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Reanalysing Eddington's 1919 Eclipse Data To Test Einstein's Gravity

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Historical Background

- In 1919, Sir Arthur Eddington and Sir Frank Dyson undertook the monstrous task of proving Einstein’s General Theory of Relativity by examining the bending of light when in proximity to the Sun.

- The scientific community at the time were staunch believers in the Newtonian theory and Einstein’s theory was treated with a lot of skepticism and hostility.

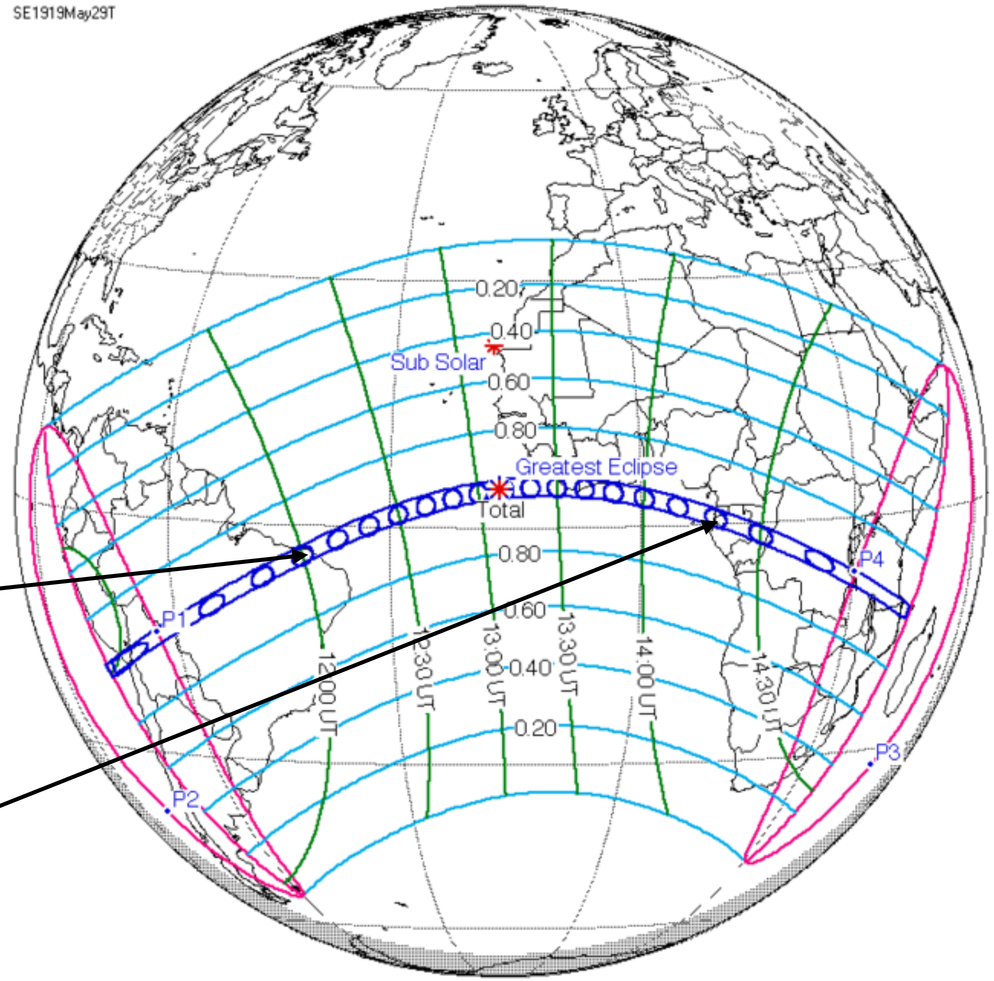
- Eddington was a minority in the Astronomy community as he strongly believed the theory proposed.

- Eddington was also a pacifist and wanted to use the proof of Einstein’s work as a opportunity for reconciliation between England and Germany, which was greatly needed after WWI.

During the spring of 1919, an eclipse took place which proved ideal for Eddington's proposed experiment. This was due to the light from the Sun being briefly obscured, which allowed light from distant stars to be observed.

Two expeditions were undertaken to locations which fitted the observational requirements of the experiment:

- The town of Sobral, in Northern Brazil, where measurements were made using an astrographic lens of about 10 inches in diameter.
- The island of Principe, off the west coast of Africa. An identical astrographic lens was brought alongside a much smaller objective lens, of about 4 inches in diameter, to produce photographic plates.



Map showing the path of the 1919 eclipse [1].

Results of the expeditions

- The plates were brought back to England to be analysed.

- Dyson presented the main results at a joint meeting of the Royal Astronomical Society and the Royal Society of London, on the 6th of November 1919.

- The expedition to Sobral yielded measurements of seven stars with good visibility. **The deflection was calculated to be $\mathbf{1.98 \pm 0.16}\ \boldsymbol{arcseconds}$ at the limb of the Sun.**

- The expedition to Principe yielded measurements of only five stars, with large amounts of error. **The deflection was calculated to be $\mathbf{1.61 \pm 0.40}\ \boldsymbol{arcseconds}$ at the limb of the Sun.**

Experimental Theory

Two possible theoretical outcomes were predicted:

1. The deviation of light incident from distant stars, in the vicinity of an object of mass M, followed Newton's laws of gravity. Hence, it can be derived that the deviation is

$$\theta_N = \frac{2GM}{rc^2} \qquad (1)$$

1. The deviation of light incident from distant stars, in the vicinity of an object of mass M, followed Einstein's Theory of General Relativity, which would yield the following result

$$\theta_E = \frac{4GM}{rc^2} \qquad (2)$$

Where in both expressions G is the gravitational constant, c is the speed of light and r is the distance of closest approach of the light to the body.

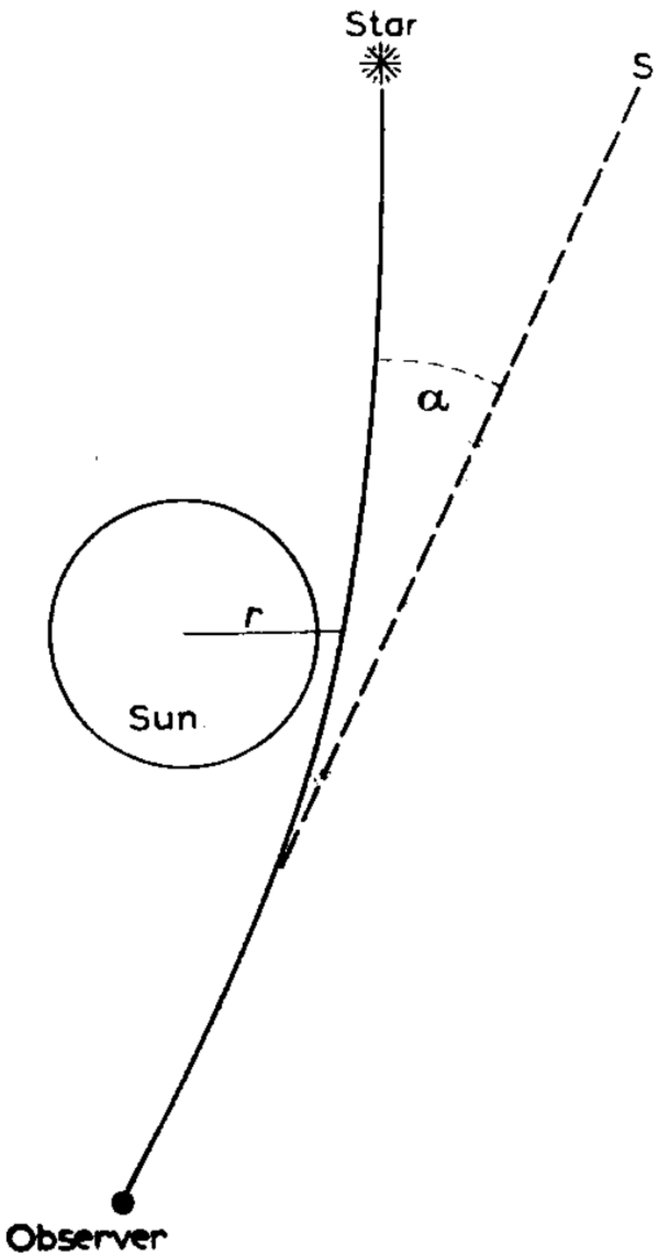
For light that has just grazed the surface of the Sun, $M = M_{\odot}$ (the Sun's mass) and $r = R_{\odot}$ (the Sun's radius), the deflections are found to be:

$$\alpha_N = 0.87\ arcseconds$$

And

$$\alpha_E = 1.74\ arcseconds$$

These theoretical predictions will be compared with our inferred result for the parameter α and the values measured by Eddington and Dyson.



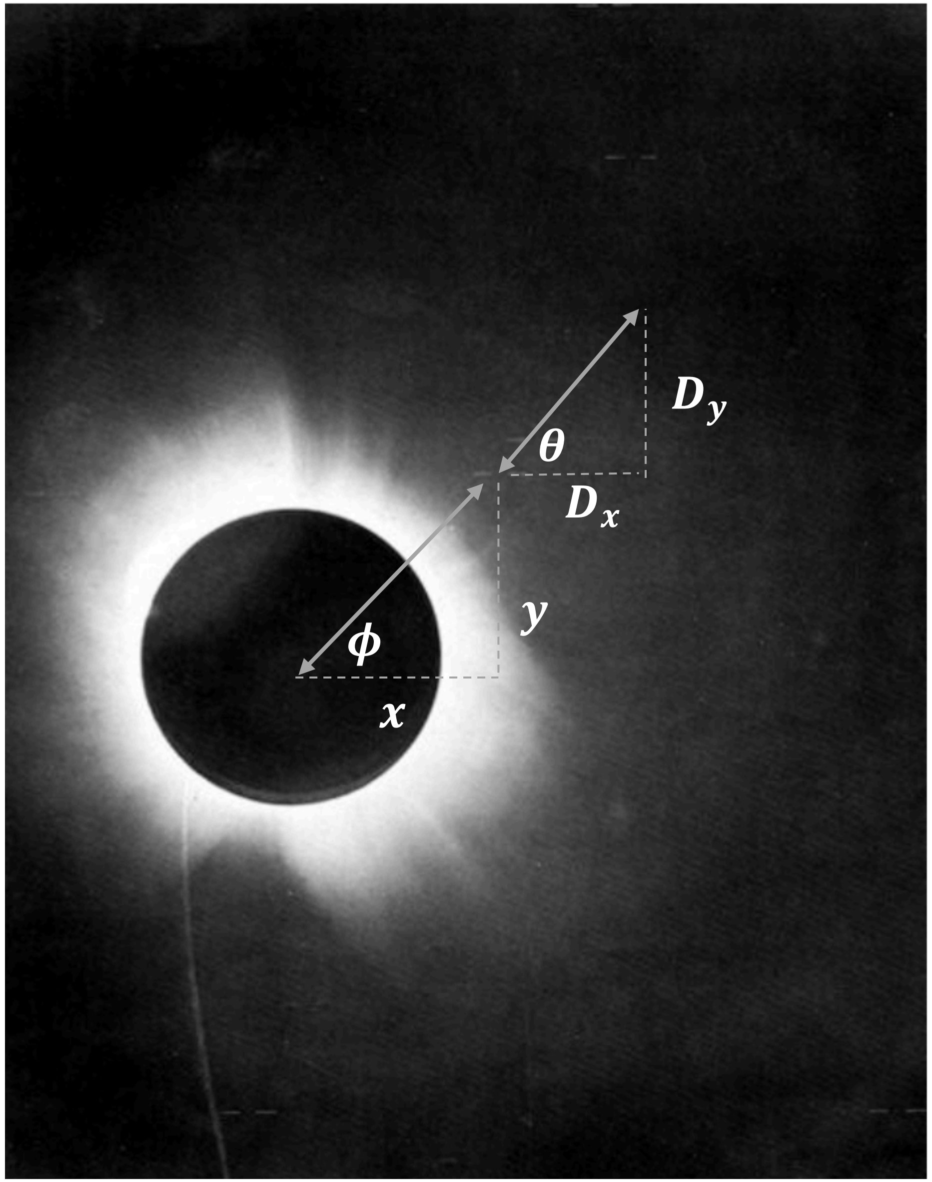
From the negative photographic plates taken during the eclipse, the deviation of the light incident from Stars, with known position (x, y), could be measured, by overlapping measurement plates with control plates.

It can be seen that

$$D_x = \theta cos\phi \quad (4)$$

$$D_y = \theta sin\phi \quad (5)$$

This could form the basis a rudimentary model.



Negative of a photographic plate used in the experiment [2]

However, due to the physical limitations of using photographic plates, nuisance parameters had to be introduced in the theoretical model. Hence, the revised model for the deflections is as followed:

$$D_x = ax + by + c + \alpha_0 E_x \quad (5)$$

$$D_y = dx + ey + f + \alpha_0 E_y \quad (6)$$

Where*:*

- c, f are corrections due to the misalignement of the superposed plates.
- a, e are corrections for the scaling of the plates, since the plates may have expanded or contracted due to temperature differences during transportation.
- b, d are corrections for the orientation of the superposed plates.
- α_0 denotes the deflection at unit distance, such that $\alpha_0 E_x$ and $\alpha_0 E_y$ are the deflections of the star in R.A. and Decl. respectively.

To infer the bending angle for light grazing the Sun in arcseconds, the parameter α_0 needs to be rescaled. We will proceed with the rescaled parameter $\alpha = 19.8\alpha_0$.

The values for x, y, E_x and E_y were known for the seven stars considered. These values as well as the predicted GR bend angles, for each star, are shown in the tables below.

No.	Right Ascension.	Declination.
11	$c-0\cdot160b-1\cdot261a-0\cdot587\alpha$	$f-1\cdot261d-0\cdot160e+0\cdot036\alpha$
5	$c-1\cdot107b-0\cdot160a-0\cdot557\alpha$	$f-0\cdot160d-1\cdot107e-0\cdot789\alpha$
4	$c+0\cdot472b+0\cdot334a-0\cdot186\alpha$	$f+0\cdot334d+0\cdot472e+1\cdot336\alpha$
3	$c+0\cdot360b+0\cdot348a-0\cdot222\alpha$	$f+0\cdot348d+0\cdot360e+1\cdot574\alpha$
6	$c+1\cdot099b+0\cdot587a+0\cdot080\alpha$	$f+0\cdot587d+1\cdot099e+0\cdot726\alpha$
10	$c+1\cdot321b+0\cdot860a+0\cdot158\alpha$	$f+0\cdot860d+1\cdot321e+0\cdot589\alpha$
2	$c-0\cdot328b+1\cdot079a+1\cdot540\alpha$	$f+1\cdot079d-0\cdot328e-0\cdot156\alpha$

Table 1: Theoretical Model used for the stars under consideration [2]

No.	Right Ascension.	Declination.
11	$c-0\cdot160b-1\cdot261a-0\cdot587\alpha$	$f-1\cdot261d-0\cdot160e+0\cdot036\alpha$
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Table 2: Undeviated position of stars and theoretical deviations due to GR [2]

Statistical Method

From Bayes' Theorem:

$$P(\underline{\theta}|Data) = \frac{L(Data|\underline{\theta})\Pi(\underline{\theta})}{P(Data)} = \frac{L(Data|\underline{\theta})\Pi(\underline{\theta})}{\int_{\Theta} P(Data, \underline{\theta})\, d\theta} \quad (7)$$

Where

- $P(\underline{\theta}|Data)$ is the *Posterior distribution*
- $L(Data|\underline{\theta})$ is the *Likelihood function*
- $\Pi(\underline{\theta})$ is the *Prior distribution on our parameters*
- $P(Data)$ is the *Evidence or Normalisation*

In our case $\underline{\theta} = (a, b, c, d, e, f, \alpha)$ and $Data = D_x, D_y$ are the respective parameters and data points under consideration.

Assumptions Made

1) The likelihood function for each measured data point, $\widehat{D_x} and\ \widehat{D_y}$, is a Gaussian distribution centered at the theoretical values $D_x\ and\ D_y$ with a given uncertainty σ.
2) Each measurement was made independently and hence the overall likelihood is the product of the individual likelihoods for each data point.
3) The prior on each parameter is uniform, to start with.

Using these assumptions we find that the form of the likelihood function for the full data set, from one plate, is as follows:

$$L(Data|\underline{\theta}) = \prod_{i=1}^{2N} \frac{1}{\sqrt{2\pi}\sigma_i} e^{-\sum_{i=1}^{2N} \frac{\left(\underline{d}_i - \underline{d}_{theory,i}\right)^2}{2\sigma_i{}^2}} \quad (8)$$

Where N is the number of stars and hence $2N$ is the number of data points, since we have measurements for $\widehat{D_x} and\ \widehat{D_y}$ for each star.

We can now make use of random walk ***Monte Carlo Markov Chain (MCMC)*** sampling techniques to bypass having to compute the evidence.

Metropolis-Hastings Algorithm:

I. **Initialise** the Markov process by choosing an arbitrary point $\underline{\theta_0} = (a_0, b_0, c_0, d_0, e_0, f_0, \alpha_0)$ in parameter space. Set $t = 0$.

II. **Iterate**:

1. Generate a new point $\underline{\theta'}$ found using a Gaussian distribution centered at $\underline{\theta_t}$ with a variable width, called the proposition width.
2. Calculate α, the ratio of the values of the posterior distribution evaluated at the two points

$$\alpha = \frac{P\left(\underline{\theta'} \middle| Data\right)}{P\left(\underline{\theta_t} \middle| Data\right)} = \frac{L\left(Data \middle| \underline{\theta'}\right)}{L\left(Data \middle| \underline{\theta_t}\right)} \quad (9)$$

3. Generate a random value $\lambda \in [0,1]$:

- If $\alpha \geq \lambda$, accept the candidate point and set $\underline{\theta_{t+1}} = \underline{\theta'}$
- If $\alpha < \lambda$, reject the candidate point and set $\underline{\theta_{t+1}} = \underline{\theta_t}$

Test for convergence

- The aim for the MH algorithm is to reach a stationary distribution which asymptotically tends towards the posterior distribution $P(\underline{\theta}|Data)$.

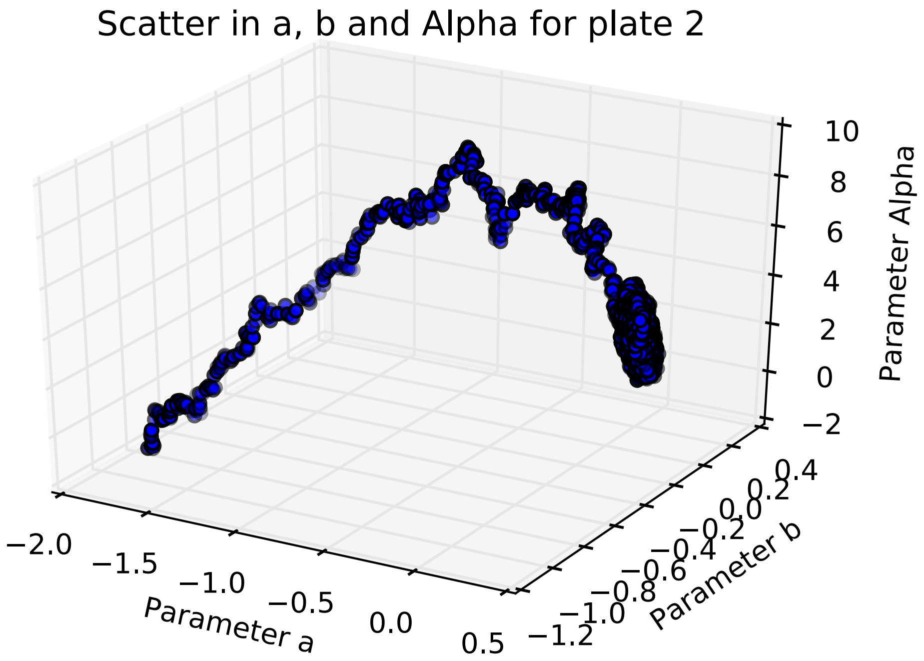
- To evaluate whether the chain has converged to the stationary distribution we must compute the **Gelman-Rubin Statistic $\boldsymbol{R}$**.

- The statistic is calculated by comparing the variance between multiple chains and the variance within each chain.

- A value of $\boldsymbol{R} \approx \boldsymbol{1}$ illustrates that the chain has converged, as the intra-chain variance should be similar to the inter-chain variance.

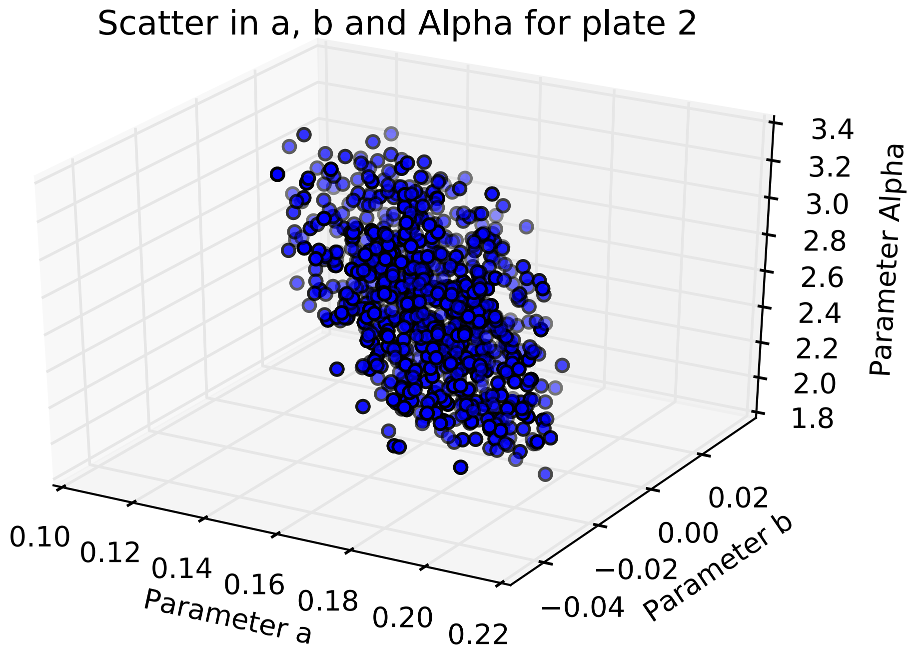
Results

- The following was found considering both measurements D_x *and* D_y for plate II, with an assumed uncertainty in each deviation measurement of $\sigma = 0.05$ and a chain length of 10000.
- A proposition width of 0.01 was chosen to be adequate to prevent slow progressions for each dimension in parameter space. This was chosen through trial and error.



Removing Burn-in:
points where

$$L(Data|\underline{\theta}) < 0.1L(Data|\underline{\theta})_{max}$$

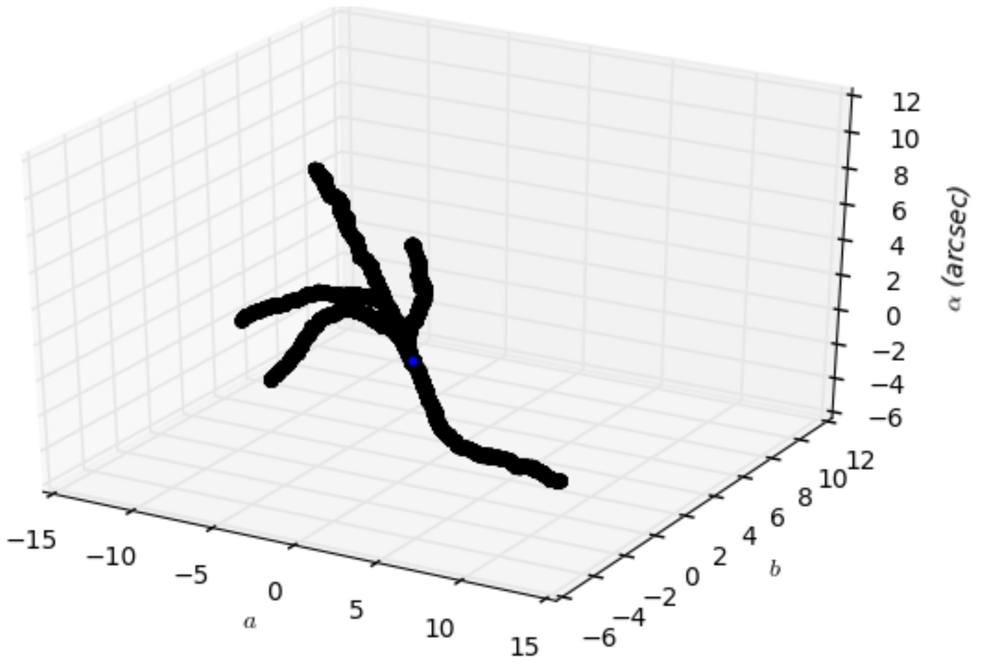


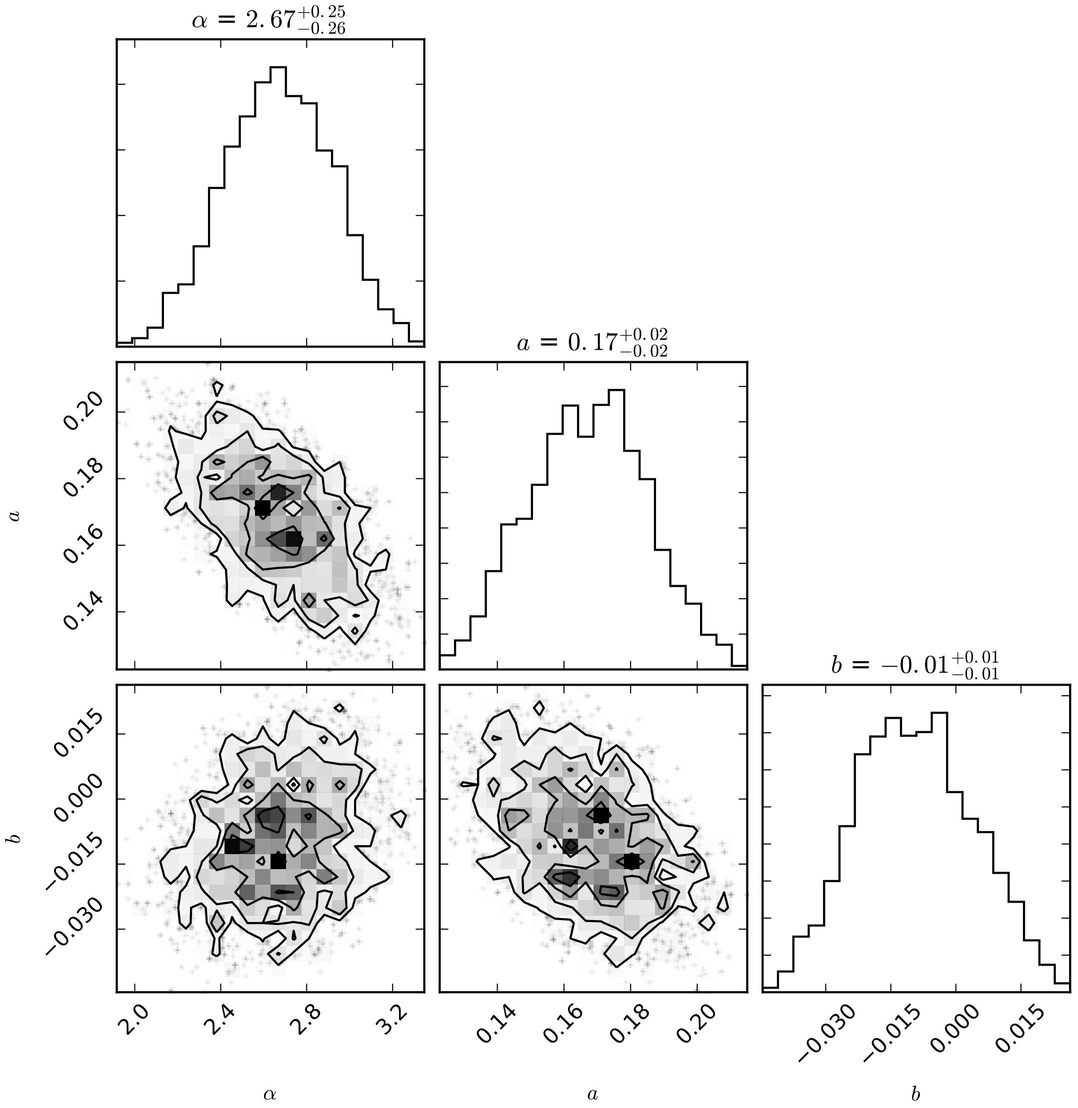
Combining the results for five chains run simultaneously, a corner plot can be made showing the marginalised distributions for parameters a, b and α, for plate II.

Convergence was assured as the **Gelman-Rubin statistic** was calculated to be

$$\mathbf{R = 1.02737\ (\approx 1)}$$

Figure showing Burn-in for 5 chains.





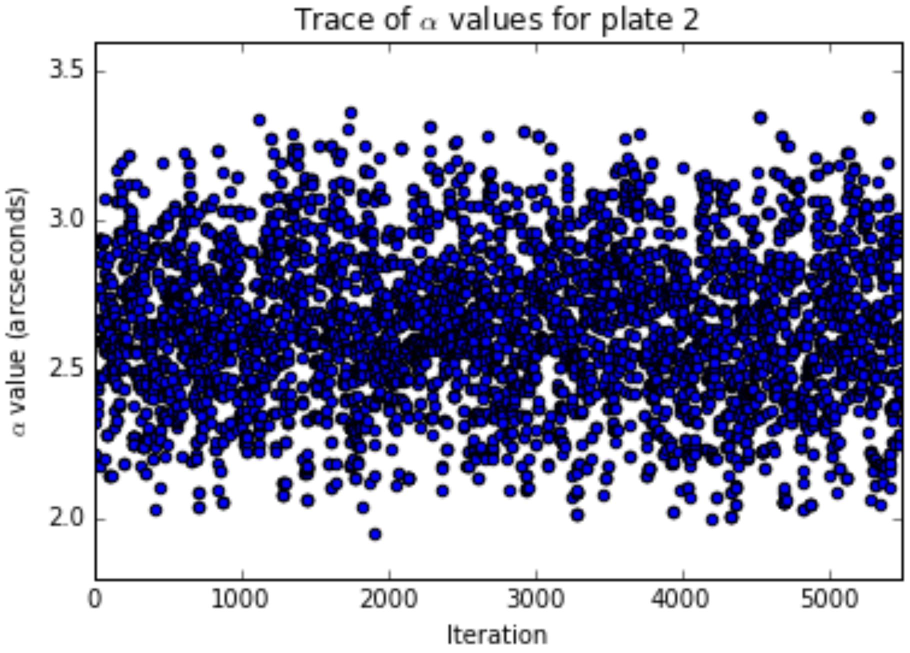
The trace for the parameter α was found, representing the variation in the value of α during the iterative process.

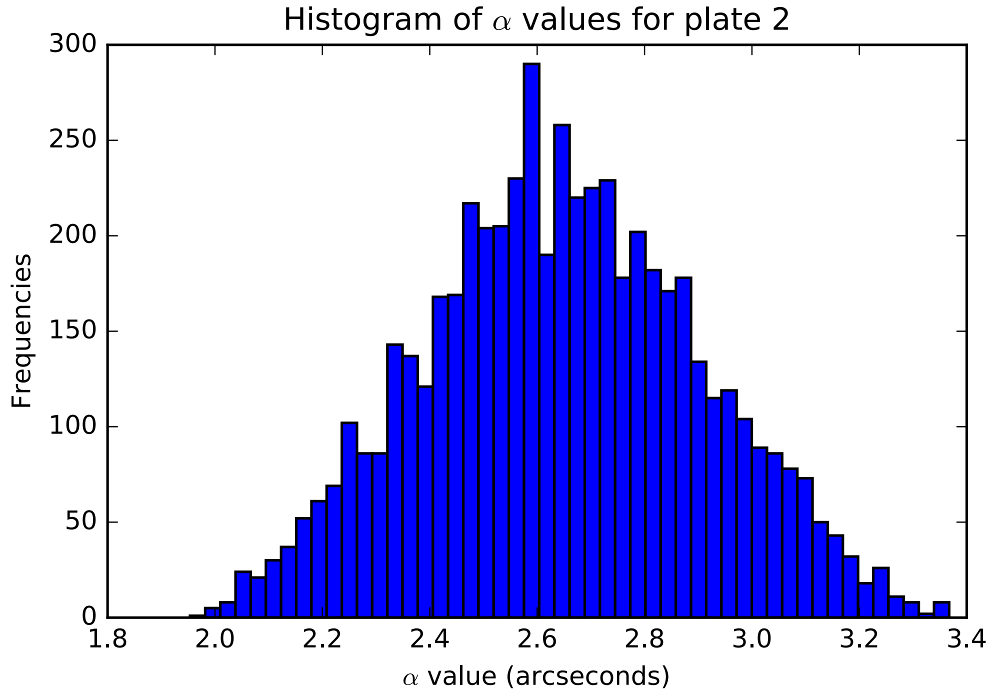
From the trace, a histogram for the distribution of α can be made, this amounts to marginalising over the nuisance parameters in our model.

By fitting a Gaussian Distribution to the normalised histogram, the mean was found:

$$\boldsymbol{Mean\ \alpha\ for\ plate\ II}$$

$$\boldsymbol{= 2.6659'' \pm 0.2661''}$$





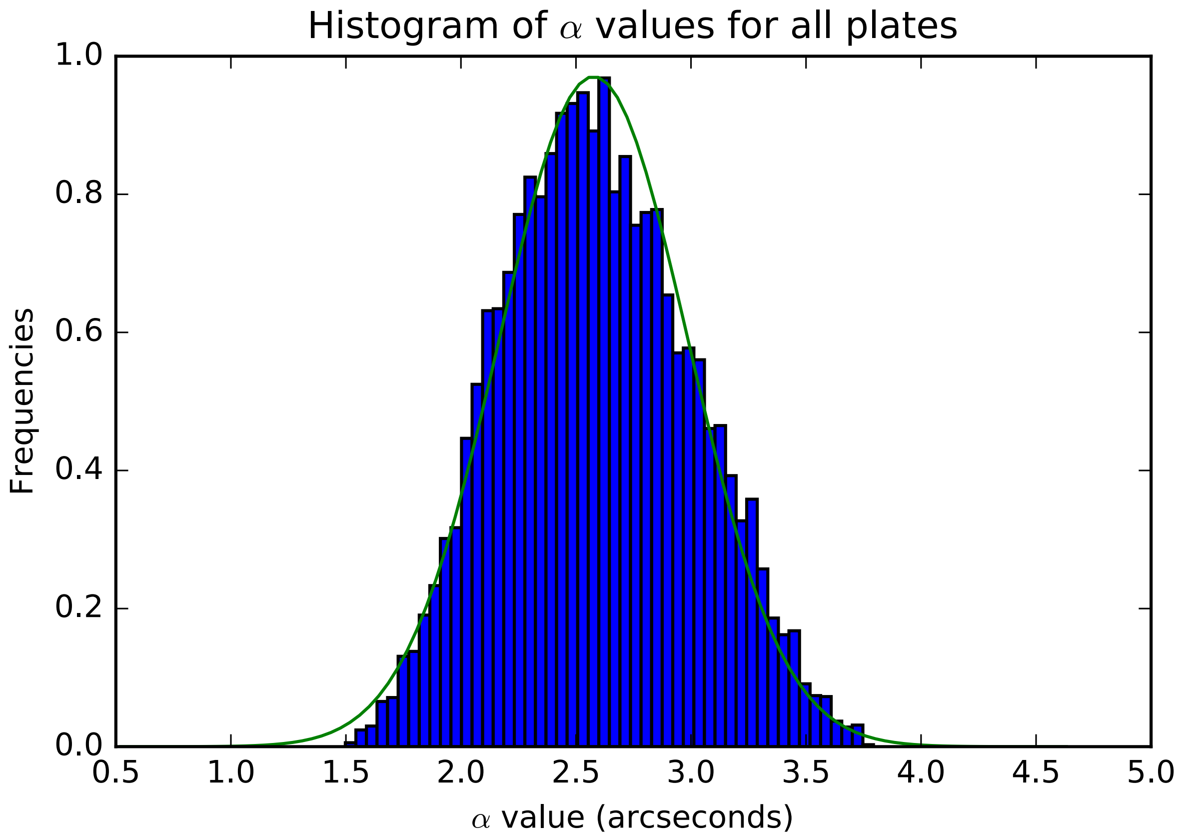
Seven photographic plates were made in total, for the Sobral expedition.

The analysis outlined was repeated for each plate, with the same conditions. Since each plate consisted of independent measurements, the overall posterior distribution α for was obtained as the product of the posterior distribution of α for each plate:

$$P(\alpha|Data) = \prod_{plate\ i=1}^{i=7} P(\alpha|Data_i)$$

$$\boldsymbol{Combined\ Mean\ of\ \alpha}$$

$$\boldsymbol{= 2.6023'' \pm 0.1294''}$$



Comparison of Results for $\boldsymbol{\alpha}$

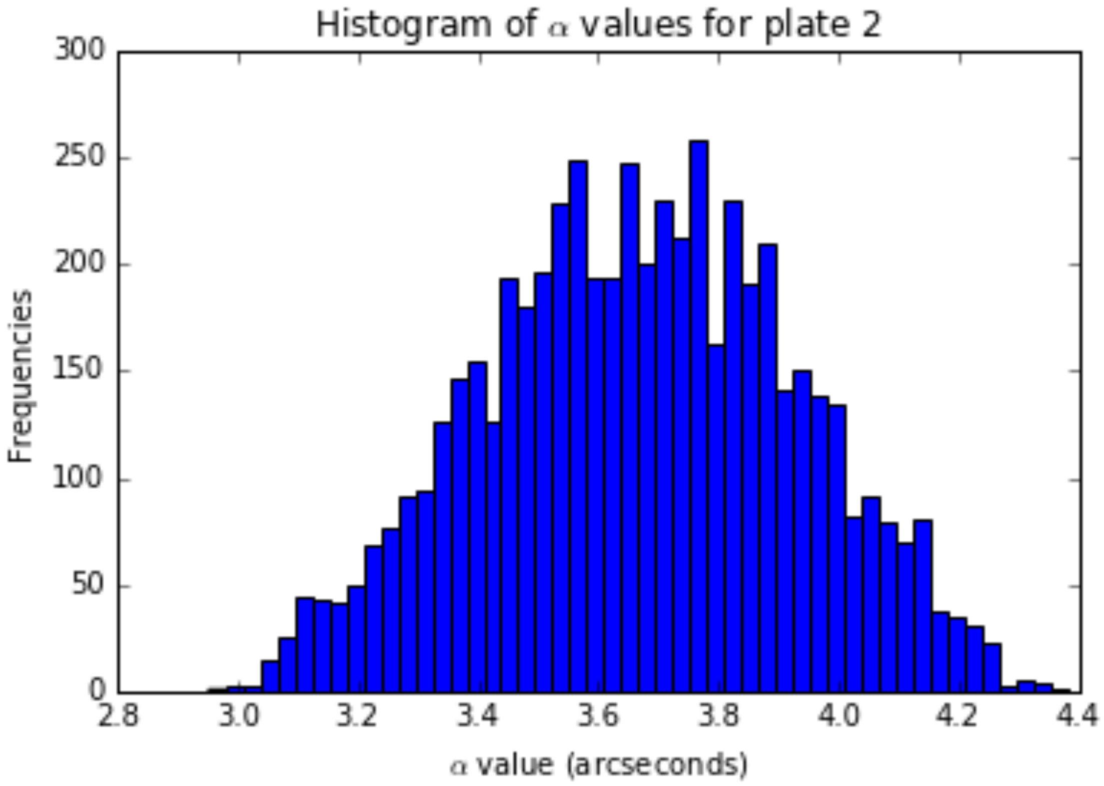
Theoretical value (arcseconds)	Measured value by Eddington and Dyson (arcseconds)	Inferred Value (arcseconds)
• $\boldsymbol{\alpha_E = 1.74}$ • $\boldsymbol{\alpha_N = 0.87}$	$\boldsymbol{\alpha_{Sobral} = 1.98 \pm 0.16}$	$\boldsymbol{\alpha = 2.6023 \pm 0.1294}$

- The value inferred is greater than both theoretical predictions by multiple standard deviations.
- It was found that $\alpha = 2.1859'' \pm 0.2579''$ for plate III and $\alpha = 3.0540'' \pm 0.2817''$ for plate I, hence the values were variable, possibly due to the physical quality of plate measurements.

- The necessity of the nuisance parameters was then tested by ignoring them in our model.

- Gaussian priors were placed on each nuisance parameter centered around 0 with narrow widths.

- Performing the same analysis, the marginalised distribution for α, for plate II, was as follows:



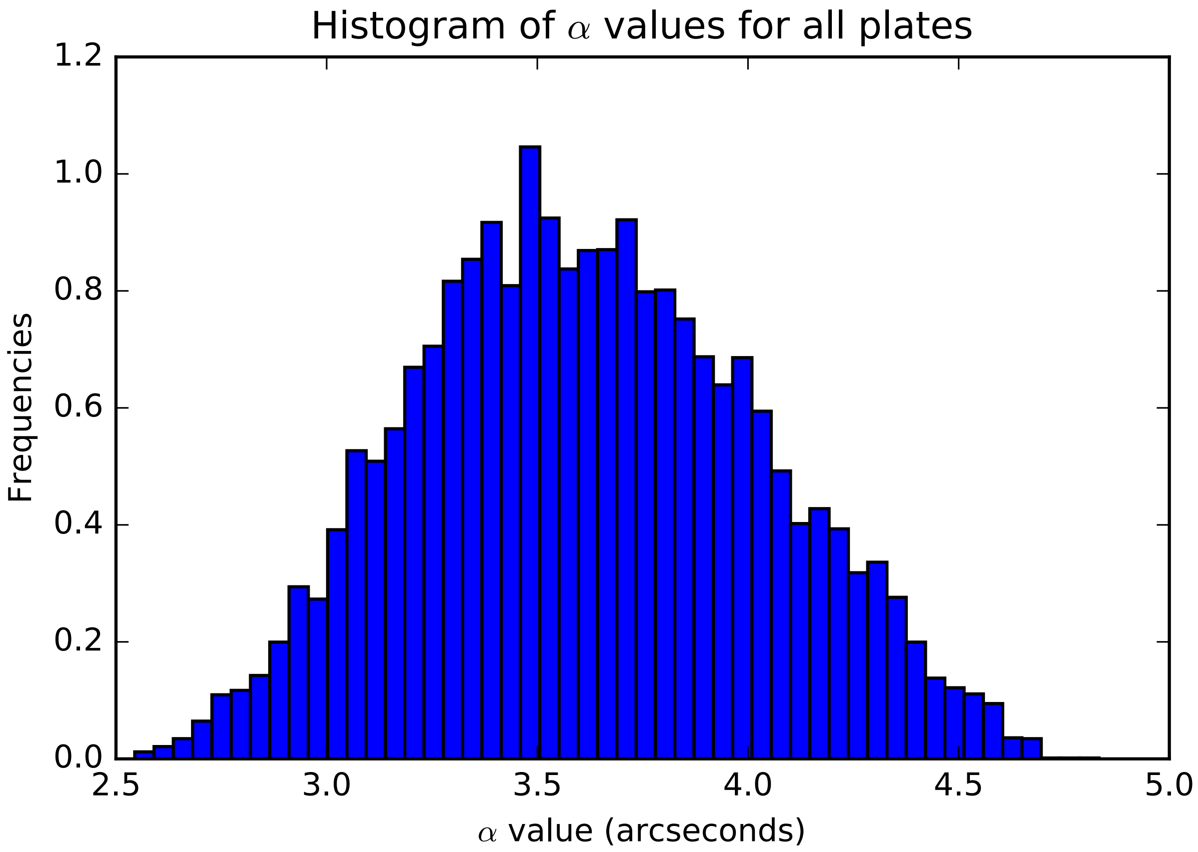
$$\boldsymbol{Mean\ \alpha\ for\ plate\ II}$$

$$\boldsymbol{= 3.6647'' \pm 0.2595''}$$

- α converges at a much greater value when nuisance parameters are not considered.

- Indicates that nuisance parameters are required.

The posterior distribution for each plate was combined as outlined previously.



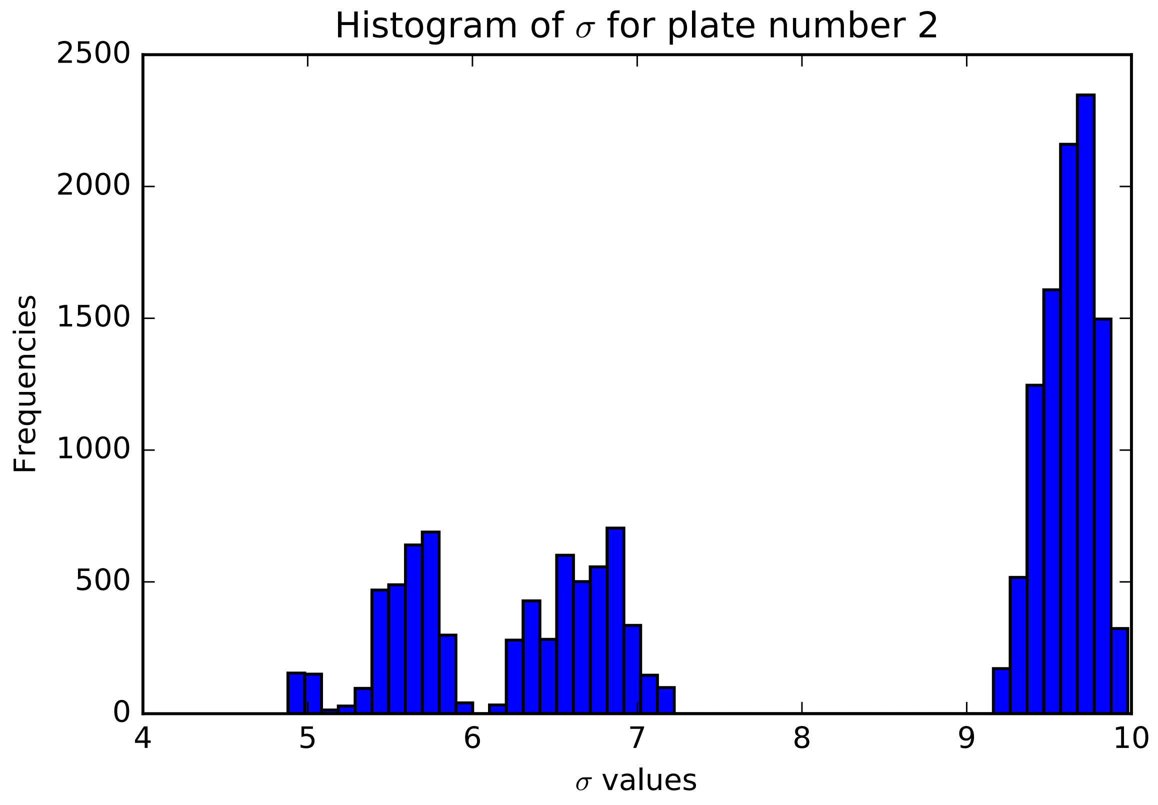
$$\textit{Combine mean of } \boldsymbol{\alpha}$$

$$= \mathbf{3.5708'' \pm 0.1315''}$$

- Value inferred for the deviation at the limb of the Sun is much greater than either theoretical predictions.

Further Discussion

- One assumption made was that the uncertainty in each deviation measurement $\sigma = 0.05$.
- The parameter σ was then taken as a parameter to be inferred and the results were explored.
- As can be seen from the combined histogram of 5 chains run simultaneously, each chain converged at various values.
- The introduction of a Gaussian prior, centered at our assumption of 0.05, on σ will be explored.
- This will prevent increasingly large values of σ to be accepted.



- If more time permits it, we will perform a model comparison analysis. This consists of computationally evaluating the Bayesian evidence and hence allowing us to find the ratio

$$\frac{P(Einstein\ Model|Data)}{P(Newton\ Model|Data)} = \frac{P(Data|Einstein\ Model)}{P(Data|Newton\ Model)} \frac{P(Einstein\ Model)}{P(Newton\ Model)} \quad (10)$$

- The first and second term on the RHS of Eq. 10, correspond to the ratio of the Bayesian evidences, which can be computed, and the ratio of the priors on the models, respectively.

- This would give a more concrete analysis of which model is favoured by the data acquired.

References

[1] Eclipse Predictions by Fred Espenak, NASA's GSFC
available from: http://eclipse.gsfc.nasa.gov/

[2] F. W. Dyson, A. S. Eddington, and C. Davidson, "A Determination of the Deflection of Light by the Sun's Gravitational Field, from Observations Made at the Total Eclipse of May 29, 1919" *Philosophical Transactions of the Royal Society of London. Series A, Containing Papers of a Mathematical or Physical Character* (1920): 291-333, on 332.