Exercises from a course on étale cohomology

Thomas Manopulo

These notes are an attempt to write up my solutions to the exercises from a course I'm currently taking on étale cohomology, taught by Johannes Anschütz.
My working surely contains many errors; if you're interested in sending any of your thoughts to me you can reach me by email at *s6thmano@uni-bonn.de*. :)

Exercise. Let $\mathcal{A}$ be an additive category and $A_\bullet : \Delta^{\mathrm{op}} \to \mathcal{A}$ a simplicial object. Define $d : A_{n+1} \to A_n$ as $d = \sum_{i=0}^{n+1} (-1)^i d_i$ as in Definition 2.4. Then $d \circ d = 0$.

Explicitly, we have

$$d(\sum_{i=0}^{n+1} (-1)^i d_i(a)) = \sum_{j=0}^{n} \sum_{i=0}^{n+1} (-1)^{i+j} d_j \circ d_i(a).$$

By definition, $d_j \circ d_i : A_{n+1} \to A_{n-1}$ is the image under $A_\bullet$ of the order-preserving map $[n-1] \to [n+1]$ whose image doesn't contain $\{a, b\} \subseteq [n+1]$, where $\{a, b\}$ are determined by the rule:

$$\{a, b\} = \begin{cases} \{i, j\} & \text{if } i > j \\ \{i, j+1\} & \text{if } j \geq i \end{cases}$$

In particular, we have the relation, valid for all $i, j > 0$:

$$d_j \circ d_i = d_{i-1} \circ d_j$$

which shows that all terms in the above sum cancel, since $(-1)^{i+j} = -(-1)^{i-1+j}$ and when $i = 0$ we get the cancellation due to $d_j \circ d_0 = d_0 \circ d_{j+1}$.

Exercise. If $K^\bullet \in \mathcal{D}(\mathbb{Z})$ is an object and $\mathcal{H}^i(K^\bullet)$ is finitely generated for each $i \in \mathbb{Z}$, then the natural map $K^\bullet \to \mathrm{RHom}_{\mathbb{Z}}(\mathrm{RHom}_{\mathbb{Z}}(K^\bullet, \mathbb{Z}), \mathbb{Z})$ is an isomorphism.

As suggested by the hint, we can apply the result in exercise 1 from sheet 10 in AG2 to conclude that we have an isomorphism of objects in the derived category

$$K^\bullet \cong \bigoplus_{i \in \mathbb{Z}} \mathcal{H}^i(K^\bullet)[-i]$$

as the category of abelian groups has homological dimension equal to one; since the functor of triangulated categories

$$\mathrm{RHom}_{\mathbb{Z}}(\mathrm{RHom}_{\mathbb{Z}}(-, \mathbb{Z}), \mathbb{Z}) : \mathcal{D}(\mathbb{Z}) \to \mathcal{D}(\mathbb{Z})$$

is exact (as it is the composition of - contravariant - exact functors) it must commute with direct sums, hence we may assume $K^\bullet = K[0]$ is concentrated in degree zero and identifies with a finitely generated abelian group. Since K splits as a tirect sum between a finite abelian group and a free one, we can consider these two cases separately:

1. if K is cyclic, then a simple (relatively standard) computation of higher Ext-groups shows that

$$\begin{aligned} \mathrm{RHom}_{\mathbb{Z}}(K, \mathbb{Z}) \cong K[-1] \implies \mathrm{RHom}_{\mathbb{Z}}(\mathrm{RHom}_{\mathbb{Z}}(K, \mathbb{Z}), \mathbb{Z}) &\cong \mathrm{RHom}_{\mathbb{Z}}(K[-1], \mathbb{Z}) \\ &\cong \mathrm{RHom}_{\mathbb{Z}}(K, \mathbb{Z})[1] \cong K. \end{aligned}$$

2. if $K \cong \mathbb{Z}$ then already $\mathrm{RHom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}$ and we're done.

Exercise. Show that if R is a valuation ring, then R is coherent as an R-module.

We use that in a valuation ring, because the set of ideals is totally ordered, every finitely generated ideal is principal (and in fact generated by a term of smallest valuation among a fixed set of generators). Thus, for any R-module homomorphism

$$\phi : R^{\oplus n} \to R$$

if we set $r_i := \phi(e_i)$ where $e_1, \ldots, e_n \in R^{\oplus n}$ are the standard generators, there exists i such that $\operatorname{im} \phi = (r_i) \implies a_j r_j = r_i$ for all $j = 1, \ldots, n$ for some set of elements $a_1, \ldots, a_n \in R$. Now, whenever $t_1 e_1 + \ldots + t_n e_n \in \ker \phi$ we have that

$$(t_1 a_1 + \ldots + t_n a_n) r_i = 0 \implies t_i = -(t_1 a_1 + \ldots + \widehat{a_i t_i} + \ldots + t_n a_n)$$

hence $\mathrm{val}(t_i) \leq \min\{\mathrm{val}(a_1 t_1), \ldots, \mathrm{val}(a_n t_n)\} \implies a_j t_j = b_j t_i$ for some elements $b_1, \ldots, b_n \in R$. In other words, $\ker \phi$ is generated by the elements

$$(e_i - a_j e_j)_j.$$

Exercise (3.46)**.** Let X be a complex analytic space. Then X is regular (i.e. each stalk $\mathcal{O}_{X,x}$ is a regular local ring) if and only if X is a complex manifold.

I really like this exercise. Suppose $\dim X = n$ and $x \in X$ is any point. By Noether normalisation for analytic algebras, we get an injective integral ring homomorphism

$$\mathbb{C}\{z_1, \ldots, z_n\} \hookrightarrow \mathcal{O}_{X,x}$$

which, passing to completions yields an injection

$$\mathbb{C}[[z_1, \ldots, z_n]] \hookrightarrow \mathcal{O}_{X,x}^{\wedge}$$

which is once again integral (I think?). Since abstractly $\mathcal{O}_{X,x}^{\wedge} \cong \mathbb{C}[[t_1, \ldots, t_n]]$ by the hypothesis on X's regularity, it follows that the above must be an isomorphism, and by Proposition 3.45 it follows that the injection $\mathbb{C}\{z_1, \ldots, z_n\} \hookrightarrow \mathcal{O}_{X,x}$ is also an isomorphism. This shows that there exists a neighbourhood of x contained in the open subset on which $z_1, \ldots, z_n$ are defined which is isomorphic as an analytic space to a unit disc in $\mathbb{C}^n$.

Exercise (3.48)**.** The assumption that f is quasi-compact in Proposition 3.47 is important. For each of the properties give an example showing that the conclusion fails if quasi-compactness of f is dropped.

We quickly recall the statement of Proposition 3.47:

Proposition (3.47)**.** *Let $f : X \to Y$ be a morphism of finite-type between schemes X and Y which are locally of finite-type over $\mathbb{C}$, and denote by $f^{an} : X^{an} \to Y^{an}$ its analytification. Then f enjoys any of the following properties if and only if f^{an} does:*

1. *being surjective,*
2. *being a closed immersion,*
3. *being proper,*
4. *being finite.*

Consider the map

$$f : \coprod_{i\in\mathbb{C}} \operatorname{Spec}\mathbb{C} \to \mathbb{A}^1_{\mathbb{C}}$$

given by the disjoint union of each inclusion of a closed points in the affine line. The corresponding morphism of complex analytic spaces is the identity on underlying topological spaces, and is thus surjective, proper and finite since (on the complex analytic side) these are all topological notions. f however is none of these, since it even lacks quasi-compactness. As for the second property, we can simply substitute f with its restriction to the closed points corresponding to the integers $\mathbb{Z} \subseteq \mathbb{A}^1_{\mathbb{C}}$ and thus f^{an} is a closed immersion, since $\mathcal{O}_{\mathbb{A}^{1,\text{an}}_{\mathbb{C}}}$ possessed the global section $\sin(2\pi i z)$, but any global section in $\mathbb{A}^1_{\mathbb{C}}$ has finitely-many zeroes.

Exercise (3.52)**.** Show that both assertions in Theorem 3.51 fail without the assumption of properness.

Theorem 3.51 is the celebrated result of Serre's:

Theorem (GAGA)**.** *Let $f : Y \to X$ be a morphism of schemes, locally of finite type over $\mathbb{C}$.*

1. *If f is proper and $\mathcal{F} \in \operatorname{Coh}_X$, then for any $i \geq 0$ the neatural map $(R^i f_*(\mathcal{F}))^{an} \to R^i f_*^{an}(\mathcal{F}^{an})$ is an isomorphism.*

2. *If $X \to \operatorname{Spec}\mathbb{C}$ is proper, then the functor $(-)^{an} : \operatorname{Coh}_X \to \operatorname{Coh}_{X^{an}}$ is an equivalence.*

Quite similarly to the previous exercise, we see that $f : \mathbb{A}^1_{\mathbb{C}} \to \operatorname{Spec}\mathbb{C}$ is a suitable counterexample for both, since

1. $(R^0 f_* \mathcal{O}_{\mathbb{A}^1_{\mathbb{C}}})^{\text{an}} \cong \mathbb{C}[T] \to R^0 f_*^{\text{an}} \mathcal{O}_{\mathbb{A}^{1,\text{an}}_{\mathbb{C}}} \cong \{\text{globally defined holomorphic functions}\}$ is not an isomorphism - for instance $g(z) = e^z$ is not polynomial (note that in this situation we used that $\operatorname{Spec}\mathbb{C}$ is isomorphic to its analytification as a locally ringed space, so $(-)^{\text{an}}$ is the identity functor).

2. $(-)^{\text{an}} : \operatorname{Coh}_{\mathbb{A}^1_{\mathbb{C}}} \to \operatorname{Coh}_{\mathbb{A}^{1,\text{an}}_{\mathbb{C}}}$ is not an equivalence because, for instance, (just as seen in the previous exercise) the (coherent) ideal sheaf $\mathcal{I} \subset \mathcal{O}_{\mathbb{A}^{1,\text{an}}_{\mathbb{C}}}$ generated by the global section $\sin(2\pi i z)$ is not algebraic, since otherwise $\mathbb{Z} \subset \mathbb{A}^1_{\mathbb{C}}$ would be a closed subscheme.

Exercise (3.54)**.** If X, Y are abelian varieties over $\mathbb{C}$, then the $\mathbb{Z}$-module of morphisms $X \to Y$ of group schemes is finite-free.

Remark 3.53.3 right after this exercise shows us how to argue: any morphism of schemes $X \to Y$ which maps the unit in X to that of Y is necessarily a homomorphism of $\mathbb{C}$-group schemes - by the cube lemma discussed in the Jacobians of Curves seminar last semester - thus $\operatorname{Hom}_{\text{Grp-Sch}/\mathbb{C}}(X, Y)$ identitfies with the family of pointed morphisms of schemes $\operatorname{Hom}_{\text{Sch}}((X, 1_X), (Y, 1_Y))$. By identifying this last abelian group with the family of closed subschemes (and thus ideal sheaves) $\Gamma \subset X \times_{\mathbb{C}} Y$ such that the projection $\Gamma \to X$ is an isomorphism and the point $(1_X, 1_Y)$ lies in Γ, we see that (as the GAGA equivalence corresponds ideal sheaves with ideal sheaves) $\operatorname{Hom}_{\text{Grp-Sch}/\mathbb{C}}(X, Y) \cong \operatorname{Hom}_{\text{Grp-c.a.s.}/\mathbb{C}}(X^{\text{an}}, Y^{\text{an}})$ and $X^{\text{an}}, Y^{\text{an}}$ are isomorphic to quotients $V_X/\Lambda_X, V_Y/\Lambda_Y$ where V_X, V_Y are finite-dimensional complex vector spaces and $\Lambda_X \subset V_X, \Lambda_Y \subset V_Y$ are lattices. We now see that the group of such morphisms is realised as that of (complex-analytic) homomorphisms of abelian groups

$$V_X \to V_Y$$

such that the composition $V_X \to V_Y \to V_Y/\Lambda_Y$ factors over Λ_X - since Λ_X contains a basis of V_X and any continuous group homomorphism from V_X to V_Y is determined by its effect on a basis, we've recognised $\operatorname{Hom}_{\text{Grp-Sch}/\mathbb{C}}(X, Y)$ as group homomorphisms $\Lambda_X \to \Lambda_Y$, which of course form a finitely generated free abelian group because $\Lambda_X \cong \mathbb{Z}^{\oplus \dim V_X}, \Lambda_Y \cong \mathbb{Z}^{\oplus \dim V_Y}$.

Exercise (4.23)**.** Let $f : T \to S$ be a compactifiable map of locally compact spaces.

1. Show that the category $\mathcal{C}$ of factorisations $\{T \xrightarrow{j} T' \xrightarrow{f'} S\}$ of f into an open immersion and a proper map is filtered.
2. Show that the definition of $Rf_!$ is independent (up to isomorphism in $\mathcal{D}(S, \underline{\mathbb{Z}})$) of a relative compactification by using the proper base change theorem.
3. Let $g : W \to T$ be another compactifiable morphism. Use the proper base change theorem to prove that there exists a natural isomorphism
$$Rf_! \circ Rg_! \cong R(f \circ g)_!.$$

1. Suppose $f' : T' \to S$ and $f'' : T'' \to S$ are two proper spaces over S endowed with open subsets $j' : T \to T', j'' : T \to T''$ isomorphic to T such that $f' \circ j' = f'' \circ j'' = f$. One can then form the fibre product over S
$$\begin{tikzcd} T' \times_S T'' \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & T' \arrow[d, "f'"] \\ T'' \arrow[r, "f''"'] & S \end{tikzcd}$$
The open immersions j' and j'' provide a map $j : T \to T' \times_S T''$ which is also immersion by definition of the (relative) product topology on $T' \times_S T''$; if $Z \subseteq T' \times_S T''$ is the closure of the image of this map then $T \hookrightarrow Z$ is an open immersion. The composition $Z \to T' \to S$ is also proper since the pre-image along $\pi : T' \times_S T'' \to T'$ of any compact subset $C \subset T'$ can be described explicitly as the set
$$\pi^{-1}(C) = \{(u, v) \in T' \times T'' \mid f'(u) = f''(v),\, u \in C\} \approx C \times_S {f''}^{-1}(f'(C))$$
which is quasi-compact since f'' is also proper, and of course intersecting with the closed subset Z also yields a quasi-compact subset.
2. As above, suppose $f = f' \circ j' = f'' \circ j''$ are two factorisations of f as a proper map together with an open immersion, and as argued (up to replacing either one with their fibre product) we may assume there exists a map between the compactifications
$$\begin{tikzcd} & T' \arrow[dd, "g"] \arrow[dr, "f'"] & \\ T \arrow[ur, "j'"] \arrow[dr, "j''"'] & & S \\ & T'' \arrow[ur, "f''"'] & \end{tikzcd}$$
and thus the two (a priori different) exceptional pushforwards of a complex $K^\bullet \in \mathcal{D}(T, \underline{\mathbb{Z}})$ are
$$\begin{aligned} &Rf''_* j''_! K^\bullet, \text{ and} \\ &Rf'_* j'_! K^\bullet \cong Rf''_* Rg_* j'_! K^\bullet \in \mathcal{D}(S, \underline{\mathbb{Z}}). \end{aligned}$$
To conclude, the goal is to compare the complexes $A_1 := j''_! K^\bullet$ and $A_2 := Rg_* j'_! K^\bullet$ in $\mathcal{D}(T'', \underline{\mathbb{Z}})$. We can apply excision respect to the open subset $T \subset T''$ to the complex A_2 to obtain the distinguished triangle
$$j''_! {j''}^* Rg_* j'_! K^\bullet \to Rg_* j'_! K^\bullet \to Ri''_* {i''}^* Rg_* j'_! K^\bullet \to j_! j^* Rg_* j'_! K^\bullet[1]$$
where $i'' : T'' \setminus T \hookrightarrow T''$ is the complementary open immersion. The first term in the above sequence is given much more explicitly by
$$j''_! {j''}^* Rg_* j'_! K^\bullet \cong j''_! {j'}^* j'_! K^\bullet \cong j''_! K^\bullet = A_1$$

so all that there is to show is the vanishing of the rightmost complex

$$Ri''_* i''^* Rg_* j'_! K^\bullet$$

which is supported on the closed subset $C'' := T'' \setminus T$. If we set $C' := T' \setminus T$ then we have a cartesian diagram

$$\begin{tikzcd} C' \arrow[r] \arrow[d, "g_{|C'}"'] \arrow[dr, phantom, "\ulcorner"] & T' \arrow[d, "g"] \\ C'' \arrow[r] & T'' \end{tikzcd}$$

where g is proper because $T' \to T''$ was chosen as the base change along $T'' \to S$ of another compactification of $T \to S$. Applying the proper base change theorem to $j'_! K^\bullet$ we see that

$$i''^* Rg_* j'_! K^\bullet \cong Rg_{|C'*} i'^* j'_! K^\bullet \cong 0$$

since $i'^* j'_! K^\bullet \cong 0$ as can be checked on stalks.

Footnote 25 *Is any separated map of locally compact spaces $f : T \to S$ compactifiable in this sense, say by some relative one-point or Stone-Čech compactification?*

I've struggled to find a smarter way of doing this, so I'll just describe what I think is the 'direct' solution, and check that the construction goes through.

To *relativise* the one-point compactification construction, we're forced to require the following properties:

1. as a set, the relative one-point compactification should be $X = T \cup S$,
2. the inclusion $T \to T \cup S = X$ should be an open immersion, and the map $g : T \cup S \to S$ given by $g_{|T} = f, g_{|S} = \mathrm{id}_S$ should be continuous and proper.

These conditions imply $S \subseteq X$ is a closed subset and $g_{|S} : S \to S$ is a homeomorphism (since it's closed, continuous and bijective), and all subsets $K \subseteq X$ which are proper over S must be closed - in analogy to the requirement that compact subsets of Hausdorff spaces are closed, and thus all compact $K \subseteq Y$ are closed in the one-point compactification $\overline{Y}$ (the difference here being that K doesn't necessarily have to be contained in T). Note that, if $K \subseteq X$ is proper over S then $K \cap S \subseteq S$ is a closed subset, since $g_{|S} = \mathrm{id}$ and proper embeddings are precisely the closed ones.

This gives us essentially what is needed to define a topology on X: we introduce the family of subsets

$$\{U \subset T \text{ open}\} \cup \{X \setminus K \mid K = C_1 \cup C_2,\ \ C_1 \subseteq T \text{ proper over } S,\ \ C_2 \subseteq S \text{ closed}\} \subseteq \mathcal{P}(X).$$

A simple check shows that this indeed defines a topology on X, that $T \hookrightarrow X$ is an open immersion and $X \to S$ is closed and continuous (the image of the complement of any of the subsets in the above list is either the whole of S or closed by definition).

Slightly more involved is that g's fibres is compact: for any $s \in S$, and open covers $f^{-1}(s) \cup \{s\} = \bigcup_\alpha U_\alpha$ where $U_\alpha = U'_\alpha \cap g^{-1}(s)$ for opens $U'_\alpha \subseteq X$, if α is such that $s \in U_\alpha$ then necessarily $U'_\alpha = X \setminus K$ is of the second form in the description of X's topology above, and $C_1 \cap f^{-1}(s) \subseteq f^{-1}(s)$ is a compact subset (because $C_1 \to S$ is proper). We thus have a finite refinement $C_1 \subseteq \bigcup_\alpha U_\alpha \implies C_1 \subseteq U_{\alpha_1} \cup \ldots \cup U_{\alpha_N} \implies g^{-1}(s) = U_\alpha \cup U_{\alpha_1} \cup \ldots \cup U_{\alpha_N}$.

Lastly, we have to check that g is separated. This is automatic because, by the conclusion in the paragraph above, each of g's fibres has to be the one-point compactification of f's corresponding fibre; since the latter is Hausdorff by f's separatedness and S and T are locally compact (and thus so are the fibres $T \to S$) this is equivalent to the Hausdorffness of its one-point compactification.

Exercise (4.25)**.** Let T be a locally compact Hausdorff space, $T = U \cup V$ an open cover and $\mathcal{F} \in \mathrm{Sh}_{\mathrm{Ab}}(T)$. Then there exists a natural long exact sequence

$$\ldots \to H^n_c(U \cap V, \mathcal{F}_{|U\cap V}) \to H^n_c(U, \mathcal{F}_{|U}) \oplus H^n_c(V, \mathcal{F}_{|V}) \to H^n_c(T, \mathcal{F}_{|T}) \to \ldots$$

If S is any locally compact topological space and $A \subseteq S$ is an open subset, then the one-point compactification $\overline{A} = A \cup \{\infty\}$ can be realised as the open subset $A \cup \{\infty\} \subseteq \overline{S} = S \cup \{\infty\}$ with its natural subspace topology; thus, if $j_{\overline{A}} : \overline{A} \hookrightarrow \overline{S}, j_S : S \hookrightarrow \overline{S}$ are the corresponding open immersions, then by independence on the compactification of compactly supported cohomology the equality

$$R\Gamma_c(A, \mathcal{F}) = R\Gamma(\overline{A}, j_{A,!}\mathcal{F}) = R\Gamma(\overline{S}, j_{\overline{A},*}j_{A,!}\mathcal{F})$$

follows, where $\mathcal{F} \in \mathrm{Sh}_{\mathrm{Ab}}(A)$ is arbitrary.
Having made these observations, the short exact sequence of sheaves on $\overline{T}$

$$0 \to j_{\overline{U\cap V},*}j_{U\cap V,!}\mathcal{F}_{|U\cap V} \to j_{\overline{U},*}j_{U,!}\mathcal{F}_{|U} \oplus j_{\overline{V},*}j_{V,!}\mathcal{F}_{|V} \to j_!\mathcal{F} \to 0$$

yields, by means of the long exact sequence in cohomology, the desired Mayer-Vietoris sequence for the cover $T = U \cup V$.

Exercise (4.31)**.** Let $X \to \operatorname{Spec}\mathbb{C}$ be a locally of finite type $\mathbb{C}$-scheme and let $Z \subseteq X$ be a closed subscheme. Let $f : Y = \mathrm{Bl}_Z X \to X$ be the blow-up of X in Z and let $E \subseteq Y$ be the exceptional divisor. Show the existence of a natural long exact sequence

$$\ldots \to H^n(X^{\mathrm{an}}, \underline{\mathbb{Z}}) \to H^n(Y^{\mathrm{an}}, \underline{\mathbb{Z}}) \oplus H^n(Z^{\mathrm{an}}, \underline{\mathbb{Z}}) \to H^n(E^{\mathrm{an}}, \underline{\mathbb{Z}}) \to \ldots$$

Since we have the equality $f^{\mathrm{an},-1}\underline{\mathbb{Z}}_{X^{\mathrm{an}}} = \underline{\mathbb{Z}}_{Y^{\mathrm{an}}}$, the unit of the adjunction $Lf^{\mathrm{an},-1} = f^{\mathrm{an},-1} \dashv Rf^{\mathrm{an}}_*$ yields a morphism

$$\underline{\mathbb{Z}}_{X^{\mathrm{an}}} \to Rf^{\mathrm{an}}_*\underline{\mathbb{Z}}_{Y^{\mathrm{an}}}$$

which, by applying the proper base change theorem on the cartesian diagram of $\mathbb{C}$-schemes

$$\begin{tikzcd}
\mathrm{Bl}_Z X^{\mathrm{an}} \setminus E^{\mathrm{an}} \arrow[r] \arrow[d, "\cong"'] \arrow[dr, phantom, "\ulcorner"] & \mathrm{Bl}_Z X^{\mathrm{an}} \arrow[d, "f"] \\
X^{\mathrm{an}} \setminus Z^{\mathrm{an}} \arrow[r] & X^{\mathrm{an}}
\end{tikzcd}$$

yields, upon restriction to $X^{\mathrm{an}} \setminus Z^{\mathrm{an}}$

$$\underline{\mathbb{Z}}_{X^{\mathrm{an}}|X\setminus Z} \cong \underline{\mathbb{Z}}_{X^{\mathrm{an}}\setminus Z^{\mathrm{an}}} \to (Rf^{\mathrm{an}}_*\underline{\mathbb{Z}}_{Y^{\mathrm{an}}})_{|X\setminus Z} \cong Rf^{\mathrm{an}}_{|X\setminus Z,*}\underline{\mathbb{Z}}_{Y\setminus E} \cong \underline{\mathbb{Z}}_{|X\setminus Z}$$

since $f_{|Y\setminus E}$ is an isomorphism. On the other hand, restricting to Z^{an} we see

$$\underline{\mathbb{Z}}_{Z^{\mathrm{an}}} \to (Rf^{\mathrm{an}}_*\underline{\mathbb{Z}}_{Y^{\mathrm{an}}})_{|Z} \cong Rf^{\mathrm{an}}_{|E,*}\underline{\mathbb{Z}}_{E^{\mathrm{an}}}$$

once again, by the proper base change theorem; TO BE FINISHED OFF

Exercise. Check that the homological dimension of $\mathbb{R}^d$ is precisely d.

The compactly supported singular cohomology is non-zero in degree d - the one-point compactification of $\mathbb{R}^d$ is homeomorphic to the d-sphere S^d via stereographic projection, and we have a short exact sequence of sheaves on S^d given by

$$0 \to j_! \underline{\mathbb{Z}}_{\mathbb{R}^d} \to \underline{\mathbb{Z}}_{S^d} \to i_{\infty,*}\underline{\mathbb{Z}} \to 0$$

where $i_\infty : \{\infty\} \hookrightarrow S^d$ is the inclusion of $\mathbb{R}^d \subset S^d$'s complement. Looking at the long exact sequence in cohomology, we see that in the top degree

$$H^d_c(\mathbb{R}^d, \underline{\mathbb{Z}}) \to H^d(S^d, \underline{\mathbb{Z}}) \cong \mathbb{Z} \to 0 = H^d(*, \underline{\mathbb{Z}})$$

which shows that $\dim \mathbb{R}^d \geq d$.

Exercise (4.40)**.** For a morphism $Y \to T$ of topological spaces let $\underline{Y}_T$ be the sheaf $U \mapsto \mathrm{Hom}_T(U, Y)$. Show that the functor

$$\{\text{local isomorphisms } Y \to T\} \to \mathrm{Sh}(T), Y \mapsto \underline{Y}_T$$

is an equivalence (the morphisms of the left hand side are morphisms over T), and its inverse maps $\mathcal{P}$ to P as described in Lemma 4.39.

As for essential surjectivity, in the mentioned lemma to a sheaf $\mathcal{F} \in \mathrm{Sh}(T)$ a local isomorphism $F \to T$ is associated, whose corresponding 'functor of points' sheaf recovers $\mathcal{F}$.
To argue faithfulness, suppose $f : Y_1 \to Y_2$ is a morphism of covering spaces over T and let $y \in Y_1$ be an arbitrary point; if $U \subseteq Y_1$ is an open subset such that the restriction of the projection $U \to T$ is a homeomorphism onto its image V, then the inverse $g : V \xrightarrow{\cong} U$ is a section in $\underline{Y_1}_T(V)$ which is sent to $f \circ g : V \to f(U) \subset Y_2$. $f \circ g$ can be evaluated at the point given by the image of y in T, yielding the image of y along $f \implies f$ is completely determined by $\underline{Y_1}_T \to \underline{Y_2}_T$.
Lastly, if $\phi : \underline{Y_1}_T \to \underline{Y_2}_T$ is any morphism of sheaves and $y \in Y_1$ is an arbitrary point together with $y \in U \subset Y_1$ a trivialising neighbourhood, then once again the inverse $g : V \xrightarrow{\cong} U$ is a section in $\underline{Y_1}(V)$, sent to a continuous map $\phi(g) : V \to Y_2$ compatible with the canonical maps to T. We can thus define $f : Y_1 \to Y_2$ by setting $f(y) := \phi(g)(t)$ where t is $y \in Y_1$'s image in T. f of course induces ϕ at the level of functors of points and is a continuous map of topological spaces because this can be checked locally on Y_2 and both Y_1 and Y_2 are locally isomorphic to T - if $Y_2 \cong T$ then the induced map is just the canonical projection to T, hence f is 'locally' this projection and thus continuous.

Exercise (4.42)**.** Let T be a locally compact space and $f : T \times [0,1] \to T$ the projection. Then for any $K \in \mathcal{D}^+(T, \underline{\mathbb{Z}})$ the narual map

$$K^\bullet \to Rf_* f^* K^\bullet$$

is an isomorphism.

By the canonical truncation triangle we can of course assume $K^\bullet \cong \mathcal{F}$ is concentrated in degree zero and identifies with a sheaf of abelian groups (passing from a potentially unbounded complex $K^\bullet$ to a bounded one can be done by realising $K^\bullet$ as a limit of its finite truncations, and noticing that the above morphism, being the unit of the adjunction $f^* \dashv Rf_*$, is functorial and commutes with limits of complexes). f is of course proper since it's the base change along $T \to \{*\}$ of the map $[0,1] \to \{*\}$ which is proper by $[0,1]$'s compactness. This allows us to compute the stalks for the described map directly: for any $x \in T$ we have

$$\mathcal{F}_x \to (Rf_* f^* \mathcal{F})_x \cong R\Gamma(\{x\} \times [0,1], f^*_{|\{x\} \times [0,1]} \mathcal{F}_x) \cong R\Gamma([0,1], \underline{\mathcal{F}_x}) \cong \mathcal{F}_x$$

since $\{x\} \times [0,1] \cong [0,1]$ and Lemma 4.41 applies.

Exercise (4.43)**.** Let T be a locally compact Hausdorff space, which is a finite union of open contractible subsets, e.g. a compact real manifold. Prove that for each $i \geq 0$ the group $H^i(T, \underline{\mathbb{Z}})$ is finitely generated.

We argue by descending induction on i, the base case being for $i >> d = \dim X$ (note that T is finite dimensional since it is locally compact Hausdorff). If we suppose then that $H^{i+1}(T, \underline{\mathbb{Z}})$ is finitely generated for any T satisfying the hypothesis, to argue the same for $H^{i-1}(T, \underline{\mathbb{Z}})$ we proceed by induction on the number of contractible open subsets needed to cover T. We may appeal to the Mayer-Vietoris sequence: expressing T as $T = U \cup V$ where $U \subseteq T$ is a contractible open subset and $V \subseteq T$ can be covered by just $n-1$ contractible opens, we have the long exact sequence

$$\ldots \to H^i(U \cap V, \underline{\mathbb{Z}}) \to H^i(V, \underline{\mathbb{Z}}) \to H^i(T, \underline{\mathbb{Z}}) \to H^{i+1}(U \cap V, \underline{\mathbb{Z}}) \to \ldots$$

which shows that $H^i(T, \underline{\mathbb{Z}})$ is finitely generated since $H^i(V, \underline{\mathbb{Z}})$ is by our second induction hypothesis, and $H^{i+1}(U \cap V, \underline{\mathbb{Z}})$ is by our first (note that $U \cap V$ can also be covered by finitely many contractible opens by some point-set topological argument which I'm still trying to think of...).

Exercise (4.44)**.** Let T be a finite CW-complex. Show that for each $i \geq 0$ the group $H^i(T, \underline{\mathbb{Z}})$ is finitely generated.

By hypothesis, T admits a finite filtration by its cells:

$$T_0 \subseteq T_1 \subseteq T_2 \subseteq \ldots \subseteq T_{n-1} \subseteq T_n = T.$$

By definition of the pushout topology, the subset $U := T \setminus T_{n-1} \subseteq T$ is open and homeomorphic to a disjoint union of n-dimensional open discs. By applying excision we may induct on n: if $j : U \hookrightarrow T$ is the corresponding open immersion and $i : T_{n-1} \hookrightarrow T$ its complementary closed immersion, then we have an exact sequence of sheaves on T

$$0 \to j_! \underline{\mathbb{Z}}_U \to \underline{\mathbb{Z}} \to i_* \underline{\mathbb{Z}} \to 0$$

which yields the exact sequence on singular cohomology groups

$$\ldots \to H^i_c(U, \underline{\mathbb{Z}}) \to H^i(T, \underline{\mathbb{Z}}) \to H^i(T_{n-1}, \underline{\mathbb{Z}}) \to \ldots$$

(the left term being compactly supported cohomology since T is compact) which yields our claim, as $H^i_c(U, \underline{\mathbb{Z}})$ is non-zero for $i \neq 0, n$ and is otherwise finitely generated (by the explicit computation of compactly supported cohomology of $\mathbb{R}^n$) and the induction hypothesis applies to T_{n-1}.

Exercise (5.4)**.** Let X be a complex analytic space and define étale sheaves on X exactly as in Definition 5.2. Show that $\mathrm{Sh}(X_{\text{ét}})$ is equivalent to the usual category of sheaves on $|X|$.

Exercise (5.32)**.**

1. Let $* \in \mathcal{C}$ be a terminal object and assume that the canonical morphism $Y \to *$ is a monomorphism. Then the natural maps

$$\mathcal{G} \to j^* j_! \mathcal{G}, \;\; j^* j_* \mathcal{G} \to \mathcal{G}$$

are isomorphisms for any $\mathcal{G} \in \mathrm{Sh}(\mathcal{C})$.