Talk 2 - Integral Models of Tori

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Corrections, comments and suggestions are all much appreciated! :) my email: thomasmanopulo@gmail.com

Today's talk will be on the construction of some '*canonical*' integral models of tori, specialising a few of the techniques we discussed in the first talk to this particular setting.

We denote by K a fixed local, non-archimedean field (such as a finite extension of $\mathbb{Q}_p$) with discrete valuation val, $\mathcal{O}_K$ its ring of integers with maximal ideal $\mathfrak{m} = (\varpi)$ and $k = \mathcal{O}_K/(\varpi)$ its residue field. We fix $\overline{K}/K$ a separable closure of K and denote by $K^{\text{un}}/K, K^{\text{un}} \subseteq \overline{K}$ the maximal unramified extension. K^{un} is also discretely valued, and its residue field is the algebraic closure $\overline{k}$ of K's residue field. Note that since K is complete val is well-defined on $\overline{K}$.

Let X/k be an affine variety. Recall that in the first talk we constructed a functor

$$\{\text{finite type, smooth integral models } \mathcal{X}/\mathcal{O}_K \text{ of } X/k\} \longrightarrow \{\text{open bounded subsets } U \subset X(K^{\text{ur}})\}$$

mapping each integral model $\mathcal{X}/\mathcal{O}_K$ to its set of $\mathcal{O}_{K^{\text{ur}}}$-valued points $\mathcal{X}(\mathcal{O}_{K^{\text{ur}}})$, which we may identify as a subset of $X(K^{\text{ur}}) = X(\mathcal{O}_{K^{ur}} \otimes_{\mathcal{O}_K} K)$ as $\mathcal{O}_{K^{\text{ur}}}$ is flat over $\mathcal{O}_K$. The main result from last week's talk established that this functor is a fully faithful embedding, whose essential image we called the collection of "*schematic subsets* $U \subset X(K^{\text{ur}})$".

Fix T/K is a torus - an affine algebraic group such that its base change $T_{\overline{K}}$ is isomorphic to a product of finitely many copies of $\mathbb{G}_{m,\overline{K}}$; before wondering how we might construct integral models for T, it's natural to ask which potentially schematic subsets $U \subset T(K^{\text{ur}})$ we might focus our attention to. This is where the constructions from last term's course become relevant, a couple of which we readily recall since it's been a while :p

We denote by $G_K = \text{Gal}(\overline{K}/K)$ the absolute Galois group of K.

Definition. *1. The valuation homomorphism for T is the group homomorphism*

$$\begin{aligned} \omega_T : T(K) &\longrightarrow \text{Hom}_{\mathbb{Z}}(X^*(T)^{G_K}, \mathbb{Z}) \\ t &\longmapsto (\omega_T(t) : \chi \mapsto -\text{val}(\chi(t))). \end{aligned}$$

2. The kernel

$$T(K)^1 = \ker \omega_T = \left\{ t \in T(K) \mid \text{val}(\chi(t)) = 0, \text{ for all } \chi \in X^*(T)^{G_K} \right\}$$

is the maximal bounded subgroup of the p-adic group $T(K)$.

3. For any finite Galois extension L'/L of fields L and L' extending K, we denote by $\text{Nm}_{L'/L}$ the norm map on L'-points of T:

$$\begin{aligned} \text{Nm}_{L'/L} : T(L') &\longrightarrow T(L) \\ g &\longmapsto \prod_{\sigma \in \text{Gal}(L'/L)} \sigma(g) \end{aligned}$$

4. Let L/K be a finite field extension of K which splits T. Denote by $T(K^{ur})^0$ the image of the norm map restricted to the maximal bounded subgroup

$$\text{Nm}_{L^{ur}/K^{ur}} : T(L^{ur})^1 \longrightarrow T(K^{ur})^1$$

We call the group $T(K^{ur})^0$ the - K^{ur}-points of the - Iwahori of T.

An example should help recall the essential behaviour of $T(K)^0$ and $T(K)^1$.

Example: Set $K = \mathbb{Q}_p$ and T be the torus defined by

$$\begin{aligned} T : \mathbf{Alg}_{\mathbb{Q}_p} &\longrightarrow \mathbf{Grp} \\ R &\longmapsto \left\{ \begin{pmatrix} a & b \\ db & a \end{pmatrix} \in \mathrm{SL}_2(R) \right\} \end{aligned}$$

where $d \in \mathcal{O}_K$ is some square-free integer. We have an isomorphism $T_{\mathbb{Q}_p(\sqrt{d})} \cong \mathbb{G}_{m,\mathbb{Q}_p(\sqrt{d})}$ given by the map

$$\begin{aligned} \phi_R : T_{\mathbb{Q}_p(\sqrt{d})}(R) &\longrightarrow R^\times \\ \begin{pmatrix} a & b \\ db & a \end{pmatrix} &\longmapsto a + \sqrt{d}b \end{aligned}$$

for $\mathbb{Q}_p(\sqrt{d})$-algebras R. ϕ's functor of points evidently defines a group homomorphism, and note that ϕ_R is an isomorphism since for any element $r \in \mathbb{G}_{m,\mathbb{Q}_p(\sqrt{d})}(R) = R^\times$, if we want to express r as $a + \sqrt{d}b$ where $a^2 + db^2 = 1$ we must have $r^{-1} = r - 2\sqrt{d}b$ which forces

$$b = \frac{r - r^{-1}}{2\sqrt{d}}, \quad a = r - \sqrt{d}b.$$

Thus the lattice of characters $X^*(T)$ is generated by the map

$$T_{\overline{\mathbb{Q}}_p} \xrightarrow{\phi_{\overline{\mathbb{Q}}_p}} \mathbb{G}_{m,\overline{\mathbb{Q}}_p}$$

and the Galois action, which evidently factors through $\mathrm{Gal}(\mathbb{Q}_p(\sqrt{d})/\mathbb{Q}_p) = \{1, \sigma\}$, is defined by

$$\sigma(\phi_{\overline{\mathbb{Q}}_p}) = -\phi_{\overline{\mathbb{Q}}_p}$$

seeing as $\sigma(\sqrt{d}) = -\sqrt{d}$. This implies the only $G_{\mathbb{Q}_p}$-invariant character is $\chi = 0$ and thus $T(K)^1 = T(K)$ (and indeed $T(K^{\mathrm{ur}})^1 = T(K^{\mathrm{ur}})$) - this occurs because the condition $\det \begin{pmatrix} a & b \\ db & a \end{pmatrix} = 1$ ensures that a and b are bounded; indeed, note that $1 = a^2 - db^2 = (a + \sqrt{d}b)(a - \sqrt{d}b)$ implies that $a + \sqrt{d}b \in \mathcal{O}_{\mathbb{Q}_p(\sqrt{d})}$ - as can be seen by expressing $a + \sqrt{d}b$ as a product $r \cdot \omega^n$ where $n = \mathrm{val}(a + \sqrt{p}b)$ and $\varpi \in \mathbb{Q}_p(\sqrt{d})$ is a uniformiser; whence we have $a + \sqrt{d}b \in \mathbb{Z}_p[\sqrt{d}]$ if either $d - 3$ or $d - 2$ is a multiple of 4, or $a + \sqrt{d}b \in \mathbb{Z}_p[\frac{1+\sqrt{d}}{2}]$ if $d - 1$ is.

Since $L = \mathbb{Q}_p(\sqrt{d})$ is a splitting field for T, we may use the norm map of its maximal unramified extension to compute the Iwahori; if $L/\mathbb{Q}_p$ is unramified then of course $T(\mathbb{Q}_p^{\mathrm{ur}})^0 = T(\mathbb{Q}_p^{\mathrm{ur}})^1$ as the norm map $\mathrm{Nm}_{L^{\mathrm{ur}}/\mathbb{Q}_p^{\mathrm{ur}}} : T(L^{\mathrm{ur}}) \to T(\mathbb{Q}_p^{\mathrm{ur}})$ reduces to the identity. However, if for example $d = p$ then $\mathbb{Q}_p(\sqrt{p})/\mathbb{Q}_p$ does acquire ramification, since the discriminant is either p or $4p$, and the norm map on maximal unramified extensions becomes

$$\begin{aligned} \mathrm{Nm}_{\mathbb{Q}_p^{\mathrm{ur}}(\sqrt{p})/\mathbb{Q}_p^{\mathrm{ur}}} : T(\mathbb{Q}_p^{\mathrm{ur}}(\sqrt{p})) &\longrightarrow T(\mathbb{Q}_p^{\mathrm{ur}}) \\ \begin{pmatrix} a & b \\ pb & a \end{pmatrix} &\longmapsto \begin{pmatrix} a & b \\ pb & a \end{pmatrix} \begin{pmatrix} \sigma(a) & \sigma(b) \\ p\sigma(b) & \sigma(a) \end{pmatrix} \\ &= \begin{pmatrix} a\sigma(a) + pb\sigma(b) & a\sigma(b) + b\sigma(a) \\ p(a\sigma(b) + b\sigma(a)) & a\sigma(a) + pb\sigma(b) \end{pmatrix}; \end{aligned}$$

expressing a, b as

$$\begin{aligned} a &= a_1 + \sqrt{p}a_2, \\ b &= b_1 + \sqrt{p}b_2 \end{aligned}$$

for $a_1, a_2, b_1, b_2 \in \mathbb{Q}_p^{\mathrm{ur}}$ yields the expression

$$\mathrm{Nm}_{\mathbb{Q}_p^{\mathrm{ur}}(\sqrt{p})/\mathbb{Q}_p^{\mathrm{ur}}} \begin{pmatrix} a & b \\ pb & a \end{pmatrix} = \begin{pmatrix} a_1^2 - pa_2^2 + pb_1^2 - p^2b_2^2 & 2(a_1b_1 - pa_2b_2) \\ 2p(a_1b_1 - pa_2b_2) & a_1^2 - pa_2^2 + pb_1^2 - p^2b_2^2 \end{pmatrix}$$

and since

$$1 = a^2 - pb^2 = a_1^2 + pa_2^2 + 2\sqrt{p}a_1a_2 - pb_1^2 - p^2b_2^2 - 2p\sqrt{p}b_1b_2 \implies 1 = a_1^2 + pa_2^2 - pb_1^2 - p^2b_2^2$$

we have

$$\mathrm{Nm}_{\mathbb{Q}_p^{\mathrm{ur}}(\sqrt{p})/\mathbb{Q}_p^{\mathrm{ur}}} \begin{pmatrix} a & b \\ pb & a \end{pmatrix} = \begin{pmatrix} 1 - 2p(a_2^2 + b_1^2) & 2(a_1b_1 - pa_2b_2) \\ 2p(a_1b_1 - pa_2b_2) & 1 - 2p(a_2^2 + b_1^2) \end{pmatrix}.$$

By the equality $\mathbb{O}_{\mathbb{Q}_p^{\mathrm{ur}}(\sqrt{p})} = \mathbb{O}_{\mathbb{Q}_p^{\mathrm{ur}}}[\sqrt{p}]$ and our previous discussion, we have that the coordinates $x = 1 - 2p(a_2^2 + b_1^2)$ and $y = 2(a_1 b_1 - pa_2 b_2)$ lie in $\mathbb{O}_{\mathbb{Q}_p^{\mathrm{ur}}}$, and by the equality $x^2 - py^2 = 1$ it also follows that $x - 1$ is a multiple of p; reducing mod p we see

$$\mathrm{Nm}_{\mathbb{Q}_p^{\mathrm{ur}}(\sqrt{p})/\mathbb{Q}_p^{\mathrm{ur}}}\begin{pmatrix} a & b \\ pb & a \end{pmatrix} = \begin{pmatrix} 1-2p(a_2^2+b_1^2) & 2(a_1b_1 - pa_2b_2) \\ 2p(a_1b_1 - pa_2b_2) & 1-2p(a_2^2+b_1^2) \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \mod p$$

showing that the image of the norm map is contained in the first congruence subgroup

$$\left\{ g \in T(\mathbb{Q}_p^{\mathrm{ur}}) = T(\mathbb{Z}_p^{\mathrm{ur}}) \mid g \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \mod p \right\},$$

and this $T(\mathbb{Q}_p^{\mathrm{ur}})^0 \subsetneq T(\mathbb{Q}_p^{\mathrm{ur}})^1 = T(\mathbb{Q}_p^{\mathrm{ur}})$ since evidently $-\mathrm{id}$ doesn't lie in $T(\mathbb{Q}_p^{\mathrm{ur}})^0$.

We now discuss the construction of integral models, focusing our attention on the mentioned open bounded subsets

$$T(K^{\mathrm{ur}})^0, T(K^{\mathrm{ur}})^1 \subseteq T(K^{\mathrm{ur}}),$$

in an attempt to construct smooth integral models corresponding to them.

As a first attempt, note that if $T \cong \mathbb{G}_{m,K} \times \cdots \times \mathbb{G}_{m,K}$ is split, then we have an obvious choice for an integral model, given simply by the product of as many copies of $\mathbb{G}_{m,\mathbb{O}_K}$, but taken over the ring of integers $\mathbb{O}_K$; explicitly, if $T \cong \operatorname{Spec} K[T, T^{-1}]$ then we choose $\mathcal{T}_{\mathbb{O}_L} = \operatorname{Spec} \mathbb{O}_K[T, T^{-1}]$.
If T is non-split, then there exists a finite field extension L/K such that T_L is split, and we can construct an integral model $\mathcal{T}_{\mathbb{O}_L}$ for T_L over $\mathbb{O}_L$ as described. By applying Weil restriction of scalars, we thus have an integral model $R_{\mathbb{O}_L/\mathbb{O}_K}\mathcal{T}_{\mathbb{O}_L}$ for the induced torus $R_{L/K}T_L$ which is not (!) in general equal to T unfortunately. The adjunction

$$(-) \times_K \operatorname{Spec} L \dashv R_{L/K}(-)$$

between base change L/K and Weil restriction of scalars provides a unit map

$$T \to R_{L/K}T_L$$

which turns out to be a closed immersion, by the explicit construction of Weil restriction. If we denote by R the ring of global sections of $R_{L/K}T_L$, $A \subseteq R$ the $\mathbb{O}_K$-subalgebra corresponding to the global sections of the integral model $R_{\mathbb{O}_L/\mathbb{O}_K}\mathcal{T}_{\mathbb{O}_L}$ and $I \subseteq R$ the ideal corresponding to the close subscheme $T \subseteq R_{L/K}T_L$, then the closed subscheme

$$\mathcal{T}^{std} := \operatorname{Spec} A/(I \cap A) \subseteq R_{\mathbb{O}_L/\mathbb{O}_K}\mathcal{T}_{\mathbb{O}_L}$$

is the scheme-theoretic closure of T in $R_{\mathbb{O}_L/\mathbb{O}_K}\mathcal{T}_{\mathbb{O}_L}$ via the natural map $R_{L/K}T_L \to R_{\mathbb{O}_L/\mathbb{O}_K}\mathcal{T}_{\mathbb{O}_L}$.

Definition. *In the previous construction, $\mathcal{T}^{std}$ is the standard model of T.*

Example: We fix $K = \mathbb{Q}_p$ and take T to be the torus from earlier, where we assume d is congruent to 3 modulo 4 for simplicity. As shown, T's base change to $L = \mathbb{Q}_p(\sqrt{d})$ is isomorphic to $\mathbb{G}_{m,L}$, and we can thus take $\mathbb{G}_{m,\mathbb{O}_L} = \operatorname{Spec} \mathbb{O}_L[t, t^{-1}] = \operatorname{Spec} \mathbb{O}_L[t_1, t_2]/(t_1t_2 - 1)$ as its integral model. The free $\mathbb{O}_K$-module $\mathbb{O}_L$ (by our assumptions on d) has $1, \sqrt{d}$ as a basis, and making the substitutions

$$t_1 = a_1 + \sqrt{d}b_1,$$
$$t_2 = a_2 + \sqrt{d}b_2$$

yields the expression for the Weil restriction

$$\begin{aligned} R_{\mathbb{O}_L/\mathbb{O}_K}G_{m,\mathbb{O}_L} &= \operatorname{Spec} \mathbb{O}_K[a_1, a_2, b_1, b_2]/((a_1 + \sqrt{d}b_1)(a_2 + \sqrt{d}b_2) - 1) \\ &= \operatorname{Spec} \mathbb{O}_K[a_1, a_2, b_1, b_2]/(a_1a_2 + db_1b_2 - 1, a_1b_2 + a_2b_1). \end{aligned}$$

The closed immersion $T \hookrightarrow R_{L/K}\mathbb{G}_{m,L}$ is given by the projection map

$$\begin{aligned} K[a_1, a_2, b_1, b_2]/(a_1a_2 - db_1b_2 - 1, a_1b_2 + a_2b_1) &\twoheadrightarrow K[a, b]/(a^2 - db^2 - 1) \\ (a_1, b_1, a_2, b_2) &\mapsto (a, b, a, -b) \end{aligned}$$

whose kernel is the ideal $(a_1 - a_2, b_1 + b_2)$. We thus have that the saught for integral model is

$$\mathcal{T}^{std} = \operatorname{Spec} \mathcal{O}_K[a,b]/(a^2 - db^2 - 1).$$

Note that although $T \to \operatorname{Spec} K$ might be smooth, the integral model $\mathcal{T}^{std} \to \operatorname{Spec} \mathcal{O}_K$ need not be; for instance, if $d = p = 2$ then the special fibre is given by

$$\mathcal{T}^{std}_{\mathbb{F}_2} = \operatorname{Spec} \mathbb{F}_2[a,b]/(a^2 - 1)$$

which isn't reduced.

Returning to the general case, we now attempt to understand which (potentially) schematic subset the integral model $\mathcal{T}^{std}$ corresponds to. Suppose for the sake of simplifying notation that T is one-dimensional.
Note that, since $T \hookrightarrow T' := R_{L/K}\mathbb{G}_{m,L}$ is a closed immersion, the analytic topology on $T(K^{\mathrm{ur}})$ is induced by that of $(R_{L/K}\mathbb{G}_{m,L})(K^{\mathrm{ur}}) = (L^{\mathrm{ur}})^\times$. Thus $T(K^{\mathrm{ur}})$'s maximal bounded subgroup $T(K^{\mathrm{ur}})^1$ must be the preimage via the closed immersion (of rigid analytic spaces) $T(K^{\mathrm{ur}}) \hookrightarrow T'(K^{\mathrm{ur}})$ of T''s maximal bounded subgroup $T'(K^{\mathrm{ur}})^1$, which is given explicitly by

$$T'(K^{\mathrm{ur}})^1 = \mathcal{O}_{L^{\mathrm{ur}}}^\times.$$

If we denote by $\mathcal{T}' = R_{\mathcal{O}_L/\mathcal{O}_K}\mathbb{G}_{m,\mathcal{O}_L}$ the integral model we chose for T', then we have that

$$\mathcal{T}'(\mathcal{O}_{K^{\mathrm{ur}}}) = \mathbb{G}_{m,\mathcal{O}_L}(\mathcal{O}_{L^{\mathrm{ur}}}) = \mathcal{O}_{L^{\mathrm{ur}}}^\times = T'(K^{\mathrm{ur}})^1.$$

Whence our discussions yield

$$T(K^{\mathrm{ur}})^1 = T'(K^{\mathrm{ur}})^1 \cap T(K^{\mathrm{ur}}) = \mathcal{T}'(\mathcal{O}_{K^{\mathrm{ur}}}) \cap T(K^{\mathrm{ur}}) = \mathcal{T}^{std}(\mathcal{O}_{K^{\mathrm{ur}}})$$

where the last equality follows from the fact that any morphism $\operatorname{Spec} \mathcal{O}_K \to \mathcal{T}'$ necessarily has closed image, and thus factors through the scheme theoretic closure.

Definition. *Let T be a torus over k. The smoothening of the standard integral model $\mathcal{T}^{std}$ is called the ft-Néron model and is denoted by $\mathcal{T}^{\mathrm{ft}}$.*

By the properties of the smoothening process we discussed last time, $\mathcal{T}^{\mathrm{ft}}(\mathcal{O}_{K^{\mathrm{ur}}}) = \mathcal{T}^{std}(\mathcal{O}_{K^{\mathrm{ur}}}) = T(K^{\mathrm{ur}})^1$ and thus $T(K^{\mathrm{ur}})$'s maximal bounded subgroup is schematic. Seeing as $\mathcal{T}^{\mathrm{ft}}$ is smooth by construction, these remarks yield the description of $\mathcal{T}^{\mathrm{ft}}$'s ring of global sections

$$\Gamma(\mathcal{T}^{\mathrm{ft}}, \mathcal{O}_{\mathcal{T}^{\mathrm{ft}}}) = \left\{ f \in \Gamma(T, \mathcal{O}_T) \mid f(T(K^{\mathrm{ur}})^1) \subseteq \mathcal{O}_{K^{\mathrm{ur}}} \right\},$$

which, surprisingly enough, doesn't even depent on the construction of $\mathcal{T}^{std}$.

Example: We continue with the previous example, but add a twist to make the computations more intriguing - we replace our ground field with another enriched with wild ramification over $\mathbb{Q}_2$: set $K = \mathbb{Q}_2(\sqrt{2})$, $\varpi = \sqrt{2}$ and let T be the torus defined by

$$T(R) = \left\{ \begin{pmatrix} a & b \\ \varpi b & a \end{pmatrix} \in \mathrm{SL}_2(R) \right\};$$

once again, T splits over the ramified extension $L = K(\sqrt{\varpi})$ and its standard integral model is given by

$$\mathcal{T}^{std} = \operatorname{Spec}(\mathcal{O}_K[a,b]/(a^2 - \varpi b^2 - 1))$$

via the same computations discussed previously.
Note that in this case the equation $a^2 - \varpi b^2 - 1$ where $a, b \in \mathcal{O}_{K^{\mathrm{ur}}}$ implies that b is a multiple of ϖ, since reducing mod ϖ yields $(a-1)^2 \equiv 0 \implies a = 1 + \varpi a', a' \in \mathcal{O}_{K^{\mathrm{ur}}}$ and substituting into our equation shows

$$\varpi b^2 = 2a'^2 + 2\varpi a' \implies b^2 \equiv 0 \implies b \equiv 0.$$

This shows the image of the reduction map

$$\mathcal{T}^{std}(\mathcal{O}_{K^{\mathrm{ur}}}) \to \mathcal{T}^{std}(\overline{\mathbb{F}}_2)$$

is given by the trivial subgroup, whose defining ideal sheaf over $\mathcal{T}^{std}$ is given by

$$(\varpi, a - 1, b)$$

which implies the relevant dilatation $\mathcal{T}^{(1)} := \operatorname{Spec} R$ is the spectrum of the coordinate ring of $\mathcal{T}^{std}$ adjoined the elements $x := \frac{a-1}{\varpi}, y := \frac{b}{\varpi}$, thought of as a sub-$\mathcal{O}_K$-algebra of $k[a,b]/(a^2 - \varpi b^2 - 1)$. Since x and y satisfy

$$\varpi x + 1 = a,$$
$$\varpi y = b,$$

we get the relation

$$2x^2 + 2\varpi x - 2\varpi y^2 = 2(x^2 + \varpi x - \varpi y^2) = 0$$

and thus a surjection

$$\phi : \mathcal{O}_K[x,y]/(2(x^2 + \varpi x - \varpi y^2)) \twoheadrightarrow R;$$

evidently $x^2 + \varpi x - \varpi y^2$ is mapped to zero via ϕ since the equation in R

$$2(x^2 + \varpi x - \varpi y^2) = 0$$

implies $x^2 + \varpi x - \varpi y^2 = 0$ as R is an integral domain by construction. Thus ϕ factors as a map

$$\phi : \mathcal{O}_K[x,y]/(x^2 + \varpi x - \varpi y^2) \twoheadrightarrow R$$

which must be an isomorphism since $(x^2 + \varpi x - \varpi y^2) \subseteq R$ is a prime ideal - if the kernel were bigger, it would be a prime strictly containing $(x^2 + \varpi x - \varpi y^2)$, thus implying that R has dimension one, contradicting the fact that the special fibre T of $\mathcal{T}^{(1)}$ is one dimensional.
Note that, as opposed to the example discussed in the previous talk, $\mathcal{T}^{(1)}$ isn't smooth since the reduction modulo ϖ is again non-reduced.

By means of a similar computation to the one done earlier, we can check the equation $x^2 + \varpi x - \varpi y^2 = 0$ for elements $x, y \in \mathcal{O}_{K^{\mathrm{ur}}}$ implies both x and y are multiples of ϖ, thus the scheme-theoretic image of

$$\mathcal{T}^{(1)}(\mathcal{O}_{K^{\mathrm{ur}}}) \to \mathcal{T}^{(1)}(\overline{\mathbb{F}}_2)$$

is defined by the ideal (x, y, ϖ). Adjoining the elements $\frac{x}{\varpi}, \frac{y}{\varpi}$ to R gives us - applying the same arguments just discussed - that the coordinate ring of $\mathcal{T}^{(2)}$ is

$$\mathcal{O}_K[u,v]/(u^2 + u - \varpi v^2)$$

which indeed is smooth over $\mathcal{O}_K$ and has two copies of $\mathbb{G}_{a,\mathbb{F}_2}$ as its special fibre.

We can now move onto discussing integral models of tori more generally, in a more abstract manner.

Definition. *Let T be a K-torus and $\mathcal{T} \to \operatorname{Spec} \mathcal{O}_K$ be a smooth $\mathcal{O}_K$-scheme equipped with an isomorphism $\mathcal{T}_K \xrightarrow{\cong} T$ of K-schemes (note that $\mathcal{T}$ is not assumed to be of finite type).*

1. *$\mathcal{T}$ is an lft-Néron model of T if for every smooth $\mathcal{O}_K$-scheme $\mathcal{X}$ together with a morphism of generic fibres $\mathcal{X}_K \to T$ there exists a unique lift $\mathcal{X} \to \mathcal{T}$.*
2. *$\mathcal{T}$ is an ft-Néron model of T if for every smooth $\mathcal{O}_K$ scheme of finite type $\mathcal{X}$ together with a morphism of generic fibres $f : \mathcal{X}_K \to T$ satisfying $f(\mathcal{X}(\mathcal{O}_{K^{ur}})) \subseteq T(K^{ur})^1$ there exists a unique lift $\mathcal{X} \to \mathcal{T}$.*
3. *$\mathcal{T}$ is a connected Néron model of T if $\mathcal{T}_k$ is connected and for every smooth $\mathcal{O}_K$-scheme $\mathcal{X}$ such that $\mathcal{X}_K, \mathcal{X}_k$ are connected together with a morphism $f : \mathcal{X}_K \to T$ satisfying $f(\mathcal{X}(\mathcal{O}_{K^{ur}})) \subseteq T(K^{ur})^0$ there exists a unique lift $\mathcal{X} \to \mathcal{T}$.*

By construction, crucially using smoothness, it follows that $\mathcal{T}^{\mathrm{ft}}$ if an ft-Néron model in the sense of the definition above.
The construction of the lft-Néron model and the connected one amount to considering an enlargement or a shrinking of $\mathcal{T}^{\mathrm{ft}}$ respectively; the former requires to sacrifice the fact that our integral models are of finite type over the base, and the latter requires us to substitute the special fibre $\mathcal{T}^{\mathrm{ft}}_k$ with its identity component.

Proposition. *Let G be an affine algebraic group over K and let $U \subset V \subset G(K^{ur})$ be two subgroups such that U is normal in V, and that U is schematic with smooth integral model $\mathcal{X}$.*

1. *There exists a smooth separated group scheme $\mathcal{Y}$ with generic fibre G such that $\mathcal{Y}(\mathcal{O}_{K^{ur}}) = V$,*
2. *the induced morphism $\mathcal{X} \to \mathcal{Y}$ is an open immersion and induces an isomorphism between relative identity components.*
3. *If the index of U in V is finite, then $\mathcal{Y}$ is affine and thus a smooth integral model of G (i.e. V is also schematic).*

Theorem. *Let T be a torus over K. Then an lft-Néron model and a connected Néron model for T exist.*

Proof. (outline) By using the above result, one can construct an integral model $\mathscr{T}^{\mathrm{lft}}$ such that $\mathscr{T}^{\mathrm{lft}}(\mathcal{O}_{K^{\mathrm{ur}}}) = T(K)$, as the inclusion $T(K)^1 \subseteq T(K)$ is of finite index. Note that, crucially, the lft-Néron model lifting property isn't satisfied automatically since, although $\mathscr{T}^{\mathrm{lft}}$ might be smooth over $\mathcal{O}_K$, it **isn't of finite type in general** (globally), and thus lifts may be constructed only locally - this happens because in the main result from last lecture, we crucially used that the set of $\overline{k}$-rational points of the special fibre of the considered integral model forms a dense subset, which needn't be the case if this base change isn't of finite type!
To argue that the lft-Néron model property holds for $\mathscr{T}^{\mathrm{lft}}$ one has to argue locally, as mentioned: if $\mathscr{Y}$ is a smooth scheme over $\mathcal{O}_K$ provided with a morphism of K-schemes $\mathscr{Y}_K \to T$, then one can construct lifts locally on $\mathscr{Y}$ **and** $\mathscr{T}^{\mathrm{lft}}$, over a collection of open subsets of $\mathscr{Y}$ which intesect non-trivially only on points of the special fibre $\mathscr{Y}_F$. This way, the local morphisms glue together (since they're all lifts of the base change $\mathscr{Y}_F \to \mathscr{T}^{std}$). Although the open cover on which the morphism from $\mathscr{Y}$ to $\mathscr{T}^{\mathrm{lft}}$ might not exhaust $\mathscr{Y}$, one argue that the complement of the open subset where $\mathscr{Y} \dashrightarrow \mathscr{T}^{\mathrm{lft}}$ is defined has codimention less than two, and can thus appeal to the theorem following this proof to conclude.

As for the connected Neron model, taking the relative identity component $\mathscr{T}^0$ of $\mathscr{T}^{\mathrm{lft}}$, which equals the relative identity component of $\mathscr{T}^{\mathrm{ft}}$ by construction, all one has to do is show that $\mathscr{T}^0(\mathcal{O}_{K^{\mathrm{ur}}}) = T(K^{\mathrm{ur}})^0$ and rely on last lecture's main theorem once again. To achieve this, we use that for the induced K-torus $T' := R_{L/K}T_L$ where L is a splitting field for T, the Iwahori $T'(K^{\mathrm{ur}})^0$ and the maximal bounded subgroup coincide $T'(K^{\mathrm{ur}})^1$, and are both equal to $\mathcal{O}_{L^{\mathrm{ur}}}^{\times}$, which in turn coincides with $\mathscr{T}'^0(\mathcal{O}_{K^{\mathrm{ur}}})$. By using the norm map $\mathrm{Nm} : T' \to T$, we obtain that restricting $\mathrm{Nm}(K^{\mathrm{ur}})$ to the maximal bounded subgroup $T'(K^{\mathrm{ur}})^0 = \mathscr{T}'^0(\mathcal{O}_{K^{\mathrm{ur}}})$ we obtain that the image is both $\mathscr{T}^0(\mathcal{O}_{K^{\mathrm{ur}}})$ and $T(K^{\mathrm{ur}})^0$ by construction. ■

Theorem. *Let S be a normal Noetherian scheme, and $u : Z \dashrightarrow G$ be an S-morphism from a smooth S-scheme Z to a smooth and separated S-group scheme G. If the open subset of Z where u is defined has codimension greater 1, then u admits an extension defined on the whole of Z.*

References

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