EDHEC BUSINESS SCHOOL

FEN_842 Risk Measurement

Lecture 3
Risk Measure Estimation

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Make an impact

Outline of lecture 3

Risk measure estimation

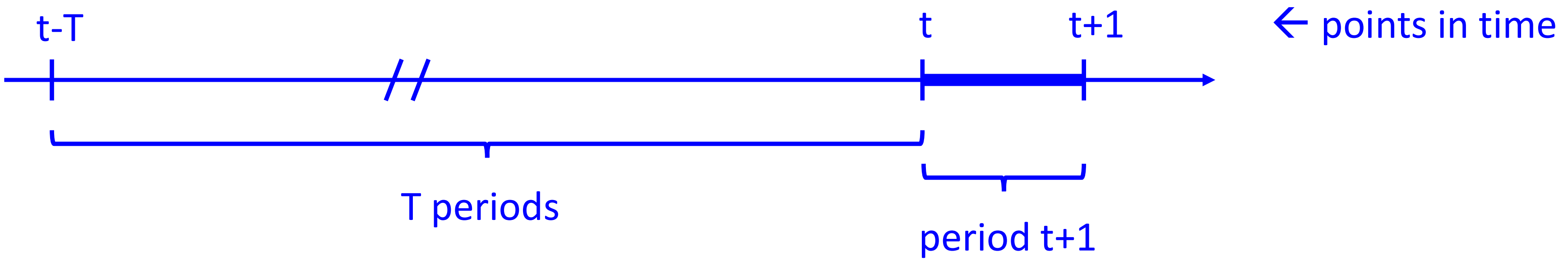
- **Pre-requisites**
- Volatility estimation : Equally Weighted
GARCH
EWMA
- Parametric VaR & ES
- Non-parametric VaR & ES : Historical Simulation
Filtered HSim
Weighted HSim
- Monte Carlo Simulation
- Recap, ex-ante vs ex-post

Individual series estimates

Various parameters

At time t, estimate the risk measure for period *t*+1

Time line : distinguish between points in time & time periods



To be determined :

- **forecast horizon :** length of period *t*+1
- **lookback period :** length of estimation period *T*
- **return / observation interval :** length of period over which returns are calculated
- how to use the observations within *T*

Security returns

Arithmetic vs geometric

- **total returns** : price appreciation plus cash disbursements

compounding :

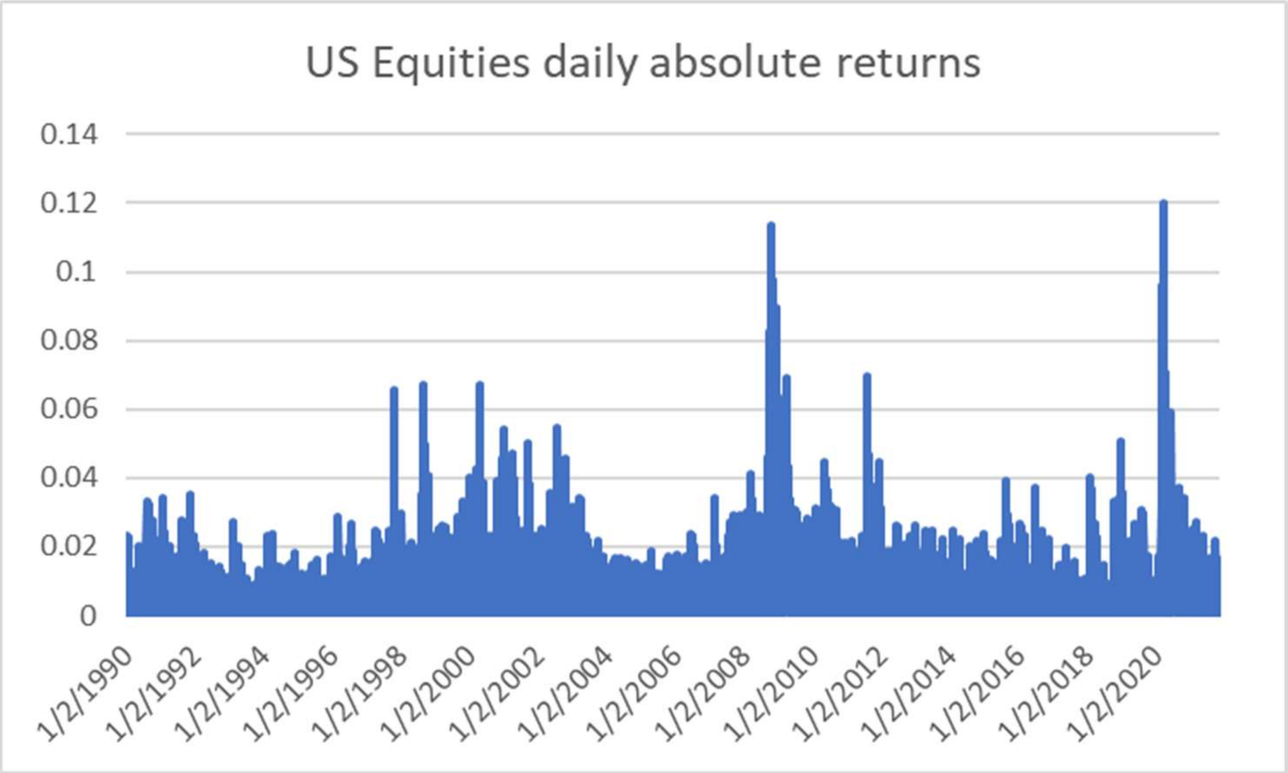
- discrete / arithmetic / percentage returns : $r_t = \dfrac{P_{t+1} - P_t + Div_t}{P_t}$
- continuous / geometric / log returns : $R_t = \ln(1 + r_t)$

- continuously compounded returns are mostly used in derivatives context : "continuous time", log returns aggregate **over time (TS)**
- discrete returns should be used in a portfolio context : aggregate **in cross-section (XS)**

- if returns are small (daily, e.g.), the difference is small,
  but henceforth we always use **discrete returns** so as to be consistent with a portfolio context

The variability of financial market returns

- Kenneth French data library, US equity market factor
- daily total returns, 2-Jan-1990 – 29-Oct-2021 (8,021 observations)
- we plot the absolute values of the returns, $|r|$, as a proxy for **instantaneous variability**



Q : *what do we observe ?*

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Volatility, equally-weighted (EW)

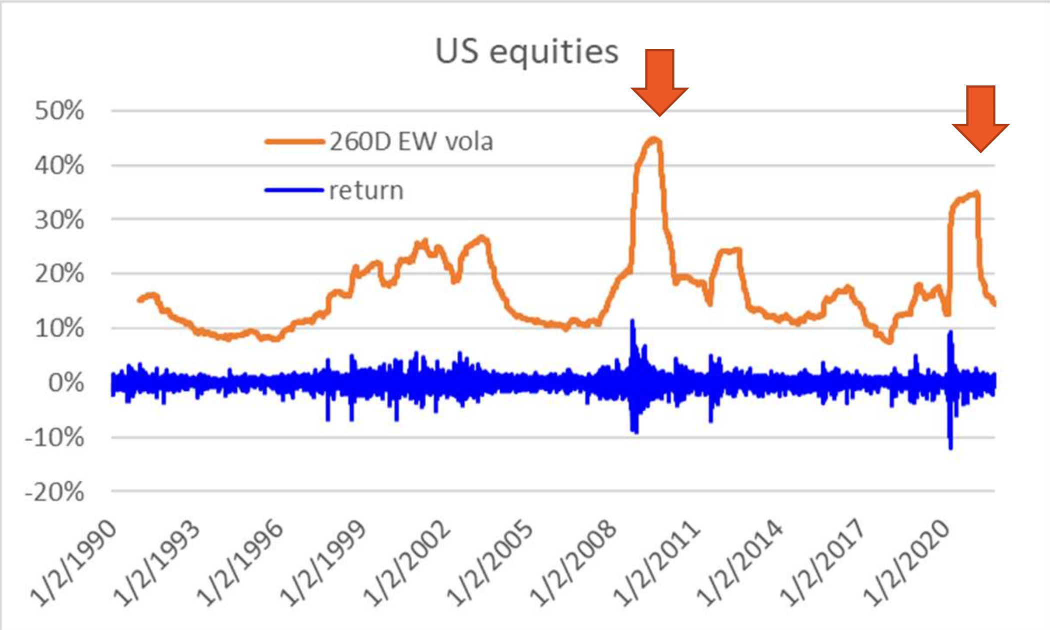
- estimate volatility from variance
- security characteristics do not change over time → historical observations are representative
- EW variance : $\sigma_{t+1}^2 = \sum_{\tau=1}^{T} w_\tau (r_\tau - \bar{r})^2$ where : $\bar{r} = \sum_{\tau=1}^{T} w_\tau r_\tau$ and : $\boxed{w_\tau = \frac{1}{T}, \forall \tau \in T}$

  proper weights : $\sum_{\tau=1}^{T} w_\tau = 1$
- iff estimated mean $\bar{r}$ is used, so apply **Bessel correction** : multiply variance with $\dfrac{T}{T-1}$
  → relevant if T is small
- for daily returns : **ignore mean return**, hence no Bessel correction
  1. mean return notoriously difficult to estimate → introduces estimation noise in variance
  2. daily mean return is very small anyway

Volatility EW

Example

- Kenneth French data library, US equity market factor
- daily total returns, 2-Jan-1990 – 29-Oct-2021 (8,021 observations)
- T=260 : 1-year EW volatility, no mean-adjustment, annualized



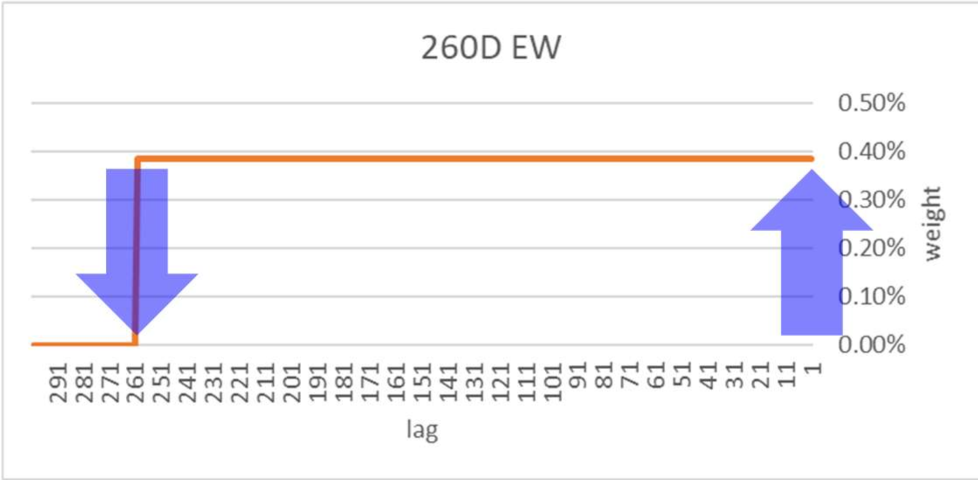
- volatility changes over time
- Global Financial Crisis (GFC) Aug2008 and covid crisis Mar2020 stand out
- vola plummets one year later
- **always first plot the return series !**

Volatility EW

Drawbacks

EW, so each week :

- a new observation is added
- the oldest observation drops out of the estimation window



Hence, the EW estimator has two serious shortcomings :

1. it is not adaptive in the front end ⇔ there is **no information decay**
   each week has a constant weight of 1/260 = 0.38%
   → *assign more weight to more recent observations*

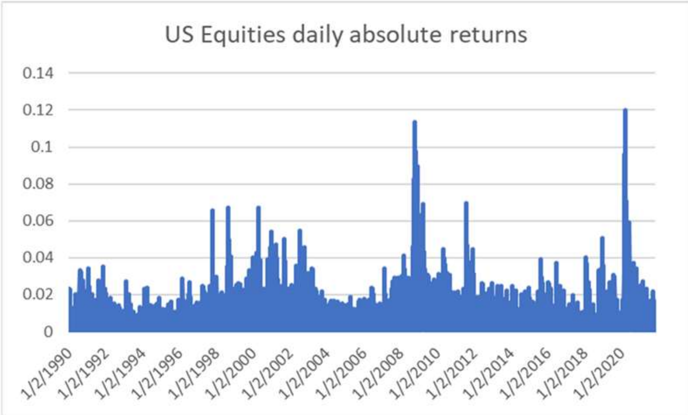
2. it suffers from the **ghost effect** in the rear end :
   1Y after the GFC & covid crises we observe “miraculous” decreases in vola estimates…
   → *let the weight of past observations decrease smoothly over time*

Financial markets volatility

Towards an adequate ex-ante vola estimator

Stylized facts about financial market volatility :

1. time-varying → adaptive, responsive risk measure
2. time clustering → persistence in a risk measure
3. long-run mean-reversion → high (low) volas tend to go down (up) to some long-run volatility level



This suggests the following **desirable properties** of a volatility measure :

1. **short-term adaptiveness** : allowing for timely risk assessments, especially when risks surge
2. **volatility clustering** : shorter-run auto-regressive behavior, "stickyness"
3. **long-run mean reversal** : for not forgetting about the mean-reversion level, ("long memory" property), especially in tranquil periods

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ARCH variance models

Variance modeling

- ARCH : AutoRegressive Conditional Heteroskedasticity, Engle [1982]
- based on two observations :

  heteroskedasticity : variance changes over time → hence conditional estimate

  squared returns $\{r_t^2\}$ are serially correlated → autoregressive process

- this suggests that the cond'l variance can be modeled as an autoregressive process
- this gives the **ARCH(q)** process :

$$\mathrm{var}(r_{t+1}|t) = \sigma_{t+1}^2 = \alpha_0 + \alpha_1 \cdot r_t^2 + \alpha_2 \cdot r_{t-1}^2 + \ldots + \alpha_q \cdot r_{t+1-q}^2 \qquad \textit{with} \quad \sum_{i=0}^{q} \alpha_i = 1$$

- empirical applications : many lags are required (i.e. large q) → computational problems
- hence, moving average (MA) terms were added to subsume higher order AR terms

Note : We here assume that the conditional mean return is zero. In case of non-zero mean, square mean-adjusted returns.

GARCH(1,1) variance

Volatility work horse

- benchmark conditional volatility model in academic finance literature
- GARCH : Generalized Autoregressive Conditional Heteroskedasticity, Bollerslev [1986]

- **GARCH(p,q)** : q = # auto-regressive lags, ARCH terms : past squared returns → $\alpha_{i=1,\ldots,q}$
  p = # moving-average lags, GARCH terms : past cond'l variances → $\beta_{j=1,\ldots,p}$

- p = q = 1 performs very well in financial markets → **GARCH(1,1)**

Next period's **conditional variance** is weighted average of 3 terms :

$$\mathrm{var}(r_{t+1}|t) = \sigma_{t+1}^2 = \gamma \cdot \sigma_L^2 + \alpha \cdot r_t^2 + \beta \cdot \sigma_t^2 \quad \textit{with} \quad \boxed{\gamma + \alpha + \beta = 1}$$

where σ_L^2 is the long-run or **unconditional variance**

Note : We here assume that the conditional mean return is zero. In case of non-zero mean, square mean-adjusted returns.

GARCH(1,1) volatility

Details

GARCH(1,1) : $\boxed{\sigma_{t+1}^2 = \gamma \cdot \sigma_L^2 + \alpha \cdot r_t^2 + \beta \cdot \sigma_t^2}$

For **estimating parameters**, conditional variance equation is rewritten as :

$$\sigma_{t+1}^2 = \omega + \alpha \cdot r_t^2 + \beta \cdot \sigma_t^2$$

Use **Law of Iterated Expectations** to find unconditional variance :

$$E\left(\sigma_{t+1}^2\right) = \omega + \alpha \cdot E\left(r_t^2\right) + \beta \cdot E\left(\sigma_t^2\right) \qquad \Rightarrow \quad \sigma_L^2 = \omega + \alpha \cdot \sigma_L^2 + \beta \cdot \sigma_L^2$$

Solving yields : $\sigma_L^2 = \dfrac{\omega}{1 - \alpha - \beta}$

so we require $\boxed{\alpha + \beta < 1}$ otherwise the long-run variance will explode

GARCH(1,1) volatility

Explanation of parameters or “weights”

$$\sigma^2_{t+1|t} = \gamma \cdot \sigma^2_L + \alpha \cdot r^2_t + \beta \cdot \sigma^2_t \quad with \quad \gamma + \alpha + \beta = 1$$

- **mean-reversion** driven by $\gamma = 1 - (\alpha + \beta)$: the higher γ, the stronger the mean-reversion

- hence, $\alpha + \beta$ relates to “persistence” or the continuation of differences from the LT variance,
  in practice : $\alpha + \beta < 1$ but close to unity

- β indicates persistence or **stickyness**,
  in the sense that a high β makes conditional variance and deviations from LT variance very persistent → less variation of conditional variance over time
  β is usually large, close to unity

- α is a responsiveness parameter, controls **updating** from innovations
  if α is large, then a large return causes an immediate large change in conditional variance
  → more variation of conditional variance over time

GARCH(1,1) volatility

Maximum Likelihood estimation of parameters

Non-linear model → maximum likelihood estimation → numerical optimization

Assume that returns r_t are zero-mean and **normally distributed** conditional on variance

Likelihood of observing specific return :

$$\frac{1}{\sqrt{2\pi\sigma_t^2}}\exp\left(\frac{-r_t^2}{2\sigma_t^2}\right)$$

of observing all T independent returns :

$$\prod_{t=1}^{T}\frac{1}{\sqrt{2\pi\sigma_t^2}}\exp\left(\frac{-r_t^2}{2\sigma_t^2}\right)$$

Maximizing Likelihood $\Leftrightarrow$ maximizing Log Likelihood, hence we estimate the parameters by :

$$\max_{\{parameters\}}\ \sum_{t=1}^{T}\left[-\ln(\sigma_t^2)-\frac{r_t^2}{\sigma_t^2}\right]$$

GARCH(1,1) parameter estimation

Example in Excel

use in Solver

use in conditional variance

For estimating parameters, write conditional variance equation as : $\sigma_{t+1}^2 = \omega + \alpha \cdot r_t^2 + \beta \cdot \sigma_t^2$

	A	B	C	D	E	F
1			parameters :	scaled	unscaled	
2			omega	0.018930	0.000002	
3			alpha	0.105427	0.105427	
4			beta	0.087828	0.878278	
5						
6						sum :
7						67398.12
8	date	day	return	return^2	cond'l var	LogLikelih
9	1/2/1990	1	0.01466	0.00021	0.00021	
10	1/3/1990	2	-0.00034	0.00000	0.00021	8.4522
11	1/4/1990	3	-0.00684	0.00005	0.00019	8.3252
12	1/5/1990	4	-0.00824	0.00007	0.00017	8.2696
	:	:	:	:	:	:
	:	:	:	:	:	:
8028	10/28/2021	8020	0.01140	0.00013	0.00006	7.4671
8029	10/29/2021	8021	0.00220	0.00000	0.00006	9.5711

=D2/10,000

=D4*10

Solver :
maximize F7
by changing cells D2:D4

squared returns

=sum(F10:F8029)

initialize : $\sigma_1^2 = r_1^2$

=E$2+E$3*D9+E$4*E9

calculate log likelihood F from D & E

GARCH(1,1) volatility

US equity data example cont'd

- from estimated parameters :

LT var	alpha	beta
0.00012	0.105	0.878

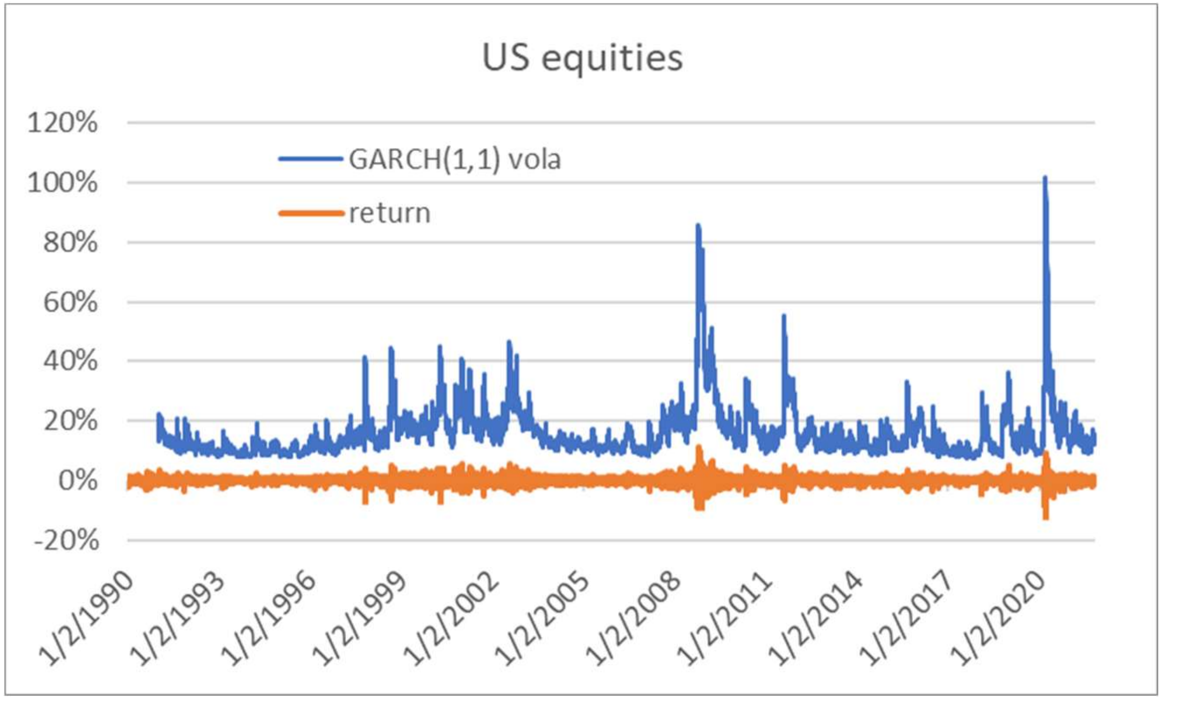
- alpha + beta = 0.984

  → persistent variance

  → **LT vola** = 17.38% p.a.

- averaging cond'l variances gives 18.26% vola p.a.

- when calculating **full-sample vola** :
  no mean-adjustment : 18.48%



GARCH(1,1) for confidence intervals

Conditional normality ?

- use GARCH volatilities to construct 95% confidence intervals on daily returns

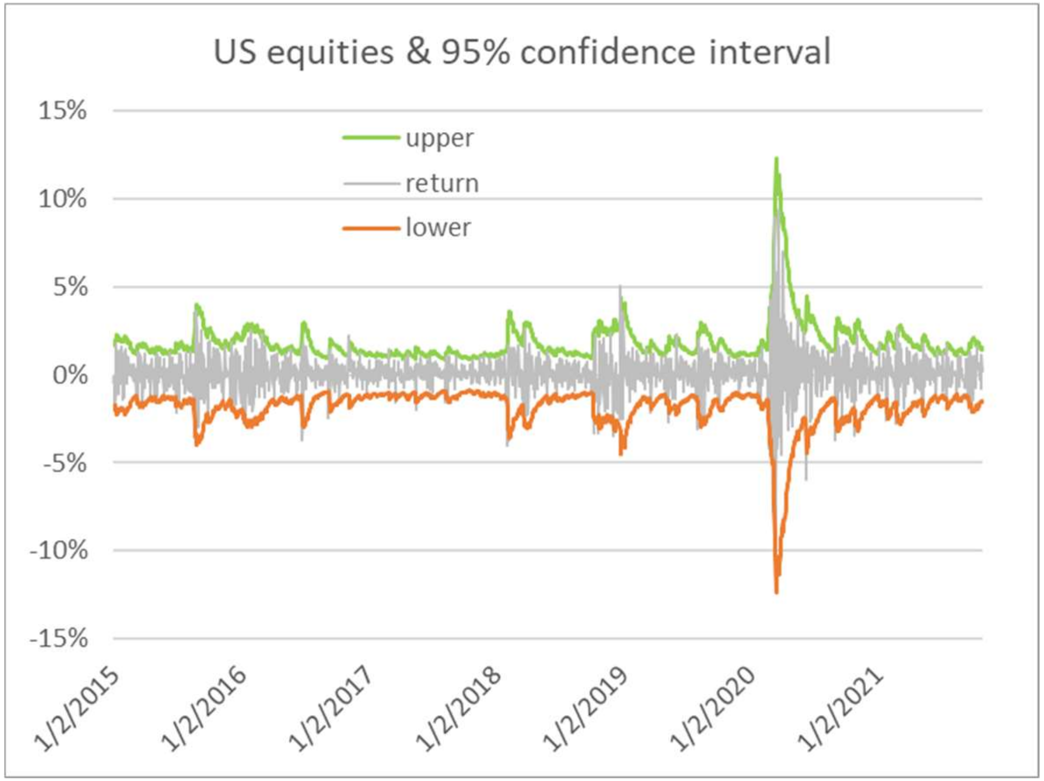
- assume : $\tilde{r}_{t+1} \sim N(0, \hat{\sigma}_{t+1|t})$

  so 95% confidence interval for r_{t+1} is

  $$\pm 1.96 \cdot \hat{\sigma}_{t+1|t}$$

- fraction of **exceedances** :
  upper : 2.20% ≈ 2.5% expected
  lower : 3.32% = too high

- reveals stylized fact #4 :
  volatility asymmetry :
  volatility is higher in declining markets than in rising markets



Time-varying volatility and fat tails

- time-varying volatility :
  each period, a return is drawn from a distribution with a different volatility
  → cf. **mixture distribution**
- this induces "**fat tails**" or **excess kurtosis**
- in our US equity data example, the overall excess kurtosis of daily returns is : **10.51**
- use GARCH(1,1) volatilities to scale returns = make ***z*-scores** : $\boxed{z_{t+1} = \dfrac{r_{t+1}}{\sigma_{t+1|t}}}$ $\sim N(0,1)$
- if GARCH captures time-varying volatility, then :
  1. the standard deviation of the z-scores should be close to 1 : stdev(z) = 1.00 !
  2. the excess kurtosis of the z-scores should be much lower : kurt(z) = **1.86**
- GARCH(1,1) does a pretty good job ! Note : we here evaluate GARCH estimates **in-sample** !

GARCH(1,1) as ARCH(∞)

A moving average of squared returns

- GARCH(1,1) is a recursive model
- expand : $\sigma_{t+1}^2 = \omega + \alpha \cdot r_t^2 + \beta \cdot \sigma_t^2$

substitute → $= \omega + \alpha \cdot r_t^2 + \beta \cdot \left[\omega + \alpha \cdot r_{t-1}^2 + \beta \cdot \sigma_{t-1}^2 \right]$

collecting terms → $= \omega\left[1+\beta\right] + \alpha \cdot \left[r_t^2 + \beta \cdot r_{t-1}^2 \right] + \beta^2 \cdot \sigma_{t-1}^2$

substitute → $= \omega\left[1+\beta\right] + \alpha \cdot \left[r_t^2 + \beta \cdot r_{t-1}^2 \right] + \beta^2 \cdot \left[\omega + \alpha \cdot r_{t-2}^2 + \beta \cdot \sigma_{t-2}^2 \right]$

collecting terms → $= \omega\left[1+\beta+\beta^2\right] + \alpha \cdot \left[r_t^2 + \beta \cdot r_{t-1}^2 + \beta^2 \cdot r_{t-2}^2 \right] + \beta^3 \cdot \sigma_{t-2}^2$ etc.

- note the specific pattern !

$$\sigma_{t+1}^2 = \frac{\omega}{1-\beta} + \alpha \cdot \sum_{i=0}^{\infty} \beta^i \cdot r_{t-i}^2$$

→ **exponential function of past squared returns**
= restricted ARCH(∞)

Evaluating GARCH(1,1)

From theory to practice ...

- benchmark model for conditional volatility in academic finance literature
- does fit data very well
- cumbersome to use in practice : requires 3 parameter estimates per security volatility;
  cumbersome in a multivariate context (covariance matrix)

solution :

- retain exponentially declining weights → information decay + no ghost effect
- ignore mean-reversion → ignore LT variance level
- let the coefficients for past squared return & variance sum to unity → one parameter only

→ **Exponentially Weighted Moving Average (EWMA)** volatility

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EWMA variance

Exponential weighting scheme

- exponentially declining weights of squared returns :

$$\sigma_{t+1}^2 \leftarrow r_t^2 + \lambda \cdot r_{t-1}^2 + \lambda^2 \cdot r_{t-2}^2 + \lambda^3 \cdot r_{t-3}^2 + \ldots \;\; = \sum_{i=0}^{\infty} \lambda^i \cdot r_{t-i}^2$$

- sum of weights : $\sum_{i=0}^{\infty} \lambda^i = \frac{1}{1-\lambda}$, hence normalize weights to sum to unity :

$$\sigma_{t+1}^2 = (1-\lambda) \cdot r_t^2 + (1-\lambda) \cdot \lambda \cdot r_{t-1}^2 + (1-\lambda) \cdot \lambda^2 \cdot r_{t-2}^2 + \ldots$$

$$\boxed{\sigma_{t+1}^2 = (1-\lambda) \cdot \sum_{i=0}^{\infty} \lambda^i \cdot r_{t-i}^2}$$

Note : We here assume that the conditional mean return is zero. In case of non-zero mean, square mean-adjusted returns.

EWMA variance

Make recursive

- EWMA variance : $\sigma_{t+1}^2 = (1-\lambda)\cdot\left[r_t^2 + \lambda\cdot r_{t-1}^2 + \lambda^2\cdot r_{t-2}^2 + \lambda^3\cdot r_{t-3}^2 + \ldots\right]$

subtract : $\lambda\cdot\sigma_t^2 = (1-\lambda)\cdot\left[\qquad \lambda\cdot r_{t-1}^2 + \lambda^2\cdot r_{t-2}^2 + \lambda^3\cdot r_{t-3}^2 + \ldots\right]$

to obtain : $\sigma_{t+1}^2 - \lambda\cdot\sigma_t^2 = (1-\lambda)\cdot r_t^2$

- or : $\boxed{\sigma_{t+1}^2 = (1-\lambda)\cdot r_t^2 + \lambda\cdot\sigma_t^2}$: simple recursive updating formula

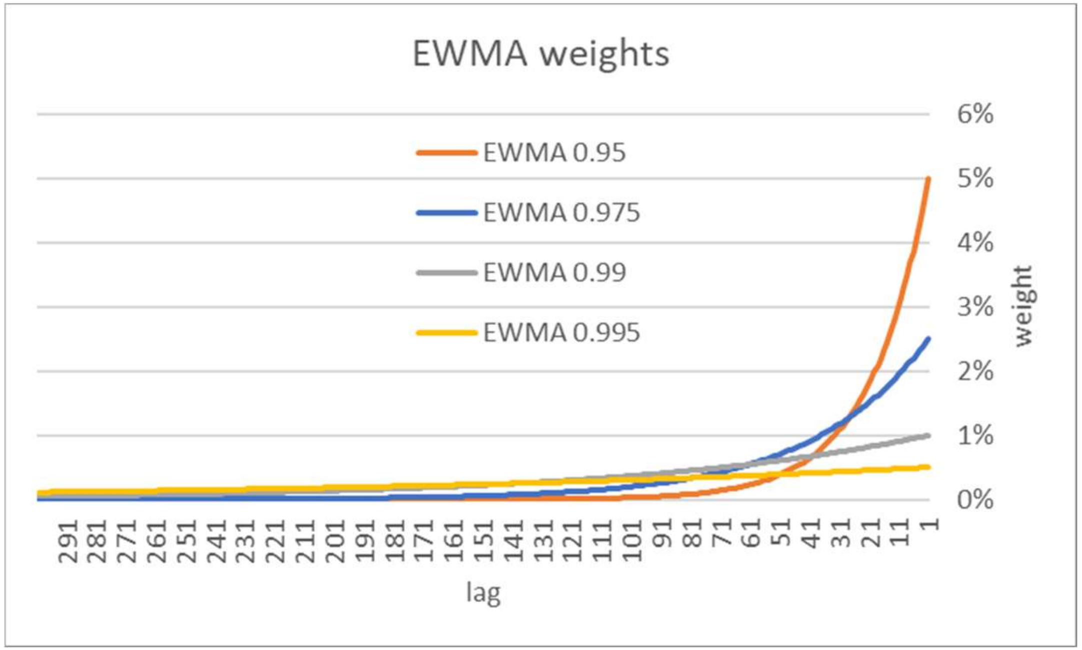
parameter λ :

- **persistence parameter** : higher λ implies higher variance persistence, sticky variance (new estimate stays close to previous), small impact / low weight of innovation
- decay parameter : higher λ implies slower decay, weights decrease only slowly, requires long lookback period

EWMA volatilities

The impact of high / low persistence λ

- compare persistence parameters :



Q : *what do you observe ?*

EWMA volatility

Determining the λ parameter

- use maximum likelihood, or minimum RMS of forecast errors to derive specific parameter value

- popular "ex cathedra" choices for persistence parameter λ :

  MSCI *RiskMetrics* : **0.97 for monthly data**
  0.94 for daily data

  . overall parameters, optimized over various asset classes & time periods
  . compare realized squared return and estimated variance
  . showed minimum Root Mean Squares (RMS) of differences

- pick your own λ :

  . to match desired half-life *H*
  . to choose "effective history" so as to match EW estimate

see EWMA statistics later

EWMA 0.94 vs GARCH(1,1)

Comparison

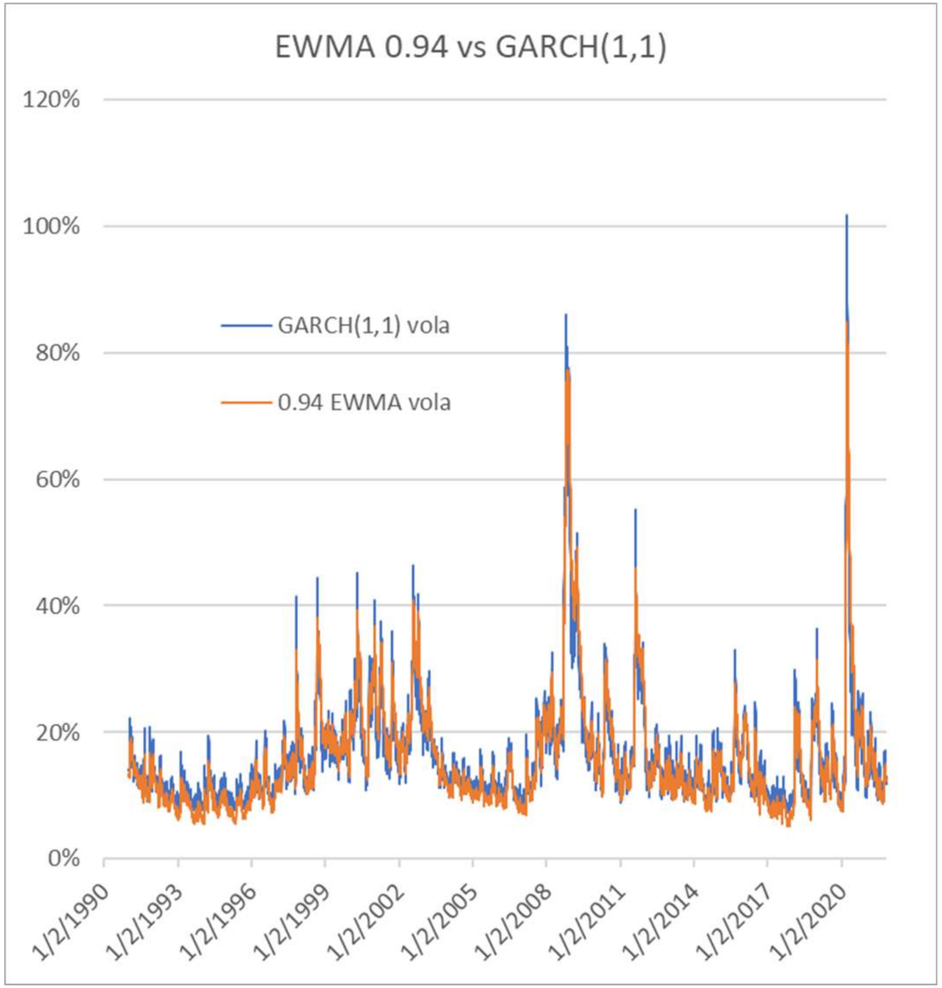
- daily data, very similar patterns

vola p.a.	avge	min	max
EWMA 0.94	15.9%	5.1%	84.9%
GARCH(1,1)	16.1%	7.2%	101.6%

- GARCH vola somewhat higher because part of weight goes to higher LT volatility of 17.38%

Q : *why is average cond'l volatility smaller than full-sample / overall vola :   18.48%   or*
LT / uncond'l vola :   17.38%   ?

- sqrt(E[var]) :   EWMA 18.48%
  GARCH 18.26%



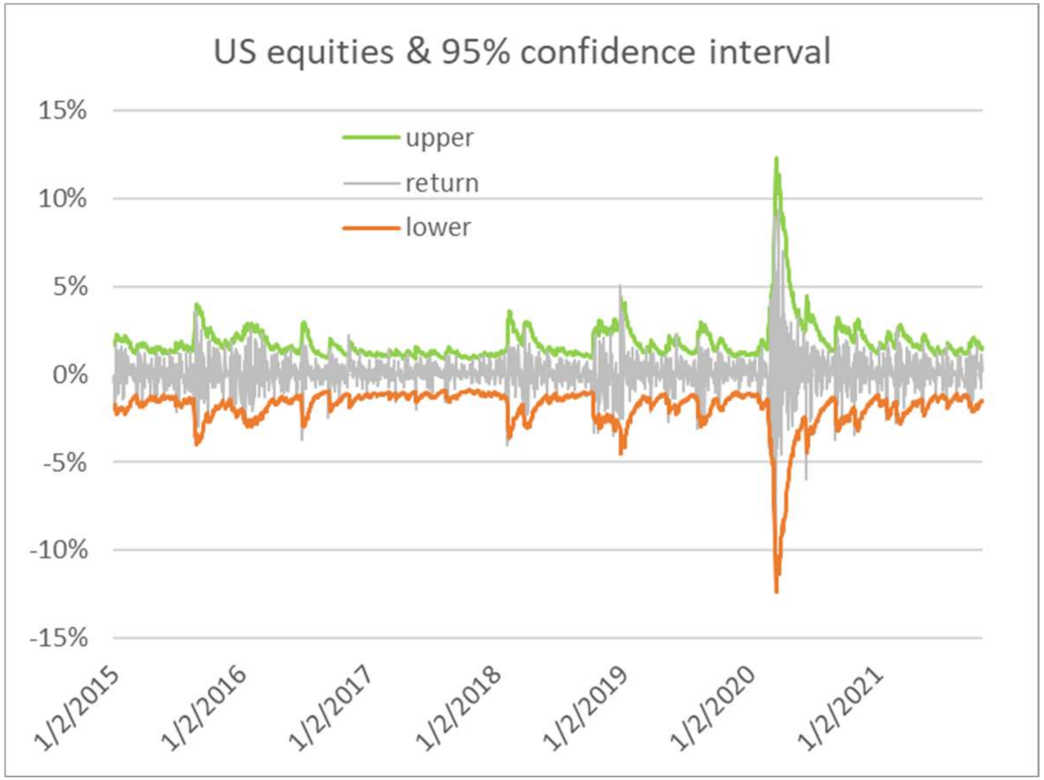
EWMA for confidence intervals

Conditional normality ?

- use EWMA volatilities to construct 95% confidence intervals on daily returns

- fraction of **exceedances** :

exceedances	**lower**	**upper**
EWMA 0.94	3.47%	2.94%
GARCH(1,1)	3.32%	2.20%



- note : EWMA is responsive,
  but always “one day late”,
  as is GARCH

- confirms stylized fact #4 :
  volatility asymmetry

EWMA 0.94 vs GARCH(1,1)

- time-varying volatility → "**fat tails**", **excess kurtosis**
- again, use conditional volatility forecasts to make z-scores : $z_{t+1} = \dfrac{r_{t+1}}{\sigma_{t+1|t}} \quad \sim N(0,1)$

		z-scores	
	initial	EWMA 0.94	GARCH(1,1)
exc kurt	10.51	2.68	1.86
stdev		1.05	1.00

- GARCH(1,1) is somewhat better : kurtosis reduction
  EWMA slightly **under**-estimates vola

- what if we use Maximum Likelihood estimate of EWMA persistence ?
  same procedure as for GARCH !
  $\lambda^* = 0.9338$ → very close to 0.94, **in this particular case**
  z-score results above do **not** change

EWMA versus GARCH(1,1)

- EWMA variance : $\sigma_{t+1}^2 = \quad (1-\lambda)\cdot r_t^2 + \lambda \cdot \sigma_t^2$

- Compare with GARCH(1,1) : $\sigma_{t+1}^2 = \gamma \cdot \sigma_L^2 + \alpha \cdot r_t^2 + \beta \cdot \sigma_t^2$

For EWMA :

- $\alpha + \beta = 1$
- $\gamma = 0$ → no long-run mean-reversion

EWMA = IGARCH(1,1) with zero intercept

Hence, EWMA variance is **non-stationary**, and in theory can wander off ("explode"),
but we do not observe this in practice.

EWMA works very well in practice despite this non-stationarity issue.

EWMA statistics

Some practical statistics : half-life

- important characteristic : "**half life**" ***H***
- defined as the period over which the EWMA weight has decreased by 50%

- current time t
- first (most recent) weight : $w_t = (1-\lambda)$ & weight at half-life lag H : $w_{t-H} = (1-\lambda)\cdot\lambda^H$

- for half-life, we must have : $\dfrac{w_{t-H}}{w_t} = \dfrac{(1-\lambda)\cdot\lambda^H}{(1-\lambda)} = \lambda^H = ½$

- hence : $$\boxed{H = \frac{\ln(½)}{\ln(\lambda)} \quad \Leftrightarrow \quad \lambda = (½)^{1/H}}$$

λ	**H**
0.995	138
0.99	69
0.975	27
0.95	14
0.90	7

EWMA statistics

Some practical statistics : Weighted Average Lag, and number of periods

- look for equivalence with EW estimate : equate "**effective history**"
  how far do you look back ?
  how large is the weight of distant observation ?
- combine lag & weight to calculate **Weighted Average Lag (*WAL*)** :

$$WAL_{EW} = \sum_{t=1}^{N} w_t \cdot t = \sum_{t=1}^{N} \frac{1}{N} \cdot t = \frac{1}{N} \cdot \left[\tfrac{1}{2} N(N+1) \right] = \tfrac{1}{2}(N+1)$$

$$WAL_{EWMA} = \sum_{t=1}^{\infty} (1-\lambda)\lambda^{t-1} \cdot t = \frac{1}{1-\lambda}$$

- hence : $\boxed{\lambda = \frac{N-1}{N+1} \Leftrightarrow N = \frac{1+\lambda}{1-\lambda}}$ **"*N*" = number of periods**

N	λ	H	H/N
520	0.996	180	0.35
260	0.992	90	0.35
130	0.985	45	0.35
65	0.970	23	0.35
32	0.940	11	0.35
21	0.909	7	0.35

Half-life ≈ one-third of N

EWMA volatility over fixed windows

Normalizing weights over fixed lookback period T

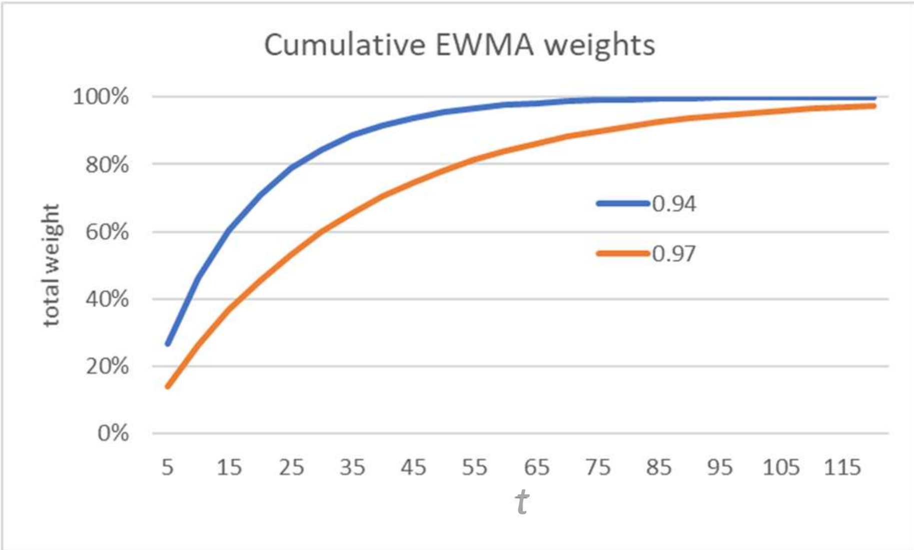
- **recursive** EWMA : volas in first part of sample depend on EWMA starting value
- in statistical analyses, a **fixed lookback** window of T periods can be desirable

solution : **fixed window EWMA** → re-scale weights to sum to 1

- total weight in $[1,T]$ =

$$\sum_{\tau=1}^{T} w_\tau = \sum_{t=1}^{T} (1-\lambda)\cdot\lambda^{\tau-1} = (1-\lambda)\frac{1-\lambda^T}{1-\lambda} = 1-\lambda^T$$

- rescaled weights analytically : $w_\tau = \dfrac{1-\lambda}{1-\lambda^T}\cdot\lambda^{\tau-1}$



EWMA statistics

only background info (assignment), not for exam

Half-life revisited

- for half-life, we must have : $$\frac{w_{t-H}}{w_t} = \frac{(1-\lambda)\cdot\lambda^H}{(1-\lambda)} = \lambda^H = ½$$

  → the weight has halved for period *H*

- total weight comprised in most recent T periods : $$\sum_{\tau=1}^{T} w_\tau = 1-\lambda^T$$

  hence, over half-life : $$\sum_{\tau=1}^{H} w_\tau = 1-\lambda^H = 50\%$$

- so over the half-life : the **individual weight** has decreased by 50%, and
  the **sum of the weights** has reached 50%

RiskMetrics settings revisited

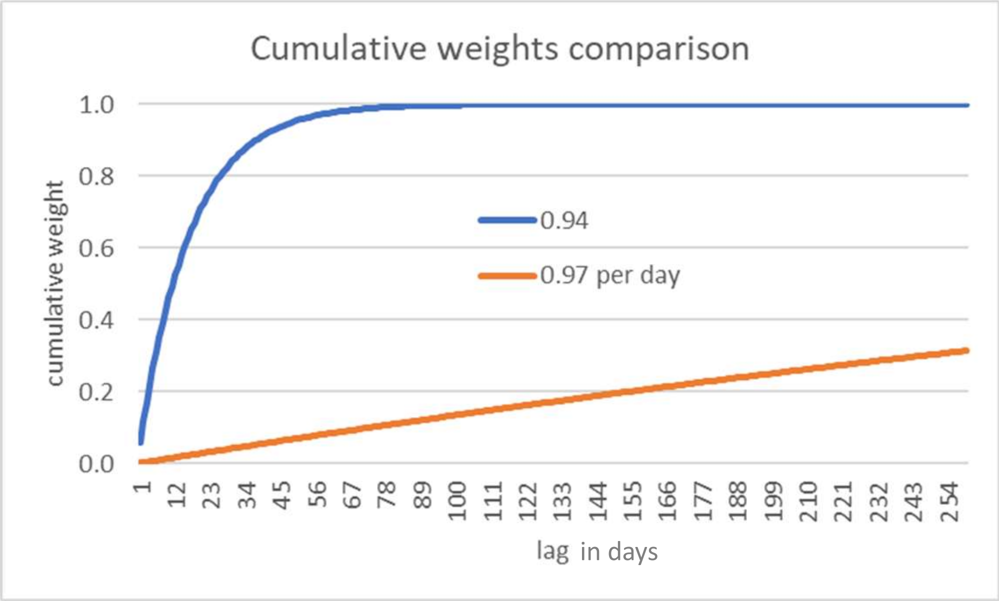
Monthly versus daily

- "0.94 daily & 0.97 monthly persistences are inconsistent"
- **theoretical reason :** EWMA = IGARCH(1,1) : $\lambda + (1 - \lambda) = 1$ → non-stationary variance ...

- **practical perspective :**
  compare implied weights for each trading day

- as expected, the faster 0.94 EWMA vola allocates much larger weights to recent days than the slower 0.97 monthly EWMA vola

- not wrong or "inconsistent", but **different**
- obviously, **daily & monthly returns have different volatility processes** ...



Outline of lecture 3

Risk measure estimation

- Pre-requisites
- Volatility estimation : Equally Weighted, GARCH, EWMA
- **Parametric VaR & ES**
- Non-parametric VaR & ES : Historical Simulation, Filtered HSim, Weighted HSim
- Monte Carlo Simulation
- Recap, ex-ante vs ex-post

Estimating parametric VaR & ES

Parametrics risk measures : assume a return distribution

- assume normality
- given our volatility estimate for period t+1, we can calculate the parametric VaR & ES :

$$VaR_{t+1}(c) = N^{-1}(c) \cdot \sigma_{t+1} \qquad\qquad ES_{t+1}(c) = \frac{n\left(N^{-1}(c)\right)}{1-c}\sigma_{t+1}$$

- 1-day VaR → daily vola
  95% confidence → c = 0.95 $z_c = N^{-1}(0.95) = 1.645$ = NORM.S.INV(0.95)
  $n(1.645) = 0.1031$ = NORM.S.DIST(1.645,FALSE)

Example :

- using close of 29-Oct-2021, estimate the **95% 1-day VaR & ES** of US Equities
- lookback = 260 days, starts on 20-Oct-2020 → daily EW vola = 0.0089%
- parametric VaR = 1.645 * 0.0089% = **1.46%**
- parametric ES = 0.1031/(1-0.95) * 0.0089% = **1.83%**

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Risk measure estimation

- Pre-requisites
- Volatility estimation :
  - Equally Weighted
  - GARCH
  - EWMA
- Parametric VaR & ES
- **Non-parametric VaR & ES :**
  - **Historical Simulation**
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Estimating non-parametric VaR & ES

What if we're not willing or able to specify a return distribution ?

Non-parametric → do not make any explicit distribution assumption

→ use empirical distribution (historical frequency distribution)

Q : *Is the empirical distribution the "actual" distribution ?*

Historical simulation (HSim)

- select lookback window T and return interval (ideally matching the forecast horizon), say : daily returns, over 260 days
- construct a frequency distribution, or simply rank the returns
- the VaR_c is given by the (1-c) percentile of this distribution
- the ES_c is given by the average of returns up to the (1-c) percentile of this distribution

Historical simulation

Example US Equity cont'd

- using close of 29-Oct-2021, estimate the **95% 1-day VaR** of US Equities
- lookback = 260 days, starts on 20-Oct-2020
- rank returns from lowest to highest
- select 5% percentile = 13^{th} lowest observation = 1.43% →

parametric VaR = 1.46%

- **95% 1-day ES** is average of 13 lowest returns = 2.06% →

parametric ES = 1.83%

date	rank	returns
1/4/2021	15	-0.0141
7/19/2021	14	-0.0141
10/4/2021	13	-0.0143
10/30/2020	12	-0.0145
3/3/2021	11	-0.0157
3/4/2021	10	-0.017
1/29/2021	9	-0.0184
10/26/2020	8	-0.0185
9/20/2021	7	-0.0187
3/18/2021	6	-0.0188
9/28/2021	5	-0.0218
5/12/2021	4	-0.0234
1/27/2021	3	-0.0253
2/25/2021	2	-0.0274
10/28/2020	1	-0.0341

HSim

Evaluation

strong points :

- simple to use : historical data & ranking
- light on **explicit** assumptions : no need to specify or assume an underlying distribution

weak points :

- do the observations in the lookback window capture the distribution that is relevant "now" ?
  → under-responsive to changes in conditional risk
- for trading portfolios : asymmetric risk response : large losses increase risk, but large gains not
- uses only (1-*c*) of the observations to estimate VaR and ES → large estimation error

So the **trade-off** in using HSim or parametric is :

- sufficient confidence in (normal) distribution → parametric, uses all observations to estimate
- fat tails or non-linear securities → HSim, use empirical distribution

HSim refinements

Is the historical distribution representative for the next day ?

- stylized facts → **time-varying** volatility

- in our example : the EWMA daily vola is on average **0.89%** over the lookback period, but **0.76%** on 29-Oct-2021 = time *t*
  so there is also a **level** difference

1. Filtered HSim (or volatility-weighted HSim) **:**

- estimate conditional volatilities : EWMA, GARCH,...
- first normalize each return with its forecasted vola into *z*-scores : $z_\tau = \dfrac{r_\tau}{\sigma_{\tau|\tau-1}}$
- next multiply *z*-scores with most recent vola to get rescaled returns : $r_\tau' = r_\tau \cdot \dfrac{\sigma_{t+1|t}}{\sigma_{\tau|\tau-1}}$
- these **rescaled returns** reflect tomorrow's (t+1) forecasted vola level

Filtered HSim : FHSim

Example US Equity cont'd

- using close of 29-Oct-2021, estimate the **95% 1-day VaR** of US Equities
- lookback = 260 days, starts on 20-Oct-2020
- rescale returns & rank
- select 5% percentile = 13th lowest observation = 1.26%

parametric VaR = 1.46%
HSim VaR = 1.43%

- **95% 1-day ES** is average of 13 lowest returns = 1.89%

parametric ES = 1.83%
HSim ES = 2.06%

date	rank	rescaled r
10/4/2021	15	-0.0124
3/18/2021	14	-0.0124
8/18/2021	13	-0.0126
10/26/2020	12	-0.0130
5/10/2021	11	-0.0135
6/18/2021	10	-0.0147
1/29/2021	9	-0.0150
1/4/2021	8	-0.0152
7/19/2021	7	-0.0173
5/12/2021	6	-0.0221
9/28/2021	5	-0.0232
10/28/2020	4	-0.0234
2/25/2021	3	-0.0243
9/20/2021	2	-0.0251
1/27/2021	1	-0.0265

Weighted HSim

A hybrid approach

- allow for information decay when using historical percentiles
- combine exponential smoothing with HSim → hybrid approach

2. Weighted HSim :

- determine EWMA weights over lookback period T
- rank returns, as in HSim
- instead of using equal 1/T frequencies, use EWMA weights to determine :
  the percentile for the VaR, and
  the weighted average return for the ES

Weighted HSim

Example US Equity cont'd

- using close of 29-Oct-2021, estimate the **95% 1-day VaR**
- lookback = 260 days, starts on 20-Oct-2020
- rank returns & cumulate EWMA frequency
- select 5% percentile, interpolate betw 21&22 = 1.03%

parametric VaR = 1.46%
HSim VaR = 1.43%
Filtered HSim VaR = 1.26%

- **95% 1-day ES** is wghted avge of lowest returns = 1.69%

parametric ES = 1.83%
HSim ES = 2.06%
Filtered HSim ES = 1.89%

date	rank	returns	weight	cum wght
9/30/2021	22	-0.0102	0.0164	0.0602
12/9/2020	21	-0.0103	0.0000	0.0439
3/2/2021	20	-0.0105	0.0000	0.0439
2/22/2021	19	-0.0113	0.0000	0.0439
3/23/2021	18	-0.0117	0.0000	0.0439
6/18/2021	17	-0.0122	0.0002	0.0439
5/10/2021	16	-0.0136	0.0000	0.0437
1/4/2021	15	-0.0141	0.0000	0.0437
:	:	:	:	:
:	:	:	:	:
9/28/2021	5	-0.0218	0.0145	0.0145
5/12/2021	4	-0.0234	0.0000	0.0000
1/27/2021	3	-0.0253	0.0000	0.0000
2/25/2021	2	-0.0274	0.0000	0.0000
10/28/2020	1	-0.0341	0.0000	0.0000

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Simulation : HSim or Monte Carlo

HSim :

- uses **implied empirical** distribution
- scenarios are drawn from history
- limited number of data points …
- … but perhaps more relevant

Monte Carlo (MC) simulation :

- the joint distibutions of assets / underlying risk factors are either
  . **assumed** or
  . **estimated** from historical data
- scenarios are generated from these distributions
- unlimited number of draws possible
- … but perhaps specification error in joint distributions

→ **same flexibility** in using the scenarios to :

. reprice instruments (full-revaluation of non-linear derivatives, e.g.)

. estimate statistics

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Ex-ante ex-post : clean vs dirty returns

Using the portfolio composition

- "**clean returns**" : use fixed **current portfolio composition** with historical returns data :

$$\left\{w_{i,t}\right\}_{i \in p, t} \quad and \quad \left\{r_{i,\tau}\right\}_{i \in p, \tau \in T}$$

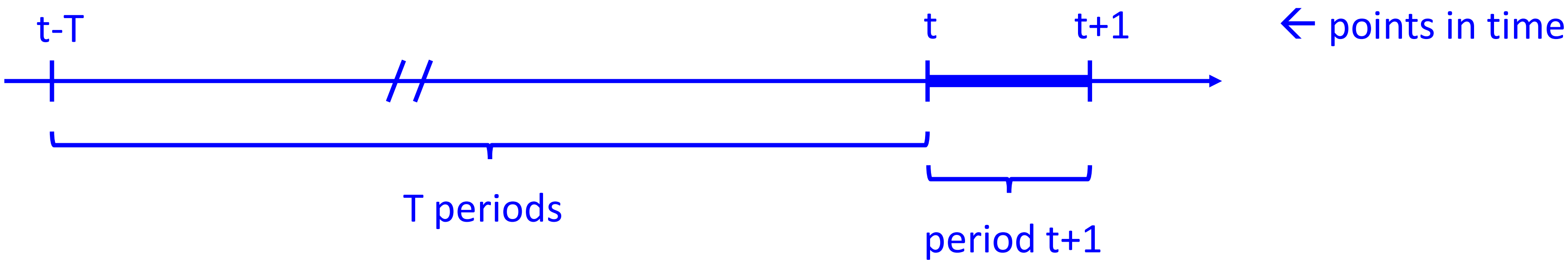
- "**dirty returns**" : realized portfolio returns, includes effect of **time-changing portfolio composition** :

$$\left\{w_{i,t}\right\}_{i \in p, t \in T} \quad and \quad \left\{r_{i,t}\right\}_{i \in p, i \in T}$$

Q : *What can you say about the volatility of the dirty versus clean portfolio returns ?*

- for risk analyses, we are interested in the risk of the **current** portfolio
  → **ex-ante** perspective
  → use **clean returns**
- **ex-post** perspective :
  1. clean returns : "portfolio simulation"
  2. dirty returns : descriptive, performance for investors

Ex-ante ex-post : cond'l & uncond'l



ex-ante :

. use portfolio composition as per time *t*

. use information up to time *t* to make forecast for period *t*+1

. market conditions change over time → use **adaptive risk estimate**

. this is by definition a **conditional forecast**

ex-post :

. descriptive use only
. use EW metrics → **unconditional estimates**