The Geometric Satake Equivalence

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Contents

These are notes from the first talk in this semester's Kleine AG organised by Louis Jaburi and Felix Zillinger, which aim to cover the content from Richarz's work [Ri] on the Satake Equivalence. The goal of this talk is to describe the classic Satake isomorphism, its link with the unramified local Langlands correspondence for the general linear group GL_n, and how the geometric Satake equivalence is a categorification of this result. We introduce the Satake Category of ℓ-adic perverse sheaves on the affine Grassmannian Gr_G and describe its monoidal structure induced its the convolution product of perverse sheaves.

These notes are likely teeming with errors and misleading intuition from my part - please feel absolutely free to send me an email with any corrections or remarks :)[1] I wholeheartedly thank Louis Jaburi for several meetings and helping me understand the content, as well as my good friends Thiago Solovera and Aaron Wild for their help and support.

1 The Satake Isomorphism

We start by reviewing the classic Satake Isomorphism, more or less following Gross' notes [Gr] and Prof. Caraiani's course from last term.

We fix a non-archimedean local field K of residue characteristic p, $\mathcal{O}_K$ its ring of integers with uniformiser $\varpi \in \mathcal{O}_K$ and a split reductive group G/K. Recall that there exists a canonical integral model $\mathcal{G}/\mathcal{O}_K$ whose $\mathcal{O}_K$-points

$$\mathcal{G}(\mathcal{O}_K) \subseteq \mathcal{G}(K) = G(K)$$

form a maximal open compact subgroup, where $G(K)$ is endowed with the analytic topology, making it into a locally profinite group (for instance and concreteness, via a closed immersion $G \hookrightarrow \mathrm{GL}_{n,K}$) admitting a topological basis made up by its open compact subgroups together with their cosets. $\mathcal{G}(\mathcal{O}_K)$ will be denoted by U_0 and is called the *hyper-special maximal compact subgroup*, in light of

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the existence of hyper-special points on G's associated Bruhat-Tits building. As an example, one can picture the setting

$$\mathrm{GL}_n(\mathbf{Z}_p) \subseteq \mathrm{GL}_n(\mathbf{Q}_p).$$

We're interested in studying the *smooth* representation theory of $G(K)$ - the category of (possibly infinite-dimensional) complex vector spaces V endowed with a linear action $\rho : G(K) \to \mathrm{GL}(V)$ so that the stabiliser of any vector

$$\mathrm{Stab}_{G(K)}(v) \subseteq G(K)$$

is an open subgroup (in other words, the map $g \in G(K) \mapsto g \cdot v \in V$ for every fixed vector $v \in V$ is continuous when V is thought of as having the discrete topology). This peculiar choice of compatibility between the topology on $G(K)$ and its representation theory arises from the theory of automorphic forms - representations of these sorts are the local picture of automorphic representations for groups of the form $\widetilde{G}(\mathbf{A}_F)$ where $\widetilde{G}/F$ is a reductive group over a number field $F/\mathbf{Q}$; in this global setting it's very important that these representations locally factor over open compact subgroups $U \subseteq \widetilde{G}(\mathbf{A}_F^{(\infty)})$ - this choice is motivated by the *adélic* proof of the finiteness of the class group (at least in my mind...) [Con].

Definition 1.1. G's *Hecke algebra* is the (non-unital) ring $\mathcal{H}(G)$ of compactly supported complex-valued functions $f : G(K) \to \mathbf{C}$, whose product is given by *convolution of functions.* For a fixed compact subgroup $U \subseteq G(K)$, the *relative Hecke algebra* $\mathcal{H}(G, U)$ is its (*unital*) subring of functions which are constant on the double cosets $UgU \subseteq G(K)$ for $g \in G(K)$.

Remark 1.2. Note that, implicitly, defining the convolution product of functions on $G(K)$ requires the choice of fixed Haar μ measure on $G(K)$ - this can always be done and μ is uniquely determined up to a scalar; to make this choice canonical we require the measure of the hyperspecial maximal compact subgroup U_0 to be 1.

Remark 1.3. Because the compact open subgroups generate $G(K)$'s topology and functions in $\mathcal{H}(G)$ are locally constant, we can express $\mathcal{H}(G)$ as a filtered colimit of the relative Hecke algebras

$$\mathcal{H}(G) = \bigcup_{U \subseteq G(K)} \mathcal{H}(G, U).$$

For any smooth $G(K)$-representation V, by definition we have

$$V = \bigcup_{U \subseteq G(K)} V^U$$

where $V^U = H^0(U, V)$ denotes the U-fixed vectors of V and $U \subseteq G(K)$ varies among $G(K)$'s open compact subgroups.

Our motivation for introducing the Hecke algebra $\mathcal{H}(G)$ is that it plays the same role group algebras do for finite groups and their representation theory.

Proposition 1.4.

1. *The category of smooth representations of $G(K)$ is equivalent to the category of smooth regular modules V over $\mathcal{H}(G)$ - i.e. such that $\mathcal{H}(G) \cdot V = V$ and each vector is fixed under the action of an indicator function $\mathbf{1}_U$ for an open subgroup $U \subseteq G(K)$.*

2. *For every fixed open compact subgroup $U \subseteq G$ the map*

$$\begin{aligned}\{\textit{irreducible representations of } G(K) \textit{ such that } V^U \neq 0\} &\longrightarrow \{\textit{simple modules over } \mathcal{H}(G, U)\}\\ V &\longmapsto V^U\end{aligned}$$

is a bijection.

The proof of Proposition 1.4 is essentially a readaptation to the smooth representation theory setting of the classical equivalence of categories mentioned above for finite groups - the remarkable thing about this situation is that, *so long as we restrict to studying irreducible representations*, we can focus our attention to representatios of the relative Hecke algebras $\mathcal{H}(G, U)$. Evidently, the larger the open compact subgroup U, the simpler the representation theory (by Proposition 1.4). The Satake isomorphism concerns the *easiest* possible case in this sense.

Definition 1.5. The relative Hecke algebra $\mathcal{H}(G, U_0)$ is called the *spherical Hecke algebra over G*. A smooth $G(K)$-module V is called *unramified* if $V^{U_0} \neq 0$; in other words, under the equivalence from Proposition 1.4, V is unramified if it corresponds to a module over the spherical Hecke algebra $\mathcal{H}(G, U_0)$.

One of the consequence of the Satake Isomorphism is that the spherical Hecke algebra $\mathcal{H}(G, U_0)$ is always abelian; thus the left hand side in Proposition 1.4 is only made up of characters when $U = U_0$.

The use of the word unramified here suggests a relationship with unramified Galois representations - recall that a continuous representation V for the Galois group $\mathrm{Gal}(\overline{K}/K)$ is called *unramified* if its action factors over the inertia subgroup $I_K \subseteq \mathrm{Gal}(\overline{K}/K)$ (in other words V is a representation of the group $\mathrm{Gal}(\overline{k}/k) \cong \widehat{\mathbf{Z}}$ which is completely determined by the Frobenius action Frob_k). The relationship here will be elucidated by the Satake isomorphism.

Let $B \subset G$ be a Borel subgroup and $T \subseteq B$ a maximal torus, which as mentioned above we assume $T \cong \mathbf{G}_{m,K}^n$ to be split; we denote by $T_0 = \mathcal{T}(\mathcal{O}_K)$ the hyperspecial maximal compact subgroup of G's torus T, viewed as a reductive group in it of itself (here $\mathcal{T}$ can be chosen to be a Néron model [Tom]). Since $T(K)$ is abelian, an irreducible unramified representation of $T(K)$ will be determined by a one-dimensional character

$$x : \mathcal{H}(T, T_0) \cong \mathbf{C}[T(K)/T_0] \to \mathbf{C}$$

and we have an isomorphism

$$\begin{gathered} \mathbf{C}[X_\bullet(T)] \xrightarrow{\cong} \mathbf{C}[T(K)/T_0] \\ \lambda \mapsto \lambda(\varpi) \end{gathered}$$

since we're assuming T is split and we have isomorphism of abelian groups

$$X_\bullet(T) \cong T(K)/T_0 \cong \mathbf{Z}^{\oplus n}$$

- because $T \cong \mathbf{G}_m^n$, we see that $T_0 \cong (\mathcal{O}_K^\times)^n$ and a coset in $T(K)/T_0$ only depends on the valuations of its components. This allows us to interpret the relative spherical Hecke algebra of the torus $\mathcal{H}(T, T_0)$ as functions on the affine complex algebraic diagonalisable group scheme

$$\widehat{T} = \operatorname{Spec} \mathbf{C}[X_\bullet(T)].$$

Since T is isomorphic to $\operatorname{Spec} K[X^\bullet(T)]$ as a scheme, $\widehat{T}$ is called T's *dual torus* and is a torus in G's dual (complex) reductive group

$$\widehat{G}/\mathbf{C}.$$

In particular, the character $x : \mathcal{H}(T, T_0) \to \mathbf{C}$ can be thought of as a point $x \in \widehat{T}(\mathbf{C})$.

Example 1.6. Suppose $K = \mathbf{Q}_p$ and $G = \mathrm{GL}_2/\mathbf{Q}_p$, one of G's tori T is given by the subgroup scheme of diagonal matrices. Suppose $\chi : T(K) \cong K^\times \times K^\times \to \mathbf{C}^\times$ is an unramified character, which can be expressed as $\chi = \chi_1 \boxtimes \chi_2$ where $\chi_1, \chi_2 : K^\times \to \mathbf{C}$ are both unramified. We consider χ as a representation of the Borel $B(\mathbf{Q}_P) \subset \mathrm{GL}_2(\mathbf{Q}_p)$ of upper-triangular matrices by letting (the $\mathbf{Q}_p$-points of) its unipotent radical act trivially. We can then induct χ to a representation of GL_2 (up to a normalisation factor, which I omit from discussing because it really is just a technical detail, albeit a rather deep one)

$$I_\chi := \text{n-Ind}_{B(\mathbf{Q}_p)}^{\mathrm{GL}_2(\mathbf{Q}_p)} \chi.$$

If I_χ is irreducible - which is always the case unless χ is chosen among a set of three examples, which include the trivial character for instance - then, by the equivalence in Proposition 1.4 and the aforementioned consequence of the Satake isomorphism, it follows that

$$(I_\chi)^{U_0} \subseteq I_\chi$$

is a one-dimensional representation of the spherical Hecke-algebra. The Iwasawa decomposition

$$\mathrm{GL}_2(\mathbf{Q}_p) = B(\mathbf{Q}_p) \cdot U_0 = N(\mathbf{Q}_p)T(\mathbf{Q}_p)\,\mathrm{GL}_2(\mathbf{Z}_p)$$

implies that the vector $v_\chi \in I_\chi$ which corresponds to the function

$$\phi_{v_\chi} : tnu_0 \mapsto \delta_B^{\frac{1}{2}}(t) \cdot \chi(t)$$

is fixed by U_0, where $\delta_B^{\frac{1}{2}}$ is this normalisation factor I mentioned I wasn't interested in talking about (the beautiful article by Kazhdan and Varshavsky [KazVar] explains its geometric meaning). Since we know v_χ explicitly, we can compute the scalar action of an element $f \in \mathcal{H}(G, U_0)$ on it:

$$\begin{aligned} f \cdot v_\chi &= \int_{N(\mathbf{Q}_p)} \int_{T(\mathbf{Q}_p)} \int_{U_0} f(tnu_0)\phi_{v_\chi}(tnu_0)\, du_0\, dn\, dt \\ &= \int_{T(\mathbf{Q}_p)} \chi(t) \int_{N(\mathbf{Q}_p)} f(tn)\delta_B^{\frac{1}{2}}(t)\, dn\, dt. \end{aligned}$$

We arrive at the general definition.

Definition 1.7. Let G/K be a reductive group, $B = N \rtimes T \subset G$ a Borel with a fixed Levi decomposition. Given an element in the spherical Hecke algebra $f \in \mathcal{H}(G, U_0)$ we set

$$\begin{aligned} \mathcal{S}(f) : T(K) &\longrightarrow \mathbf{C} \\ t &\longmapsto \int_{N(K)} f(tn)\delta_B^{\frac{1}{2}}(t)\, dn. \end{aligned}$$

$\mathcal{S}(f)$ is said to be the *Satake transform* of the function f.

A simple check shows that $\mathcal{S}$ defines a map between spherical Hecke algebras

$$\mathcal{S} : \mathcal{H}(G, U_0) \longrightarrow \mathcal{H}(T, T_0).$$

where here the hyperspecial maximal compact subgroup T_0 is the intersection $U_0 \cap T \subset T$ (note that the base change $T \times_G \mathcal{G} := \mathcal{T}$ defines an integral model for T).

Theorem 1.8 (The Satake Isomorphism)**.** *The Satake Transform $\mathcal{S}$ defines a $\mathbf{C}$-algebra isomorphism between the spherical Hecke algebra $\mathcal{H}(G, U_0)$ and the subalgebra $\mathcal{H}(T, T_0)^W \subseteq \mathcal{H}(T, T_0)$ of W-invariant functions in $\mathcal{H}(T, T_0)$, where $W = N_G(T)/T$ is G's Weyl group.*

A detailed proof can be found in [Gr]; the idea is to reduce the result to a relatively straightforward linear algebra computation: if $G = \mathrm{GL}_n$ we can use the Cartan decomposition

$$\mathrm{GL}_n(K) = \coprod_{\lambda \in X_\bullet(T)^+} U_0 \cdot \lambda(\varpi_K) \cdot U_0$$

to produce a basis for $\mathcal{H}(\mathrm{GL}_n, \mathrm{GL}_n(\mathcal{O}_K))$ on the left hand side given by the indicator functions of the double-cosets $\{\mathbf{1}_{U_0\lambda(\varpi_K)U_0}\}_\lambda \subseteq \mathcal{H}(G, U_0)$, and for the right hand side we can use the discussed isomorphism $\mathcal{H}(T, T_0) \cong \mathbf{C}[X^\bullet(\widehat{T})]$ and $\widehat{G} = \mathrm{GL}_n /\mathbf{C}$'s representations via $\widetilde{T}$'s highest-weight theory - the characters associated to the finite dimensional $\mathrm{GL}_n /\mathbf{C}$-representations $V_\lambda = H^0(\mathrm{GL}_n /B, \mathcal{L}_\lambda)$ form a set of generators for the W-invariant characters on $\widehat{T}$, where λ varies among $\widehat{T}$'s integral dominant weights. Now all we have to do is express $\mathcal{S}(\mathbf{1}_{U_0\lambda(\varpi_K)U_0})$ in terms of these characters and check that the relations expressing the image of one basis by a means of the other via $\mathcal{S}$ define an invertible transformation (indeed, it turns out to be upper triangular if we order things correctly). These relations are described by the Kazhdan-Lusztig polynomials.

Corollary 1.9 (The Unramified Local Langlands Correspondence)**.** *There exists a natural bijection between smooth, irreducible unramified representations of* $\mathrm{GL}_n(K)$ *and n-dimensional Frobenius-semisimple unramified Weil-Deligne representations* $W_K \to \mathrm{GL}_n(\mathbf{C})$ *(which correspond, via Grothendieck's Monodromy theorem, to representations of* $\mathrm{Gal}(\overline{K}/K)$ *which factor over inertia).*

Proof. By the Satake isomorphism, the left hand side is given by one-dimensional modules over the ring $\mathbf{C}[X^\bullet(\widehat{T})]^W$ - i.e. characters $x : \mathbf{C}[X^\bullet(\widehat{T})]^W \to \mathbf{C}$, which of course correspond to collections of diagonal matrices whose diagonal entries are agree up to permutation. The right hand side, on the other hand, by the isomorphism $W_K/I_K \cong \mathbf{Z}$ is completely determined by the *set* of eigenvalues of the Frobenius element $\mathrm{Frob}_K \in W_K$ (which by assumption acts as a diagonal matrix). ■

2 Categorifying the Satake Isomorphism

As mentioned in our sketch of the proof of Theorem 1.8, the ring $\mathcal{H}(T, T_0)^W$ is effectively being treated as the Grothendieck ring of the category of finite-dimensional representations of the complex reductive group $\widehat{G}$, since all finite dimensional representations of $\widehat{G}$ are direct sums of ones of the form V_λ for $\lambda \in X^\bullet(\widehat{T})$ dominant integral. The Satake transform can thus be thought of as an isomorphism of $\mathbf{C}$-algebras

$$\mathcal{S} : \{G(\mathcal{O}_K)\text{-invariant functions on the quotient } G(K)/G(\mathcal{O}_K)\} \xrightarrow{\cong} K_0(\mathrm{Rep}(\widehat{G})).$$

The aim of the game now is to reinterpret both sides as being abelian groups which arise from geometric categories; from this point onwards, we set ourselves in the *function field setting*, where $K = F(\!(t)\!)$ and $\mathcal{O}_K = F[\![t]\!]$ for a fixed field F. The right hand side of course can be *categorified* via the representation category $\mathrm{Rep}(\widehat{G})$, whereas for the left hand side it's tempting to say $\mathcal{H}(G, U_0)$ might be thought of as the ring of a particular family of invariant functions on an F-variety, since in the setting from the classic Satake Isomorphism they literally are functions on the rigid analytic space $G(K)$. The strategy will be to construct a geometric space Gr_G /F (which will turn out to live in the category of filtered colimits of projective schemes - i.e. *ind-projective ind-schemes*) whose F-rational points are equiped with a $G(\mathcal{O}_K)$ action, and our categorification of the ring $\mathcal{H}(G, U_0)$ will end up being the category of equivariant perverse sheaves on this space. The way to get from the category of perverse sheaves to $\mathcal{H}(G, U_0)$ will be to identify Gr_G's F-rational points as

$$\mathrm{Gr}_G(F) \cong G(K)/G(\mathcal{O}_K)$$

and apply the Frobenius-trace map, which to a constructible sheaf $\mathcal{F} \in \mathrm{Sh}_c(\mathrm{Gr}_G)$ associates the function

$$(x : \mathrm{Spec}\, F \to \mathrm{Gr}_G) \in \mathrm{Gr}_G(F) \longmapsto \mathrm{tr}(\mathrm{Frob}_F \mid H^0(\mathrm{Spec}\, K, x^*\mathcal{F})) \in \mathbf{Q}_\ell$$

and can then be extended to perverse sheaves by requiring it to be additive on the bounded derived category $D^b_c(\mathrm{Gr}_G, \mathbf{Q}_\ell)$. To prove the equivalence between the category of equivariant ℓ-adic perverse sheaves P on Gr_G and $\mathrm{Rep}(\widehat{G})$ we'll be basing our argument on a *Tannakian formalism*: we introduce a symmetric monoidal structure on P endowed with a *fibre functor* to the category of $\mathbf{Q}_\ell$-vector spaces.

This data gives us an equivalence between this category of perverse sheaves and the representation category of an algebraic group - part of our work will be devoted to recognising this group as being precisely the Langlands dual $\widehat{G}$. The curious consequence of this argument is that it realises the Langlands dual group intependently of G's root datum; this strategy removes the combinatorics from $\widehat{G}$'s definition entirely.

To motivate the definition of Gr_G, let us ponder on the case $G = \mathrm{GL}_n / K$ and think about how one should construct $\mathrm{Gr}_{\mathrm{GL}_n}$ by the requirement

$$\mathrm{Gr}_{\mathrm{GL}_n}(F) = \mathrm{GL}_n(F(\!(t)\!))/ \mathrm{GL}_n(F[\![t]\!]).$$

Via the natural action of $\mathrm{GL}_n(F(\!(t)\!))$ on the standard lattice $F[\![t]\!]^{\oplus n}$ in the vector space $F(\!(t)\!)^{\oplus n}$, we can think of this quotient as being identified with the family of lattices

$$\mathrm{Gr}_{\mathrm{GL}_n}(F) := \left\{\text{locally free } F[\![t]\!]\text{-submodules } \Lambda \subset F(\!(t)\!)^{\oplus n} \mid \Lambda\left[t^{-1}\right] \cong F(\!(t)\!)^{\oplus n}\right\}.$$

Which brings us to the definition

$$\begin{aligned} \mathrm{Gr}_{\mathrm{GL}_n} : (F\text{-algebras}) &\longrightarrow (\text{Sets}) \\ R &\longmapsto \left\{\text{locally free } R[\![t]\!]\text{-submodules } \Lambda \subset R(\!(t)\!)^{\oplus n} \mid \Lambda\left[t^{-1}\right] \cong R(\!(t)\!)^{\oplus n}\right\}. \end{aligned}$$

Since locally free $R[\![t]\!]$-modules correspond to torsors for the sheaf of groups $\mathrm{GL}_n / R[\![t]\!] \in \widetilde{\mathrm{Spec}\, R[\![t]\!]}_{\text{ét}}$, we can describe

$$\mathrm{Gr}_G(R) \cong \begin{cases} \mathcal{F} \in H^1(\mathrm{Spec}\, R[\![t]\!], \mathrm{GL}_n), \\ \beta \in \Gamma(\mathrm{Spec}\, R(\!(t)\!), \mathcal{F}) \end{cases}$$

where β here is a trivialisation of the torsor $\mathcal{F}$ on the open subscheme $\mathrm{Spec}\, R(\!(t)\!) \subset \mathrm{Spec}\, R[\![t]\!]$ - note that in this case it isn't necessary to specify the topology under which $\mathcal{F}$ is a torsor, since quasi-coherent sheaves of finite $\mathcal{O}_{R[\![t]\!]}$-modules are Zariski-locally free if and only if they're fpqc-locally free by faithfully flat descent; this is not the case for other reductive groups (for instance, PGL_n-torsors assemble to the Brauer group, which would be trivial if it were defined over the Zariski topology) and in general we'll chose torsors which are sheaves for the étale topology - if G is smooth then these are equivalent to the category of fpqc-torsors via faithfully flat descent of quasi-coherent algebras.

The space we'll work with concretely is a mild generalisation of $\mathrm{Gr}_{\mathrm{GL}_n}$ which replaces GL_n with an arbitrary reductive group over F, the open unit disc $\mathrm{Spec}\, F[\![t]\!]$ with an arbitrary curve X and its open subset $\mathrm{Spec}\, F(\!(t)\!) \subset \mathrm{Spec}\, F[\![t]\!]$ with the complement of an arbitrary relative Cartier divisor (where in this setting the divisor was fixed as the zero locus $V(t) \subset \mathrm{Spec}\, R[\![t]\!]$) - the key difference will be that this divisor won't be fixed!

The reason we're interested in this more general Grassmannian will be of rather technical nature: in proving that the convolution product defines a Tannakian structure on the category of perverse sheaves, it'll be important to have nice behaviour of these objects with respect to particular base change diagrams, and for this we'll consider an embedding of Gr_G's category of perverse sheaves into $\mathcal{G}r_G$'s category of *universally locally acyclic* sheaves - the Tannakian structure will be easier to work with in this setting and we can then deduce the relevant properties for Gr_G.

3 The Beilinson-Drinfeld Grassmanian

We fix a smooth geometrically connected curve X over F and a reductive group G over F; denote by Σ the fppf-sheaf

$$R \in (F\text{-algebras}) \longmapsto \{\text{effective Cartier divisors } D \subset X_R = X \times_F R \text{ s.t. } D \to \mathrm{Spec}\, R \text{ is flat}\}$$

which is isomorphic to the disjoint union $\coprod_{d \geq 1} \mathrm{Hilb}^d_{X/F}$ of Hilbert schemes.

Definition 3.1. The Beilinson-Drinfeld Grassmannian associated to the pair G, X is the functor

$$\mathcal{G}r_G : R \in (F\text{-algebras}) \longmapsto \begin{cases} D \in \Sigma(R), \\ \mathcal{F} \in H^1(X_R, G), \\ \beta \in \Gamma(X_R \setminus D, \mathcal{F}_{|X_R \setminus D}) \end{cases}$$

Remark 3.2. For $G = \mathrm{GL}_n$, it's not hard to show that $\mathcal{G}r_G$ is a filtered colimit of the subsheaves

$$\mathcal{G}r_G^{(m)} : R \longmapsto \{J \subset \mathcal{O}_{X_R}(-mD)^{\oplus n}/\mathcal{O}_{X_R}(mD)^{\oplus n} \mid J \in \mathrm{Coh}_{X_R} \text{ and } \mathcal{O}_{X_R}(-mD)^{\oplus n}/J \text{ is flat over } R\} \subset \mathcal{G}r_G(R)$$

each of which can be embedded via a closed immersion to a (projective) Hilbert scheme. Thus $\mathcal{G}r_G$ can be realised as an ind-projective ind-scheme. In the general case one can argue the same by using a closed immersion $G \hookrightarrow \mathrm{GL}_n$.

We give an alternate construction of the Beilinson-Drinfeld Grassmannian, by means of G's *loop groups.*

Definition 3.3. Let $D \in \Sigma(R)$ be a relative Cartier divisor on R and $\mathcal{I} = \mathcal{O}_{X_R}(-D)$ its corresponding ideal sheaf. Denote by $\widehat{\mathcal{O}}_{X,D}$ the formal completion of the sheaf of algebras $\mathcal{O}_{X_R}$ at $\mathcal{I}$ and denote by $\widehat{D} := \underline{\mathrm{Spec}}_{\mathcal{O}_{X_R}} \widehat{\mathcal{O}}_{X_R,D}$ the *affine* formal neighbourhood, and let $\widehat{D}^\circ := \widehat{D} \setminus D$ be the open complement of D in $\widehat{D}$. The *global loop group* is the functor of groups

$$\mathcal{L}G : R \in (F\text{-algebras}) \longmapsto \left\{(s, D) \mid D \in \Sigma(R), s \in G(\widehat{D}^\circ)\right\}$$

and the *global positive loop group* is its sub-functor

$$\mathcal{L}^+G : R \in (F\text{-algebras}) \longmapsto \left\{(s, D) \mid D \in \Sigma(R), s \in G(\widehat{D})\right\}.$$

Lemma 3.4. *1. $\mathcal{L}G$ is representable by an ind-group scheme over Σ and is isomorphic to the sheaf*

$$\mathcal{L}G : R \in (F\text{-}algebras) \longmapsto \left\{(D, \mathcal{F}, \beta, \sigma) \mid D \in \Sigma(R), \mathcal{F} \in H^1(X_R, G), \beta \in \Gamma(X_R \setminus D, \mathcal{F}_{|X_R \setminus D}), \sigma \in \Gamma(\widehat{D}, \mathcal{F}_{|\widehat{D}})\right\}.$$

of G-torsors over X_R with a fixed trivialisation outside of D and at the affine formal neighbourhood $\widehat{D}$.

2. *$\mathcal{L}^+G$ is representable by an affine group scheme over Σ with geometrically connected fibres.*

3. *The projection map*

$$\begin{aligned} \mathcal{L}G &\longrightarrow \mathcal{G}r_G \\ (D, \mathcal{F}, \beta, \sigma) &\longmapsto (D, \mathcal{F}, \beta) \end{aligned}$$

defines an $\mathcal{L}^+G$-torsor over $\mathcal{G}r_G$ and thus induces an isomorphism of fpqc-sheaves over Σ

$$\mathcal{L}G/\mathcal{L}^+G \xrightarrow{\cong} \mathcal{G}r_G.$$

Recall the Beauville-Laszlo descent lemma [BeLa]:

Theorem 3.5 (Un lemme de descente)**.** *Let A be a ring, $f \in A$ a non-zero divisor and $\widehat{A}$ its f-adic completion. Suppose given the following data:*

- *an $A[f^{-1}]$-module F,*
- *an $\widehat{A}$-module G such that* $\ker(G \xrightarrow{\cdot f} G) = 0$.

- *an isomorphism of $\widehat{A}[f^{-1}]$-modules*

$$\phi : F \otimes_{A[f^{-1}]} \widehat{A}[f^{-1}] \xrightarrow{\cong} G \otimes_{\widehat{A}} \widehat{A}[f^{-1}].$$

Then there exists an A-module M which induces the triple (F, G, ϕ) via base change which is unique up to unique isomorphism. Furthermore, M is finite/projective/flat if and only if F and G are finite/projective/flat.

With this result we can tackle the moduli description of the global loop group in Lemma 3.4.

Proof. 1. Zariski-locally, a relative Cartier divisor $D \subset X_R$ is given by the vanishing locus of a regular section $f \in \Gamma(X_R, \mathcal{O}_{X_R})$ - the data $\mathcal{F}, \beta$ and σ from the description in the proposition then coincide with the gluing data from Theorem 3.5. Since all torsors for an affine group scheme are representable and $\mathcal{G}r_G$ is an ind-scheme as discussed in Remark 3.2, we see that points 2. and 3. imply the remaining first portion of part 1.

2. By the equality
$$\widehat{D} = \varprojlim_{i \geq 0} \underbrace{\operatorname{Spec}_{\mathcal{O}_{X_R}} \mathcal{O}_{X_R}/\mathcal{O}_{X_R}(-iD)}_{:= D^{(i)}}$$
we see that $\mathcal{L}^+G$ is a projective limit of the fpqc sheaves
$$R \in (F\text{-algebras}) \longmapsto \left\{ (D, s) \mid D \in \Sigma(R), s \in \Gamma(\operatorname{Res}_{D^{(i)}/X_R} G, \mathcal{O}_{\operatorname{Res}_{D^{(i)}/X_R} G}) \right\}$$
each of which is isomorphic to a closed subscheme of the Hilbert scheme of X_R. Since the transition maps for these are affine and they all have geometrically connected fibres over Σ as these are given by $\varprojlim \operatorname{Res}_{D^{(i)}/X_R} G$, we see that $\mathcal{L}^+G$ is also an affine scheme with geometrically connected fibres.

3. First, we claim that every G-torsor $\mathcal{F} \in H^1(\widehat{D}, G)$ is trivial after a faithfully flat affine base change $\operatorname{Spec} R' \to \operatorname{Spec} R$. To be finished.

■

Theorem 3.6 (Grothendieck algebrisation)**.** *Let A be a Noetherian ring which is complete with respect to the ideal $I \subset A$ and set $S = \operatorname{Spec} A, S_n = \operatorname{Spec} A/I^n$. Consider a sequence of pullback diagrams of the form*

$$\begin{tikzcd} X_1 \arrow[r, "i_1"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & X_2 \arrow[r, "i_2"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & X_3 \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \dots \\ S_1 \arrow[r] & S_2 \arrow[r] & S_3 \arrow[r] & \dots \end{tikzcd}$$

Let $(\mathcal{L}_n\, \phi_n)_{n \geq 1}$ be a collection of torsors $\mathcal{L}_n \in H^1_{ét}(X_i, G)$ together with a compatible system of isomorphisms $\phi_n : i_n^* \mathcal{L}_{n+1} \xrightarrow{\cong} \mathcal{L}_n$, where G is a reductive algebraic group over A. If $X_1 \to S_1$ is proper and $\mathcal{L}_1 \in \operatorname{Pic} X_1$ is ample[2]*, then there exists an ample line bundle $\mathcal{L} \in \operatorname{Pic} X$ whose restrictions to each X_i are the given $\mathcal{L}_i$'s.*

Remark 3.7. As one may expect, taking fibres along the projection $\mathcal{G}r_G \to \Sigma$ produces the classical Grassmannian described in Section 2: if $x \in X(F)$ is a fixed F-rational point and $D_x \in \Sigma(F)$ is the corresponding relative effective Cartier divisor on X, then since X is smooth we have $\widehat{D}_x \cong \operatorname{Spec} F[\![t]\!]$ and thus isomorphisms of fpqc-sheaves

$$\mathcal{L}G_x \cong LG,$$
$$\mathcal{L}^+G_x \cong L^+G,$$
$$\mathcal{G}r_{G,x} \cong \operatorname{Gr}_G.$$

[2]On the Stacks Project (tag 0898) I found this result stated for $G = \mathbf{G}_m$ with the assumption that $\mathcal{L}_1$ corresponds to an ample line bundle; how is one supposed to replace this condition for a general reductive group?

4 The Convolution Product

We're interested in equipping a certain category of sheaves on $\mathcal{G}r_G$ with a convolution product, as to define a Tannakian category. This goal sets two main problems to tackle: we first focus on considering the correspondence between Gr_G and $\mathrm{Gr}_G \times \mathrm{Gr}_G$

$$\begin{tikzcd} & LG \times \mathrm{Gr}_G \arrow[dl, "p"'] \arrow[dr, "q"] & & \\ \mathrm{Gr}_G \times \mathrm{Gr}_G & & LG \times^{L^+G} \mathrm{Gr}_G \arrow[dr, "m"] & \\ & & & \mathrm{Gr}_G \end{tikzcd}$$

in using the results from Lemma 3.4; afterwards we study which category of sheaves we should work with as to allow this convolution product to satisfy the desired properties needed to define a Tannakian structure.

With a correspondence as above at hand, it'll be possible to construct a symmetric monoidal structure on the category of L^+G-equivariant ℓ-adic perverse sheaves on Gr_G: given L^+G-equivariant sheaves $\mathcal{A}_1, \mathcal{A}_2$ on Gr_G we construct their box product

$$\mathcal{A}_1 \boxtimes \mathcal{A}_2 := p_1^* \mathcal{A}_1 \otimes^L p_2^* \mathcal{A}_2$$

which is a perverse sheaf on the product $\mathcal{G}r_G \times \mathcal{G}r_G$. By using $\mathcal{A}_2$'s equivariance, we deduce the existence of a perverse sheaf $\mathcal{A}_1 \widetilde{\boxtimes} \mathcal{A}_2$ on $LG \times^{L^+G} \mathcal{G}r_G$ satisfying

$$p^*(\mathcal{A}_1 \boxtimes \mathcal{A}_2) = q^*(\mathcal{A}_1 \widetilde{\boxtimes} \mathcal{A}_2).$$

We then set $\mathcal{A}_1 \star \mathcal{A}_2 := m_*(\mathcal{A}_1 \widetilde{\boxtimes} \mathcal{A}_2)$.

Remark 4.1. By using the trace map mentioned at the end of Section 2, it's not hard to show that the trace map associated to the convolution of two sheaves is the actual convolution between the associated trace maps of the two individual sheaves - in this sense, the convolution really is the *correct* tensor product structure to consider if we want to categorify the Hecke algebra.

A global version of this construction can be done fairly directly.

Definition 4.2. The k-fold convolution Grassmanian $\widetilde{\mathcal{G}r}_{G,k} = \widetilde{\mathcal{G}r}_k$ is the functor on F-algebras

$$R \in (F\text{-algebras}) \longmapsto \begin{cases} D_1, \dots, D_k \in \Sigma(R), \\ \mathcal{F}_1, \dots, \mathcal{F}_k \in H^1(X_R, G), \\ \beta_i : \mathcal{F}_{i|X_R \setminus D_i} \xrightarrow{\cong} \mathcal{F}_{i-1|X_R \setminus D_{i-1}} \end{cases}$$

where $\mathcal{F}_0 \in H^1(X_R, G)$ is the trivial torsor.

Remark 4.3. Via similar arguments to the ones above, one can show that $\widetilde{\mathcal{G}r}_{G,k}$ is representable by an ind-scheme, ind-proper over Σ^k.

Definition 4.4. We denote by m_k the *k-fold convolution map*

$$\begin{aligned} m_k : \widetilde{\mathcal{G}r}_{G,k} &\longrightarrow \mathcal{G}r_G \\ (D_i, \mathcal{F}_i, \beta_i)_i &\longmapsto (D,\, \mathcal{F}_k,\, \beta_{1|X_R \setminus D} \circ \ldots \circ \beta_{k|X_R \setminus D}) \end{aligned}$$

where $D = D_1 + \ldots + D_k \in \Sigma(R)$ is the sum of the divisors $D_1, \dots, D_k$.

This defines for us the analogue of the map m from above. As for p and q, we construct global versions of the products $LG \times \mathrm{Gr}_G$ and $LG \times^{L^+G} \mathrm{Gr}_G$.

Definition 4.5. We define $\widetilde{\mathcal{L}G}_k$ as the functor

$$\widetilde{\mathcal{L}G}_k : R \in (F\text{-algebras}) \longmapsto \begin{cases} D_1, \dots, D_k \in \Sigma(R), \\ \mathcal{F}_1, \dots, \mathcal{F}_k \in H^1(X_R, G), \\ \beta_i : \mathcal{F}_{i|X_R \setminus D_i} \xrightarrow{\cong} \mathcal{F}_{0|X_R \setminus D_i}, \\ \sigma_i : \mathcal{F}_{0|\widehat{D}_i} \xrightarrow{\cong} \mathcal{F}_{i-1|\widehat{D}_i} \end{cases}$$

and we then define the projections

$$\begin{aligned} p_k : \widetilde{\mathcal{L}G}_k &\longrightarrow \mathcal{G}r^k \\ (D_i, \mathcal{F}_i, \beta_i, \sigma_i) &\longmapsto (D_i, \mathcal{F}_i, \beta_i) \end{aligned}$$

and

$$\begin{aligned} q_k : \widetilde{\mathcal{L}G}_k &\longrightarrow \widetilde{\mathcal{G}r}_k \\ (D_i, \mathcal{F}_i, \beta_i, \sigma_i) &\longmapsto (D_i, \mathcal{F}'_i, \beta'_i) \end{aligned}$$

where $\mathcal{F}_1 = \mathcal{F}'_1$ and $\mathcal{F}'_i$ is defined inductively by gluing $\mathcal{F}_{i|X_R \setminus D_i}$ to $\mathcal{F}'_{i-1|\widehat{D}_i}$ along $\sigma_{i|\widehat{D}_i^\circ} \circ \beta_{i|\widehat{D}_i^\circ}$.

We thus have the correspondence

$$\begin{tikzcd} & \widetilde{\mathcal{L}G}_k \arrow[dl, "p_k"'] \arrow[dr, "q_k"] & & \\ \mathcal{G}r_G^k & & \widetilde{\mathcal{G}r}_k \arrow[dr, "m"] & \\ & & & \mathcal{G}r_G \end{tikzcd}$$

which will allow us to define the convolution just as above, with some work. We also remark that taking the fibre over a point $x \in X(F) \subseteq \Sigma$ along the map $\mathcal{G}r_G \to \Sigma$ and setting $k = 2$ yields the correspondence we orginally started with above.

5 Universally locally acyclic sheaves

In this section I follow [FarSch] for the notion of universally locally acyclic sheaves, because I find their exposition of the topic a little more digestible than Richarz coverage.

Definition 5.1. 1. Let $f : X \to S$ be a morphism of schemes, $A \in D^b_c(X_{\text{ét}}, \Lambda)$ where Λ is some n-torsion sheaf of rings on $X_{\text{ét}}$ where $n \in \mathcal{O}_S^\times(S)$. Then A is *f-locally acyclic* if for all geometric points $\overline{x} \in X$ with image $\overline{s} = f(\overline{x})$ and geometric generalisations $\overline{t} \rightsquigarrow s$ the canonical map

$$A_{\overline{x}} = R\Gamma(X^{\mathrm{sh}}_{\overline{x}}, A) \longrightarrow R\Gamma(X_{\overline{x}} \times_{S_{\overline{s}}} \overline{t}, A)$$

is an isomorphism.

2. A is *f-universally locally acyclic* if for any base change

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

the sheaf $g'^* A$ is f'-locally acyclic.

Example 5.2. 1. If f is smooth then Λ is universally locally acyclic.

2. A complex $A \in D^b_c(X_{\text{ét}}, \Lambda)$ is id_X-universally locally acyclic if and only if it is locally constant.

3. If f is proper and A if f-universally locally acyclic then Rf_*A is locally constant.

4. For every proper morphism of S-schemes $g : X' \to X$ the pushforward of a ULA sheaf (over S) on X' is ULA on X (again, over S).

5. If A is f-universally locally acyclic we have a version of Poincaré duality

$$\mathbf{D}_{X/S}(A) \otimes f^*B \xrightarrow{\cong} RHom(A, Rf^!B)$$

for all complexes $B \in D^b_c(S_{\text{ét}}, \Lambda)$.

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