Exercises
Computational Intelligence Lab
SS 2019

Machine Learning Institute
Dept. of Computer Science, ETH Zürich
Prof. Dr. Thomas Hofmann
Web http://cil.inf.ethz.ch/

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(Dictionary Learning and Compressed Sensing)

Problem 1 (Sparse coding with overcomplete dictionary):

Given a signal $\mathbf{x} = (3, 1, -2) \in \mathbb{R}^3$ and an overcomplete dictionary $\mathbf{U} = [\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4] \in \mathbb{R}^{3\times 4}$,

$$\mathbf{U} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix},$$

find the sparse representation $\mathbf{z}$ of the signal $\mathbf{x}$ with $\|\mathbf{z}\|_0 \leq 2$.

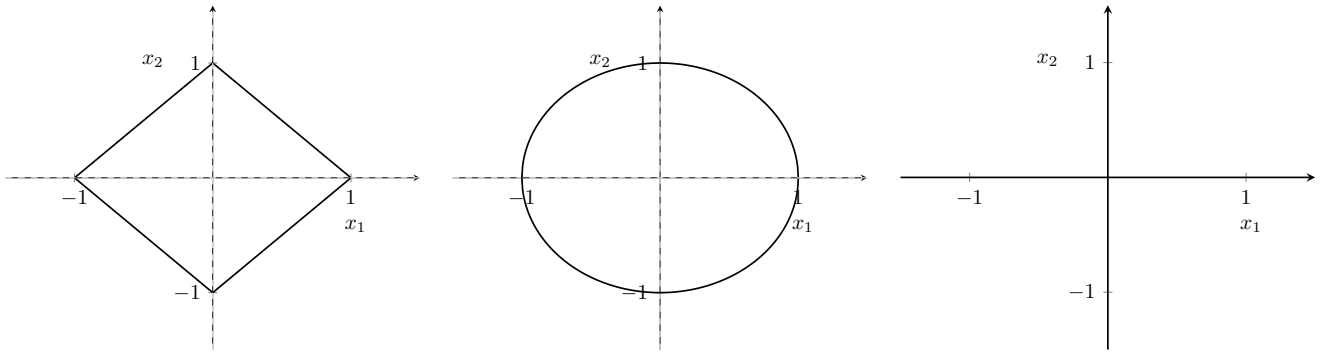
a. Find the atom $\mathbf{u}^{(1)}$ that minimize the reconstruction error $\|\mathbf{x} - \hat{\mathbf{x}}^{(0)}\|_2^2$ where $\hat{\mathbf{x}}^{(0)} = z^{(1)}\mathbf{u}^{(1)}$, and compute the residual $\mathbf{r}^{(1)} = \mathbf{x} - \hat{\mathbf{x}}^{(0)}$.

b. Find the atom $\mathbf{u}^{(2)}$ that minimize the reconstruction error $\|\mathbf{r}^{(1)} - \hat{\mathbf{x}}^{(1)}\|_2^2$ where $\hat{\mathbf{x}}^{(1)} = z^{(2)}\mathbf{u}^{(2)}$.

c. Write down the sparse representation $\mathbf{z}$ of signal $\mathbf{x}$.

Problem 2 (Compressed Sensing):

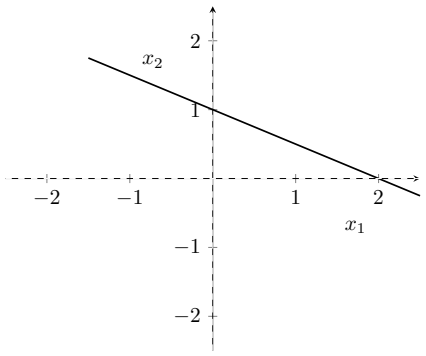
a. Map each of the three equations $\|\mathbf{x}\|_2 = 1$, $\|\mathbf{x}\|_1 = 1$, and $\|\mathbf{x}\|_0 = 1$ to a plot among a., b., or c. on the following figure. Note that $\mathbf{x}$ is s 2D vector with coordinates x_1 and x_2 (i.e. $\mathbf{x} = \Big[x_1, x_2\Big]$).



a. . . . . b. . . . . c. . . . . .

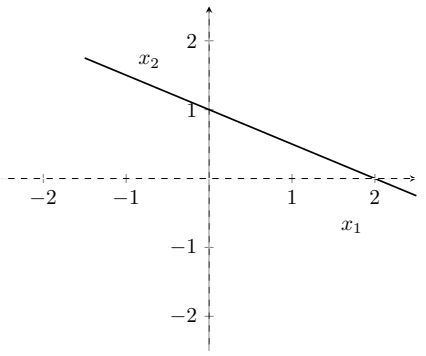
b. Show the solution of each optimization problem on plots a., b., and c. of the following figure.

$\min \|\mathbf{x}\|_2$
Subject to $\frac{1}{2}x_1 + x_2 = 1$



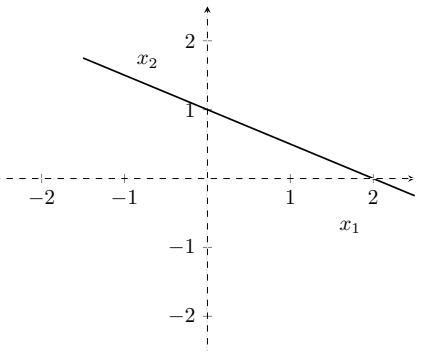
a.

$\min \|\mathbf{x}\|_1$
Subject to $\frac{1}{2}x_1 + x_2 = 1$



b.

$\min \|\mathbf{x}\|_0$
Subject to $\frac{1}{2}x_1 + x_2 = 1$



c.

c. We can formulate the above three optimization problem as

$$\min \|\mathbf{x}\|_p$$
$$\text{subject to } \frac{1}{2}x_1 + x_2 = 1,$$

where $p \in \{0, 1, 2\}$. Mark the right sentence using your previous answers.

[ ] Solutions of the constrained problems have intersection for $p = 1$ and $p = 0$.

[ ] Solutions of the constrained problems have intersection for $p = 2$ and $p = 0$.

Problem 3 (Compressed Sensing):

Please find the iPython notebook ex3.ipynb from

github.com/dalab/lecture_cil_public/tree/master/exercises/ex11

answer the question in this file.

Problem 4 (Matching pursuit algorithm):

In the last tutorial session, you have seen that the matching pursuit algorithm converges. In this exercise, you will show some limitations of the algorithm.

a. Find an overcomplete dictionary and a vector $\mathbf{x}$ such that the approximation $\hat{\mathbf{x}}$ resulting from the matching pursuit algorithm will never exactly equal $\mathbf{x}$ no matter the number of iterations. Note that this implies that at least one atom will be selected multiple times.

b. Find an instance where the sparse representation returned by matching pursuit (assuming that after some number of iterations, the approximation is perfect) is not optimal, i.e. there is a different representation for which the 0-norm is strictly smaller.[1]

[1] Hint: You have to think in at least three dimensions.