Exercises
Computational Intelligence Lab
SS 2019

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Problem 1 (Signal denoising):

Let $\mathbf{x} \in \mathbb{R}^D$ be a one dimensional signal:

$$\mathbf{x} = \sum_{k=1}^{L} z_k \mathbf{u}_k + \boldsymbol{\epsilon} = \mathbf{U}\mathbf{z} + \boldsymbol{\epsilon},$$

where $\mathbf{U} = [\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_L]$ is a $D \times L$ matrix ($L \geq D$) containing the elements of a (known) dictionary $\mathcal{U}$ with $|\mathcal{U}| = L$ and $\boldsymbol{\epsilon}$ is a small perturbation.

In addition, we know that $\mathbf{x}$ is *sparse* in $\mathcal{U}$, i.e. $\|\mathbf{z}\|_0 \ll L$. Equivalently, we can say that there exist a permutation σ^* of the indices $\{1, \cdots, L\}$ such that for $\tilde{K} \ll L$,

$$\mathbf{x} = \sum_{k=1}^{\tilde{K}} z_{\sigma^*(k)} \mathbf{u}_{\sigma^*(k)} + \boldsymbol{\epsilon}.$$

Our task is to recover the permutation σ^* and the vector $\mathbf{z}$. To make the task easier we make the following assumptions:

- the atoms (dictionary elements) of $\mathcal{U}$ are orthonormal[1];
- ϵ is a zero-mean Gaussian noise.

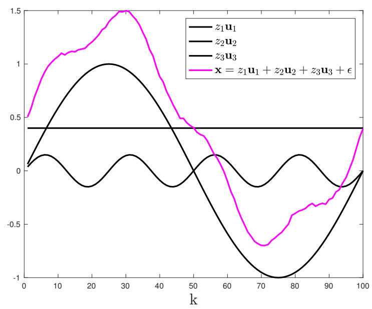


Figure 1: $\mathbf{x} \in \mathbb{R}^{100}$ has a sparse representation but is corrupted by noise.

1) Let $\hat{\mathbf{x}}_\sigma$ be the reconstruction using the estimate σ of σ^* and $\hat{K}$ atoms. Find the solution to the minimization problem

$$\hat{\sigma} = \underset{\sigma}{\operatorname{argmin}} \|\mathbf{x} - \hat{\mathbf{x}}_\sigma\|_2^2. \tag{P}$$

2) Justify why this choice of $\hat{\sigma}$ makes sense.

Problem 2 (1D signal compression and Haar wavelets):

You get now an opportunity to check your intuition from the last exercise using Haar wavelets. Please find the iPython notebook `ex2.ipynb` from

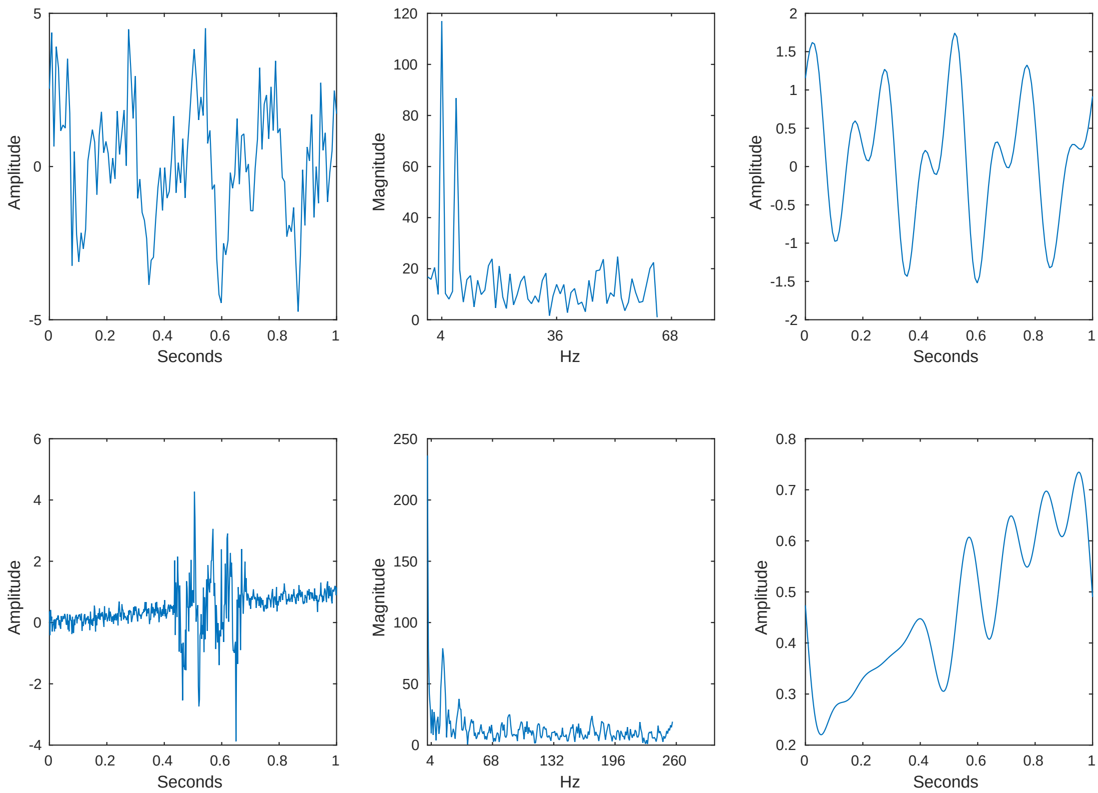
github.com/dalab/lecture_cil_public/tree/master/exercises/ex10/ex2.ipynb,

and answer the questions.

Problem 3 (Choice of dictionary is crucial):

The figure below shows two different 1-D signals (left column) with their corresponding spectrum obtained using the FFT (middle column). In the right column, we show a signal obtained by discarding part of the frequencies in the spectrum.

[1] This implies that each element in $\mathcal{U}$ is "equally important" (key assumption for simple decoding, why?) and that we can represent any element $\mathbf{x} \in \mathbb{R}^D$ in a unique way through $\mathbf{z}$.



- (1) Write down the formula to obtain the spectrum in the middle column of the previous figure, in terms of linear transformation or change of basis (assuming a given basis, i.e. the orthogonal matrix $\mathbf{U}$) applied to the original signal $\mathbf{x}$.
  (2) Write down the inverse formula to obtain the reconstructed signal in the right column in terms of linear transformation (change of basis) applied to the filtered spectrum $\hat{\mathbf{z}}$.
- What part of the signal would you discard to obtain the reconstructed signal? Draw a rectangle on each spectrum in the middle column where everything inside the rectangle is kept for the reconstruction.
- **true/false** The Wavelet transform is a better choice than Fourier for the first signal (top row).
- **true/false** The Wavelet transform is a better choice than Fourier for the second signal (bottom row).
- Looking at the middle figure in the top row, what do the first peaks in the spectrum correspond to?

Problem 4 (Image compression):

Please find the iPython notebook `ex4.ipynb` from

github.com/dalab/lecture_cil_public/tree/master/exercises/ex10/ex4.ipynb,

and answer the questions.