Notes on Galois Representations Arising From Modular Forms

Thomas Manopulo

Contents

These are some notes from a portion of a course taught by Ana Caraiani in the WS22/23; I found this part of the theory quite challenging and very enlightening, so I thought it might be a nice exercise to gather what I managed to make of the lectures. An important disclaimer I should make is that in no way do I claim these notes are a faithful transcription of the topics discussed in the lecture - any mistake you might (or certainly) find is almost definitely of my making and due to my lack of experience in an attempt to write up maths rather beyond my wingspan.
The main goal is to discuss the key ideas behind the following result:

Theorem (Serre-Deligne, Eichler-Shimura)**.** *Let $f \in S_k(\Gamma_1(N), \chi, \mathbf{C})$ be a normalised eigenform of weight $k \geq 1$, with Fourier expansion*

$$f(z) = \sum_{n=1}^{\infty} a_n q^n;$$

if we set $K_f := \mathbf{Q}[a_1, a_2, \ldots]$, then $K_f/\mathbf{Q}$ is a number field (in particular, each a_n is algebraic) and for every prime $\lambda \mid \ell$ in K_f there exists a continuous irreducible Galois representation

$$\rho_{f,\lambda} : G_{\mathbf{Q}} \to \mathrm{GL}_2(K_{f,\lambda})$$

completely characterised by the following property: for all primes $p \nmid N \cdot \ell$ *the representation* $\rho_{f,\lambda}|_{G_{\mathbf{Q}_p}}$ *is unramified and the (well-defined up to conjugation) -absolute Frobenius element* $\mathrm{Frob}_p^{-1} \in G_{\mathbf{Q}_p}$ *has the characteristic polynomial*

$$\det(X - \rho_{f,\lambda}(\mathrm{Frob}_p^{-1})) = X^2 - a_p X + p^{k-1}\chi(p).$$

1 Representations of real groups

In this section we discuss representations of algebraic groups over infinite places, following chapters 3 and 4 from [GeHa]. For this section, the field k will be either $\mathbf{R}$ or $\mathbf{C}$ and G will be an affine algebraic k-groups with Lie algebra $\mathfrak{g}$.

1.1 Hilbert-Space representations

We start by defining the main object of study in the representation theory of the Lie group $G(k)$.

Definition 1.1. A Hilbert space representation of the Lie group $G(k)$ consists of a Hilbert space V together with a group homomorphism $\pi : G(k) \to \mathrm{GL}(V)$ such that the induced map $G \times V \to V$ is continuous, where $\mathrm{GL}(V)$ is the group of continuous linear maps $V \to V$ with a continuous inverse.

A typical source of representations of $G(k)$ arises when considering locally compact Hausdorff spaces X on which $G(k)$ acts; such spaces always admit (right/left-) Haar measures which give rise to the Hilbert space $L^2(X, dx)$ on which $G(k)$-acts. To single out the smooth functions in $L^2(X, dx)$ (such as modular forms, for instance) we need develop the notion of *smooth vectors*.

Definition 1.2. Let (π, V) be a Hilbert space representation of $G(k)$. We define the action of an element $X \in \mathfrak{g}$ in $G(k)$'s Lie algebra on a vector $v \in V$ via the formula

$$\pi(X) \cdot v := \frac{d}{dt}\pi(\exp(t \cdot X)v)_{|t=0}$$

if the limit exists. If $\pi(X) \cdot v$ is well defined for every element $X \in \mathfrak{g}$ then v is said to be a C^1-vector. Inductively, we say the vector $v \in V$ is C^j if $\pi(X) \cdot v$ is C^{j-1} for all $X \in \mathfrak{g}$, and *smooth* if v is C^j for all $j \geq 1$.

In the situation from just before the definition, we have that $L^2(G(k), \mu)_{\mathrm{sm}}$ where μ is a right-invariant Haar measure on $G(k)$ coincides with the family of square-integrable smooth functions $G(k) \to \mathbf{C}$.

If $X \in \mathfrak{g}, g \in G$ and $v \in V$ then

$$\pi(X) \cdot (\pi(g) \cdot v) = \pi(g)\pi(\mathrm{Ad}(g^{-1}X) \cdot v)$$

by the relation

$$g^{-1}\exp(X)g = \exp(\mathrm{Ad}(g^{-1})X);$$

this implies $\pi(X)$ is defined on $\pi(g) \cdot v$ whenever it is on $v \implies V_{\mathrm{sm}} \subseteq V$ is a subrepresentation.

It turns out that V_{sm} is infact a *Lie algebra representation* of $\mathfrak{g}$: this follows from a reduction to considering the case $V = C^\infty(G(k), \mathbf{C}) = V_{\mathrm{sm}}$ by means of the Riesz representation theorem.

From now on, we denote by $U_\infty \subseteq G(k)$ a maximal compact subgroup (which, by the existence of principal $G(k)$-homogeneous spaces - see the course notes on Bruhat-Tits theory) is unique up to conjugation. For instance, $\mathrm{GL}_n(\mathbf{R})$ admits $O_n \subseteq \mathrm{GL}_n(\mathbf{R})$ as a maximal compact subgroup, and for $\mathrm{GL}_n(\mathbf{C})$ we have U_n. By the classical representation theory of compact Lie groups, every irreducible representation of U_∞ is finite dimensional, and we denote by $\widehat{U}_\infty$ the collection of isomorphism classes of such irreducible representations.

Definition 1.3. If (π, V) is a Hilbert-space representation of $G(k)$, then for each irreducible (finite-dimensional) representation $\sigma \in \widehat{U}_\infty$ we denote by

$$V(\sigma) := \mathrm{im}(\mathrm{Hom}_{U_\infty}(\sigma, V) \otimes_{\mathbf{C}} \sigma \to V) \subseteq V$$

the σ-isotypic subspace and $V_{\mathrm{fin}} := \bigoplus_\sigma V(\sigma)$ the subspace of U_∞-finite vectors.
(π, V) is called admissible if each isotypical summand $V(\sigma)$ is finite dimensional.

We have a result giving a sufficient condition for admissibility.

Theorem 1.4 (Harish-Chandra). *If G/k is a reductive algebraic group and V is an irreducible unitary $G(k)$-Hilbert space representation then V is admissible.*

It turns out that one can develop much of the theory by sacrificing the action of $G(k)$, and just restricting to an analysis of how $\mathfrak{g}$ and U_∞ behave on $G(k)$'s representations: in studying automorphic representations it'll be necessary to make this generalisation:

Definition 1.5. A $(\mathfrak{g}, U_\infty)$-module consists of a vector space $V/\mathbf{C}$ together with:

1. a Lie algebra homomorphism $\pi : \mathfrak{g} \to \mathrm{End}(V)$,
2. a group homomorphism $\pi : U_\infty \to \mathrm{GL}(V)$ making V into a locally finite and continuous U_∞-representation.
3. the following compatibility relations between the above two homomorphisms:

$$\pi(g)\pi(X)\pi(g^{-1}) = \pi(\mathrm{Ad}(g)(X)),$$
$$\pi(X) \cdot v = \frac{d}{dt}\pi(\exp(tX)v)_{|t=0},$$

for all $v \in V, g \in U_\infty, X \in \mathfrak{g}$.

The following theorem already motivates the above step towards generalisation:

Theorem 1.6 (Harish-Chandra). *An admissible representation (π, V) of $G(k)$ is irreducible if and only if the underlying $(\mathfrak{g}, U_\infty)$-module V_{fin} is irreducible.*

Whenever (π, V) is an irreducible $(U_\infty, \mathfrak{g})$-module, by Schur's lemma we get a well-defined central character

$$\chi_\pi : Z(G(k)) \to \mathbf{C}$$

which extends the character $\chi_{\pi|Z(G(k))\cap U_\infty} = \pi_{|Z(G(k))\cap U_\infty}$ and $d\chi_\pi = \pi_{|\mathrm{Lie}\, Z(G(k))}$.

1.2 Classification of irreducible admissible $(\mathfrak{gl}_2, O(2))$-modules

We quickly recall the Harish-Chandra description of the centre of the universal enveloping algebra of a complex reductive Lie algebra $\mathfrak{g}$; to fit the following result into the previously described picture, one is forced to replace $\mathfrak{g}$ with its complexification - this is necessary so we can access an explicit description of the *infinitesimal characters* $d\chi_\pi$.

Theorem 1.7 (Harish-Chandra isomorphism). *Suppose $\mathfrak{g}$ is a reductive Lie algebra over $\mathbf{C}$, $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra (which may be taken to be the complexification of the Lie algebra of the Lie algebra of any of G's tori), and denote by $\mathfrak{z} = Z(U(\mathfrak{g}))$ the centre of $\mathfrak{g}$'s universal enveloping algebra. There exists an isomorphism*

$$\gamma_{HC} = \gamma : \mathfrak{z} \to \mathrm{Sym}(\mathfrak{t}_{\mathbf{C}})^{W_\mathfrak{g}}$$

where $W_\mathfrak{g}$ is $\mathfrak{g}$'s Weyl group, acting on $\mathfrak{h}$.

Example 1.8. Set $G = \mathrm{GL}_2$ and $B = T \rtimes N$ the standard Borel of upper-triangular matrices, together with its standard Levi decomposition. The Cartan corresponding to T in $\mathfrak{gl}_2$ is generated by

$$z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and the action of the Weyl group $W_{\mathfrak{g}} \cong \mathbf{Z}/(2)$ generated by w_0 is given by $w_0(h) = -h$ and $w_0(z) = z$. Thus

$$\mathrm{Sym}(\mathfrak{t}_{\mathbf{C}})^{W_{\mathfrak{g}}} \cong \mathbf{C}[z, h^2].$$

On the other hand, the centre of the universal enveloping algebra $U(\mathfrak{gl}_2)$ is generated by the elements z and the Casimir element

$$\underline{\omega} = (h-1)^2 + 4ef \in \mathfrak{z}.$$

If V is an irreducible admissible $(\mathfrak{g}, U_\infty)$-module, then $\underline{\omega}$ and z will act by scalars $\omega, \mu \in \mathbf{C}$ respectively. We take $U_\infty = O(2)$ as our maximal compact subgroup and denote by $U_\infty^0 = SO(2) \cong U(1) = \{e^{it}\}_{t \in \mathbf{R}}$. The whole subgroup U_∞^0 will act simultaneously diagonalisably on V and we thus get a decomposition

$$V = \bigoplus_{n \in \mathbf{Z}} V_n$$

where e^{it} acts on V_n by the scalar $e^{int} \in \mathbf{C}$. From the above one can completely classify irreducible admissible $(\mathfrak{g}, U_\infty^0)$-modules as being those appearing in the following list of four:

1. the finite dimensional representations, of the form

   $$V_\lambda = \det^{\lambda_1} \otimes \mathrm{Sym}^{\lambda_1 - \lambda_2}(\mathrm{Std})$$

   where $\lambda = (\lambda_1, \lambda_2)$ is a dominant integral weight.

2. the 'principal series representations' defined by the condition $\underline{\omega} \neq m^2$ for any integer $m \in \mathbf{Z}$. We have a distinction to make here:

   - $V = \bigoplus_{n \in 2\mathbf{Z}} \mathbf{C} \cdot v_n$,
   - $V = \bigoplus_{n \in 2\mathbf{Z}+1} \mathbf{C} \cdot v_n$.

3. the 'discrete series representations', which are infinite dimensional and $\underline{\omega} = m^2, m \in \mathbf{Z}$ is a square. I struggled a few minutes to figure out how to index my direct sum to express the weight space decomposition and failed to find a nice way to write it down... I think this is a good occasion for you to try figure it out for yourself >:)

2 Cuspidal automorphic representations

We now focus on the representation theory arising from an attempt to 'patch' into one picture the finite and the infinite, by studying the adelic points of the reductive group G we started with.

2.1 Modular forms

The following basis for $\mathfrak{gl}_2 = \mathfrak{gl}_{2,\mathbf{C}}$ will turn out to be a lot more suitable for our computations:

$$Z = \mathrm{id}_{2\times 2}, \quad H = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad E = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \quad F = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}.$$

The triple (E, F, H) satisfies the standard $\mathfrak{sl}_2$-relations, and the centre of $\mathfrak{gl}_2$'s centre is given by

$$Z(U(\mathfrak{g})) = \mathfrak{z} = \mathbf{C}[Z, H^2 - 2H + 4EF].$$

Denote by $\mathcal{B} = \mathrm{Isom}_{\mathbf{R}}(\mathbf{R}^2, \mathbf{C})$ the set of bases of $\mathbf{C}$ as a real vector space. We have a transitive free right action of $\mathrm{GL}_2(\mathbf{R})$ on $\mathcal{B}$ by pre-composition, and thus $\mathcal{B}$ inherits $\mathrm{GL}_2(\mathbf{R})$'s smooth manifold structure. Similarly, $\mathrm{GL}_{\mathbf{R}}(\mathbf{C})$ admits a free transitive left action on $\mathcal{B}$ by post-composition. A more concrete way of visualising $\mathcal{B}$ is via the isomorphism

$$\begin{aligned} \mathcal{B} &\xrightarrow{\cong} (\mathbf{C} \setminus \mathbf{R}) \times \mathbf{C}^\times \\ (z_1, z_2) &\mapsto (\frac{z_1}{z_2}, z_2). \end{aligned}$$

which turns the right $\mathrm{GL}_2(\mathbf{R})$-action mentioned above into

$$(\tau, z) \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left(\frac{a\tau + c}{b\tau + d}, (b\tau + d)z \right).$$

Furthermore, thanks to this description, $\mathcal{B}$ is endowed with the structure of a complex analytic manifold, isomorphic to an open subset of $\mathbf{C}^2$. The left restricted action of $\mathbf{C}^\times \subset \mathrm{GL}_{\mathbf{R}}(\mathbf{C})$ turns into

$$\alpha \cdot (\tau, z) = (\tau, \alpha z).$$

If we fix a basis $\iota \in \mathcal{B}$, the isomorphism of complex analytic varieties $\mathrm{GL}_2(\mathbf{R}) \xrightarrow{\cong} \mathcal{B}$ given by mapping g to $\iota \cdot g$ gives rise to an isomorphism of complexified tangent spaces

$$\mathfrak{gl}_{2,\mathbf{C}} \cong \mathfrak{gl}_{2,\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C} = T_{\mathrm{id}_{2\times 2}} \mathrm{GL}_2(\mathbf{R}) \otimes_{\mathbf{R}} \mathbf{C} \xrightarrow{\cong} T_\iota \mathcal{B}$$

and since $\mathcal{B}$ is complex analytic, the tangent space $T_\iota\mathcal{B}$ decomposes into holomorphic and anti-homolorphic summands

$$T_\iota \mathcal{B} = T_\iota^{\mathrm{hol}} \mathcal{B} \oplus \overline{T_\iota^{\mathrm{hol}} \mathcal{B}}.$$

If we denote by $V_\iota \subset \mathfrak{gl}_{2,\mathbf{C}}$ the preimage along the above isomorphism of the holomorphic germs $T_\iota^{\mathrm{hol}}\mathcal{B}$ then V_ι can be computed explicitly:

Exercise 2.1. Let $\iota \in \mathcal{B}$ be the basis $(i, 1)$. Then V_ι is generated as a $\mathbf{C}$-vector space by the elements $Z + H, E \in \mathfrak{gl}_{2,\mathbf{C}}$, and $\overline{V_\iota}$ is generated by $Z - H, F$.

We denote by $\mathcal{L}$ the set of rank 2 lattices contained in $\mathbf{C}$, and we have a natural map

$$\begin{aligned} \mathcal{B} = \mathrm{Isom}_{\mathbf{R}}(\mathbf{R}^2, \mathbf{C}) &\longrightarrow \mathcal{L} \\ \iota &\longmapsto \iota(\mathbf{Z}^2) \end{aligned}$$

which induces a bijection $\mathcal{B}/\mathrm{GL}_2(\mathbf{Z}) \cong \mathcal{L}$. Since $\mathrm{GL}_2(\mathbf{Z})$ acts properly discontinuously on $\mathcal{B}$ we get that $\mathcal{L}$ induces a complex analytic manifold structure this way.

For an integer $N \in \mathbf{Z}_{\geq 1}$ we denote by $\mathcal{L}(N)$ the set of rank two lattices in $\mathbf{C}$ equipped with a level N structure:

$$\mathcal{L}(N) = \left\{ (L, j) \mid L \in \mathcal{L}, j : (\mathbf{Z}/N\mathbf{Z})^2 \xrightarrow{\cong} L/NL \right\}$$

which evidently admits a surjective map $\mathcal{L}(N) \to \mathcal{L}$ whose fibres are isomorphic to $\mathrm{GL}_n(\mathbf{Z}/N\mathbf{Z})$.

We have a bijection

$$[\mathcal{B} \times \mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z})]/\mathrm{GL}_2(\mathbf{Z}) \xrightarrow{\cong} \mathcal{L}(N)$$

which sends the pair $(\iota : \mathbf{R}^2 \xrightarrow{\cong} \mathbf{C}, g : \mathbf{Z}/N\mathbf{Z} \xrightarrow{\cong} \mathbf{Z}/N\mathbf{Z})$ to $(\iota(\mathbf{Z}^2), \iota_N \circ g^{-1})$ where ι_N is the reduction of ι modulo N, which induces on $\mathcal{L}(N)$ the structure of a complex analytic manifold, rendering the map above $\mathcal{L}(N) \to \mathcal{L}$ a covering map with monodromy $\mathrm{GL}_n(\mathbf{Z}/N\mathbf{Z})$.

Lastly, note that there's also an action of $\mathrm{GL}_{\mathbf{R}}(\mathbf{C})$ on $\mathcal{L}(N)$ given by

$$g \cdot (L, j) := (g(L), g_N \circ j).$$

Particularly crucial to our definition of modular forms is the action of $\mathbf{C}^\times \subseteq \mathrm{GL}_{\mathbf{R}}(\mathbf{C})$.

Definition 2.2. A modular form of weight $k \in \mathbf{Z}$ and level N is a holomorphic function $F : \mathcal{L}(N) \to \mathbf{C}$ such that:

1. F is holomorphic,
2. $F(\alpha(L, j)) = \alpha^{-k} F((L, j))$ for all scalars $\alpha \in \mathbf{C} \subset \mathrm{GL}_{\mathbf{R}}(\mathbf{C})$,
3. F satisfies a growth condition (cfr. Remark 2.1).

Remark 2.3. By unraveling the above bijections describing explicitly what the complex structures at play are, a modular form F as in Definition 2.2 identifies with a holomorphic function

$$f : (\mathbf{C} \setminus \mathbf{R}) \times \mathrm{GL}_n(\mathbf{Z}/N\mathbf{Z}) \to \mathbf{C}$$

satisfying an analogous growth condition, such that the equation

$$f\left(\frac{a\tau + c}{b\tau + d}, g\gamma\right) = (b\tau + d)^{-k} f(\tau, g)$$

holds for all $(\tau, g) \in (\mathbf{C} \setminus \mathbf{R}) \times \mathrm{GL}_n(\mathbf{Z}/N\mathbf{Z})$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbf{Z})$.

Remark 2.4. We'll state the growth condition properly just below, but for now one can take it to be a requirement needed to bound the values of F near the *cusps*, so that one can think of F as a global section of some line bundle on a compactification of $\mathcal{L}(N)$.

For a modular form f as in Remark 2.3

$$f : (\mathbf{C} \setminus \mathbf{R}) \times \mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z}) \to \mathbf{C}$$

we have $f(\tau + N, g) = f(\tau, g)$ by setting $\gamma = \begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix}$ thus $f(-, 1)$ admits a Fourier expansion

$$f(\tau, 1) = \sum_{n=-\infty}^{\infty} a_n(f) e^{2\pi i n \tau / N}.$$

The growth condition mentioned in Definition 2.2 now becomes

$$a_n(g \cdot f) = 0, \text{ for all } n < 0, g \in \mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z})$$

which is equivalent to requiring $|f(\tau, g)| \leq C|\operatorname{im}(\tau)|^r$ for some positive constants $C, r \in \mathbf{R}_{>0}$.

2.2 The $\mathrm{GL}_2(\mathbf{A}_f)$ action on modular forms

We denote by $\mathcal{M}_k(N)$ the complex vector space of modular forms of weight k and level N, and by $\mathcal{M}_k$ the colimit

$$\mathcal{M}_k = \varinjlim_N \mathcal{M}_k(N)$$

where the colimit is taken along the maps $\mathcal{M}_k(N) \to \mathcal{M}_k(N')$ by means of the pullback along the canonical morphism $\mathcal{L}(N') \to \mathcal{L}(N)$ whenever $N \mid N'$.

The right action of $\mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z})$ on $\mathcal{L}(N)$ turns into a left action on $\mathcal{M}_k(N)$, and by means of the projections $\mathrm{GL}_2(\widehat{\mathbf{Z}}) := \varprojlim_M \mathrm{GL}_2(\mathbf{Z}/(M)) \to \mathrm{GL}_n(\mathbf{Z}/(N))$ for each N we get an action of $\mathrm{GL}_2(\widehat{\mathbf{Z}})$. By use of Hecke correspondences on $(\mathcal{M}_k(N))_N$, which we'll discuss later on, one can extend the action to $\mathrm{GL}_2(\mathbf{A}_f) \supseteq \mathrm{GL}_2(\widehat{\mathbf{Z}})$, $\mathbf{A}_f = \mathbf{Q} \otimes_{\mathbf{Z}} \widehat{\mathbf{Z}}$.

Define the subgroup

$$\widetilde{\Gamma}(N) := \left\{ g \in \mathrm{GL}_2(\widehat{\mathbf{Z}}) \mid g \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N \right\}.$$

By the equalities

$$\begin{cases} \mathrm{GL}_2(\mathbf{Q}) \cdot \mathrm{GL}_2(\widehat{\mathbf{Z}}) = \mathrm{GL}_2(\mathbf{A}_f) \\ \mathrm{GL}_2(\mathbf{Q}) \cap \mathrm{GL}_2(\widehat{\mathbf{Z}}) = \mathrm{GL}_2(\mathbf{Z}) \end{cases}$$

we get diffeomorphisms

$$\begin{array}{ccc} \mathrm{GL}_2(\mathbf{Q})\backslash \mathrm{GL}_2(\mathbf{A}_{\mathbf{Q}}) / \widetilde{\Gamma}(N) & \xrightarrow{\cong} & \mathrm{GL}_2(\mathbf{Z})\backslash \mathrm{GL}_2(\mathbf{R}) \times \mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z}) \\ \downarrow \cong & & \downarrow \cong \\ \mathcal{L}(N) & \xleftarrow{\cong} & \mathcal{B} \times \mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z}) / \mathrm{GL}_2(\mathbf{Z}) \end{array}$$

After introducing automorphic forms, we'll see that the above description of $\mathcal{L}(N)$ will help us identify modular forms $\mathcal{M}_k(N)$ of weight k and level N with automorphic forms for $\mathrm{GL}_2(\mathbf{A}_{\mathbf{Q}})$.

2.3 Smooth representations of $G(\mathbf{A}_{F,f})$

Let F be a number field and G/F a split reductive group with an integral model $\mathbf{G}/\mathcal{O}_F$ such that the base change to $\mathcal{O}_F[N^{-1}]$ is reductive for some integer $N \in \mathbf{Z}_{\geq 1}$. The $\mathbf{A}_{F,f}$-valued points of G acquire a topology, given by the subspace topology when G is included into $\mathbf{G}^n_{a,F}$ as a closed subscheme for some n; for instance, $\mathbf{G}_{m,F}(\mathbf{A}_{F,f}) = \mathbf{A}^\times_{F,f}$ has the induced topology when included into $\mathbf{G}^2_{a,F}(\mathbf{A}_{F,f}) = \mathbf{A}^2_{F,f}$ via the isomorphism $\mathbf{G}_{m,F} \cong \operatorname{Spec} F[t, t^{-1}] \cong \operatorname{Spec} F[x,y]/xy - 1 \hookrightarrow \operatorname{Spec} F[x,y] \cong \mathbf{G}^2_{a,F}$.

If $U \subset G(\mathbf{A}_{F,f})$ is a compact open subgroup, then there exists a finite set S of finite places of F containing all places above N such that $U = U^S \times U_S$ where we use the notation

$$U^S := \prod_{v \notin S} \mathcal{G}(\mathcal{O}_{F_v}),$$
$$U_S := \prod_{v \in S} G(F_v).$$

Recall that the maximal compact subgroup $\mathcal{G}(\mathcal{O}_{F_v}) \subseteq G(F_v)$ is called, in light of Bruhat-Tits theory, the *hyperspecial* maximal compact subgroup - when considering $G_{F_v} = G \times_F \operatorname{Spec} F_v$'s Bruhat-Tits building, $\mathcal{G}(\mathcal{O}_{F_v})$ is the stabiliser of a hyperspecial point, i.e. one whose local building contains hyperplanes parallel to all other hyperplanes in the corresponding affine building.

Definition 2.5. A *smooth* representation of $G(\mathbf{A}_F)$ consists of a complex vector space $V/\mathbf{C}$ together with

1. a group homomorphism $\pi : G(\mathbf{A}_{F,f}) \times U_\infty \to GL(V)$ where $U_\infty \subseteq G(F_\infty)$ is a fixed maximal compact subgroup, and
2. a Lie algebra homomorphism $\pi : \mathfrak{g} = \operatorname{Lie} G(F_\infty) \otimes_{\mathbf{R}} \mathbf{C} \to \operatorname{End}(V)$ from $G(F_\infty)$'s complexified Lie algebra,

satisfying the following conditions:

- for all vectors $v \in V$ the stabiliser $\mathrm{Stab}_{G(\mathbf{A}_{F,f})}(v) \subseteq G(\mathbf{A}_{F,f})$ is an open subgroup (i.e. $\pi_{|G(\mathbf{A}_{F,f})}$ is a smooth representation of the locally profinite group $G(\mathbf{A}_{F,f})$),
- $\pi_{|U_\infty}$ is locally finite and continuous,
- for all $X \in \mathrm{Lie}\, U_\infty$ we have the equalities

$$\pi(X)v = \frac{d}{dt}\pi(\exp(tX))_{|t=0},$$
$$\pi(g)\pi(X)\pi(g^{-1}) \cdot v = \pi(\mathrm{Ad}(g_\infty)X) \cdot v$$

  for all $g = (g_f, g_\infty) \in G(\mathbf{A}_{F,f}) \times U_\infty, X \in \mathfrak{g}$ and $v \in V$.

The representation (π, V) is admissible is for all open subgroups $U \subseteq G(\mathbf{A}_{F,f})$ and irreducible representations $\rho \in \widehat{U}_\infty$ the U-fixed vectors of the isotypical summand

$$V^U(\rho) \subseteq V$$

is finite dimensional.

2.4 Automorphic forms

We can now give the somewhat abstract definition of automorphic forms, which - although long and possibly quite confusing - should be compared to the very concrete classical interpretation of modular forms as holomorphic functions defined on the complex analytic variety of rank-2 lattices $L \subset \mathbf{C}$ together with a trivialisation $L/NL \cong \mathbf{Z}/(N) \oplus \mathbf{Z}/(N)$ (see Section **??** on modular forms).

Definition 2.6. The space of automorphic forms for G/F, denoted by

$$\mathcal{A} = \mathcal{A}(G(F)\diagdown G(\mathbf{A}_F)),$$

is the complex vector space of functions $f : G(F)\diagdown G(\mathbf{A}_F) \to \mathbf{C}$ satisfying

1. $f(gu) = f(g)$ for all $g \in G(\mathbf{A}_F)$ and $u \in U$ where $U \subseteq G(\mathbf{A}_{F,f})$ is some open subgroup.
2. for all $g_f \in G(\mathbf{A}_{F,f})$ the map

$$\begin{aligned} G(F_\infty) &\to \mathbf{C} \\ g_\infty &\mapsto f(g_f g_\infty) \end{aligned}$$

   is a smooth function of smooth manifolds.
3. f is U_∞ and $\mathfrak{z}$-finite, where $\mathfrak{z} \subset U(\mathfrak{g})$ is the centre of $\mathfrak{g}$'s universal enveloping algebra.
4. f has 'moderate growth'.

Remark 2.7. We make a few comments on the above definition.

- Elements in the Lie algebra $X \in \mathfrak{g}$ act on automorphic forms $f \in \mathcal{A}$ via the formula

$$(Xf)(g) := \frac{d}{dt} f(g \exp(tX))_{|t=0}.$$

- If $U \subseteq G(\mathbf{A}_{F,f})$ is an open subgroup, then the double coset space

$$G(F)\diagdown G(\mathbf{A}_{F,f}) \diagup U$$

  is actually finite (this is a generalisation of the *finiteness of the class number*, see [Con]). This means that when f is restricted to the finite adeles, its image is actually finite!

- The last growth condition depends on the following norm on $G(\mathbf{A}_F)$: if we fix a closed immersion $\iota' : G \hookrightarrow \mathrm{GL}_{n,K}$ and denote by $\iota : G(\mathbf{A}_F) \to \mathrm{GL}_{2n,K}(\mathbf{A}_F)$ the map defined by
$$\iota(g) := \begin{pmatrix} \iota'(g) & 0 \\ 0 & \iota'(g^{-1})^t \end{pmatrix}$$
we set
$$||g|| := \prod_v \max ||\iota(g)_{i,j}||_v.$$
The moderate growth condition in Definition 2.6 can be stated as the requirement
$$|f(g)| \leq c \cdot ||g||^r$$
for some constants $c, r > 0$, for all $g \in G(\mathbf{A}_F)$.

The space of automorphic forms is a smooth representation of $G(\mathbf{A}_F)$ essentially by construction. $\mathcal{A}$ is far to large to work with conretely; in particular, it's far from being admissible; an *automorphic representation* is one which arises from $\mathcal{A}$:

Definition 2.8. An automorphic representation of $G(\mathbf{A}_F)$ is an irreducible admissible $G(\mathbf{A}_{F,f}) \times (\mathfrak{g}, U_\infty)$-module isomorphic to a subquotient of $\mathcal{A}$.

Definition 2.9. The space of cusp forms $\mathcal{A}_0 = \mathcal{A}_0(G(F)\diagdown G(\mathbf{A}_F))$ is the subspace of $\mathcal{A}$ consisting of automorphic forms $f : G(F)\diagdown G(\mathbf{A}_F) \to \mathbf{C}$ such that for all proper parabolic subgroups $P = L \rtimes N \subset G$ we have
$$\int_{N(F)\diagdown N(\mathbf{A}_F)} f(ng)\, dn = 0.$$

For any continuous character $\chi : Z(G)(\mathbf{A}_F)\diagup Z(G)(F) \to \mathbf{C}^\times$ we denote by
$$\mathcal{A}_0(G(F)\diagdown G(\mathbf{A}_F), \chi) \subset \mathcal{A}_0(G(F)\diagdown G(\mathbf{A}_F))$$
the subspace of cusp forms on which $Z(G)(\mathbf{A}_F)$ acts by the central character χ. It turns out that $\mathcal{A}_0(G(F)\diagdown G(\mathbf{A}_F), \chi)$ is a semisimple $(G(\mathbf{A}_{F,f}), (\mathfrak{g}, U_\infty))$-module; this ends up being one of the ways to construct automorphic representations:

Definition 2.10. A cuspidal automorphic representation of G is an automorphic representation isomorphic to an irreducible constituent of $\mathcal{A}_0(\chi) = \mathcal{A}_0(G(F)\diagdown G(\mathbf{A}_F), \chi)$ for some χ.

2.5 $\mathcal{M}_k(N)$ as a space of automorphic forms

In subsection 2.2 we concluded with the commutative diagram of diffeomorphisms

$$\begin{tikzcd}
\mathrm{GL}_2(\mathbf{Q})\diagdown \mathrm{GL}_2(\mathbf{A}_\mathbf{Q})\diagup \widetilde{\Gamma}(N) \arrow[r, "\cong", "p"'] \arrow[d, "\cong"] & \mathrm{GL}_2(\mathbf{Z})\diagdown \mathrm{GL}_2(\mathbf{R}) \times \mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z}) \arrow[d, "q"', "\cong"] \\
\mathcal{L}(N) & \mathcal{B} \times \mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z})\diagup \mathrm{GL}_2(\mathbf{Z}) \arrow[l, "\cong"']
\end{tikzcd}$$

where p is induced by the map

$$\begin{aligned}
\mathrm{GL}_2(\mathbf{A}_\mathbf{Q}) = \mathrm{GL}_2(\mathbf{R}) \times (\mathrm{GL}_2(\mathbf{Q}) \cdot \mathrm{GL}_2(\widehat{\mathbf{Z}})) &\longrightarrow \mathrm{GL}_2(\mathbf{Z})\diagdown \mathrm{GL}_2(\mathbf{R}) \times \mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z}) \\
(g_\infty,\ g_\mathbf{Q} \cdot g_{\widehat{\mathbf{Z}}}) &\longmapsto \mathrm{GL}_2(\mathbf{Z}) \cdot (g_\infty, \overline{g_{\widehat{\mathbf{Z}}}})
\end{aligned}$$

and q by simply identifying $\mathcal{B}$ with $\mathrm{GL}_2(\mathbf{R})$ through the basepoint $\iota_0 = (i, 1)$ as discussed in section **??**.

Proposition 2.11. *The pullback along the identification*

$$\mathrm{GL}_2(\mathbf{Q})\diagdown^{\mathrm{GL}_2(\mathbf{A_Q})}\diagup\widetilde{\Gamma}(N) \xrightarrow{\cong} \mathcal{L}(N)$$

of a modular form $f : \mathcal{L}(N) \to \mathbf{C}$ of weight k and level N is an automorphic form ϕ of level N; this pullback an isomorphism

$$\mathcal{M}_k(N) \cong \{\phi \in \mathcal{A}(\mathrm{GL}_2(\mathbf{A_Q})) \mid Z\phi = H\phi = k\phi,\ F\phi = 0,\ \phi \textit{ has level } \widetilde{\Gamma}(N)\} \subseteq \mathcal{A}(\mathrm{GL}_2(\mathbf{Q})\diagdown \mathrm{GL}_2(\mathbf{A_Q})).$$

Proof. Since $f : \mathcal{L}(N) \to \mathbf{C}$ is holomorphic, we get that the corresponding map

$$\phi : \mathrm{GL}_2(\mathbf{Q})\diagdown \mathrm{GL}_2(\mathbf{A_Q}) \to \mathbf{C}$$

must be annihilated by the antiholomorphic bundle of the complexified Lie algebra of $\mathrm{GL}_2(\mathbf{R}) \implies (Z - H)\phi = 0$ and $F\phi = 0$.
Differentiating the equality

$$f(\alpha(L, j)) = \alpha^{-k} f(L, j),\ \ \alpha \in \mathbf{C}^\times$$

we see that $Z\phi = H\phi = k$, since the (complexified) Lie algebra of the subgroup $\mathbf{C}^\times \subseteq \mathrm{GL}_2(\mathbf{R})$ is generated by Z and H. Lastly, the growth condition is verified by the last point in Remark 2.7 and the equivalence described in 2.1.

The last two conditions missing are that ϕ is both $\mathfrak{z} = \mathbf{C}[Z, (H-1)^2 + 4EF]$ and $U^0_\infty = SO(2)$-finite. For the first bit, note that ϕ is annihilated by the elements

$$Z - k, H^2 - 2H + 4EF - k^2 + 2k \in \mathfrak{z}$$

and the ideal these generate has finite codimension (because $\mathfrak{z}$ is a two-dimensional polynomial ring). Lastly, we have that the equality

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \phi = (a + bi)^k \phi,\ \ a + bi \in \mathbf{C}^\times$$

shows that in fact the $SO(2)$ subrepresentation generated by ϕ is infact one-dimensional. ■

2.6 The Hecke operators

Consider a newform $f \in S_k(\Gamma_1(N), \mathbf{C})$, where $k \geq 1$ and

$$\Gamma_1(N) = \left\{ \gamma \in \mathrm{GL}_2(\mathbf{Z}) \mid \gamma \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \mod N \right\}.$$

We also assume $a_1 = 1$ (for some reason? :/).

The relative Hecke algebras $\mathcal{H}(\mathrm{GL}_2(\mathbf{Q}_p), \Gamma_1(N))$ act on the space of automorphic forms $\mathcal{A}(\mathrm{GL}_2(\mathbf{Q})\diagdown \mathrm{GL}_2(\mathbf{A_Q}))$ of level $\Gamma_1(N)$ via the map $\mathbf{Q}_p \to \mathbf{A_Q}$, and in particular we get an action of the '*Hecke Operators*'

$$T_p \in \mathcal{H}(\mathrm{GL}_2(\mathbf{Q}_p), \Gamma_1(N))$$

which are the locally constant functions associated to the double coset $\Gamma_1(N) \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \Gamma_1(N)$.

We say that f is an *eigenform* if it's an eigenvector for T_p for all $p \nmid N$. If we express f in terms of its Fourier series

$$f(z) = q + \sum_{n=2}^{\infty} a_n q^n$$

then the Hecke operator action is explicitly given by

$$T_p(f)(z) = q + \sum_{n=2}^{\infty} a_{np} q^n + p^{k-1} \sum_{n=1}^{\infty} b_n q^{pn} = a_p f(z)$$

hence the Fourier coefficient a_p can be read as the eigenvalue of T_p. By this description, a hands-on computation shows that for primes p, q the operators T_p and T_q commute.

Remark 2.12. Let $\chi : (\mathbf{Z}/N\mathbf{Z})^\times \to \mathbf{C}^\times$ be a Dirichlet character. The modular form f is said to have *nebentypus* χ if the equation

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^k f(z)$$

holds for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$.
We have a vector space decomposition

$$S_k(\Gamma_1(N), \mathbf{C}) = \bigoplus_{\chi} S_k(\Gamma_1(N), \chi, \mathbf{C}).$$

2.7 Eigenforms and newforms

In an attempt to diagonalise spaces of modular forms with respect to the Hecke operators, we define introduce an inner product on $S_k(\Gamma_1(N), \mathbf{C})$.

Definition 2.13 (The Petersson Inner Product)**.** For $k, N \geq 1$ we define

$$\begin{aligned} (-,-) : S_k(\Gamma_1(N), \mathbf{C}) \times S_k(\Gamma_1(N), \mathbf{C}) &\longrightarrow \mathbf{C} \\ f, g \longmapsto (f,g) := \int_{D(\Gamma_1(N))} & f(x+iy) \cdot \overline{g(x+iy)} \cdot y^{k-2}\, dxdy \end{aligned}$$

where $D(\Gamma_1(N)) \subseteq \mathbf{C} \setminus \mathbf{R}$ is a fundamental domain for the action of $\Gamma_1(N)$ on $\mathbf{C} \setminus \mathbf{R}$.

Theorem 2.14. *The Petersson inner product is a non-degenerate hermitian form on the space of cusp forms $S_k(\Gamma_1(N), \mathbf{C})$, and T_p is an adjoint operator on $S_k(\Gamma_1(N), \mathbf{C})$ for all primes p not dividing N. In particular, $S_k(\Gamma_1(N), \mathbf{C})$ admits an orthogonal basis of simultaneous eigenvectors for the family of commuting self-adjoint operators $\{T_p\}_{p \nmid N}$.*

Interestingly enough, there are some cusp forms which *'come from'* lower levels, and there's a natural way of relating these.

Lemma 2.15. *Let $f \in S_k(\Gamma_1(N), \mathbf{C})$ be a cusp form of weight N and level k. Then for any integer $d \geq 1$ we have that the function $f(d \cdot z)$ is a modular form of weight $d \cdot N$ and for all primes p not dividing $d \cdot N$ we have*

$$(T_p g)(z) = (T_p f)(d \cdot z).$$

Remark 2.16. The operators

$$S_p = \langle p \rangle : S_k(\Gamma_1(N), \mathbf{C}) \to S_k(\Gamma_1(pN), \mathbf{C})$$

for $p \nmid N$ are called the *diamond operators*, and the above lemma shows that they commute with the Hecke operators T_q for $q \nmid pN$.

Definition 2.17. For any $N, k \geq 1$ we define the family of *old forms* as

$$S_k(\Gamma_1(N), \mathbf{C})^{\text{old}} = \text{Span}_{\mathbf{C}}(f(d \cdot z) \mid f \in S_k(\Gamma_1(M), \mathbf{C}), dM = N)$$

and the family of *new forms*

$$S_k(\Gamma_1(N), \mathbf{C})^{\text{new}} := (S_k(\Gamma_1(N), \mathbf{C})^{\text{old}})^{\perp}$$

as the othogonal complement of the space of old forms with respect to Petersson inner product.

Note that, since the diamond operators commute with Hecke operators, we see that the above is a $\mathbf{T}$-equivariant decomposition, and thus eigenforms split up into old and new ones.

We mention a nice characterisation of newforms, which follows from the harmonic analysis of modular forms.

Theorem 2.18. *Suppose $f \in S_k(\Gamma_1(N), \mathbf{C})$ is such that its n-th Fourier coefficient $a_{n,f}$ vanishes whenever n is coprime to N. Then f is an old form.*

3 Modular Curves and Eichler-Shimura

We recall the basic constructs for arithmetic moduli; the proper reference is [KatMaz]. My favourite reference is the handwritten notes from Mihatsch's course.

3.1 Elliptic curves and their moduli

Definition 3.1. Let K be a field; an *elliptic curve* E/K is a connected smooth projective curve of genus 1 together with a fixed K-rational point $0 \in E(K)$.

Remark 3.2. The main results on elliptic curves we use are the following, mostly discussed in the seminar on Jacobians from last term.

- E is isomorphic to its Jacobian, and inherits from it the structure of a commutative group scheme over K, such that $0 : \operatorname{Spec} K \to E(K)$ is the identity section.
- if $\mathrm{char}(K) \neq 2, 3$, E is isomorphic to a projective curve of the form

  $$V(Y^2Z - X^3 - aXZ^2 - bZ^3) \subseteq \mathbf{P}^2_K$$

  for some set of elements $a, b \in K$ such that

  $$\Delta(a, b) := -16(4a^3 + 27b^2) \neq 0.$$

Remark 3.3. More generally, for an arbitrary scheme S one defines an elliptic curve (or a *family of elliptic curves*) over S to be a proper smooth morphism

$$E \to S$$

with geometrically connected fibres of genus one, together with an S-valued point $(0 : S \to E) \in E(S)$. E once again inherits the structure of an abelian variety via its relative Jacobian. We furthermore have the following property: the subscheme $E[N] \subseteq E$ is a finite flat group scheme over S; it is furthermore étale if and only if N is invertible in $\Gamma(S, \mathcal{O}_S)$.

Remark 3.4. The moduli of such spaces (endowed with some mild additional structure in order to guarantee representability) and their integral models will turn out to be our main instrument to constructing the Galois representations in the Deligne-Serre theorem. To get a grasp on how these moduli spaces behave, we quickly describe the quite accessible case where $K = \mathbf{C}$.

As discussed in the first part of Johannes Anschütz's course on étale cohomology, the $\mathbf{C}$-rational points $E(\mathbf{C})$ form a complex abelian Lie group isomorphic to the quotient

$$E(\mathbf{C}) \cong \operatorname{Lie} E(\mathbf{C})/\Lambda \cong \mathbf{C}/\Lambda$$

where $\Lambda = H_1(E(\mathbf{C}), \mathbf{Z}) \subseteq \mathbf{C}$ is a rank-2 lattice. As mentioned in Remark 3.2, any elliptic curve is projective and thus by GAGA (also discussed in [Ja]), we get an equivalence of categories

$$\{\text{ell. curves over } \mathbf{C}\} \xrightarrow{\cong} \{\text{one dimensional complex tori}\}.$$

By the identification $E(\mathbf{C})[N] = (\frac{1}{N}\Lambda)/\Lambda \subseteq \mathbf{C}/\Lambda$ we get isomorphisms

$$\mathfrak{h}^{\pm}/\Gamma_1(N) \cong \{\text{elliptic curves } E/\mathbf{C} \text{ with a point } Q \in E(\mathbf{C})[N] \text{ of exact order } N\}$$
$$\mathfrak{h}^{\pm}/\Gamma_0(N) \cong \{\text{elliptic curves } E/\mathbf{C} \text{ with a subgroup } C \subseteq E(\mathbf{C})[N],\ \ C \cong \mathbf{Z}/N\mathbf{Z}\}$$

which just follow from the identification of an elliptic curve with its first homology group $\Lambda \subseteq \mathbf{C}$, and the fact that en entire function $\mathbf{C} \to \mathbf{C}$ sends a lattice Λ to another Λ' only if $\Lambda' = a\Lambda$ for some complex number $a \in \mathbf{C}^{\times}$.

Theorem 3.5. *Let N be a an integer ≥ 3 and p a prime not diving N.*

1. *The moduli problems*

$$\mathcal{P}_{\widetilde{\Gamma}_1(N)} : S/\operatorname{Spec}\mathbf{Z}[1/N] \longmapsto \{E/S,\ Q \in E[N](S) \textit{ a point of exact order } N\},$$
$$\mathcal{P}_{\widetilde{\Gamma}(N)} : S/\operatorname{Spec}\mathbf{Z}[1/N] \longmapsto \left\{E/S,\ \alpha : \underline{\mathbf{Z}/N\mathbf{Z} \oplus \mathbf{Z}/N\mathbf{Z}} \xrightarrow{\cong} E[N]\right\}$$

are representable by smooth affine curves over $\operatorname{Spec}\mathbf{Z}[1/N]$.

2. *The moduli problem*

$$\mathcal{P}_{\widetilde{\Gamma}_1(N)\cap\widetilde{\Gamma}_0(p)} : S/\operatorname{Spec}\mathbf{Z}[1/N] \longmapsto \{E/S, Q \in E[N](S) \textit{ a point of exact order } N, \rho : E \to E' \textit{ degree } p \textit{ isogeny}\}$$

is representable by an affine curve $X_{\widetilde{\Gamma}_1(N)\cap\widetilde{\Gamma}_0(p)}$ which is flat over $\mathbf{Z}[1/N]$ *and smooth outside of p.*

Remark 3.6. Quite crucially, the canonical maps

$$\begin{tikzcd} & X_{\widetilde{\Gamma}_1(N)\cap\widetilde{\Gamma}_0(p)} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X_{\widetilde{\Gamma}_1(N)} & & X_{\widetilde{\Gamma}_1(N)} \end{tikzcd}$$

given by $\pi_1(E, Q, \rho) = (E, Q)$ and $\pi_2(E, Q, \rho) = (E', \rho(Q))$ induce a correspondence between the modular curve $X_{\widetilde{\Gamma}_1(N)}$ and itself.

Remark 3.7. By invariance of the described moduli problems via flat base change, we get an isomorphism of complex analytic varieties

$$X_{\widetilde{\Gamma}_1(N)}(\mathbf{C}) = \mathfrak{h}^{\pm}/\Gamma_1(N)$$

as mentioned in Remark 3.4.

3.2 Geometric modular forms

Much like the addition of finitely many cusps to the complex modular curve in order to obtain a compactification, a very important result we'll rely upon is the existence of a '*projectivisation*' of $X_{\widetilde{\Gamma}_1(N)}$: we have a commutative diagram of smooth $\mathbf{Z}[1/N]$-schemes

$$\begin{tikzcd} \mathcal{E} \arrow[r] \arrow[d] & \mathcal{E}^* \arrow[d] \\ X_{\widetilde{\Gamma}_1(N)} \arrow[r] & X^*_{\widetilde{\Gamma}_1(N)} \end{tikzcd}$$

where $\mathcal{E} \to X_{\widetilde{\Gamma}_1(N)}$ is the universal elliptic curve and $X^*_{\widetilde{\Gamma}_1(N)} \to \operatorname{Spec} \mathbf{Z}[1/N]$ is projective.

We have an identity section $e : X^*_{\widetilde{\Gamma}_1(N)} \to \mathcal{E}^*$ extending the universal section in $\mathcal{E}(X_{\widetilde{\Gamma}_1(N)})$, and we will denote by $\underline{\omega}$ the '*universal relative differentials*'

$$\underline{\omega} := e^* \underline{\omega}^1_{\mathcal{E}^*/X^*_{\widetilde{\Gamma}_1(N)}} \in \operatorname{Pic}(X^*_{\widetilde{\Gamma}_1(N)}).$$

Remark 3.8. The complement $C := X^*_{\widetilde{\Gamma}_1(N)} \setminus X_{\widetilde{\Gamma}_1(N)}$ is a simple-normal-crossings-divisor in $X^*_{\widetilde{\Gamma}_1(N)}$, called the *divisor of cusps* and we have an isomorphism

$$\underline{\omega}^{\otimes 2} \cong \underline{\omega}^1_{X^*_{\widetilde{\Gamma}_1(N)}} (\log C)$$

where $\underline{\omega}^1_{X_{\widetilde{\Gamma}_1(N)}} (\log C)$ is the sheaf of differentials with logarithmic poles on C.

The main result we want to discuss now is the following identification

Theorem 3.9. *We have isomorphisms*

$$\mathcal{M}_k(\widetilde{\Gamma}_1(N), \mathbf{C}) \cong H^0(X^*_{\widetilde{\Gamma}_1(N),\mathbf{C}}, \underline{\omega}^{\otimes k}),$$
$$S_k(\widetilde{\Gamma}_1(N), \mathbf{C}) \cong H^0(X^*_{\widetilde{\Gamma}_1(N),\mathbf{C}}, \underline{\omega}^{\otimes k}(-C)).$$

Proof. (*idea*) The union of upper and lower half-planes $\mathfrak{h}^{\pm}$ is the moduli space of elliptic curves $E/\mathbf{C}$ with a fixed isomorphism

$$\mathbf{Z} \oplus \mathbf{Z} \xrightarrow{\cong} H_1(E, \mathbf{Z}).$$

An fixed basis for the rank 2 lattice $H_1(E, \mathbf{Z})$ as above induces $e^* \underline{\omega}^1_{E/\mathbf{C}} \cong \mathbf{C}$ a trivialisation of the relative differentials. Then, if one check the $\mathrm{GL}_2(\mathbf{Z})$-action is as expected we win. ■

Since the above isomorphism is only valid after a base-change to $\mathbf{C}$, this encourages the definition:

Definition 3.10. For any $\mathbf{Z}[1/N]$-algebra Λ, we can define

$$\mathcal{M}_k(\Gamma_1(N), \Lambda) := H^0(X^*_{\widetilde{\Gamma}_1(N)} \times_{\mathbf{Z}[1/N]} \Lambda, \underline{\omega}^{\otimes k}),$$
$$S_k(\Gamma_1(N), \Lambda) := H^0(X^*_{\widetilde{\Gamma}_1(N)} \times_{\mathbf{Z}[1/N]} \Lambda, \underline{\omega}^{\otimes k}(-C))$$

3.3 Hecke operators as correspondences

For all integers $N \geq 3$ and primes p not diving N, the moduli problem $X_{\widetilde{\Gamma}_1(N) \cap \widetilde{\Gamma}_0(p)}$ also admits a compactification $X^*_{\widetilde{\Gamma}_1(N)}$ which is projective over $\operatorname{Spec} \mathbf{Z}[1/N]$, and we have extensions of the projections described in Remark 3.6

$$\begin{tikzcd} & X^*_{\widetilde{\Gamma}_1(N) \cap \widetilde{\Gamma}_0(p)} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X^*_{\widetilde{\Gamma}_1(N)} & & X^*_{\widetilde{\Gamma}_1(N)} \end{tikzcd}$$

so that π_1 and π_2 are finite flat morphisms of degree $p + 1$. We thus obtain Hecke correspondences T_p between the compactified modular curve $X^*_{\widetilde{\Gamma}_1(N)}$ an itself; furthermore, there exists a canonical isomorphism of sheaves

$$\pi_1^* \underline{\omega} \cong \pi_2^* \underline{\omega}$$

which is only defined generically, (i.e. over $\operatorname{Spec}\mathbf{Q}$), and a trace map

$$\operatorname{Tr}\pi_1 : \pi_{1,!}\pi_1^*\underline{\omega} \to \underline{\omega}.$$

Taking global sections and composing we obtain

$$H^0(X^*_{\widetilde{\Gamma}_1(N)}, \underline{\omega}^{\otimes k})_{\mathbf{Q}} \xrightarrow{\pi_2^*} H^0(X^*_{\widetilde{\Gamma}_1(N)\cap\widetilde{\Gamma}_0(p)}, \pi_2^*\underline{\omega}^{\otimes k})_{\mathbf{Q}} \cong H^0(X^*_{\widetilde{\Gamma}_1(N)\cap\widetilde{\Gamma}_0(p)}, \pi_1^*\underline{\omega}^{\otimes k})_{\mathbf{Q}} \xrightarrow{\operatorname{Tr}\pi_1} H^0(X^*_{\widetilde{\Gamma}_1(N)}, \underline{\omega}^{\otimes k})_{\mathbf{Q}}.$$

and we call T_p the above composition multiplied by $\frac{1}{p}$. We state a few results about T_p.

1. T_p extends to a map defined over $\mathbf{Z}[1/N]$,
2. the base change of T_p to $\mathbf{C}$ identifies with the Hecke operators introduced in Subsection 2.6 given by

$$\begin{aligned} T_p : \mathcal{M}_k(\Gamma_1(N), \mathbf{C}) &\longrightarrow \mathcal{M}_k(\Gamma_1(N), \mathbf{C}) \\ f(z) = \sum_{n=0}^{\infty} a_n q^n &\longmapsto \sum_{n=0}^{\infty} a_{np} q^n + \sum_{n=0}^{\infty} p^{k-1}\chi(p) a_n q^{pn} \end{aligned}$$

   where χ is the nebetypus of f.

3. the correspondence T_p (thought of as a map from the Picard scheme of $X^*_{\widetilde{\Gamma}_1(N)}$ to itself) can be described as a map on divisors as follows

$$T_p : (E, Q) \longmapsto \sum_{E \xrightarrow{\alpha} E',\, \deg\alpha = p} (E', \alpha(Q)) \cdot \frac{1}{p}.$$

We also have another notion of a Hecke operator S_p, constructed from the diagram

$$\begin{tikzcd} & X^*_{\widetilde{\Gamma}_1(N)} \arrow[dl, "p"'] \arrow[dr, "\mathrm{id}"] & \\ X^*_{\widetilde{\Gamma}_1(N)} & & X^*_{\widetilde{\Gamma}_1(N)} \end{tikzcd}$$

where p is defined on the moduli problem via

$$p : (E, Q) \longmapsto (E, pQ).$$

Composing with the canonical isomorphism of sheaves $p^*\underline{\omega} \cong \underline{\omega}$ on $X^*_{\widetilde{\Gamma}_1(N)}$ we get a pullback map

$$H^0(X_{\widetilde{\Gamma}_1(N)}, \underline{\omega}^{\otimes k}) \xrightarrow{p^*} H^0(X_{\widetilde{\Gamma}_1(N)}, \underline{\omega}^{\otimes k}).$$

Upon a base-change to complex coefficients and thinking of the above vector spaces as modular and cusp forms, the map $S_p = p^*$ identifies with the *diamond bracket operator* $\langle p \rangle$.

Definition 3.11. The *abstract Hecke algebra* $\mathbf{T}$ is the $\mathbf{Z}$ algebra generated by the elements T_p, S_p for $p \nmid N$ prime. We thus have natural maps

$$\begin{aligned} \mathbf{T} &\to \operatorname{End}_\Lambda(S_k(\Gamma_1(N), \Lambda)), \\ \mathbf{T} &\to \operatorname{End}_\Lambda(T_k(\Gamma_1(N), \Lambda)). \end{aligned}$$

Remark 3.12. If $f \in S_k(\Gamma_1(N), \chi, \mathbf{C})$ is a normalised eigenform of nebentypus χ and weight $k \geq 1$, then $T_p(f) = a_{p,f} f$ and $S_p(f) = \chi(p) f$ for all p not dividing N.

3.4 The Eichler-Shimura Isomorphism

Eichler-Shimura theory is the core idea behind the Deligne-Serre theorem, since it provides us precisely with the Galois representation of $G_{\mathbf{Q}}$ we search for, so long as the weight k is at least 2.

Theorem 3.13 (The Eichler-Shimura isomorphism). *Suppose $k \in \mathbf{Z}$ is greater than or equal to 2. There exists a local system θ_k on the open modular curve $X_{\widetilde{\Gamma}_1(N)}(\mathbf{C})$ so that we have a $\mathbf{T}$-equivariant isomorphism*

$$H^1(X_{\widetilde{\Gamma}_1(N)}(\mathbf{C}), \vartheta_k) \otimes_{\mathbf{Q}} \mathbf{C} \cong \mathcal{M}_k(\Gamma_1(N), \mathbf{C}) \oplus \overline{S_k(\Gamma_1(N), \mathbf{C})}.$$

Remark 3.14. The Eichler-Shimura isomorphism is a generalisation of the following simple argument, only valid for the case there we consider the weight $k = 2$: suppose X is a proper smooth complex variety. The de Rham isomorphism gives a realisation of the singular cohomology groups $H^*(X(\mathbf{C}), \mathbf{C})$ with complex coefficients as the cohomology of the de Rham complex of differentials

$$\Omega^\bullet_{X/\mathbf{C}} = [\mathcal{O}_X \to \Omega^1_{X,\mathbf{C}} \to \Omega^2_{X/\mathbf{C}} \to \dots] \cong \underline{\mathbf{C}}$$

and, in particular, applying the Hodge decomposition we see that

$$H^1(X(\mathbf{C}), \mathbf{C}) \cong H^0(X, \Omega^1_{X/\mathbf{C}}) \oplus H^1(X, \mathcal{O}_C) \cong H^0(X, \Omega^1_{X/\mathbf{C}}) \oplus \overline{H^0(X, \Omega^1_{X/\mathbf{C}})}$$

where in the last isomorphism we applied Serre duality. If $j : U \hookrightarrow X$ is an open subset whose complement $C = X \setminus U$ is divisor with normal crossings, then the complex of differentials with log poles along C

$$\Omega^\bullet_{X/\mathbf{C}}(\log C) = \left\{ \omega \in j_*\Omega^i_{X/\mathbf{C}} \mid \text{both } \omega \text{ and } d\omega \text{ have poles of order at most 1 along } C \right\}$$

then we may compute $U(\mathbf{C})$'s singular cohomology groups by replacing $\Omega^\bullet_{X/\mathbf{C}}$:

$$H^*(U(\mathbf{C}), \mathbf{C}) \cong H^*_{\mathrm{dR}}(X, \Omega^\bullet_{X/\mathbf{C}}(\log C)).$$

By applying the Kodaira-Spencer isomorphism, described in Remark 3.8, we can specialise these considerations to the open modular curve $U = X_{\widetilde{\Gamma}_1(N)}$ included in the compactified one $X^*_{\widetilde{\Gamma}_1(N)}$ and obtain

$$\mathcal{M}_2(\Gamma_1(N), \mathbf{C}) \cong H^0(X^*_{\widetilde{\Gamma}_1(N)}(\mathbf{C}), \underline{\omega}^{\otimes 2}) \cong H^0(X^*_{\widetilde{\Gamma}_1(N)}(\mathbf{C}), \underline{\omega}^{\otimes 2}(-C))$$

and by the isomorphism

$$H^*(X_{\widetilde{\Gamma}_1(N)}, \mathbf{C}) \cong H^*_{\mathrm{dR}}(X_{\widetilde{\Gamma}_1(N)}, \underline{\omega}^\bullet(\log C))$$

we see that the space of weight 2 modular forms appears in computing the *singular* cohomology (i.e. with constant coefficients) of the open modular curve. For higher weight, one has to replace the constant sheaf $\underline{\mathbf{C}}$ with a local system of $\mathbf{C}$-vector spaces; we refrain from discussing the construction of this local system any further, but an important aspect to remark is that ϑ_k does *not* in general (i.e. for $k > 2$) extend to a local system defined on the compactified modular curve $X^*_{\widetilde{\Gamma}_1(N)}$. For more details we reference Deligne's article [Del].

Having expressed modular forms in terms of the cohomology of a local system on the open modular curve, we introduce the Galois group $G_{\mathbf{Q}}$ into the picture. Suppose $f \in S_k(\Gamma_1(N), \chi, \mathbf{C})$ is a normalised eigenform of weight $k \geq 2$ and nebentypus χ. Thus we have the relations

$$T_p f = a_p f, \ S_p f = \chi(p) f, \ p \nmid N$$

and by strong multiplicity one, f is determined by these values. By abstractly chosing an isomorphism $\mathbf{C} \cong \overline{\mathbf{Q}_\ell}$ the values $a_p, \chi(p)$ can be thought of as a system of eigenvalues of f in $\overline{\mathbf{Q}}_\ell$, and thus a maximal ideal $\mathfrak{m}_f \subset \mathbf{T} \otimes_{\mathbf{Z}} \overline{\mathbf{Q}}_\ell$ such that

$$H^1(X_{\widetilde{\Gamma}_1(N)}(\mathbf{C}), \vartheta_k)_{\mathfrak{m}_f}$$

is a two-dimensional $\overline{\mathbf{Q}}_\ell$-vector space, by Theorem 3.13. To introduce an action of $G_{\mathbf{Q}}$, we want to define an ℓ-adic model of ϑ_k on the étale site of $X_{\widetilde{\Gamma}_1(N)}/\operatorname{Spec}\mathbf{Z}[1/N]$. For this, all we say is that this can be achieved by chosing

$$\vartheta_k := \operatorname{Sym}^{k-2}(R^1 f_* \underline{\mathbf{Q}_\ell})$$

where $f : \mathcal{E} \to X_{\widetilde{\Gamma}_1(N)}$ is the universal elliptic curve over $X_{\widetilde{\Gamma}_1(N)}$. We can now apply the comparison between sheaf cohomology of the complex analytic variety $X_{\widetilde{\Gamma}_1(N)}(\mathbf{C})$ and the étale cohomology of ϑ_k - cfr. the final section of Anschütz's notes [Ja], *applications of the proper base change theorem* - and obtain a $\mathbf{T}$-equivariant isomorphism

$$H^1(X_{\widetilde{\Gamma}_1(N)}(\mathbf{C}), \vartheta_k \otimes \overline{\mathbf{Q}}_\ell)_{\mathfrak{m}_f} \cong H^1_{\text{ét}}(X_{\widetilde{\Gamma}_1(N)} \times_{\mathbf{Z}[1/N]} \overline{\mathbf{Q}}, \vartheta_k \otimes \overline{\mathbf{Q}}_\ell)_{\mathfrak{m}_f}$$

and on the latter we have a natural action of the Galois group $G_{\mathbf{Q}}$.
The dual of this vector space will be our representation $\rho_{f,\lambda}$ for $\lambda \mid \ell$.

3.5 Special fibres of the modular curve and the Eichler-Shimura relation

In this subsection, having constructed the sought-for representation $\rho_{f,\lambda}$, we check that for $p \nmid N\ell$ the representation $\rho_{|G_{\mathbf{Q}_p}}$ is unramified and the conjugacy class of Frobenius elements have characterstic polynomial

$$\det(X - \rho_{f,\lambda}(\operatorname{Frob}_p)) = X^2 - a_p X + p\chi(p).$$

For simplicity, we assume $k = 2$ from now onwards and thus $\vartheta_k = \underline{\mathbf{C}} \cong \underline{\mathbf{Q}_\ell}$. Since for all primes p not dividing $N \cdot \ell$ we have that the modular curve $X_{\widetilde{\Gamma}_1(N)}$ has good reduction at ℓ, the smooth and proper base change theorems yield an isomorphism of $G_{\mathbf{Q}_p}$-representations

$$H^1_{\text{ét}}(X^*_{\widetilde{\Gamma}_1(N)} \times_{\mathbf{Z}[1/N]} \overline{\mathbf{Q}}_p, \overline{\mathbf{Q}}_\ell)_{\mathfrak{m}_f} \cong H^1_{\text{ét}}(X^*_{\widetilde{\Gamma}_1(N)} \times_{\mathbf{Z}[1/N]} \overline{F}_p, \overline{\mathbf{Q}}_\ell)_{\mathfrak{m}_f}$$

and thus $(\rho_{f,\lambda})_{|G_{\mathbf{Q}_p}}$ is unramified (if the weight is greater than two, more subtle arguments are required since ϑ_k doesn't extend to a local system on $X^*_{\widetilde{\Gamma}_1(N)}$ and we don't have the proper base change theorem at our disposal). The desired expression for the characteristic polynomial $\det(X - \rho_{f,\lambda}(\operatorname{Frob}_p))$ will follow from the *Eichler-Shimura relation*:

Theorem 3.15 (Eichler-Shimura relation)**.** *We have an equality of endomorphisms of the abelian variety* $\operatorname{Pic}^0(X^*_{\widetilde{\Gamma}_1(N)} \times_{\mathbf{Z}[1/N]} \mathbf{F}_p)$

$$T_p = F_* + \langle p \rangle F^*$$

where F is the absolute Frobenius of the projective $\mathbf{F}_p$-variety $X^*_{\widetilde{\Gamma}_1(N)} \times_{\mathbf{Z}[1/N]} \mathbf{F}_p$.

Remark 3.16. This will allow us to conclude since we have an isomorphism

$$H^1_{\text{ét}}(X_{\widetilde{\Gamma}_1(N)} \times_{\mathbf{Z}[1/N]} \overline{\mathbf{F}}_p, \mathbf{Q}_\ell) \cong V_\ell(\operatorname{Pic}^0(X^*_{\widetilde{\Gamma}_1(N)} \times_{\mathbf{Z}[1/N]} \overline{\mathbf{F}}_p))^\vee$$

which follows since $\operatorname{Pic}^0(X^*_{\widetilde{\Gamma}_1(N)})$ is an abelian variety - here we denote by $V_\ell A$ the ℓ-adic Tate module of an abelian variety A (this Stack Overflow post helped clear things up for me a little). The geometric Frob_p acting on the first ℓ-adic étale cohomology group $H^1_{\text{ét}}(X_{\widetilde{\Gamma}_1(N)} \times_{\mathbf{Z}[1/N]\overline{\mathbf{F}_p}}, \mathbf{Q}_\ell)$ satisfies the polynomial

$$X^2 - T_p X + p\langle p \rangle.$$

To prove the Eichler-Shimura relation we study the geometry of the special fibres of the compactified modular curves $X^*_{\widetilde{\Gamma}_1(N)}$ and $X^*_{\widetilde{\Gamma}_1(N)\cap\widetilde{\Gamma}_0(p)}$. We denote by $\overline{X}^*_{\widetilde{\Gamma}_1(N)}$ and $\overline{X}^*_{\widetilde{\Gamma}_1(N)\cap\widetilde{\Gamma}_0(p)}$ these special fibres; by the theory of elliptic curves over $\overline{\mathbf{F}}_p$ we deduce the existence of a decomposition

$$\overline{X}_{\widetilde{\Gamma}_1(N)} = \overline{X}^{\text{ord},*}_{\widetilde{\Gamma}_1(N)} \amalg \overline{X}^{\text{s}}_{\widetilde{\Gamma}_1(N)}$$

where both pieces have four different but equivalent moduli-theoretic descriptions in terms of the open modular curve $\overline{X}_{\widetilde{\Gamma}_1(N)} \subseteq \overline{X}^*_{\widetilde{\Gamma}_1(N)}$ and the cusp of divisors $\overline{C} \subseteq \overline{X}^*_{\widetilde{\Gamma}_1(N)}$ naturally lies in the ordinary locus:

1. The *p-rank stratification*: a closed point $x : \operatorname{Spec} \kappa \hookrightarrow \overline{X}_{\widetilde{\Gamma}_1(N)}$ lying in the open modular curve corresponding to the pair $(E, Q)/\kappa$ satisfies

$$\begin{cases} x \in \overline{X}^{\text{ord}}_{\widetilde{\Gamma}_1(N)} \iff \#E[p](\overline{\kappa}) = p \\ x \in \overline{X}^{\text{s}}_{\widetilde{\Gamma}_1(N)} \iff \#E[p](\overline{\kappa}) = 1 \end{cases}$$

2. The *Ekedahl-Oort stratification*:

$$\begin{cases} x \in \overline{X}^{\text{ord}}_{\widetilde{\Gamma}_1(N)} \iff E_{\overline{\kappa}}[p] \cong \mu_p \oplus \underline{\mathbf{Z}/p\mathbf{Z}} \\ x \in \overline{X}^{\text{s}}_{\widetilde{\Gamma}_1(N)} \iff E_{\overline{\kappa}}[p] \text{ is biconnected} \end{cases}$$

3. The *Newton stratification*:

$$\begin{cases} x \in \overline{X}^{\text{ord}}_{\widetilde{\Gamma}_1(N)} \iff E_{\overline{\kappa}}[p^\infty] \text{ is isogenous to } \mu_{p^\infty} \oplus \underline{\mathbf{Q}_p/\mathbf{Z}_p} \\ x \in \overline{X}^{\text{s}}_{\widetilde{\Gamma}_1(N)} \iff E_{\overline{\kappa}}[p^\infty] \text{ is supersingular - i.e. a connected, one-dimensional } p\text{-divisible group of height } 2 \end{cases}$$

4. The last description, which will be our main tool, is in terms of the *Hasse invariant:* in positive characteristic, the universal elliptic curve

$$\overline{\mathcal{E}} \to \overline{X}^*_{\widetilde{\Gamma}_1(N)}$$

admits a special isogeny, called the *Vershibung isogeny*

$$\text{Ver} : \overline{\mathcal{E}}^{(p)} \to \overline{\mathcal{E}}$$

where $\overline{\mathcal{E}}^{(p)}$ is the base-change of $\mathcal{E}$ along the absolute Frobernius $\text{Frob}_p : \overline{X}^*_{\widetilde{\Gamma}_1(N)} \to \overline{X}^*_{\widetilde{\Gamma}_1(N)}$. The pullback induces a morphism of coherent sheaves

$$(\text{Ver})^* : \underline{\omega} \to \underline{\omega}^{\otimes p}$$

which thus corresponds to a global section

$$\text{Ha} \in H^0(\overline{X}^*_{\widetilde{\Gamma}_1(N)}, \underline{\omega}^{\otimes p-1}).$$

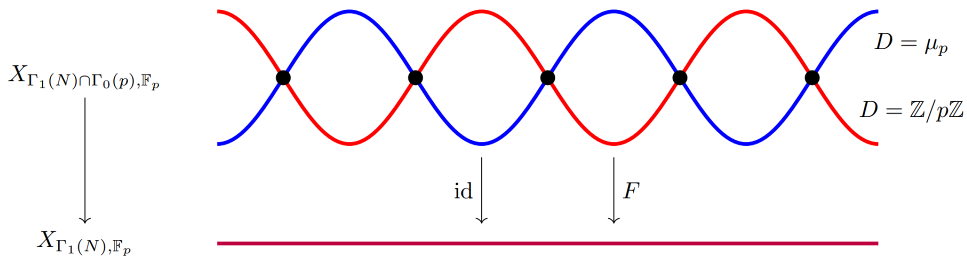
We now have the description

$$\overline{X}^{\text{ord},*}_{\widetilde{\Gamma}_1(N)} = D(\text{Ha}),\ \overline{X}^{\text{s}}_{\widetilde{\Gamma}_1(N)} = V(\text{Ha})$$

which can easily be seen to coincide with the p-rank stratification since E is ordinary if and only if $\text{Ver} : E^{(p)} \to E$ is finite étale $\iff$ $\text{Ver}^*_{|E}$ is an isomorphism.

To access a description of the mod-p Hecke operators, we also require a description of the geometry of $\overline{X}^*_{\widetilde{\Gamma}_1(N)\cap\widetilde{\Gamma}_0(p)}$.

Theorem 3.17 (The Deligne-Rapoport model)**.** *The special fibre of the modular curve $\overline{X}^*_{\widetilde{\Gamma}_1(N)\cap\widetilde{\Gamma}_0(p)}/\mathbf{F}_p$ inherits the Newton stratification, as above. It is the union of two copies of $\overline{X}^*_{\widetilde{\Gamma}_1(N)}$ intersecting transversally at the (finitely many) supersingular points.*



The restriction of the projection map

$$\begin{aligned}\pi : \overline{X}^*_{\widetilde{\Gamma}_1(N)\cap\widetilde{\Gamma}_0(p)} &\longrightarrow \overline{X}^*_{\widetilde{\Gamma}_1(N)}\\ (E,E',Q) &\longmapsto (E,Q)\end{aligned}$$

to each of the two copies $\overline{X}^{\mathrm{can}}, \overline{X}^{\mathrm{anti}}$ of $\overline{X}^*_{\widetilde{\Gamma}_1(N)} \subseteq \overline{X}^*_{\widetilde{\Gamma}_1(N)}$ identifies with the identity on $\overline{X}^{\mathrm{can}}$ and the absolute Frobenius $\mathrm{Frob}_{\overline{X}^*_{\widetilde{\Gamma}_1(N)}}$ on $\overline{X}^{\mathrm{anti}}$. We have the moduli theoretic description of each component:

$$\begin{aligned}\overline{X}^{\mathrm{can}} &= \left\{(E,Q,D) \in \overline{X}_{\widetilde{\Gamma}_1(N)\cap\widetilde{\Gamma}_0(p)} \mid D \cong \mu_p \subseteq E[p]\right\}\\ \overline{X}^{\mathrm{anti}} &= \left\{(E,Q,D) \in \overline{X}_{\widetilde{\Gamma}_1(N)\cap\widetilde{\Gamma}_0(p)} \mid D \cap \mu_p = \{0\}\right\}\end{aligned}$$

where D is the kernel of the isogeny $E \to E'$ from the moduli-theoretic description of $X_{\widetilde{\Gamma}_1(N)\cap\widetilde{\Gamma}_0(p)}$ in Theorem 3.5.

We now use the description of the Hecke correspondence in 3 to conclude:

$$T_p((E,Q)) = (E^{(p)},Q^{(p)}) + p(E^{(p^{-1})}, pQ^{(p^{-1})}) = F(E,Q) + \langle p\rangle pF^{-1}(E,Q)$$

which induces the equality

$$T_p = F_* + \langle p\rangle F^*$$

on $\mathrm{Pic}^0(\overline{X}^*_{\widetilde{\Gamma}_1(N)})$.

4 The Deligne-Serre theorem

Having discussed the main ideas behind the proof of Theorem for weights k at least 2, we discuss the strategy for proving the analogous result for weight 1 cusp forms, since the method involves very new techniques: instead of constructing the representation through the ℓ-adic cohomology of the modular curve, we consider its coherent cohomology.

Theorem 4.1 (Deligne-Serre)**.** *Let $f \in S_1(N,\chi,\mathbf{C})$ be a weight 1 normalised eigenform of level $\Gamma_1(N)$ and nebentypus χ.*
Then there exists a uniquely determined continuous irreducible representation

$$\rho_f : G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{C})$$

such that for all p not dividing N the restriction $(\rho_f)_{|G_{\mathbf{Q}_p}}$ is unramified and the characteristic polynomial of Frob_p^{-1} *is given by*

$$\det(X - \rho_f(\mathrm{Frob}_p^{-1})) = X^2 - a_p X + \chi(p).$$

The strategy of the proof is as follows - if I have time some day in the future I should edit this pdf and add some more details which were discussed in the lectures.

1. Fix λ a prime in K_f that doesn't divide 6; we want to apply the Eichler-Shimura theorem on existence of Galois representations to a cusp form of weight strictly greater than 1 - to do this, we multiply f by a form of higher weight. From the classical theory of modular forms, we have an *Eisenstein series* of weight $\ell - 1$ given explicitly by

$$E_{\ell-1}(q) = 1 - \frac{2(\ell-1)}{B_{\ell-1}} \sum_{n=1}^{\infty} \sigma_{\ell-2}(n) q^n$$

   where $B_{\ell-1}$ is the $\ell - 1$-th Bernoulli number and

$$\sigma_{\ell-2}(n) := \sum_{k|n} k^{\ell-2}.$$

   We see that $E_{\ell-1}(q) \equiv 1$ modulo ℓ. $E_{\ell-2}$ can also be seen to be a lift to characteristic zero of the Hasse invariant $\mathrm{Ha} \in H^0(\overline{X}^*_{\widetilde{\Gamma_1(N)}}, \underline{\omega}^{\ell-1}) = S_{\ell-1}(\Gamma_1(N), \mathcal{O}_{K_f,\lambda}/m_{K_f,\lambda})$ mentioned above in 4. Thus multiplying f by $E_{\ell-1}$ produces

$$f_\ell = f \cdot E_{\ell-1} \in S_\ell(\Gamma_1(N), \chi, \mathcal{O}_{K_f})$$

   a cusp form congruent to f modulo λ, but not necessarily an eigenform in characteristic zero - Eicherler Shimura can't be applied just yet.

2. Apply the *Deligne-Serre lifting lemma:* there exists an *eigenform* $\widetilde{f_\ell} \in S_\ell(\Gamma_1(N), \chi, K'_f)$ such that $\widetilde{a_p} \equiv a_p$ modulo λ' for all $p \nmid N\ell$.

3. Now that we have an eigenform, we can apply Eichler-Shimura to $\widetilde{f_\ell}$ and produce a Galois representation

$$\rho_{\widetilde{f_\ell},\lambda'} : G_{\mathbf{Q}} \to \mathrm{GL}_2(K'_{f,\lambda'})$$

   so that for all $p \nmid N\ell$ the induced $G_{\mathbf{Q}_p}$ representation is unramified and Frob_p satisfies the equation

$$\det(X - \rho_{\widetilde{f_\ell},\lambda'}(\mathrm{Frob}_p)) = X^2 - \widetilde{a}_p X + \chi(p)p^{\ell-1}.$$

4. We now go back down to positive characteristic: consider the semisimplification of $\rho_{\widetilde{f_\ell},\lambda'}$'s reduction:

$$\overline{\rho}_{f,\lambda} := \overline{\rho}^{\mathrm{ss}}_{\widetilde{f_\ell},\lambda} : G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathcal{O}_{K'_f}/(\lambda'))$$

   which again satisfies

$$\det(X - \overline{\rho}_{f,\lambda}(\mathrm{Frob}_p)) = X^2 - \overline{a_p} X + \chi(p)$$

   for all $p \nmid N\ell$.

5. Use the *Rankin-Selberg method*, which in particular involves showing that the images of the representations for varying ℓ are uniformly bounded in ℓ, to lift to a $\mathrm{GL}_2(\mathbf{C})$-valued representation $\rho_f : G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{C})$.

References

[GeHa] Jayce R. Getz, Heekyoung Hahn (2022), *An Introduction to Automorphic Representations*, online pdf.

[Con] Brian Conrad, *Finiteness of Class Numbers for Algebraic Groups*, online pdf.

[Eme] Matthew Emerton, *Classical Modular Forms as Automorphic Forms*, online pdf.

[KatMaz] Nicholas M. Katz, Barry Mazur, *Arithmetic Moduli of Elliptic Curves*.

[Del] Pierre Deligne, *Formes Modulaires et Représentations ℓ-adiques*.

[Ja] Johannes Anschütz, *Lecture notes on étale cohomology*, online pdf.

[JaLa] Johannes Anschütz, *Lecture notes for an Introductory Course on The Langlands Program*, online pdf.