Chapter 10
Logic and Learning

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Abstract Learning and learnability have been long standing topics of interests within the linguistic, computational, and epistemological accounts of inductive inference. Johan van Benthem's vision of the "dynamic turn" has not only brought renewed life to research agendas in logic as the study of information processing, but likewise helped bring logic and learning in close proximity. This proximity relation is examined with respect to learning and belief revision, updating and efficiency, and with respect to how learnability fits in the greater scheme of dynamic epistemic logic and scientific method.

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10.1 Learning and the Dynamic Turn in Logic

For well over a decade, Johan van Benthem has been pushing the agenda of the *dynamic turn* in logic forward:

> …over the past decades computer science has also begun to influence the research agenda of logic. Traditionally, logic is about propositions and inference. Its account of this is declarative, in terms of languages and semantic models that represent information. But inference is in the first place an information-generating *process*, and just one among many at that. […]These days, in the same spirit, modern logic is undergoing a *Dynamic Turn*, putting activities of inference, evaluation, belief revision or argumentation at centre stage, not just their products like proofs or propositions. [6, p. 503]

The classical conception of logic as the study of propositions, valid arguments, and information representation may be extended to logic as the study of inference broadly conceived and of correct information processing. Once this step is made, logic may serve as the gateway for studying, modelling and optimizing belief revision processes, strategies in games, procedures for decision, deliberation and action, rational agent interaction, and . . . learning [10].

The dynamic turn from deduction and representation to active inference and information-generating processes comes from the influence computer science has exercised on logic. But from computer science come also the first ideas of formal learning theory. The concept of identification in the limit has been introduced as a computational counterpart of the process of language acquisition [25]. It inspired a group of mathematicians and computer scientists, and led to a number of results concerning (learning of) recursively enumerable sets. This culminated in the book *Systems that Learn* (Osherson et al. [44], later extended to Jain et al. [31]). From the perspective of linguistics, a promising line was given by Angluin [2] to Gold's scheme bending it to learning recursive languages generated by traditional types of grammars like context-free grammars.

In general, formal learning theory is about *reliable processes* for information acquisition as Kevin T. Kelly, explains:

> A learning problem specifies (1) what is to be learned, (2) a range of relevant possible environments in which the learner must succeed, (3) the kinds of inputs these environments provide to the learner, (4) what it means to learn over a range of relevantly possible environments, and (5) the sorts of learning strategies that will be entertained as solutions. A learning strategy solves a learning problem just in case it is admitted as a potential solution by the problem and succeeds in the specified sense over the relevant possibilities. A problem is solvable just in case some admissible strategy solves it. [34, p. 1]

"Learning problem", "possible environments", "learner", "success", "strategies" and "solvability" sound like computer science terms, but they feature prominently in the dynamic turn of logic as well. And for good reason too. Formal learning theory takes its point of departure with the problem of finding true or empirically adequate, general theories from an ongoing stream of particular, empirical data. The basic idea is to seek procedural justification in terms of reliable truth-tracking performance, rather than in philosophical intuition or other more or less unregimented prescriptions for scientific inquiry—reliable or not. For example, one of the first publications in formal learning theory involved Putnam's computational critique of the learning power of Carnap's confirmation theory [47].[1] Putnam's thread has been taken up by Glymour, Kelly, Schulte and again Osherson et al., and has been applied to more traditional epistemological issues. Such issues include explications of empirical underdetermination and simplicity, critiques of Bayesianism, Ockham's razor, justification of inductive inference, causal discovery, belief revision, and epistemic logic.

Many of these concerns and applications are congruent with the agenda of the dynamic turn in logic. By way of example, the axioms of belief revision [20] may be interpreted as prescriptions for methods of learning [35, 40] for which both their respective learning powers and the relative merits in terms of efficiency and speed may be assessed [32]. How belief revision fairs with respect to learning understood as conditioning and lexicographic revision (central to dynamic epistemic logic) has been investigated in [4, 22]. Similarly, the classical axioms of epistemic logic may be viewed as epistemic learning goals. The learning problem to be settled is then what sort of learners will be able to converge to, and in what sense, the validity of these axioms. Bridging logic and learning is about adding a long-run perspective to epistemic logic in which agents are taken to be mechanisms that learn over time. This is achieved by merging branching alethic-temporal logic with possible environments, learners, success, and strategies, all concepts from formal learning theory. On top of the model of all branching empirical data streams a formal language is introduced that includes epistemic modalities whose indices are learning mechanisms. The idea is then to look for reflections between epistemic axioms in the logic and the structural features of the learning mechanisms [27, 28]. Pursuing the line of formal languages one can formulate conditions for limiting learning in dynamic epistemic and doxastic logic [21].

In general, the connection between formal learning theory and dynamic epistemic logic benefits both paradigms. On the one hand, learning theory receives the fine-structure of well-motivated local learning actions and qualitative logical perspective, which in the long run offers a chance of generic reasoning calculi about inductive learning. On the other hand, dynamic epistemic logic gets a long-term 'horizon' which it missed, criteria for choosing appropriate update rules, and adequate learnability conditions.

Logic and learning are now being brought into close proximity. A decade ago these close encounters were already on the horizon of van Benthem's vision of the dynamic turn uniting logic, computation and learning:

[1] The terms "identifiability", "learnability", and "solvability" are often used interchangeably in formal learning theory. Preferring one over the others is usually determined by the wider, often philosophical, methodological, or technical context. "Identifiability" is used in technical contexts, concerned with choosing (identifying) one among many possibilities (e.g., Turing machines or grammars). "Learnability" is a broader quasi-psychological notion often assumed to be (accurately) modelled by identifiability. Finally, "solvability" occurs in more logic-oriented works, and denotes the possibility of deciding on an issue, e.g., whether a hypothesis is true or false. Obviously, the latter can also be viewed as a kind of identifiability.

> Update, revision, and learning form a coherent family of issues, going upward from short term to long-term behaviour. [6, p. 510]

No better occasion than this to discuss how exactly these central concepts from logic and learning fit together; belief revision and reliability, updating and efficiency, epistemic logic in relation to expressibility and logic and scientific method.

10.2 Belief Revision and Learning

Consider the following scenario. An agent faces uncertainty about the actual state of affairs. She wants to come up with a conjecture that (if not completely, then at least substantially) describes the phenomenon she is confronted with. The progress of this inquiry is driven not only by internal deliberations, but also by observations, outcomes of experiments, those performed by herself and those communicated to her by others. The incoming information triggers occasional changes in her beliefs. It seems natural to assume that we, humans, are naturally equipped with cognitive mechanisms that make such changes possible. The way to mathematically model these mechanisms arises via adopting a high-level perspective of studying the long term belief evolution and its effects—studying not only learning, but also the possible *success* of the learning process. In this context several simple questions have been intensively researched. How to distinguish some policies of changing-ones-mind as more "desirable" than others? How *reliable* are possible belief-revision and knowledge-update policies?

Logical theories of belief revision construct models for belief states in ways that make the latter amenable to changes triggered by appropriately represented information. They propose ways in which the new information gets incorporated into and changes an agent's belief state. Several such theories have already been investigated in light of inductive inference. Here are three note-worthy attempts, of which the most recent one [4, 22] will be dealt with in more detail. Of the other two attempts the first one uses formal learning theory to evaluate belief revision policies and is due to [39, 40]. They rely on a first-order framework for inductive inquiry and within this setting a special class of learners that mimic a belief-revising agent is introduced. The belief revision procedure is that of the AGM paradigm [1], and thus contraction driven. It has been demonstrated, among other things, that a revision method that strongly resembles AGM revision is not universal, i.e., there are problems that are solvable (learnable) in the limit, but cannot be solved by any AGM-learner. The second approach [32, 33, 35] is concerned with the reliability of some belief revision policies, this time for the possible worlds interpretations of belief, given by a variety of authors [14, 15, 26, 43, 48]. The inductive inquiry framework adopted here is that of prediction: the successive data received by the agent are true reports of successive outcomes of some discrete, sequential experiment. The goal of learning is to arrive at a sufficiently informative belief state that allows predicting how the sequence will evolve in the unbounded future. The investigation of the learning power of the procedures listed above indicates that the simple conditioning-based revision may be found among the most powerful.

In the remainder of this section focus is on some examples of belief revision methods and their convergence properties—the truth-tracking power. Before going here attention is directed towards a specification of the basic setting. The remaining part of this section summarises the results of [4, 22], the reader is referred to those for more details and proofs.

10.2.1 Epistemic Spaces, Belief Revision, and Learning

An agent's uncertainty is, as usual, represented by an epistemic space (S, Φ) consisting of a set S of epistemic possibilities, or possible worlds, together with a family of propositions $\Phi \subseteq \mathscr{P}(S)$. As in epistemic logic these propositions represent facts or observables being true or false in any of the possible worlds under consideration. The agent will receive information about a possible world (the actual one), and this stream of data is modelled as an open-ended (infinite) sequence of propositions. For now, abstract away from any time or memory restrictions, so it is assumed that the information keeps on arriving indefinitely in a piecemeal fashion. Such an infinite stream $\varepsilon = (\varepsilon_1, \varepsilon_2 \ldots)$ of successive propositions from Φ will be called a stream for $s \in S$ just in case the set $\{\varepsilon_n : n \in \mathbb{N}\}$ of all propositions in the stream coincides with the set $\{P \in \Phi : s \in P\}$ of all propositions that are true in the given world.

Given such representation a learning method is a function L that on input of an epistemic space (S, Φ) and a finite sequence of observations $\sigma = (\sigma_0, \ldots, \sigma_n)$ outputs a hypothesis. The hypothesis is then a set of possible worlds, i.e., a proposition. In other words, $L((S, \Phi), \sigma) \subseteq S$. Now we can define a condition of learning that closely resembles identification in the limit.

Definition 10.1 Let us take an epistemic space (S, Φ).

A world $s \in S$ is *learnable in the limit by a method* L if, for every observational stream ε for s, there exists a finite stage n such that $L((S, \Phi), \varepsilon_0, \ldots, \varepsilon_k) = \{s\}$ for all $k \geq n$.

The epistemic space (S, Φ) is said to be *learnable in the limit by* L if all its worlds are learnable in the limit by L.

Finally, the epistemic space (S, Φ) is *learnable in the limit* just in case there is a learning method that can learn it in the limit.

The above notion of learning is additionally motivated by the fact that the epistemic state resulting from a *successful* learning process need not be as strong as irrevocable knowledge, i.e., the S5 type of knowledge. It rather matches the defeasible type of knowledge proposed by Lehrer [37, 38] and others, formalized by Stalnaker [49] and rediscovered in modal logics under the name of 'safe belief'. The strength of safety is in the guarantee that it provides: a safe belief is not endangered by new veritistic observations. In other words, defeasible knowledge emerges when stability is reached. The need for such a notion appeared in many different frameworks: from reaching an agreement in a conversational situation (see, e.g., [37, 38]) to considerations in philosophy of science pertaining to infallible scientific knowledge (see, e.g., [28]).

The above described learning method outputs conjectures aiming at one that would uniquely describe the actual world. Such a definition does not however give any insight into the details of the underlying deliberation process—what makes the learning method choose one conjecture over another? To address this question we will, in a manner of speaking, be 'plugging-in a belief-revision engine'.

In order to make an epistemic space account for beliefs one may enrich it with a plausibility order. The idea is that although remaining uncertain between several options, the agent holds some of them most entrenched—those seem to her simply more plausible or more probable, or more elegant than other options—and hence she *believes* in what is true in all those best states. We set a plausibility space to be $(S, \Phi, \leq)$, where $\leq$ is a total preorder on S, called plausibility order.

Three qualitative belief revision policies have received substantial attention in dynamic epistemic logic: conditioning, lexicographic revision, and minimal revision. In particular, the first may be related (via its eliminative nature) to public announcement logic [46], the remaining two have been given a logical treatment and a complete axiomatisation by van Benthem [8].

Definition 10.2 Take a plausibility space $(S, \Phi, \leq)$ and a proposition $p \in \Phi$. Below we will call any $s \in p$, a 'p-world'.

1. *Conditioning* of the plausibility space $(S, \Phi, \leq)$ with the proposition p results in removing all inconsistencies with p, i.e., the operation gives a new plausibility space $(S', \Phi', \leq')$, where S' includes only the p-worlds and Φ' as well as $\leq'$ are cut down to the new domain, S'.
2. *Lexicographic revision* of the plausibility space $(S, \Phi, \leq)$ with the proposition p results in keeping the same states in S but promoting all the p-worlds to be more plausible than all those that are not p-worlds, and within the two clusters the order remains unchanged.
3. *Minimal revision* of the plausibility space $(S, \Phi, \leq)$ with the proposition p results in promoting the most plausible p-worlds to be the most plausible overall, the rest of the order remaining the same. As in the case of lexicographic revision, S stays the same throughout the process.

A belief-revision method is then a function R that upgrades a plausibility state, i.e., it associates to any plausibility space $(S, \Phi, \leq)$ and any sequence $\sigma = (\sigma_0, \ldots, \sigma_n)$, some new plausibility space $R((S, \Phi, \leq), \sigma) := (S^\sigma, \Phi^\sigma, \leq^\sigma)$, with $S^\sigma \subseteq S$, $\Phi^\sigma = \{P \cap S^\sigma : P \in \Phi\}$, and $\leq^\sigma$ is $\leq$ revised by method R under sequence σ.

Now we are ready to merge the learning function and the revision function into a belief revision based learning method. First, take an epistemic space (S, Φ) together with a prior-plausibility assignment given by some $\leq_S$. From the resulting plausibility space and a belief revision function R obtain in a canonical way the learning method L^R, given by:

Table 10.1 Universality of belief revision policies under different kinds of data

	Conditioning	Lexicographic	Minimal
Positive	Yes	Yes	No
Positive and negative	Yes	Yes	No
Fair	No	Yes	No

$$L^R((S,\Phi),\sigma) := \min\ R((S,\Phi,\leq_S),\sigma),$$

where $\min(S,\Phi,\leq)$ is defined to be the set of all the least elements of S with respect to $\leq$ (if such least elements exist) or $\emptyset$, otherwise.

Definition 10.3 An epistemic space (S,Φ) is *learnable in the limit by a belief-revision method* R if there exists some prior plausibility order $\leq_S$ such that (S,Φ) is learnable in the limit by the canonical learning method $L^R(S,\Phi,\leq_S)$.

Learning methods differ in their learning power. One may investigate the issue of the learnability range by looking for the most powerful among them, those that are universal—those that can learn any epistemic state that is learnable by any other method.

10.2.2 Learning Power of Belief Revision

The results of universality of the aforementioned belief revision policies are summarised in Table 10.1. For the sake of completeness, here we report on the universality of learning by belief revision policies under three different conditions. The first one is learning from streams of positive data, which for any possible worlds s enumerate only propositions true in s. The second is learning from streams of positive and negative data, where data streams enumerate propositions and negations of propositions true in s. Finally, fair streams represent unfriendly conditions—when some observational errors may occur. For this, we give up soundness of data streams, i.e., the condition that in the data stream for a possible world s only the information that is true in s can occur, and replace it by a "fairness" assumption: errors occur only finitely often and are always eventually corrected. Unsurprisingly, this can be destructive for conditioning. If erroneous observations are possible, then eliminating worlds that do not fit the observations is risky business.

For the above-listed universality results a non-standard setting, allowing non-well-founded plausibility orders, is essential, i.e., neither of those methods is universal with respect to well-founded prior plausibility orders.[2]

[2] In general some of the proofs require a construction of an appropriate prior plausibility order. For this some classical learning-theoretic concepts and results are used, i.e., locking sequences introduced by Blum and Blum [13], as well as finite tell-tale sets and the simple non-computable version of Angluin's theorem [2], see also the next section.

The results summarized in this section and developed further in the original work provide additional insight with respect to the motivation for belief-revision operators in epistemic logic. In particular, the different capabilities conditioning and lexicographic have to deal with errors lends formal justification to the intuition that the intention of performing lexicographic revision means having less trust in the source of information. On the other hand, the results on minimal revision challenge its popular characteristics as the safest revision policy. Moreover, the most popular approach to modelling beliefs in possible worlds semantics, namely by guaranteeing the well-foundedness of the underlying preorders turn out to be restrictive for learnability.

The above setting and its results are learning-theoretic in spirit, but they also contribute to the study of truth-tracking and truth approximation within the dynamic epistemic logic tradition. The inductive inference perspective leads to studying new relevant features of iterated revision: data-retention, conservatism, history-independence and ways in which these influence the learning process (see [4, 22]). Note that some limit phenomena within iteration scenarios in doxastic-epistemic logic have been studied before in the context of game theory, involving plausibility changes in games in a learning process with active agents trying to both 'learn' and 'teach' (see [9]), and in the context of belief revision, where one can observe a trade-off between initial plausibility order and plausibility order built up from local cues during the learning process (see [5]).

10.3 Conclusive Update and Efficiency

In formal learning theory the particular way of learning is not prescribed but usually supposed to obey certain computability constraints whereas in dynamic epistemic logic there are intuitively clear, determinate manners in which models are updated disregarding computability. It is interesting to compare the two aspects of determinateness and computability. In the previous section we introduced the basic (non-effective) version of convergence. In formal learning theory, learning is commonly studied as an effective procedure and learners are taken to be recursive functions.

Let us see how this may work in the present setting. Again, consider the epistemic space (S, Φ). Firstly, assume that it can consist of at most countably many possibilities in S and countably many relevant propositions given in the set Φ. Moreover, we will introduce the condition of uniform decidability of the epistemic space. An epistemic space is uniformly decidable just in case there is a computable function f that for each pair consisting of a possible world and a proposition decides whether the proposition is true or false in the possible world.

Definition 10.4 An epistemic space (S, Φ) is *uniformly decidable* just in case there is a computable function $f : S \times \Phi \to \{0, 1\}$ such that:

$$f(w, p) = \begin{cases} 1 & \text{if } s \in p, \\ 0 & \text{if } s \notin p. \end{cases}$$

In epistemic logic it is common to assume that checking whether or not an atomic proposition holds within a possible world is treated as primitive and its complexity is left out. Hence, the assumption of uniform decidability does not seem to be restrictive with respect to the traditional setting. It does however seem non-trivial in the analysis of some epistemic situations, e.g., scientific scenarios, where performing such an atomic test may be hard. This simple and appealing condition is used to investigate the properties of convergence to knowledge.

Since the overall number of possibilities is at most countable we can name them with natural numbers. Similarly, the set of propositions Φ is countable. So, assume that for an epistemic space (S, Φ), $S = \{s_1, s_2, s_3, \ldots\}$ and $\Phi = \{p_1, p_2, p_3, \ldots\}$. In this context it is easy to see that the function f that gives the uniform decidability of an epistemic space can may thought of as a number theoretic function $f : \mathbb{N} \times \mathbb{N} \to \{0, 1\}$. Throughout this section unless specified otherwise, assume the epistemic spaces to be uniformly decidable. Moreover, for sake of simplicity assume that in S there are no multiple worlds that make exactly the same propositions true (this assumption is not essential, see [23]).

In this new setting one may easily define the effective version of learnability in the limit.

Definition 10.5 Take an epistemic space (S, Φ).

A world $s_m \in S$ is *effectively learnable in the limit by a function* L if L is recursive and for every observational stream ε for s, there exists a finite stage n such that $L((S, \Phi), \varepsilon_0, \ldots, \varepsilon_k) = \{m\}$ for all $k \geq n$.

The epistemic space (S, Φ) is said to be *effectively learnable in the limit by* L if L is recursive and all the worlds in S are learnable in the limit by L.

Finally, the epistemic space (S, Φ) is *effectively learnable in the limit* just in case there is a recursive learning function that can learn it in the limit.

The remaining part of this section summarizes the results of Gierasimczuk [22], Gierasimczuk and de Jongh [23], the reader should consult those for proofs and a more detailed and rigorous presentation.

10.3.1 Conclusive Update

The above notion of learnability in the limit guarantees the existence of a method that allows for *convergence* to a correct hypothesis. Observe, that the exact moment at which a correct hypothesis has been reached is not known and in general can be uncomputable. Things are different if we require learning to be conclusive, i.e., if the learner is supposed to definitely decide on one answer after a finite amount of information. We can think of this condition as of one in which the learning function is allowed to answer only once—the gameshow case. Clearly, such a conjecture has to be based on certainty. In other words, the learner must know that the answer she gives is true as there is no chance of a change of mind later. In order to define such convergence we will extend the range of learning function L by $\uparrow$, the answer corresponding to the output "I do not know". In the definition below by $\varepsilon\restriction n$ we mean the initial segment of ε of length n, i.e., the sequence $(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{n-1})$.

Definition 10.6 Learning function L is (at most) *once defined* on (S, Φ) iff for any stream ε for any world in S and any $n, k \in \mathbb{N}$ such that $n \neq k$ it holds that $L(\varepsilon\restriction n) = \uparrow$ or $L(\varepsilon\restriction k) = \uparrow$.

Accordingly, define conclusive learnability[3] in the following way (note the important difference in the main condition when compared to Definition 10.5).

Definition 10.7 Take an epistemic space (S, Φ).

A world $s_m \in S$ is *conclusively learnable in an effective way by a function* L if L is recursive, once-defined, and for every observational stream ε for s, there exists a finite stage n such that $L((S, \Phi), \varepsilon_0, \ldots, \varepsilon_k) = \{m\}$.

The epistemic space (S, Φ) is said to be *conclusively learnable in an effective way by* L if L is recursive and all its worlds in S are conclusively learnable in an effective way by L.

Finally, the epistemic space (S, Φ) is *conclusively learnable in an effective way* just in case there is a recursive learning function that can conclusively learn it in an effective way.

The necessary and sufficient condition for conclusive learnability (finite identifiability) involves a modified, stronger notion of finite tell-tale [2], the *definite finite tell-tale set*, (DFTT, for short) [36, 42]. Here we give a version adapted to our needs.

Definition 10.8 Let (S, Φ) be an epistemic space. A set $D_i \subseteq \Phi$ is a *definite finite tell-tale set* (DFTT) for s_i in S if:

1. D_i is finite,
2. $s_i \in \bigcap D_i$, and
3. for any $s_j \in S$, if $s_j \in \bigcap D_i$ then $s_i = s_j$.

Theorem 10.1 *An epistemic space* (S, Φ) *is conclusively learnable in an effective way just in case there is a recursive function* $f : \mathbb{N} \to \mathscr{P}^{<\omega}(\Phi)$ *such that for each* $n \in \mathbb{N}$, $f(n)$ *is a finite definite tell-tale set for* s_n.

Hence, a possible world is conclusively learnable just in case it makes a finite conjunction of propositions true that is not true in any other possible world.

10.3.2 Eliminative Power and Complexity

Following the idea of propositional update, knowing that one hypothesis is true clearly means being able to exclude all other possibilities. This leads to the qualitative notion of *eliminative power* of a proposition, which stands for the set of possibilities that this proposition excludes.

[3] We will use the name "conclusive learnability" interchangeably with "finite identifiability" which is also sometimes referred to as "identification with certainty".

Definition 10.9 Consider a uniformly decidable epistemic space (S, Φ), and a proposition $x \in \Phi$. The *eliminative power* of x with respect to (S, Φ) is determined by a function $El_{(S,\Phi)} : \Phi \rightarrow \mathscr{P}(\mathbb{N})$, such that:

$$El_{(S,\Phi)}(x) = \{i \,|s_i \notin x \;\&\; s_i \text{ in } S\}.$$

Additionally, for $X \subset \Phi$ we write $El_{(S,\Phi)}(X)$ for $\bigcup_{x \in X} El_{(S,\Phi)}(x)$.

In other words, function $El_{(S,\Phi)}$ takes x and outputs the set of indices of all the possible worlds in (S, Φ) that are inconsistent with x, and therefore, in the light of x, can be "eliminated". If one were to link this notion to the epistemic logic terminology, eliminative power of a proposition is the complement of its extension in the epistemic space. This idea applies in a similar way to any formula of any (epistemic) modal language. We may now characterise finite identifiability in terms of eliminative power.

Proposition 10.1 *A set D_i is a definite tell-tale of s_i in S iff*

1. *D_i is finite, and*
2. *$El_{(S,\Phi)}(D_i) = \mathbb{N} - \{i\}$.*

We will now proceed to analyze the computational complexity of finding DFTTs. In order to do so attention is restricted to finite collections of finite sets. One may question the purpose of further reduction of sets that are already finite. As a matter of fact, if a finite collection of finite sets is finitely identifiable, then each set is already its own DFTT, but obviously finite sets can be much larger than their minimal DFTTs. A simple observation to start with:

Proposition 10.2 *Let (S, Φ) be such that S and Φ are both finite. For any s_i in S, $El_{(S,\Phi)}\{x \mid s_i \in x\}$ can be computed in polynomial time w.r.t. the size of epistemic space (i.e., $card(\Phi) \times card(S)$).*

Then the computational problem of conclusive learnability of an epistemic space is defined in the following way.

Definition 10.10 (FIN- ID *Problem*)
Instance: A finite epistemic space (S, Φ), a world s_i in S.
Question: Is s_i conclusively learnable within (S, Φ)?

Theorem 10.2 FIN- ID *Problem is in P.*

This result does not settle the issue of the efficiency of conclusive learning. In this context an interesting notion is the minimality of DFTTs. Finding the minimal, and even better, the minimal-size DFTTs may be viewed as the task of an efficient teacher, who looks for an optimal sample that allows conclusive learning. There are two nonequivalent ways in which DFTTs can be minimal. Call D_i a *minimal* DFTT of s_i in (S, Φ) just in case all the elements of the sets in D_i are essential for finite identification of s_i in (S, Φ), i.e., taking any element out of the set D_i will decrease the set's eliminative power with respect to (S, Φ), in such a way that it will no longer be a DFTT. In other words, minimal DFTTs of a state contain sufficient information to exclude other possibilities and involve no redundant data.

Definition 10.11 Take an epistemic space (S, Φ), and s_i in S. A *minimal DFTT of* s_i *in* (S, Φ) is a D_i, such that:

1. D_i is a DFTT for s_i in (S, Φ), and
2. for all $X \subset D_i$, X is not a DFTT for s_i in (S, Φ).

Proposition 10.3 *Let* (S, Φ) *be a finitely identifiable finite epistemic space. Finding a minimal DFTT of* s_i *in* (S, Φ) *can be done in polynomial time w.r.t.* $card(\{x \mid s_i \in x\})$.

Note that a possible world may well have many minimal DFTTs of different cardinalities. This is enough reason to introduce a second notion of minimality—*minimal-size* DFTT. Minimal-size DFTTs are the minimal DFTTs of smallest cardinality.

Definition 10.12 Consider an epistemic space (S, Φ), and s_i in S. A *minimal-size DFTT of* s_i *in* (S, Φ) is a D_i, such that

1. D_i is a DFTT for s_i in (S, Φ), and
2. there is no DFTT $D_i{}'$ for s_i such that $card(D_i{}') < card(D_i)$.

How hard is it to find minimal-size DFTTs? In order to answer this question we will first specify the corresponding computational problem.

Definition 10.13 (MIN- SIZE *DFTT Problem*)

Instance: A finite epistemic space (S, Φ), a possible world $s_i \in S$ and a positive integer $k \leq card(\{p \mid s_i \in p\})$.

Question: Is there a DFTT $X_i \subseteq S_i$ of size $\leq k$?

Theorem 10.3 *The* MIN- SIZE DFTT *Problem is NP-complete.*

10.3.3 Preset Learning and Fastest Learning

Attention is now devoted to learners who can be seen as taking a more prescribed course of action by basing their conjectures on symptoms, i.e., on their knowledge of (some) DFTTs. Of course, if an s_i has a DFTT, it will have many, usually infinitely many DFTTs, e.g., each finite set of propositions true in s_i, which is a superset of a DFTT for s_i is a DFTT for s_i as well. Hence, it is more useful to express the learner's access to DFTTs by means of a so-called *dftt-function*. Such a function, let us call it f_{dftt}, is supposed to decide on an input of a finite set $X \subset \Phi$ and an $i \in \mathbb{N}$, whether it considers X to be a DFTT for s_i ($f_{dftt}(X, i) = 1$) or not ($f_{dftt}(X, i) = 0$). In case of a finitely identifiable epistemic space (S, Φ) there exists a dftt-function that recognizes for each $i \in \mathbb{N}$ at least one X as a DFTT for $s_i \in S$.

A learner that uses such a dftt-function in the process of identification is called a *preset learner*. Intuitively speaking, each time the learner receives a new input, and all the answers before have been $\uparrow$, the learner looks for the first world that accounts for the propositions listed in the sequence enumerated so far. Assume that this world's index is i. Then among the content of the sequence observed so far the learner looks for a subset X for which $f_{dftt}(X, i) = 1$. If the learner finds one, it answers with i, otherwise it answers with another $\uparrow$. It has been shown that if an epistemic space is finitely identifiable at all it is finitely identifiable by a preset learner; it is also the case that the preset learners are exactly those learners that react solely to the set-theoretic content of the information received, disregarding the order and multiplicity of the information.

If a preset learner L is based on a dftt-function that recognizes all DFTTs for all S_i, then the learner will make the proper conjecture always at the earliest possible stage of inquiry. Refer to such a learner as a *fastest learner*. It is clear that the procedure of the fastest learner is closely related to the DEL approach. There is an interesting question whether the fastest learner is always a recursive one. In [23] it has been shown that this is not the case.

Intuitively, *fastest learner* finitely identifies a world s_i as soon as objective 'ambiguity' between languages has been lifted. In other words, define the extreme case of a finite learner who settles on the right language as soon as *any* DFTT for it has been enumerated. We will characterize such a fastest learner as a preset learner based on the collection of all DFTTs.

Take again a finitely identifiable epistemic space (S, Φ), and s_i in S. Now, consider the collection $\mathbb{D}_i$ of all DFTTs of s_i in (S, Φ). For any sequence of data σ, $set(\sigma)$ stands for the set of propositions occurring in σ.

Definition 10.14 (S, Φ) is *finitely identifiable in the fastest way* if and only if there is a learning function L such that, for each ε and for each $i \in \mathbb{N}$,

$$\begin{aligned} L(\varepsilon \restriction n) = i \ \text{ iff } \ & \exists D_i^j \in \mathbb{D}_i \ (D_i^j \subseteq \text{set}(\varepsilon \restriction n)) \& \\ & \neg \exists D_i^k \in \mathbb{D}_i \ (D_i^k \subseteq \text{set}(\varepsilon \restriction n - 1)). \end{aligned}$$

Refer to such L as a *fastest learning function*.

Theorem 10.4 *There is a uniformly decidable epistemic space that is finitely identifiable, but for which no recursive function F exists such that for each i, $F(i)$ is the set of all minimal DFTTs for s_i.*

Proposition 10.4 *There is a uniformly decidable epistemic space that is finitely identifiable, but for which no recursive function F exists such that for each i, $F(i)$ is the set of all minimal-size DFTTs for S_i.*

Time to turn to the more general question whether every finitely identifiable class has a fastest learner. The answer is negative—there are finitely identifiable classes of languages which cannot be finitely identified in the fastest way.

Theorem 10.5 *There is a uniformly decidable epistemic space that is finitely identifiable, but is not finitely identifiable in the fastest way.*

Theorem 10.5 shows that fastest finite identifiability is properly included in finite identification and hence also in preset finite identification. Therefore, we have demonstrated the existence of yet another kind of learning, even more demanding than finite identification. Speaking in terms of conclusive update, our considerations show that in some cases, even if computable convergence to certainty is possible, it is not computable to reach that certainty the moment in which objective ambiguity disappears.

In the light of these discoveries about preset learning there is an additional computational justification for introducing multi-agency to this setting. It is interesting to switch the perspective from the single agent, learning-oriented view, to the two agent game of learner and *teacher* (see [3, 24]). The responsibility of effective learning, in the line with natural intuitions, is in the hands of the teacher, whose computational task is to find samples of information that guarantee optimal learning. Intuitively, it is not very surprising that the task of finding such minimal samples can be more difficult than the complexity of the actual learning. As such, computing the minimal(size) DFTTs seems to go beyond the abilities of the learner and is not necessary in order to be rational or successful. However, such a task is naturally performed by a teacher.

10.4 Epistemic Logic and Learning

This section devotes more attention to the syntactic counterparts of the logical approach to learnability. The previously chosen semantics may be reflected in an appropriate syntax for knowledge, belief, and their changes over time, both in dynamic and temporal settings.

The approach to inductive learning in light of dynamic epistemic and epistemic temporal logic is as follows: Take the initial class of sets to be possible worlds in an epistemic model, which mirrors the learner's initial uncertainty over the range of sets. The incoming pieces of information are taken to be events that modify the initial model. We will show that iterated update on epistemic models based on finitely identifiable classes of sets is bound to lead to the emergence of irrevocable knowledge. In a similar way identifiability in the limit leads to the emergence of safe (truthful and stable) belief. From here we consider a general temporal representation of learning in the limit. The relationship between dynamic epistemic logics and temporal epistemic logics has been studied (see [11, 12]). Given this correspondence, the study of convergence brings about new interesting problems.

10.4.1 Learning and Dynamic Epistemic Logic

The uncertainty range of the agent is revised as new pieces of data (in the form of propositions) are received. The information comes from a completely trusted source, and as such causes the agents to eliminate the worlds that do not satisfy it. In learning theory the truthfulness of incoming data is often assumed, and therefore, in principle, it is justified to use propositional and epistemic update as a way to conduct inquiry (for such interpretation of update see [7]). The following assumes basic knowledge of dynamic epistemic logic (for an overview see Chap. 6 of this book).

Epistemic states may be transformed into epistemic models, and plausibility states into doxastic models in order to deal with the epistemic languages. The initial learning model is a simple single-agent epistemic model whose structure corresponds to the initial epistemic space.

Definition 10.15 Let us take an epistemic space (S, Φ). For every proposition in $p_n \in \Phi$ we take a symbol $\mathtt{p_n} \in \textsc{Prop}$. Moreover, we will use the set $\textsc{Nom}$, which contains a nominal symbol $\mathtt{i}$ for every $i \in \mathbb{N}$. *The initial learning model* $\mathscr{M}_{(S,\Phi)}$ is a triple:

$$\langle W, \sim, V\rangle,$$

where $W := S$, $\sim := W \times W$, $V : \textsc{Prop} \cup \textsc{Nom} \to \mathscr{P}(W)$, such that $s_i \in V(\mathtt{p_n})$ iff $s_i \in p_n$ in (S, Φ), and for any $\mathtt{i} \in \textsc{Nom}$ we set $V(\mathtt{i}) = \{s_i\}$.

Similarly, every epistemic plausibility space $(S, \Phi, \leq)$ can be straightforwardly turned into an epistemic doxastic model $\mathscr{M}_{(S,\Phi,\leq)} = \langle W, \sim, \leq, V\rangle$.

On such models, as on other epistemic models one may interpret epistemic and doxastic logic languages in a standard way. Dynamic versions of such logics include some additional operators that allow describing changes taking place within a model. One particular logic of this type is public announcement logic (PAL, see [46]), where basic epistemic logic is extended to account for update with a specific 'φ-announcement' expression, written as $!\varphi$.

Definition 10.16 (*Syntax of* $\mathscr{L}_{\mathrm{PAL}}$) The syntax of epistemic language $\mathscr{L}_{\mathrm{PAL}}$ is defined as follows:

$$\begin{aligned} \varphi &:= \ p \mid \neg\varphi \mid \varphi \vee \varphi \mid K_a\varphi \mid [A]\varphi \\ A &:= \ !\varphi \end{aligned}$$

where $p \in \textsc{Prop}$, $a \in \mathscr{A}$, where $\mathscr{A}$ is a set of agents.

Definition 10.17 (*Semantics of* $\mathscr{L}_{\mathrm{PAL}}$) For the epistemic fragment $\mathscr{L}_{\mathrm{EL}}$ the interpretation is as usual (see Chap. 6). The remaining clause of $\mathscr{L}_{\mathrm{PAL}}$ is as follows.

$$\mathscr{M}, w \models [!\varphi]\psi \text{ iff if } \mathscr{M}, w \models \varphi \text{ then } \mathscr{M} \mid \varphi, w \models \psi$$

It has been shown that epistemic update performed on finitely identifiable class of sets leads to irrevocable knowledge.

Theorem 10.6 *The following are equivalent:*

1. *An epistemic space* (S, Φ) *is finitely identifiable.*
2. *For every* $s_i \in S$ *and every data stream* ε *for* s_i *there is an* $n \in \mathbb{N}$ *such that for all* $m \geq n$, $\mathscr{M}_{(S,\Phi)}, s_i \models [!(\bigwedge set(\varepsilon \restriction m))]\, K\, \mathtt{i}$.

A similar in spirit, but more complex result may be obtained for identifiability in the limit and doxastic version of public announcement logic whose language includes the expression $B_a\varphi$, interpreted on epistemic doxastic models in the following way.

Definition 10.18 (*Semantics of* $\mathscr{L}_{\text{DOX- PAL}}$)

$$\begin{array}{ll} \mathscr{M}, w \models B_a\phi \text{ iff} & \text{there is } v \in W \text{ such that } v \sim_a w \text{ and } v \leq_a w \\ & \text{and for all } s \in W \text{ such that } s \leq_a v \text{ and } s \sim_a w \text{ it holds that} \\ & \mathscr{M}, s \models \varphi \end{array}$$

Theorem 10.7 *The following are equivalent:*

1. (S, Φ) *is identifiable in the limit.*
2. *There is a plausibility preorder* $\leq \subseteq S \times S$ *such that for every* $s_i \in S$ *and every data stream* ε *for* s_i *there is* $n \in \mathbb{N}$ *such that for all* $m \geq n$, $\mathscr{M}_{(S,\Phi,\leq)}, s_i \models [!(\bigwedge set(\varepsilon \restriction m))]B\, \mathtt{i}$.

The results on the universality of lexicographic revision in [4, 22] allow drawing a corollary that the above theorem will also hold for the dynamic logic of lexicographic upgrade, in which case the update operator ! is replaced with $\Uparrow$. However, such results for the dynamic logic of minimal upgrade (also known as elite change, with the operator $\uparrow$) cannot be obtained.

On the grounds of different results from [4, 22], the plausibility preorder mentioned in Theorem 10.7 sometimes must be non well-founded, allowing models without minimum words according to $\leq$. Decision is thus necessary as to how to interpret the belief operator, $B_a\varphi$. Above, a more general, limiting interpretation of belief operator has been introduced—the agent believes that ϕ in a world w just in case she considers a more plausible world v such that all words that are more plausible than v satisfy φ [22, Chap. 6]).

The last remark concerns the meaning of $K\,\mathtt{i}$ ($B\,\mathtt{i}$) in the characterising formulas—they stand for the knowledge (belief) of what is the actual state. Going back to our original motivation, that of formal learning theory, it is contingent on what ones take to being the right, finite, generatively complete description of the world. In terms of propositional knowledge and belief this corresponds to the following: whatever is true in the actual world I know (believe) that it is true and vice versa. In other words, we may say: $K\,\mathtt{i}$ iff for any $p \in Prop$ such that $s_i \in p$ we have that $p \leftrightarrow Kp$, and similarly for $B\,\mathtt{i}$.

10.4.2 Learning and Temporal Logic

May one achieve a more complete description of learning in the limit with modal logic? To that end, a logic that allows quantifying over time and over possible histories is needed. Temporal logic offers such a view.[4]

First, consider what could serve as candidate for a temporal unfolding of an possible world s_i in an epistemic space (S, Φ). Instead of viewing a world as a set of atomic propositions, we can represent it as a set of (infinite) histories—possible streams of observations that could take place provided s_i is the actual state. What are the possible streams of information within s_i depends on, let us say, the "nature" of the possibility. Are the events that it generates sequential, is it possible that they permute, will some of them repeat, must everything that is true eventually occur? The learning-theoretic paradigm considered here offers a particular set of answers to those questions. Data streams of a given possibility s_i enumerate all and only the propositions true in s_i, the order of propositions does not matter, and repetitions can occur without restrictions.

Hence, thinking of the propositions true in s_i as events that might occur in s_i, build a temporal structure describing possible future evolutions at s_i. The learning theoretic paradigm requires that those are finite prefices of certain infinite data streams. The latter are infinite sequences of propositions true at s_i which enumerate all and only those propositions, possibly with repetitions. The finite sequences that may be observed are hence determined by a "protocol" that permits certain infinite streams at s_i. Each possible word in our initial uncertainty range can be assigned such a temporal representation. Note that for different possibilities we get mutually disjoint protocols. However, in each point of time the agent observes only a finite sequence of events and obviously such a finite prefix of an infinite data stream can be consistent with more than one possibility. Observing such sequences would not give the agent enough information to distinguish between the two worlds. In such case the agent is uncertain between the two finite sequences not only with respect to how the finite sequence will develop in the future, but also unsure as to the original possibility that generated the sequence. The temporal forest is transformed into an epistemic temporal forest, where the uncertainty relation of the agent will relate identical finite sequences, and the valuation is copied from the initial epistemic space. The resulting structure is an epistemic temporal model (see [18, 19, 45]) that represents all possible evolution of a learning scenario. Below we will call such a structure $\texttt{For}((S, \Phi), P)$, a forest built from an epistemic space (S, Φ) and a protocol P.

A relevant epistemic temporal language $\mathscr{L}_{\mathrm{ETL}^*}$ contains the following expressions:

$$\varphi := p \mid \neg\varphi \mid \varphi \vee \varphi \mid K\varphi \mid F\varphi \mid A\varphi$$

where p ranges over a countable set of proposition letters PROP. $K\varphi$ reads: 'the agent knows that φ'. Symbol F stands for future, and we define G to mean $\neg F \neg$. $A\varphi$ means: 'in all infinite continuations conforming to the protocol, φ holds'.

[4] This section overviews the approach given in [17, 22].

$\mathscr{L}_{\mathrm{ETL}^*}$ is interpreted over epistemic temporal frames, $\mathscr{H}$, and pairs of the form (ε, h), the former being a maximal, infinite history in our trees, and the latter a finite prefix of ε (see [41, 45]) in the usual way. The modality 'A' refers to the particular infinite sequences that belong to the chosen protocol. It may be viewed as an operator that performs a global update on the overall temporal structure, 'accepting' only those infinite histories that conform to the protocol.

To give a temporal characterization of finite identifiability the following idea must be expressed: In the epistemic temporal forest, for any starting, bottom node s_i it is the case that for all infinite data streams in the future there will be a point after which the agent will know that she started in s_i, which means that she will remain certain about the part of the forest she is in. The designated propositional letters from $\textsc{Prop}_{\textsc{Nom}}$ correspond to the partitions, which can also be viewed as underlying theories that allow predicting further events.[5] Formally, with respect to finite identifiability of sets, the following theorem holds.

Theorem 10.8 *The following are equivalent:*

1. (S, Φ) *is finitely identifiable.*
2. $\texttt{For}((S, \Phi), P) \models \texttt{i} \rightarrow AFGK\texttt{i}$.

In order to give a temporal characterization of identifiability in the beliefs of the learner must be expressed. Therefore, the temporal forests should include a plausibility ordering. Conditioning (update) is a universal learning method from truthful data. In other words, in the case of identifiability in the limit, eliminating the worlds of an epistemic plausibility model is enough to reach stable and true belief. This allows for considering very specific temporal structures that result from updating a doxastic epistemic model with purely propositional information.[6] The epistemic temporal models need to be extended with a doxastic preorder on infinite data streams, and the epistemic temporal language needs to be extended with a doxastic counterpart, the belief operator B, that works in a way similar to dynamic doxastic epistemic logic.

As in the case of finite identifiability we will now provide a formula of doxastic epistemic temporal logic that characterises identifiability in the limit.

Theorem 10.9 *The following are equivalent:*

1. (S, Φ) *is identifiable in the limit.*
2. *There exists a plausibility preorder* $\leq \subseteq S \times S$ *s.t.* $\texttt{For}((S, \Phi, \leq), P) \models \texttt{i} \rightarrow AFGB\texttt{i}$.

The above results show that the two prominent approaches, learning theory and epistemic modal-temporal logics, may be joined together to describe the notions of belief and knowledge involved in inductive inference. Bridging the two approaches benefits both sides. For formal learning theory, to create a logic for it is to provide

[5] The characterisation involving designated propositional letters can be replaced with one that uses nominals as markers of bottom nodes. For such an approach see Dégremont and Gierasimczuk [16].

[6] For more complex actions performed on plausibility models in the context of the comparison between dynamic doxastic and doxastic temporal logic see van Benthem and Dégremont [11].

additional syntactic insight into the process of inductive learning. For logics of epistemic and doxastic change, it enriches their present scope with different learning scenarios, i.e., not only those based on the incorporation of new data but also on generalisation.

The temporal logic based approach to inductive inference gives a straightforward framework for analyzing various domains of learning on a common ground. In terms of protocols, sets may be seen as classes of specific histories—their permutation-closed complete enumerations. Functions, on the other hand, may be viewed as 'realities' that allow only one particular infinite sequence of events. We can think of many intermediate concepts that may be the object of learning. Interestingly, the identification of protocols, that seems to be a generalization of the set-learning paradigm provides what has been the original motivation for epistemic temporal logic from the start: identifying the current history that the agent is in, including its order of events, repetitions, and other constraints.

10.5 Logic, Learning, and Scientific Method

Logic and learning theory may also be bridged by considering the methodological merits of learners for identifying classical axioms of epistemic logic. This means treating epistemic axioms as learning goals and then considering the methodological constraints on learners for converging to the truth of such axioms (Hendricks [27, 28] and Kevin T. Kelly's chapter, Chap. 11). Axioms T, K, 4, 5 for instance, all present different learning problems for definitions of knowledge based on limiting convergence for both assessment and discovery methods. Assessment methods take as inputs finite evidence sequences and hypothesis, and map them to onto truth or falsity, while discovery methods conjecture hypotheses (sets of possible worlds) in response to incoming evidence. It turns out that the validity of canonical axioms of epistemic logic may be acutely sensitive to the methodological constraints enforced on the methods of scientific inquiry whether based on assessment or discovery. A method of scientific discovery may be consistent in the sense that it only conjectures something consistent with current evidence, consistently expectant insofar as it conjectures something consistent with the evidence and expects to see more of the same, or may be infallible in the sense that it only conjectures something which is entailed by the evidence observed so far. Now, these different methodological constraints come into play once one attempts to validate axioms of epistemic logic. No amount of methodology is going to help validating the axiom of negative introspection (5) and the reason is intuitively this: Not knowing means not having converged, which does not entail—not even for the infallible learner—knowledge of lack of convergence to the truth. Similarly, arguments since the times of American pragmatism have conveyed that positive introspection and limiting convergence are irreconcilable since positive introspection demands knowledge of the modulus of convergence to knowledge yet limiting definition entails exactly this very modulus of convergence may not be known—there just is a time, such that for each later time, the method has settled for the truth and will not oscillate again, but it may not necessarily be known when this time will be. However, there is a way to circumvent this conclusion—and thus eat the cake and have it. If the axiom of positive introspection is allowed a diachronic interpretation such that the consequent of knowing that one knows either happens later than the antecedent of knowing or would have obtained later even had things been otherwise, then it is possible to validate the axiom 4 assuming that the discovery method in question is consistently expectant [28, 29]. This result has two significant horns: It demonstrates that axioms of epistemic logic may have a temporal dimension of importance given the dichotomy between synchronic and diachronic interpretations of the axioms, and that their very validity is contingent upon what the method of inquiry decides to do. Additionally, methods of inquiry may work together—assessment methods are definable in terms of discovery methods and vice versa which turns out to being an important feature when considering the transmissibility of knowledge from one agent to another [28]. This is all as it should be. One of the important methodological benefits of treating agent indices of epistemic logic as learning functions is to activate agents in such a way that they play crucial roles in validating the epistemic axioms apparently describing the very rationality of epistemic agency for single agents and multiple agents interacting [30].[7]

The same goes for other methodological recommendations to be found in contemporary literature of formal epistemology. Axioms of belief revision may likewise be interpreted as recommendations for learners and the question then becomes how well these recommendations fare with respect to convergence to the truth—sometimes they do quite well, sometimes they create truth-tracking disasters and inductive amnesia [32, 35].

Combining logic and learning provides a stronghold for epistemological and methodological agent-interactive studies. Such studies, earlier on reserved for either epistemic logic or formal learning theory telling respectively their partial stories, are now given a chance to make for the full story. Computational epistemology is strictly speaking not about knowledge but about learning, but of course learning is about knowledge acquisition. And there you have it as Johan van Benthem would have it: learning as part of the dynamic turn in logic.

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