Divisors on The Curve and the Hilbert Diamond

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In this pdf I gather what I've been trying to learn about the space Div^d of divisors on the Fargues-Fontaine curve. We follow and readapt parts of Chapter 2 in [FS] and Section 2 in [Cdc].

1 The curve and its associated diamond

We begin with a study of the space Div^1 of degree-one divisors on the Fargues-Fontaine curve. We follow and readapt parts of Chapter 2 in [FS] and Section 2 in [Cdc].

Fix E a p-adic local field, which can be either a finite extension of $\mathbf{Q}_p$ or of $\mathbf{F}_p(\!(\pi)\!)$ - we will refer to these two cases as the settings of *mixed characteristic* and *equal characteristic* respectively. We denote by $\pi \in E^{\circ\circ}$ a fixed uniformiser and let q be the cardinality of the residue field E°/π. Sometimes the notation $\mathcal{O}_E = E^\circ$ for the ring of integers and $\mathfrak{m}_E = E^{\circ\circ} = (\pi)$ its unique maximal ideal will be used.

Given an affinoid perfectoid space $S = \mathrm{Spa}(R, R^+) \in \mathrm{Perf}_{\mathbf{F}_q}$, there exists a unique π-complete torsion free $\mathcal{O}_E$-algebra whose mod π-reduction is R^+: if E is of mixed characteristic then one can take the $\mathcal{O}_E$-algebra of Witt-vectors $W_{\mathcal{O}_E}(R^+)$, and for the case $p = 0 \in E$ the power series ring $R^+[\![\pi]\!]$. In either case, we denote by $\mathbf{A}_{\mathrm{inf}}(R, R^+)$ this Huber ring and consider the pre-adic space

$$Y_S := \mathrm{Spa}(\mathbf{A}_{\mathrm{inf}}(R, R^+), \mathbf{A}_{\mathrm{inf}}(R, R^+)) \setminus V([\varpi]\pi)$$

where $[-] : R^+ \to \mathbf{A}_{\mathrm{inf}}(R, R^+)$ is the Teichmüller map (a multiplicative section of the projection $\mathbf{A}_{\mathrm{inf}}(R, R^+) \twoheadrightarrow R^+$, which in the equal characteristic case is just the inclusion of constant terms) and $\varpi \in R^{\circ\circ} \cap R^\times$ is a pseudo-uniformiser - note that Y_S doesn't depend on the choice of ϖ since for any two pseudouniformisers ϖ and ϖ' in R^+ we have $\varpi^{p^k} \mid \varpi'$ and $\varpi'^{p^k} \mid \varpi$ in R^+ for k large enough.

$\mathbf{A}_{\mathrm{inf}}(R, R^+)$ is endowed with a '*relative*'[1] Frobenius map

$$\varphi : \mathbf{A}_{\mathrm{inf}} \xrightarrow{\cong} \mathbf{A}_{\mathrm{inf}}$$
$$\sum_{n=0}^{\infty} [a_n]\pi^n \longmapsto \sum_{n=0}^{\infty} [a_n^q]\pi^n$$

[1] The quotation marks are because in the mixed-characteristic setting $\mathbf{A}_{\mathrm{inf}}(R, R^+)$ isn't an algebra over R^+ since R^+ lives over $\mathbf{F}_q$ whereas $\mathbf{A}_{\mathrm{inf}}(R, R^+)$ is an $\mathcal{O}_E$-algebra. However one can basically pretend this is the case by passing to the category of diamonds, as we'll soon explore.

which is an isomorphism since S is perfectoid.

Proposition 1.1. *The structure presheaf on Y_S is a sheaf, and thus Y_S defines an adic space over* $\operatorname{Spa} E$. *Y_S is analytic and the base change*

$$Y_S \times_{\operatorname{Spa} E} \operatorname{Spa} E_\infty \to \operatorname{Spa} E_\infty$$

is perfectoid, where $\operatorname{Spa} E_\infty \to \operatorname{Spa} E$ *is the pro-étale cover of* $\operatorname{Spa} E$ *given by* $E_\infty = \operatorname{Frac}(\mathcal{O}_E[\pi^{1/q^\infty}])^\wedge_\pi$. *The tilt*

$$(Y_S \times_{\operatorname{Spa} E} \operatorname{Spa} E_\infty)^\flat \in \operatorname{Perf}_{\mathbf{F}_q}$$

is isomorphic to the perfectoid punctured open unit disc $\widetilde{\mathbf{D}}^*_S$.

To tackle Proposition 1.1, we require the notion of sousperfectoid rings.

Definition 1.2. A complete Tate algebra R over $\mathbf{Z}_p$ is called *sousperfectoid* if there exists a perfectoid Tate ring $\widetilde{R}$ and an injection $R \hookrightarrow \widetilde{R}$ admitting a continuous section $\widetilde{R} \twoheadrightarrow R$ as R-modules.

Lemma 1.3. *Let (R, R^+) be a Huber pair with R a Tate sousperfectoid-algebra over $\mathbf{Z}_p$. Then*

1. *R is uniform and,*
2. *for all rational open subsets $U \subset \operatorname{Spa}(R, R^+)$ the ring $\mathcal{O}_X(U)$ is sousperfectoid.*

In particular $\operatorname{Spa}(R, R^+)$ *is stably uniform and thus* $\operatorname{Spa}(R, R^+)$ *is an adic space*[2].

Proof. 1. Let $\iota : R \hookrightarrow \widetilde{R}$ be as in Definition 1.2; since $\widetilde{R}^\circ$ is bounded, it'll be sufficient to see that $\iota^{-1}(\widetilde{R}^\circ) = R^\circ$ so that we have an injection $R^\circ \hookrightarrow \widetilde{R}^\circ$.

Given an element $f \in R$ so that $\iota(f) \in \widetilde{R}^\circ$ and a continuous ring homomorphism $R \to K$ to a non-archimedean field K, the splitting $\widetilde{R} \twoheadrightarrow R$ provides a section of the map

$$K \to \widetilde{R} \otimes_R K$$

hence $\widetilde{R} \otimes_R K \neq 0$. Given now a continuous ring homomorphism $\widetilde{R} \otimes_R K \to L$ to another non-archimedean field L, by commutativity of the diagram

$$\begin{tikzcd} R \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \widetilde{R} \arrow[d] \\ K \arrow[r] & K \otimes_R \widetilde{R} \arrow[d] \\ & L \end{tikzcd}$$

we have that the image of f in L lies in L°, and since $K \to L$ is a continuous injective morphism of Banach-algebras, f's image in K must lie in $K^\circ \implies f \in R^\circ$ as $R \to K$ was arbitrary.

2. If we consider the corresponding continuous map $f : \widetilde{X} := \operatorname{Spa}(\widetilde{R}, \widetilde{R}^+) \to X$ (where we set $\widetilde{R}^+$ to be the integral closure of $R^+ \subset \widetilde{R}$; this defines a ring of integral elements by the inclusion $R^\circ \subset \widetilde{R}^\circ$ discussed in the previous part) then because $\widetilde{X}$ is a perfectoid adic space, the ring $\mathcal{O}_{\widetilde{X}}(f^{-1}(U))$ is perfectoid and the canonical map

$$\mathcal{O}_X(U) \to \mathcal{O}_{\widetilde{X}}(f^{-1}(U))$$

admits a continuous splitting as $\mathcal{O}_X(U)$-modules, induced by the splitting $\widetilde{R} \twoheadrightarrow R$ - this follows from the explicit construction of $\mathcal{O}_X(U), \mathcal{O}_{\widetilde{X}}(f^{-1}(U))$ as '*Tate-localisations*' of the rings R and $\widetilde{R}$ respectively).

■

[2]This result is covered in many references and can be argued by considering the classical proof of Tate's acyclicity theorem - see for instance Lecture 2 in Kedlaya's lecture notes [SheStaShtu] or the Buzzard-Verberkmoes article [BuVe].

At this point, it's rather clear how the second result in Proposition 1.1 can be related to the first. We can now discuss the proof, with the aid of some results covered in [BMS]:

Proof. A tool of fundamental importance to studying the geometry of Y_S is the *radius map*: a continuous function $\kappa : |Y_S| \to (0,\infty)$ measuring the relative position of the values the global sections $[\varpi], \pi \in \mathbf{A}_{\mathrm{inf}}(R, R^+)$ take at each point.
To construct κ, firstly note that an open prime ideal of $\mathbf{A}_{\mathrm{inf}}(R, R^+)$ must contain the topologically nilpotent element $[\varpi] \in \mathbf{A}_{\mathrm{inf}}(R, R^+)$ since $\mathbf{A}_{\mathrm{inf}}$ is endowed with the $[\varpi]$-adic topology; this implies that all points in $|Y_S|$ are analytic as the zero-locus $V(\pi[\varpi])$ was duly excluded. Hence each point $x \in Y_S$ admits a unique rank-one generalisation η_x, which we may thus assume takes values (as a function $|\cdot(\eta_x)|$ on $\Gamma(Y_S, \mathcal{O}_{Y_S})$) in the real numbers $\mathbf{R}$. The map

$$\begin{aligned} \kappa : |Y_S| &\longrightarrow (0,\infty) \subset \mathbf{R} \\ x &\longmapsto \frac{\log |[\varpi](\eta_x)|}{\log |\pi(\eta_x)|} \end{aligned}$$

is well-defined[3] and continuous, since the preimage of any bounded interval $[a, a'] \subset (0,\infty)$ with rational endpoints $a = \frac{m}{n}, a' = \frac{m'}{n'} \in \mathbf{Q}_{>0}$ is given by the rational open

$$Y_S \supset Y_{S,[a,a']} := \left\{ x \mid mn' \leq \frac{\log |[\varpi]^{nn'}(\eta_x)|}{\log |\pi(\eta_x)|} \leq m'n \right\} = \left\{ x \mid |\pi^{m'n}(x)| \leq |[\varpi]^{nn'}(x)| \leq |\pi^{mn'}(x)| \right\}.$$

Since $Y_{S,[a,a']}$ is in fact a rational open subset of the quasi-compact space $|\operatorname{Spa}(\mathbf{A}_{\mathrm{inf}}(R, R^+), \mathbf{A}_{\mathrm{inf}}(R, R^+))|$, we see that $Y_{S,[a,a']}$ is quasi-compact and Y_S is covered by the family of rational open subsets $\left\{ Y_i := Y_{S,[q^{i-1}, q^i]} \right\}_{i \in \mathbf{Z}}$, so we restrict our attention to each of these. As the Frobenius isomorphism φ induces an isomorphism of preadic spaces $Y_i \xrightarrow{\cong} Y_{i+1}$ it'll be sufficient to study

$$Y_0 = \{ x \in Y_S \mid |\pi^q(x)| \leq |[\varpi^q](x)| \leq |\pi(x)| \}.$$

Its sections are given by the ring

$$B = \mathbf{A}_{\mathrm{inf}}(R, R^+) \left\langle \frac{\pi}{[\varpi]}, \frac{[\varpi^q]}{\pi} \right\rangle,$$

endowed with the topology making $B^+ \subset B$ an open subring, where B^+ is the integral closure of

$$\mathbf{A}_{\mathrm{inf}}(R, R^+) \left[\frac{\pi}{[\varpi]}, \frac{[\varpi^q]}{\pi} \right] \subset B$$

which itself has the $[\varpi]$-adic topology. Appealing to Lemma 1.3 it'll be sufficient to show that $\widetilde{B} := B \widehat{\otimes}_E E_\infty = \Gamma(\widetilde{Y}_0, \mathcal{O}_{\widetilde{Y}_0})$ is a perfectoid Tate ring, where $\widetilde{Y}_0 := Y_0 \times_{\operatorname{Spa} E} \operatorname{Spa} E_\infty$: the canonical injection

$$B \to B \otimes_E E_\infty$$

evidently admits a section $\sigma : B \otimes_E E_\infty \to B$ as topological B-modules - if $(b_i)_{i \in I}$ is a basis for the E-Banach algebra B and $(e_j)_{j \in J}$ a basis for the extension of fields E_∞ / E then $(b_i \otimes e_j)_{i,j}$ is a basis for $B \otimes_E E_\infty$ and one can take $b_i \otimes e_j \mapsto b_i$ as a section - so by continuity one can extend σ to a section[4]

$$\widehat{\sigma} : B \widehat{\otimes}_E E_\infty \to B.$$

[3]Equivalent rank-one valuations have proportional logarithms. Furthermore, $|\pi(\eta_x)|, |[\varpi](\eta_x)|$ always lie in the open interval $(0,1)$ for $x \in |Y_S|$ since π and $[\varpi]$ are topologically nilpotent and $|\cdot(\eta_x)|$ is continuous; thus the fraction makes sense and its value is always positive.

[4]By the Hahn-Banach theorem such an extension exists, and since $B \otimes_E E_\infty \subset B \widehat{\otimes}_E E_\infty$ is dense $\widehat{\sigma}$ is indeed a splitting.

The ring of integral elements $\widetilde{B}^+ = \Gamma(\widetilde{Y}_1, \mathcal{O}^+_{\widetilde{Y}_1})$ is given by the integral closure of $B^+ \widehat{\otimes}_{\mathcal{O}_E} \mathcal{O}_{E_\infty}$ in $\widetilde{B}$, and we see directly that all the elements

$$\left(\frac{\pi}{[\varpi]}\right)^{1/q^n}, \left(\frac{[\varpi^q]}{\pi}\right)^{1/q^n} \in \widetilde{B}$$

lie in $\widetilde{B}^+$, so that we have an inclusion

$$\widetilde{B}^+_0 := (\mathbf{A}_{\inf}(R, R^+) \widehat{\otimes}_{\mathcal{O}_E} \mathcal{O}_{E_\infty}) \left[\left(\frac{\pi}{[\varpi]}\right)^{1/q^\infty}, \left(\frac{[\varpi^q]}{\pi}\right)^{1/q^\infty} \right]^\wedge_{[\varpi]} \subset \widetilde{B}^+$$

and evidently $\widetilde{B} = \widetilde{B}^+_0[[\varpi]^{-1}]$. If we mod out by $[\varpi]$ (and consequently by π as well) we see

$$\widetilde{B}^+_0/[\varpi] \cong \varinjlim_{T_0 \mapsto T_0^q, T_1 \mapsto T_1^q} \left(\mathbf{A}_{\inf}(R, R^+)[T_1, T_2]\right) / ([\varpi], [\varpi]T_1^{q^n} - \pi, \pi T_2^{q^n} - [\varpi^q]) \cong R^+/\varpi[T_1^{1/q^\infty}, T_2^{1/q^\infty}]$$

which is a perfect ring $\implies$ $\widetilde{B}^+_0$ is integral perfectoid[5] so that (cfr. [BMS] Lemma 3.21) the Tate algebra $\widetilde{B}$ is perfectoid in the usual sense. If we take the inverse limit of $\widetilde{B}^+_0/p$ along the Frobenius map thanks to the above description of $\widetilde{B}^+_0/[\varpi]$ we see that the tilt of $\widetilde{B}$ is given by

$$R\left\langle \left(\frac{t}{\varpi}\right)^{1/q^\infty}, \left(\frac{\varpi^q}{t}\right)^{1/q^\infty} \right\rangle$$

and so (keeping track of the rings of integral elements as well) the perfectoid space $\widetilde{Y}_1^\flat$ is the annulus

$$\{|\varpi^q(x)| \leq |t(x)| \leq |\varpi(x)|\} \subset \widetilde{\mathbf{D}}^*_R.$$

As the Frobenius map φ scales down the radii in $\widetilde{\mathbf{D}}^*_R$ (modulo π we have $\varphi(\varpi) = \varpi^q$) the perfectoid spaces $\widetilde{Y}_i$ assemble to form a cover of $\widetilde{\mathbf{D}}^*_R$ by annuli as we wanted to show. ■

The diamond associated with Y_S and its purely simple description in the following proposition are evidence that Y_S is both a rather knotty space (for instance, its global sections are difficult to get a hold of: one has to take a filtered colimit on the sections on the open subsets $Y_{S,[q^{-i},q^i]}, i \in \mathbf{Z}_{\geq 0}$ and then complete with respect to the family of norms corresponding to each of these) and of tangible nature.

Proposition 1.4. *There's a natural isomorphism of diamonds*

$$Y_S^\diamond \cong \operatorname{Spd} E \times S.$$

In other words, given a perfectoid space $T \in \mathrm{Perf}_{\mathbf{F}_q}$ over $\mathbf{F}_q$, we have a natural equivalence

$$\underbrace{\left\{T^\# \in \mathrm{Perfd}_{Y_S}, \iota : T^{\#\flat} \xrightarrow{\cong} T\right\}}_{:=Y_S^\diamond(T)} \xrightarrow{\cong} \underbrace{\left\{T^\# \in \mathrm{Perfd}_E, \iota : T^{\#\flat} \xrightarrow{\cong} T, f : T \to S\right\}}_{:=\operatorname{Spd} E(T) \times S(T)}.$$

Proof. Since both sides are sheaves (all that's needed here is the analytic topology) we may assume $T = \operatorname{Spa}(A, A^+) \in \mathrm{Perf}_{\mathbf{F}_q}$ is an affinoid perfectoid. Now, given a morphism from an untilt $T^\# = \operatorname{Spa}(A^\#, A^{\#+}) \to Y_S$, we have a continuous ring homomorphism

$$\mathbf{A}_{\inf}(R, R^+) \to A^{\#+}$$

[5] i.e. a quotient of its algebra of unramified q-typical Witt-vectors by a distinguished element of degree one.

such that the images of the sections $[\varpi]$ and π are invertible elements in $A^{\#}$. Upon tilting[6] and applying the isomorphism ι we get a morphism

$$(\mathbf{A}_{\inf}(R, R^+))^\flat \cong R^+ \to A^+$$

which induces a continuous ring homomorphism $R \to A$ and thus a morphism of adic spaces $f : T \to S$. This construction yields a morphism of diamonds $Y_S^\diamond \to \operatorname{Spd} E \times S$.

Its inverse can be described as follows: given a morphism $T \to S$ and an untilt $T^{\#}, \iota : T^{\#\flat} \xrightarrow{\cong} T$, ι realises $A^{+\#}$ as a quotient of $\mathbf{A}_{\inf}(A, A^+)$ by a distinguished element of degree one. Precomposing the projection $\mathbf{A}_{\inf}(A, A^+) \twoheadrightarrow A^{\#+}$ with the morphism $\mathbf{A}_{\inf}(R, R^+) \to \mathbf{A}_{\inf}(A, A^+)$ induced by $T \to S$ yields a desired morphism

$$T^{\#} \to Y_S.$$

■

Recall that for a diamond $\mathcal{D} \in \widetilde{\operatorname{Perf}}_{\mathbf{F}_q,\text{proét}}$ with presentation $\mathcal{D} \cong X/R$ where X is a perfectoid space over $\mathbf{F}_q$ and $R \rightrightarrows X$ is a pro-étale equivalence relation on X, we define its *underlying topological space* as the quotient space

$$|X|/|R|$$

where by definition $|R| \rightrightarrows |X|$ defines an equivalence relation. The following two properties are quite relevant to our study.

Proposition 1.5 ([EtD], sections 11 and 15). *Let $\mathcal{D} \in \widetilde{\operatorname{Perf}}_{\mathbf{F}_q,\text{proét}}$ be a diamond.*

1. *We have an equivalence between the category* $\operatorname{Ouv}(|\mathcal{D}|)$ *of open subsets of the topological space $|\mathcal{D}|$ and the category of open immersions of diamonds $U \hookrightarrow \mathcal{D}$ mapping each open immersion $U \hookrightarrow \mathcal{D}$ to the open subset $|U| \subset |\mathcal{D}|$.*

2. *Given an analytic adic space X over $\mathbf{Z}_p$, there exists a natural homeomorphism*

$$|X^\diamond| \approx |X|.$$

We refrain from discussing details about these constructions and diamonds in general since the writer hasn't found the time to study them well enough (and will likely still need quite some time and experience before he stops being scared of them).

Thanks to Proposition 1.5 we see that there's a natural projection map

$$w_S : |Y_S| \approx |Y_S^\diamond| = |S \times \operatorname{Spd} E| \to |S|.$$

The map w_S is functorial in S and allows us to relate the analytic adic spaces Y_S for varying $S \in \operatorname{Perf}_{\mathbf{F}_q}$.

Proposition 1.6. *Let $S' \subset S$ be an affinoid open of the perfectoid space $S \in \operatorname{Perf}_{\mathbf{F}_q}$ (in particular, S' is also perfectoid). Then $Y_{S'} \to Y_S$ is an open immersion of analytic adic spaces and we have the equality*

$$w_S^{-1}(|S'|) = |Y_{S'}| \subset |Y_S|.$$

Proof. By the commutative diagram

$$\begin{tikzcd} {|Y_{S'}|} \arrow[r] \arrow[d, "w_{S'}"] & {|Y_S|} \arrow[d, "w_S"] \\ {|S'|} \arrow[r] & {|S|} \end{tikzcd}$$

[6] Recall that the the unit of the adjunction $W_{\mathcal{O}_E}(-) \dashv (-)^\flat$ is an isomorphism.

we get that $Y_{S'}$ maps to $w_S^{-1}(|S'|)$, and since $w_S^{-1}(|S'|) \subset Y_S$ is an open subset this gives rise to a factorisation

$$\begin{tikzcd} Y_{S'} \arrow[rr] \arrow[dr] & & Y_S \\ & w_S^{-1}(|S'|) \arrow[ur] & \end{tikzcd}$$

as analytic adic spaces. Since $\mathrm{Spa}\, E_\infty \to \mathrm{Spa}\, E$ is faithfully flat[7], checking that $Y_{S'} \to w_S^{-1}(|S'|)$ is an isomorphism can be done upon base change to $\mathrm{Spa}\, E_\infty$, and here this follows from the Tilting equivalence together with Proposition 1.4:

$$Y^\flat_{S',E_\infty} \cong Y^\diamond_{S',E_\infty} \cong \mathrm{Spa}\, E^\flat_\infty \times S' \cong w_S^{-1}(S')^\flat_{E_\infty}.$$

■

We can now generalise the construction of Y_S to arbitrary perfectoid spaces $S \in \mathrm{Perf}_{\mathbf{F}_q}$: given an analytic cover $\{S_\alpha\}_\alpha$ of S by affinoid perfectoids, one can construct Y_{S_α} for each term in the cover and then use descent - Proposition 1.6 shows that $Y_{S_\alpha \times_S S_\beta} \subset Y_{S_\alpha}$ is an open immersion and these inclusions satisfy the cocycle condition because of the functoriality of Y_-. The obtained analytic adic space Y_S satisfies $Y_S^\diamond \cong S \times \mathrm{Spd}\, E$ and is so that $Y_S \times_{\mathrm{Spa}\, E} \mathrm{Spa}\, E_\infty$ is perfectoid, since both these properties can be checked locally.

2 Divisors and line bundles

By the isomorphism $Y_S^\diamond \cong \mathrm{Spd}\, E \times S$ from Proposition 1.4 we conclude that given an untilt $S^\# \in \mathrm{Perfd}_E, S^{\#\flat} \cong S$ there is a corresponding morphism $S^\# \to Y_S$. The following proposition shows that this map realises $S^\#$ as a Cartier divisor in Y_S, which is our first glance at behaviour of one-dimensional ilk Y_S exhibits: if $S = \mathrm{Spa}\, C$ is a perfectoid field over $\mathbf{F}_q$ then its $C^\#$-rational points for varying untilts $C^\#$ - all of which are perfectoid fields - are the vanishing loci of a single regular section on Y_S.

Proposition 2.1. *Let $S \in \mathrm{Perf}_{\mathbf{F}_q}$ and $S \to Y_S^\diamond$ a morphism of diamonds, corresponding to an untilt $S^\# \in \mathrm{Perfd}_E, \iota : S^{\#\flat} \xrightarrow{\cong} S$ and a morphism $i : S^\# \to Y_S$. Then $S^\# \hookrightarrow Y_S$ is a closed immersion presenting $S^\#$ as a Cartier divisor in Y_S.*

Proof. Assume for simplicity that we work in the mixed-characteristic setting; for the equal characteristic case one can argue analogously. As Cartier divisors can be detected locally, we can assume $S = \mathrm{Spa}(R, R^+)$ is affinoid perfectoid and thus so is its fixed untilt $S^\# = \mathrm{Spa}(R^\#, R^{\#+})$; the isomorphism $R^{\#+\flat} \cong R^+$ and R's Fontaine map realise $R^{\#+}$ as a quotient of the form

$$R^{\#+} \cong W_{\mathcal{O}_E}(R^+)/\xi$$

where $\xi = \pi - a[\varpi] \in W_{\mathcal{O}_E}(R^+)$ is a distinguished element of degree one. The goal is now to show the sequence of sheaves on Y_S

$$0 \to \mathcal{O}_{Y_S} \xrightarrow{\cdot\xi} \mathcal{O}_{Y_S} \to i_*\mathcal{O}_{S^\#} \to 0$$

is exact, so that $S^\#$ is precisely the vanishing locus of the regular section $\xi \in W_{\mathcal{O}_E}(R^+) \to \Gamma(Y_S, \mathcal{O}_{Y_S})$ and has the locally ringed space structure induced by that of Y_S.
Exactness is guaranteed at points lying outside $V(\xi)$ by the above expression for $R^\#$ in terms of $W_{\mathcal{O}_E}(R^+)$, so let $x \in V(\xi)$ and $U \subset Y_S$ an affinoid open subset containing x - we show that the above sequence is exact upon taking sections on U; since U is arbitrary this will guarantee exactness of the sequence as sheaves.

[7]Does this work for adic spaces as it does for schemes...?

Because the preimage $i^{-1}(U) \subset S^\#$ is perfectoid, via the Tilting equivalence it corresponds to an open subset $i^{-1}(U)^\flat \subset S$ which is again perfectoid; via Proposition 1.6 we may replace S by $i^{-1}(U)^\flat$ and assume $S = i^{-1}(U)^\flat$ so that $i(S^\#) \subset U$.
Since ξ is a global section, we have an inclusion

$$\{x \mid |\xi(x)| \leq |[\varpi]^n(x)|\} \subset U$$

for some $n > 0$; hence we may assume U is given precisely by this rational open. We've now set ourselves up for studying sections on U in a somewhat explicit situation. The sequence at hand is

$$A \xrightarrow{\cdot \xi} A \to W_{\mathcal{O}_E}(R^+)/\xi$$

where $A := \mathcal{O}_{Y_S}(\{|\xi(x)| \leq |[\varpi]^n(x)|\})$. We have by definition

$$A = W_{\mathcal{O}_E}(R^+)\left\langle \frac{\xi}{[\varpi^n]} \right\rangle$$

endowed with the topology such that the subring $A^+ \subset A$ given by the completed integral closure of

$$W_{\mathcal{O}_E}(R^+)\left[\frac{\xi}{[\varpi^n]}\right]$$

with respect to the $[\varpi]$-adic topology. Since $R^\#$ is the completion of the quotient A/ξ, all we have to do is show that the ideal ξ generated in A is closed, as then A/ξ will already be complete. For this, I still have to fathom the argument... something something spectral norm? [FS] Proposition II.1.4. ■

If $S = \operatorname{Spa} C \in \mathrm{Perf}_{\mathbf{F}_q}$ for a perfectoid algebraically closed field C, then each untilt $\operatorname{Spa}(C^\#, C^{\#+})$ of $\operatorname{Spa}(C, \mathcal{O}_C)$ defines a closed geometric point (since $C^\#$ will also be a perfectoid field) in Y_C which is an effective Cartier divisor.

Definition 2.2. Let $S = \operatorname{Spa} C \in \mathrm{Perf}_{\mathbf{F}_q}$ for an algebraically closed perfectoid field C. Then we denote by $|Y_C|^{\mathrm{cl}} \subset |Y_C|$ the subset of closed points corresponding to Cartier divisors arising from untilts as in Proposition 2.1. This is the collection of *classical points* on Y_S.

And now we may define our central object of study.

Definition 2.3. Let S be a perfectoid space over $\mathbf{F}_q$ and Y_S the analytic adic space considered hitherto. The *Fargues-Fontaine curve* - or, occasionally *the curve* - is the analytic adic space given by the quotient

$$Y_S/\varphi^{\mathbf{Z}}$$

where $\varphi : Y_S \xrightarrow{\cong} Y_S$ is the Frobenius automorphism.

Note that Definition 2.3 is well posed, in the sense that thanks to the radius map $\kappa : |Y_S| \to \mathbf{R}$ described in the proof of 1.1 we see that

$$\kappa(\varphi(x)) = \frac{\log |[\varpi^q](\eta_x)|}{\log |\pi(\eta_x)|} = q\kappa(\varphi(x))$$

so that φ acts properly discontinuously on $|Y_S|$ and one can form the quotient as one usually would; see for instance the construction of the Tate curve.

Note that, just as our study with Y_S, we see that the Frobenius on $\mathbf{A}_{\mathrm{inf}}$ induces, upon passing to the associated diamonds and running through the isomorphism in Proposition 1.4, the Frobenius on the base S and thus X_S's associated diamond is given by

$$X_S^\diamond \cong S/\varphi_S^{\mathbf{Z}} \times \operatorname{Spd} E.$$

Since the absolute Frobenius on $X_S^\diamond$, given by the composition $\varphi_S \circ \varphi_{\mathrm{Spd}\,E}$ is the identity on underlying topological spaces (since it preserves all valuations) we see that we have a natural homeomorphism

$$|X_S| \approx |S/\varphi_S^{\mathbf{Z}} \times \mathrm{Spd}\,E| \approx |S \times \mathrm{Spd}\,E/\varphi_{\mathrm{Spd}\,E}^{-\mathbf{Z}}|$$

and thus a continuous 'structure' map

$$\tau : |X_S| \to |S|$$

Our aim is to study line bundles on X_S. For this it is necessary to develop the foundations of descent theory in an appropriate topology; while the pro-étale topology would suffice for our analysis, the usual approach taken in the literature (which starts from [PAG] think) is via the v-topology.

Definition 2.4. Let Y be a perfectoid space. A collection of morphisms $\{f_i : Y_i \to X\}_{i\in I}$ in Perfd/X is a *cover of X in the v-topology on* Perfd if for all quasi-compact open subsets $V \subset X$ a finite collection of the maps $f_{i_1}, \dots, f_{i_n}$ are so that the subsets $f_{i_j}(V_{i_j}) \subset X$ cover V for some $V_{i_j} \subset Y_{i_j}$ quasi compact.

Remark 2.5. Although the v-topology might seem too fine to effectively be used in any meaningful way, we mention that (contrary to what a first impression might suggest) it is *subcanonical*, meaning that objects $S \in \mathrm{Perf}_{\mathbf{F}_q}$ define v-sheaves via their functors of points. This can be argued by a means of replacing S with some totally disconnected perfectoid cover of it; we won't go into the details of the v-topology since it isn't quite relevant and not exactly totally necessary for our purposes.

With Remark 2.5 at hand we can give a description of the classical points on Y_S, which will be handy for some results in the next section.

Lemma 2.6. *Let* $\mathrm{Spa}\,C' \to \mathrm{Spa}\,C$ *be a morphism in* $\mathrm{Perf}_{\mathbf{F}_q}$*, where C'/C is an extension of perfectoid algebraic closed fields.*

1. *A point $x \in Y_C$ is classical if and only if its preimage in $Y_{C'}$ is classical.*
2. *If a rank-one point $x \in |Y_C|$ is not classical, then its preimage along $Y_{C''} \to Y_C$ contains a non-empty open subset of $Y_{C''}$ for some extension C''/C of perfectoid algebraically closed fields.*
3. *For every section $f \in B$, $V(f) \subset U$ only contains classical points lying in U.*
4. *Suppose* $\mathrm{Spa}(B, B^+) \cong U \subset Y_S$ *is a connected affinoid open subset. For each maximal ideal* $\mathfrak{m} \in \mathrm{Sp}\,B$ *the residue field $B/\mathfrak{m}$ is non-archimedean, and thus defines a point in $|U|$. The map thus defined* $\mathrm{Sp}\,B \to |U|^{\mathrm{cl}} = |Y_C|^{\mathrm{cl}} \cap U \subset U$ *- factoring through the inclusion $|U|^{\mathrm{cl}} \subset U$ by the previous part - is a bijection (and thus a homomorphism if one considers* $\mathrm{Sp}\,B$ *as a rigid analytic variety).*

Proof. 1. Note that if $x \in |Y_C|^{\mathrm{cl}}$ is a classical point, then $Y_C' \times_{Y_C} \{x\} \subset Y_C$ also defines a classical point, since the morphism $\mathbf{A}_{\inf}(C, \mathcal{O}_C) \to \mathbf{A}_{\inf}(C', \mathcal{O}_{C'})$ maps distinguished elements of degree one to distinguished elements of degree one. Conversely, if $x \in |Y_C|$ is a rank one-point defined by a map

$$x : \mathrm{Spa}(K(x), K(x)^+) \to Y_C,$$

at the level of diamonds we see

$$x^\diamond : \mathrm{Spd}(K(x), K(x)^+) \to \mathrm{Spa}\,C \times \mathrm{Spd}\,E$$

thanks to Proposition 1.4. If x's preimage in $Y_{C'}$ is classical, then the morphism given by the composition

$$x_{C'}^\diamond : \mathrm{Spd}(K(x), K(x)^+) \times_{Y_C^\diamond} Y_{C'}^\diamond \to \mathrm{Spa}\,C' \times \mathrm{Spd}\,E \to \mathrm{Spa}\,C'$$

is an isomorphism, since by hypothesis $\mathrm{Spd}(K(x), K(x)^+) \times_{Y_C^\diamond} Y_{C'}^\diamond$ is represented by the tilt of the adic spectrum of an untilt of C'. As $\mathrm{Spa}\,C' \to \mathrm{Spa}\,C$ is a v-cover, by Remark 2.5 we can conclude that the composition

$$x^\diamond : \mathrm{Spd}(K(x), K(x)^+) \to \mathrm{Spa}\,C \times \mathrm{Spd}\,E \to \mathrm{Spa}\,C$$

is an isomorphism so that the morphism x defines a Cartier divisor corresponding to an untilt of C as was claimed.[8]

2. Recall that $E_\infty := E(\pi^{1/q^\infty})^\wedge$ is a perfectoid field, proétale over E, whose tilt is the perfectoid field $\mathbf{F}_q(\!(t^{1/p^\infty})\!)$ and, as discussed in Proposition 1.1, the base change $Y_C \times_{\operatorname{Spa} E} \operatorname{Spa} E_\infty$ is perfectoid. On underlying topological spaces, this provides us with a surjective open map

$$|Y_C \times_{\operatorname{Spa} E} \operatorname{Spa} E_\infty| = |\widetilde{\mathbf{D}}_C^*| \approx |\mathbf{D}_C^*| \twoheadrightarrow |Y_C|$$

such that the preimage of $|Y_C|^{\mathrm{cl}}$ is the open subset $|\mathbf{D}_c|^{\mathrm{cl}} := \{0 < |x| < 1\}$ - indeed, at the level of underlying topological spaces, a classical point just corresponds to a map $|\operatorname{Spa} C| \to |Y_C|$ and the *classical points of the punctured open disc* $|\mathbf{D}_C^*|$[9] are precisely these; it's thus sufficient to show the analogous statement for the punctured unit disc. If $x \in |\mathbf{D}_C|^*$ is a rank-one point which doesn't lie in the open subset $\{0|x| < 1\}$, then by the classification of points on the unit disc x must be a Gauß point $x_{p,r}$ for a point $p \in \mathbf{D}_C^*(C)$ and radius $r \in (0,1)$. If C'' is the completed algebraic closure of r, then it'll be perfectoid and the extenion C''/C is so that the preimage of $x_{p,r}$ along $\mathbf{B}_{C''} \to \mathbf{B}_C$ contains the disc $\mathbf{B}_{C''}(p,r) = \{|x-p| \leq r\} \subset \mathbf{B}_{C''}$.

3. Suppose $V(f)$ contains a point which isn't classical; since Y_C is analytic, we may assume it to be rank-one. By the previous part, there exists an extension of algebraically closed perfectoid fields C''/C such that the preimage of $V(f)$ along $Y_{C''} \to Y_C$ contains some open subset $U' \subset U_{C''}$; thus f is mapped to zero via the restriction map $\mathcal{O}_{Y_{C''}}(U_{C''}) \to \mathcal{O}_{Y_S}(U')$ and hence - upon replacing C by C'' and U by $U_{C''}$ - it'll be sufficient to show that for all open subsets $U' \subset U$ the restriction map $\mathcal{O}_{Y_S}(U) \to \mathcal{O}_{Y_S}(U')$ is injective. Since E_∞/E is flat, we pass to the perfectoid base change $Y_C \times_E E_\infty$ and prove the analogous result there.

   If $g \in \mathcal{O}_{Y_C \times_E E_\infty}(U)$ is mapped to zero along $\mathcal{O}_{Y_C \times_E E_\infty}(U) \to \mathcal{O}_{Y_C \times_E E_\infty}(U')$ then (just as before) U' is contained in the vanishing locus $V(g)$. Since in the tilting equivalence vanishing loci of ideals match [EtD] Theorem 5.8, we see that passing to $(Y_C \times_E E_\infty)^\flat \cong \widetilde{\mathbf{D}}_C^*$ we have a similar setup: the open subsets $U'^\flat \subset U^\flat$ are so that $U'^\flat \subset Z(I)$ for some ideal sheaf $I \subset \mathcal{O}_{\widetilde{\mathbf{D}}_C^*|U}$. By the homomorphism $|\mathbf{D}_C^*| \approx |\widetilde{\mathbf{D}}_C^*|$ which holds in both the Zariski and analytic topologies (since $\widetilde{\mathbf{D}}_C^*$ is the perfection of $\mathbf{D}_C^*$), we may also assume we work with the punctured open unit disc $\mathbf{D}_C^*$ as opposed to its perfectoid cover $\widetilde{\mathbf{D}}_C^*$. We can then conclude because injectivity of the restriction maps on $\mathbf{D}_C^*$ follows from the Weierstraß preparation lemma for the Tate algebra $C\langle T\rangle$.

4. By definition, any classical point in U corresponds to a map $\operatorname{Spa}(C^\#, C^{\#+}) \to U$ where $C^\#$ is a perfectoid field whose tilt is C, thus classical points in U all arise as maximal ideals in $B = \Gamma(U, \mathcal{O}_{Y_S})$: the induced map $|U|^{\mathrm{cl}} \hookrightarrow \operatorname{Sp} B$ in an injection. As for showing its surjectivity, given any maximal ideal $\mathfrak{m} \subset B$ the vanishing locus $V(\mathfrak{m}) \subset B$ can only contain one point, which must be classical by the previous part.

■

Lemma 2.6 yields for us a central result of the geometry of Y_C.

Theorem 2.7. *Let $\operatorname{Spa}(B, B^+) \cong U \subset Y_C$ be a connected affinoid open. Then B is a pribncipal ideal domain.*

---

[8] Given a diamond $D \in \widetilde{\mathrm{Perf}_{\mathbf{F}_q}}$, its structure sheaf $\mathcal{O}_D$ (which is induced from any presentation of D as a quotient of a perfectoid space) is a v-sheaf (note that this doesn't follow from Remark 2.5 since $\mathbf{A}^{1\diamond}_{\mathbf{F}_q}$ is not a perfectoid space). Thus $\operatorname{Spd}(K, K^+) \cong \operatorname{Spa}(C, C^+)$ as diamonds $\implies$ K is a perfectoid field and $K^\flat \cong C$ by the tilting equivalence.

[9] Hence the name.

Proof. Since all maximal ideals in B define closed Cartier divisors by Lemma 2.6, they're all generated by some distinguished element of degree one in $\mathbf{A}_{\inf}(C, \mathcal{O}_C)$ and are thus principal.
Suppose given now an element $f \in B$. If $x \in |U|^{\text{cl}}$ is so that $f(x) = 0$ and $\xi \in B$ is a generator of the maximal ideal corresponding to x, we claim that $f = \xi^n g$ for some $n \geq 0$ and g such that $|g(x)| \neq 0$; we endow B with the spectral norm $|| - ||$ and considering the finitely many points on the Shilov boundary of U, we may swap ξ for some multiple of it so that $||\xi^n(x)|| \geq 1$ for all n. If $|\frac{f}{\xi^n}| \neq 0$ for all n then on the rational open $U_x := \{|\xi| \leq |[\varpi]|\}$ we see

$$||f||_{U_x} = \sup \{|f(y)| \mid y \in U_x\} \leq |[\varpi]|^n ||f||$$

hence $||f|| = 0$ since n is arbitrary and $[\varpi]$ is topologically nilpotent. Thus f vanishes on the whole open subset U_x and this contradicts Lemma 2.6.
Now that we may express f as $\xi^n g$ for some g not vanishing at x, we see that g won't vanish on some open subset containing x and thus $V(f)$ is discrete and quasi-compact $\implies$ $V(f)$ is finite and f can be expressed as

$$f = g\xi_1^{d_1} \dots x_n^{d_n}$$

where the xi_i's generate maximal ideals in B and g is a unit. ∎

The proof of Theorem 2.7 is effectively just an adic re-adaptation of the implication from elementary commutative algebra '$\dim R = 1, R$ UFD $\implies R$ PID'.

Lemma 2.8. *The category fibred in groupoids* BGL_n *over* Perfd*, defined by*

$$\mathrm{BGL}_n(X) = \{\textit{Groupoid of locally free sheaves on } X \textit{ of rank } n\}.$$

for $X \in$ Perfd*, is a stack in the* v*-topology. In other words, vector bundles of rank* n *satisfy descent for the* v*-topology on* Perfd*.*

Proof. (*sketch*) Let X be a perfectoid space; as vector bundles satisfy descent in the analytic topology by definition, we can localise on X and assume $X = \mathrm{Spa}(R, R^+)$ affinoid perfectoid. Then, since a disjoint union of finitely many affinoid perfectoids is an affinoid perfectoid space, we reduce to showing that the descent category for vector bundles on a perfectoid space $\widetilde{X}$ along a surjective map of affinoids

$$f : \widetilde{X} = \mathrm{Spa}(\widetilde{R}, \widetilde{R}^+) \twoheadrightarrow X = \mathrm{Spa}(R, R^+), X, \widetilde{X} \in \mathrm{Perfd}$$

is equivalent to the category of vector bundles on X; in this affinoid setting, we may work with the categories of finite projective $\widetilde{R}$ and R modules instead, by the 'adic Serre-Swan' theorem.
First one shows that the functors defined by by

$$X \in \mathrm{Perfd} \mapsto H^0(X, \mathcal{O}_X),$$
$$X \in \mathrm{Perfd} \mapsto H^0(X, \mathcal{O}_X^+)$$

are v-sheaves, whose higher cohomology groups vanish/resp. almost vanish on affinoids - this can be achieved by reducing to the case of totally disconnected perfectoid spaces, since any perfectoid space admits a proétale cover by a disjoint union of such (these statements are true for the proétale topology via the Tate-acyclicity argument). Then, fully faithfulness of the pullback functor

$$\mathrm{FinProjMod}_R \to \mathrm{Desc}_{R \to \widetilde{R}}(\mathrm{FinProjMod}_{\widetilde{R}})$$

follows since it admits a right adjoint - sending a finite projective $\widetilde{R}$-module M with descent data $\alpha : M \otimes_{\widetilde{R}, \iota_1} \left(\widetilde{R} \widehat{\otimes}_R \widetilde{R}\right) \xrightarrow{\cong} M \otimes_{\widetilde{R}, \iota_2} \left(\widetilde{R} \widehat{\otimes}_R \widetilde{R}\right)$ to the R-module given by the equaliser

$$\mathrm{eq}\left(M \rightrightarrows M \otimes_{\widetilde{R}, \iota_2} \left(\widetilde{R} \widehat{\otimes}_R \widetilde{R}\right)\right)$$

- such that the unit is an isomorphism (this is true for the trivial vector bundle since the structure sheaf is a v-sheaf as we mentioned, and for the general case we have have the unit is an isomorphism if and only if such is true locally).

So the only tricky part is essential surjectivity: given a finite projective module M over $\widetilde{R}$ with descent data α along $R \to \widetilde{R}$, the goal is to produce an R-module whose base change to $\widetilde{R}$ is M.

Assume first that R is a perfectoid field, so that then it is non-archimedean and $\widetilde{R}$ is an R-Banach algebra. From the theory of Banach spaces over non-archimedean fields [BGR] one can reduce to the case where $\widetilde{R}$ is topologically free as a Banach R-algebra (which in this setting just means that $\widetilde{R}$ is topologically generated by a countable basis over R), meaning in particular that $-\widehat{\otimes}_R\widetilde{R}$ is an exact functor which reflects exact sequences - i.e. $\operatorname{Spa}(\widetilde{R}, \widetilde{R}^+) \to \operatorname{Spa}(R, R^+)$ behaves as does a faithfully flat map in algebraic geometry. One can now apply the same proof as with classic faithfully flat descent: to check the equivalence of categories one has to verify exactness of the sequence

$$0 \to M \to \widetilde{R}\widehat{\otimes}_R M \xrightarrow{\beta} \widetilde{R}\widehat{\otimes}_R\widetilde{R}\widehat{\otimes}_R M$$

for any finite projective[10] R-module. This can be checked upon applying $-\widehat{\otimes}_R\widetilde{R}$ after which one has the aid of a section of the structure morphism, given by multiplication $\widetilde{R}\widehat{\otimes}_R\widetilde{R} \xrightarrow{\mu} \widetilde{R}$ (which is well defined by $\widetilde{R}$'s completeness).

Finally, suppose R is arbitrary and let $x \in \operatorname{Spa}(R, R^+)$ be any point, which corresponds to a map

$$\operatorname{Spa}(\kappa(x), \kappa(x)^+) \to \operatorname{Spa}(R, R^+).$$

By the separate case just discussed, the descent data α base changed to $\kappa(x)$ yields a sub vector space of dimension n in $M_{\kappa(x)} := M\widehat{\otimes}_{\widetilde{R}}(\widetilde{R}\widehat{\otimes}_S\kappa(x))$ which generates $M_{\kappa(x)}$ as an $\widetilde{R} \otimes_R \kappa(x)$-module freely (because $\operatorname{Spa}(\kappa(x), \kappa(x)^+)$ only has the trivial vector bundle and $M_{\kappa(x)}$ must be the base change of such). By the Nakayama lemma, this yields a rank n free sub-$\mathcal{O}_X(U)$-module of M_U which freely generates M_U as a $\mathcal{O}_{\widetilde{X}}(f^{-1}(U))$-module; once again, since we may restrict to open subsets of X, we can assume that in fact $U = \operatorname{Spa}(R, R^+)$.

By possibly restricting on $\operatorname{Spa}(R, R^+)$ further, we may pick a $\kappa(x)$-basis of $M_{\kappa(x)}$ which lifts to a free generating set of M such that $\alpha - 1$ expressed with respect to this basis is an $n \times n$ invertible matrix with coefficients in $\varpi \cdot \left(\widetilde{R}^+\widehat{\otimes}_{R^+}\widetilde{R}^+\right)$: given an arbitrary basis, one can divide out by some power of ϖ to ensure that the coefficients of the matrix representing α lie in $\widetilde{R}\widehat{\otimes}_R\widetilde{R}$, and then if one chooses the basis so that $\alpha_{\kappa(x)} - 1$'s coefficients are multiples of ϖ, then this condition extends to an open neighbourhood of x by considering rational opens defined by inequalities in terms of these coefficients.

It then follows that the $n \times n$ matrix with coefficients in $\widetilde{R}^+ \otimes_R \widetilde{R}^+$ representing $\frac{\alpha-1}{\varpi}$ defines a twist of the sheaf $\mathcal{O}_X^+$ in the v-topology (???) and thus a class in $H_v^1(X, \mathcal{O}_X^+)$. Since this cohomology is $\sqrt{(\varpi)}$-torsion, we can successively approximate and obtain a basis such that α is represented by the identity, and thus M descends to a finite free R-module. ∎

Example 2.9. I feel the need to include this (fairly obvious) non-example of a v-cover, since it took me an embarrassingly long time to realise it didn't satisfy the quasi-compactness condition and hence doesn't provide a counter-example to the descent result in Lemma 2.8: the following are *not* examples

[10] Note that the finiteness of M here is quite important to study these completed tensor products, since then M is endowed with the quotient topology via a surjection of the form $\widetilde{R}^{\oplus n} \twoheadrightarrow M$.

of v-covers:

$$\operatorname{Spa}(K\langle T^{1/p^\infty}\rangle, \mathcal{O}_K\langle T^{1/p^\infty}\rangle) = \widetilde{\mathbf{B}}_K = \{0\} \cup \left(\widetilde{\mathbf{B}}_K \setminus \{0\}\right),$$
$$\widetilde{\mathbf{B}}_K \setminus \{0\} = \bigcup_{n\geq 0} \{|\varpi|^n \leq |T| \leq |\varpi|^{n-1}\}$$

for any perfectoid field K with pseudouniformiser $\varpi \in K^{\circ\circ} \cap K^\times$.

Corollary 2.10. *The functor fibred in groupoids defined by*

$$\mathrm{Bun}_n : S \in \mathrm{Perf}_{\mathbf{F}_q} \longmapsto \{\textit{Groupoid of vector bundles on } X_S \textit{ of rank } n\}$$

is a (small) v-stack.

Proof. Again we have to prove a descent result, but here it follows from Proposition 1.1: by construction the gropoid of vector bundles on X_S is equivalent to that of $\varphi^{\mathbf{Z}}$-equivariant vector bundles on Y_S, and given any v-cover $\{U_\alpha \to S\}_\alpha$ the corresponding collection of morphisms on $Y_S \times_E E_\infty$

$$\{Y_{U_\alpha} \times_{\operatorname{Spa} E} \operatorname{Spa} E_\infty \to Y_S \times_{\operatorname{Spa} E} \operatorname{Spa} E_\infty\}_\alpha$$

define a v-cover of the perfectoid space $Y_S \times_{\operatorname{Spa} E} \operatorname{Spa} E_\infty$ thanks to Proposition 1.6, for which we have descent of ($\mathbf{Z}$-equivariant) vector bundles by Lemma 2.8. ■

If we fix an affinoid perfectoid space $S = \operatorname{Spa}(R, R^+) \in \mathrm{Perf}_{\mathbf{F}_q}$, the map $Y_S \to X_S$ allows us to study vector bundles on X_S to some detail, because Y_S is a *Stein space*: it admits the cover

$$Y_S = \bigcup_{n\geq 0} \underbrace{Y_{S,[q^{-n},q^n]}}_{:=\kappa^{-1}([q^{-n},q^n])}$$

where each open subset Y_{S,q^{-n},q^n} is rational and affinoid, so that each inclusion of successive opens provides a map on sections

$$\mathcal{O}_{Y_S}(Y_{S,[q^{-n}],q^n}) \to \mathcal{O}_{Y_S}(Y_{S,[q^{-n-1},q^{n+1}]})$$

which is a continuous ring homomorphism of Banach E-algebras with dense dense, by the direct description mentioned in the proof of Proposition 1.1.
Suppose now $\mathcal{E}$ is a vector bundle on X_S and let $\mathcal{F} = \mathcal{E}_{|Y_S}$ be the pullback. Because $Y_{S,[q^{-n},q^n]}$ is affinoid we see that the cohomology group

$$H^i(Y_{S,[q^{-n},q^n]}, \mathcal{F}) = 0$$

vanishes for $i > 0$, and the (for lack of better terminology) '*topological Mittag-Leffler condition*' (see Remark 13.2.4 in chapter 0 from [EGA]) implies

$$R^1 \varprojlim_{n\geq 0} \mathcal{F}(Y_{S,[q^{-n},q^n]}) = 0.$$

Because $R\Gamma(Y_S, \mathcal{F}) = R\varprojlim_{n\geq 0} R\Gamma(Y_{S,[q^{-n},q^n]}, \mathcal{F})$ we conclude that $R\Gamma(Y_S, \mathcal{F}) \cong \mathcal{F}(Y_S)[0]$, and thus the étale cover $Y_S \to X_S$ provides a Cech cover computing the cohomology of $\mathcal{E}$, so that

$$R\Gamma(X_S, \mathcal{E}) \cong \left[\mathcal{F}(Y_S) \xrightarrow{\mathrm{id}-\varphi} \mathcal{F}(\underbrace{Y_S \times \mathbf{Z}}_{\cong Y_S \times_{X_S} Y_S})\right]$$

and, in particular, $H^i(X_S, \mathcal{E}) = 0$ for $i > 2$.

Furthermore, we have a straightforward way of producing vector bundles on X_S: one can simply pick any φ-semilinear automorphism of the trivial vector bundle $\mathcal{O}_{Y_S}^{\oplus n}$ over Y_S, providing $\mathcal{O}_{Y_S}^{\oplus n}$ with the descent data needed to produce a vector bundle of rank n on the curve X_S. This is where the connection to isocrystals is drawn; if we fix an algebraic closure $\overline{\mathbf{F}}_q$ of $\mathbf{F}_q$ and let $S \in \mathrm{Perf}_{\overline{\mathbf{F}}_q}$, we obtain a morphism

$$Y_S \to \mathrm{Spa}(W_{\mathcal{O}_E}(\overline{\mathbf{F}}_q), W_{\mathcal{O}_E}(\overline{\mathbf{F}}_q)) \setminus V(\pi) \cong \mathrm{Spa}(\breve{E}, \mathcal{O}_{\breve{E}}) := \mathrm{Spa}\, \breve{E}$$

where $\breve{E}$ is the completion of the maximal unramified extension of E (and similarly for the mixed characteristic case). $\mathrm{Spa}\, \breve{E}$ is also endowed with its Frobenius automorphism φ, and there's a rich theory of φ-semilinear vector spaces (i.e. $\varphi^{\mathbf{Z}}$-equivariant vector bundles on $\mathrm{Spa}\, \breve{E}$ whose underlying vector bundle is trivial) which historically arose in an article by Mazur studying algebraic properties of the crystalline cohomology of varieties over finite fields and their zeta functions [Maz]. These are classified up to isomorphism as follows: we fix a rational number $\lambda = s/r$ so that $r > 0$ and $(r, s) = (1)$; then the map $E^r \to E^r$ defined by the matrix

$$\begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & & \ddots & 1 \\ \pi^{-s} & & & & 0 \end{pmatrix} \in M_{r\times r}(\breve{E})$$

precomposed with the (diagonal) Frobenius map φ, defines an $\varphi^{\mathbf{Z}}$-equivariant vector bundle on $\mathrm{Spa}\, \breve{E}$ called the *isocrystal of slope* $-\lambda = -s/r$; we denote by $\mathcal{O}(\lambda)$ the corresponding vector bundle of tank r on X_S, obtained by pulling this isocrystal back to a $\varphi^{\mathbf{Z}}$-equivariant vector bundle on Y_S and using the corresponding descent data as mentioned.

This classification theorem, originally proven by Dieudonné and Manin, is argued by means of techniques from the algebraic geometry of vector bundles on curves, even though at its core it's but a linear algebra problem - the strategy is to introduce a Harder-Narasimhan formalism on the category of isocrystals, and show that all filtrations split into direct sums of semistable isocrystals, so that one only has to focus on the stable case; then one can appeal to basic linear algebra methods as the corresponding vector space is cyclic with respect to the φ-semilinear automorphism (the matrix above is really a companion matrix).

The geometry of Bun_n is very rich; upon this stack the connection to the Langlands correspondence is drawn, and many conjectures in the local Langlands programme have been translated to questions about the étale cohomology of Bun_n. The starting point of this theory is the classification of vector bundles on $X_{\mathrm{Spa}\, C}$ where C is a perfectoid field over $\mathbf{F}_q$, which is based on both homological-algebra techniques and the properties of isocrystals. I hope to eventually write up what I can about this topic, but for the time being I'll limit the exposition to only stating the key result we require to continue.

Theorem 2.11. *1. Let C be an algebraically closed perfectoid field over $\mathbf{F}_q$ and consider the curve X_C. Every line bundle is isomorphic to $\mathcal{O}(d)$ for a unique integer $d \in \mathbf{Z}$.*

2. If $S \in \mathrm{Perf}_{\mathbf{F}_q}$ is a perfectoid space over $\mathbf{F}_q$ and $\mathcal{L} \in \mathrm{Bun}_1(S)$ is a line bundle on the curve X_S such that for all geometric points $\mathrm{Spa}\, C \to S$ the restriction $\mathcal{L}_{|X_{\mathrm{Spa}\, C}}$ is isomorphic to $\mathcal{O}(d)$ for a fixed integer d, then there exists a proétale cover $\{U_\alpha \to S\}_\alpha$ such that the restriction $\mathcal{L}_{|U_\alpha}$ is isomorphic to $\mathcal{O}(d)$; in other words, $\mathrm{Isom}(\mathcal{L}, \mathcal{O}(d)) \in \widetilde{S}_{pro\acute{e}t}$ is a proétale $\underline{E}^\times = \underline{\mathrm{Aut}}(\mathcal{O}(d))$-torsor[11].

To keep notations consistent with our main reference [Cdc], we denote by $\mathscr{P}$ic the v-stack Bun_1.

[11] As with any line bundle, we have that $\underline{\mathrm{Aut}}(\mathcal{O}(d)) \cong \underline{H^0(X_S, \mathcal{O}_{X_S}^\times)} \cong \underline{H^0(Y_S, \mathcal{O}_{Y_S})^{\varphi=1}} = \underline{E}^\times$ by $\mathbf{A}_{\mathrm{inf}}$'s π-adic separatedness. Furthermore, note that the passage to a proétale cover of S is really necessary here: unless S lives over $\overline{\mathbf{F}}_q$, it doesn't even make sense to talk about $\mathcal{O}(d)$ and isocrystals since X_S might not live over $\mathrm{Spa}\, \breve{E}$.

Corollary 2.12. *The stack* $\mathscr{P}\mathrm{ic}$ *is a disjoint union*

$$\mathscr{P}\mathrm{ic} = \coprod_{d\in\mathbf{Z}} \mathscr{P}\mathrm{ic}^d$$

where $\mathscr{P}\mathrm{ic}^d \cong [*/\underline{E}^\times] \subset \mathscr{P}\mathrm{ic}$ *is the closed and open substack of line bundles whose geometric fibres restrict to* $\mathcal{O}(d)$ *as in Theorem 2.11.*

3 Descriptions of the space of divisors and the Abel-Jacobi morphism

We've now developed all the instruments needed to study the relationship between divisors and line bundles on the curve. As per usual, the beauty is in studying the geometry of the space of our objects of interest.

Definition 3.1. For a fixed integer $d \in \mathbf{Z}$, we denote by Div^d the v-sheaf on $\mathrm{Perf}_{\mathbf{F}_q}$ defined by

$$S \in \mathrm{Perf}_{\mathbf{F}_q} \mapsto \left\{ (\mathcal{L}, u) \mid \begin{array}{c} \mathcal{L} \in \mathscr{P}\mathrm{ic}^d(S), u \in H^0(X_s, \mathcal{L}) \text{ such that} \\ u_{|X_{\mathrm{Spa}\, C}} \neq 0 \text{ for all geometric points } \mathrm{Spa}\, C \to S \end{array} \right\} / \sim$$

where $(\mathcal{L}, u) \sim (\mathcal{L}', u')$ if there exists an isomorphism $\mathcal{L} \xrightarrow{\cong} \mathcal{L}'$ mapping the section u to u'.

Note that the only automorphism of a line bundle $\mathcal{L}$ fixing a predetermined non-zero global section $u \in H^0(X_S, \mathcal{L}) \setminus \{0\}$ is the identity, so that Div^d is a v-sheaf by Proposition 2.8.

Furthermore, since $Y_S \to X_S$ is a local isomorphism by construction, note that Lemma 2.6 and Theorem 2.7 transport to the Fargues-Fontaine curve X_C over perfectoid algebraically closed fields $C/\mathbf{F}_q$.

Lemma 3.2. *Let* $(\mathcal{L}, u) \in \mathrm{Div}^d(S)$*. Then the morphism of line bundles* u *defines*

$$u : \mathcal{O}_{X_S} \to \mathcal{L}$$

is injective.

Proof. If $S = \mathrm{Spa}(C, C^+) \in \mathrm{Perf}_{\mathbf{F}_q}$ where C is an algebraically closed perfectoid field over $\mathbf{F}_q$, then the vanishing locus of u is a discrete subset in X_C by Theorem 2.7, so the morphism u defines $\mathcal{O}_{X_S} \to \mathcal{L}$ is either an isomorphism or injective at stalks, depending on whether the point in consideration is outside $V(u)$ or an isolated point in $V(u)$.
For the general case, if $f \in \ker(u : \mathcal{O}_{X_S} \to \mathcal{L})$ then for each point $x \in X_S$ with image $\tau(x) \in S$, we may consider a geometric point $\mathrm{Spa}(C, \mathcal{O}_C) \to S$ whose image is $\tau(x)$ and by the case just analysed the pullback of f vanishes along the map $X_C \to X_S$. Since the image of $X_C \to X_S$ contains the intersection U_x of all open subsets containing x (as can be seen by considering the underlying topological spaces of the associated diamonds) and thus f vanishes on U_x. Since x was arbitrary, this implies $f = 0$. ■

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