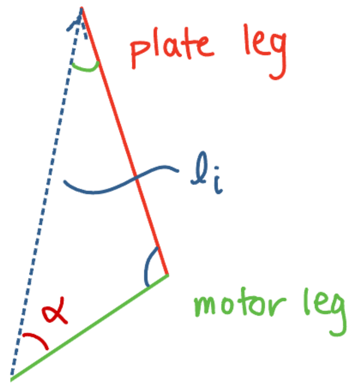


$$l_{plate}^2 = l_i^2 + l_{motor}^2 - 2 l_i l_{motor} \cos\alpha$$

$$\alpha = \cos^{-1}\left(\frac{l_i^2 + l_{motor}^2 - l_{plate}^2}{2 l_i l_{motor}}\right)$$

Figure 1: Cosine law.

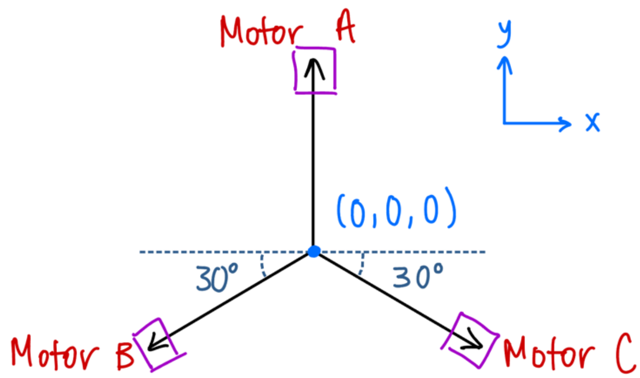


Figure 2: Motor orientations.

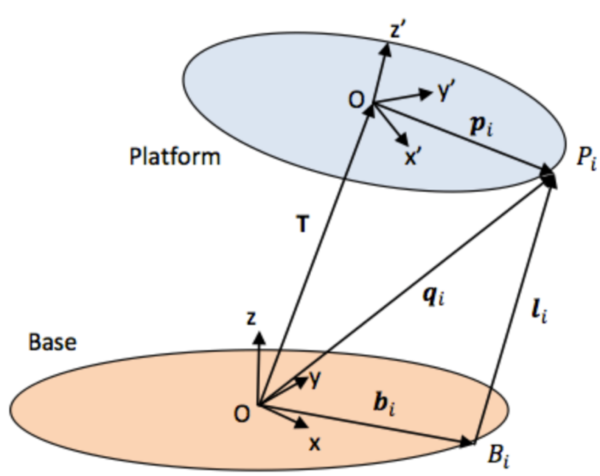


Figure 3: Diagram describing the plate vectors.

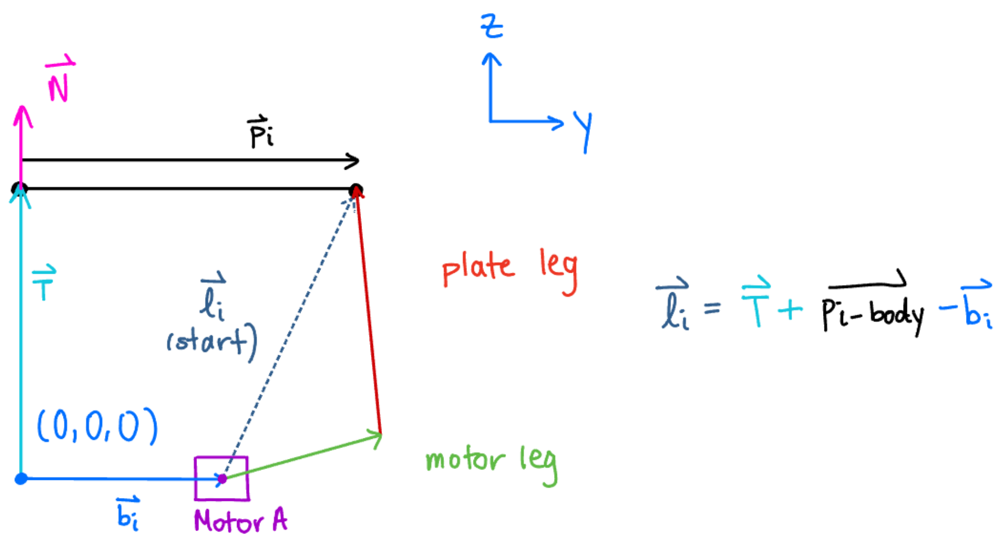


Figure 4: Motor A at rest.

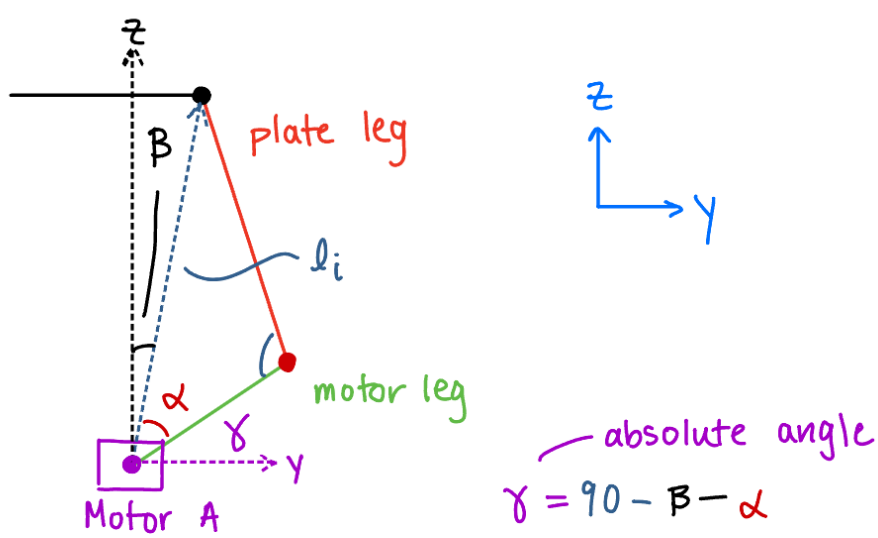


Figure 5: Gamma absolute motor angle calculation.

Define the rotation matrix for a tilt about the X and Y axis.

$$ {}^{Body}_{plat}R = Y_{rot} * X_{rot} \rightarrow \text{Rotation of platform w.r.t. body} $$

$$ Y_{rot} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} $$

$$ X_{rot} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\emptyset & -\sin\emptyset \\ 0 & \sin\emptyset & \cos\emptyset \end{bmatrix} $$

$$ {}^{B}_{p}R = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} * \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\emptyset & -\sin\emptyset \\ 0 & \sin\emptyset & \cos\emptyset \end{bmatrix} $$

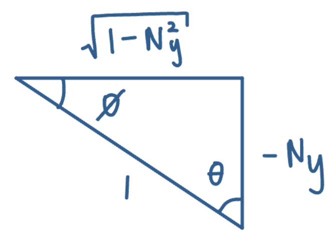
$$ {}^{B}_{p}R = \begin{bmatrix} \cos\theta & \sin\theta\sin\emptyset & \sin\theta\cos\emptyset \\ 0 & \cos\emptyset & -\sin\emptyset \\ -\sin\theta & \cos\theta\sin\emptyset & \cos\emptyset\cos\theta \end{bmatrix} $$

Backwards-solve the rotation matrix that will transform a unit $\hat{k}$ vector to the plane's normalized normal vector.

$$ \begin{bmatrix} \cos\theta & \sin\theta\sin\emptyset & \sin\theta\cos\emptyset \\ 0 & \cos\emptyset & -\sin\emptyset \\ -\sin\theta & \cos\theta\sin\emptyset & \cos\emptyset\cos\theta \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \hat{N} = \begin{bmatrix} N_x \\ N_y \\ N_z \end{bmatrix} $$

$$ \begin{bmatrix} \sin\theta\cos\emptyset \\ -\sin\emptyset \\ \cos\emptyset\cos\theta \end{bmatrix} = \begin{bmatrix} N_x \\ N_y \\ N_z \end{bmatrix} $$

$$ \emptyset = \sin^{-1}(-N_y) $$



$$ \theta = \sin^{-1}\left(\frac{N_x}{\cos\emptyset}\right) $$

$$ \theta = \sin^{-1}\left(\frac{N_x}{\sqrt{1-N_y{}^2}}\right) $$

Once that is solved, substitute in $\emptyset$ and θ into the matrix. Then, apply that rotation to the $\overrightarrow{p_{\iota-plane}}$ vector to solve for the magnitude and direction of the rotated vector $\overrightarrow{p_{\iota-body}}$ to use in the $\overrightarrow{l_\iota}$ equation.

$$ \overrightarrow{p_{\iota-body}} = {}^{Body}_{plat}R * \overrightarrow{p_{\iota-plat}} $$

$$ \overrightarrow{p_{\iota-body}} = \begin{bmatrix} \cos\theta & \sin\theta\sin\emptyset & \sin\theta\cos\emptyset \\ 0 & \cos\emptyset & -\sin\emptyset \\ -\sin\theta & \cos\theta\sin\emptyset & \cos\emptyset\cos\theta \end{bmatrix} * \overrightarrow{p_{\iota-plat}} $$

For angles $\theta_y = 5°$ and $\phi_x = 5°$, we can calculate $\vec{N}$ as follows.

$$\begin{bmatrix} \sin 5° \cos 5° \\ -\sin 5° \\ \cos 5° \cos 5° \end{bmatrix} = \begin{bmatrix} N_{x-norm} \\ N_{y-norm} \\ N_{z-norm} \end{bmatrix} = \begin{bmatrix} 0.08682 \\ -0.08716 \\ 0.99240 \end{bmatrix}$$

For angles $\theta_y = 5°$ and $\phi_x = 5°$, we can calculate $\vec{l_a}$ as follows for Motor A. Given $\overrightarrow{p_{a-plat}} = \begin{bmatrix} 0 \\ 15 \\ 0 \end{bmatrix}$,

$$\overrightarrow{p_{a-body}} = \begin{bmatrix} \cos 5°, & \sin 5°, \sin 5°, & \sin 5°, \cos 5°, \\ 0 & \cos 5°, & -\sin 5°, \\ -\sin 5°, & \cos 5°, \sin 5°, & \cos 5°, \cos 5°, \end{bmatrix} * \begin{bmatrix} 0 \\ 15 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.11394 \\ 14.94293 \\ 1.30236 \end{bmatrix}$$

Recall, $\vec{l_a} = \vec{T} + \overrightarrow{p_{a-body}} - \vec{b_a}$. Given $\vec{T} = \begin{bmatrix} 0 \\ 0 \\ 8 \end{bmatrix}$ and $\vec{b_a} = \begin{bmatrix} 0 \\ 12 \\ 0 \end{bmatrix}$,

$$\vec{l_a} = \begin{bmatrix} 0 \\ 0 \\ 8 \end{bmatrix} + \begin{bmatrix} 0.11394 \\ 14.94293 \\ 1.30236 \end{bmatrix} - \begin{bmatrix} 0 \\ 12 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.11394 \\ 2.94293 \\ 9.30236 \end{bmatrix}$$

For a known $\vec{l_a} = \begin{bmatrix} 0.11394 \\ 2.94293 \\ 9.30236 \end{bmatrix}$, we can calculate the relative motor angle $\alpha_{plate-leg}$ as follows. Given $l_{motor-leg} = 3$ and $l_{plate-leg} = 6$,

$$\left|\vec{l_a}\right| = \sqrt{0.11394^2 + 2.94293^2 + 9.30236^2} = 9.7574$$

$$\alpha_{plate-leg} = \cos^{-1}\left(\frac{{l_a}^2 + {l_{motor-leg}}^2 - {l_{plate-leg}}^2}{2 l_a l_{motor-leg}}\right) = \cos^{-1}\left(\frac{9.7574^2 + 5^2 - 8^2}{2 \times 9.7574 \times 5}\right) = 54.8268°$$

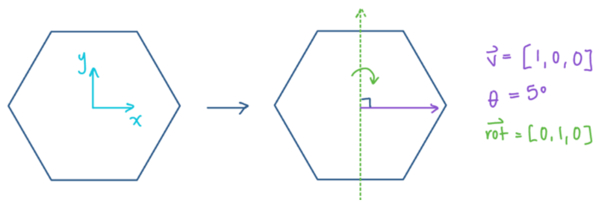
After obtaining the relative motor angle, we can find the angle β_a of $\vec{l_a}$ with respect to the unit $\hat{k}$ vector. See `absolute_motor_angle.png` for more details.

$$\beta_a = \cos^{-1}\left(\frac{\hat{k} \cdot \vec{l_a}}{\left|\hat{k}\right|\left|\vec{l_a}\right|}\right) = \cos^{-1}\left(\frac{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0.11394 \\ 2.94293 \\ 9.30236 \end{bmatrix}}{9.7574}\right) = 17.5670°$$

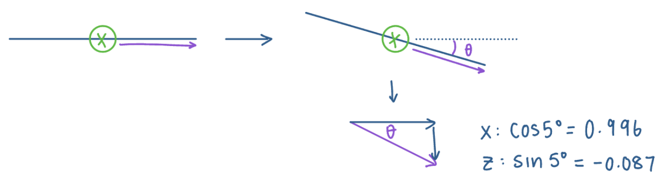
Subtracting $\alpha_{plate-leg}$ and β_a from 90° yields the absolute motor angle γ_a w. r. t. the horizontal.

$$\gamma_a = 90° - \alpha_{plate-leg} - \beta_a = 90° - 54.8268° - 17.5670 = 17.61°$$

For a direction vector of $\vec{v} = \begin{bmatrix}1\\0\\0\end{bmatrix}$ and a tilt magnitude of $\theta = 5°$, the normal vector of the plate can be calculated. First, the axis of rotation is determined to be $\vec{v}_{rot-axis} = \begin{bmatrix}0\\1\\0\end{bmatrix}$, which is the orthogonal vector to the direction vector in the x-y plane.



Then, the plate is tilted about this axis in the direction of the direction vector, about magnitude θ. Vector decomposition is used to determine the new, rotated direction vector.



$$\vec{v}_{rot} = \begin{bmatrix}0.996\\-0.087\\0\end{bmatrix}$$

Given the rotated direction vector and the axis of rotation, these are crossed to find the third orthogonal vector, which is the normal vector to the plane.

$$\vec{N} = \vec{v}_{rot-axis} \times \vec{v}_{rot} = \begin{bmatrix}0\\1\\0\end{bmatrix} \times \begin{bmatrix}0.996\\-0.087\\0\end{bmatrix} = \begin{bmatrix}-0.087\\0\\-0.996\end{bmatrix}$$

The vector is multiplied by -1 to ensure the Z-direction is positive.

$$\vec{N} = \begin{bmatrix}0.087\\0\\0.996\end{bmatrix}$$