The de Rham comparison for rigid-analytic varieties

SoSe 2023, lectured by Johannes Anschütz

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Contents

These are notes taken from this semester's course on the de Rham comparison theorem for rigid analytic varieties, taught by Johannes Anschütz. They are merely an attempt at typing up the contents of the course and by no means do they exhaust what was discussed in the lectures (especially with what concerns intuition and the significance of theorems). I, the note-writer, take full responsibility for any inaccuracies herein, in the results and their interpretations; should you find any mistakes or have suggestions for improving the exposition I'd very much appreciate an email with your remarks :) [1]

I wholeheartedly thank Johannes Anschütz for thoroughly reviewing these and helping me understand this (rather stunning) theory.

1 6th of April, motivation and main aim of the course

Today we'll be discussing some motivation for the theorem in the title. Suppose X is Kähler compact complex manifold (for instance, a closed subvariety of $\mathbf{C}P^n$). In this setting we have many different cohomology theories for X and comparison isomorphisms between them.

1. We have the comparison between *singular* and *sheaf* cohomology with constant coefficients

$$H^*_{\mathrm{sing}}(X, \mathbf{Z}) \cong H^*(X, \underline{\mathbf{Z}}).$$

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This follows from arguing that the *sheafification* of the complex of singular cochains forms a flasque resolution of the constant sheaf $\underline{\mathbf{Z}}$ on X, discussed in the exercise sheets for Algebraic Geometry 2.

2. Sheaf cohomology turns out to be isomorphic to X's *smooth de Rham cohomology* when we pass to real coefficients, by X's structure as a *real* manifold

$$H^*(X,\underline{\mathbf{Z}}) \otimes_{\mathbf{Z}} \mathbf{R} \cong H^*(X,\underline{\mathbf{R}}) \cong H^*_{\mathrm{dR},\mathbf{R}}(X) := \mathcal{H}^*(\Gamma(X,\mathcal{A}^{\bullet}_{X,\mathbf{R}}))$$

following from the Poincaré lemma, and the fact that sheaves of $C^\infty(X,\mathbf{R})$-modules are always flasque by the existence of partitions of unity. The above isomorphism works similarly for complex coefficients (with no holomorphic condition imposed) - as seen in last semester's course on étale cohomology.

3. The smooth de Rham cohomology can be compared to *holomorphic de Rham cohomology:* if

$$\Omega^{\bullet}_{X/\mathbf{C}} := [\Omega^0_{X/\mathbf{C}} \xrightarrow{d} \Omega^1_{X/\mathbf{C}} \xrightarrow{d} \Omega^2_{X/\mathbf{C}} \to \dots]$$

denotes the complex of sheaves of holomorphic differentials on X, then the canonical inclusion

$$\Omega^{\bullet}_{X/\mathbf{C}} \to \mathcal{A}^{\bullet}_{X,\mathbf{C}}$$

is a quasi-isomorphism - thus

$$H^*_{\mathrm{dR}}(X) := \mathcal{H}^*(R\Gamma(X,\Omega^{\bullet}_{X/\mathbf{C}})) \cong H^*_{\mathrm{dR},\mathbf{R}}(X) \otimes_{\mathbf{R}} \mathbf{C}.$$

In particular, we get a quasi isomorphism

$$\underline{\mathbf{C}} \cong \Omega^{\bullet}_{X/\mathbf{C}}.$$

The proof, found in [Voi], uses the exactness of the sequence

$$0 \to \Omega^p_{X/\mathbf{C}} \to \mathcal{A}^{p,0}_{X,\mathbf{C}} \xrightarrow{\overline{\partial}} \dots \to \mathcal{A}^{p,q}_{X,\mathbf{C}}$$

where $\mathcal{A}^{p,q}_{X,\mathbf{C}}$ is the sheaf of smooth (p,q)-forms, which is a *holomorphic version of the Poincaré lemma*; then one can use the spectral sequence associated to the double complex $(\mathcal{A}^{p,q}_{X,\mathbf{C}})_{p,q}$ defined by the differentials ∂ and $\overline{\partial}$, whose total complex is the (smooth) de Rham complex $\mathcal{A}^{\bullet}_{X,\mathbf{C}}$.

4. The following comparison is particularly profound; we have a comparison between holomorphic de Rham cohomology and Hodge cohomology.

Theorem 1.1 (Hodge decomposition)**.** *If X is as above, then for $n \geq 0$ there exists a natual decomposition*

$$H^n_{\mathrm{dR}}(X) \cong \bigoplus_{i+j=n} H^i(X,\Omega^j_{X/\mathbf{C}}).$$

The *Hodge filtration* on $H^*_{\mathrm{dR}}(X)$ is defined by

$$\mathrm{Fil}^k(H^n_{\mathrm{dR}}(X)) = \mathrm{Im}(\mathcal{H}^n(0 \to \dots \to 0 \to \Omega^k_{X/\mathbf{C}} \xrightarrow{d} \Omega^{k+1}_{X/\mathbf{C}} \to \dots) \to \underbrace{\mathcal{H}^n(\Omega^{\bullet}_{X/\mathbf{C}})}_{:=H^n_{\mathrm{dR}}(X)}),$$

i.e. as the image in $H^n_{\mathrm{dR}}(X)$ of the cohomology of the k-th ('*stupid*') truncation of the de Rham complex. Then by the Hodge decomposition this identifies with

$$\bigoplus_{i+j=n,\, j\geq k} H^i(X,\Omega^j_{X/\mathbf{C}});$$

in particular,

$$\mathrm{gr}^k(H^n_{\mathrm{dR}}(X)) \cong H^{n-k}(X, \Omega^k_{X/\mathbf{C}}).$$

This decomposition gives access to techniques which are extensively used to study complex varieties, since the integral singular cohomology maps to the one with complex coefficients (for instance, via the sheaf-theoretic interpretations) and the kernel is given precisely by the torsion $\mathrm{Tor}(H^n(X,\mathbf{Z}))$ - the inclusion $H^n(X,\mathbf{Z})/\mathrm{Tor}(H^n(X,\underline{\mathbf{Z}})) \subseteq H^n(X,\underline{\mathbf{C}})$ defines a lattice and the Hodge decomposition provides a *Hodge structure* on the vector spaces $H^n(X,\underline{\mathbf{C}})$; all these additional structures on X's singular cohomology provide methods for considering X's *deformation theory*, since one can study how the points on the flag variety on the vector space $H^n(X,\underline{\mathbf{C}})$ corresponding to the Hodge filtration above vary as X varies in its deformation class, for instance - indeed, it turns out that for certain classes of Kähler varieties, such as K3 surfaces or abelian varieties, variations of these Hodge filtrations *correspond* to deformations of X.

The goal is to see if one can deduce results akin to these comparisons in the setting of rigid analytic varieties over p-adic fields K - the interesting new character which appears in the picture is the action of the absolute Galois group $G_K = \mathrm{Gal}(\overline{K}/K)$.

Definition 1.2. 1. A *non-archimedean field* is a non-discrete complete topological field K whose topology is induced by a non-archimedean norm

$$|-| : K \to \mathbf{R}_{\geq 0}$$

satisfying

- $|x| = 0 \iff x = 0, |1| = 1,$
- $|xy| = |x| \cdot |y|,$
- $|x+y| \leq \max\{|x|,|y|\}.$

2. A p-adic field is a complete, discretely valued field K of characteristic zero whose residue field is perfect of characteristic p.

Example 1.3. • $\mathbf{Q}_p$ and all its finite extensions are p-adic,

- if K is any non-archimedean field and L/K is algebraic, there's a unique extension of the $|-| : K \to \mathbf{R}_{\geq 0}$ to L defining a non-archimedean field structure on the completion of L.
- By the previous part, if K is p-adic then the completion of K's maximal unramified extension K^{ur} is also p-adic, whereas the completion of $\overline{K}$ is just non-archimedean since it isn't discretely valued.

A major goal of this course will be a (sketch of the) proof of the following theorem, an analogue of Theorem 1.1.

Theorem 1.4 (p-adic Hodge decomposition). *If K is a p-adic field, X a proper, smooth scheme over K and $n \geq 0$ then there exists a narual G_K-equivariant isomorphism*

$$H^n_{\text{ét}}(X_{\overline{K}}, \mathbf{Z}_p) \otimes_{\mathbf{Z}_p} C \cong \bigoplus_{i+j=n} H^i(X, \Omega^j_{X/K}) \otimes_K C(-j)$$

where $C := \widehat{\overline{K}}$ and $G_K = \mathrm{Gal}(\overline{K}/K)$.

Remark 1.5. 1. The definition of the étale cohomology group on the left is as the inverse limit

$$H^*_{\text{ét}}(X_{\overline{K}}, \mathbf{Z}_p) := \varprojlim_k H^*_{\text{ét}}(X_{\overline{K}}, \mathbf{Z}/p^k)$$

where each component $H^*_{\text{ét}}(X_{\overline{K}}, \mathbf{Z}/p^k)$ is actually a finite group (recall that the constant sheaves $\mathbf{Z}/p^k$on X's étale site are constructible and the lower-shriek functor preserves constructibility - in particular, the *proper pushforward* map $X \to \operatorname{Spec} K$ preserves constructible sheaves, since it agrees with the lower-shriek functor; constructible sheaves on $\operatorname{Spec} K$ are of course just finite abelian groups with a continuous G_K-action).

2. The Tate-twists on the right describe the Galois action:

$$\mathbf{Q}_p/\mathbf{Z}_p \cong \mu_{p^\infty}(C) := \left\{ x \in C \mid x^{p^k} = 1 \text{ for some } k \geq 1 \right\}.$$

The above set is stable under the natural G_K action of course and thus we get a continuous Galois-character $\chi : G_K \to \operatorname{Aut}(\mu_{p^\infty}(C)) \cong \mathbf{Z}_p^*$.
Equivalently, the Tate twist can be defined by the Tate-module

$$\mathbf{Z}_p(1) = T_p(\mu_{p^\infty}(C)) := \varprojlim_n (\ldots \to \mu_{p^n}(C) \xrightarrow{p} \mu_{p^{n-1}}(C) \to \ldots \xrightarrow{p} \mu_p(C)) \cong \mathbf{Z}_p$$

(where the last isomorphism is as abelian groups) and thus we get a continuous character $\chi : G_K \to \operatorname{Aut}(\mathbf{Z}_p(1)) \cong \mathbf{Z}_p^*$, which agrees with the one defined above. We then set $\mathbf{Z}_p(j) := \mathbf{Z}_p(1)^{\otimes j}$ for $j \geq 0$ and $\mathbf{Z}_p(-j) := \operatorname{Hom}_{\mathbf{Z}_p}(\mathbf{Z}_p(j), \mathbf{Z}_p)$.

3. If we take $X = \mathbf{P}^1_K, n = 2$, then we have an isomorphism of G_K-representations

$$H^2_{\text{ét}}(X_{\overline{K}}, \mathbf{Z}_p) \cong \mathbf{Z}_p(-1)$$

because $H^2_{\text{ét}}(X_{\overline{K}}, \mathbf{Z}_p) \otimes_{\mathbf{Z}_p} \mathbf{Z}_p(1) \cong H^2_{\text{ét}}(X_{\overline{K}}, \mathbf{Z}_p(1)) \xrightarrow{\cong} \mathbf{Z}_p$ by the Kummer exact sequence. More generally, we have the equality

$$H^{2d}_{\text{ét}}(X_{\overline{K}, \mathbf{Z}_p}) \cong \mathbf{Z}_p(-d)$$

whenever X is a geometrically connected K-variety of dimension d. Whereas on the right hand-side of the p-adic Hodge decomposition for $X = \mathbf{P}^1_K$ we see the only non-trivial component is

$$H^1(\mathbf{P}^1_K, \Omega^1_{X/K}) \cong K.$$

So the Galois actions indeed line up when the correct Tate twists are applied.

It's also important to mention the fundamental computation of Tate:

Theorem 1.6 (Tate). *We have an isomorphism*

$$H^i_{\text{cts}}(G_K, C(j)) = \begin{cases} K & j = 0, i = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

and, in particular, $C(j) \cong C(j')$ only when $j = j'$. Hence the Tate twists are required *in the p-adic Hodge decomposision.*

The course will be to develop the framework to prove the p-adic Hodge decomposition in the setting of rigid analytic varieties. We're interested in comparing $H^*_{\text{ét}}(X_{\overline{K}}, \mathbf{Z}_p)$ endowed with its natural continuous G_K-representation with $H^*_{\text{dR}}(X)$ as a filtered K-vector space. At the moment, it seems rather unclear how one should jump from objects which are inherently just *continuous G_K-representations* - the étale cohomology groups $H^*_{\text{ét}}(X_{\overline{K}}, \mathbf{Z}_p)$ - to *filtered K-vector spaces* - the de Rham cohomology K-vector spaces $H^*_{\text{dR}}(X)$ (note that the Hodge filtration is defined in this setting in the same way). The way this will be argued is by an idea of Fointaine: we introduce a field B_{dR}, called the *field of p-adic periods*, which is functorially attached to C (and thus G_K has an action on B_{dR}). This field will satisfy the properties

- $B_{\mathrm{dR}} \cong C(\!(u)\!)$ (albeit non G_K-equivariantly, when G_K acts on $C(\!(t)\!)$ via the action on coefficients of Laurent series),
- it possesses a G_K-stable *lattice* $B_{\mathrm{dR}}^+ \subset B_{\mathrm{dR}}$ which is a complete discrete valuation ring (non G_K-equivariantly) isomorphic to $C[\![u]\!]$.
- We have an isomorphism of G_K-representations $\mathrm{gr}^* B_{\mathrm{dR}} \cong \bigoplus_{j \in \mathbf{Z}} C(j)$,
- $(B_{\mathrm{dR}})^{G_K} = K$.

If we define, for any $\mathbf{Z}_p$-linear continuous G_K-representation V (such as the étale cohomology groups above)

$$D_{\mathrm{dR}}(V) := (V \otimes_{\mathbf{Z}_p} B_{\mathrm{dR}})^{G_K}$$

we get a filtered K-vector space by all the structure B_{dR} described by the properties above.

The main aim of the course will be the proof of the following result.

Theorem 1.7 (the de Rham comparison). *Let X/K be a proper smooth rigid-analytic variety and $n \geq 0$. Then there exists a natural filtered G_K-equivariant isomorphism*

$$H^n_{\text{ét}}(X_C, \mathbf{Z}_p) \otimes_{\mathbf{Z}_p} B_{\mathrm{dR}} \cong H^n_{\mathrm{dR}}(X) \otimes_K B_{\mathrm{dR}}.$$

Remark 1.8. Because the isomorphism is as *filtered* G_K-representations, by passing to the associated graded vector spaces one recovers the p-adic Hodge decomposition from Theorem 1.4.

Remark 1.9. An obstruction to proving this lemma is the total loss of a direct analogue of the Poincaré lemma: for instance, if $K\langle T\rangle$ is the Tate algebra of analytic functions on the closed unit disc $\mathbf{B}_K = \mathrm{Spa}(K\langle T\rangle, \mathcal{O}_K\langle T\rangle)$ then the de Rham complex becomes

$$\Omega^\bullet_{\mathbf{B}_K/K} = [K\langle T\rangle \xrightarrow{d} K\langle T\rangle\, dT]$$

where

$$d(\sum_{i=0}^{\infty} a_i T^i) = \sum_{i=1}^{\infty} i \cdot a_i T^{i-1}$$

and, in an attempt to *integrate* these analytic functions to prove the (classic) equality of closed one-forms with the exact ones, we're forced to consider power series of the form

$$\sum_{i=0}^{\infty} \frac{b_i}{i} T^i$$

where $\sum_{i=0}^{\infty} b_i T^i dT$ is an - automatically closed - one-form. Although in classic complex geometry this produces no problem, in this setting we have huge convergence issues since dividing the i-th term of the power series $\sum_{i=0}^{\infty} b_i T^i$ by i makes the terms far larger (since $p^{-m} \to \infty$ as $m \to \infty$).

To prove an analogue of the Poincaré lemma in this setting we'll be forced to go pass the p-adic period ring B_{dR} and perfectoid spaces.

2 13th of April, Huber rings

Remark 2.1. In Remark 1.8 we mentioned that the de Rham comparison theorem implies the p-adic Hodge decomposition from Theorem 1.4: explicitly put, by taking the k-th graded component of the right hand side we see

$$\mathrm{gr}^k(H^n_{\text{ét}}(X_C, \mathbf{Z}_p) \otimes_{\mathbf{Z}_p} B_{\mathrm{dR}}) \cong H^n_{\text{ét}}(X_{\mathbf{C}}, \mathbf{Z}_p) \otimes \underbrace{\mathrm{gr}^k B_{\mathrm{dR}}}_{\cong C(k)}$$

whereas the filtration on left hand side is explicitly given by

$$\mathrm{fil}^k(H^n_{\mathrm{dR}}(X) \otimes_K B_{\mathrm{dR}}) := \sum_{i+j=k} \mathrm{fil}^i(H^n_{\mathrm{dR}}(X)) \otimes_K \mathrm{fil}^j(B_{\mathrm{dR}})$$

hence

$$\mathrm{gr}^k(H^n_{\mathrm{dR}}(X) \otimes_K B_{\mathrm{dR}}) \cong \bigoplus_{i+j=k} \mathrm{gr}^i(H^n_{\mathrm{dR}}(X)) \otimes_K \mathrm{gr}^j(B_{\mathrm{dR}}) \cong \bigoplus_{i+j=k} H^{n-i}(X, \Omega^i_{X/K}) \otimes_K C(j).$$

Setting $k = 0$ thus yields

$$H^n_{\text{ét}}(X_C, \mathbf{Z}_p) \otimes_{\mathbf{Z}_p} C \cong \bigoplus_i H^{n-i}(X, \Omega^i_{X/K}) \otimes_K C(-i).$$

Today we discuss the algebraic setup for the sorts of geometric spaces concerned in Theorem 1.7.

Definition 2.2. A *Huber ring* is a topological ring A admitting an open subring $A_0 \subset A$ such that the subspace topology on A_0 is I-adic for a finitely generated ideal $I \subset A_0$. A morphism between Huber rings A and B is a continuous ring homomorphism $A \to B$.

Remark 2.3. If A is Huber, then there exists a fundamental system of neighbourhoods of 0 which are *subgroups* - note that these aren't necessarily ideals in A.

Example 2.4.

1. Any ring A can be endowed with the discrete topology making it Huber, since one can take $A = A_0$ and $I = (0)$.

2. If A is any ring and $I \subset A$ is a finitely-generated ideal then A is a Huber ring when endowed with the I-adic topology.

3. Let A_0 be any ring, $g \in A_0$ a non-zero divisor and set $A := A_0[g^{-1}]$. Then there exists a unique topology on A such that A is a topological ring and $A_0 \subset A$ is an open subring endowed with the g-adic topology (indeed, since the topology is determined by requiring $A_0 \subseteq A$ to be open and its topology g-adic, showing that this defines a topological ring structure boils down to arguing that the multiplication-by-g map defines a homeomorphism $A \to A$).

4. As a concrete example of the above situation, one can take $A = \mathbf{Q}_p \supset A_0 = \mathbf{Z}_p$ and $g = p \in \mathbf{Z}_p$. More generally, one can take K a non-archimedean field and set $A = K \supset A_0 = \mathcal{O}_K$ and $g = \pi$ where $\pi \in \mathcal{O}_K$ is such that $0 < |\pi| < 1$.

5. If $(A, |-|)$ is a K-Banach algebra, we can set $A_0 \subset A$ as the unit ball in A (which is a subring since K is non-archimedean) and let $g \in K^\times$ be any non-zero element of norm strictly less than 1 (note that this forces the injection $A_0[g^{-1}] \hookrightarrow A$ to be an isomorphism).

6. For any Huber ring A we can define its *completion* $\widehat{A}$; explicitly, $\widehat{A} := A \otimes_{A_0} \widehat{A}_0$ where $A_0 \subset A$ is a fixed ring of definition and its completion is taken with respect to an ideal of definition.

7. If K is a non-archimedean field, and $\pi \in \mathfrak{m}_K \setminus \{0\}$ we can consider the polynomial ring

$$A_0 := \mathcal{O}_K[T_1, \dots, T_n]$$

with the π-adic topology and $A = A_0[\pi^{-1}] = K[T_1, \dots, T_n]$ as in example 3. above. Its completion is the *Tate algebra over K*, explicitly described by

$$\mathcal{O}_K\langle T_1, \dots, T_n\rangle = \left\{ \sum_{i_1, \dots, i_n \geq 0} a_{i_1, \dots, i_n} T_1^{i_1} \dots T_n^{i_n} \mid a_{i_1, \dots, i_n} \in \mathcal{O}_K, |a_{i_1, \dots, i_n}| \to 0 \text{ as } |i_1, \dots, i_n| \to \infty \right\},$$

$$K\langle T_1, \dots, T_n\rangle = \left\{ \sum_{i_1, \dots, i_n \geq 0} a_{i_1, \dots, i_n} T_1^{i_1} \dots T_n^{i_n} \mid a_{i_1, \dots, i_n} \in K, |a_{i_1, \dots, i_n}| \to 0 \text{ as } |i_1, \dots, i_n| \to \infty \right\}.$$

Exercise 2.5. Set $|\sum_i a_{i_1, \dots, i_n} T_1^{i_1} \dots T_n^{i_n}| := \max |a_{i_1, \dots, i_n}|$ on the Tate algebra $K\langle T_1, \dots, T_n\rangle$. Show that this defines a multiplicative norm, defining the topology on $K\langle T_1, \dots, T_n\rangle$.

The exercise shows that Tate algebras indeed fall under the family of those described in Example 5.

Remark 2.6. One of the interests in introducing adic spaces is that one can construct a fully faithful embedding from the categories of schemes and a suitable subcategory of the category of formal schemes into that of adic spaces - examples 1. and 2. from Example 2.4 describe the essential images of the *affine* objects of these two categories respectively.

Definition 2.7. Let A be a Huber ring and $S \subset A$ a subset. Then $S \subset A$ *bounded* if for all open neighbourhoods U of 0 in A there exists an open neighbourhood V of 0 such that $S \cdot V \subset U$.

In the above definition we use the notation

$$S \cdot V := \left\{ \sum_i \pm s_i \cdot v_i \mid s_i \in S, v_i \in V \right\}.$$

We remark that a priori it's unclear if, in the setting of example 5. from Example 2.4, the notion of *bounded subsets* agrees with that from the defining structure of Banach algebras - namely, in a Banach algebra A a subset $X \subset A$ is *bounded* if the norms of its elements form a bounded subset of $\mathbf{R}_{\geq 0}$. The following lemma settles this potential mismatch.

Lemma 2.8. *Let $A = A_0[g^{-1}]$ be a Huber ring as in Example 3. Then $S \subset A$ is bounded if and only if $S \subset g^{-n} \cdot A_0$ for some $n \geq 0$.*

Proof. Suppose first that $S \subset A$ is bounded; then, taking $U = A_0$ in Definition 2.7 we can refine V further to be $g^n \cdot A_0$ for some n, since $\{g^n \cdot A_0\}_{n \in \mathbf{Z}}$ is a fundamental system of neighbourhoods of $0 \in A$. Then

$$g^n \cdot A_0 \cdot S \subseteq U \implies S \subseteq g^{-n} \cdot A_0.$$

For the converse we have the following more general lemma (which, in particular, shows that *all* rings of definition in a Huber ring are bounded); from it we see that $g^{-n} \cdot A_0$ is bounded for each n and thus $S \subseteq g^{-n} \cdot A_0$ is bounded. ∎

Lemma 2.9. *Let A be a Huber ring, $A_0 \subset A$ an arbitrary subring. Then the following are equivalent:*

1. *A_0 is a ring of definition,*

2. *A_0 is open and its topology is I-adic for some ideal $I \subset A_0$,*

3. A_0 *is open and bounded.*

Proof. That 1. implies 2. is by definition, and suppose A_0 is open and endowed with the I-adic topology $\implies A$ has a fundamental system of neighbourhoods of 0 that are ideals in A_0. This implies A_0 is bounded by definition of these being *ideals.*

Lastly, suppose that A_0 is open and bounded. Then, given an arbitrary ring of definition $B \subset A$ with finitely generated ideal of definition

$$J = (\pi_1, \ldots, \pi_n) \subseteq B$$

we set $T = \{\pi_1, \ldots, \pi_n\}$ and for $k \geq 1$, $T(k) := \{t_1 \cdot \ldots \cdot t_k \mid t_i \in T\}$ the set of arbitrary products of k elements in T; hence, by definition, $J^{k+1} = T(k) \cdot J$ for every $k \geq 1$.
Since A_0 is open and $\left\{J^k\right\}$ is a fundamental system of neighbourhoods of $0 \in A$ we see that $J^k \subset A_0$ for some $k \geq 0$. To conclude, we argue that A_0 has the $I := T(k) \cdot A_0$-adic topology for this choice of k - i.e. that the ideals $I^n \subseteq A_0$ are all open and they form a fundamental system of neighbourhoods of 0. Since we have the inclusions

$$I^n = T(n \cdot k) \cdot A_0 \supseteq T(n \cdot k) \cdot J^\ell = J^{n \cdot k + \ell}$$

where ℓ is any integer such that $J^\ell \subseteq A_0$, we see that I^n is indeed open for all n. To see that $\{I^n\}_{n \geq 1}$ is a fundamental system of neighbourhoods, let $U \subset A_0$ be an arbitrary open subset containing 0. Hence there exists $m \geq 1$ such that $J^{m \cdot k} \cdot A_0 \subset U$ because A_0 is bounded. Then

$$I^m = T(m \cdot k) \cdot A_0 \subseteq T(m \cdot k - 1) \cdot J \cdot A_0 = J^{m \cdot k} \cdot A_0 \subseteq U$$

whence every open neighbourhood of 0 contains I^n for some n. ■

The following definition describes the main family of Huber rings we'll be interested in throughout the course.

Definition 2.10. Let A be a Huber ring. A *topologically nilpotent element* $g \in A$ is an element such that $g^n \to 0$ for $n \to \infty$ (i.e. g^n lies in any open neighbourhood of $0 \in A$ provided n is chosen large enough). A is called *Tate* if there exists a topologically nilpotent unit $g \in A^\times$. Topologically nilpotent units in A are called *pseudo-uniformisers.*

Example 2.11.

1. The elements $p^n \in \mathbf{Q}_p$ for $n \geq 1$ are all pseudo-uniformisers.
2. If $A = A_0[g^{-1}]$ as in example 3. from Example 2.4, then $g \in A$ is a pseudo-uniformiser.
3. For any morphism of Huber rings $A \to B$, if A is Tate then so will B be (since any continuous map must respect topologically-nilpotent elements).

As most of the algebras we'll encounter in this course will be built from algebras over $\mathbf{Q}_p$, part 3. of Example 2.11 shows that most Huber rings we'll focus on will indeed be Tate. The following lemma illustrates how the condition on having a topologically nilpotent unit constrains the structure of a Huber ring.

Lemma 2.12. *Let A be a Tate ring (i.e. a Huber ring which is Tate), $\pi \in A$ a pseudo-uniformiser and $A_0 \subset A$ a ring of definition. Then there exists $n \geq 1$ such that $g := \pi^n \in A_0$; furthermore, for such a fixed integer n we have $A = A_0[g^{-1}]$ and A_0 has the g-adic topology.*

As a consequence, consider the topological ring $\mathbf{Z}_p[\![T]\!]$ with the (p, X)-adic topology; then there can be no topology on the localisation $\mathbf{Z}_p[\![T]\!][p^{-1}]$ making the subring $\mathbf{Z}_p[\![T]\!] \subset \mathbf{Z}_p[\![T]\!][p^{-1}]$ open, since $p \in \mathbf{Z}_p[\![T]\!]$ is a topologically nilpotent unit in $\mathbf{Z}_p[\![T]\!][p^{-1}]$ yet $\mathbf{Z}_p[\![T]\!]$'s topology is *not* p-adic - indeed, the sequence $\{X^n\}_{n \geq 0}$ accumulates to 0 with the (p, X)-adic topology but not for the p-adic one!

Proof. Since A_0 is open we see that $\pi^n \in A_0$ for some $n \geq 1$; as powers of pseudo-uniformisers are pseudo-uniformisers we can replace π with π^n and assume that π lies in A_0 without loss of generality. For any ideal of definition $I \subset A_0$, since I is open, we may again assume without loss of generality that $\pi \in I$. Because π is a unit we have that the multiplication-by-π map $\pi : A \to A$ is a homeomorphism and thus $\pi \cdot A_0 \subset A_0$ is an open ideal. There then exists $m \geq 1$ such that $I^m \subseteq \pi \cdot A_0$ and thus $\pi^m \cdot A_0 \subseteq I^m \subseteq \pi \cdot A_0 \subset I$ so the I-adic and π-adic topologies on A_0 coincide.

Lastly, to prove the equality $A_0[g^{-1}] = A$ note that for any $x \in A$ the sequence $\{\pi^n \cdot x\}_{n\geq 1}$ converges to 0 for $n \to \infty$ whence $\pi^n x \in A_0$ for n large enough as A_0 is open. ■

Definition 2.13. An element $a \in A$ in a Huber ring A is called *power-bounded* if the set of its powers $\{x^n\}_{n\geq 1}$ forms a bounded subset. We denote by $A^\circ \subset A$ the subring of power-bounded elements.

Remark 2.14. We'll soon prove that A° indeed forms a ring - for now we take it for granted and observe this is the case in the following examples.

Example 2.15.

1. If K is non-archimedean and $A = K\langle T_1, \ldots, T_n\rangle$ then

$$A^\circ = \mathcal{O}_K\langle T_1, \ldots, T_n\rangle$$

   as can be checked by using the multiplicativity of the Gauß norm from Exercise 2.5.

2. For a Huber ring A we define its set of topologically-nilpotent elements $A^{\circ\circ} := \{a \in A \mid a^n \to 0 \text{ as } n \to \infty\}$, which form an ideal in A°.

3. If $A = \mathbf{Q}_p[T]/T^2$ with ring of definition $A_0 := \mathbf{Z}_p[T]/T^2$ then $p \in A$ is a pseudouniformiser and the power-bounded elements are given by the subring $A^\circ = \mathbf{Z}_p \oplus \mathbf{Q}_p \cdot T$ (because $T^2 = 0$ in A!). In particular, we see that A° is *not* bounded, and hence can't be a ring of definition.

3 20th of April, continuous valuations

Today we discuss *continuous valuations on Huber rings.* To finish last time's discussion on power-bounded elements, we first show that these indeed form a ring.

Lemma 3.1. *Let A be a Huber ring.*

1. *If A_0 is a ring of definition in A, then all elements in A_0 are power-bounded.*

2. *The set of power-bounded elements A° is the filtered union of A's rings of definition (in particular, A° is a ring).*

Proof.

1. Since A_0 is bounded by Lemma 2.9 each $x \in A_0$ is power-bounded since $\{x^n\}_{n\geq 1} \subset A_0$.

2. Given rings of definition $A_0, A_0' \subset A$ we first show $B := A_0 \cdot A_0'$ is also a ring of definition. Again, we argue via Lemma 2.9: B is evidently open since it contains the open subring A_0, and to check that B is bounded let $U \subseteq A$ be an open subgroup; since A_0, A_0' are bounded there exist open subgroups $V_1, V_2 \subset A$ satisfying

$$V_1 \cdot A_0' \subset U,$$
$$V_2 \cdot A_0 \subset V_1.$$

   Thus $V_2 \cdot B \subset V_1 \cdot A_0' \subset U$ which implies B is bounded. If we apply the same argument replacing the set $A_0' \subset A$ by $S = \{x^n\}_{n\geq 0}$ for an arbitrary power-bounded element $x \in A^\circ$ and keeping

A_0 some ring of definition, we see that $A_0[x] \subset A$ is also a ring of definition, i.e. every power-bounded element is contained in some ring of definition $\implies$ A° is the filtered union of A's rings of definition.

■

Definition 3.2. We say a Huber ring A is *uniform* if A° is bounded (i.e. A° is a ring of definition).

Being uniform is a strong restriction on A's structure: for instance, if A is uniform then this forces its ring of power-bounded to have an *adic* topology, in contrast to example 3 in Example 2.15. The following lemma illustrates further properties of uniform Huber rings.

Lemma 3.3. *Let A be a Huber ring; if A is separated, uniform and Tate then A is reduced.*

Recall that being separated means that A's topology is Hausdorff (equivalently, that all convergent sequences have a *unique* limit point).

Proof. Suppose $x \in A$ is a nilpotent element and $\pi \in A$ a pseudo-uniformiser. Let $n \geq 1$; as $\pi^{-n} \cdot x$ is nilpotent, it must lie in A° by definition. Because the intersection

$$\bigcap_{n \geq 1} \pi^n \cdot A^\circ = \{0\}$$

is trivial - A°'s topology is π-adic by Lemma 2.12 and A is separated by assumption - we see that $x = 0$. ■

Definition 3.4.

1. Suppose A is a Huber ring. A subring $A^+ \subset A$ is called *a ring of integral elements* if $A^+ \subset A^\circ$ and A^+ is open and integrally closed in A.
2. A *Huber pair* is a pair (A, A^+) where A is a Huber ring and $A^+ \subset A$ is a ring of integral elements.
3. A *morphism of Huber pairs* $(A, A^+) \to (B, B^+)$ is a continuous ring homomorphism $A \to B$ mapping A^+ to B^+.

Exercise 3.5.

1. Show that $A^\circ \subset A$ is always a ring of integral elements (i.e. that A° is integrally closed in A).
2. For a Huber pair (A, A^+) the set of topologically nilpotent elements $A^{\circ\circ}$ is an ideal in A^+.
3. The map

$$A^+ \mapsto A^+/A^{\circ\circ}$$

defines a bijection between subrings of integral elements $A^+ \subset A$ and integrally closed subrings of $A^\circ/A^{\circ\circ}$.

As a particular instance of Exercise 3.5, suppose K is non-archimedean and A is the Tate algebra $K\langle T\rangle$; denote by k the residue field $k = \mathcal{O}_K/\mathfrak{m}_K$. Then the subring of power-bounded elements is given by $A^\circ = \mathcal{O}_K\langle T\rangle$ which is a ring of definition as A is uniform, we have $A^{\circ\circ} = \mathfrak{m}_K\langle T\rangle$ and

$$A^\circ/A^{\circ\circ} \cong k[T].$$

So, for example, we may take $k[T]$ or even just k in $A^\circ/A^{\circ\circ}$ as integrally closed subrings - these correspond to the rings of integral elements A° and

$$A^+ := \left\{ \sum_{i=0}^{\infty} a_i T^i \mid a_0 \in \mathcal{O}_K, |a_i| < 1 \text{ for all } i > 0 \right\} \subset K\langle T\rangle$$

respectively.

We now pass to discussing continuous valuations, which will form the *points* in the building blocks for Huber's category of adic spaces, playing the role akin to *prime ideals* for spectra of rings.

Definition 3.6. A continuous valuation on a topological ring A is a map

$$|-|: A \to \Gamma \cup \{0\},$$

where Γ is a totally ordered abelian group (for instance, one could take $\Gamma = \mathbf{R}_{>0}$ or more generally $\Gamma = \prod_{i=1}^{n} \mathbf{R}_{>0}$ with the lexicographical ordering) satisfying:

1. $|a \cdot b| = |a| \cdot |b|$,
2. $|a+b| \leq \max\{|a|, |b|\}$,
3. $|1| = 1$,
4. $|0| = 0$,
5. the set $\{a \in A \mid |a| < \gamma\} \subset A$ is open,

for all elements $a, b \in A$ and $\gamma \in \operatorname{im}(|-|) \subset \Gamma$.

Remark 3.7.

1. Note that the ordering on Γ naturally extends to $\Gamma \cup \{0\}$ by decreeing 0 to be a minimum in the set $\Gamma \cup \{0\}$.
2. Contrary to what one might think, the terminology is rather misleading since not all elements of valuation 0 must be zero - indeed, the set

$$\operatorname{supp}(|-|) := \{f \in A \mid |f| = 0\} \subset A$$

is a closed prime ideal, and often non-zero.

Definition 3.8. For a Huber pair (A, A^+) we define $\operatorname{Spa}(A, A^+)$ to be the set of equivalences classes of continuous valuations $|-|: A \to \Gamma \cup \{0\}$ such that $|A^+| \leq 1$.
We equip $\operatorname{Spa}(A, A^+)$ with the topology induced by requiring the subsets

$$U\left(\frac{f}{g}\right) := \left\{x \in \operatorname{Spa}(A, A^+) \mid |f(x)| \leq |g(x)| \neq 0\right\}$$

to be open, for $f, g \in A$ where we use the notation from Remark 3.9.

Remark 3.9. In Definition 3.8, we say two continuous valuations $|-|, |-|'$ are *equivalent* if the equivalence

$$|a| \leq |b| \iff |a|' \leq |b|'.$$

holds for all $a, b \in A$. We also remark that we have a very illuminating piece of notation, similar to what one introduces when studying schemes: valuations in $\operatorname{Spa}(A, A^+)$ are often denoted by letters $x, y, z, \ldots$ and for an element $g \in A$ we denote by $|g(x)|$ the element $x(g)$ as an element in x's valuation group (it's important to note that this depends on x's choice of equivalence class - part of this ambiguity is lost when one notices that the subgroups generated by g's valuations in the different representatives of x's equivalence class are all mapped to each other): although the element $|g(x)|$ a priori depends on the chosen value group Γ representing $|-|$, the condition $|f(x)| \leq |g(x)| \neq 0$ is well defined.
Lastly, note the relation between the topology on $\operatorname{Spa}(A, A^+)$ and the Zariski topology: the subsets $D(f) = \{x \in \operatorname{Spa}(A, A^+) \mid |f(x)| \neq 0\}$ are all open. In other words, the *support map*

$$\operatorname{supp}: \operatorname{Spa}(A, A^+) \to \operatorname{Spec} A$$

is continuous.

Example 3.10.

1. If A is discrete, $\mathrm{Spa}(A, A^+)$ is in bijection with the set of pairs $(\mathfrak{p}, R)$ where $\mathfrak{p} \in \mathrm{Spec}\, A$ and $R \subset \kappa(\mathfrak{p})$ a valuation subring such that A^+ maps to R under the projection map $A \to \kappa(\mathfrak{p})$.

2. If k is a field and $X \to \mathrm{Spec}\, k$ is a proper normal connected curve then $\mathrm{Spa}(K, \widetilde{k})$ is homeomorphic to X, where K is X's function field and $\widetilde{k}$ is k's integral closure in K.

3. To expand on the previous example, and to illustrate the role of the ring of integral elements A^+ in Definition 3.8, if $U = \mathrm{Spec}\, B \subset X$ is an open subset then $\mathrm{Spa}(K, B)$ is homeomorphic to U; for example $\mathrm{Spa}(K, \widetilde{k[f]}) = \{x \mid |f(x)| \leq 1\}$.

4. By Ostrowski's theorem, $\mathrm{Spa}(\mathbf{Q}, \mathbf{Z})$ is given by $\{\mathbf{Q} \to \{0, 1\}\} \cup \{|-|_p : \mathbf{Q} \to \mathbf{R}_{\geq 0} \mid p \in \mathbf{Z} \text{ prime}\}$ where $|-|_p$ are the p-adic valuations. In particular $\mathrm{Spa}(\mathbf{Q}, \mathbf{Z}) \approx \mathrm{Spec}\, \mathbf{Z}$ (albeit not via the support map).

We have a (difficult) theorem describing the topological space $\mathrm{Spa}(A, A^+)$.

Theorem 3.11 (Huber)**.** *For any Huber pair (A, A^+) the topological space $\mathrm{Spa}(A, A^+)$ is spectral.*

Furthermore, the topology on $\mathrm{Spa}(A, A^+)$ doesn't depend on A's completeness.

Proposition 3.12. *If (A, A^+) is a Huber pair, then the canonical mophism of Huber pairs $(A, A^+) \to (\widehat{A}, \widehat{A}^+)$ induces a homeomorphism*

$$\mathrm{Spa}(\widehat{A}, \widehat{A}^+) \to \mathrm{Spa}(A, A^+).$$

Before concluding today's lecture, we give an intricate description of the topological space $\mathrm{Spa}(K\langle T\rangle, \mathcal{O}_K\langle T\rangle)$ for a non-archimedean field K, as it'll turn out to be of quite some relevance to the theory we develop henceforth. Because $\mathcal{O}_K\langle T\rangle$ is the p-adic completion of the polynomial ring $\mathcal{O}_K[T]$, Proposition 3.12 allows us to simplify our problem to considering continuous valuations on $K[T]$ which have values at most 1 on the power-bounded elements $\mathcal{O}_K[T]$.

We start by restricting to rank-one valuations

$$|-| : K\langle T\rangle \to \mathbf{R}_{\geq 0}$$

and, because $|-|$ must be continuous when restricted to the subring $K \subset K\langle T\rangle$ and K is Tate, it follows that $|-|_{|K}$ is equivalent to K's canonical valuation: without loss of generality we may thus assume $|-|$ restricts to K's norm. Interestingly enough, this assumption allows us to conclude that $|-|$ is induced norm from K's valuation, as the following lemma shows.

Lemma 3.13.

1. *Let $c \in \mathcal{O}_K$ and $r \in [0, 1] \subset \mathbf{R}$. Then the map*

$$\begin{aligned} |-|_{c,r} : K[T] &\longrightarrow \mathbf{R}_{\geq 0} \\ \sum_{i=0}^{n} x_i (T-c)^i &\longmapsto \max |x_i| r^i \end{aligned}$$

defines a point $x_{c,r} \in \mathrm{Spa}(K[T], \mathcal{O}_K[T])$. Furthermore $|-|_{c,r}$ can also be described via the expression

$$|-|_{c,r} = \sup\left\{|f(x)| \mid x \in K, |x - c| \leq r\right\}.$$

2. *Suppose* $x = |-(x)| \in \mathrm{Spa}(K[T], \mathcal{O}_K[T])$. *Then there exists a (co-)filtered family of discs* $\{\mathbf{B}(c_i, r_i)\}_{i\in I}$ *where* $c_i \in \mathcal{O}_K$ *and* $r_i \in [0,1]$ *such that* x *is given by*

$$|f(x)| = \inf_{i\in I} \{|f(x_{c_i,r_i})|\}$$

where the points $x_{c_i,r_i} \in \mathrm{Spa}(K[T], \mathcal{O}_K[T])$ *are as in part* 1.

Here $\mathbf{B}(c,r)$ for $c \in \mathcal{O}_K$ and $r \in [0,1]$ denotes the closed disc

$$\mathbf{B}(c,r) := \{x \in K \mid |x-c| \leq r\}\,.$$

For concreteness, the '*main*' example of the points $x_{c,r}$ is when $r = 1$; in this case all the points $\{x_{c,1}\}_{c\in\mathcal{O}_K}$ agree (recall that $\mathbf{B}(c,1) = \mathbf{B}(0,1)$ for all $c \in \mathcal{O}_K$) and they're the point defined by $K[T]$'s - Gauß - norm inducing its topology. This can be thought of as the *origin* of the adic unit disc $\mathrm{Spa}(K\langle T\rangle, \mathcal{O}_K[T])$.

Proof.

1. This is essentially the same computation as in Exercise 2.5.

2. We sketch the main idea: consider the family of discs

$$\{\mathbf{B}(c, r_c)\}_{c\in\mathcal{O}_K}\,,\ \ r_c := |(\Delta_c)(x)|$$

where $\Delta_c(T) := T - c \in \mathcal{O}_K[T]$ for $c \in \mathcal{O}_K$. These are nested by the non-archimedean inequality $|-(x)|$ satisfies. A continuity argument shows that $|-(x)|$ agrees with the infimum of these norms, as claimed.

■

Lemma 3.13 allows us to distinguish four cases: fix a rank-one point $|-|' : K[T] \to \mathbf{R}_{\geq 0}$ in $\mathrm{Spa}(K[T], \mathcal{O}_K[T])$ and the filtered sequence of discs $\{\mathbf{B}(c_i, r_i)\}_{i\in I}$ as in the lemma. We assume $I = \mathbf{N}_{\geq 1}$.

1. $r_n \to 0$ as $n \to \infty$ and $\bigcap_n \mathbf{B}(c_n, r_n) = \{c\}$ for some $c \in \mathcal{O}_K$. In this case, $|-(x)|'$ is given by valuation at the point x:
$$|f(x)|' = |f(c)|$$
where on the right hand side we have the K-norm of the actual valuation of the polynomial $f \in K[T]$ at the point c.

2. If $r_n \to r > 0$ as $n \to \infty$ for some real number $r \in \mathbf{R}_{\geq 0}$ lying in K's value group, i.e. there exists $c \in \mathcal{O}_K$ such that $|c| = r$, then $|-|'$ coincides with $|-|_{c,r}$ in part 1 from Lemma 3.13.

3. $r_n \to r > 0$ as $n \to \infty$ and $r \notin |K^\times|$. These are called the 'generalised Gauß points' and once again are just given by $|-|_{c,r}$ where $c \in K$ is any point lying in $\bigcap_n \mathbf{B}(c_i, r_i)$ (note that this intersection is non-empty, since the sequence $\{r_n\}_{n\geq 1}$ converges to a positive number and K's value group is dense in $\mathbf{R}_{\geq 0}$ because K is algebraically closed).

4. The last case is for when the real numbers $r_n \to 0$ tend to zero as $n \to \infty$ and the intersection of the nested discs $\bigcap_{n\geq 1} \mathbf{B}(c_i, r_i) = \emptyset$ contains no points in K (if this never occurs for any sequence of nested discs in $\mathcal{O}_K$ we say that K is *spherically complete*; for instance $\mathbf{C}_p$ is not spherically complete).

The discrepancy in expressing x as $|-|_{c,r}$ is cleared up by the equivalence

$$|-|_{c,r} = |-|_{c',r'} \iff r = r' \text{ and } |c - c'| = r$$

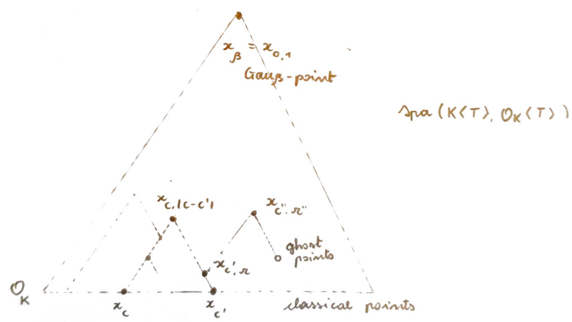
which again follows from the fact that K isn't discretely valued and '*in non-archimedean topologies, all points in a disc are its centre*'. We can now picture the rank-one points geometrically, by drawing the '*classical*' points the first type in the previous list as a copy of $\mathcal{O}_K$, and then all other points form branches - or upside down trees, really - which ramify above these. Note that by thinking of $\mathcal{O}_K$ as a subset of $\mathrm{Spa}(K[T], \mathcal{O}_K[T])$ as these classical points indeed makes $\mathcal{O}_K$ into a topological subspace of $\mathrm{Spa}(K[T], \mathcal{O}_K[T])$, the topology in $\mathcal{O}_K$ is adic and thus its rational open subsets form a basis. It's crucial to note however that $\mathcal{O}_K \subset \mathrm{Spa}(K[T], \mathcal{O}_K[T])$ is not closed if K isn't spherically complete - its points can accumulate to missing *'ghost'* points as described in the fourth point just above.

The points of higher rank are described in the following lemma.

Lemma 3.14. *Let $|-|_{c,r}$ be a point in $\mathrm{Spa}(K[T], \mathcal{O}[T])$ as in point 2 of the above classification of rank 1 points, which isn't the 'Gauß point' $|-|_{0,1}$. The points which $|-|_{c,r}$ specialises to in $\mathrm{Spa}(K[T], \mathcal{O}_K[T])$ are in bijection with $\mathbf{P}^1(k)$ where $k := \mathcal{O}_K/\mathfrak{m}_K = K^\circ/K^{\circ\circ}$ is K's residue field. Similarly, the points $|-|_{0,1}$ specialises to are in bijection with $\mathbf{A}^1(k)$. These exhaust all points of higher rank. They are given explicitly by*

$$\begin{aligned} |-|_{c,r,\epsilon} : K[T] &\longrightarrow \mathbf{R}_{\geq 0} \times \gamma^{\mathbf{Z}} \\ \sum_{i=0}^{m} a_i (T-c)^i &\longmapsto \max_i \left\{ |a_i| \cdot r^i \gamma^{\epsilon \cdot i} \right\} \end{aligned}$$

for $r \in (0,1]$ and $\epsilon = \pm 1$. If $r = 1$ then the above formula defines the points the Gauß point specialises to if $\epsilon = 1$ (for otherwise we'd get $|T|_{0,1,-1} = \gamma^{-1} > 1$ so $|-|_{0,1,-1}$ doesn't satisfy the due inequality from Definition 3.8).



4 27th of April, adic spaces

We start today's lecture by discussing a slight variation of the example we treated at the end of last lecture. Consider the Huber pair

$$(A, A^+) = (K\langle T\rangle, \mathcal{O}_K + \mathfrak{m}_K\langle T\rangle)$$

and, in particular, note that $T \notin A^+$ in contrast to when we studied the adic unit disc $\mathrm{Spa}(K\langle T\rangle, \mathcal{O}_K\langle T\rangle)$. Since $A^+ \subset \mathcal{O}_K\langle T\rangle$ we have a natural inclusion of topological spaces

$$\mathrm{Spa}(A, A^+) \supset \mathrm{Spa}(K\langle T\rangle, \mathcal{O}_K\langle T\rangle)$$

which is actually *strict*, since $\mathrm{Spa}(A, A^+)$ includes the continuous valuation defined by

$$|-|_{0,1,-1} : A \to \mathbf{R}_{\geq 0} \times \gamma^{\mathbf{Z}}$$
$$|\sum_{i=0}^{\infty} a_i T^i|_{0,1,-1} := \max\left\{|a_i|\gamma^{-i}\right\}$$

which doesn't lie in the adic unit disc as $|T|_{0,1,-1} = \gamma^{-1} > 1$.

Today we aim at constructing a (pre-)sheaf of rings on the topological space $\mathrm{Spa}(A, A^+)$. A crucial technical restriction we must make to start is introducing a slight replacement for the natural basis for the topology on $\mathrm{Spa}(A, A^+)$ introduced in Definition 3.8.

Definition 4.1. Let (A, A^+) be a Huber pair. Assume $f_1, \ldots, f_n, g \in A$ are such that the ideal $A \cdot f_1 + \ldots + A \cdot f_n \subset A$ is open. Then we call the corresponding open subset

$$U\left(\frac{f_1, \ldots, f_n}{g}\right) \subset \mathrm{Spa}(A, A^+)$$

a *rational* open subset. We often denote $U\left(\frac{f_1,\ldots,f_n}{g}\right)$ by $U\left(\frac{T}{g}\right)$ where $T = \{f_1, \ldots, f_n\}$.

Lemma 4.2. *Rational subsets form a basis for the topology on* $\mathrm{Spa}(A, A^+)$ *which is stable under finite intersections.*

Proof. For finite subsets $T, T' \subset A$ and elements $s, s' \in A$ such that $T \cdot A, T' \cdot A \subset A$ are open, we have the equality

$$U\left(\frac{T}{s}\right) \cap U\left(\frac{T'}{s'}\right) = \{x \in \mathrm{Spa}(A, A^+) \mid |t \cdot t'(x)|, |t \cdot s'(x)|, |t' \cdot s(x)| \leq |s \cdot s'(x)| \neq 0 \text{ for all } t \in T, t' \in T'\}$$

and furthermore the set $T \cdot T' \subset A$ is also open since there must exist some ideal of definition $I \subset A$ contained in both $T \cdot A$ and $T' \cdot A' \implies I^2$ is contained in $(T \cdot A) \cdot (T' \cdot A)$.
We're yet to show that all open subsets of the form

$$U\left(\frac{f_1, \ldots, f_n}{g}\right) \subset \mathrm{Spa}(A, A^+)$$

for *arbitrary* $f_1, \ldots, f_n, g \in A$ are covered by rational open subsets. Given a point $x \in U\left(\frac{f_1,\ldots,f_n}{g}\right)$, as $|g(x)| \neq 0$ and $|-|$ is continuous, there must exist an ideal of definition $I \subset A$ with finitely many generators $h_1, \ldots, h_n \in I$ such that

$$x \in U\left(\frac{f_1, \ldots, f_n, h_1, \ldots, h_m}{g}\right) \subset U\left(\frac{f_1, \ldots, f_n}{g}\right)$$

where $U\left(\frac{f_1,\ldots,f_n,h_1,\ldots,h_m}{g}\right)$ is now rational since $I \subset (h_1, \ldots, h_m, f_1, \ldots, f_n)$ by construction. ■

As a good example of a non-rational subset, we can consider $A = K\langle u \rangle$ and the adic unit disc $\mathrm{Spa}(A, A^+)$; then $U\left(\frac{0}{u}\right) = \{x \mid |u(x)| \neq 0\}$ isn't rational by the main result from today's lecture, which will follow - we aim to show that for rational open subsets $U\left(\frac{T}{s}\right) \subset \mathrm{Spa}(A, A^+)$ there's a homeomorphism $U\left(\frac{T}{s}\right) \approx \mathrm{Spa}(B, B^+)$ for an appropriate Huber pair (B, B^+) (then $U\left(\frac{0}{u}\right)$ from our example can't be rational since it isn't quasi-compact, as can be checked using the explicit description of the points in $\mathrm{Spa}(K\langle T \rangle, \mathcal{O}_K\langle T \rangle)$ discsussed last lecture by means of an open cover for $U\left(\frac{0}{u}\right)$ with the filtered sequence of annuli $U\left(\frac{\pi^n}{u}\right)$).

Proposition 4.3. *Suppose* (A, A^+) *is a complete Huber pair. Then*

1. $A \neq 0 \implies \mathrm{Spa}(A, A^+) \neq \emptyset$,
2. $A^+ = \{f \in A \mid |f(x)| \leq 1$ *for all* $x \in \mathrm{Spa}(A, A^+)\}$,
3. *an element* $f \in A$ *is invertible in* A *if and only if* $|f(x)| \neq 0$ *for all* $x \in \mathrm{Spa}(A, A^+)$.

Proof. For points 1. and 2. we refer to Huber's text [Hub]. As for part 3., suppose $|f(x)| \neq 0$ for all $x \in \mathrm{Spa}(A, A^+)$. Then we see that the Huber pair $(A/(f), \overline{A}^+)$ must have an empty adic spectrum where $\overline{A}^+$ is the integral closure of the image of A^+ in $A/(f)$; thus, by Proposition 3.12 and part 1. we see that the completion $A/(f)^\wedge = 0$ is the zero ring, and since A is complete we have an isomorphism $A/(f)^\wedge \cong A/\overline{(f)}$ where $\overline{f}$ is the closure of the ideal (f) in A. We then have the implication

$$1 \in \overline{(f)} \implies 1 - g \cdot f \in A^{\circ\circ}$$

for some $g \in A$. Thus $1 - (1 - f \cdot g) \in A^\times$ and f is invertible. ■

We can now tackle our main goal for today's lecture:

Theorem 4.4. *Suppose (A, A^+) is a complete Huber pair and let $U \subset X = \mathrm{Spa}(A, A^+)$ be a rational open subset. Then there exists a complete Huber pair $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ over (A, A^+) such that the following properties hold:*

1. *for all complete Huber pairs (B, B^+) over (A, A^+) such that the induced map $\mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$ factors over the open subset U there exists a unique factorisation of Huber pairs*

$$\begin{tikzcd} (\mathcal{O}_X(U), \mathcal{O}_X^+(U)) \arrow[r] \arrow[dr] & (B, B^+) \\ & (A, A^+) \arrow[u] \end{tikzcd}$$

2. *the map $\mathrm{Spa}(\mathcal{O}_X(U), \mathcal{O}_X^+(U)) \to \mathrm{Spa}(A, A^+)$ induces a homeomorphism $\mathrm{Spa}(\mathcal{O}_X(U), \mathcal{O}_X^+(U)) \approx U$.*

Proof. We sketch the salient parts of the argument: suppose $T \subset A$ and $s \in A$ are so that $U = U\left(\frac{T}{s}\right)$ and $T \cdot A \subset A$ is open, and fix a ring of definition $A_0 \subset A$ with an ideal of definition $I \subset A_0$. If $\mathrm{Spa}(B, B^+) \to X$ factors over U, then the ring homomorphism $A \to B$ induces a ring homomorphism $A[s^{-1}] \to B$ by Theorem 4.3 so that the element t/s is mapped to B^+ for all $t \in T$. As we require the pair $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ to be a complete Huber pair, our goal is to set

$$(\mathcal{O}_X(U), \mathcal{O}_X^+(U)) = (A[s^{-1}]^\wedge, \widetilde{A}^+[t/s \mid t \in T])$$

where $\widetilde{A}^+[t/s \mid t \in T]$ denotes the integral closure of $A[t/s]_{t \in T} \subset A[s^{-1}]^\wedge$. To this end, we're forced to equip $A[1/s]$ with the topology making $A_0[t/s \mid t \in T] \subset A[1/s]$ an open subring, endowed with the $I \cdot A_0[t/s \mid t \in T]$-adic topology. The condition that T generated an *open* ideal in A now becomes relevant, because of the following claim:

Claim: $T \cdot A \subset A$ open $\implies$ $A[1/s]$ is a topological ring (i.e. it's natural ring structure is compatible with the topology described above).

For this, it'll be sufficient to show that the multiplication-by-s^{-1} map

$$s^{-1} : A[s^{-1}] \to A[s^{-1}]$$

is continuous, i.e. that $s^{-1} I^n \subset A_0[t/s \mid t \in T]$ for some n. Since $T \cdot A$ is open, there exists an integer $n \geq 1$ such that $I^n \subset T \cdot A$ which (up to replacing I with I^n) we may assume is equal to 1. If $f_1, \dots, f_k \in A$ are a (finite!) set of generators for I, then there exists a finite subset $M \subset A$ such that the elements $f_1, \dots, f_k$ all lie in $T \cdot M$ (by just taking an expression following the inclusion $\{f_1, \dots, f_k\} \subset T \cdot A$). As the elements in I are topologically nilpotent, we have $M \cdot I^n \subset A_0$ for some $n \geq 1$, hence

$$f_i \cdot I^n \subset T \cdot M \cdot I^n \subset T \cdot A_0$$

for all $i = 1, \dots, k$ and thus $I^{n+1} \subset T \cdot A_0$ as we desired. The homeomorphism $U \approx \mathrm{Spa}(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ follows from Theorem 3.12. ∎

This theorem allows us to define a presheaf of rings on $\mathrm{Spa}(A, A^+)$.

Definition 4.5. We define the presheaf of pairs of topological rings $(\mathcal{O}_X, \mathcal{O}_X^+)$ on $X = \mathrm{Spa}(A, A^+)$ by requiring $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ to be the Huber pair as in Theorem 4.4 for all rational open subsets $U \subset \mathrm{Spa}(A, A^+)$. We call the Huber pair (A, A^+) *sheafy* if $(\mathcal{O}_X, \mathcal{O}_X^+)$ is a sheaf of pairs of topological rings.

Unfortunately, not all Huber pairs are sheafy; however, they luckily turn out to be in most of the cases we're interested in for studying rigid analytic geometry, thanks to the following result of Huber.

Theorem 4.6 (Huber)**.** *Suppose (A, A^+) is a complete Huber pair. Then $\mathcal{O}_X$ and $\mathcal{O}_X^+$ are sheaves in the following cases:*

1. *if A is discrete,*

2. *if A is finitely generated of a Noetherian ring of definition,*

3. *if A is Tate and strongly Noetherian (i.e. that the algebras $A\langle T_1, \ldots, T_n\rangle$ are all noetherian for $n \geq 1$).*

For instance, if K is a non-archimedean field $\implies$ K is strongly Noetherian [BGR].

We can now introduce the category of adic spaces we'll be focusing on.

Definition 4.7. 1. Consider the category (V) whose objects are the triples

$$(X, \mathcal{O}_X, \{|-(x)|\}_{x\in X})$$

where X is a topological space, $\mathcal{O}_X$ is a sheaf of topological rings on X and $\{|-(x)|\}_{x\in X}$ is a family of continuous valuations indexed by the points in X, so that $|-(x)|$ is a valuation on the stalk $\mathcal{O}_{X,x}$ for each $x \in X$; and a morphism of triples $(Y, \mathcal{O}_Y, \{|-(y)|\}_{y\in Y}) \to (X, \mathcal{O}_X, \{|-(x)|\}_{x\in X})$ is the data of a morphism of topologically-ringed spaces $f : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ such that for all $y \in Y$ the diagram

$$\begin{tikzcd}
\mathcal{O}_{X,f(y)} \arrow[r] \arrow[d, "|-(x)|"] & \mathcal{O}_{Y,y} \arrow[d, "|-(y)|"] \\
\Gamma_x \cup \{0\} \arrow[r] & \Gamma_y \cup \{0\}
\end{tikzcd}$$

commutes up to equivalence of norms.

2. For a sheafy Huber pair (A, A^+) the triple

$$(\mathrm{Spa}(A, A^+), \mathcal{O}_{\mathrm{Spa}(A,A^+)}, \{|-(x)|\}_{x\in \mathrm{Spa}(A,A^+)})$$

is an object in (V) called the *affinoid adic space* associated to (A, A^+).

3. An *adic space* is an object in (V) locally isomorphism to a sheafy affinoid adic space.

The following proposition gives us access to many of the tools one uses when studying algebraic geometry through schemes.

Proposition 4.8. *Let X be an adic space and (A, A^+) a sheafy Huber pair. Then the canonical map*

$$\mathrm{Hom}_{(V)}(X, \mathrm{Spa}(A, A^+)) \xrightarrow{\Gamma} \mathrm{Hom}_{\textit{cont-pairs}}((A, A^+), (\mathcal{O}_X(X), \mathcal{O}_X^+(X)))$$

is a bijection.

Proof. We again reference [Hub]. ■

Definition 4.9. Let K be a non-archimedean field. A *rigid analytic variety* over K is an adic space X over $\mathrm{Spa}(K, \mathcal{O}_K)$ which is locally isomorphic to $\mathrm{Spa}(A, A^\circ)$ where A is a *Tate algebra.*

5 4th of May, properties of morphisms

Today we define different properties of morphisms between adic spaces, imitating the analogous notions from the setting of schemes.

Definition 5.1. Let (A, A^+) and (B, B^+) be Huber pairs and $f : (A, A^+) \to (B, B^+)$ a morphism.

1. f is a *quotient map* if:

   (a) the ring homomorphism $f : A \to B$ is surjective, continuous and open (in particular, B's topology is the quotient topology on $A/\ker f$),

   (b) B^+ is the integral closure of $f(A^+)$.

2. If A is Tate (whence B is as well), then f is said to be of *topologically of finite type* if there exists a quotient map
$$(A\langle T_1, \dots, T_n\rangle, A^+\langle T_1, \dots, T_n\rangle) \twoheadrightarrow (B, B^+).$$

3. Let K be a non-archimedean field. An adic space X which is locally topologically of finite type over $\mathrm{Spa}(K, \mathcal{O}_K)$ (i.e. such that X admits an open cover of affinoids $X = \bigcup_\alpha \mathrm{Spa}(A_\alpha, A_\alpha^+)$ so that the induced morphism of Huber pairs $(K, \mathcal{O}_K) \to (A, A^+)$ is topologically of finite type) is called an *rigid analytic variety over* K.

Remark 5.2. We remark that quasi separated rigid-analytic varieties are equivalent to the quasi-separated rigid-analytic varieties in Tate's sense [Ta].

For rigid analytic varieties we have a far stronger restriction on the ring of integral elements from Definition 3.8.

Lemma 5.3. *Let $X = \mathrm{Spa}(A, A^+)$ be a rigid analytic variety over $\mathrm{Spa}(K, \mathcal{O}_K)$. Then $A^+ = A^\circ$.*

Proof. This follows from the crucial result discussed in [Hub]: by the fact that X is locally of finite type over K, there exists a *global* quotient map
$$(K\langle T_1, \dots, T_n\rangle, \mathcal{O}_K\langle T_1, \dots, T_n\rangle) \twoheadrightarrow (A, A^+)$$
which a priori only exists locally in Definition 5.1. Then we have the implication
$$A^\circ \text{ integral over } f(\mathcal{O}_K\langle T_1, \dots, T_n\rangle) \implies A^+ = A^\circ.$$
Which can be argued as follows: as the map $K\langle T_1, \dots, T_n\rangle \to A$ must preserve power-bounded elements, we get that the image of $\mathcal{O}_K\langle T_1, \dots, T_n\rangle$ in A lies in A°; but then $A^+ \subset A^\circ$ is an integral ring extension as $f(\mathcal{O}_K\langle T_1, \dots, T_n\rangle) \subset A^+$ is by hypothesis. Since A is Tate and $A = \mathrm{Frac}\, A^+ = A^+[\varpi^{-1}]$ for any pseudo-unformiser $\varpi \in A^\circ$ and A° is integrally closed, we can conclude that $A^\circ = A^+$. ■

Example 5.4.

1. The closed unit n-ball $\mathbf{B}_K^n := \mathrm{Spa}(K\langle T_1, \dots, T_n\rangle, \mathcal{O}_K\langle T_1, \dots, T_n\rangle)$ whose functor of points can also be explicitly described by
$$\mathbf{B}_K^n(A, A^+) = \mathrm{Hom}_{(K, \mathcal{O}_K)}((K\langle T_1, \dots, T_n\rangle, \mathcal{O}_K\langle T_1, \dots, T_n\rangle), (A, A^+)) = (A^+)^n.$$

2. The n-torus over K is given by the rational open subset
$$\mathbf{T}_K^n := \mathrm{Spa}(K\langle T_1^{\pm 1}, \dots, T_n^{\pm 1}\rangle, \mathcal{O}_K\langle T_1^{\pm 1}, \dots, T_n^{\pm 1}\rangle) = \{x \in \mathbf{B}_K^n \mid |T_i(x)| = 1\}$$
which is rational since it's the intersection of the rational opens $U\left(\frac{T_1, \dots, T_n, \varpi}{1}\right)$ and $U\left(\frac{1}{T_1}\right), \dots, U\left(\frac{1}{T_n}\right)$ where the ideals generated by the 'numerators' (1) and $(T_1, \dots, T_n, \varpi)$ are open in $K\langle T_1, \dots, T_n\rangle$

(note that the inequality $|\varpi| \leq 1$ is redundant as ϖ is topologically nilpotent). We also have an equality

$$\mathbf{B}^1_K = \mathbf{T}^1_K \cup (-1 + \mathbf{T}^1_K)$$

where $-1 + \mathbf{T}^1_K$ is the translate of $\mathbf{T}^1_K$ by 1 (note that $\mathbf{B}^1_K$ is an *adic-group* by its functor of poins description) since $|T(x)| < 1 \implies |1 + T(x)| = 1$ for any $x \in \mathbf{B}^1_K$, by the non-archimedean inequality.

3. As mentioned last time, the locally ringed space $\mathrm{Spa}(K\langle T\rangle, \mathcal{O}_K + \mathfrak{m}_K\langle T\rangle)$ (which is sheafy by Theorem 4.6) can't be a rigid-analytic space since its ring of power-bounded elements is $\mathcal{O}_K\langle T_1, \dots, T_n\rangle$ and not $\mathcal{O}_K + \mathfrak{m}_K\langle T\rangle$, contradicting Lemma 5.3.

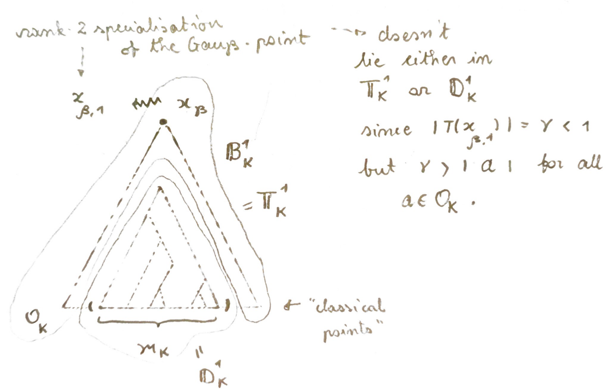
4. The *open unit disc* is defined as the filtered union of rational opens

$$\mathbf{D}^1_K = \bigcup_{m \geq 1} \left\{ x \in \mathbf{B}^1_K \mid |T^m(x)| \leq |\varpi| \neq 0 \right\};$$

$\mathbf{D}^1_K$ represents the functor $\mathbf{D}^1_K(A, A^+) = A^{\circ\circ}$. Recall that this isn't affinoid since it isn't quasi compact. We have a strict inclusion

$$\mathbf{D}^1_K \cup \mathbf{T}^1_K \subsetneq \mathbf{B}^1_K.$$

since the rank-two point $x_{0,1,1}$ described in Lemma 3.14 doesn't lie in either.



This can be compared - and, in principle, 'solves' the problem of having to introduce the G-topology of admissible covers admitting finite refinements on the category of rigid spaces - to the setting of rigid analytic varieties in the sense of Tate [Ta] where the analogues of these two open subsets do indeed cover the unit disc.

5. We define the *adic affine line*, given by the filtered union

$$\mathbf{A}_K^{1,\mathrm{ad}} = \bigcup_{n \geq 1} \operatorname{Spa}(K\langle \varpi^n T \rangle, \mathcal{O}_K\langle \varpi^n T \rangle).$$

representing the functor

$$\mathbf{A}_K^{1,\mathrm{ad}} : X/\operatorname{Spa}(K, \mathcal{O}_K) \longmapsto \mathcal{O}_X(X) = \Gamma(X, \mathcal{O}_X).$$

This can be seen directly by the equality

$$A = \bigcup_{n \geq 1} \varpi^{-n} \cdot A^+$$

for any affinoid adic space $\operatorname{Spa}(A, A^+)$ over $\operatorname{Spa}(K, K^+)$.
In particular, we get a fibre product (in the sense of functors of points)

$$\mathbf{A}_K^{1,\mathrm{ad}} \cong \operatorname{Spa}(K, \mathcal{O}_K) \times_{\operatorname{Spa}(\mathbf{Z},\mathbf{Z})} \operatorname{Spa}(\mathbf{Z}[T], \mathbf{Z}).$$

Moreover, we have isomorphisms $\mathbf{B}_K^1 \cong \operatorname{Spa}(K, \mathcal{O}_K) \times_{\operatorname{Spa}(\mathbf{Z},\mathbf{Z})} \operatorname{Spa}(\mathbf{Z}[T], \mathbf{Z}[T])$ and $\mathbf{D}_K^1 \cong \operatorname{Spa}(K, \mathcal{O}_K) \times_{\operatorname{Spa}(\mathbf{Z},\mathbf{Z})} \operatorname{Spa}(\mathbf{Z}[\![T]\!], \mathbf{Z}[\![T]\!])$.

Generalising the last of the above examples, if $Y \to \operatorname{Spec} K$ is a locally of finite type K-scheme, then there exists an *'analytification'* of Y denoted by Y^{ad} which is a rigid analytic space satisfying

$$\operatorname{Hom}_{\operatorname{Spa}(K,\mathcal{O}_K)}(Z, Y^{\mathrm{ad}}) \cong \{\text{morphisms } Z \to Y \text{ of locally ringed spaces over } K\}.$$

The construction follows the same lines as the classic analytification functor for locally of finite-type schemes over the complex numbers, discussed for example in last semester's lecture course on étale cohomology.

We now proceed to studying morphisms of adic spaces. We first mention that some of the definitions we introduce differ from the ones described in [Hub] by certain finiteness conditions we omit. The conventions we follow are the ones adopted in [EtD].

Definition 5.5. An adic space X is called *analytic* if X is locally isomorphic to $\operatorname{Spa}(A, A^+)$ with A a Tate ring.

Definition 5.6. Let $f : Y \to X$ be a morphism of adic spaces.

1. f is *finite étale* if:
   (a) there exists an open affinoid cover
   $$X = \bigcup_{i \in I} U_i$$
   where $U_i \cong \operatorname{Spa}(A_i, A_i^+)$ such that $f^{-1}(U_i) \cong \operatorname{Spa}(B_i, B_i^+)$ and the induced ring homomorphism $A_i \to B_i$ is finite étale,
   (b) B_i has the natural topology induced from it being a finitely generated A_i-module, and
   (c) B_i^+ is the integral closure of A_i^+ in B_i.

2. f is *étale* if for all $y \in Y$ there exist open neighbourhoods U and V of y and $f(y)$ respectively such that $U \to V$ is finite étale.

3. f is *smooth* if locally on Y the morphism f factors as in the diagram

$$\begin{tikzcd} Y \arrow[r, "\text{ét}"] \arrow[dr, "f"'] & \mathbf{B}^n_X \cong X \times_{\mathrm{Spa}(\mathbf{Z},\mathbf{Z})} \mathrm{Spa}(\mathbf{Z}[T],\mathbf{Z}) \arrow[d] \\ & X \end{tikzcd}$$

assuming the fibre product defining $\mathbf{B}^n_X$ exists in the category of adic spaces.

4. f is *separated* if the fibre product $Y \times_X Y$ exists and the image of the diagonal map

$$\Delta_f : Y \to Y \times_X Y$$

is closed.

5. X is an (analytic) *affinoid field* if $X \cong \mathrm{Spa}(L, L^+)$ where L is a non-archimedean field and the subring of integral elements $L^+ \subset L$ is an open and bounded valuation subring (note that we have a homeomorphism $|\mathrm{Spa}(L, L^+)| \cong |\mathrm{Spec}(L^+/\varpi)|$ for any pseudouniformiser ϖ in L^+; in particular, $|\mathrm{Spa}(L, L^+)|$ is a linearly ordered chain of points and plays the same role spectra of valuation rings do in the valuative criteria Tag 0BX4).

6. f is said to be *partially proper* if f is separated and for all affinoid fields $\mathrm{Spa}(L, L^+)$ fitting into the commutative square

$$\begin{tikzcd} \mathrm{Spa}(L, \mathcal{O}_L) \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\ \mathrm{Spa}(L, L^+) \arrow[r] \arrow[ur, dashed] & X \end{tikzcd}$$

there exists a (necessarily unique) lift $\mathrm{Spa}(L, L^+) \to Y$.

7. f is *proper* if it is quasi-compact and partially proper.

Remark 5.7. Note that the definition of étale maps in definition 5.6 doesn't translate to the context of schemes: not all étale morphisms of schemes $f : Y \to X$ are *Zariski-locally finite étale*! Indeed, consider the morphism

$$\begin{aligned} f : \mathbf{A}^1_{\mathbf{C}} \setminus \{0,1\} &\longrightarrow \mathbf{A}^1_{\mathbf{C}} \setminus \{0\} \\ x &\longmapsto x^2 \end{aligned}$$

which is étale but not finite (since it isn't proper). The point $-1 \in \mathbf{A}^1_{\mathbf{Q}} \setminus \{0,1\}$ has no (Zariski) neighbourhood on which f's restriction is finite étale - algebraically speaking, all the ring extensions

$$\mathbf{C}[T^{\pm 2}, (T^2 - a_1^2)^{\pm 1}, \ldots, (T^2 - a_m^2)^{\pm 1}] \subset \mathbf{C}[T, (T-1)^{\pm 1}, (T - a_1)^{\pm 1}, \ldots, (T - a_m)^{\pm 1}]$$

aren't integral for $a_1, \ldots, a_m \neq -1$.
We also remark that the definition above for smooth morphisms runs parallel to the *uniformising parameters* theorem for smooth morphisms of schemes, discussed in Algebraic Geometry 2.

Example 5.8.

1. $\mathbf{D}^1_K$ is partially proper over K as can be seen by directly analysing the functor of points.

2. The analytification of any separated, locally of finite type scheme $Y \to \operatorname{Spec} K$ over K is partially proper because of the canonical isomorphisms

$$\operatorname{Hom}_{(K,\mathcal{O}_K)}(\operatorname{Spa}(L,L^+),(\operatorname{Spec} B)^{\mathrm{ad}}) \cong \operatorname{Hom}(B,\underbrace{\Gamma(\operatorname{Spa}(L,L^+),\mathcal{O}_{\operatorname{Spa}(L,L^+)})}_{\cong L})$$

for any affinoid field $\operatorname{Spa}(L,L^+)$ and affinoid adic space $\operatorname{Spa}(B,B^+)$ over $\operatorname{Spa}(K,\mathcal{O}_K)$: the left hand side doesn't depend on L^+ and thus the map

$$\operatorname{Hom}_{(K,\mathcal{O}_K)}(\operatorname{Spa}(L,L^+),Y) \to \operatorname{Hom}_{(K,\mathcal{O}_K)}(\operatorname{Spa}(L,\mathcal{O}_L),Y)$$

is automatically bijective!

3. If $Y \to \operatorname{Spec} K$ is proper then $Y^{\mathrm{ad}} \to \operatorname{Spa}(K,\mathcal{O}_K)$ is proper by Chow's lemma Tag 02O2. For example, by the equality

$$\mathbf{P}_K^{1,\mathrm{ad}} = \mathbf{A}_K^{1,\mathrm{ad}} \cup_{\mathbf{G}_{m,L}^{\mathrm{ad}}} \mathbf{A}_K^{1,\mathrm{ad}} = \mathbf{B}_K^1 \cup_{\mathbf{T}_K^1} \mathbf{B}_K^1$$

we see that the projective line $\mathbf{P}_K^{1,\mathrm{ad}}$ is quasi-compact and thus proper over $\operatorname{Spa}(K,\mathcal{O}_K)$.

4. The unit disc $\mathbf{B}_K^1$ is not partially proper over $\operatorname{Spa}(K,\mathcal{O}_K)$, whereas it's '*compactification*' $\overline{\mathbf{B}}_K^1 = \operatorname{Spa}(K\langle T\rangle,\mathcal{O}_K + \mathfrak{m}_K\langle T\rangle)$ is.

We now can finally begin to study the de Rham complex and the analogue of the Poincaré lemma we aim to find.

Definition 5.9. Let K be a non-archimedean field and $X \to \operatorname{Spa}(K,\mathcal{O}_K)$ a smooth rigid analytic variety. The *de Rham complex* of X is the complex of sheaves on X given by

$$\mathcal{O}_X \to \Omega^1_{X/K} \xrightarrow{d} \Omega^2_{X/K} \xrightarrow{d} \dots$$

where $\Omega^i_{X/K}$ is the sheaf of continuous differential i-forms.

Recall that, as mentioned in the discussion on the first lecture in Remark 1.9, the de Rham complex for $\operatorname{Spa}(K\langle T\rangle,\mathcal{O}_K\langle T\rangle)$ is

$$K\langle T\rangle \xrightarrow{d} K\langle T\rangle\, dT$$

which has a very large cokernel because of relevant convergence issues of the '*naïve integral*'.
The local theory of adic spaces is governed by that of perfectoids, which brings us to the following definition.

Definition 5.10.

1. A complete Tate ring R is *perfectoid* if it is
   (a) uniform (i.e. R° is a ring of definition),
   (b) there exists a pseudo-unifomiser $\varpi \in R^\circ$ such that $\varpi^p \mid p$ in R°, and
   (c) the Frobenius map
   $$\Phi : R^\circ/\varpi \to R^\circ/\varpi^p$$
   given by $\Phi(\overline{f}) = \overline{f}^p$, is an isomorphism.

2. A Huber pair (A,A^+) is *perfectoid* if A is perfectoid.

3. A perfectoid space is an adic space X which is locally isomorphic to $\operatorname{Spa}(R,R^+)$ for a perfectoid Huber pair (R,R^+).

Remark 5.11. Remarkably, for any perfectoid Huber pair (R, R^+) the structure pre-sheaf on $X = \mathrm{Spa}(R, R^+)$ is always a sheaf - this follows from the (non obvious) fact that all rational open subsets in $X = \mathrm{Spa}(R, R^+)$ are also adic spectra of perfectoid Huber pairs; in particular, they're adic spectra of *uniform* Huber pairs.

In the next lecture, we'll discuss some examples and motivate this notion.

6 11th of May, perfectoid spaces

We fix a prime $p \in \mathbf{Z}$. In the last lecture we introduced the following notion.

Definition 6.1.

1. A complete Tate ring R is *perfectoid* if R is uniform and there exists a pseudouniformiser $\varpi \in R^\circ$ such that ϖ^p divides p in R° and the Frobenius map
$$\Phi : R^\circ/\varpi \longrightarrow R^\circ/\varpi^p$$
defined by $x \mapsto x^p$ is an isomorphism.
2. A Huber pair (R, R^+) is *perfectoid* if R is perfectoid.
3. An adic space is *perfectoid* if it admits an open cover of affinoid adic spaces of the form $\mathrm{Spa}(R, R^+)$ where R is perfectoid.

The condition on the Frobenius in Definition 6.1 is really a condition on surjectivity, as the following Lemma illustrates.

Lemma 6.2. *Let R be a complete uniform Tate ring and $\varpi \in R^{\circ\circ} \cap R^\times$ a pseudouniformiser such that $\varpi^p \mid p$ in R°. Then:*

1. *The Frobernius map*
$$\Phi : R^\circ/\varpi \to R^\circ/\varpi^p$$
is injective.
2. *Φ is surjective if and only if the mod-p Frobenius map*
$$\begin{aligned} \Phi_p : R^\circ/p &\longrightarrow R^\circ/p \\ x &\longmapsto x^p \end{aligned}$$
is surjective. In particular, R is perfectoid if and only if the Frobenius Φ is an isomorphism for any pseudouniformiser ϖ whose p-th power divides p.

Proof.

1. Suppose the image of $x \in R^\circ$ in R°/ϖ lies in $\ker \Phi$ and we can thus express x's p-th power as
$$x^p = y \cdot \varpi^p, \;\; y \in R^\circ.$$
We then have
$$\left(\frac{x}{\varpi}\right)^p = y \in R^\circ$$
and since R° is integrally closed it follows that $\frac{x}{\varpi} \in R^\circ$ and thus x is multiple of ϖ in R°.
2. Since R°/p surjects onto R°/ϖ^p it immediately follows that if Φ_p is surjective so must Φ be. As for the converse, suppose Φ is surjective and fix $x \in R^\circ$. Since R is uniform, R° is a ring of definition and thus ϖ-adically complete. This implies we can express x as a power series
$$x = x_0^p + \varpi^p \cdot x_1^p + \varpi^{2p} \cdot x_2^p + \dots$$

by iteratively lifting a p-th root of

$$\frac{1}{\varpi^{kp}} \cdot (x - x_0^p - \ldots - \varpi^{(k-1)\cdot p} \cdot x_{k-1}^p) \in R^\circ/\varpi^p$$

to R°. Then the image of

$$x_0 + \varpi \cdot x_1 + \varpi^2 \cdot x_2 + \ldots \in R^\circ$$

in R°/p maps to x via Φ_p by construction.

■

Example 6.3. 1. $\mathbf{Q}_p$ is not perfectoid since its valuation group is $\mathbf{Z}$ and $\mathrm{val}(p) = 1 \implies$ there can be no elements whose p-th power divides p. More generally, a non-archimedean discretely valued field K can't be perfectoid since $\mathcal{O}_K/\varpi$ and $\mathcal{O}_K/\varpi^p$ have different legths as $\mathcal{O}_K/\varpi$-modules for any element $\varpi \in \mathcal{O}_K \setminus \{0\}$.

2. In contrast, an example of a perfectoid ring is the field $K = \mathbf{Q}_p^{\mathrm{cycl}}$ defined as the p-adic completion of $\mathbf{Q}_p(\mu_{p^\infty})$ which is the splitting field over $\mathbf{Q}_p$ of the family of polynomials $\{T^{p^n} - 1\}_{n\geq 0}$: its ring of integers $\mathbf{Z}_p^{\mathrm{cycl}} = \mathbf{Q}_p^{\mathrm{cycl},\circ}$ must be bounded since its the unit ball of the Banach algebra $\mathbf{Q}_p^{\mathrm{cycl}}$, and as its value group is given by the colimit

$$\mathrm{val}(\mathbf{Q}_p^{\mathrm{cycl}}) = \mathrm{val}(\mathbf{Q}_p(\mu_{p^\infty})) = \bigcup_{n\geq 1} \underbrace{\frac{1}{(p-1)p^n}\mathbf{Z}}_{=\mathrm{val}(\mathbf{Q}_p(\mu_{p^n}))} = \frac{1}{p-1}\mathbf{Z}\left[p^{-1}\right]$$

which is p-divisible, we see that there exists a pseudouniformiser ϖ whose p-power is equal to p up to some unit. To check Φ's surjectivity, we appeal to Lemma 6.2: we have $\mathbf{Z}_p^{\mathrm{cycl}} = \mathbf{Z}_p[\mu_{p^\infty}]_p^\wedge$ thus

$$\mathbf{Z}_p^{\mathrm{cycl}}/p \cong \varinjlim_{X\mapsto X^p} \mathbf{F}_p[X]/\psi_{p^n}$$

where $\psi_d \in \mathbf{F}_p[X]$ is the d-th cyclotomic polynomial for $d \geq 1$. Note that we have a surjective map

$$\mathbf{F}_p[X]/(X^{p^m} - 1) = \mathbf{F}_p[X]/(X-1)^{p^m} \to \mathbf{F}_p[X]/\psi_{p^n - 1}$$

which is compatible with the transition maps from the directed limit above; since the mod-p Frobenius map Φ_p on

$$\varinjlim_n \mathbf{F}_p[X]/(X^{p^n} - 1) \cong \mathbf{F}_p[X^{1/p^\infty}]/X - 1$$

is surjective, it follows that it must also be on $\mathbf{Z}_p^{\mathrm{cycl}}/p$.

3. Similar arguments as above show that the fields $\mathbf{Q}_p(p^{1/p^\infty})^\wedge$ and $\mathbf{F}_p(\!(t^{1/p^\infty})\!) := \left(\varinjlim_n \mathbf{F}_p(\!(t^{1/p^n})\!)\right)^\wedge$ are also perfectoid.

4. In general, any p-adic non-archimedean algebraically closed field K is perfectoid, K itself has all its p-roots.

5. Possibly the most important example for our purposes is the *perfectoid Tate algebra*, defined by

$$K\langle T_1^{1/p^\infty}, \ldots, T_n^{1/p^\infty}\rangle := \left(\mathcal{O}_K\left[T_1^{1/p^\infty}, \ldots, T_n^{1/p^\infty}\right]_\varpi^\wedge\right)[\varpi^{-1}].$$

A somewhat remarkable result is that perfectoid rings in positive characteristic coincide with perfect Tate rings - the condition that R° is bounded is forced upon as soon as one requires the characteristic-p ring R to be perfect.

Lemma 6.4. *Let R be a complete Tate ring such that $p = 0$ in R. Then R is perfectoid if and only if it is perfect, i.e. $\Phi_p : R \to R$ is an isomorphism.*

Proof. Suppose first that R is perfectoid, and thus Φ_p is surjective by Lemma 6.2. However, since R° is bounded it must also be reduced by Lemma 3.3 $\implies$ $\Phi_{p|R^\circ}$ is injective and hence Φ_p must also be as $R = R^\circ[\varpi^{-1}]$.

Conversely, suppose Φ_p is an isomorphism and, as mentioned, by Lemma 6.2 the only condition from Definition 6.1 which doesn't directly follow is that R° is bounded.
Let $R_0 \subset R$ be a ring of definition and by the Banach Open Mapping Theorem it follows that $\Phi_p(R_0) \subset R$ is open $\implies$ $\varpi R_0 \subset \Phi_p(R_0)$ for some pseudo-uniformiser $\varpi \in R^{\circ\circ}$. Since Φ_p is bijective this implies $\Phi_p^{-1}(R_0) \subset \varpi^{-1/p} \cdot R_0$ and inductively

$$\Phi_p^{-n}(R_0) \subset \varpi^{-\sum_{i=1}^{n} p^{-i}} \cdot R_0.$$

Since $\sum_{i=0}^{n} p^{-i} < 1$ we have $\Phi_p^{-n}(R_0) \subset \varpi^{-1} R_0$ whence the subring

$$R_0' := \bigcap_n \Phi_p^{-n}(R_0) \subset \varpi^{-1} \cdot R$$

is bounded - because it's contained in $\varpi^{-1} R_0$ - and open, as $R_0 \subset \Phi_p^{-n}(R_0)$ for every n because Φ_p is a bijection on R. We can thus conclude that R_0' is a ring of definition in R.
If $x \in R$ is a topologically nilpotent element then $x^{p^n} \in R_0'$ for some n because R_0' is open $\implies$ $x \in R_0'$ by applying Φ_p^{-n}, since R_0' is Φ_p-stable by construction. Seeing as $\varpi \cdot x$ is topologically nilpotent for any power-bounded element $x \in R^\circ$ we have the inclusions

$$R^\circ \subset \varpi^{-1} \cdot R^{\circ\circ} \subset \varpi^{-1} \cdot R_0'$$

which show that R° must be bounded as was claimed. ■

Definition 6.5. A *perfectoid field* is a non-archimedean field K which is perfectoid as a Tate ring.

We now come to our main motivation for introducing perfectoid spaces: they serve as a *local model* for rigid spaces, much like spectra of local rings do in the Zariski topology for schemes, or spectra of strictly henselian rings in étale cohomology; in this setting we have to introduce a new topology which makes this notion precise - the following proposition illustrates the descibed phenomenon.

Proposition 6.6. *Let A be a Tate algebra over $\mathbf{Q}_p$. Then there exists a filtered family of finite étale faithfully flat A-algebras*

$$A \to A_i, \;\; i \in I,$$

such that the colimit $A_\infty = \varinjlim_i A_i$ has no non-split finite étale faithfully flat algebras $A_\infty \to B$.

Although we won't discuss the proof of this result, we will sketch why the non-existence of non-split A_∞ algebras implies that the completion $\widehat{A}_\infty := R$ is a perfectoid ring. In Bhatt's notes [Bha] it's shown that $\widehat{A}_\infty$ is uniform, whereas the condition on the Frobenius can be argued as follows: given a pseudo-uniformiser $\varpi_0 \in R$ we have $\varpi_0 \mid p^n$ for some large n (because $p^n \in \varpi_0 \cdot R^\circ$ for $n >> 0$). Thus the R-algebra

$$R' = R[x]/(x^{p^n} - \varpi_0 \cdot x - \varpi_0)$$

is finite étale and faithfully flat. If ϖ_0 was chosen in A_∞ and thus R' is the base change along $A_\infty \to \widehat{A}_\infty$ of finite étale faithfully flat algebra over A_∞, then we get that $R \to R'$ admits a section and thus there exists $\varpi \in R$ such that

$$\varpi^{p^n} - \varpi_0 \cdot \varpi - \varpi_0 = 0.$$

Thus $\varpi^{p^n} \mid \varpi_0$ in R° and hence ϖ is a unit. Furthermore we have

$$\left(\frac{p}{\varpi^p}\right)^n = \varpi^{p^n - p\cdot n} \cdot \frac{p^n}{\varpi^{p^n}} = \varpi^{p^n - p\cdot n} \cdot \frac{p^n}{\varpi_0} \cdot \frac{1}{1+\varpi} \in R^\circ$$

and since R° is integrally closed it follows that $\varpi^p \mid p$ in R°.

Lastly, to check that $\Phi_p : R/p \to R/p$ is surjective we can argue as in the construction of ϖ: by approximation, it'll be sufficient to show that all elements in R/p which are images of elements in A_∞ lie in the image of Φ_p, and for any $f \in A_\infty^\circ$ the R-algebra

$$R_f := R[X]/(x^p - \varpi^p \cdot x - f)$$

is the base change along $A_\infty \to R$ of a faithfully flat finite étale A_∞-algebra, and thus admits a splitting. The corresponding element $f' \in R$ satisfying the equation

$$f'^p - \varpi^p \cdot f' = f$$

lies in R° by the same arguments and maps to the image of f in R°/p via the composition

$$R^\circ \to R^\circ/p \xrightarrow{\Phi_p} R^\circ/p.$$

Proposition 6.6 proves that, in passing to a pro-étale cover (i.e. a projective limit of étale covers), any rigid analytic adic space X is locally isomorphic to a perfectoid space. For example, we have a pro-étale map

$$\mathbf{Q}_p\langle T^{\pm 1}\rangle \to \mathbf{Q}_p^{\mathrm{cycl}}\langle T^{\pm 1/p^\infty}\rangle$$

which shows that the punctured unit disc over $\mathbf{Q}_p$ is pro-étale-locally isomorphic to the punctured perfectoid unit disc over $\mathbf{Q}_p^{\mathrm{cycl}}$.

7 25th of May, tilts

Today we study how over Perfectoid spaces certain objects, particularly relevant to the de Rham comparison isomorphism (such as the period ring B_{dR} and the version of the Poincaré lemma we're after), become accessible. This relies on Tilting - a bridge between characteristic 0 perfectoid rings and characteristic p perfect rings.

Definition 7.1. Let R be any ring. We define its *tilt* by

$$\mathrm{Tlt}(R) := \varprojlim_{\Phi_p : R/p \to R/p} R/p := \left\{(x_0, x_1, x_2, \ldots) \mid x_i \in R/p, x_i = x_{i+1}^p\right\}$$

where $\Phi_p : R/p \to R/p, \Phi_p(x) = x^p$ is the mod-p Frobenius.

Note that since Φ is a ring homomorphism $R/p \to R/p$ we have that $\mathrm{Tlt}(R)$ is a perfect characteristic p ring.

Example 7.2.

1. We have $\mathbf{Z}/p \cong \mathbf{Z}_p/p \cong \mathbf{F}_p$ which is perfect, thus

$$\mathrm{Tlt}(\mathbf{Z}_p) \cong \mathrm{Tlt}(\mathbf{Z}) \cong \mathbf{F}_p.$$

2. For the Tate algebra over $\mathbf{Z}_p$ we have

$$\mathrm{Tlt}(\mathbf{Z}_p\langle T\rangle) = \varprojlim_{\Phi} \mathbf{F}_p[T] = \bigcap_{n\geq 0} \mathbf{F}_p[T^{p^n}] = \mathbf{F}_p.$$

For Noetherian rings in characteristic p, tilts are quite tame:

Exercise 7.3. Let R be a Noetherian $\mathbf{F}_p$-algebra. Then $\operatorname{Tlt}(R)$ is a finite product of fields, as many as there are connected components in $\operatorname{Spec} R$.

We present another few examples which produce more intriguing tilts.

Example 7.4. 1. If we take $R = \mathbf{Z}_p[p^{1/p^\infty}]_p^\wedge$ then

$$\operatorname{Tlt}(R) = \varprojlim_{\Phi} \mathbf{F}_p[T^{1/p^\infty}]/T \cong \varprojlim(\mathbf{F}_p[T^{1/p^\infty}]/T \twoheadleftarrow \mathbf{F}_p[T^{1/p^\infty}]/T^p \twoheadleftarrow \mathbf{F}_p[T^{1/p^\infty}]/T^{p^2} \twoheadleftarrow \dots) \cong \mathbf{F}_p[\![T^{1/p^\infty}]\!]$$

where $\mathbf{F}_p[T^{1/p^\infty}]/T^{p^i} \twoheadrightarrow \mathbf{F}_p[T^{1/p^\infty}]/T^{p^{i-1}}$ is the projection map. In particular, note that $\operatorname{Tlt}(R)$ and R both have dimension one.

2. Similarly, if we consider $R = \mathbf{Z}_p[\mu_{p^\infty}]$ (its completion, which we've encountered before as $\mathbf{Z}_p^{\mathrm{cycl}} = \mathbf{Q}_p^{\mathrm{cycl},\circ}$, has the same tilt since this only depends on the quotient modulo p) we get

$$\operatorname{Tlt}(R) = \varprojlim_{\Phi} \mathbf{F}_p[U^{1/p^\infty}]/(U-1) \cong \varprojlim_{\Phi} \mathbf{F}_p[T^{1/p^\infty}]/T \cong \mathbf{F}_p[\![T^{1/p^\infty}]\!]$$

(recall that the p^n-th cyclotomic polynomial is $\Phi_p(T^{p^{n-1}}) = 1 + x^{p^{n-1}} + \ldots + x^{(p-1)p^{n-1}}$) where we make the substitution $T = U - 1$ and thus $T^{1/p^n} = U^{1/p^n} - 1$ in characteristic p.

The following simple lemma turns out to be key for many constructions regarding tilting.

Lemma 7.5. *Let R be any ring and $I \subseteq R$ an ideal containing p. Then if $a \equiv b$ modulo I for $a, b \in R$ we have*

$$a^{p^k} \equiv b^{p^k} \mod I^k$$

for all $k \geq 0$.

Proof. We argue by induction on k, where the base case is our hypothesis on a and b. By replacing a and b with a^{p^k}, b^{p^k} it'll be equivalent to show

$$a \equiv b \mod I^k \implies a^p \equiv b^p \mod I^{k+1}.$$

If we express $b = a + c$ with $c \in I^k$ we have

$$b^p = a^p + \underbrace{\binom{p}{1} a^{p-1} c + \ldots + \binom{p}{1} a c^{p-1}}_{\in p \cdot I^k \subseteq I^{k+1}} + \overbrace{c^p}^{\in I^{pk}} \equiv a^p \mod I^{k+1}.$$

■

Corollary 7.6. *Let R be any ring and $p \in I \subset R$ an ideal such that R is I-adically complete. Then the map of (multiplicative) monoids*

$$\varprojlim_{x \mapsto x^p} R \xrightarrow{\cong} \operatorname{Tlt}(R/I)$$

is an isomorphism. In particular, $\varprojlim R$ can be endowed with a ring structure induced by that of $\operatorname{Tlt}(R/I)$*; the sum of two elements $(x_i)_i, (y_i)_i \in \varprojlim R$ is explicitly given by*

$$(x_0, x_1, x_2, \ldots) + (y_0, y_1, y_2, \ldots) = \left(\lim_{n\to\infty} (x_n + y_n)^{p^n}, \lim_{n\to\infty} (x_{n+1} + y_{n+1})^{p^n}, \lim_{n\to\infty} (x_{n+2} + y_{n+2})^{p^n}, \ldots \right)$$

Proof. Given

$$\overline{x} = (\overline{x}_0, \overline{x}_1, \ldots) \in \mathrm{Tlt}(R/I)$$

and arbitrary lifts $x_i \in R$ of $\overline{x}_i \in R/I$ for each i, we have that by Lemma 7.5 the sequence $\{x_i^{p^i}\}_{i\geq 0}$ is convergent in R with respect to the I-adic topology, since the difference of two successive terms

$$x_i^{p^i} - x_{i+1}^{p^{i+1}}$$

lies in I^{i+1} as $x_i \equiv x_{i+1}^p \mod I$ by definition. If we pick a different collection of lifts $x'_0, x'_1, \ldots$ then again by Lemma 7.5 we have

$$x_i \equiv x'_i \mod I \implies x_i^{p^i} \equiv x_i'^{p^i} \mod I^{i+1}$$

meaning that the sequences $\{x_i^{p^i}\}_i, \{x_i'^{p^i}\}_i$ converge to the same element in R. Thus $\overline{x}^{\#} := \lim_{i\to\infty} x_i^{p^i} \in R$ defines a well-posed multiplicative map

$$\mathrm{Tlt}(R/I) \to R$$

and clearly given

$$x = (x_0, x_1, \ldots) \in \varprojlim_{x\mapsto x^p} R$$

we see that applying $(-)^{\#}$ to x's image in $\mathrm{Tlt}(R/I)$ recovers the element x_0 by construction (since we're free to choose any set of lifts of the components of x's image in $\mathrm{Tlt}(R/I)$, we can pick $x_0, x_1, x_2, \ldots$ and then the sequence defining $\overline{x}^{\#}$ is in fact constant and equal to x_0). This implies the map

$$\begin{aligned} \mathrm{Tlt}(R/I) &\to \varprojlim_{x\mapsto x^p} R \\ \overline{x} &\mapsto (\overline{x}^{\#}, (\overline{x}^{1/p})^{\#}, (\overline{x}^{1/p^2})^{\#}, \ldots) \end{aligned}$$

is an inverse of $\varprojlim R \to \mathrm{Tlt}(R/I)$. The formula for the addition in the lemma follows by the fact that the inverse of the Frobenius map in $\mathrm{Tlt}(R/I)$ is given by

$$\begin{aligned} \Phi_p^{-1} : \mathrm{Tlt}(R/I) &\longrightarrow \mathrm{Tlt}(R/I) \\ (\overline{x}_0, \overline{x}_1, \overline{x}_2, \ldots) &\longmapsto (\overline{x}_1, \overline{x}_2, \overline{x}_3, \ldots) \end{aligned}$$

■

Definition 7.7. Let A be a perfectoid Tate ring. We define A's tilt by

$$A^{\flat} := \varprojlim_{x\mapsto x^p} A$$

endowed with the ring structure defined by

$$(x_0, x_1, x_2, \ldots) + (y_0, y_1, y_2, \ldots) = \left(\lim_{n\to\infty}(x_n + y_n)^{p^n}, \lim_{n\to\infty}(x_{n+1} + y_{n+1})^{p^n}, \lim_{n\to\infty}(x_{n+2} + y_{n+2})^{p^n}, \ldots\right)$$

for elements $(x_i)_i, (y_i)_i \in \varprojlim A$.

Lemma 7.8. *Let A be a perfectoid Tate ring.*

1. *There exists a pseudouniformiser $\varpi \in A^{\times} \cap A^{\circ\circ}$ such that $\varpi^p \mid p$ admitting a compatible system of p^i-th roots of unity $(\varpi^{1/p^i})_{i\geq 0}$ in A.*

2. $A^\flat$ *is perfectoid and the element*

$$\varpi^\flat = (\varpi, \varpi^{1/p}, \varpi^{1/p^2}, \ldots),$$

defined by part 1*, is a pseudo-uniformiser. Furthermore,*

$$\mathrm{Tlt}(A^\circ) \cong A^{\flat,\circ}, \text{ and } A^\circ/\varpi \cong A^{\flat,\circ}/\varpi^\flat.$$

Proof. 1. Given $\varpi_0 \in A^\times \cap A^{\circ\circ}$ satisfying $\varpi_0^p \mid p$, we may take $I = (\varpi_0^p)$ as in Corollary 7.6

$$\varprojlim_{x \mapsto x^p} A^\circ \xrightarrow{\cong} \mathrm{Tlt}(A^\circ/\varpi_0^p) \twoheadrightarrow A^\circ/\varpi_0^p$$

where the last map is the projection onto the zero-th component of the inverse limit - in this case this map is surjective as A is perfectoid and thus

$$A^\circ/\varpi_0^p \underset{\cong}{\xrightarrow{\Phi}} A^\circ/\varpi_0^{p^2} \twoheadrightarrow A^\circ/\varpi_0^p.$$

If we lift $\varpi_0 \in A^\circ/\varpi_0^p$ to $\varprojlim A^\circ$ we get our desired pseudo-uniformiser with a compatible system of its p^n-th roots of unity.

2. We start by showing that $A^\flat \cong A^{\flat,\circ}\left[(\varpi^\flat)^{-1}\right]$. The inclusion $A^\circ \hookrightarrow A$ produces an injection of multiplicative monoids

$$\begin{tikzcd}
\varprojlim\limits_{x \mapsto x^p} A^\circ \arrow[r, hook] \arrow[dr, hook] & \varprojlim\limits_{x \mapsto x^p} A \\
& \left(\varprojlim\limits_{x \mapsto x^p} A^\circ\right)\left[(\varpi^\flat)^{-1}\right] \arrow[u, hook, "\alpha"']
\end{tikzcd}$$

where

$$\left(\varprojlim_{x \mapsto x^p} A^\circ\right)\left[(\varpi^\flat)^{-1}\right] := \varinjlim\left(\varprojlim_{x \mapsto x^p} A^\circ \xrightarrow{\cdot\varpi^\flat} \varprojlim_{x \mapsto x^p} A^\circ \xrightarrow{\cdot\varpi^\flat} \ldots\right)$$

is a directed (co)limit of multiplicative monoids, and α's existence follows; α is also surjective because given

$$a = (a_0, a_1, a_2 \ldots) \in \varprojlim_{x \mapsto x^p} A$$

and $n > 0$ such that $\varpi \cdot a_0 \in A^\circ$ we see that $\varpi^{n/p^i} \cdot a_i \in A^\circ$ for every $i \geq 0$ since A° is integrally closed. Thus α applied to the fraction $(\varpi^\flat)^n \cdot a/(\varpi^\flat)^n \in \left(\varprojlim A^\circ\right)[(\varpi^\flat)^{-1}]$ recovers a.

By definition of the inverse limit topology, the subring $A^{\circ,\flat} = \varprojlim A^\circ \subset A^\flat$ is open, so we're interested in showing that $A^{\circ,\flat} \subset A^\flat$ is endowed with the $\varpi^\flat$-adic topology; this can be seen directly since

$$\varpi^{\flat,n} \cdot A^{\circ,\flat} = \left\{(a_0, a_1, \ldots) \in A^{\circ,\flat} \mid \varpi^n \mid a_0\right\}$$

(just as above, $\varpi^n \mid a_0 \implies \varpi^{n/p^i} \mid a_i$ for every $i \geq 1$) - since A°'s topology is ϖ-adic our claim follows, by continuity of the projection onto the zero-th component $A^{\circ,\flat} \to A^\circ$.

Moreover, the ring of definition $A^{\circ,\flat}$ in fact coincides with the ring of power bounded elements $A^{\flat,\circ}$, because

$$x = (x_i)_{i \geq 0} \in A^\flat, \{x^n\}_{n \geq 0} \text{ bounded } \implies \{x^n\}_n \subseteq (\varpi^\flat)^{-m} \cdot A^{\circ,\flat} \text{ for some } m \geq 0$$

but this means that $x_i^n \in \varpi^{-m/p^i} \cdot A^\circ$ for every i. Since A° is bounded as A is perfectoid by assumption, it follows that $x_i \in A^\circ$ for every i. Thus the Tate ring $A^\flat$ is uniform.

Finally, to conclude that $A^\flat$ is perfectoid we check that $(-)^\#$ defines an isomorphism

$$A^{\circ,\flat}/\varpi^\flat \xrightarrow[\cong]{(-)^\#} A^\circ/\varpi.$$

Since A is perfectoid, $(-)^\# : A^{\circ,\flat} \to A^\circ/\varpi$ is surjective, and its kernel is given by elements $x = (x_0, x_1, \ldots) \in A^{\circ,\flat}$ such that $\varpi \mid x_0 \implies \varpi^{1/p^i} \mid x_i$ and thus x is a multiple of $\varpi^\flat$ by definition.

■

We conclude today's lecture with a quick introduction to Witt vectors, which we'll finish off next time. Observe that the functor

$$\mathrm{Tlt} : \{p\text{-adically complete rings } R\} \longrightarrow \{\text{perfect } \mathbf{F}_p\text{-algebras}\}$$

commutes with arbitrary limits $\implies$ by abstract principles there must exists a left adjoint W from the category of perfect $\mathbf{F}_p$-algebras to p-adic complete rings.

Definition 7.9. Let R be any ring. The ring of *p-typical Witt vectors over R* is the ring $W(R)$ defined by

$$W(R) = R^{\mathbf{N}} = \{(r_0, r_1, r_2, \ldots) \mid r_i \in R\}$$

with addition and multiplication defined and characterised by the following property: $R \mapsto W(R)$ is a functor from the category of rings to itself making the *ghost map*

$$\begin{aligned} &\mathrm{gh}_R : W(R) \longrightarrow R^{\mathbf{N}} \\ &(r_0, r_1, \ldots) \longmapsto (r_0\,,\ r_0^p + pr_1, r_0^{p^2} + pr_1^p + p^2 r_2\,,\ \ldots, r_0^{p^n} + pr_1^{p^{n-1}} + \ldots + p^n r_n\,,\ \ldots) \end{aligned}$$

a natural transformation between the functors from W to $(-)^{\mathbf{N}}$, the later being defined by the product ring structure.

It isn't hard to show by induction that multiplication and addition between elements in $W(R)$ exist, by imposing the n-th term of $(r_i)_i + (r'_i)_i$ (resp. $(r_i)_i \cdot (s_i)_i$) to be a polynomial S_n (resp. P_n) in the elements $r_0, \ldots, r_n, r'_0, \ldots, r'_n$.

Example 7.10. We list some first observations which follow straight from Definition 7.9.

1. If $p \in R^\times$, then the ghost map gh_R is evidently bijective (inductively on the components one can produce the inverse of any element $(r_0, r_1, \ldots) \in R^{\mathbf{N}}$). Thus $W(R)$ is actually isomorphic to $R^{\mathbf{N}}$ via the *ring homomorphism* gh_R by construction

2. Similarly, if R is p-torsion free then gh_R is injective.

3. In contrast, if $p \cdot R = 0$ then the ghost map gh_R only depends on r_0 since

$$\mathrm{gh}_R(r_0, r_1, r_2, \ldots) = (r_0, r_0^p, r_0^{p^2}, \ldots)$$

   - since one is mainly interested in constructing W for algebras over $\mathbf{F}_p$ by our discussion preceding Definition 7.9, this explains why gh is called the ghost map, since it doesn't *'see'* the components in degrees higher than zero in positive characteristic.

4. The first two components of addition give relatively tame formulas, for higher degrees things get a little more tricky: given $(r_i)_i, (s_i)_i \in W(R)$ we have

$$(r_0, r_1, \ldots) + (s_0, s_1, \ldots) = \left(r_0 + s_0, r_1 + s_1 - \sum_{i=0}^{p-1} \binom{p}{i} r_0^i s_0^{p-i}, \ldots \right).$$

8 15th of June, Witt vectors and the ring B_{dR}^+

Last time we discussed the following result, expressing the ring of Witt vectors via their universal property.

Proposition 8.1. *There exists a unique ring structure on the functor*

$$\begin{aligned} W(-) : (\text{Rings}) &\longrightarrow (\text{Sets}) \\ R &\longmapsto R^{\mathbf{N}} \end{aligned}$$

such that the map

$$\begin{aligned} \mathrm{gh} : W(R) &\longrightarrow R^{\mathbf{N}} \\ (a_0, a_1, \ldots) &\longmapsto (a_0, a_0^p + pa_1, \ldots) \end{aligned}$$

defines a natural ring homomorphism in R.

Remark 8.2. Note that the ghost map gh defines a bijection between $W(R)$ and $R^{\mathbf{N}}$ if $p \in R$ is a unit, as can be seen inductively on the components. In contrast, if $p = 0$ in R then the ghost maps only depend on the zero-th coefficient:

$$\mathrm{gh}\,(a_0, a_1, a_2, \ldots) = (a_0, a_0^p, a_0^{p^2}).$$

The general strategy to prove facts about the Witt vectors ring $W(R)$ is to translate statements to the ghost components and prove them functorially as maps $R^{\mathbf{N}} \to R^{\mathbf{N}}$, reducing to the case where $p \in R^\times$.

Lemma 8.3. *1. There exists a unique natural transformation $F : W(-) \to W(-)$ of functors* (Rings) $\to$ (Rings) *lifting the Frobenius map $R \to R$ on $\mathbf{F}_p$-algebras R, such that for all R the following diagram*

$$\begin{tikzcd} W(R) \arrow[r, "F"] \arrow[d, "\mathrm{gh}"] & W(R) \arrow[d] \\ R^{\mathbf{N}} \arrow[r, "\text{shift}"'] & R^{\mathbf{N}} \end{tikzcd}$$

commutes, where $\text{shift}(b_0, b_1, \ldots) = (b_1, b_2, \ldots)$.

2. *There exists a unique additive map $V : W(R) \to W(R)$ natural in R making the following diagram commute*

$$\begin{tikzcd} W(R) \arrow[r, "V"] \arrow[d, "\mathrm{gh}"] & W(R) \arrow[d, "\mathrm{gh}"] \\ R^{\mathbf{N}} \arrow[r, "v_{\mathrm{gh}}"] & R^{\mathbf{N}} \end{tikzcd}$$

where $v_{\mathrm{gh}}\,(b_0, b_1, b_2, \ldots) = (0, pb_0, pb_1, \ldots)$ and such that $F \circ V = \cdot p$ (in other words, $V(a_0, a_1, \ldots) = (0, a_0, a_1, \ldots)$).
Furthermore, we have the relation $xV(y) = V(F(x)y)$ for all $x, y \in W(R)$ - in particular, the image of V^n is an ideal in $W(R)$.
Setting $W_n(R) := W(R)/V^n(W(R)$ the ring of 'n-truncated Witt-vectors', yields an expression of $W(R)$ as the projective limit

$$W(R) = \varprojlim_n W_n(R).$$

3. *The map defined by $[-] : R \to W(R), [r] := (r, 0, 0, \ldots)$ - which on ghost components becomes* $\mathrm{gh}\,[r] = (r, r^p, r^{p^2}, \ldots)$ *- defines a natural multiplicative map.*

4. *Each $x \in W(R)$ can uniquely be expressed as $x = \sum_{n=0}^{\infty} V^n[r_n]$ with $r_n \in R$.*

5. *If $p = 0$ in R and $\Phi : R \to R$ denotes the Frobenius map (so that then $W(\Phi) = F$ as in part 1), then we furthermore have $F \circ V = V \circ F = \cdot p$.*

Remark 8.4. Intuitively, the functor $W(-)$ serves (non-literally) as an expression of the base change of an affine scheme $\operatorname{Spec}(R) \to \operatorname{Spec} \mathbf{Z}_p$ along the map $\operatorname{Spec} \mathbf{Z}_p \to \operatorname{Spec} \mathbf{Z}$.
Also, if R is a perfect $\mathbf{F}_p$-algebra (which will be our main case of interest) F defines an isomorphism $W(R) \to W(R)$ so that we can express

$$V = p \cdot F^{-1}$$

and thus every $x \in W(R)$ is expressed as

$$x = \sum_{n=0}^{\infty} p^n [r_n^{1/p^n}].$$

In this sense our previous intuition furthermore becomes "$\operatorname{Spec} R \widehat{\times} \operatorname{Spf} \mathbf{Z}_p = \operatorname{Spf} W(R)$" (again, non-literally). This can be made precise in the language of Diamonds [PAG].

The Witt vectors will serve their purpose in this course thanks to the following Proposition, expressing them as a left-adjoint.

Proposition 8.5. *The functor*

$$\mathrm{Tlt} : (p\text{-complete rings } R) \longrightarrow (\text{perfect } \mathbf{F}_p\text{-algebras})$$

admits $W(-)$ as a left adjoint. Moreover,

1. *The unit of the adjunction, for a perfect $\mathbf{F}_p$-algebra S*

$$S \cong \varprojlim_{\Phi} S \xrightarrow{\cong} \varprojlim_{\Phi} W(S)/p = \mathrm{Tlt}(W(S))$$

   is an isomorphism, and thus $W(-)$ is fully faithful; $W(-)$'s essential image is the (full) subcategory of p-complete, p-torsion free rings such that R/p is perfect.

2. *The counit*

$$\theta : W(\mathrm{Tlt}(R)) \to R$$

   is given by

$$\sum_{n=0}^{\infty} [a_n] p^n \longmapsto a_n^{\#} p^n.$$

   θ is called the associated Fontaine map.

We won't discuss the proof of this statement, but we will mention that the crucial point is in showing that θ indeed defines a ring homomorphism, which can be done by means of the commutative diagram through the n-truncated Witt-vectors

$$\begin{tikzcd} W_{n+1}(R) \arrow[rr, "\theta_n"] \arrow[dr] & & R/p^{n+1} \\ & W_{n+1}(R/p) \arrow[ur] & \end{tikzcd}$$

where $\theta_n(a_0, a_1, \ldots, a_n) = a_0^{p^n} + \ldots + p^n a_n$ is a ring homomorphism by construction of the truncated Witt-vectors. Then one can check that the relevant triangle diagrams commute, given that the unit and counit are explicit.

We now have the tools to discuss the relationship with perfectoid spaces.

Proposition 8.6. *Let (R, R^+) be a perfectoid Huber pair. Then*

1. *The Fontaine map $\theta : W(R^{+\flat}) \to R^+$ is surjective.*

2. *The kernel $\ker\theta$ is generated by a non-zero divisor of the form $\xi = p + [\eta]\alpha$ with $\eta \in R^{+\flat}$ a pseudo-uniformiser and $\alpha \in W(R^{+\flat})$.*

Proof. Let $\varpi \in R^\times \cap R^{\circ\circ}$ be a pseudo-uniformiser such that $\varpi^p \mid p$ and ϖ admits a compatible system of p^n-th roots of unity, defining a pseudo-uniformiser $\varpi^\flat \in R^{+\flat}$, as in Lemma 7.8.

1. Firstly, note that $W(R^{+\flat})$ is $[\varpi^\flat]$-adically complete: we have

$$W(R^{+\flat}) \cong \varprojlim W_{n+1}(R^{+\flat}),$$

so $W(R^{+\flat})$ is complete as soon as all the rings of truncates Witt vectors are, and for this one can argue directly by induction on n - $R^{+\flat}$ is of course $\varpi^\flat$-adically complete and we have the short exact sequence

$$0 \to V^n(W(R^{+\flat}))/V^{n+1}(W(R^{+\flat})) \cong W_0(R^{+\flat}) \to W_{n+1}(R^{+\flat}) \to W_n(R^{+\flat}) \to 0.$$

By commutativity of the diagram

$$\begin{tikzcd} W(R^{+\flat}) \arrow[r, "\theta"] & R^\flat/\varpi \\ R^{+\flat} \arrow[u, "{[-]}"] \arrow[r] & R^{+\flat}/\varpi^\flat \arrow[u, "\cong"'] \end{tikzcd}$$

we see that θ is surjective modulo $[\varpi^\flat] \implies \theta$ is surjective by $[\varpi^\flat]$-completeness.

2. Any ξ of the form $p + [\eta]\alpha$ is a non-zero divisor since

$$\xi \cdot \sum_{n=0}^{\infty} [x_n]p^n = 0 \implies \sum_{n=0}^{\infty} [x_n]p^{n+1} \equiv 0 \mod [\eta]$$

and thus $\eta \mid x_n$ for all $n \geq 0$. Replacing x_n by $\frac{x_n}{\eta}$ it follows by induction that $[\eta]^m \mid \sum\limits_{n=0}^{\infty} [x_n]p^n$ for all $m \implies \sum\limits_{n=0}^{\infty} [x_n]p^n = 0$.

As for the existence of ξ, note first that there exists $f \in \varpi^\flat \cdot R^{+\flat}$ such that $p \equiv f^\sharp$ modulo $p\varpi R^+$: since $\varpi \mid p$ in R^+ there exists $\beta \in R^{+\flat}$ such that $\beta^\# \equiv \frac{p}{\varpi} \mod p \cdot R^+$ (by part 2 of Lemma 7.8 $R^+ \twoheadrightarrow R^+/\varpi \xrightarrow{(-)^\#, \cong} R^{+\flat}/\varpi^\flat$); then we may set f to be $\varpi^\flat\beta \in \varpi^\flat \cdot R^{+\flat}$. If we now let $x \in W(R^{+\flat})$ be so that

$$p - f^\# = p\varpi\theta(x)$$

we get that $\xi = p - [f] - [\varpi^\flat]x \in \ker\theta$ by construction.

To show that the kernel $\ker\theta$ is generated by ξ we argue that $([\varpi^\flat], \xi)$ is a regular sequence in $W(R^{+\flat})$ (cfr. Tag 0AUH [Stacks]), and note that this is equivalent to the regularity of the sequence $([\varpi^\flat], p)$ as $\xi \equiv p$ in $W(R^{+\flat})/[\varpi^\flat]$; this will be sufficient to conclude since then θ induces an isomorphism

$$W(R^{+\flat})/(\xi, [\varpi^\flat]) \cong R^{+\flat}/\varpi^\flat \xrightarrow{\cong} R^+/\varpi$$

and thus by $W(R^{+\flat})/\xi$'s $[\varpi^\flat]$-completeness it follows that $\theta : W(R^{+\flat})/\xi \xrightarrow{\cong} R^+$.

Again by $[\varpi^\flat]$-adic completeness, $([\varpi^\flat], p)$ is regular if and only if $(p, [\varpi^\flat])$ is[2]. This however hardly requires an argument since $W(R^{+\flat})/p \cong W_0(R^{+\flat}) \cong R^{+\flat}$.

■

Example 8.7. Consider the perfectoid field $\mathbf{F}_p(\!(t^{1/p^\infty})\!)$ and its ring of integers $\mathbf{F}_p[\![t^{1/p^\infty}]\!]$. The Huber pair $(\mathbf{F}_p(\!(t^{1/p^\infty})\!), \mathbf{F}_p[\![t^{1/p^\infty}]\!])$ admits the untilt $(\mathbf{Q}_p(p^{1/p^\infty})^\wedge, \mathbf{Z}_p(p^{1/p^\infty})^\wedge_p)$. By unraveling the isomorphism $\left(\mathbf{Z}_p(p^{1/p^\infty}))^\wedge_p\right)^\flat \cong \mathbf{F}_p[\![t^{1/p^\infty}]\!]$ we see that

$$t^\# = \lim_{n\to\infty} (p^{1/p^n})^{p^n} = p$$

so that $p - [t] \in \ker \theta_{\mathbf{Q}_p(p^{1/p^\infty})^\wedge} \subseteq W(\mathbf{F}_p[\![t^{1/p^\infty}]\!])$.

As discussed in Example 7.4, another untilt of $(\mathbf{F}_p(\!(t^{1/p^\infty})\!), \mathbf{F}_p[\![t^{1/p^\infty}]\!])$ is the perfectoid field

$$(\mathbf{Q}_p^{\mathrm{cycl}}, \mathbf{Z}_p^{\mathrm{cycl}}) = (\mathbf{Q}_p(\mu_{p^\infty})^\wedge, \mathbf{Z}_p[\mu_{p^\infty}]^\wedge_p).$$

If we define the element $\epsilon = (1, \zeta_p, \zeta_{p^2}, \ldots) \in (\mathbf{Z}_p^{\mathrm{cycl}})^\flat$, which under the isomorphism from Example 7.4 corresponds to $t+1 \in \mathbf{F}_p[\![t^{1/p^\infty}]\!]$, we see that

$$\theta_{\mathbf{Q}_p^{\mathrm{cycl}}}\left(1 + [\epsilon^{1/p}] + \ldots + [\epsilon^{1-1/p}]\right) = 1 + \zeta_p + \ldots + \zeta_p^{p-1} = 0.$$

Since the image of $1 + [\epsilon^{1/p}] + \ldots + [\epsilon^{1-1/p}]$ in $W(\mathbf{F}_p[\![t^{1/p^\infty}]\!]/\mathfrak{m}_{\mathbf{F}_p[\![t^{1/p^\infty}]\!]}) \cong W(\mathbf{F}_p) = \mathbf{Q}_p$ is given by $1+1+\ldots+1 = p$ we see that $1 + [\epsilon^{1/p}] + \ldots + [\epsilon^{1-1/p}]$ is also a distinguished element of degree one.

To check that these two elements generate the kernels of the respective Fontaine maps, we may appeal to the following lemma.

Lemma 8.8. *Let (R, R^+) and $\varpi \in R^+, \varpi^\flat \in R^{+\flat}$ be as in Proposition 8.6, and suppose*

$$\xi = p + [\eta]\alpha,$$
$$\xi' = p + [\eta']\alpha'$$

are two elements which are distinguished of degree 1 (i.e. as in Proposition 8.6). If we have an inclusion

$$(\xi) \subseteq (\xi') \subseteq W(R^{+\flat})$$

then the ideals $(\xi) = (\xi')$ coincide.

Proof. Deine the '*p-differential map*' (cfr. these lectures)

$$\delta : W(R^{+\flat}) \longrightarrow W(R^{+\flat})$$
$$f \longmapsto \frac{\varphi(f) - f^p}{p}$$

where φ is the Frobenius map from Proposition 8.3, satisfying the relation

$$\delta(f \cdot g) = f^p \cdot \delta(g) + g^p \cdot \delta(f) + p \cdot \delta(f) \cdot \delta(g)$$

for all $f, g \in W(R^{+\flat})$. We see that

$$\delta(\xi) = \frac{p + [\eta^p]\varphi(\alpha) - (p + [\eta]\alpha)^p}{p} \longmapsto 1 - p^{p-1} \in W(R^{+\flat}/(R^{+\flat})^{\circ\circ})^\times.$$

[2]Exercise: Let A be a ring and $a, b \in A$ two element. Then if A is a-adically complete the sequence (a, b) is regular if and only if (b, a) is.

and since $W((R^{+\flat})^{\circ\circ}) = \ker(W(R^{+\flat}) \to W(R^{+\flat}/(R^{+\flat})^{\circ\circ}) \subset W(R^{+\flat})$ is made up of topologically nilpotent elements with respect to the $[\varpi^\flat]$-adic topology (once again, $[\varpi^\flat]$-adic completeness here is very important) we can conclude that $\delta(\xi)$ - and $\delta(\xi')$ likewise - is a unit.

Expressing ξ as $u\xi'$ for some $u \in W(R^{+\flat})$ yields

$$\delta(\xi) = u^p\delta(\xi') + \underbrace{\xi'^p\delta(u) + p\delta(u)\delta(\xi')}_{\in (p,[\varpi^\flat])} \in W(R^{+\flat})^\times$$

and so $u^p = \delta(\xi')^{-1}\delta(\xi) + O$ where $O \in (p, [\varpi^\flat])$. Because $W(R^{+\flat})$ is $(p, [\varpi^\flat])$-adically complete, we see that $u \in W(R^{+\flat})^\times$ and the game is won. ■

Proposition 8.6 allows us to control (and effectively '*parametrise*') the untilts of a perfectoid Huber pair (S, S^+) over $\mathbf{F}_p$. A fairly straightforward consequence is the following important equivalence of categories.

Corollary 8.9 (Tilting Equivalence)**.** *Let (R, R^+) be a perfectoid Huber pair. The tilt*

$$\left\{\textit{Perfectoid Huber pairs over } (R, R^+)\right\} \xrightarrow{\cong} \left\{\textit{Perfectoid Huber pairs over } (R^\flat, R^{+\flat})\right\}$$
$$(S, S^+) \longmapsto (S^\flat, S^{+\flat})$$

is an equivalence of categories, whose inverse is given by

$$(T, T^+) \longmapsto (W(T^+) \otimes_{W(R^{+\flat})} R\, ,\, W(T^+) \otimes_{W(R^{+\flat})} R^+).$$

Proof. If $\ker\theta_R$ is generated by $\xi \in W(R^{+\flat})$ as in Lemma 8.6, we have that for any perfectoid Huber pair (S, S^+) over (R, R^+) the image of ξ in $W(S^{+\flat})$ wil generate $\ker\theta_S$ since it'll of course be distinguished of degree 1 and Lemma 8.8 applies. This implies

$$S \cong \left(W(S^{+\flat})/\xi\right)\left[[\varpi]^{-1}\right] \cong W(S^{+\flat}) \otimes_{W(R^{+\flat})} R$$

where $\varpi \in R^+$ is any pseudo-uniformiser. As for showing that the converse composition is isomorphic to the identity, all that there is to check is that for any Huber parir (T, T^+) over $(R^\flat, R^{+\flat})$ the ring $W(T^+) \otimes_{W(R^{+\flat})} R$ is indeed perfectoid as then

$$(W(T^+) \otimes_{W(R^{+\flat})} R)^\flat \cong (W(T^+)/\xi)^\flat \left[[\varpi]^{-1}\right] \cong T$$

by Proposition 8.5. ■

We are now at the point where we can introduce the period ring we're after, briefly mentioned in the first lecture.

Definition 8.10. Let (R, R^+) be a perfectoid Huber pair. We define

$$B^+_{\mathrm{dR}}(R, R^+) := \left(W(R^{+\flat})\left[[\varpi^\flat]^{-1}\right]\right)^\wedge_{\ker\theta_R},$$

and $B_{\mathrm{dR}}(R, R^+) := B^+_{\mathrm{dR}}\left[\xi^{-1}\right]$ where $\ker\theta = (\xi)$.

Remark 8.11. The most important case for us will be when $(R, R^+) = (C, \mathcal{O}_C)$ for some algebraically closed non-archimedean field $C/\mathbf{Q}_p$. In this case we see that the ideal $\ker\theta \subset B^+_{\mathrm{dR}} := B^+_{\mathrm{dR}}(C, \mathcal{O}_C)$ is maximal (since $B^+_{\mathrm{dR}}/\ker\theta \cong C$ is a field) and by $\ker\theta$-adic completeness we also see that B^+_{dR} is local. Since $\ker\theta$ is principal we can apply Tag 00PD and conclude that actually B^+_{dR} is a discrete valuation ring.

A crucial observation to make is that the construction of $B_{\mathrm{dR}}^+(R, R^+)$ is functorial in (R, R^+) by Lemma 8.8 and so B_{dR}^+ acquires an action of the Galois group $G = \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$, so that the Fontaine map (being a natural transformation)

$$\theta : B_{\mathrm{dR}}^+ \longrightarrow C$$

is G-equivariant.

9 22th of June, some computations in B_{dR}^+ and almost purity

We start off today's lecture by exploring B_{dR}^+ through the medium of some simple computations. Fix a non-archimedean algebraically closed field $C/\mathbf{Q}_p$ over $\mathbf{Q}_p$ and a system of p^n-th roots of p

$$p, p^{1/p}, p^{1/p^2}, \ldots \in C$$

defining thus a pseudo-uniformiser $p^\flat \in \mathcal{O}_C^\flat$. Recall that in Definition 8.10 we introduced the ring

$$B_{\mathrm{dR}}^+ = \left(W(\mathcal{O}_C^\flat)\left[[p^\flat]^{-1}\right]\right)_\xi^\wedge$$

where $\xi = p - [p^\flat]$ just as in Example 8.7. Given a compatible system of p^n-th roots of unity

$$1, \zeta_p, \zeta_{p^2}, \ldots \in C$$

we have a corresponding element $\epsilon \in \mathcal{O}_C^\flat$. The ideal $\ker\theta$ can also be expressed in terms of ϵ, via the relation

$$\ker\theta = \left(\frac{[\epsilon]-1}{\left[\epsilon^{1/p}\right]-1}\right) = \left(1 + \left[\epsilon^{1/p}\right] + \left[\epsilon^{2/p}\right] + \ldots + \left[\epsilon^{1-1/p}\right]\right)$$

since

$$\theta\left(1 + \left[\epsilon^{1/p}\right] + \left[\epsilon^{2/p}\right] + \ldots + \left[\epsilon^{1-1/p}\right]\right) = 1 + \zeta_p + \ldots + \zeta_p^{p-1} = 0$$

and $1 + \left[\epsilon^{1/p}\right] + \ldots + \left[\epsilon^{1-1/p}\right]$ is also a distinguished element of degree one, so that Lemma 8.8 applies - indeed we have

$$1 + \left[\epsilon^{1/p}\right] + \left[\epsilon^{2/p}\right] + \ldots + \left[\epsilon^{1-1/p}\right] \longmapsto 1 + 1 + \ldots + 1 = p \in W(\mathcal{O}_C^\flat/\mathfrak{m}_{\mathcal{O}_C^\flat}).$$

Define $\mu := [\epsilon] - 1 \in \ker\theta$; since $\theta(\left[\epsilon^{1/p}\right] - 1) = \zeta_p - 1 \in C^\times$ is a unit, we see that

$$\mu = \frac{[\epsilon]-1}{\left[\epsilon^{1/p}\right]-1} \cdot \left(\left[\epsilon^{1/p}\right] - 1\right)$$

also generates $\ker\theta$ - i.e. is a uniformiser for the discrete valuation ring B_{dR}^+ - and can be used to more directly study the (one dimensional) Galois representation

$$\ker\theta / \ker\theta^2 \in \mathrm{Rep}_{\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)}$$

which is the cotangent space of B_{dR}^+. We see that

$$g \in \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p) \implies g([\epsilon]-1) = [g(\epsilon)] - 1 = [\epsilon]^{\chi(g)} - 1 = (1+\mu)^{\chi(g)} - 1 \equiv \chi(g)\cdot\mu \mod \mu^2 = \ker\theta^2$$

where $\chi : \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p) \to \mathbf{Z}_p^\times$ is the cyclotomic character. We thus get

$$\ker\theta / \ker\theta^2 \cong C(1) \in \mathrm{Rep}_{\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)}$$

and more generally $\ker \theta^n / \ker \theta^{n+1} \cong C(n)$ for $n \in \mathbf{Z}_{\geq 0}$.

As a side remark, note that the vanishing locus $V(\mu) \subset \mathrm{Spec}(\mathbf{A}_{\inf})$ is actually made up of infinitely many points, corresponding to the orbit of μ via the Frobenius map - indeed

$$\mu = [\epsilon] - 1 = \underbrace{\frac{[\epsilon]-1}{\left[\epsilon^{1/p}\right]-1}}_{\in \mathbf{A}_{\inf}^{\times}} \underbrace{\left(\left[\epsilon^{1/p}\right]-1\right)}_{=\varphi^{-1}(\mu)} = \underbrace{\frac{[\epsilon^{1/p}]-1}{\left[\epsilon^{1/p^2}\right]-1}}_{\in \mathbf{A}_{\inf}^{\times}} \underbrace{\left(\left[\epsilon^{1/p^2}\right]-1\right)}_{=\varphi^{-2}(\mu)} = \dots$$

We now move onto discussing the form Corollary 8.9 takes on the geometric side of the story. The first consequence we draw is a result on the vanishing of all higher cohomology groups for affinoid perfectoid spaces, akin to Serre's theorem in algebraic geometry.

Theorem 9.1 (Scholze, [Perf] Theorem 1.8)**.** *Let (R, R^+) be a perfectoid Huber pair and $(R^\flat, R^{+\flat})$ its tilt. We denote by $X, X^\flat$ the topological spaces*

$$X = \mathrm{Spa}(R, R^+), X^\flat = \mathrm{Spa}(R^\flat, R^{+\flat}).$$

1. *The map*

$$\begin{aligned} \alpha : X &\longrightarrow X^\flat \\ x &\longmapsto x^\flat := (f \in R^\flat \mapsto |f^\#(x)|) \end{aligned}$$

defines a homeomorphism $X \approx X^\flat$ which identifies rational opens.

2. *For all rational open subsets $U \subset X$ and corresponding rational open $\alpha(U) \subset X^\flat$, the pair $(\mathcal{O}_X(U), \mathcal{O}_X(U)^+)$ is perfectoid whose tilt is $(\mathcal{O}_{X^\flat}(U^\flat), \mathcal{O}_X^(U^\flat)^+)$ - in particular, $U^\flat \approx \alpha(U)$.*

3. *$\mathcal{O}_X$ and $\mathcal{O}_{X^\flat}$ are sheaves, and the higher comohomology groups*

$$H^p(X, \mathcal{O}_X), H^p(X, \mathcal{O}_X^+), \; p > 0$$

are zero and almost zero respectively[3]*.*

Proof. We give some indications on how the proof goes; most of what isn't discussed is mainly just a technical verification.

1. We see that α is well defined since $|-(x^\flat)|$ satisfies the strong triangle inequality by

$$\begin{aligned} |(f+g)(x^\flat)| := \lim_{n\to\infty} |\left((f^{1/p^n})^\# + (g^{1/p^n})^\#\right)(x)|^{p^n} &\leq \lim_{n\to\infty} \max\{|(f^{1/p^n})^\#(x)|, |(g^{1/p^n})^\#(x)|\}^{p^m} \\ &= \max\{|f^\#(x)|, |g^\#(x)|\} \end{aligned}$$

where

$$\begin{aligned} f &= (f^\#, (f^{1/p})^\#, \dots), \\ g &= (g^\#, (g^{1/p})^\#, \dots) \in R^\flat. \end{aligned}$$

For any rational open subset

$$V = U\left(\frac{f_1, \dots, f_n}{g}\right) \subset X^\flat$$

we see that by construction

$$\alpha^{-1}(V) = U\left(\frac{f_1^\#, \dots, f_n^\#}{g^\#}\right) \subset X$$

[3]A module M over R^+ is said to be *almost zero* if it's $R^{\circ\circ}$-torsion.

is also rational. To see that rational opens in X correspond to rational opens in $X^\flat$ one can use approximation techniques and argue that actually all rational open subsets $U \subset X$ are of the form

$$U = U\left(\frac{f_1^\#, \dots, f_n^\#}{g^\#}\right)$$

for some elements $f_1, \dots, f_n, g \in R^\flat$.
This shows that α is continuous, open and also surjective. What remains to be shown is that α is injective, but this also follows from the fact that α is a bijection on rational opens since the topological space X is spectral by Theorem 3.11 and thus T_0.

2. This can be achieved by explicitly checking that for any rational open $U = U\left(\frac{f_1,\dots,f_n}{g}\right) \subset X^\flat$, the rings of sections for the presheaves $\mathcal{O}_X, \mathcal{O}_X^+$ on $\alpha^{-1}(U) = U\left(\frac{f_1^\#,\dots,f_n^\#}{g^\#}\right)$ are given by the construction in Theorem 4.4.

3. Assume first that $p = 0$ in R, so that R is an algebra over the perfectoid field $K = \mathbf{F}_p(\!(\varpi^{1/p^\infty})\!)$. This implies (Lemma 6.13 in [Perf]) that R is the completion of a filtered colimit of reduced topologically of finite type K-algebras (S_i, S_i^+)[4]. It'll be sufficient to argue the exactness/almost exactness of the Cech complexes corresponding to an arbitrary finite cover $X = \bigcup_{j=1}^n U_j$ by rational open subsets, which we may assume are pulled back from opens $U_j^i \subseteq X_i := \mathrm{Spa}(S_i, S_i^+)$ - indeed, $H^i(X, \mathcal{O}_X)$ and $H^i(X, \mathcal{O}_X^+)$ are the filtered colimit of the cohomology of such Cech complexes; we can consider the corresponding Cech complex

$$0 \to \mathcal{O}_{X_i}(X_i) \to \prod_j \mathcal{O}_{X_i}(U_j^i) \to \prod_{j_1,j_2} \mathcal{O}_{X_i}(U_{j_1,j_2}^i) \to \dots$$

which is exact by Tate's acyclicity theorem - indeed, the X_i's are affinoid rigid analytic varieties in the sense of Definition 4.9, and we have a comparison result between the cohomology of such adic spaces and their associated counterparts from the theory developed by Tate [Ta] (cfr. [Hub2] Proposition 4.3). Since the S_i's are reduced (applying Lemmas 5.3 and then 3.3) it follows that the Cech complex of power-bounded elements

$$0 \to \mathcal{O}_{X_i}(X_i)^+ \to \prod_j \mathcal{O}_{X_i}(U_j^i)^+ \to \prod_{j_1,j_2} \mathcal{O}_{X_i}(U_{j_1,j_2}^i)^+ \to \dots$$

has cohomology groups which are $R^{\circ\circ}$-torsion. Banach's open mapping theorem applied to the surjection of K-vector spaces

$$\prod_{j_1,\dots,j_p} \mathcal{O}_{X_i}(U_{j_1,\dots,j_p}) \twoheadrightarrow \ker(d_{p+1})$$

where d_{p+1} is the $p+1$-th Cech differential, yields that the cohomology group $H^p(X, \mathcal{O}_X^+)$ is killed by some fixed power of K's pseudo-uniformiser ϖ; since the Cech complex is finite, this means all cohomology groups vanish when multiplied by ϖ^N for some fixed integer $N \in \mathbf{Z}_{\geq 0}$.

[4]In [Perf], Definition 6.9, the notion of *p-finiteness* for perfectoid Huber pairs (R, R^+) over $(K, \mathcal{O}_K)$ is introduced; Lemma 6.13 actually implies that the filtered colimit here is given by

$$(S, S^\circ) \xrightarrow{\text{Frob}} (S, S^\circ) \xrightarrow{\text{Frob}} (S, S^\circ) \xrightarrow{\text{Frob}} \dots$$

for a fixed reduced topologically of finite type K-algebra S.

Taking the directed limit along the Frobenius maps yields that the perfected Cech complex

$$0 \to \mathcal{O}_{X_i}(X_i)^{+,\text{perf}} \to \prod_j \mathcal{O}_{X_i}(U_j^i)^{+,\text{perf}} \to \dots$$

is also almost exact - indeed, given a p-boundary $x, d(x) = 0$, we have that $\varpi^N \cdot x^{p^n}$ can be expressed as dy for some $p-1$-cycle y, but then $d(y^{1/p^n}) = \varpi^{N/p^n} \cdot x \implies$ the p-th cohomology group is ϖ^N-torsion (by construction, the Frobenius map on the perfections commutes with the differentials). Passing to the filtered colimit along the X_i's yields almost exactness of the complex

$$0 \to \varinjlim_i \mathcal{O}_{X_i}(X_i)^{+,\text{perf}} \to \varinjlim_i \prod_j \mathcal{O}_{X_i}(U_j^i)^{+,\text{perf}} \to \dots.$$

And then upon ϖ-adic completion, which also preserves almost exactness, we see that $H^p(X, \mathcal{O}_X^+)$ is almost zero for all $p > 0$, and thus by inverting ϖ it follows that $H^p(X, \mathcal{O}_X) = 0$.

The case of zero characteristic can be deduced by means of tilting.

■

A much more important result is the equivalence of étale sites, giving us access to a comparison between the étale cohomology of a perfectoid space and that of its tilt.

Theorem 9.2 (Purity and Almost Purity). *Let R be a perfectoid Tate ring with and $R^\flat$ its tilt. Then*

1. *all finite étale R-algebras S are also perfectoid,*
2. *tilting defines an equivalence of categories*

$$\{\textit{finite étale } R\textit{-algebras}\} \xrightarrow{\cong} \left\{\textit{finite étale } R^\flat\textit{-algebras}\right\}$$
$$S \longmapsto S^\flat,$$

3. *if $R \to S$ is finite étale then $R^\circ \to S^\circ$ is* almost *finite étale*[5].

Proof. Once again, we only sketch the main ideas of the argument for when $p = 0$ in R. In this setting we have an analogous result for schemes:

Lemma 9.3. *Let $f : Z \to Y$ be an étale morphisms of schemes over $\mathbf{F}_p$ and suppose that Y is perfect. Then Z is also perfect*

Proof. Let $\Phi_Y : Y \xrightarrow{\cong} Y$ be Y's Frobenius and let $f^{(1)} : Z^{(1)} \to Y$ be the base change of f along Φ_Y

$$\begin{tikzcd} Z^{(1)} \arrow[r] \arrow[d, "f^{(1)}"'] \arrow[dr, phantom, "\ulcorner", very near start] & Z \arrow[d, "f"] \\ Y \arrow[r, "\Phi_Y"] & Y \end{tikzcd}$$

The pairs of morphisms $\Phi_Z : Z \to Z$ and $f : Z \to Y$ induce the *relative Frobenius map* $\Phi_{Z/Y} : Z \to Z^{(1)}$. Since $Z^{(1)} \to Z$ is an isomorphism, Φ_Z is an isomorphism if and only if $\Phi_{Z/Y}$ is. By the commutative triangle

$$\begin{tikzcd} Z \arrow[rr, "\Phi_{Z/Y}"] \arrow[dr, "f"'] & & Z^{(1)} \arrow[dl, "f^{(1)}"] \\ & Y & \end{tikzcd}$$

[5] We won't discuss the notion of almost finite étale maps; such a definition can be achieved by making sense of the *almost non-degeneracy* of the trace pairing $\mathrm{tr} : S^\circ \otimes_{R^\circ} S^\circ \to R^\circ$.

we see that $\Phi_{Y/S}$ is étale - as $f, f^{(1)}$ are - and is thus an isomorphism if and only if it is bijective. To check this we can thus replace Y by $\operatorname{Spec} L$ where L is a perfect field over $\mathbf{F}_p$, and then is Z must be a disjoint union of spectra of finite field extensions of L, which are of course perfect. ■

In conclusion, we see that for part 1, S is a perfect Tate algebra over $R \implies S$ is perfectoid by Proposition 6.4. ■

Corollary 9.4. *If $X = \operatorname{Spa}(R, R^+)$ is an affinoid perfectoid space and $X^\flat = \operatorname{Spa}(R^\flat, R^{+\flat})$ is its tilt, we have an equivalence of étale sites*

$$X_{\text{ét}} \cong X^\flat_{\text{ét}}$$

induced by the equivalence from Theorem 9.2. Furthermore, $\mathcal{O}_X$ and $\mathcal{O}_X^+$ are sheaves on the étale site of X and $H^p(X_{\text{ét}}, \mathcal{O}_X^+)$ is almost zero for $p > 0$.

A fairly unexpected consequence is the isomorphism between absolute Galois groups

$$\operatorname{Gal}(\overline{\mathbf{Q}}_p^{\text{cycl}}/\mathbf{Q}_p^{\text{cycl}}) \cong \operatorname{Gal}(\overline{\mathbf{F}_p(\!(t^{1/p^\infty})\!)}/\mathbf{F}_p(\!(t^{1/p^\infty})\!)) \cong \operatorname{Gal}(\overline{\mathbf{F}_p(\!(t)\!)}/\mathbf{F}_p(\!(t)\!)).$$

We conclude today's lecture by illustrating the phenomena of *almost finite maps*, hinting at how one might formally introduce the apt definition.

Example 9.5. Let $K = \mathbf{Q}_p(p^{1/p^\infty})^\wedge$ where $p \neq 2$ and let L/K be the finite field extension $L = K(\sqrt{p})$. If we denote by K_n the local field

$$K_n = \mathbf{Q}_p(p^{1/p^n})$$

and $L_n = K_n(\sqrt{p}) = \mathbf{Q}_p(p^{1/2p^n})$. We have that the ring of integers is as expected

$$\mathcal{O}_{L_n} = \mathcal{O}_{K_n}[p^{1/2p^n}] \cong \mathcal{O}_{K_n}[t]/t^2 - p^{1/p^n}$$

so that the different ideal δ_{L_n/K_n} is then generated by $\frac{d}{dt}(t^2 - p^{1/p^n})_{|t=p^{1/2p^n}} = 2p^{1/2p^n}$ and '*in the limit*'

$$\varinjlim \mathcal{O}_{K_n}[p^{1/2p^n}] = \mathcal{O}_L$$

the ramification $|\delta_{L_n/K_n}| = |2p^{1/2p^n}| \to 1$ '*vanishes*'.

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