Angle Trisection Using Origami

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Introduction

- Origami is a technique of paper folding.
- Mathematical application of these techniques were discovered in the 20th century.
- Most of the applications of origami is related to arts and crafts.
- Origami can solve the classical problem of angle trisection which is not possible using purely a compass and an unmarked straight edge.

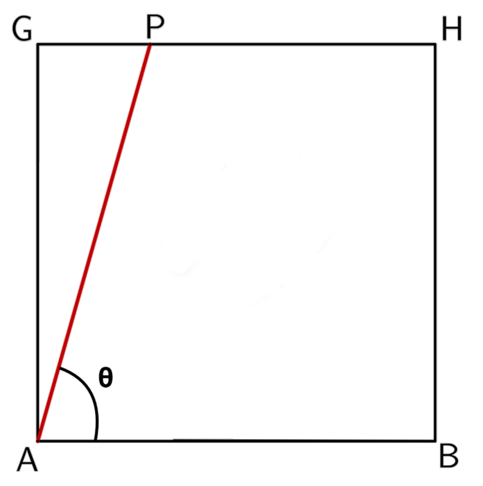
Origami procedures

- **O1:** Given two non identical points $P$and Q, one can fold the unique straight line $l = PQ$ containing both points.
- **O2:** Given two non parallel straight lines l_1 and l_2 one can determine their unique point of intersection $P = l_1 \cap l_2$.
- **O3:** Given two parallel straight linesl_1 and l_2 one can fold the line m parallel to and equidistant from them.
- **O4:** Given two intersecting straight lines l_1 and l_2 one can fold their angle bisectors.
- **O5:** Given two non identical points P and Q one can fold the unique perpendicular bisector b of the line segment PQ.

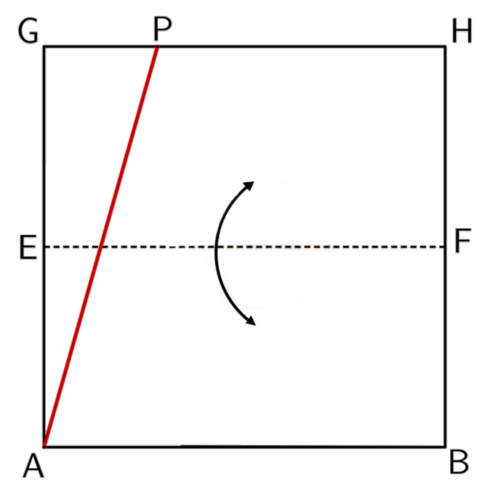
Origami Proof

The demonstration of the proof requires a paper with a horizontal base and sides perpendicular to the base.

- Make an arbitrary angle θ with the bottom edge AB using origami procedure O1. Fold a straight line AP from point A to any point P on the paper [Step 1].



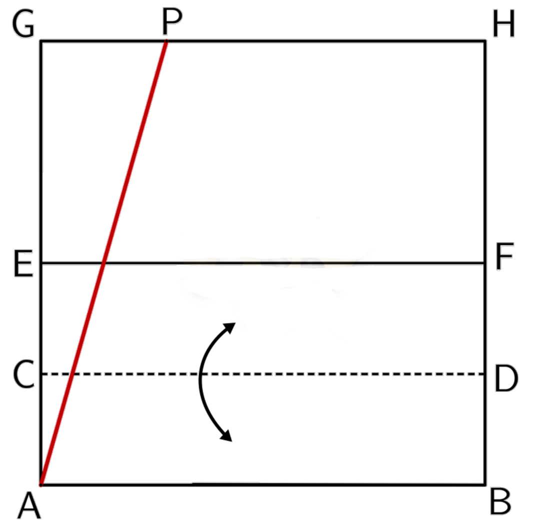
Step 1



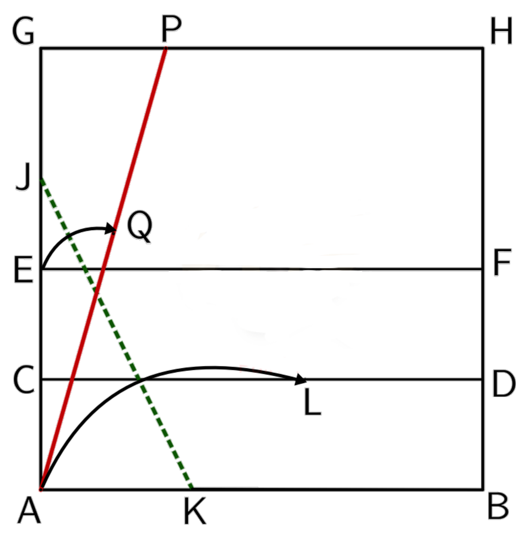
Step 2

We will show that the angle between AB and AP can be trisected.

- With origami procedure O3, fold a straight line EF which passes through AP and parallel to the base[Step 2]. For small angles fold EF passing through AP.



Step 3



Step 4

- Fold base AB up to line EF and unfold, procedure O3, creating line CD which is parallel and at equal distance from AB and EF.[Step 3].

- Next, fold the bottom left corner up, so that point E touches the line AP, and point A touches the line CD [Step 4]. $EA = QL$.

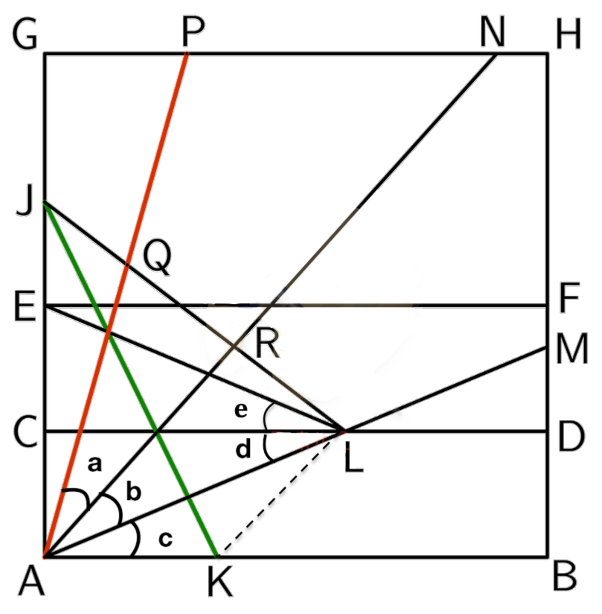
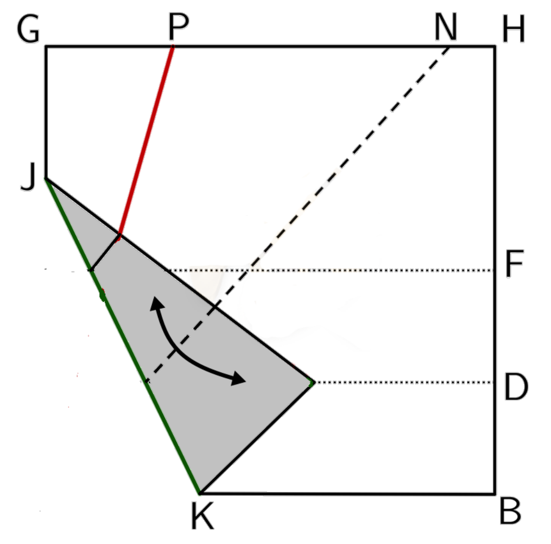
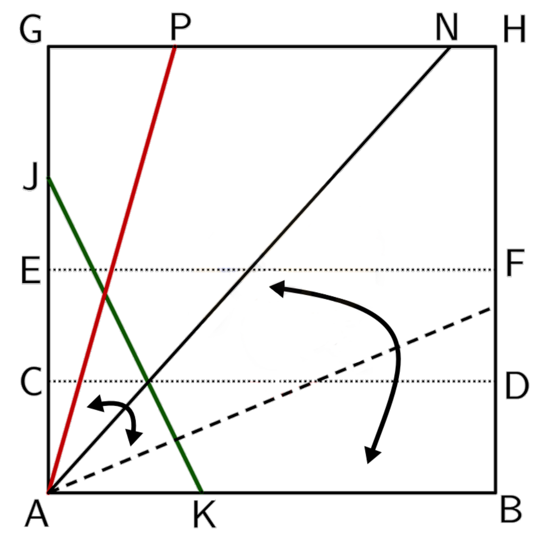


Figure A: Trisecting an angle

In figure A consider the triangle EAL, we know that $EC = CA$ and CL is perpendicular EA. Hence, CL divides the triangle EAL into two equal halves. Triangle EAL is an isosceles triangle, and angle d=e.



Step 5



Step 6

- With the corner still up, fold both layers to continue the crease that ends at point R all the way to N [Step 5], then unfold, procedure O5.

The mirror image of triangle EAL when reflected on the green line is the triangle QAL; by combining origami procedure O4 and O5 [Figure B]. Hence the triangle QAL is also an isosceles triangle and angles $a = b = d = e$ in Figure A.

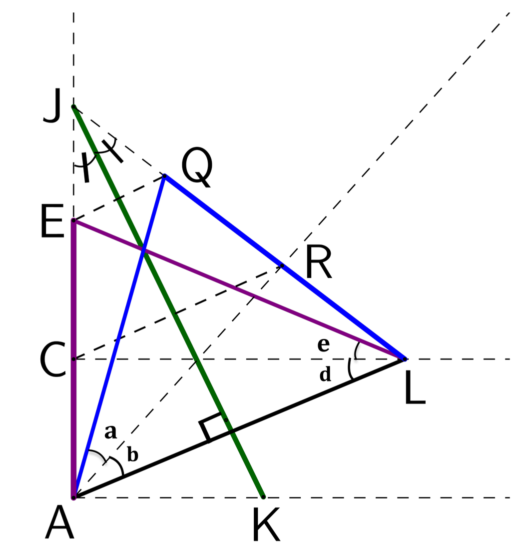
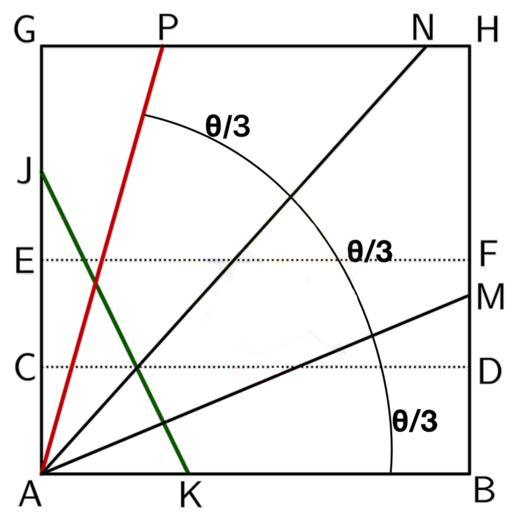


Figure B: Triangle reflection



Step 7

- Finally, fold the bottom edge AB up to line AN and unfold [Step 6], procedure O4.
- The two creases AN and AM divide the original angle PAB into three equal parts [Step 7].

The line CD is parallel to the bottom edge AB, and AM is a transversal producing $d = c$.

- From Step 5, we have $d = a = b$.
- Therefore, $a = b = c$.

Conclusion

- Origami procedures O1-O5 can be constructed using euclidean tools.
- Origami has proved that angle trisection is possible.
- Some origami techniques applied to Euclidean geometry can produce angle trisection.