spans $\mathscr{L}(\lambda)_\mu$ over $k = \mathbb{Q}(q)$ it follows that there exists r such that $q^r x \in \mathscr{L}(\lambda)$ (by definition of localisation). If we pick r to be minimal, then $r > 0 \implies \widetilde{E}_i(q^r x) = q^r \widetilde{E}_i(x) \in q\mathscr{L}(\lambda)$ for all i thus $\widetilde{E}_i(\overline{q^r x}) = 0 \in \mathscr{L}(\lambda)/q\mathscr{L}(\lambda)$. By last time's lemma it follows that $\overline{q^r x} = 0$ or $q^r x \in HW(L(\lambda)_\mu)$.
   Since we are assuming $(\mathscr{L}(\lambda), B(\lambda))$ *is a crystal basis we get that* $HW(\mathscr{L}(\lambda)_\mu) = \emptyset$ thus necessarily $q^r x = 0$ which contradicts $r > 0$. $\square$

Lemma (Admissible lattices are determined by their highest weights): Assume $\mathscr{L}$ is an A-submodule of $L(\lambda)$ where $\lambda \in X^+$.
Assume $\mathscr{L}_\lambda = Av_\lambda$. Then

1. $\widetilde{F}_i \mathscr{L} \subseteq \mathscr{L}$ for all $i \implies \mathscr{L}(\lambda) \subseteq \mathscr{L}$.
2. $\widetilde{E}_i \mathscr{L} \subseteq \mathscr{L}$ for all $i \implies \mathscr{L} \subseteq \mathscr{L}(\lambda)$.

Proof: $L(\lambda) = Av_\lambda = L_\lambda$ by construction of $\mathscr{L}(\lambda)$ so:

1. Clear by the definition of $\mathscr{L}(\lambda)$.
2. We argue by induction on the partial order of the weights. $x \in \mathscr{L}_\mu \implies \widetilde{E}_i x \in \mathscr{L}_{\mu+\alpha_i}$ and by induction we have $\mathscr{L}(\lambda)_{\mu+\alpha_i}$ for all i. Thus by the previous lemma we have that $x \in \mathscr{L}(\lambda)_\mu$.$\square$

Now we have a result which - in the prof's words - is *a bit strange*.

Theorem: ***If*** (!!) $L(\lambda)$ has a crystal basis, then $(\mathscr{L}(\lambda), B(\lambda))$ is a crystal basis.

Proof: We have the following - slightly more general - claim

Let $M \cong \mathscr{L}(\lambda)^{\oplus r}$ *for* $r \geq 0$ *and suppose* $(\mathscr{L}, B)$ *is a crystal basis for* M*. If* $HW(B) = B_\lambda$ *then there exists some* U_q*-homomorphism* $\phi_j : L(\lambda) \to M$ *for* $1 \leq j \leq r$ *such that*

- $M = \bigoplus_{j=1}^r \phi_j(L(\lambda))$,
- $\mathscr{L} = \bigoplus_{j=1}^r \phi_j(\mathscr{L}(\lambda))$,
- $B = \bigsqcup_{j=1}^r \overline{\phi_j}(B(\lambda))$.

Assuming the claim we have that by the previous lemma each $(\phi_j(\mathscr{L}(\lambda), \overline{\phi_j}(B(\lambda)))$ is a crystal basis of $\mathrm{im}\phi_j$ for all $j = 1, \ldots, r$.
Now since $\phi_j : L(\lambda) \to \mathrm{im}\phi_j$ is an isomorphism of U_q-modules we get that $(\mathscr{L}(\lambda), B(\lambda))$ is a crystal basis of $L(\lambda)$.
Sketch of proof of claim (Jantzen 9.9)

1. $r = \dim M_\lambda = (B_\lambda)$ and let $B_\lambda = \{b_1,\dots,b_r\}$. Pick lifts $v_1,\dots,v_r \in M_\lambda$. By the Nakayama lemma (applied to the ring $\mathbb{Q}[q]_{(q)}$) we have that $v_1,\dots,v_r$ are a basis of M_λ. These define U_q-homomorphisms $L(\lambda) \to U_q v_j$ for $1 \le j \le r$ mapping v_λ to v_j for each j.

2. evidently we have $\phi_j(\mathscr{L}(\lambda)) \subseteq \mathscr{L}$ and induction of the weights shows equality.

3. "$\subseteq$": Let $b \in B_\mu$. We have the following *claim:* $b \in \bigcup_{1\le j\le r} \overline{\phi_j}(B(\lambda))$. For $\mu = \lambda$ this is clear, so we can suppose $\mu < \lambda$.
   We can express $b = \widetilde{F}_i \widetilde{E}_i b$ at least one i since $(\mathscr{L}, B)$ is a crystal basis, so $\widetilde{E}_i b \in \bigcup_{1\le j\le r} \overline{\phi_j}(B(\lambda))$ by induction and so applying $\widetilde{F}_i$ yields $b = \widetilde{F}_i \widetilde{E}_i b \in \widetilde{F}_i(\bigcup_{1\le j\le r} \overline{\phi_j}(B(\lambda))) \subseteq \bigcup_{1\le j\le r} \overline{\phi_i}(B(\lambda))$ by definition of $B(\lambda)$ and $\widetilde{F}_i$'s action on $B(\lambda)$.
   "$\supseteq$:" Let $x \in \bigcup_{1\le j\le r} \overline{\phi_j}(B(\lambda)_\mu)$. Just like before we have that if $\lambda = \mu$ then evidently $x \in B$ by definition; and if $\mu < \lambda$, so there exists i such that $y \in B(\lambda)_{\mu+\alpha_i} \implies \widetilde{F}_i y = x$ and by induction we have $\overline{\phi_j}(y) \in B \implies \widetilde{F}_i(\overline{\phi_j}(y)) = \overline{\phi_j}(\widetilde{F}_i(y)) = \overline{\phi_j}(x) \in B$. $\square$

The following proposition has a similar proof kinda.

Proposition: Let M be a finite dimensional U_q-module, λ a maximal weight and $(\mathscr{L}, B)$ a crystal basis of M. If $M = N_1 \oplus N_2$ with N_i =isotypical component of $L(\lambda)$ and set $\mathscr{L}_i := \mathscr{L} \cap N_i, B_i = B \cap \mathscr{L}_i / q\mathscr{L}_i$. Then

1. $\mathscr{L} = \mathscr{L}_1 \oplus \mathscr{L}_2$ and $B = B_1 \cup B_2$,

2. there exists an isomorphism $L(\lambda) \oplus \cdots \oplus L(\lambda) \xrightarrow{\cong} N_i$ mapping "our crystal basis" (namely $(\mathscr{L}(\lambda), B(\lambda))$) to the crystal basis $(\mathscr{L}_i, B_i)$.

Proof: omitted.

Theorem (uniqueness of the crystal basis): Let $\lambda \in X^+$, and assume $(\mathscr{L}(\lambda), B(\lambda))$ is a crystal basis of $L(\lambda)$. Let $(\mathscr{L}, B)$ also be a crystal basis. Then there exists $0 \neq a \in \mathbb{Q}(q)$ such that $\mathscr{L} = a\mathscr{L}(\lambda)$ and

$$\mathscr{L}(\lambda)/q\mathscr{L}(\lambda) \xrightarrow{\cong} \mathscr{L}/q\mathscr{L}$$

$$x \longmapsto ax$$

sending $B(\lambda)$ to B (set-theoretically).

Proof: If $M = L(\lambda)$ then $N_1 = L(\lambda)$ as in the previous proposition, so we have an isomorphism $L(\lambda) \to N_1 \cong L(\lambda)$ of U_q-modules mapping $(\mathscr{L}(\lambda), B(\lambda))$ to $(\mathscr{L}, B)$ which *must* be an isomorphism given by multiplication by some scalar $a \in \mathbb{Q}(q)$. $\square$

Example: Take $\mathfrak{g} = \mathfrak{sl}_2, V = L(q)$ the two-dimensional representation of type 1. As discussed, V has a the "standard" crystal basis $\mathscr{L}(q), B(q)$. ***Problem:*** $(\mathscr{L}(\lambda) \otimes \mathscr{L}(\lambda), B(\lambda) \otimes B(\lambda))$ - *even in this super basi example* - is **not** a crystal for $V \otimes V$ any longer :/
The reason is because (!!!!) $\Delta(F) = F \otimes K^{-1} + 1 \otimes F \implies \Delta(F)(v_0 \otimes v_0) = q^{-1} v_0 \otimes v_0 + v_0 \otimes v_0$ but we want $F(v_0 \otimes v_0) = \widetilde{F}(v_0 \otimes v_0) \in (\mathscr{L}(\lambda) \otimes \mathscr{L}(\lambda))_0$ for this to be a crystal basis and $(\mathscr{L}(\lambda) \otimes \mathscr{L}(\lambda))_0 \subseteq A v_0 \otimes v_0 + A v_0 \otimes v_1$ which of course goes against $q^{-1} \in A$ (which is basically the whole point of this contruction).

"Easy" solution: Change comultiplication!
Consider the antiautomorphism $\tau : U \to U$ given by $E_i \mapsto F_i, F_i \mapsto E_i, K_i \mapsto K_i$ and set $\Delta' = (\tau \otimes \tau) \circ (\Delta \circ \tau)$ so $\Delta'(F_i) = \tau \otimes \tau(E_i \otimes 1 + K_i \otimes E_i) = F_i \otimes 1 + K_i \otimes F_i$ which solves the problem since there's no K_i^{-1} popping up :)

Remark: Weight spaces are independent of the choice of comultiplication Δ or Δ', and we actually have an isomorphism

$$V \otimes W \xrightarrow{\cong} V \otimes W$$

of U_q-modules, where the module on the left is defined by the usual comultiplication Δ and the on the right is with Δ' (which is of course not the identity).

Definition: Let $(\mathscr{L}, B)$ be a crystal basis of M a finite-dimensional U_q-module, and let $b \in B$. Define the functions (these shouldn't be anything new):

1. $\epsilon_i(b) := \max\{r \ge 0 \mid \widetilde{E}_i^r b \neq 0\}$,

2. $\phi_i(b) := \max\{r \ge 0 \mid \widetilde{F}_i b \neq 0\}$.

Theorem: Let M_1 and M_2 be finite-dimensional U_q-modules, and let $(\mathscr{L}_i, B_i)$ be a crystal basis of M_i. Then $(\mathscr{L}_1 \otimes \mathscr{L}_2, B_1 \otimes B_2)$ is a crystal basis for $M_1 \otimes M_2$ with the above U_q-module structure and

$$\widetilde{F}_i(b \otimes b') = \begin{cases} \widetilde{F}_i b \otimes b' & \text{if } \phi_i(b) > \epsilon_i(b') \\ b \otimes \widetilde{F}_i(b') & \text{if } \phi_i(b) \leq \epsilon_i(b') \end{cases}$$

$$\widetilde{E}_i(b \otimes b') = \begin{cases} \widetilde{E}_i b \otimes b' & \text{if } \phi_i(b) \geq \epsilon_i(b') \\ b \otimes \widetilde{E}_i(b') & \text{if } \phi_i(b) < \epsilon_i(b') \end{cases}$$

Proof (idea): For fixed i this is just an $\mathfrak{sl}_2$-calculation, and we use the isomorphism $(M \oplus N) \otimes P \cong (M \otimes P) \oplus (N \otimes P)$ to reduce to M_i irreductible. Then we simply compute :p $\square$

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In our last lecture we introduced the following crystal basis for tensor products of modules whose crystal bases we know:

Theorem: Let M_1 and M_2 be finite-dimensional U_q-modules, and let $(\mathscr{L}_i, B_i)$ be a crystal basis of M_i. Then $(\mathscr{L}_1 \otimes \mathscr{L}_2, B_1 \otimes B_2)$ is a crystal basis for $M_1 \otimes M_2$ with the above U_q-module structure and

$$\widetilde{F}_i(b \otimes b') = \begin{cases} \widetilde{F}_i b \otimes b' & \text{if } \phi_i(b) > \epsilon_i(b') \\ b \otimes \widetilde{F}_i(b') & \text{if } \phi_i(b) \leq \epsilon_i(b') \end{cases}$$

$$\widetilde{E}_i(b \otimes b') = \begin{cases} \widetilde{E}_i b \otimes b' & \text{if } \phi_i(b) \geq \epsilon_i(b') \\ b \otimes \widetilde{E}_i(b') & \text{if } \phi_i(b) < \epsilon_i(b') \end{cases}$$

Proof (idea): The following identities are relatively easy to verify:

$$\mathbb{Q}(q) \otimes_A (\mathscr{L}_1 \otimes_A \mathscr{L}_2) \cong M_1 \otimes_{\mathbb{Q}(q)} M_2$$

$$\mathscr{L}_1 \otimes_A \mathscr{L}_2 = \bigoplus_{v \in X, \lambda + \mu = v} (\mathscr{L}_1)_\lambda \otimes_A (\mathscr{L}_2)_\mu$$

$$B_1 \otimes B_2 \subseteq \mathscr{L}_1 \otimes_A \mathscr{L}_2 / q(\mathscr{L}_1 \otimes_A \mathscr{L}_2) \text{ is a } \mathbb{Q}\text{-basis of } \mathscr{L}_1 \otimes_A \mathscr{L}_2 / q(\mathscr{L}_1 \otimes_A \mathscr{L}_2) \cong \mathscr{L}_1 / q\mathscr{L}_1 \otimes_{\mathbb{Q}} \mathscr{L}_2 / q\mathscr{L}_2$$

where by definition (which is implicit in the theorem's statement) $B_1 \otimes B_2 = \bigcup_{v \in V} \bigcup_{\lambda+\mu=v} (B_1)_\lambda \otimes (B_2)_\mu$. What's missing is to show the theorem for $U_q(\mathfrak{sl}_2)$ (via long calculations) and then restrict to $U_{q_i}(\mathfrak{sl}_2)$.
Then we need to show $\widetilde{E}_i \mathscr{L} \subseteq \mathscr{L}, \widetilde{E}_i B \subseteq B \cup \{0\}$ and analogously $\widetilde{F}_i \mathscr{L} \subseteq \mathscr{L}, \widetilde{F}_i B \subseteq B \cup \{0\}$: this is again an $\mathfrak{sl}_2$-computation. $\square$

This theorem is extremely important because it works as the foundation for the existence of crystal bases for arbitrary modules, via the *Grand Loop Argument* (Jantzen). Definitely take a look at this from the book :)

Examples: insert image

Observation (lemma): Let $(\mathscr{L}(\lambda), B(\lambda))$ be the "nice" crystal basis of $L(\lambda)$. Let $(\mathscr{L}, B)$ be the crystal basis of M some finite-dimensional U_q-module. Then $HW(B(\lambda) \otimes B_\mu) = \{\overline{v}_\lambda \otimes b' \mid \widetilde{E}_i^{\langle \lambda, \alpha_i^\vee \rangle + 1} b' = 0\}$.

Proof: Assume $b \otimes b' \in HW(B(\lambda) \otimes B_\mu)$. Then $\widetilde{E}_i(b \otimes b') \in \{\widetilde{E}_i b \otimes b', b \otimes \widetilde{E}_i b'\}$; so if $\widetilde{E}_i(b \otimes b') = b \otimes \widetilde{E}_i b'$ then $\epsilon_i(b') > \phi_i(b) \implies \epsilon_i(b') > 0 \implies b \otimes \widetilde{E}_i b' \neq 0$ which is contradiction since $b \otimes b'$ is of "kashiwara highest weight".
So we must necessarily have $0 = \widetilde{E}_i b \otimes b' = \widetilde{E}_i b \otimes b'$ for all i (since i was arbitrary in the previous remark) and thus $b \in \operatorname{span} \overline{v_\lambda}$. Now, $\epsilon_i(b') \leq \phi_i(b) = \langle \lambda, \alpha_i^\vee \rangle$ so we're done :) $\square$

Remark: We showed that if M is a U_q-module and $M \cong L(\lambda_1) \oplus \cdots \oplus L(\lambda_r)$, then any crystal basis for M is isomorphic to $(\mathscr{L}(\lambda_1) \oplus \cdots \oplus \mathscr{L}(\lambda_r), B(\lambda_1) \oplus \cdots \oplus B(\lambda_r))$.
So if M is irreducible then the crystal graph must be connected, and between summands there is no connection.
We thus have that in general components of the crystal graph of M correspond to irreducible direct summands of M.

Warning(!): The above is just a *bijection of sets*.

Missing part in my notes: at this point in the lecture I was bewildered by so many cool examples and graphs :) I was so in awe that my fingers literally stopped typing without me noticing.

Abstract crystals

Goal: Define "*abstract crystal graphs*". Let $\mathfrak{g}$ be a semisimple Lie algebra with Cartan Matrix A, $\pi = \{\alpha_1, \dots, \alpha_n\}$ its set of simple roots, $X = \sum_{i=1}^n \mathbb{Z}\omega_i$ the set of integral dominant weights.

Definition: An (abstract) crystal is a set B together with maps $\mathrm{wt}: B \to X$ for $i = 1, \dots, n$, $\phi_i, \epsilon_i : B \to \mathbb{Z} \cup \{-\infty\}$ and $\widetilde{E}_i, \widetilde{F}_i : B \to B \cup \{0\}$ which satisfy

(C1) $\phi_i(b) = \epsilon_i(b) + \langle \mathrm{wt}(b), \alpha_i^\vee \rangle$,

(C2) $\mathrm{wt}(\widetilde{E}_i b) = \mathrm{wt}(b) + \alpha_i$ and $\mathrm{wt}(\widetilde{F}_i b) = \mathrm{wt}(b) - \alpha_i$,

(C3) $\epsilon(\widetilde{E}_i b) = \epsilon_i(b) - 1$, $\phi_i(\widetilde{F}_i b) = \phi_i(b) - 1$, $\epsilon_i(\widetilde{F}_i b) = \epsilon_i(b) + 1$,$\phi_i(\widetilde{E}_i b) = \phi_i(b) + 1$,

(C4) $b, b' \in B$ implies $b' = \widetilde{F}_i b \iff b = \widetilde{E}_i b'$,

(C5) $b \in B, \phi_i(b) = -\infty \implies \widetilde{E}_i(b) = 0 = \widetilde{F}_i(b)$ and $\epsilon_i(b) = -\infty$.

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Example: Today we started off computing the crystal graph for the tensor product of representations of $\mathfrak{sp}_7$, $V_{sp} \otimes V$ where $V_{sp} = L(\omega_\infty)$ and $V = L(\epsilon_1)$ is the natural representation.

Back to abstract crystals: recall that the setup was we had a semisimple Lie algebra $\mathfrak{g}$ and a set B, and functions $\widetilde{e_i}, \widetilde{f_i} : B \to B \cup \{0\}, \phi_i, \epsilon_i : B \to \mathbb{Z} \cup \{-\infty\}$ and $\mathrm{wt}: B \to X$ where X is the set of integral weights of $\mathfrak{g}$.

Definition: An abstract crystal is of *finite type* if $-\infty$ doesn't occur as a value for the elements in B. The abstract crystal is *seminormal* if $\phi_i(b) = \max\{k \mid \widetilde{f_i}^k b \neq 0\}$ and $\epsilon_i(b) = \max k \mid \widetilde{e_i}^k b \neq 0$.

Lemma: Given an abstract crystal of finite type, for all $b \in B$ we have

$$\mathrm{wt}(b) = \sum_{i=1}^n (\phi_i(b) - \epsilon_i(b))\omega_i$$

where ω_i is the i-th fundamental weight.

Proof:

$$\langle \mathrm{wt}(b) - \sum_{i=1}^n (\phi_i(b) - \epsilon_i(b))\omega_i \,,\, \alpha_j^\vee \rangle = \langle \mathrm{wt}(b), \alpha_j^\vee \rangle - (\phi_j(b) - \epsilon_j(b)) \overset{(C3)}{=} 0$$

by the abstract crystals axioms.

Examples:

1. $(\mathscr{L}(\lambda), B(\lambda))$ crystal basis of $L(\lambda)$ irreducible and finite-dimensional for $U_q(\mathfrak{g})$ defines a seminormal abstract crystal basis of finite type.
2. (the *"stupid crystal"*) $T(\lambda)$ for $\lambda \in X$ is defined by $B = \{*\}, \mathrm{wt}(*) = \lambda, \widetilde{e_i}(*) = 0 = \widetilde{f_i}(*)$ and $\epsilon_i(*) = -\infty = \phi_i(*)$ for all i, which is neither of finite type nor seminormal.
3. $B(\infty)$ of type A_1 is defined by the infinite crystal graph

$$b_0 \xrightarrow{1} b_{-1} \xrightarrow{1} b_{-2} \xrightarrow{1} \dots$$

where $\epsilon_1(b_{-j}) = j$, $\phi_1(b_{-j}) = -j$ and $\mathrm{wt}(b_{-j}) = -2j$.

Remark: We have a notion of "*tensor products*" of abstract crystals: $\widetilde{e_i}$ and $\widetilde{f_i}$ are defined on the product of the two bases B_1 and B_2 as $\widehat{E_i}$ and $\widehat{F_i}$ were defined in the standard setting and furthermore

$$\mathrm{wt}(b \otimes b') = \mathrm{wt}(b) + \mathrm{wt}(b')$$

$$\phi_i(b \otimes b') = \max\{\phi_i(b), \phi_i(b') + \langle \mathrm{wt}(b), \alpha_i^\vee \rangle\}$$

$$\epsilon_i(b \otimes b') = \max\{\epsilon_i(b), \epsilon_i(b) - \langle \mathrm{wt}(b'), \alpha_i^\vee \rangle\}.$$

Where do crystals pop up in real life?

Littelmann's path model

Construction: Let X be the integral weight lattice for some semisimple Lie algebra $\mathfrak{g}$, and define $X_{\mathbb{Q}} := \mathbb{Q} \otimes_{\mathbb{Z}} X$ and

$$\widetilde{\pi} := \{\text{continuous piecewise linear maps } \phi : [0,1] \to X_{\mathbb{Q}}, \phi(0) = 0\}$$

and $\mathbb{Z}\pi$ the free $\mathbb{Z}$-module with basis π where $\pi = \widetilde{\pi}/\sim$ for an equivalence relation $\sim$ which we'll define later.

Ideal: Define on $\mathbb{Z}\pi$ actions of $\widetilde{e_i}$ and $\widetilde{f_i}$ to get an abstract crystal. For each $\lambda \in X$ we can construct the element $\phi_\lambda \in \widetilde{\pi}$ the straight-line path from 0 to λ. Let $M(\lambda) = \mathscr{E}$-submodule of $\mathbb{Z}\pi$ generated by ϕ_λ where $\mathscr{E} \subseteq \mathrm{End}_{\mathbb{Z}}(\mathbb{Z}\pi)$ generated by $\widetilde{e_i}, \widetilde{f_i}$.

Theorem: Let $B(\lambda) = M(\lambda) \cap \pi$ paths in $M(\lambda)$. For every $\lambda \in X^+$ we have

$$\sum_{\phi \in B(\lambda)} e^{\phi(1)} = \mathrm{char} L(\lambda)$$

The above theorem is our objective: we're interested in expressing the character $\mathrm{char} L(\lambda)$ in terms of this loop construction; we can now return and fill in the blanks in our definitions.

Definition: $\phi_1, \phi_2 \in \widetilde{\pi}$ are *($\sim$-)equivalent* is $\exists \gamma : [0,1] \to [0,1]$ a piecewise linear continuous surjective map such that $\phi_2 = \phi_1 \circ \gamma$.

Pick now $\alpha = \alpha_1$ a simple root; we're interested in defining $\widetilde{e_\alpha}$ and $\widetilde{f_\alpha}$.

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Remark (Generalised Littlewood-Richardson rule): In terms of the Littelmann path model, given $\phi_1, \phi_2 \in \Pi$ with image in the closure of the dominand Weyl chamber, let $\phi_1(1) = \lambda$, $\phi_2(1) = \mu$. We have the formula

$$L(\lambda) \otimes L(\mu) = \bigoplus_{\pi \in W} L(\pi(1))$$

where W is the set of paths in Π such that the image is in closure of dominant Weyl chamber and $\pi = \phi * \eta$ where η is contained among the paths attached to $L(\mu)$.

Geralities on Geometric Realisations: Fix a field k; given any set X we denote by $\mathrm{Func}_k(X)$ the set $\{\text{maps } f : X \to k\}$. Assume $\pi : A \to B$ is a map of sets and define $\pi^* : \mathrm{Func}_k(B) \to \mathrm{Func}_k(A)$ as $f \mapsto f \circ \pi$.
If π has finite fibres then define $\pi_! : \mathrm{Func}(A) \to \mathrm{Func}(B)$ as $f \mapsto (\pi_!(f) : b \mapsto \sum_{a \in \pi^{-1}(b)} f(a)$.

Appendix: things that came before the 24th of November

Unfortunately I began to work on these notes relatively late after the course started, so - in preparation for my oral exam in a few days - I collect in this appendix a few propositions and theorems which were discussed before; most of the arguments here are either from my memory and some references which Prof. Stroppel didn't quite follow by the letter or by a sheer replication of my friend Lukas' notes.

Representation theory for $U_q(\mathfrak{sl}_2)$

Definition: k be any field and $q \in k^\times$ a non-zero element whose square isn't one. Define the *quantum universal enveloping algebra for* $\mathfrak{sl}_2$as the k-algebra generated by elements E, F, K, K^{-1} with the relations

$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}},\ KEK^{-1} = q^2 E,\ KFK^{-1} = q^{-2}F,\ KK^{-1} = K^{-1}K = 1.$$

We have a natural $\mathbb{Z}$-grading for $U_q(\mathfrak{sl}_2)$ defined by

$$\deg E = 1,\ \deg F = -1,\ \deg K = \deg K^{-1} = 0$$

which is well defined by the above prescriptions since $U_q(\mathfrak{sl}_2)$'s defining relations are homogeneous.
The following theorem is pretty much essential for all practical computations:

Theorem: The set of monomials $S = \{F^a K^b E^c \mid a, c \in \mathbb{Z}_{\geq 0}, b \in \mathbb{Z}\} \subseteq U_q(\mathfrak{sl}_2)$ forms a basis for $U_q(\mathfrak{sl}_2)$ over k.

Proof: The set $\operatorname{Span} S$ is stable under multiplication by E, F and $K^{\pm 1}$ by the following computations

$$F \cdot F^a K^b E^c = F^{a+1} K^b E^c$$

$$K \cdot F^a K^b E^c = q^{-2a} F^a K^{b+1} E^c$$

$$K^{-1} \cdot F^a K^b E^c = q^{2a} F^a K^{b-1} E^c$$

and lastly, by induction we can show the following very important relation (from now on referred to as the *EF relation*):

$$EF^a = F^a E + [a]_q F^{a-1}[K; 1-a]$$

where $[K;n]$ is defined by $[K;n] := \frac{q^n K - q^{-n} K^{-1}}{q - q^{-1}}$: the base case $n = 1$ follows straight from $U_q(\mathfrak{sl}_2)$'s construction, and if we suppose

$$EF^a = F^a E + [a]_q F^{a-1} \frac{q^{1-a}K - q^{a-1}K^{-1}}{q - q^{-1}}$$

then multiplying on the left by F yields

$$\begin{aligned} EF^{a+1} &= F^a EF + [a]_q F^{a-1} \frac{q^{1-a}K - q^{a-1}K^{-1}}{q - q^{-1}} \cdot F = F^{a+1}E + F^a \cdot \frac{K - K^{-1}}{q - q^{-1}} + [a]_q F^a \frac{q^{-a-1}K - q^{a+1}K^{-1}}{q - q^{-1}} \\ &= F^{a+1}E + F^a \cdot \Big(\frac{K - K^{-1} + (q^{a-1} + q^{a-3} + \cdots + q^{3-a} + q^{1-a}) \cdot (q^{a+2}K - q^{-a-2}K^{-1})}{q - q^{-1}}\Big) \\ &= F^{a+1}E + [a+1]_q F^a \frac{q^{-a}K - q^a K^{-1}}{q - q^{-1}}. \end{aligned}$$

From this it follows that

$$E \cdot F^a K^b E^c = q^{2b} F^a K^b E^{c+1} + [a]_q \frac{q^a F^{a-1} K^{b+1} E^c - q^{-a} F^{a-1} K^{b-1} E^c}{q - q^{-1}}$$

which shows our claim. Since evidently $1 \in \operatorname{Span} S$ it follows that $\operatorname{Span} S = U_q(\mathfrak{sl}_2)$. To show linear independence, we construct a representation V for $U_q(\mathfrak{sl}_2)$: let V be the k-vector space given by the localised polynomial ring $k[x, y^{\pm 1}, z]$, and define $U_q(\mathfrak{sl}_2)$'s action on V reflecting our above calculations:

$$F \cdot x^a y^b z^c := x^{a+1} y^b z^c$$

$$K \cdot x^a y^b z^c := q^{-2a} x^a y^{b+1} z^c$$

$$E \cdot x^a y^b z^c := q^{2b} x^a y^b z^{c+1} + [a]_q \frac{q^a x^{a-1} y^{b+1} z^c - q^{-a} x^{a-1} y^{b-1} z^c}{q - q^{-1}}$$

If we thus suppose we have a zero-linear combination of the elements in S

$$0 = \sum_{a,b,c} \lambda_{a,b,c} F^a K^b E^c$$

then applying this linear combination to $1 \in V$ yields

$$0 = \sum_{a,b,c} \lambda_{a,b,c} x^a y^b z^c$$

which of course shows that each coefficient $\lambda_{a,b,c}$ must be zero since the monomials $x^a y^b z^c$ form a basis for V. □

Remark: Note that $U_q(\mathfrak{sl}_2)$ also possesses another decomposition, aside from the grading described earlier: the linear endomorphism $u \in U_q(\mathfrak{sl}_2) \mapsto KuK^{-1} \in U_q(\mathfrak{sl}_2)$ is indeed diagonalisable by our previous theorem (indeed, every element in S is an eigenvector) and the element $u = F^a K^b E^c$ has eigenvalue q^{2c-2a}; thus, if q is not a root of unity each of these eigenvalues is distinct from any other and thus the decomposition for $U_q(\mathfrak{sl}_2)$ into eigenspaces of conjugation by K coincides with our initial grading. When $q^l = 1$ for some $l \in \mathbb{Z}$ however matters get a little more complicated; this motivates why we begin by analysing $U_q(\mathfrak{sl}_2)$'s representation theory when $q^l \neq 1$ for all $l \in \mathbb{Z}$.

Construction: Let $U_q(\mathfrak{sl}_2)^{\geq 0} \subseteq U_q(\mathfrak{sl}_2)$ be the subalgebra generated by E, K and K^{-1} - i.e. the analogue of the Borel subalgebra in the classical setting. $U_q^{\geq 0}(\mathfrak{sl}_2)$ possesses for every $\lambda \in k^*$ a one-dimensional representation k_λ spanned over k by v defined by

$$Ev := 0$$

$$Kv := \lambda v$$

$$K^{-1}v := \lambda^{-1} v$$

The induced representation

$$\mathrm{Ind}_{U_q^{\geq 0}(\mathfrak{sl}_2)}^{U_q(\mathfrak{sl}_2)}(k_\lambda) = U_q(\mathfrak{sl}_2) \otimes_{U_q^{\geq 0}(\mathfrak{sl}_2)} k_\lambda := M(\lambda)$$

is called the *Verma module of highest weight* λ or the *standard cyclic module of highest weight* λ. By the previous theorem it's easy to see that a basis for $M(\lambda)$ is given by the following vectors

$$\{v, Fv, F^2 v, F^3 v, \dots\}$$

and furthermore, by the "*restrictions / extensions*" adjunction, $M(\lambda)$ satisfies the so-called "*Verma-module universal property*" (a name which I'm come to dislike a little :/): for every $U_q(\mathfrak{sl}_2)$-module V we have the following natural bijection

$$\mathrm{Hom}_{U_q(\mathfrak{sl}_2)}(M(\lambda), V) \cong \mathrm{Hom}_{U_q^{\geq 0}(\mathfrak{sl}_2)}(k_\lambda, \mathrm{Res}_{U_q(\mathfrak{sl}_2)}^{U_q^{\geq 0}(\mathfrak{sl}_2)}(V)) \cong \{v \in V \mid Kv = \lambda v \text{ and } Ev = 0\}.$$

Theorem: Let $\lambda \in k$ be any scalar, and suppose $q \in k\times$ isn't a root of unity.

1. If λ is $\lambda \notin \{\epsilon q^n \mid \epsilon \in \{\pm 1\}, n \in \mathbb{Z}_{\geq 0}\}$ (i.e. λ is not *dominant integral*) then $M(\lambda)$ is an irreducible $U_q(\mathfrak{sl}_2)$-module.
2. if $\lambda = \epsilon q^n$ for some $\epsilon \in \{\pm 1\}$ and $n \in \mathbb{Z}_{\geq 0}$ then $M(\lambda) = M(\epsilon q^n)$ has precisely one submodule N isomorphic to $M(\epsilon q^{-(n+2)})$ generated over $U_q(\mathfrak{sl}_2)$ by $F^{n+1}v$ and the quotient $M(\epsilon q^n)/N$ is finite dimensional.

Proof: We first make a very general claim, which has an analogue in pretty much every setting in this course:

Let M be a (possibly infinite-dimensional) $U_q(\mathfrak{sl}_2)$-module which has a decomposition $M = \bigoplus_\lambda M_\lambda$ into eigenspaces for K (i.e. into weight spaces). Then every submodule $N \subseteq M$ also admits a decomposition into weight spaces.

Proof of claim: Given any $n \in N$ expressed as a sum of weight vectors

$$n = \sum_\lambda m_\lambda, \ m_\lambda \in M_\lambda$$

we show that each m_λ lies in $N_\lambda = M_\lambda \cap N$. If by contradiction there existed some $n \in N$ such that, when expressed as before, not all m_λ lie in N, then we certainly have that the number of such terms which don't lie in N must be greater strictly than one - if m_{λ_0} is the only term not in N we would have

$$m_{\lambda_0} = n - \sum_{\lambda \neq \lambda_0} m_\lambda \in N.$$

However, if we take $n \in N$ such that the number $\#\{\lambda \mid m_\lambda \notin N\}$ is minimal, then for any λ_0 such that $m_\lambda \notin N$ we have

$$Kn - \lambda_0 n = \sum_\lambda \lambda m_\lambda - \lambda_0 m_\lambda$$

which is an element in N that has precisely one term less than n not lying in N. This contradicts n's minimality. $\square$

If we now go back to discussing our theorem, if $N \subseteq M(\lambda)$ is any submodule, then it must be a direct sum of the one-dimensional weight spaces $\mathrm{Span} F^n v$. However, note that by the *EF-relation* we have that $EF^n v = [n]_q \frac{q^n\lambda - q^{-n}\lambda^{-1}}{q-q^{-1}} F^{n-1}v$ which is a *non-zero* element of the weight space $\mathrm{Span} F^{n-1}v$ for every n if λ isn't of the form $\pm q^n$ for some $n \geq 0$ - we have that $[n]_q \neq 0$ for all n since q isn't a root of unity, and $q^n\lambda = q^{-n}\lambda^{-1}$ only if $\lambda = \pm q^n$. This implies that in case 1. (iteratively applying E to any element in $N \cap \mathrm{Span} F^n v \neq 0$) N contains some non-zero scalar multiple of v and must thus coincide with N.
As for case 2. it follows that $M(\pm q^n)$'s only submodule is spanned by $F^{n+1}v$ its unique highest weight vector different to v, which is evidently isomorphic to $M(\pm q^{-(n-2)})$ (since $F^{n+1}v$ has weight $\pm q^{-(n-2)}$) and by the PBW basis for $M(\pm q^{-(n-2)})$ discussed above. $\square$

The finite dimensional module $M(\pm q^n)/M(\pm q^{-(n+2)})$ is denoted by $L(q^n, \pm)$ or $L(n, \pm)$ and is crucial for the classification of $U_q(\mathfrak{sl}_2)$-modules.

Proposition: Let q be a non-root of unity and M a finite-dimensional $U_q(\mathfrak{sl}_2)$-module. Then M decomposes into a direct sum of weight spaces.

Proof: Suppose $r \geq 0$ is such that $F^r M = 0$. Define the following elements in $U_q(\mathfrak{sl}_2)$ for $s = 0, \dots, r$:

$$h_s := \prod_{j=-(s-1)}^{s-1} [K; s-r+j] = \prod_{j=-(s-1)}^{s-1} \frac{q^{s-r+j}K - q^{r-s-j}K^{-1}}{q-q^{-1}}.$$

For $s = 0$ we have $h_0 = 1$ and thus $F^r h_0 M = 0$. By induction on s we have that $F^{r-s} h_s M = 0$:

$$m \in M,\ F^{r-s} \prod_{j=-(s-1)}^{s-1} \frac{q^{s-r+j}K - q^{r-s-j}K^{-1}}{q-q^{-1}} \cdot m = 0$$

$\implies$ ... I thought this would be swifter but it had me invested much more than expected.

For $s = r$ we thus obtain that $h_r M = 0$ and thus K satisfies the polynomial equation

$$\prod_{j=-(r-1)}^{r-1} \frac{q^j t - q^{-j}t^{-1}}{q-q^{-1}}$$

whose roots are all distinct and equal to $\pm q^j$ since q isn't a root of unity. $\square$ The following simple fact comes from one of the course's exercise sheets.

Proposition: Let $q \in k^\times$ be a non-root of unity, M any finite-dimensional $U_q(\mathfrak{sl}_2)$-module. Then E and F act as nilpotent endomorphisms on M.

Proof: Let $p(T) \in k[T]$ be the minimal polynomial of the endomorphism given by the action of K on M. If $p(T) = p_1^{d_1}(T)\dots p_s^{d_s}(T)$ is its factorisation into irreducible factors then M decomposes as a direct sum $M = M_1 \oplus \dots \oplus M_s$ where $M_i = \ker(q_i^{d_i}(K))$. Note that $EK = q^2KE$ implies $f(q^2K)E = Ef(K)$ for every polynomial $f \in k[T]$, hence $E \ker f(K) \subseteq \ker(f(q^2K))$ and in general $E^j \ker f(K) \subseteq \ker(f(q^{2j}K))$. However, since $p_i^{d_i}(q^{2j}T)$ is coprime to $p_i^{d_i}(T)$ for every i, j because $q \neq \sqrt{1}$, they have the same degree and the same non-zero constant term (note that K is invertible and thus $T \nmid p(T)$). It follows that for every i we have that either $EM_i \subseteq M_j$ for some $j \neq i$ - if $p_i(q^2T) = p_j(T)$ - or $EM_i = 0$. Since the M decomposes as the direct sum of the M_i's and these are fintely many, it follows that $E^r M = 0$ for some $r \geq 1$. Applying the Cartan involution yields that $\omega(E)^r = F^r$ also acts as zero on M. $\square$

Remark: From these simple results it follows that all finite-dimensional irreducible $U_q(\mathfrak{sl}_2)$-modules M are isomorphic to $L(n, \epsilon)$ for some $n \in \mathbb{Z}_{\geq 0}$ and $\epsilon \in \{\pm 1\}$: whenever $m \in M$ is a weight vector of weight $\pm q^n$, we have that $Em \in M$ is also a weight vector, of weight q^{n+2}; since M decomposes as a direct sum of weight spaces and E acts nilpotently on M, it follows that (iteratively applying E) there must exist $m \in M_{\pm q^n}$ for some $n \geq 0$ such that $Em = 0$. Thus by $M(\pm q^n)$'s property we have a surjective (since M is irreducible) $U_q(\mathfrak{sl}_2)$-homomorphism $M(\pm q^n) \twoheadrightarrow M$ which must necessarily factor through to an isomorphism $L(n, \pm) \xrightarrow{\cong} M$ because M is finite dimensional and $M(\pm q^n)$ only has one quotient.

Definition: We define the *Casimir element* in $U_q(\mathfrak{sl}_2)$ to be

$$C = FE + \frac{[K;1]}{q-q^{-1}} = FE + \frac{qK - q^{-1}K}{(q-q^{-1})^2}.$$

A pretty quick computation shows that C commutes with K, K^{-1}, E and F thus $C \in Z(U_q(\mathfrak{sl}_2))$.

Remark: C is a handy took for *separating* representations: C must act as a scalar on $M(\lambda)$ for all λ since $M(\lambda)$ is generated over $U_q(\mathfrak{sl}_2)$ by just one vector, and suppose C acts as the same scalar on $M(\lambda)$ and $M(\mu)$ for two scalars $\lambda, \mu \in k^\times$.
These scalars are easily computed to be (via C's action on the highest weight vector)

$$\frac{q\lambda - q^{-1}\lambda^{-1}}{(q-q^{-1})^2} \text{ on } M(\lambda) \text{ and}$$

$$\frac{q\mu - q^{-1}\mu^{-1}}{(q-q^{-1})^2} \text{ on } M(\mu)$$

so we must have either $\lambda = \mu$ or $\lambda = q^{-2}\mu^{-1}$. Note that if either λ or μ is dominant integral then we necessarily have that **the other isn't**: C acts as the same scalar on $L(n,\epsilon)$ and $L(m,\epsilon')$ if and only if $n = m$ and $\epsilon = \epsilon'$.
For each $\lambda \in k^\times$ denote by $c(\lambda) = \frac{q\lambda - q^{-1}\lambda^{-1}}{(q-q^{-1})^2}$ this scalar.

Theorem: Suppose $q \in k^\times$ isn't a root of unity and M is a finite-dimensional $U_q(\mathfrak{sl}_2)$-module. Then M is a direct sum of irreducible $U_q(\mathfrak{sl}_2)$-modules.

Proof: Let $M_0 = (0) \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_r = M$ be a composition series for M, so each quotient M_i/M_{i-1} for $i = 1, \ldots, r$ is isomorphic to some $L(\lambda_i)$ for some $\lambda_i \in X^+ := \{\pm q^n \mid n \in \mathbb{Z}_{\geq 0}\}$, on which C acts as $c(\lambda_i)$ (since $L(\lambda_i)$ is a quotient of $M(\lambda_i)$). Thus the polynomial $(t - c(\lambda_1))\ldots(t - c(\lambda_r))$ annihilates C and hence M is a direct sum of C's generalised eigenspaces $M_{(\nu)} = \{m \in M \mid (C - \nu\,\mathrm{id}_M)^k \cdot m = 0 \text{ for some } k \geq 0\}$ for $\mu \in k$ (if each of the λ_i's were different then C would be diagonalisable).
Since $C \in U_q(\mathfrak{sl}_2)$ is central it follows that each $M_{(\nu)}$ is actually a $U_q(\mathfrak{sl}_2)$-submodule of M's, so to argue M's semisimplicity it's enough to show each $M_{(\nu)}$ is a direct sum of its simple $U_q(\mathfrak{sl}_2)$-submodules. Assume thus without loss of generality that $M = M_{(\nu)}$, and hence that the Casimir $C \in U_q(\mathfrak{sl}_2)$ acts as the same scalar on each of M's composition factors, thus M only has one composition factor by our previous remarks: $\lambda_i = \lambda$ for all $i = 1, \ldots, r$.
By the previous proposition we have that all submodules of M are spanned by M's weight vectors, therefore, since each quotient $M_i/M_{i-1} \cong L(\lambda)$ has a one-dimensional weight space relative to the weight λ, it follows that M_λ has dimension precisely r the number of M's composition factors. Let $v_1, \ldots, v_r \in M_\lambda$ be a basis for M's λ-weight space and we thus have a homomorphism of $U_q(\mathfrak{sl}_2)$-modules $\phi : \bigoplus_{i=1}^r L(\lambda) \to M$ mapping surjectively onto M's $U_q(\mathfrak{sl}_2)$-subspace generated by $v_1, \ldots, v_r$. ϕ must be injective since its kernel must be a $U_q(\mathfrak{sl}_2)$-submodule of $\bigoplus_{i=1}^r L(\lambda)$'s, and hence - if non-zero - it must contain a highest weight vector $v \in \ker\phi$ which must of course be a linear combination of each of the highest weight vectors for the direct summands $L(\lambda), \ldots, L(\lambda)$; since these r vectors are send to the linearly independent vectors $v_1, \ldots, v_r$, we get that $\ker\phi$ can have no such highest weight vector and thus $\ker\phi = 0$.
As for surjectivity, note that the $U_q(\mathfrak{sl}_2)$-module $M/\mathrm{im}\,\phi$ must only have copies of $L(\lambda)$ as its composition facts, since it is a quotient of M; however the weight space $(M/\mathrm{im}\,\phi)_\lambda$ is zero dimensional, which indicates by our previous remarks that $M/\mathrm{im}\,\phi$ can't have any composition factors isomorphic to $L(\lambda)$; thus $M/\mathrm{im}\,\phi = 0$ and ϕ is an isomorphism. □

We can now consider the case in which $q^l = 1$ for some $l \geq 0$. Just as we utilised the Casimir's existence in $Z(U_q(\mathfrak{sl}_2))$ to study $U_q(\mathfrak{sl}_2)$'s representation theory previously, in this case it turns out to be quite useful to introduce other distinguished elements in $U_q(\mathfrak{sl}_2)$'s centre.

Remark: We have that $q^l = 1 \implies [l]_q = \frac{q^l - q^{-l}}{q - q^{-1}} = 0$ since $q^{2l} = 1$ and in fact we actually have $[l']_q = 0$ where

$$l' = \begin{cases} \frac{l}{2} & l \text{ even} \\ l & l \text{ odd} \end{cases}$$

This implies by the EF-relations that the elements $E^{l'}, F^{l'}, K^{\pm l'}$ lie in $Z(U_q(\mathfrak{sl}_2))$.
By the PBW theorem it follows that $U_q(\mathfrak{sl}_2)$ has an abelian polynomial subalgebra $P = k[E^{l'}, F^{l'}, K^{\pm l'}] \subseteq U_q(\mathfrak{sl}_2)$ such that $U_q(\mathfrak{sl}_2)$ is a finite P-module (generated by elements $\{E, E^2, \ldots, E^{l'-1}, F, F^2, \ldots, F^{l'-1}, K^{\pm 1}, \ldots, K^{\pm(l'-1)}\}$).

Construction: Let $\lambda \in k^\times$ be any element and consider the Verma module $M(\lambda)$ generated by the highest weight vector $v_\lambda \in M(\lambda)$. The EF-formula implies that $F^{l'}v$ is also highest weight (once again, because $[l']_q = 0$), so we have submodule $N = U_q(\mathfrak{sl}_2)\cdot(F^{l'}v_\lambda - bv_\lambda) \subseteq M(\lambda)$ for any $b \in k$, which is proper since $v_\lambda \notin N$ (in particular, if q is a root of unity $M(\lambda)$ is *never* irreducible). Define the *baby Verma module* as the finite dimensional quotient $M(\lambda,b) := M(\lambda)/N$.
$M(\lambda,b)$ has a weight space decomposition induced by that of $M(\lambda)$ with weights $W(M(\lambda)) = \{\lambda, q^{-2}\lambda, \dots, q^{-2(l'-1)}\lambda\}$ such that each weight space $M(\lambda,b)_{q^{-2i}\lambda}$ is one dimensional and spanned by F^iv_λ. We have the following description of how E,F and $K^{\pm1}$ act on this weight space:

$$E\cdot v_\lambda = 0, \qquad E\cdot F^iv_\lambda = [i]_q\frac{q^{1-i}\lambda - q^{i-1}\lambda^{-1}}{q-q^{-1}}F^{(i-1)}v_\lambda \qquad \text{for } i = 1,\dots,l'-1$$

$$F\cdot F^{l'-1}v_\lambda = bv_\lambda, \qquad F\cdot F^iv_\lambda = F^{i+1}v_\lambda \qquad \text{for } i = 0,\dots,l'-2,$$
$$K^{\pm1}\cdot F^iv_\lambda = (q^{-2i}\lambda)^{\pm1}F^iv_\lambda.$$

From this we can see that if $q^{1-i}\lambda \neq q^{i-1}\lambda^{-1}$ for all $i = 1,\dots,l'$ (i.e. $\lambda \neq \pm q^i$ for any i) then any submodule of $M(\lambda,b)$'s must contain the highest weight vector v_λ and must thus be equal to $M(\lambda,b)$.
Analogously (this time appealing to F's action rather than E's) we have that $b \neq 0$ also implies all submodules of $M(\lambda,b)$ contain v_λ.
So the only case in which $M(\lambda,b)$ could possibly not be irreducible is when $\lambda = \pm q^i$ and $b = 0$. In this case the above formulas show that $F^{i+1}v_\lambda$ is highest weight and generated a submodule isomorphic to $L(q^iq^{-2i-2},\pm) = L(q^{-i-2},\pm)$ whereas the quotient $M(\pm q^i,0)/U_q(\mathfrak{sl}_2)F^{i+1}v_\lambda$ is isomorphic to $L(q^i,\pm)$; note that the sequence

$$0 \to L(q^{-i-2},\pm) \to M(\pm q^i,0) \to L(q^i,\pm) \to 0$$

is non-split.

Classification of irreducible $U_q(\mathfrak{sl}_2)$-modules: Suppose k is algebraically closed, $q^l = 1$ and M a finite-dimensional irreducible $U_q(\mathfrak{sl}_2)$-module. Since each element $E^{l'}, F^{l'}, K^{\pm l'}$ acts on M as a scalar, we have a homomorphism of rings from the polynomial algebra $k[E^{l'}, F^{l'}, K^{\pm l'}] \to k$ to our ground field k, which by Hilbert's Nullstellensatz has some maximal ideal of the form $(E^{l'} - a, F^{l'} - b, K^{\pm l'} - c) \subseteq k[E^{l'}, F^{l'}, K^{\pm l'}]$ as its kernel, with $a, b, \in k$ and $c \in k^\times$.

1. If $a = 0$ then since M has a weight space decomposition the non-zero subspace (because E acts nilpotently) $\ker E$ contains a non-zero weight vector $m \in M$ of weight λ. We thus have a surjection $\phi : M(\lambda) \twoheadrightarrow M$ mapping v_λ to m. Since $F^{l'} - b$ acts as zero on M it follows that ϕ factors through to a homomorphism $M(\lambda,b) = M(\lambda)/(F^{l'}v_\lambda - bv_\lambda)M$ which shows M is a quotient of the baby Verma $M(\lambda,b)$. However, by our previous analysis we have that $M(\lambda,b)$ only has irreducible quotients (including, in some cases, the quotient by the zero subspace) isomorphic to either baby Verma's or standard cyclic irreducible quotients $L(q^i,\pm)$.

2. If $a \neq 0$ and $b = 0$ then the twisted module ${}^{\omega}M$ (where $\omega \in \mathrm{Aut}(U_q(\mathfrak{sl}_2))$ is the Cartan involution) falls into the previous case, thus in this case M is isomorphic to a "*twisted*" baby Verma module ${}^{\omega}M(\lambda,a)$ (since the irreducible quotient modules $L(q^i,\pm)$ satisfy $b = a = 0$.

3. If $a, b \neq 0$ then let $m \in M$ be any weight vector, say of weight $\lambda \in k$. We thus have the l' weight vectors

$$m = \frac{1}{b}F^{l'}m, Fm, F^2m, \dots, F^{l'-1}m$$

of weights respectively $\lambda, q^{-2}\lambda, \dots, q^{-2l'+2}\lambda$ which span an l-dimensional subspace of M's. A simple computation relying on the EF-formula shows that this subspace is indeed a submodule, which must thus coincide with M by irreducibility.

$U_q(\mathfrak{sl}_2)$'s centre

As observed, we have that the element $C = FE + \frac{qK - q^{-1}K^{-1}}{q-q^{-1}} \in U_q(\mathfrak{sl}_2)$ lies in $U_q(\mathfrak{sl}_2)$'s centre. Our objective it to show that infact $Z(U_q(\mathfrak{sl}_2))$ is precisely the k-subalgebra generated by C, when q isn't a root of unity.

Remark: Let $u \in Z(U_q(\mathfrak{sl}_2))$; note that since $[u,K]=0$, we have that u must be an element of degree zero, because it's an eigenvector of eigenvalue q^0 with respect to conjugation by K. Thus expressing u in terms of our PBW theorem yields an expression of the form

$$u = \sum_{r\in\mathbb{Z}_{\geq 0}} F^r u_r E^r, \quad \text{where } h_r \in k[K^{\pm 1}].$$

If we now see how the conditions $Eu = uE$ and $Fu = uF$ reflect on the structure of the u_r's we get

$$Eu = \sum_r EF^r u_r E^r = \sum_r F^r E u_r E^r + [r]_q F^{r-1}[K;1-r]u_r E^r = \sum_r F^r \gamma(u_r) E^{r+1} + [r]_q F^{r-1}[K;1-r]u_r E^r$$

$$= \sum_r F^r \cdot (\gamma(u_r) + [r+1]_q [K;-r]u_{r+1}) \cdot E^{r+1}$$

$$\text{and on the other hand } uE = \sum_r F^r u_r E^{r+1}$$

where $\gamma : k[K^{\pm 1}] \to k[K^{\pm 1}]$ is the ring homomorphism given by $K \mapsto q^{-2}K$. So since the above are both expressions in terms of the PBW basis we have that $u_r = \gamma(u_r) + [r+1]_q[K;-r]u_{r+1}$ for every $r \in \mathbb{Z}_{\geq 0}$. *In particular*, since $q^r \neq 1$ implies $[r+1]_q \neq 0$ for all r, we can always express u_{r+1} in terms of u_r! This shows that the element u_0 totally determines u :) The following proposition follows from these remarks - the introduction of the "shift" $\gamma : k[K^{\pm 1}] \to k[K^{\pm 1}]$ to the "projection" $u = \sum_r F^r u_r E^r \mapsto u_0$ is essentially there just to make the upcoming statement a little cleaner.

Proposition: Let $\xi : Z(U_q(\mathfrak{sl}_2)) \to k[K^{\pm 1}]$ be defined by $u = \sum_r F^r u_r E^r \mapsto \gamma(u_0)$. Then ξ is a linear injective map, called the *Harish-Chandra homomorphism*.

We can now understand further what elements $u \in Z(U_q(\mathfrak{sl}_2))$ look like with the aid of the representation theory we developed.

Proposition: Let $u = \sum_r F^r u_r E^r \in Z(U_q(\mathfrak{sl}_2))$ be a central element. Then $\xi(u) \in k[K^{\pm 1}]$ is a symmetric Laurent polynomial with respect to the automorphism $K \mapsto K^{-1}$ (i.e. if we express $\xi(u) = \sum_{i\in\mathbb{Z}} a_i K^i$ then $a_i = a_{-i}$ for all $i \in \mathbb{Z}$).

Proof: Since u is central and the $U_q(\mathfrak{sl}_2)$-module $M(q^j)$ is generated by one element, u acts as a scalar on $M(q^j)$ which is of course given by $\sum_r F^r u_r E^r \cdot v_{q^j} = u_0 \cdot v_{q^j} = \sum_{i\in\mathbb{Z}} a_i q^{j\cdot i} v_{q^j}$. However, $M(q^j)$ also has a submodule generated by $F^{j+1}v_{q^j}$ isomorphic to $M(q^{-j-2})$ and just as before we get that u acts on this subspace as the scalar $\sum_{i\in\mathbb{Z}} a_i q^{i\cdot(-j-2)}$; it follows that we have an equality:

$$\sum_{i\in\mathbb{Z}} a_i q^{j\cdot i} = \sum_{i\in\mathbb{Z}} a_i q^{-ij-2i}$$

for all $j \geq 0$.
Since the (characters) group homomorphisms

$$\Phi_i : \mathbb{Z} \to k$$
$$j \mapsto q^j$$

are distinct (because q isn't a root of unity), it follows that they are linearly independent by Dirichlet's theorem and thus

$$a_i = q^{-2i} a_{-i}$$

for all i, i.e. $\xi(u) = \gamma(u_0)$ is symmetric.$\square$

Remark: The previous proposition shows that ξ's image in $k[K^{\pm 1}]$ is given by the symmetric Laurent polynomials $k[K^{\pm 1}]^s$ where $s : k[K^{\pm 1}] \to k[K^{\pm 1}]$ maps K to K^{-1}. Note that ξ must actually also be surjective on this subalgebra, because $\xi(C) = \xi(FE + \frac{qK - q^{-1}K^{-1}}{q - q^{-1}}) = \frac{q^3K - q^{-3}K^{-1}}{q-q^{-1}}$ is a degree one-term, and thus all polynomials in $k[K^{\pm 1}]$ can be obtained via the images of linear combinations of C and its powers. Since ξ is injective by the previous lemma, it follows that it is an isomorphism of algebras and thus

$$Z(U_q(\mathfrak{sl}_2)) \cong k[C].$$

Kac/Moody Lie algebras

Definition: Let $A = (a_{i,j})_{i,j} \in M_{n\times n}(\mathbb{C})$ be an $n \times n$ matrix with complex coefficients *of arbitrary rank* (this is the important bit). A *realisation* for A is a triple $(\mathfrak{h}, \pi, \pi^\vee)$ consisting of a finite dimensional complex vector space $\mathfrak{h}$ and subsets of linearly independent vectors of size n

$$\pi = \{\alpha_1, \dots, \alpha_n\} \subseteq \mathfrak{h}^*$$
$$\pi^\vee = \{\alpha_1^\vee, \dots, \alpha_n^\vee\} \subseteq \mathfrak{h}$$

satisfying $\alpha_j(\alpha_i^\vee) = a_{i,j}$.

Remark: There's always a simple way of constructing a realisation for A: up to a reordering of the elements in π and $\pi^\vee$ we can always assume that A is a matrix of the form

$$A = \begin{pmatrix} \mathbf{A}_0 & * \\ * & * \end{pmatrix}$$

where $\mathbf{A}_0$ is an $r \times r$ invertible matrix, r being A's rank.
If we now let $\mathfrak{h} = \mathbb{C}^{2n-r}$ with basis $\epsilon_1, \dots, \epsilon_{2n-r}$, its dual basis $\epsilon_1^*, \dots, \epsilon_{2n-r}^*$, and C the invertible endomorphism of $\mathfrak{h}^*$ given by the matrix (in terms of the basis $\{\epsilon_1^*, \dots, \epsilon_n^*\}$)

$$\begin{pmatrix} \mathbf{A}_0 & * & 0 \\ * & * & \mathrm{id}_{n-r} \\ 0 & \mathrm{id}_{n-r} & 0 \end{pmatrix}$$

then we can set $\alpha_1^\vee = \epsilon_1, \dots, \alpha_n^\vee = \epsilon_n$ and $\alpha_1 = C\epsilon_1^*, \dots, \alpha_n = C\epsilon_n^*$. Since C is invertible (a pretty quick inspection of C's columns shows) it follows that π and $\pi^\vee$ are both subsets of linearly independent vectors, and they evidently satisfy the condition that $\alpha_j(\alpha_i^\vee) = a_{i,j}$ since these are precisely the entries in C's top left $n \times n$ submatrix.

The constructed realisation turns out to have *minimal dimension*.

Proposition: Let $A \in M_{n\times n}(\mathbb{C})$ be any $n \times n$ square matrix with complex coefficients and let r be A's rank. If $(\mathfrak{h}, \pi, \pi^\vee)$ is any realisation for A then $\dim \mathfrak{h} \geq 2n - r$.

Proof: We may complete π and $\pi^\vee$ arbitrarily to bases for $\mathfrak{h}^*$ and $\mathfrak{h}$

$$\alpha_1, \dots, \alpha_n, \alpha_{n+1}, \dots, \alpha_m \in \mathfrak{h}^*$$
$$\alpha_1^\vee, \dots, \alpha_n^\vee, \alpha_{n+1}^\vee, \dots, \alpha_m^\vee \in \mathfrak{h}$$

and thus the matrix $R = (\alpha_j(\alpha_i^\vee))_{i,j} \in \mathcal{M}_{n\times n}(\mathbb{C})$, whose top left $n \times n$ submatrix is equal to A and has rank r, must be invertible (when thought of as a linear endomorphism, any vector in its kernel represents an element in $\mathfrak{h}$ annihilated by every linear functional in $\mathfrak{h}^*$). If we take B to be R's submatrix consisting of R's first n rows (thus including the mentioned submatrix A) we have that R has rank n and m columns, whose first n span a subspace of dimension r. It follows that the remaining $m-n$ columns must span a subspace of dimension *at least* $n-r$ since otherwise B would have rank less than n, thus

$$m - n \geq n - r$$

which is our claim. □

We can now begin to construct the Kac-Moody Lie algebra associated to a Generalised Cartan Matrix A.

Construction: Let $A \in \mathcal{M}_{n\times n}(\mathbb{Z})$ be a Generlised Cartan Matrix (i.e. a matrix with integral coefficients, whose diagonal entries are all equal to 2 and whose coefficient $a_{i,j}$ is zero if and only the reflected one $a_{j,i}$ is also zero) and let $(\mathfrak{h}, \pi, \pi^\vee)$ be a minimal realisation for A with a completed basis $\alpha_1^\vee, \dots, \alpha_n^\vee, \alpha_{n+1}^\vee, \dots, \alpha_m^\vee$ (the following can be done with no alterations even if A isn't a Generalised Cartan Matrix and this is how we introduced the notion in the lectures, but I think that's a bit of a redundant generalisation for our purposes). Define $\widetilde{\mathfrak{g}}(A)$ to be the complex Lie algebra given by generators $\widetilde{e_1}, \dots, \widetilde{e_n}, \widetilde{f_1}, \dots, \widetilde{f_n}, \widetilde{\alpha_1^\vee}, \dots, \widetilde{\alpha_m^\vee}$ subject to to the *(generalised) Chevalley relations*:

$$[\widetilde{\alpha_i^\vee}, \widetilde{\alpha_j^\vee}] = 0,$$
$$[\widetilde{e_i}, \widetilde{f_j}] = \delta_{i,j}\widetilde{\alpha_i},$$
$$[\widetilde{\alpha_i}^\vee, \widetilde{e_j}] = \alpha_j(\alpha_i^\vee)\widetilde{e_j},$$
$$[\widetilde{\alpha_i}^\vee, \widetilde{f_j}] = -\alpha_j(\alpha_i^\vee)\widetilde{f_j}.$$

We couldn't possibly hope for $\widetilde{g}(A)$ to be finite-dimensional, or even or "reasonable" size, since it contains n different copies of $\mathfrak{sl}_2(\mathbb{C})$ given by the triples $\{\widetilde{e_i}, \widetilde{f_i}, \widetilde{\alpha_i^\vee}\}$ which have no relations among them whatsoever. We thus want to consider a particular quotient of $\widetilde{\mathfrak{g}}(A)$'s which will yield our (non-quantised) central object for the course.

Construction: We now construct a representation for $\widetilde{\mathfrak{g}}(A)$ which will help us understand part of its underlying structure. Fix a weight $\lambda \in \mathfrak{h}^*$ and let V be the tensor algebra on the n-dimensional complex vector space $\mathrm{span}_{\mathbb{C}}(v_1,\dots,v_n) \cong \mathbb{C}^n$; we define $\widetilde{\mathfrak{g}}(A)$'s action on V by

$$\widetilde{f_i} \cdot (v_{i_1} \otimes \cdots \otimes v_{i_r}) := v_i \otimes v_{i_1} \otimes \cdots \otimes v_{i_r}$$

$$\widetilde{\alpha_i^\vee} \cdot (v_{i_1} \otimes \cdots \otimes v_{i_r}) = (\lambda - \alpha_{i_1} - \cdots - \alpha_{i_r})(\alpha_i^\vee) \cdot v_{i_1} \otimes \cdots \otimes v_{i_r}$$

$$\widetilde{e_i} \cdot 1 = 0$$

$$\widetilde{e_i} \cdot (v_{i_1} \otimes \cdots \otimes v_{i_r}) = \delta_{i,i_1} \widetilde{\alpha_{i_1}^\vee} \cdot (v_{i_2} \otimes \cdots \otimes v_{i_r}) + \widetilde{f_{i_1}} \widetilde{e_i} \cdot (v_{i_2} \otimes \cdots \otimes v_{i_r})$$

where in the last couple of lines we implicitly define $\widetilde{e_i}$'s action by induction on r.
The above do indeed define a representation for $\widetilde{\mathfrak{g}}(A)$ (albeit a very infinite-dimensional one) by simply checking the above relations one by one.

Remark: The above construction gives an analogue of the *triangular (Cartan) decomposition* for $\widetilde{\mathfrak{g}}(A)$: define $\widetilde{\mathfrak{n}}^+, \widetilde{\mathfrak{n}}^-, \widetilde{h}$ to be the Lie-subalgebras of $\widetilde{\mathfrak{g}}(A)$ generated by $\widetilde{e_i},\dots,\widetilde{e_n}$ for $\mathfrak{n}^+$, $\widetilde{f_1},\dots,\widetilde{f_n}$ for $\mathfrak{n}^-$ and $\widetilde{\alpha_1^\vee},\dots,\widetilde{\alpha_m^\vee}$ for $\widetilde{\mathfrak{h}}$. Our claim is that $\widetilde{\mathfrak{g}}(A) = \widetilde{\mathfrak{n}}^- \oplus \widetilde{\mathfrak{h}} \oplus \widetilde{\mathfrak{n}}^+$.
We evidently have $\widetilde{\mathfrak{g}}(A) = \widetilde{\mathfrak{n}}^- + \widetilde{\mathfrak{h}} + \widetilde{\mathfrak{n}}^+$ since the latter is a Lie-ideal (check the Chevalley relations) containing each of $\widetilde{\mathfrak{g}}(A)$'s natural generators, so all that there is to verify is that the sum is direct. If we suppose $u + h + u' = 0$ where $u \in \widetilde{\mathfrak{n}}^+, h \in \widetilde{h}, u' \in \widetilde{\mathfrak{n}}^-$ then applying the sum to the element $1 \in V$ in the above representation yields

$$0 = u(1) + h(1) + u'(1) = \lambda(h) + u'(1)$$

since $u'(1)$ has positive degree with respect to $V = T^{\otimes}(v_1,\dots,v_n)$ it follows that $\lambda(h) = 0$ and $u'(1) = 0$; since λ was arbitrary in $\mathfrak{h}^*$, we have that $h = 0$ and $u' = 0$.
We also have a nice decomposition into *root spaces* for $\widetilde{\mathfrak{g}}(A)$: we set $\widetilde{\mathfrak{g}}(A)_\alpha := \{u \in \widetilde{\mathfrak{g}}(A) \mid [h,u] = \alpha(h)u \text{ for all } h \in \widetilde{h}\}$ for every $\alpha \in Q$ where $Q := \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_n \subseteq \mathfrak{h}^*$ is the *root lattice*; we thus have that $\mathfrak{h} \subseteq \widetilde{\mathfrak{g}}(A)_0$ and $[e_i, \widetilde{\mathfrak{g}}(A)_\alpha] \subseteq \widetilde{\mathfrak{g}}(A)_{\alpha+\alpha_i}, [f_i, \widetilde{\mathfrak{g}}(A)_\alpha] \subseteq \widetilde{\mathfrak{g}}(A)_{\alpha-\alpha_i}$; thus $\widetilde{\mathfrak{n}}^+ \subseteq \bigoplus_{\alpha\in Q^+} \widetilde{\mathfrak{g}}(A)_\alpha$ and $\widetilde{\mathfrak{n}}^- \subseteq \bigoplus_{\alpha \in Q^-} \widetilde{\mathfrak{g}}(A)_\alpha$. Hence

$$\widetilde{\mathfrak{g}}(A) \supseteq (\bigoplus_{\alpha\in Q^+} \widetilde{\mathfrak{g}}(A)) \oplus (\bigoplus_{\alpha \in Q^-} \widetilde{\mathfrak{g}}(A)) \oplus \widetilde{\mathfrak{g}}(A)_0 \supseteq \widetilde{\mathfrak{n}^+} \oplus \widetilde{\mathfrak{h}} \oplus \widetilde{\mathfrak{n}^-} = \widetilde{\mathfrak{g}}(A).$$

Definition: Let A be a Generalised Cartan Matrix and $\widetilde{\mathfrak{g}}(A)$ as constructed above. Note that the collection of Lie-ideals $\{I \subseteq \widetilde{\mathfrak{g}}(A) \mid I \cap \widetilde{\mathfrak{h}} = (0)\}$ whose intersection with the cartan subalgebra is trivial has a maximal element, namely given by the sum of all such ideals. Let I be this maximal element. The Lie algebra $\mathfrak{g}(A) := \widetilde{\mathfrak{g}}(A)/I$ is called the *Kac-Moody Lie algebra associated to* A. We denote by $p : \widetilde{\mathfrak{g}}(A) \to \mathfrak{g}(A)$ the natural projection map, whose kernel is I, and I^+, I^- the intersections of I with the triangular parts $\widetilde{\mathfrak{n}}^+, \widetilde{\mathfrak{n}}^-$.

We require a few rather technical lemmas and notions. The following generalises the notion of "*highest-weight vector*", and by the following proposition it essentially serves the same purpose.

Definition: Let M be a $\mathfrak{g}(A)$-module. A *pseudoprimitive* element $v \in M$ is a weight vector such that there exists some $\mathfrak{g}(A)$-submodule $N \subseteq M$ not containing v but such that $e_i \cdot v \in N$ for all i.

In other words, v is pseudoprimitive if it is primitive (i.e. of highest weight) in some quotient of M's.

Proposition: Let M be a $\mathfrak{g}(A)$-module admitting a weight-space decomposition $M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda$ such that M's set of weights satisfy the "*mountains-condition*":

$$WT(M) \subseteq \bigcup_{\mu \in X} \{\lambda \in \mathfrak{h}^* \mid \lambda \leq \mu\}$$

for some finite set of weights $X \subseteq \mathfrak{h}^*$. If we let $P(M) \subseteq M$ denote the set of pseudoprimitive vectors in M, then $P(M)$ is a set of generators for M as a $\mathfrak{g}(A)$-module.

Proof: Let P be the $\mathfrak{g}(A)$-submodule of M's generated by $P(M)$. If $P \subsetneq M$ then M/P must contain a highest weight vector, and thus M contains a pseudoprimitive vector which doesn't lie in P. $\square$

Proposition: Let M be a $\mathfrak{g}(A)$-module admitting a weight-space decomposition which satisfies the *mountain-contidion*. Then $P(M)$ is a set of generators for *M as an $\mathfrak{n}^-$-module*.

Proof: Let $P \subseteq M$ be the $\mathfrak{n}^-$-submodule in M generated by pseudoprimitive vectors $P(M)$, and suppose by contradiction that $P \subsetneq M$ is a proper subspace. M is generated as a $\mathfrak{g}(A)$-module by pseudoprimitive vectors by the previous proposition, we must have $U(\mathfrak{g}(A))v$ contains some element $u \cdot v$ outside of P for some $v \in P(M)$. We can write $u \in U(\mathfrak{g}(A))$ as $u' + u''$ where $u' \in U(\mathfrak{n}^-)$ and $u'' = u_1 x \in U(\mathfrak{n}^-)U(\mathfrak{n}^+)\mathfrak{n}^+$ - $u_1 \in U(\mathfrak{n}^-)$ and $x \in U(\mathfrak{n}^+)\mathfrak{n}^+$ - and since $u' \cdot v \in P$ it must be that $u'' \cdot v \notin P$; also, since $u_1 \in U(\mathfrak{n}^-)$ it must also be that $x \cdot v \notin P$. Since $x \cdot v$ isn't pseudoprimitive, it follows that $x \cdot v$ must lie in the $U(\mathfrak{g}(A))$-module generated by the vectors $e_1 x \cdot v, \ldots, e_n x \cdot v$; so the $U(\mathfrak{g}(A))$-submodule given by

$$U(\mathfrak{g}(A))\mathfrak{n}^+ x \cdot v = U(\mathfrak{n}^-)U(\mathfrak{n}^+)\mathfrak{n}^+ x \cdot v \subseteq M$$

contains vectors which don't lie in P (namely $x \cdot v$) and hence $U(\mathfrak{n}^+)\mathfrak{n}^+ x \cdot v$ must contain some vector which doesn't lie in P (since P is an $\mathfrak{n}^-$-submodule), say $x' x \cdot v \notin P$. We can thus iterate this procedure and get a sequence of weight vectors

$$v, v_1, v_2, v_3, \ldots$$

where $v_i \notin P$ for every i and $v_i = x v_{i-1}$ for some $x \in U(\mathfrak{n}^+)\mathfrak{n}^+$ implying thus that v_i's weight is always strictly greater than that of v_{i+1}. By our assumption on M's weight structure, sure an infinite sequence of weight vectors can't occur. Thus $P = M$. $\square$

Definition: Let $A \in \mathcal{M}_{n\times n}(\mathbb{Z})$ be a generalised cartan matrix; A is said to be *symmetrisable* if there exists some diagonal matrix $D \in \mathcal{M}_{n\times n}(\mathbb{Z})$ with integral coefficients such that DA is symmetric.

The above notion of *symmetrisability* reflects the recoupment in the classical case of semisimple Lie algebras $\mathfrak{g}$ of the matrix $(\kappa(\alpha_i^\vee, \alpha_j^\vee))_{i,j}$ where κ is the Killing bilinear form restricted to the Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ from the matrix of Cartan integers $(\langle \alpha_i, \alpha_j \rangle)_{i,j}$: recall that to recover the Killing κ on $\mathfrak{h}$ form from $(\langle \alpha_i, \alpha_j \rangle)_{i,j} = (\frac{2\kappa(\alpha_i^\vee, \alpha_j^\vee)}{\kappa(\alpha_j^\vee, \alpha_j^\vee)})$ all one has to do it multiply the matrix of Cartan integers on the right by the diagonal matrix whose diagonal entries are $\frac{\kappa(\alpha_1^\vee, \alpha_1^\vee)}{2}, \ldots, \frac{\kappa(\alpha_n^\vee, \alpha_n^\vee)}{2}$.
In some sense, having a bilinear form on $\mathfrak{h}$ turns out to be equivalent to requiring the existence of such a matrix *"symmetriser" D*.

To work with Kac-Moody Lie algebras more concretely, we will need to understand I's structure in a more explicit manner: this is very possible by means of the mentioned bilinear form on $\mathfrak{h}$, in this extra hypothesis of *symmetrisability*.

Definition: Suppose A is a symmetrisable Generalised Cartan Matrix, D a diagonal matrix with integral coefficients $d_1, \ldots, d_n$ such that DA is symmetric. If we pick a complement $h' \subseteq \mathfrak{h}$ to the subspace $\mathfrak{h}_0$ in $\mathfrak{h}$ spanned by the roots $\alpha_1^\vee, \ldots, \alpha_n^\vee$ then we can naturally define a non-degenerate s ymmetric bilinear form on $\mathfrak{h}$ by setting

$$(\alpha_i^\vee, \alpha_j^\vee) = d_i \langle \alpha_j, \alpha_i^\vee \rangle = d_i a_{i,j},$$

$$(\alpha_i^\vee, x) = d_i \alpha_i(x), \ \ x \in \mathfrak{h}'$$

$$(x, y) = 0, \ \ x, y \in \mathfrak{h}'.$$

Proposition: Let $A \in \mathcal{M}_{n\times n}(\mathbb{Z})$ be a symmetrisable generalised cartan matrix, $\widetilde{\mathfrak{g}}(A) \twoheadrightarrow \mathfrak{g}(A)$ the natural projection with kernel $I = I^+ + I^-$ where $I^\pm = I \cap \widetilde{\mathfrak{n}}^\pm$. We have an injective homomorphism of $\mathfrak{g}(A)$-modules

$$I^-/[I^-, I^-] \xrightarrow{\cong} \bigoplus_{i=1,\ldots,n} M(-\alpha_i)$$

where $M(\lambda)$ is the Verma $\mathfrak{g}(A)$-module of highest weight $\lambda \in \mathfrak{h}^*$.

Proof: Consider the $\widetilde{\mathfrak{g}}(A)$-Verma module $\widetilde{M}(0)$ of highest weight zero, and define its suspace

$$\widetilde{J}(0) := \sum_{i=1}^n U(\widetilde{n}^-)\widetilde{f}_i \cdot v_0 \subseteq \widetilde{M}(0).$$

A simple computation shows $\widetilde{J}(0)$ is actually a $\widetilde{\mathfrak{g}}(A)$-submodule (it's evidently stable under the action of $\mathfrak{n}^-$ and $\mathfrak{h}$, and a quick induction on the number of factors shows that it is stable under the action of $U(\mathfrak{n}^+)$ as well - using that v_0's weight is zero and thus $\alpha_i^\vee \cdot v_0 = 0$ for all i). The vectors $\widetilde{f_1} \cdot v_0, \ldots, \widetilde{f_n} \cdot v_0 \in \widetilde{J}(0)$ are evidently highest weight, and thus induce a homomorphism of $\widetilde{\mathfrak{g}}(A)$-modules

$$\bigoplus_{i=1,\ldots,n} \widetilde{M}(-\alpha_i) \twoheadrightarrow \widetilde{J}(0)$$

which must be an isomorphism by the PBW theorem. Extension of scalars yields

$$\bigoplus_{i=1,\dots,n} M(-\alpha_i) \cong U(\mathfrak{g}(M)) \otimes_{U(\widetilde{\mathfrak{g}}(A))} \widetilde{J}(0).$$

So in our proposition we can actually replace this direct sum of Verma modules $\bigoplus_{i=1,\dots,n} M(-\alpha_i)$ with $U(\mathfrak{g}(M)) \otimes_{U(\widetilde{\mathfrak{g}}(A))} \widetilde{J}(0)$. Define a map (of vector spaces at the moment)

$$\phi : u \in I^-/[I^-,I^-] \mapsto 1 \otimes uv_0 \in U(\mathfrak{g}(M)) \otimes_{U(\widetilde{\mathfrak{g}}(A))} \widetilde{J}(0)$$

which is well defined since $u \in I^- \subseteq \widetilde{\mathfrak{n}}^-$ and $1 \otimes [u,u']v_0 = p(u) \otimes u'v_0 - p(u') \otimes uv_0 = 0$ since $p(u) = p(u')$ for elements $u, u' \in I^-$.

The kernel of the lifted map $\overline{\phi} : I^- \to U(\mathfrak{g}(M)) \otimes_{U(\widetilde{\mathfrak{g}}(A))} \widetilde{J}(0)$ evidently contains $[I^-,I^-]$ by our above computation, so to show ϕ's injective we need to show this kernel is precisely $[I^-,I^-]$. Let $\pi : U(\widetilde{\mathfrak{n}^-}) \to U(\mathfrak{n}^-)$ be the projection induced by the quotient map $\widetilde{\mathfrak{n}^-} \twoheadrightarrow \widetilde{\mathfrak{n}^-}/I \cong \mathfrak{n}^-$. π's kernel is, by one of the exercise sheets, the extended ideal $I^-U(\widetilde{\mathfrak{n}^-})$. If we think of I^- as being the *degree one terms* in $U(I^-)$ then a generic element in $\ker\overline{\phi}$ is of the form

$$\sum_{i=1}^n u_i f_i \in \ker\phi$$

where $\sum_{i=1}^n \pi(u_i) \otimes \widetilde{f}_i v_0 = 0$ and, under the above isomorphism $U(\mathfrak{g}(M)) \otimes_{U(\widetilde{\mathfrak{g}}(A))} \widetilde{J}(0) \cong \bigoplus_{i=1}^n M(-\alpha_i)$ we get that $\sum_{i=1}^n \pi(u_i) v_{-\alpha_i} = 0$ and thus $\pi(u_i) = 0$ for every i since the $v_{-\alpha_i}$ lie in distinct subrepresentations. So we get $u_i \in \ker\pi = U(\widetilde{n}^-)I^-$ and thus

$$\sum_{i=1}^n u_i f_i \in I^-U(\mathfrak{n}^-) \cap I^-.$$

However, this last Lie-subalgebra of I^-'s is precisely $[I^-,I^-]$:)

Lemma: Let $L' \subseteq L$ be Lie algebras, L' a subalgebra of L's. Then $L'U(L) \cap L' = [L'L']$.

Proof: Evidently $[x,y] \in L'U(L) \cap L'$ for every $x,y \in L'$ essentially by definition of $L' \subseteq L$ being a Lie-subalgebra. For the converse, let $\{x_1,\dots,x_r\} \subseteq L'$ be a basis, and we may extend to a basis for L as $\{x_1,\dots,x_r,y_{r+1},\dots,y_t\} \subseteq L$ which induces a PBW-basis for $U(L)$ given by monomials respecting the expressed order.
Let $\sum_{(a)} b_{(a)} x_1^{a_1} \dots x_r^{a_r} y_{r+1}^{a_{r+1}} \dots y_t^{a_t} \in L'U(L)L \cap L' \implies a_{r+1} = \dots = a_t = 0$ and $a_1 + \dots + a_r \geq 2$ for every (a) such that $b_{(a)}$ is non-zero. So we get

$$L'U(L)L \cap L' \subseteq ....$$

Proof of Serre relations: Let $J \subseteq I \subseteq \widetilde{\mathfrak{g}}(A)$ be the ideal generated by the Serre relations, and suppose by contradiction that I/J is a non-zero vector space. By the existence of the Cartan involution, we get that I^-/J^- is non-zero, and I^-/J^-'s weight space decomposition is made up of weights contained in Q^- (since such is true for I^-).
Furthermore, evidently the roots $-\alpha_i$ can't occur as weights for I^-/J^- since these would produce elements in the Cartan (because such a weight space would also imply $(I^+/J^+)_{\alpha_i} \neq 0$ for the same i).
Let $\alpha = \sum_i c_i \alpha_i \in Q^+$ be such that $(I^-/J^-)_{-\alpha} \neq 0$ and $\mathrm{ht}(\alpha) = \sum_i c_i \in \mathbb{Z}_{\geq 2}$ is minimal. It follows that (by the Braid group action) $(I^-/J^-)_{-s_{\alpha_i}(\alpha)} \neq 0$ for all i and thus $\mathrm{ht}(\alpha) \leq \mathrm{ht}(s_{\alpha_i}(\alpha))$. Computing these numbers explicitly yields

$$\sum_j c_j \leq \sum_j c_j) \sum_j c_j \langle \alpha_j, \alpha_i \rangle \implies \sum_j c_j \langle \alpha_j, \alpha_i \rangle \leq 0$$

thus

$$(\alpha,\alpha) = \sum_i c_i (\sum_j c_j (\alpha_j, \alpha_i)) = \sum_i \frac{c_i(\alpha_i,\alpha_i)}{2} \sum_j c_j \langle \alpha_j, \alpha_i \rangle \leq 0$$

However! We have that $2(\alpha,\rho) = \sum_j 2c_j(\alpha_i,\rho) \geq 0 \implies (\alpha,\alpha) \neq 2(\alpha,\rho)$.

References

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