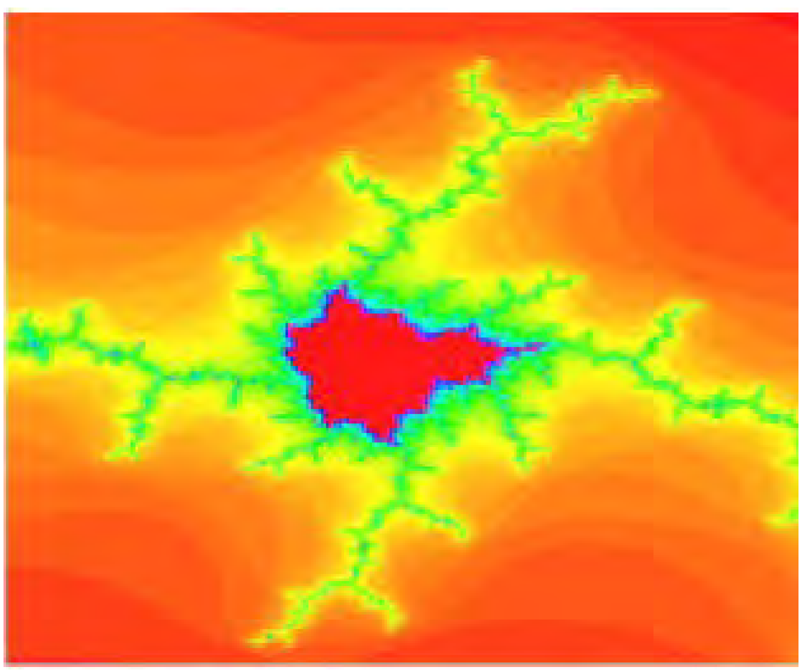
# Chaotic Rendezvous:

## Introductory Chaos Experimentation Using Maple Software and a Desktop Computer

by Thomas E. Oberst



# Chaotic Rendezvous: Introductory Chaos Experimentation Using Maple Software and a Desktop Computer

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# Chapter 11

# The Mandelbrot Set

Finally, we begin the long-anticipated study of the Mandelbrot set. Discovered in 1980 by Benoit B. Mandelbrot, the Mandelbrot set is one of the most fascinating phenomena in all of mathematics. Like the filled Julia sets of the previous chapter, the Mandelbrot set is a beautiful and infinitely complex fractal. It transcends any one filled Julia set of $Q_c(z) = z^2 + c$ because the Mandelbrot set contains information about each and every filled Julia set of $Q_c$ (and hence each and every Julia set of $Q_c$). For this reason it has been called a "dictionary of Julia sets."

Before we define the Mandelbrot set, we must review some details about Julia sets. Recall that, in Chapter 10, filled Julia sets seemed to fall into two main categories: those that were totally connected solid sets, and those that consisted of infinitely many disconnected points. The latter are sets of the Cantor set type. Furthermore, we made the observation that the transition from a completely connected set to a completely disconnected set occurs abruptly for small changes in $c$. This suggests an important fact: there is no middle ground between completely connected filled Julia sets and completely disconnected filled Julia sets. Analytically, the division between these two types of filled Julia sets can be determined using the orbit of the critical point 0 of $Q_c$.

**Theorem 11.1 (The Fundamental Dichotomy)** *Consider the orbit of the critical point 0 of the complex quadratic map $Q_c(z) = z^2 + c$.*

1. *Given a particular value of c, if the orbit of 0 under $Q_c$ is unbounded, then the filled Julia set for $Q_c$ consists of infinitely many disconnected points (and is thus a Cantor set).*

2. *Given a particular value of c, if the orbit of 0 under $Q_c$ is bounded, then the filled Julia set for $Q_c$ is a connected set.*

Now if we use the critical point 0 as our initial seed, the first iterate of $Q_c$ will be $c$. It then follows from the Escape Criterion (discussed above in Chapter 10) that if $|c| > 2$, then the orbit of 0 will escape to $\infty$. Therefore, we propose the following theorem.

**Theorem 11.2** *If $|c| > 2$, then the filled Julia set of $Q_c$ is a Cantor set. Furthermore, since the boundary of a Cantor set is equal to the Cantor set itself, the filled Julia set of $Q_c$ will be equal to the Julia set of $Q_c$.*

We are now ready to define the Mandelbrot set.

**Definition 11.1** *The* Mandelbrot set $\mathcal{M}$ *is the set of all values of* $c$ *for which the filled Julia set of* $Q_c$ *is a connected set. Or, eqivalently,* $\mathcal{M} = \{c \in \mathbb{C} : |Q_c^n(0)| \nrightarrow \infty\}$*, where* $\mathbb{C}$ *is the set of all complex numbers.*

Note that the $\mathcal{M}$ set exists in the complex $c$-plane (or, equivalently, the $ab$-plane), rather than in the complex $z$-plane (i.e., the $xy$-plane). Therefore, each point $c$ on the Mandelbrot set corresponds to a connected filled Julia set for the function $Q_c$. As discussed above, it follows from the Escape Criterion that the $\mathcal{M}$ set is contained within a circle of radius 2 in the $c$-plane:

**Theorem 11.3** *If* $c \in \mathcal{M}$*, then* $|c| \leq 2$*.*

Furthermore, it can also be shown (although we will not prove it here), that the $\mathcal{M}$ set necessarily contains all $c$-values for which $|c| \leq \frac{1}{4}$:

**Theorem 11.4** *If* $|c| \leq \frac{1}{4}$*, then* $c \in \mathcal{M}$*.*

Finally, we note that

**Theorem 11.5** *The Mandelbrot set is symmetric about the real* $a$*-axis.*

This fact is not difficult to prove, and so we exclude the proof here.

The algorithm for producing the Mandelbrot set is quite similar to those used to produce filled Julia sets in Chapter 10. The difference is that, instead of randomly sampling and plotting values of $x$ and $y$ in the complex $z$-plane, we are randomly sampling and plotting values of $a$ and $b$ in the $c$-plane. The iterating function is the same: $z^2 + c$. To plot a filled Julia set we keep $c$ constant and test as many values of $z$ as possible. To plot the Mandelbrot set we keep $z = 0$ constant and test as many values of $c$ as possible. In either of the two cases, we pick a maximum number of values to test and a maximum number of iterations to perform. If, in either case, the orbit remains within a circle of radius 2 after we have performed our maximum number of iterations, the test value is included in the set.

To speed up the plotting of the Mandelbrot set, we can make use of Theorems 11.4 and 11.5. After randomly selecting values of $a$ and $b$ from a specified range, an `if-then` statement is used to determine if $|c| = \sqrt{a^2 + b^2} < \frac{1}{4}$. If it is, then we automatically include $c = a + ib$ as a point in the Mandelbrot set and move on. This follows directly from Theorem 11.4. Additionally, for plots of the entire $\mathcal{M}$ set, we can exploit the symmetry of the $\mathcal{M}$ set about the $a$-axis (Theorem 11.5) by testing only positive values of $b$. If the corresponding $c$-value turns out to belong in $\mathcal{M}$, then we include both $c = a + ib$ and $c = a - ib$. This will either halve the time it takes to plot the $\mathcal{M}$ set or double the number of points in it, depending on how you look at it. Symmetry cannot be exploited on magnifications of the $\mathcal{M}$ set which lie entirely above or below the $a$-axis, however.

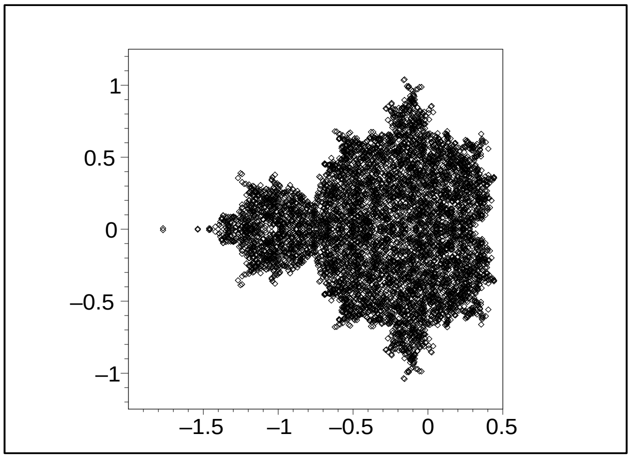
Below is a procedure I have developed for producing the $\mathcal{M}$ set and various magnifications of the set. The procedure is called `Mandelbrot`. It takes into account the time-saving shortcuts discussed above. It also uses the `rand` command for randomly selecting values of $a$ and $b$ as discussed extensively in Chapter 10. Because of its very close relation to the procedure `julia_bw` used in Chapter 10, I will not discuss the steps of the `Mandelbrot` procedure in depth. An understanding of the `julia_bw` procedure should lead to an understanding of the `Mandelbrot` procedure. So without further ado:

```
> Mandelbrot:=proc(numpoints, numiterates, amin, amax, bmin, bmax,
> decplaces)
> global myset, R1, R2;
> local P, Q, a, b, M, j, i, amin2, amax2, bmin2, bmax2:
> with(plots):
> P:=array(0..numiterates+2);
> Q:=array(0..numiterates+2);
> myset:=array(0..numpoints+2);
> myset[0]:={}:
> amin2:=trunc(amin*10^decplaces):  amax2:=trunc(amax*10^decplaces):
> bmin2:=trunc(bmin*10^decplaces):  bmax2:=trunc(bmax*10^decplaces):
> R1:=evalf(rand(amin2*10^Digits..amax2*10^Digits)/10^(Digits+decplaces
> )):
> R2:=evalf(rand(bmin2*10^Digits..bmax2*10^Digits)/10^(Digits+decplaces
> )):
> for j from 1 to numpoints do
> a:=R1(); b:=R2();
> Q[0]:=0; P[0]:=0;
> if sqrt(a^2+b^2)<=.25 then myset[j]:=(myset[j-1] union
> {[a,b],[a,-b]}):
> else if sqrt(a^2+b^2)<=2 then
> for i from 0 while sqrt((P[i]^2-Q[i]^2+a)^2+(2*Q[i]*P[i]+b)^2)<=2 and
> i<=numiterates do
> P[i+1]:=evalf(P[i]^2-Q[i]^2+a):  Q[i+1]:=evalf(2*P[i]*Q[i]+b):  od:
> if i>numiterates then myset[j]:=(myset[j-1] union
> {[a,b],[a,-b]}):
> else myset[j]:=myset[j-1] end if:
> else myset[j]:=myset[j-1] end if:  end if:
> od:
> M:=pointplot(myset[numpoints], symbolsize=10, symbol=diamond,
> scaling=constrained, color=black):
> display(M, axes=boxed, view=[(amin)..(amax),(bmin)..(bmax)]);
> end proc:
```

I will now use this procedure to plot the $\mathcal{M}$ set. I will take advantage of Benoit Mandelbrot's observation that the $\mathcal{M}$ set is contained within a rectangle with $-2 \leq a \leq 0.5$

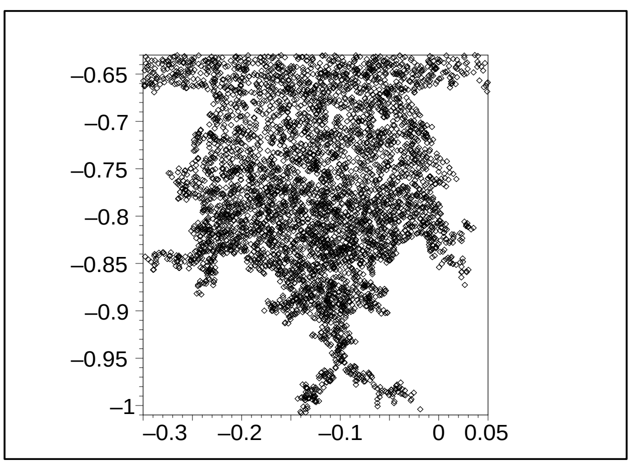
and $-1.25 \leq b \leq 1.25$. I will sample 12,000 different values of $c$ and iterate $Q_c(0)$ 15 times for each $c$ value. Because my window specifications have a maximum accuracy of only two decimal places, I will use two decimal places to distinguish values of $a$ and $b$.

```
> Mandelbrot(12000, 15, -2.00, 0.50, -1.25, 1.25, 2);
```



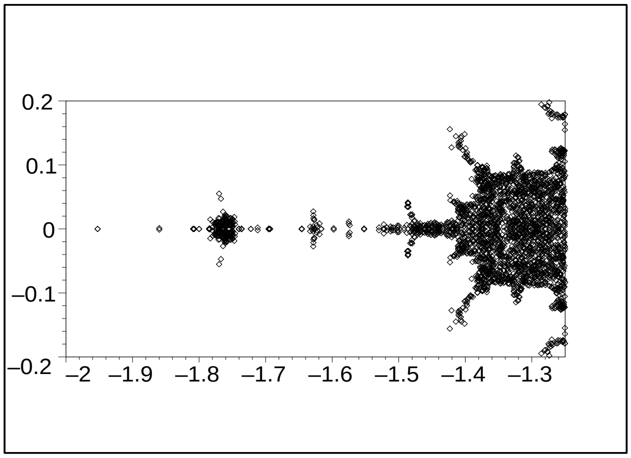

Let's zoom in on the lower (period-3) bulb protruding from the main cardioid area of the set. Notice how this magnification shows as much complexity as the entire $\mathcal{M}$ set itself! (There is less density because I have lowered the number of points sampled to 10,000 and I was also not able to exploit symmetry.)

```
> Mandelbrot(10000, 15, -0.30, 0.05, -1.01, -0.63, 2);
```

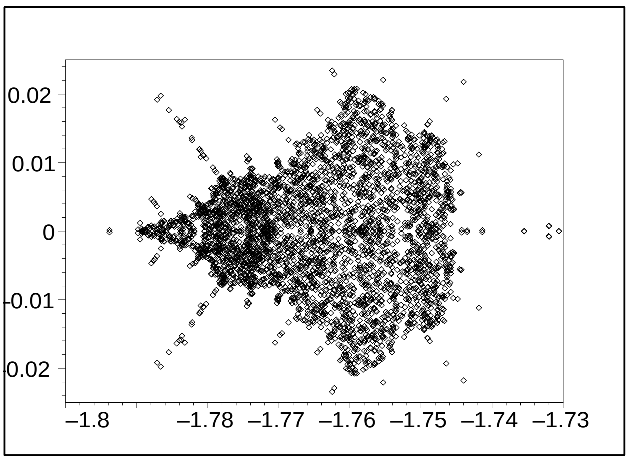



Now we will magnify the main antenna on the left-hand side of the $\mathcal{M}$ set and then the small island which is part of this antenna (the period-3 island).

```
> Mandelbrot(150000, 15, -2.00, -1.25, -0.20, 0.20, 2);
```

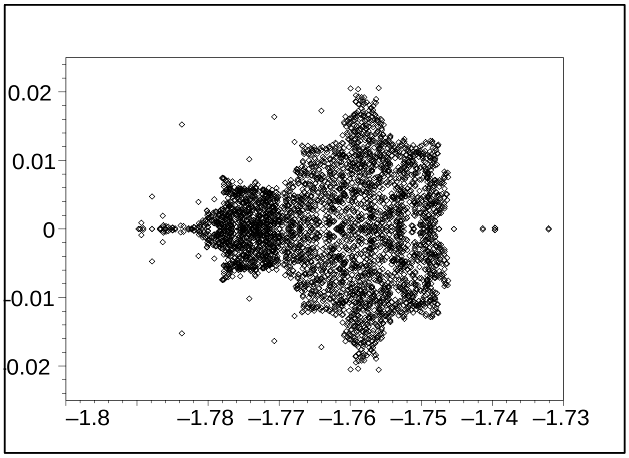



```
> Mandelbrot(8000, 20, -1.800, -1.730, -0.025, 0.025, 3);
```




Notice how this small island is remarkably similar to the entire $\mathcal{M}$ set! This is what I like to call a "baby brot"—a miniature version of the entire $\mathcal{M}$ set contained within the set itself. The similarity becomes even more clear if we increase our number of points from 8,000 to 12,000 and our number of iterations from 20 to 30:

```
> Mandelbrot(12000, 30, -1.800, -1.730, -0.025, 0.025, 3);
```




As in the case with the filled Julia set, there is a faster method for computing and plotting the $\mathcal{M}$ set which makes use of the `plot3d` command's color option. This procedure is listed below and has been named `mandelfast`. The procedure is extremely similar to the `juliafast` procedure of Chapter 10. An understanding of that procedure leads naturally to an understanding of the `mandelfast` procedure.

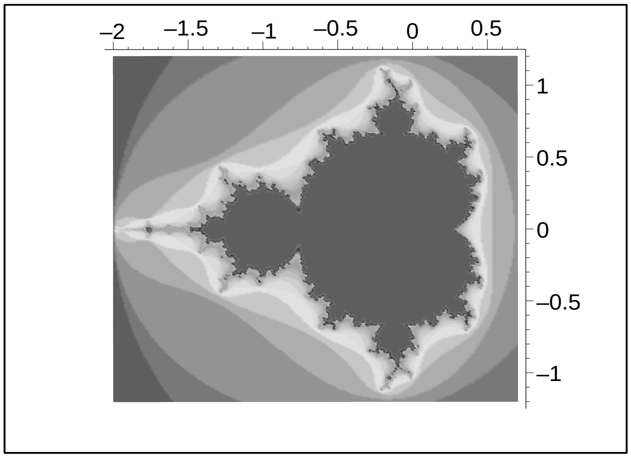
```
> mandelfast:=proc(x,y)
> local x2, x3, y2, i;
> global maxit;
> x2:=x;
> y2:=y;
> for i from 0 to maxit while evalhf(sqrt(x2^2+y2^2)) < 2 do
> x3:=x2;
> x2:=evalhf(x2^2-y2^2+x):
> y2:=evalhf(2*x3*y2+y):
> od:
> i
> end:
```

We will now use this procedure along with the `plot3d` command to plot the $\mathcal{M}$ set and several magnifications of portions of the $\mathcal{M}$ set. The window of magnification is controlled with the ranges that must be specified in the argument of `plot3d`. The most important thing to notice in the examples below is that the maximum number of iterations which must be used—called "maxit"—must be increased as the magnification factor is increased. If it is not, the results will be simplistic, uninteresting, and inaccurate. The *exact* number is not important and depends on the region of the $\mathcal{M}$ set which we are exploring and on our degree of magnification. We just need to have sufficiently many iterations to see fine detail at each level of magnification. Take a look:

```
> maxit:=30;
```

$$maxit := 30$$

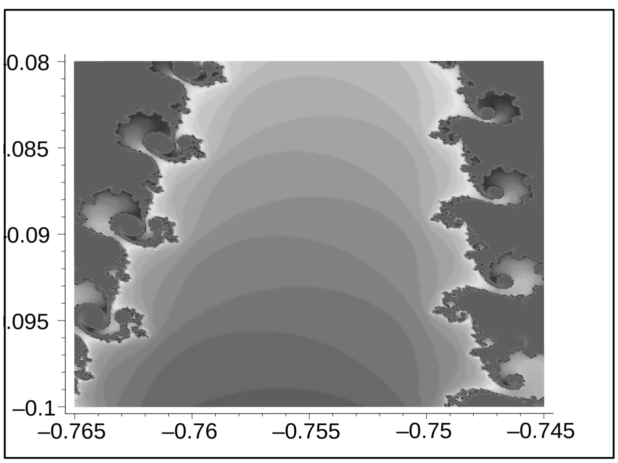
```
> plot3d(0, -2..0.7, -1.2..1.2, orientation=[-90,0], grid=[250, 250],
> style=patchnogrid, scaling=constrained, color=mandelfast, axes=frame);
```




```
> maxit:=100;
```

$$maxit := 100$$

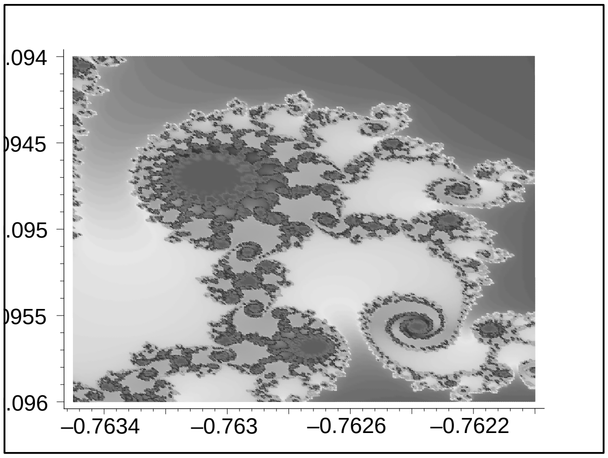
```
> plot3d(0, -.765..-.745, -.08..-.1, orientation=[-90,0], grid=[250,
> 250], style=patchnogrid, color=mandelfast, axes=normal);
```




```
> maxit:=200;
```

$$maxit := 200$$

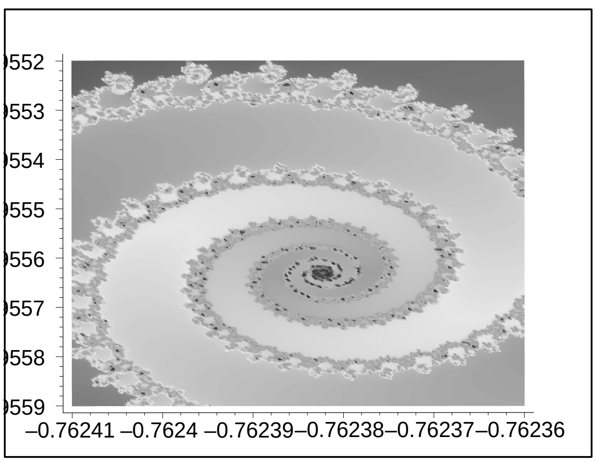
```
> plot3d(0, -.7635..-.762, -.094..-.096, orientation=[-90,0],
> grid=[250, 250], style=patchnogrid, color=mandelfast, axes=normal);
```




```
> maxit:=500;
```

$$maxit := 500$$

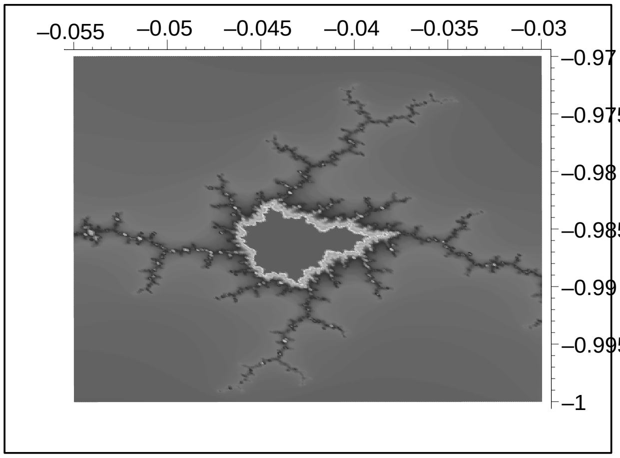
```
> plot3d(0, -.76236..-.76241, -.09559..-.09552, orientation=[-90,0],
> grid=[250, 250], style=patchnogrid, color=mandelfast, axes=normal);
```




```
> maxit:=75;
```

$$maxit := 75$$

```
> plot3d(0, -0.055...-0.03, -1...-0.97, orientation=[-90,0], grid=[250,
> 250], style=patchnogrid, color=mandelfast, axes=frame);
```




This last plot is another example of a "baby brot."

## 11.1 Experiment: Exploring the $\mathcal{M}$ Set

1. Plot the Mandelbrot set in black and white. Use a manageable number of points. This number depends on the speed of your computer and how much time you have, but it shouldn't need to exceed 5000 points. I used many more points when creating the examples shown above so that the resulting plots would be very well formed. If you are ambitious, you can try using as many points as I used or even more. The results will look great, but I should warn you that some of my example plots took around one hour to produce. The maximum number of iterations has less significance in how long it will take you to plot the Mandelbrot set. A number between 15 and 30 will do just fine. Higher numbers may cause the algorithm to slow.

2. Plot the Mandelbrot set in color using the `mandelfast` procedure. Use a manageable grid size. 250 pixels by 250 pixels works well. The maximum number of iterations has less significance in how long it will take you to plot the Mandelbrot set. A number between 15 and 30 will do just fine.

3. Pick one area along the border of the Mandelbrot set and perform three successive zoom-ins on that area. Here the maximum number of iterations is significant. For each zoom-in, you will need to increase the maximum number of iterations used in the algorithm. If you don't, your results will be simplistic, uninteresting, and inaccurate. See my examples earlier in this chapter to get a sense of how many points you will need to use. The *exact* number is not important and depends on the region of the Mandelbrot set which you are exploring and on your degree of magnification. You just need to have sufficiently many iterations to see fine detail at each level of magnification.

4. Repeat the previous step for two other areas along the border the Mandelbrot set. See if you can find"baby brot" (a miniature version of the entire Mandelbrot set contained within the set itself) in your explorations.

5. If time permits, zoom in on other regions around the border of the Mandelbrot set. Also, you may want to try to zoom in on smaller regions of the Mandelbrot set (thus achieving greater magnification). What problems do you run into as you create greater and greater magnifications?

## 11.2 Experiment: Periods of Bulbs of the $\mathcal{M}$ Set

The goal of this experiment is to determine the periodicity of the main cardioid of the $\mathcal{M}$ set as well as the periodicities of many of the bulbs and "decorations" protruding from the main cardioid of the $\mathcal{M}$ set.

The main cardioid of the $\mathcal{M}$ set, as well as every bulb and decoration of the $\mathcal{M}$ set, each have one and only one specific periodicity. To find out the periodicity of a particular region (by "region" I mean either the main cardioid or one of the bulbs or decorations) of the $\mathcal{M}$ set), we use the following method: Choose a few values of $c$ near the center of the region and compute the orbit of $z_0 = 0$ under $Q_c$. If the orbits are fixed or approach single fixed points, then assume that you are in a "period-1" region of the $\mathcal{M}$ set. If the orbits are 2-cycles or are eventually attracted towards 2-cycles, then assume that you are in a "period-2" region of the $\mathcal{M}$ set. If the orbits are 3-cycles or are eventually attracted towards 3-cycles, then assume that you are in a "period-3 region" of the $\mathcal{M}$ set. And so on. It is important that you choose points near the center of the cardioid or bulb. Points near the boundary will exercise the same asymptotic behavior as those near the center but will take much longer to do so (recall Experiment 6.4: Rates of Convergence). Note that each bulb and decoration of the $\mathcal{M}$ set is connected to the main cardioid or to other bulbs and decorations at only one single point. The "regions" of the $\mathcal{M}$ set which I talk about above should therefore be well-defined and determinable when viewing a plot of the $\mathcal{M}$ set.

1. Plot the $\mathcal{M}$ set using the `mandelfast` procedure. Find the periodicity of the main cardioid using the method described in the paragraph above. (Choose a few values of $c$ near the center of the main cardioid of the $\mathcal{M}$ set and compute the orbit of $z_0 = 0$ under $Q_c$ for each $c$-value. Observe the fate of the orbits and use this to determine the periodicity of the cardioid.)

2. Find the periodicity of all of the large bulbs which are tangent to the main cardioid in the $\mathcal{M}$ set using the method described in the above paragraph. (For each bulb, choose values of $c$ near the middle of the bulb and compute the orbit of $z_0 = 0$ under $Q_c$ for each $c$-value. Observe the fate of the orbits and use this to determine the periodicity of the bulb.) Sketch an outline of the $\mathcal{M}$ set and use this sketch to label the periodicities of the main cardioid and all of its main decorations. This sketch or additional similar sketches should also be used for the steps below.

3. Using the method described above, identify all of the period-3 decorations of the $\mathcal{M}$ set. You most likely have already found at least two of them in step 2 above! *HINT*: There are exactly three period-3 decorations on the $\mathcal{M}$ set. One of them is *not* a bulb tangent to the main cardioid.

4. Identify all of the period-4 decorations of the $\mathcal{M}$ set. Again, you may have already found them all in step 2 above. *HINT*: There are exactly six of them.

5. Identify all of the period-5 decorations of the $\mathcal{M}$ set. You probably found a few of these while performing step 2 above. *HINT*: There are 15 of them and some are very difficult to find, but I think it is an enjoyable hunt.

6. *Optional*. If you are a real zealot, try identifying all of the period-6 decorations of the $\mathcal{M}$ set. There are 27 in all![1]

## 11.3 Experiment: Spokes and Antennas of the $\mathcal{M}$ Set

The goal of this experiment is to determine the relationship between the periodicities of decorations of the $\mathcal{M}$ set and the structure of the antennae which emanate from the decorations.

1. Plot the $\mathcal{M}$ set using the `mandelfast` procedure and magnify one of the bulbs attached to the main cardioid of the $\mathcal{M}$ set. Choose a bulb whose periodicity has been determined from Experiment 11.2 above. When you magnify the bulb, be sure to include the entire antenna array protruding from the end of it. Count and record the number of "spokes" diverging from the main junction point of the antenna. Record this number along side the periodicity of the bulb.

2. Magnify another bulb tangent to the main cardioid of the $\mathcal{M}$ set whose periodicity differs from the bulb you just used in the last step. Again, count and record the number of spokes diverging from the main junction point of the antenna protruding from the bulb. Record this number along side the periodicity of the bulb.

3. Perform the procedure used in the previous two steps for bulbs of at least five different periodicities (*including* the two periodicities already used in the two steps above). What correlation, if any, do you find between the number of spokes diverging from the main junction point of the antenna and the periodicity of the bulb? Write a brief essay explaining your findings.

## 11.4 Experiment: Misiurewicz Points

1. Plot the $\mathcal{M}$ set using the `mandelfast` procedure and magnify one of the bulbs attached to the main cardioid of the $\mathcal{M}$ set. Magnify the antenna protruding from the end of the bulb and center your plot on the main junction point of the antenna from which all the spokes are diverging. Zoom in on this junction point two or three times. Determine as accurately as you can the $c$-value corresponding to this junction point.

2. Create a filled Julia set for the $c$-value determined in step 1 above. Do you notice any similarities between the filled Julia set and the region of the $\mathcal{M}$ set surrounding the junction point? Explain in a brief essay.

[1] See Devaney 1992, 257-258.

3. Repeat steps 1 and 2 above for antenna from several other decorations of the $\mathcal{M}$ set.

A $c$-value in the $\mathcal{M}$ set whose corresponding filled Julia set appears very similar in appearance and form to the region of the $\mathcal{M}$ set surrounding that $c$-value is called a *Misiurewicz point*. Misiurewicz points happen to always be $c$-values for which $z_0 = 0$ is eventually periodic.