Banach-Colmez Spaces and Div^d

Contents

1 Div^d in terms of Banach-Colmez spaces

We've now developed all the instruments needed to study the relationship between divisors and line bundles on the curve. As per usual, the beauty is in studying the geometry of the space parameterising our objects of interest.

Definition 1.1. For a fixed integer $d \in \mathbf{Z}$, we denote by Div^d the v-sheaf on $\mathrm{Perf}_{\mathbf{F}_q}$ defined by

$$S \in \mathrm{Perf}_{\mathbf{F}_q} \mapsto \left\{ (\mathcal{L}, u) \mid \begin{array}{c} \mathcal{L} \in \mathscr{P}\mathrm{ic}^d(S), u \in H^0(X_s, \mathcal{L}) \text{ such that} \\ u_{|X_{\mathrm{Spa}\, C}} \neq 0 \text{ for all geometric points } \mathrm{Spa}\, C \to S \end{array} \right\} / \sim$$

where $(\mathcal{L}, u) \sim (\mathcal{L}', u')$ if there exists an isomorphism $\mathcal{L} \xrightarrow{\cong} \mathcal{L}'$ mapping the section u to u'.

Note that the only automorphism of a line bundle $\mathcal{L}$ fixing a predetermined non-zero global section $u \in H^0(X_S, \mathcal{L}) \setminus \{0\}$ is the identity, so that Div^d is a v-sheaf by Proposition **??**.

Furthermore, since $Y_S \to X_S$ is a local isomorphism by construction, note that Lemma **??** and Theorem **??** transport to the Fargues-Fontaine curve X_C over perfectoid algebraically closed fields $C/\mathbf{F}_q$.

Lemma 1.2. *Let $(\mathcal{L}, u) \in \mathrm{Div}^d(S)$. Then the morphism of line bundles u defines*

$$u : \mathcal{O}_{X_S} \to \mathcal{L}$$

is injective.

Proof. If $S = \mathrm{Spa}(C, C^+) \in \mathrm{Perf}_{\mathbf{F}_q}$ where C is an algebraically closed perfectoid field over $\mathbf{F}_q$, then the vanishing locus of u is a discrete subset in X_C by Theorem **??**, so the morphism u defines $\mathcal{O}_{X_S} \to \mathcal{L}$ is either an isomorphism or injective at stalks, depending on whether the point in consideration is outside $V(u)$ or an isolated point in $V(u)$.
For the general case, if $f \in \ker(u : \mathcal{O}_{X_S} \to \mathcal{L})$ then for each point $x \in X_S$ with image $\tau(x) \in S$, we may consider a geometric point $\mathrm{Spa}(C, \mathcal{O}_C) \to S$ whose image is $\tau(x)$ and by the case just analysed the pullback of f vanishes along the map $X_C \to X_S$. Since the image of $X_C \to X_S$ contains the intersection U_x of all open subsets containing x (as can be seen by considering the underlying topological spaces of the associated diamonds) and thus f vanishes on U_x. Since x was arbitrary, this implies $f = 0$. ■

Having set things up this way, we have that the definition of the Abel-Jacobi map falls in place nearly on its own.

Definition 1.3. Let $d \geq 1$ be an integer. We define the Abel-Jacobi map as the morphism of small v-stacks on $\mathrm{Perf}_{\mathbf{F}_q}$

$$\begin{aligned} \mathrm{AJ}^d : \mathrm{Div}^d &\longrightarrow \mathscr{P}\mathrm{ic}^d \\ (\mathcal{L}, u) &\longmapsto \mathcal{L}. \end{aligned}$$

Whenever $\mathrm{Spa}(A, A^+) \to \mathrm{Spa}(B, B^+)$ is a finite étale cover of affinoid perfectoid spaces over $\mathbf{F}_q$, we have that the corresponding morphism $X_{(B,B^+)} \to X_{(A,A^+)}$ is also a finite étale: by π-adic completeness, the ring $W_{\mathcal{O}_E}(A^+)$ is henselian with respect to the ideal generated by π and so the categories of almost finite étale $W(A^+)/\pi \cong A^+$-algebras and almost finite étale $W(A^+)$-algebras are equivalent, so that then $W(B^+)$ is almost finite étale over $W(A^+)$ as such is the morphism $A^+ \to B^+$ by Almost Purity [aPur]; it then follows that passing to the non-vanishing locus of $[\varpi]\pi$ yields a finite-étale morphism as desired[1]. The continuous morphisms $|Y_S| \to |X_S| \to |S|$ thus extend to morphisms of topoi

$$\begin{aligned} \tau : \widetilde{X}_{S,\text{proét}} &\to \widetilde{S}_{\text{proét}}, \\ \widetilde{Y}_{S,\text{proét}} &\to \widetilde{S}_{\text{proét}} \end{aligned}$$

which for simplicity will both be denoted by τ by abuse of notation.

Our goal is to study the morphism AJ^d in order to relate $\overline{\mathbf{Q}}_\ell$-local systems on Div^d and $\mathscr{P}\mathrm{ic}^d \cong [*/\underline{E}^\times]$: the category of such on the latter is equivalent to smooth $\underline{E}^\times$-representations in $\overline{\mathbf{Q}}_\ell$-vector spaces whereas, upon studying different descriptions of Div^d, we'll find that $\overline{\mathbf{Q}}_\ell$-local systems in this case identify with continuous representations of the Weil group W_E. Ultimately we make use of the Abel-Jacobi morphism to classify one-dimensional Galois representations of the local field E and thus conclude with a geometric proof of local class field theory.

Our first description of Div^d will be in terms of Banach-Colmez spaces.

Definition 1.4.
1. We denote by $\mathbf{B}$ the pro-étale sheaf on $\mathrm{Perf}_{\mathbf{F}_q}$ defined by setting

$$\mathbf{B} : S \in \mathrm{Perf}_{\mathbf{F}_q} \mapsto \Gamma(Y_S, \mathcal{O}_{Y_S})$$

   and we let $\varphi : \mathbf{B} \xrightarrow{\cong} \mathbf{B}$ be its Frobenius automorphism, induced by that on Y_S.

2. We denote by $\mathbf{B}^{\varphi=\pi^d}$ the subsheaf of $\mathbf{B}$ on $\mathrm{Perf}_{\mathbf{F}_q,\text{proét}}$ consisting of $\pi^{-d}\varphi$-invariant sections.

Remark 1.5. If S is a perfectoid space over $\mathbf{F}_q$ then the sheaf $\mathbf{B}_S := \mathbf{B} \times S \in \widetilde{\mathrm{Perf}}_{S,\text{proét}}$ identifies with the pushforward $\tau_* \mathcal{O}_{Y_S}$ and similarly $\mathbf{B}_S^{\varphi=\pi^d} \cong \tau_* \mathcal{O}_{X_S}(d)$; in particular, we see that whenever the perfectoid base $S \in \mathrm{Perf}_{\mathbf{F}_q}$ is fixed, the pro-étale sheaves $\mathbf{B}_S$ and $\mathbf{B}_S^{\varphi=\pi^d}$ are diamonds since they're pushforwards of line bundles on Y_S and X_S respectively (which admit corresponding *geometric vector bundles*). However, their *absolute* versions $\mathbf{B}$ and $\mathbf{B}^{\varphi=\pi^d}$ are not: for instance we have an isomorphism

$$\mathbf{B}^{\varphi=1} \cong \underline{E}.$$

I thought locally profinite sets were locally spatial diamonds...? How is $\mathbf{B}^{\varphi=1}$ not a diamond then?

[1] This seems a little fishy... Does it work?

Note that, for affinoid perfectoid spaces $S = \mathrm{Spa}(R, R^+) \in \mathrm{Perf}_{\mathbf{F}_q}$, the global sections on Y_S don't depend on the ring of integral elements R^+, so we have equalities

$$\mathbf{B}(\mathrm{Spa}(R, R^+)) = \mathbf{B}(\mathrm{Spa}(R, R^\circ)),$$
$$\mathbf{B}^{\varphi=\pi^d}(\mathrm{Spa}(R, R^+)) = \mathbf{B}^{\varphi=\pi^d}(\mathrm{Spa}(R, R^\circ))$$

hence the absolute Banach-Colmez spaces $\mathbf{B}$ and $\mathbf{B}^{\varphi=\pi^d}$ are partially proper pro-étale sheaves on $\mathrm{Perf}_{\mathbf{F}_q}$, in the sense that they satisfy the valuative criteria for partial properness classically studied for adic spaces [Adic].

Because of its relevance, we quickly recall a central result discussed in the previous pdf.

Theorem 1.6. *1. Let C be an algebraically closed perfectoid field over $\mathbf{F}_q$ and consider the curve X_C. Every line bundle is isomorphic to $\mathcal{O}(d)$ for a unique integer $d \in \mathbf{Z}$.*

2. If $S \in \mathrm{Perf}_{\mathbf{F}_q}$ is a perfectoid space over $\mathbf{F}_q$ and $\mathcal{L} \in \mathrm{Bun}_1(S)$ is a line bundle on the curve X_S such that for all geometric points $\mathrm{Spa}\, C \to S$ the restriction $\mathcal{L}_{|X_{\mathrm{Spa}\, C}}$ is isomorphic to $\mathcal{O}(d)$ for a fixed integer d, then there exists a proétale cover $\{U_\alpha \to S\}_\alpha$ such that the restriction $\mathcal{L}_{|U_\alpha}$ is isomorphic to $\mathcal{O}(d)$; in other words, $\mathrm{Isom}(\mathcal{L}, \mathcal{O}(d)) \in \widetilde{S}_{pro\acute{e}t}$ is a proétale $\underline{E}^\times = \underline{\mathrm{Aut}}(\mathcal{O}(d))$-torsor.

Because of Theorem 1.6, we see that Banach-Colmez spaces provide a description of the moduli space of divisors Div^d we're interested in understanding.

Proposition 1.7. *Consider the v-sheaf*

$$\mathbf{B}^{\varphi=\pi^d}\backslash\{0\} : S \in \mathrm{Perf}_{\mathbf{F}_q} \longmapsto \left\{ s \in \mathbf{B}^{\varphi=\pi^d}(S) \mid s_{|X_{(K,K^+)}} \neq 0 \text{ for all geometric points } \mathrm{Spa}(K, K^+) \to S \right\}.$$

Then the morphism

$$\left(\mathbf{B}^{\varphi=\pi^d} \setminus \{0\}\right) / \underline{E}^\times \longrightarrow \mathrm{Div}^d$$

defined by mapping each section $s \in \mathbf{B}^{\varphi=\pi^d}(S)$ to the pair $(\mathcal{O}(d), s) \in \mathrm{Div}^d(S)$ is an isomorphism.

Proof. As both sheaves are defined in the v-topology, Theorem 1.6 applies and is just a reformulation. ■

2 The relation to untilts via Lubin-Tate theory

We now give a different description of Div^1 via the theory of Lubin-Tate formal groups. As motivation, consider the projective line $\mathbf{P}^1_k$ over a field k; its Picard group is completely described by the degree map

$$\deg : \mathrm{Pic}(\mathbf{P}^1_k) \xrightarrow{\cong} \mathbf{Z}$$

and it admits up to isomorphism only one line bundle $\mathcal{O}(1)$ of degree one. While $\mathcal{O}(1)$ can be described explicitly using descent data relative to the cover

$$\mathrm{Spec}(k[X]) \cup \mathrm{Spec}(k[X^{-1}]) \twoheadrightarrow \mathbf{P}^1_k,$$

one can also view $\mathcal{O}(1)$ as the line bundle of meromorphic functions with a single order-one pole at a given point $x : \mathrm{Spec}\, k \to \mathbf{P}^1_k$: once such a point x is fixed, there is a corresponding ideal sheaf $\mathcal{I}_x \subset \mathcal{O}_X$ so that the sequence

$$0 \to \mathcal{I}_x \to \mathcal{O}_X \to x_* \mathcal{O}_{\mathrm{Spec}\, k} \to 0$$

is an exact sequence of quasi-coherent sheaves on $\mathbf{P}^1_k$. Dualising the morphism of sheaves $\mathcal{I}_x \to \mathcal{O}_X$ we obtain a global section $s \in \Gamma(X, \mathcal{I}_x^\vee)$ whose vanishing locus is precisely the point x by construction; in particular, $\mathcal{I}_x^\vee \cong \mathcal{O}(1)$ is *the* line bundle of degree on $\mathbf{P}^1_k$ one and all non-zero global sections of $\mathcal{O}(1)$ arise this way (by reversing the construction).

We can now mimic this procedure for the Fargues-Fontaine curve: given an affinoid perfectoid space $S = \mathrm{Spa}(R, R^+) \in \mathrm{Perf}_{\mathbf{F}_q}$ and some untilt $S^\# = \mathrm{Spa}(R^\#, R^{\#+}) \in \mathrm{Perfd}$ we have that the Fontaine map provides an isomorphism

$$\theta : \mathbf{A}_{\mathrm{inf}}(R, R^+)/\xi \xrightarrow{\cong} R^{\#+}$$

where $\xi \in \mathbf{A}_{\mathrm{inf}}(R, R^+)$ is a distinguished element of degree one, realising the untilt $S^\#$ as a divisor on Y_S, as discussed in the previous pdf. The section $\xi \in \Gamma(Y_S, \mathcal{O}_S)$ whose vanishing locus is precisely $S^\# \subset Y_S$ however does not in general descent to a global section on X_S of the line bundle $\mathcal{O}(1)$, since

$$\Gamma(X_S, \mathcal{O}(1)) = \Gamma(Y_S, \mathcal{O}_{Y_S})^{\varphi=\pi}$$

and a priori there's no reason why $\varphi(\xi)$ should equal $\pi \cdot \xi$; the goal is to take a weighted sum of the sequence of global sections $\varphi^n(\xi)$ for $n \in \mathbf{Z}$, to produce a section whose vanishing locus is the *collection* of untilts given by the sequence of Frobenius-twists of $S^\#$. The formula needed to perform this averaging process arises as a logarithm from the theory of p-divisible groups.

Let $\mathcal{G}/\mathcal{O}_E$ be a Lubin-Tate formal $\mathcal{O}_E$-module of height one, admitting a section $X \in \Gamma(G, \mathcal{O}_G)$ so that abstractly

$$\mathcal{G} \cong \mathrm{Spf}\, \mathcal{O}_E[\![X]\!]$$

and its generic fibre $\mathcal{G}_E := \mathcal{G} \times_{\mathrm{Spa}\, E} \mathrm{Spa}\, \mathcal{O}_E$ is isomorphic to $\mathbf{G}_{a,E}$ (the unique height 0 formal $\mathcal{O}_E$-module) via the logarithm

$$\log_{\mathcal{G}} : \mathcal{G}_E \to \mathbf{G}_{a,E}$$

defined explicitly by the global section

$$X + \frac{1}{\pi} X^q + \frac{1}{\pi^2} X^{q^2} + \ldots \in \Gamma(\mathcal{G}_E, \mathcal{O}_{\mathcal{G}_E}).$$

We introduce $\mathcal{G}_E$'s '*universal cover*', defined by the inverse limit of formal $\mathcal{O}_E$-modules

$$\widetilde{\mathcal{G}} := \varprojlim_{\cdot \pi} \mathcal{G} = \lim \left(\mathcal{G} \xleftarrow{\cdot \pi} \mathcal{G} \xleftarrow{\cdot \pi} \ldots \right)$$

which as a formal scheme is isomorphic to $\mathrm{Spf}\, \mathcal{O}_E[\![\widetilde{X}^{1/q^\infty}]\!]$: indeed, by the equivalence π-adically complete flat $\mathcal{O}_E$-algebras with perfect residue field and perfect $\mathbf{F}_q$-algebras, its sufficient to check that $\widetilde{\mathcal{G}}$'s special fibre is isomorphic to

$$\mathcal{O}_E[\![\widetilde{X}^{1/q^\infty}]\!]/\pi = \mathbf{F}_q[\![\widetilde{X}^{1/q^\infty}]\!]$$

but this is automatic since the action of π on $\mathcal{G} \cong \mathrm{Spf}\, \mathcal{O}_E[\![X]\!]$ is by a power series of the form $X^q + \pi O(X) + o(X^q)$. If we take the generic fibre of the projection map $\widetilde{\mathcal{G}} \to \mathcal{G}$ we can then study its composition with the logarithm map

$$\widetilde{\mathcal{G}}_E \to \mathcal{G}_E \xrightarrow{\log_{\mathcal{G}}} \mathbf{G}_{a,E}$$

which is explicitly given by the section $\sum_{n \in \mathbf{Z}} \pi^{-n} \widetilde{X}^n$ - to do: write this down properly...? Can you compute the mod $\widetilde{X}^n$-reduction of this section?

If now we go back to the untilt $S^\# = \mathrm{Spa}(R^\#, R^{\#+})$, we see that the $S^\#$-valued points of $\widetilde{\mathcal{G}}_E$ are given by

$$\widetilde{\mathcal{G}}_E(S^\#) = \varprojlim_{r \mapsto r^p} R^{\#,\circ\circ} \cong R^{\circ\circ}$$

since $R^\#$'s tilt is R, and the logarithm thus becomes a morphism $R^{\circ\circ} \to R^\#$ - should this go to $R^{\#\circ\circ}$? Does the operation of taking the generic fibre of a formal scheme over $\mathcal{O}_E$ really send $\widehat{\mathbf{G}}_{a,\mathcal{O}_E}$ to $\mathbf{G}_{a,E}$?.

Proposition 2.1. *If $S = \mathrm{Spa}(R, R^+) \in \mathrm{Perf}_{\mathbf{F}_q}$ is an affinoid perfectoid and $S^\# = \mathrm{Spa}(R^\#, R^{\#+})$ an untilt of S, then the map*

$$\begin{aligned} period_{S^\#} : \widetilde{\mathcal{G}}_E(S^\#) = R^{\circ\circ} &\longrightarrow H^0(X_S, \mathcal{O}(1)) = H^0(Y_S, \mathcal{O}_{Y_S})^{\varphi=\pi} \\ r &\longmapsto \sum_{n \in \mathbf{Z}} \pi^{-n}[r]^n \end{aligned}$$

is an isomorphism; in particular, we have a commutative diagram

$$\begin{tikzcd} \widetilde{\mathcal{G}}_E(S^\#) \arrow[r, "period_{S^\#}"] \arrow[d] & H^0(X_S, \mathcal{O}(1)) \arrow[dd, hook] \\ \mathcal{G}_E(S^\#) \arrow[d, "\log_{\mathcal{G}}"] & \\ \mathbf{G}_a(S^\#) = R^\# & H^0(Y_S, \mathcal{O}_{Y_S}) \arrow[l, "\theta_{S^\#}"'] \end{tikzcd}$$

where $\theta_{S^\#}$ is the Fontaine map for the untilt $S^\#$.

The proof follows from considering the Dieudonné module corresponding to the p-divisible group $\mathcal{G}$... to do: more details; in the equal characteristic case Y_S is a punctured unit disc and things are easy!!

Corollary 2.2. *The Banach-Colmez space $\mathbf{B}^{\varphi=\pi} \in \widetilde{\mathrm{Perf}_{\mathbf{F}_q}}$ is isomorphic to the perfectoid space $\mathrm{Spf}\,\mathbf{F}_q[\![X^{1/q^\infty}]\!]$ and the map sending a fibre-wise non-vanishing section $s \in (\mathbf{B}^{\varphi=\pi} \setminus \{0\})(S)$ to the*

...... finish later

3 The morphism $\Sigma_d : (\mathrm{Div}^1)^d \to \mathrm{Div}^d$

Just as in the classical case from algebraic geometry, one can consider the morphism

$$\Sigma_d : (\mathrm{Div}^1)^d \longrightarrow \mathrm{Div}^d$$

which sums d degree-one divisors to obtain a degree-d divisor. This supplies us with a description Div^d as a symmetric power of Div^1.

Proposition 3.1. *Σ_d is a surjective quasi-pro-étale morphism of diamonds inducing an isomorphism*

$$(\mathrm{Div}^1)^d / S_d \xrightarrow{\cong} \mathrm{Div}^d$$

where S_d is the symmetric group on d symbols.

Proof. To show Σ_d is quasi-pro-étale we must argue that for any strictly totally disconnected perfectoid space $S \in \mathrm{Perf}_{\mathbf{F}_q}$ with a morphism of pro-étale sheaves $S \to \mathrm{Div}^d$ (corresponding to a degree d divisor $[(u, \mathcal{O}(d))] \in \mathrm{Div}^d(S)$) the base-change $T := S \times_{\mathrm{Div}^d} (\mathrm{Div}^1)^d$ is a perfectoid space and the morphism $(\Sigma_d)_{|S} : T \to S$ quasi-compact and pro-étale. However, $(\Sigma_d)_{|S}$ can also be expressed as the base change along $(\mathrm{Div}^1_S)^d \to \mathrm{Div}^d_S$ of the morphism $S \xrightarrow{[(u,\mathcal{O}(d))]} \mathrm{Div}^d_S$ since the diagram

$$\begin{tikzcd} \mathrm{Div}^d_S \arrow[r] \arrow[d] & \mathrm{Div}^d \arrow[d] \\ S \arrow[r] & \mathrm{Spa}\,\mathbf{F}_q \end{tikzcd}$$

is cartesian. Now we can use that Div^d_S and $(\mathrm{Div}^1_S)^d$ are represented by spatial diamonds as mentioned in Remark 1.5 and thus T is a spatial diamond (since the category of diamonds contains its fibre products). Fargues' classification of quasi-pro-étale maps can now be applied:

Proposition 3.2 (13.6, [EtD]). *Let $f : Y \to X$ be a separated morphism of v-stacks. Then f is quasi-pro-étale if and only if it is representable is locally spatial diamonds and all geometric fibres $Y \times_X \mathrm{Spa}(C, \mathcal{O}_C)$ are pro-étale over $\mathrm{Spa}(C, \mathcal{O}_C)$.*

Thus we've reduced to considering the case where $S = \mathrm{Spa}(C, \mathcal{O}_C)$; here, any degree d divisor (i.e. an element of $\Gamma(Y_{(C,\mathcal{O}_C)}, \mathcal{O}_{Y_{(C,\mathcal{O}_C)}})^{\varphi=\pi^d}$) is a product of d degree-one divisors by the following classical result, which follows from a detailed analysis of the 'classical' p-adic Hodge Theory rings.

Theorem 3.3 (6.2.1, [FF]). *The graded E-algebra*

$$\bigoplus_{d=0}^{\infty} \Gamma(Y_{(C,\mathcal{O}_C)}, \mathcal{O}_{Y_{(C,\mathcal{O}_C)}})^{\varphi=\pi^d}$$

is a unique factorisation domain.

Via the same argument, we see that the canonical morphism of diamonds

$$(\mathrm{Div}^1)^d \times S_d \longrightarrow (\mathrm{Div}^1)^d \times_{\mathrm{Div}^d} (\mathrm{Div}^1)^d$$

is an isomorphism, and thus $(\mathrm{Div}^1)^d / S_d \cong \mathrm{Div}^d$.

■

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