Stability of Modified Max Pressure Controller with Application to Signalized Traffic Networks

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***Abstract*— This work describes a type of distributed feedback control algorithm that acts on a vertical queueing network where flow dynamics may greatly outpace the rate of feedback and actuation. The modeled network has a known, finite set of feasible actuations for the binary controllers located at each network node. It also has known expected demands, split ratios, and maximum service rates. Previous work proposed the application of a max pressure controller to maximize throughput on such a network without the need for centralized computation of a control policy. Here we extend the max pressure controller to satisfy practical constraints on the frequency of switching and guarantees on proportional actuation. We fundamentally alter the formulation of max pressure to a setting where the controller may only update at a rate significantly slower than the dynamics of queue formation. Furthermore, the set of allowable controllers is extended to any convex combination of available signal phases to account for signal changes within a single signal "cycle". We show that this proposed extended max pressure controllers stabilize a vertical queueing network (queue lengths remain bounded in expectation) given slightly increased restrictions on admissible network demand flows. This work is motivated by the application of controlling traffic signals on arterial road networks. Max pressure provides an intriguing alternative to existing feedback control systems due to its distributed implementation and theoretical guarantees, but cannot be directly applied as originally formulated due to hardware and safety constraints. We ultimately apply our extension of max pressure to a simulation of an existing arterial roadway and provide comparison to the control policy that is currently deployed on this site.**

I. Introduction

This article investigates the design and stability of decentralized controller for vertical queueing networks. In a *vertical queueing network*, agents traveling across the network are stored in "point queues" which do not inhabit a "horizontal" position along the length of a network edge, but instead are considered to be stored in "vertical" stacks at each node. Such models are inherited from fields such as supply chain management or internet routing, but are also representative of signalized urban traffic networks [?][?][?]. The concept of a *stabilizing* network controller, or one which ensures that the mean length of all queues in the network remain bounded, is relevant to many applications such as communications networks [?][?][?] or industrial systems [?][?][?][?].

In the present work we examine a vertical queueing model in which only a finite set of non-conflicting turning movements (or *phases*) can be permitted to flow simultaneously across each network node. Phase actuation is dictated by a controller such as a traffic light. Specifically, we consider application of a *max pressure controller*.

Max Pressure is a distributed network control policy derived from the concept of a "back pressure controller", which was first studied in the context of routing packets through a multi-hop communications network [?]. The idea was applied to road traffic management more recently by Varaiya [?] as well as Wongpiromsarn et al. [?]. The concept of max pressure control is intuitive: at each intersection, priority is given to the signal phase which will be able to service the most traffic given knowledge of both available upstream demand and the subsequent feasibility of downstream queues. It is a particularly attractive concept for control of a signalized urban traffic network because it can be operated in a distributed manner on local controller hardware but still provides theoretical guarantees on network-wide performance. Variaya's original formulation of this controller, however, does not fully consider the practical limitations on the rate of queue measurement and signal actuation in vehicle traffic networks. For example, a standard max pressure controller has no bound on the rate of signal switches which may occur relative to the rate of modeled queue formation and dissipation in the network. In implementation, a traffic signal incurs a penalty upon every change in actuation in the form of capacity loss due to "intersection clearance time": a 2-3 second period where all movements are given a red light in order to allow traffic from the previous phase to clear the intersection before possibly conflicting traffic can be permitted to enter. Max pressure also lacks the ability to synchronize adjacent signals in a network by constraining the actuation periods of critical phases to fixed relative offsets. This feature is valued by traffic managers who wish to promote continuity of flow and limit vehicle stops on a preferred throughway. Furthermore, a standard max pressure implementation provides no explicit lower bound on the service rate of queues on minor approaches where demand may be very low relative to the main direction.

These limitations motivate a new extension of the max pressure control algorithm which bounds signal switches and can maintain timed cyclical behaviors for signal coordination and queue service equity. While a similar concept was suggested in [?], this work further extends a simple proportional phase controller to allow model dynamics to explicitly act at a faster rate than the controller update period. We then extend the stability proof of [?] to prove that our *cycle-based max pressure* controller still provides the desired guarantee of queue stability with a penalty to the theoretical bound on queue lengths due to the decreased rate of controller update.

The remainder of this article is organized as follows: Sections II-III describes the modeling framework and standard max pressure controller from [?]; Section IV formulates an extended cycle-based max pressure controller; Section V proves that this extended controller stabilizes a vertical queueing network; finally, Section VI presents numerical results provided by this controller using a microscopic traffic simulation running in the Aimsun platform.

II. Model Framework

We consider a network of arterial roads with infinite storage capacity, modeled topologically as a graph with road links being edges and intersections being vertices. An individual link $l \in \mathcal{L}$ can be either at the entry of the network ($l \in \mathcal{L}_{\text{ent}}$) or in the interior of the network ($l \in \mathcal{L} \backslash \mathcal{L}_{\text{ent}}$). The inflow on entry links is a defined entirely by a random demand d_l, while the input flows of all other links depend on queues on upstream links and the relevant set of physical flow constraints are defined within the network. We require that each link has an exit path, that is, a continuous set of subsequent links on which vehicles can travel from the link to eventually exit the network. Each link in the network model can have multiple *queues* corresponding to individual *movements*: all vehicles in a given queue on any link are intending to advance onto the same subsequent link (though not necessarily the same subsequent queue). We describe the dynamics of these queues as a discrete time dynamical model using the following notation:

- A *movement* (l, m) distinguishes an intention to travel from link l to link m, (in that case, say that $m \in Out(l)$ where $Out(l)$ is the set of links immediately downstream l);
- A *queue* $x(l, m)(t)$ is the number of vehicles on link l waiting to enter link m at timestep t, and $X(t)$ is the set (vector or matrix) of all the queue lengths on the network at timestep t;
- A *saturation flow* $c(l, m)$ is the expected number of vehicles that can travel from link l to link m per time step given maximum demand for the queue $x(l, m)$, and $C(l, m)(t)$ is the *realized saturation flow* at time t;
- The *turn ratio* $r(l, m)$ is the expected proportion of vehicles that are leaving l which are intending to enter m, and $R(l, m)(t)$ is the *realized turn ratio* at time t;
- The *demand vector* d of dimension $|\mathcal{L}_{\text{ent}}|$ specifies demands at network entry links;
- The *flow vector* f of dimension $|\mathcal{L}|$ denotes flows on all links of the network such that f_l is the flow in link l.

Note that there is intuitively a linear relationship between the expected link flow f and the boundary demand d:

$$f = dP \tag{1}$$

where the (possibly non-unique) matrix P depends on expected routing proportions within the network.

Intersection Signal Controller

A road intersection is modeled as a node in our framework. Controllers (traffic signals) are placed at every node to limit the set of queues permitted to discharge at any given time. A set of movements that can be simultaneously actuated without flow conflicts is called a *phase*. Each permissible phase for a given intersection can be represented as a binary control matrix S that is defined as follows:

$$S(l, m) = \begin{cases} 1 & \text{if movement } (l, m) \text{ is activated} \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

We denote U_n the known finite set of permissible control matrices for node n. Note that in this article, we often drop the subscript n for ease of notation.

Practically, only one phase can be actuated at any point in time: at each model time step t, a single control matrix $S(t)$ encodes which set of queues approaching the intersection are permitted to discharge during that time step. However, in this work we consider a *relaxed controller* which operates on a contiguous set of modeled time steps. This relaxed controller is defined as a matrix S^r with each element $S^r(l, m)$ representing the fraction of the operational time steps that are allocated to the movement $\{l, m\}$:

$$S^r(l, m) = \lambda_{l,m} \in [0, 1] \tag{3}$$

Such a relaxed controller can be seen as a convex combination of all possible control matrices, $S^r = \sum_{S \in U} \lambda_S S$. In general, the selection of such a controller can be based on feedback representing the state of the network queues at a previous time step.

Queue Dynamics

The evolution of network queue lengths $X(t)$ can be seen as a Markov chain: the state at time $(t + 1)$ is a function of only the state at time t and external demand d,

$$X(t + 1) = F(X(t), d) \tag{4}$$

Define $[\, a \wedge b \,] := \min\{a, b\}$. To describe queue dynamics explicitly, we must make a distinction between entry links and internal links: if $l \in \mathcal{L}_{\text{ent}}$,

$$\begin{aligned} x(l, m)(t + 1) &= x(l, m)(t) + d_l(t + 1) \\ &\quad - [C(l, m)(t + 1)S(l, m)(t + 1) \wedge x(l, m)(t)] \end{aligned} \tag{5}$$

and if $l \in \mathcal{L} \backslash \mathcal{L}_{\text{ent}}$,

$$\begin{aligned} &x(l, m)(t + 1) = x(l, m)(t) + \\ &\sum_k [C(k, l)(t + 1)S(k, l)(t + 1) \wedge x(k, l)(t)]R(l, m)(t + 1) \\ &\qquad - [C(l, m)(t + 1)S(l, m)(t + 1) \wedge x(l, m)(t)] \end{aligned} \tag{6}$$

Demand Feasibility

We focus on networks for which the boundary inflow demands $d = (d_l)_{(l \in \mathcal{L}_e)}$ are *feasible*—that is, the network is servicing a distribution of inflows for which it is possible to find a controller that allows *in average* more departures than arrivals at each link.

Define $conv(U)$ to be the convex hull of the set of permissible control matrices U. The following properties are then shown in [**?**]:

***Property** 1:* A matrix M is in $conv(U)$ if and only if $\exists$ a sequence of control matrices $\overline{S} = \{S(1), S(2), ..., S(t), ... | S(\cdot) \in U\}$ such that $\forall (l, m)$

$$M(l, m) = \liminf_{T} \frac{1}{T} \sum_{t=1}^{T} S(l, m)(t) \tag{7}$$

The element $M(l, m)$ in (**??**) can be interpreted as the long-term average proportion of intersection capacity given to movement (l, m). Hence define $M_{\overline{S}}$ to be the long-term control proportion matrix constructed as in (**??**) using a specific control sequence $\overline{S} = \{S(1), S(2), ..., S(t), ...\}$.

***Property** 2:* A demand d is *feasible* if and only if $\exists\, M_{\overline{S}} \in conv(U)$ and $\varepsilon > 0$ such that

$$c(l,m)M_{\overline{S}}(l,m) > f_l r(l,m) + \varepsilon. \tag{8}$$

where $f = dP$ as in (**??**).
Define D^0 to be the set of all average demand vectors $d = \{d_l\}$ that satisfy (**??**) and are therefore feasible.

A network is *stable* if the following quantity is bounded:

$$\frac{1}{T}\sum_{t=1}^{T} \mathbb{E}\Big\{|X(t)|_1\Big\} \tag{9}$$

where $|X|_1 = \sum_{l,m} |x(l,m)|$ and the network state evolves according to dynamics under state dynamics (**??**)-(**??**).

III. Standard Max Pressure Controller

Consider a weight assigned to each queue (l,m) as a function of all network queue lengths X:

$$w(l,m)(X(t)) = x(l,m)(t) - \sum_{p \in Out(m)} r(m,p)x(m,p)(t) \tag{10}$$

where $Out(m)$ is the set of all links receiving flow from link m. The *pressure* $\gamma(S)$ that is potentially alleviated by a control action S at time step t is defined as follows:

$$\gamma(S)(X(t)) = \sum_{l,m} c(l,m)w(l,m)(X(t))S(l,m)(t) \tag{11}$$

At each time step t, the standard max pressure controller $u^*(X(t))$ explicitly choses the phase $S^* \in U$ that maximizes $\gamma(S)(X(t))$:

$$S^*(t) = u^*(X(t)) = \arg\max\{\gamma(S)(X(t))|S \in U\} \tag{12}$$

Varaiya [**?**] shows the following stability result for the standard max pressure controller:

***Theorem** 3:* The max pressure control u^* is stabilizing whenever the average demand vector $d = \{d_l\}$ is within the set of feasible demands D^0.

This theoretical guarantee is one of the many attractive qualities of max pressure for controlling vehicular traffic in urban road networks. Yet the controller as originally formulated is not practical for application on a signalized traffic network for three reasons:

a) it does not account for capacity reductions (lost time) due to excessive signal switching,
b) it cannot enforce coordination between subsequent intersections for purposes of maximizing flow continuity, and
c) it does not provide guarantees that low-demand queues will be served within a finite time period.

These limitations motivate our extension of the standard immediate feedback max pressure control algorithm. In the following section, we define a new *cycle-based* max pressure controller which bounds the number of signal switches per fixed time period, provides capacity for standard signal coordination methods, and can easily guarantee a minimum service rate for all intersection phases. We then show that the application of this controller yields a similar stability guarantee to that shown by Varaiya for the standard controller given slightly weaker conditions on demand flow. The structure of this proof is as follows:

i. First, we formalize a calculation of the *lost time* incurred by signal switching actions.
ii. Then we introduce a formulation of the cycle-based max pressure algorithm and briefly describe how it inherently rectifies issues a), b), and c) above.
iii. Next, we introduce the concept of a τ-*updated* controller and we show that switching control only once every τ time steps does not impact the set of feasible demands.
iv. We finally show that queue stability holds with a cycle-based max pressure controller consisting of *τ-updated sequences of relaxed control matrices with minimum proportion constraints*.

IV. A Cycle-based Max Pressure Controller

For safety reasons, an intersection controller cannot switch signal phase actuation immediately. Instead, it must incorporate a pause of $R \approx 2.5$ seconds in which all signal phases have a red light. This *clearance time* allows all vehicles in the previously actuated phase to clear the intersection before a conflicting phase can be permitted to use the intersection. In the standard formulation of max pressure, the controller chooses an appropriate action based feedback received at every time step of the modeled dynamics. To accurately capture queuing behaviors observed on arterial roadways, a model would need to operate with a time discretization of $\Delta t < 10$ seconds. A signal switch at every time step could therefore result in more than 25% loss of intersection service capacity, which is not considered in the theoretical examination presented in [**?**].

In this work, we explicitly specify the number of signal switches that occur in a fixed number of model time steps using the familiar concept of a *signal cycle*. As typical with modern traffic signals, the *cycle-based max pressure controller* rotates through all available signal phases within a known time period. We define *cycle time* τ as a predefined number of model time steps and require that each controller phase S must be green for some proportion $\lambda_S \geq \kappa_S$ of the τ steps, where the *minimum green splits* $\kappa_S \in (0,1)\ \forall\ S \in U$ are parameters selected by a network manager to enforce equity in movement actuation.

The selection of a cycle time τ intuitively affects intersection capacity. Our proof of network stability in the following sections relies on the fact that road links are *undersaturated*: that is, the expected demand is served (on average) within a signal cycle. To avoid link saturation, we pose the following convex optimization problem (extended from that in [**?**]) to determine minimum constrained feasible actuation time Λ^*:

$$\begin{aligned} \Lambda^* = \min_{\lambda=\{\lambda_S\}} \quad & \sum_{S \in U} \lambda_S \\ \text{subject to} \quad & \lambda_S \geq \kappa_S\ \forall\ S \in U \\ & f_l r(l,m) < \sum_S \lambda_S c(l,m)S(l,m) \end{aligned} \tag{13}$$

where $\kappa_S \in [0,1]\ \forall S \in U$ and $\sum_S \kappa_S < 1$. If $\Lambda^* > 1$, the demand is not feasible under the set of $\{\kappa_S\}$ for any cycle length. If $\Lambda^* < 1$, then we can define a cycle length for which flow is admissible without link saturation. However, this cycle length τ must be significantly greater than Λ^* to account for clearance times. If we define $L = \lceil (\frac{R}{\Delta t} \cdot |U|) \rceil$ to be the total number of *lost time steps* per cycle, a feasible cycle length τ must satisfy the following condition:

$$\tau > \tfrac{L}{1-\Lambda^*} \tag{14}$$

Given an appropriate τ which satisfies (**??**), the cycle-based max pressure controller is a relaxed control matrix that is constructed as follows:

$$S^{r*}(t) = u^{c*}(X(t)) = \sum_{S \in U} \lambda^*_S S, \quad \text{where} \tag{15}$$

$$\{\lambda^*_S\} = \underset{\lambda_1,\ldots,\lambda_{|U|}}{\arg\max} \sum_{S \in U} \lambda_S \gamma(S)\big(X(\lfloor t/\tau \rfloor)\big) \tag{16}$$

$$\text{subject to} \quad \lambda_S \geq \kappa_s, \quad \sum_S \lambda_S \leq 1 - \tfrac{L}{\tau}$$

At time step $t = n\tau$ for integer n, the controller u^{c*} uses feedback measurements $x(t)$ to select a relaxed control matrix S^{r*} with components λ^*_S that satisfy (**??**). This relaxed controller is then applied for the subsequent τ time steps $\{t, t+1, \ldots, t+\tau-1\}$ before the controller is updated.

Note that this controller is modeled such that all phases in an intersection are simultaneously actuated at some proportion of their maximum flow capacity. This is not possible in practice, as many phases will have to make conflicting use of the same intersection resources. Hence individual phases S will have to be actuated in series, with each having a duration corresponding to a number of "time units" that are equal to cycle proportions ($\lambda_S \tau \cdot \Delta t$). Feedback measurements will then be a measure of "average" cycle queue length acquired over a set of measurements spanning the previous cycle.

Because cycle-based max pressure can be implemented such that phases occur in a predictable order, a controller running cycle-based max pressure can be synchronized with neighboring controllers to enforce a "green-wave progression" as is standard practice in existing traffic signal control design.

V. Stability of Cycle-Based Max Pressure

Cycle-based max pressure is fundamentally different from the standard max pressure formulation in [**?**] in two ways: first, it only updates the controller once every signal cycle (or τ model time steps); second, it applies a relaxed phase actuation (which is some convex combination of standard phase actuations). This section will address how each of these modifications individually affects the stability of the resulting controlled networks.

Properties of a τ-updated controller

Suppose that we impose that the control actuation $S^*(t)$ can only be updated every τ model time steps. A resulting *τ-updated control sequence* is composed of a single control matrix repeated for τ time steps of the model dynamics:

$$S(n\tau + 1) = S(n\tau + 2) = \ldots = S((n+1)\tau) \tag{17}$$

In Appendix **??**, we prove that the set of demands that can be accommodated using τ-updated control sequences is the same set of feasible flows as in (**??**). This equivalence becomes intuitive when one considers that our definition of feasible flows considers only the long-term (more precisely, infinite-term) average of both demand and service rates, and any infinite control sequence with limited admissible phases can be re-arranged to form a τ-updated sequence for some τ. As we will show in the following sections, the only additional impact of occasional updating will be an increased bound on queue lengths relative to the standard max pressure setting.

Stability of cycle-based max pressure

Here we examine the stability of the cycle-based max pressure controller using τ-updated sequences of relaxed controllers with fixed minimum phase proportion constraints, as formulated in (**??**)-(**??**).

Define $conv_\kappa$ as the set of convex combinations of control matrices with coefficients larger than κ:

$$conv_\kappa = \Big\{ \sum_S \lambda_S S \big| \; \lambda_S > \kappa_S \; \forall S \in U \Big\} \tag{18}$$

Also define a set of *undersaturated* admissible demands D_κ with elements d such that $f = dP$ and

$$f_l r(l,m) < c(l,m) S^r(l,m) \tag{19}$$

This condition (also seen in (**??**)) ensures that a demand $d \in D_\kappa$ can in average be served *within a single cycle* by a relaxed control matrix that maintains a specified minimum time allocation for each phase.

***Theorem** 4:* The cycle-based max pressure controller defined in (**??**)-(**??**) stabilizes a network whenever the demand is within a set of feasible undersaturated demands D^κ.

The remainder of this section proves Theorem **??** by finding a bound on (**??**) given a cycle-based max pressure controller. Begin by considering the expectation of the following function of queue state perturbation conditioned on the past queue state:

$$\begin{aligned} |X(t+1)|^2 - |X(t)|^2 &= |X(t) + \delta(t)|^2 - |X(t)|^2 \\ &= 2X(t)^T \delta(t) + |\delta(t)|^2 = 2\alpha(t) + \beta(t) \end{aligned} \tag{20}$$

with $\delta(t) = X(t+1) - X(t)$, $\alpha(t) = X(t)^T \delta(t)$, and $\beta(t) = |\delta(t)|^2$. We continue by addressing bounds on β and α separately.

First we consider $\beta(t) = |\delta(t)|^2$.

***Lemma** 5:*

$$\beta(t) = \big|\delta(t)\big|^2 \leq NB^2 \tag{21}$$

where $B = \big\{\overline{C}(l,m), \sum_k \overline{C}(k,l), \overline{d}(l,m)\big\}$, N is the number of queues in the network, $\overline{C}(l,m)$ is the maximum value of the random service rate $C(l,m)(t)$, and $\overline{d}(l,m)$ is the maximum value of random demand $d(l,m)$.

The proof of Lemma **??** is exactly the same as presented in [**?**] and will therefore not be repeated here. Note that because these bounds hold for any arbitrary $S(l,m)(t) \in [0,1]$, this bound on β is trivially extended to any convex combination of control matrices; hence it is still valid in our extension.

Now we examine a bound on $\alpha(t) = X(t)^T \delta(t)$. Again following [?], we define additional sub-terms:

$$\mathbb{E}\{\alpha(t)|X(t)\} = \sum_{l\in\mathcal{L},m} w(l,m)(t)\cdot \tag{22}$$

$$\Big[f_l r(l,m) - \mathbb{E}\Big\{\big[C(l,m)(t+1)S(l,m)(t)\wedge x(l,m)(t)\big]\big|X(t)\Big\}\Big]$$
$$= \alpha_1(t)+\alpha_2(t), \quad \text{with}$$

$$\alpha_1(t) = \sum_{l\in\mathcal{L},m}\left[f_l r(l,m) - c(l,m)S(l,m)(t)\right]w(l,m)(t) \tag{23}$$

$$\alpha_2(t) = \sum_{l\in\mathcal{L},m} S(l,m)(t)w(l,m)(t)\cdot \tag{24}$$

$$\Big[c(l,m) - \mathbb{E}\Big\{\big[C(l,m)(t+1)\wedge x(l,m)(t)\big]\big|X(t)\Big\}\Big] \tag{25}$$

***Lemma** 6:* For all l, m, t,

$$\alpha_2(t) \le \sum_{l\in\mathcal{L},m} c(l,m)\overline{C}(l,m) \tag{26}$$

The proof of Lemma **??** again directly follows that presented in [?]; an extension from a binary controller $S\in\{0,1\}$ to a relaxed controller $S^r\in[0,1]$ is trivial.

In fact, the extension made here only affects the $\alpha_1(t)$ term. To demonstrate a bound on $\alpha_1(t)$ given application of a cycle-based max pressure controller u^{c*}, we first examine the stability of a standard max pressure controller using relaxed controllers with minimum phase proportion constraints and a stricter limitation on network demands. We will then show that a τ-updated cycle-based max pressure controller also stabilizes a network, but results in a larger bound on queue lengths.

Define an intermediate "relaxed max pressure" policy in which relaxed controllers are applied at the standard max pressure update rate (once per time step of the model dynamics). This situation was suggested in [?] to simulate enforcing minimum phase proportions in a cycle formulation of max pressure. Yet it still unrealistically models "cycle" updates at the same rate as the model of queueing and discharging behaviors on a realistic traffic network (hence the introduction of the τ-updated formulation in this work). Nonetheless, we use this intermediate formulation to demonstrate that queue stability is still achieved upon use of a relaxed controller.

***Lemma** 7:* If a "relaxed" max pressure control policy S^{r*} is updated and applied at each time step t and the demand d is in the set of feasible undersaturated demands D^κ, then there exists an $\varepsilon>0$, $\eta>0$ such that

$$\alpha_1(t) \le -\varepsilon\eta\big|X(t)\big| \tag{27}$$

Proof: Consider the relaxed max pressure control matrix S^{r*} defined in (**??**) for $\tau=1$. By construction, $\forall\ S^r\in conv_\kappa$

$$\sum_{l,m} c(l,m)w(l,m)(X(t))S^r(l,m)$$
$$\le \sum_{l,m} c(l,m)w(l,m)(X(t))S^{r*}(l,m) \tag{28}$$

with equality only if $S^r=S^{r*}$. Thus $\forall\ (S^r\in conv_\kappa)\neq S^{r*}$,

$$\sum_{l,m}\big[f_l r(l,m) - c(l,m)S^{r*}(l,m)(t)\big]w(l,m)(X(t))$$
$$< \sum_{l,m}\big[f_l r(l,m) - c(l,m)S^r(l,m)\big]w(l,m)(X(t)) \tag{29}$$

If the demand flow is admissible according to (**??**), then $\exists\ \hat{S}\in conv_\kappa$ and some small $\varepsilon>0$ such that

$$c(l,m)\hat{S}(l,m) = \begin{cases} f_l r(l,m)+\varepsilon & \text{if } w(l,m)(X(t))>0 \\ 0 & \text{otherwise}\end{cases}$$

Therefore,

$$\alpha_1(t) = \sum_{l,m}[f_l r(l,m) - c(l,m)S^{r*}(l,m)(t)]w(l,m)(X(t))$$
$$< \sum_{l,m}\big[f_l r(l,m) - c(l,m)\hat{S}(l,m)(t)\big]w(l,m)(X(t))$$
$$= -\varepsilon\sum_{l\in\mathcal{L},m}\max\{w(l,m)(X(t)),0\} \tag{30}$$
$$+ \sum_{l\in\mathcal{L},m} f_l r(l,m)\min\{w(l,m)(X(t)),0\}$$

We assume that by our choice of $\hat{S}$, $f_l r(l,m)>\varepsilon$. Hence $\alpha_1(t) < -\varepsilon\sum_{l,m} w(l,m)(t)$. Given the linearity of (**??**) and the known properties of $r(l,m)(t)$, it can be show that $\sum_{l,m} w(l,m)(t)\ge\eta|X(t)|$ for some $\eta>0$. This completes the derivation of (**??**). ■

For ease of notation, now combine (**??**), (**??**) and (**??**) to obtain the following expression given application of the "relaxed max pressure" controller:

$$\mathbb{E}\Big\{|X(t+1)|^2-|X(t)|^2|X(t)\Big\} = \mathbb{E}\Big\{2\alpha(t)+\beta(t)\Big\}$$
$$< -2\varepsilon\eta\,|X(t)| + 2\sum_{l\in\mathcal{L},m}[c(l,m)\overline{C}(l,m)] + NB^2 \tag{31}$$

where N and B are as in (**??**). Next we establish a bound on queue growth in a single time step between controller updates.

***Lemma** 8:* Assuming a cycle-based max pressure controller with an cycle steps τ beginning at time t, the following bound on state perturbation holds for all $p\in[0,\tau-1]$:

$$\mathbb{E}\Big\{|X(t+p+1)|^2-|X(t+p)|^2\big|X(t)\ldots X(t+p)\Big\}$$
$$< Y + h(p) - 2\varepsilon\eta|X(t+p)| \tag{32}$$

$$\text{for}\quad Y = 2\sum_{l,m} c(l,m)\overline{C}(l,m) + NB^2 \quad\text{and} \tag{33}$$

$$h(p) = 2pNB\Big(\varepsilon\eta + \sum_{l,m}\big[f_l r(l,m)+c(l,m)\big]\Big) \tag{34}$$

Proof: As in Lemmas **??**-**??** above, begin by dividing the argument of (**??**) into three parts: $|X(t+p+1)|^2-|X(t+p)|^2 = 2(\alpha_1(t+p)+\alpha_2(t+p))+\beta(t+p)$, where β, α_1 and α_2 are quantities that depend on the controller applied at $(t+p)$:

$$\beta(t+p) = |X(t+p+1)-X(t+p)|^2 \tag{35}$$

$$\alpha_1(t+p) = w(l,m)(X(t+p))\cdot \tag{36}$$
$$\sum_{l,m}\Big(f_l r(l,m) - c(l,m)S(l,m)(t)\Big)$$

$$\alpha_2(t+p) = w(l,m)(X(t+p))\cdot\sum_{l,m}\Big(c(l,m)S(l,m)(t) \tag{37}$$

$$-\mathbb{E}\Big\{\big[C(l,m)(t+p+1)\wedge x(l,m)(t+p)\big]\big|X(t+p)\Big\}\Big)$$

Bounds on the expectations of $\beta(\cdot)$ and $\alpha_2(\cdot)$ were previously established for any binary or relaxed control matrix in (**??**) and (**??**), respectively. Thus we already know that:

$$\mathbb{E}\Big\{|X(t+p+1)|^2-|X(t+p)|^2|X(t)\dots X(t+p-1)\Big\} < 2\sum_{l,m}c(l,m)\overline{C}(l,m)+NB^2+\mathbb{E}\Big\{2\alpha_1(t+p)\Big\} \tag{38}$$

The remainder of the bound proposed in (**??**) originates from the $2\alpha_1(t+p)$ term, which is directly dependent on the explicit form of the controller S. Rewrite $2\cdot\alpha_1$ from (**??**) as follows:

$$\begin{aligned}
&2\sum_{l,m}w(l,m)(X(t+p))[f_l r(l,m)-c(l,m)S(l,m)(t)]\\
&=2\sum_{l,m}w(l,m)(X(t))[f_l r(l,m)-c(l,m)S(l,m)(t)]\\
&\quad+2\sum_{l,m}\Big\{w(l,m)\Big(X(t+p)-X(t)\Big)\cdot\\
&\qquad[f_l r(l,m)-c(l,m)S(l,m)(t)]\Big\} \qquad (39)\\
&=\xi_1(t,p,S)+\xi_2(t,p,S)\quad\text{for}
\end{aligned}$$

$$\xi_1(t,S)=2\sum_{l,m}w(l,m)(X(t))[f_l r(l,m)-c(l,m)S(l,m)(t)]$$

$$\xi_2(t,p,S)=2\sum_{l,m}\Big\{w(l,m)\Big(X(t+p)-X(t)\Big)\cdot [f_l r(l,m)-c(l,m)S(l,m)(t)]\Big\}$$

By Lemma **??** we know that $\xi_1(t,S) < -2\varepsilon\eta|X(t)|$. Because $|X(t)| = |X(t+p)-(X(t+p)-X(t))| > |X(t+p)|-|X(t+p)-X(t)|$, we find that

$$\begin{aligned}
\xi_1(t,p,S) &< -2\varepsilon\eta\big(|X(t+p)|-|X(t+p)-X(t)|\big)\\
&< -2\varepsilon\eta|X(t+p)|+2\varepsilon\eta\sum_{i=1}^{p}|X(t+i)-X(t+i-1)|\\
&= -2\varepsilon\eta|X(t+p)|+2\varepsilon\eta\sum_{i=1}^{p}|\delta(t+i-1)| \qquad (40)
\end{aligned}$$

So by (**??**) and (**??**),

$$\begin{aligned}
\xi_1(t,S) &< -2\varepsilon\eta|X(t+p)|\\
&\quad+2\varepsilon\eta p\sum_{l,m}\max\left\{\overline{C}(l,m),\sum_k\overline{C}(k,l),\overline{d}(l,m)\right\}\\
&=2\varepsilon\eta\cdot\Big(pNB-|X(t+p)|\Big) \qquad (41)
\end{aligned}$$

To bound ξ_2, we study the term

$$\begin{aligned}
&w(l,m)(X(t+p))-w(l,m)(X(t))\\
&=\sum_{n=1}^{p}w(l,m)(X(t+n))-w(l,m)(X(t+n-1))\\
&=\sum_{n=1}^{p}\Big\{x(l,m)(t+n)-x(l,m)(t+n-1)\\
&\quad-\sum_{s\in Out(m)}[x(m,s)(t+n)-x(m,s)(t+n-1)]r(m,s)\Big\}\\
&=\sum_{n=1}^{p}w(l,m)(\delta(t+n-1)) \qquad (42)
\end{aligned}$$

By (**??**) and the fact that $w(\cdot)$ is linear,

$$|w(l,m)(\delta(t+n-1))|<NB \tag{43}$$

Plugging (**??**) back into the definition of ξ_2, we obtain

$$\begin{aligned}
\xi_2(t,p,S)&=2\Big(\sum_{l,m}\big([f_l r(l,m)-c(l,m)S(l,m)]\cdot\\
&\qquad\sum_{n=1}^{p}w(l,m)(\delta(t+n-1))\big)\Big)\\
&<2\sum_{n=1}^{p}\sum_{l,m}[f_l r(l,m)-c(l,m)S(l,m)]\cdot\\
&\qquad\sum_{u,v}\max\left\{\overline{C}(u,v),\sum_k\overline{C}(k,u),\overline{d}(u,v)\right\}\\
&=2NB\sum_{n=1}^{p}\sum_{l,m}[f_l r(l,m)-c(l,m)S(l,m)] \qquad (44)
\end{aligned}$$

Also note that

$$\Big|\sum_{n=1}^{p}\sum_{l,m}[f_l r(l,m)-c(l,m)S(l,m)]\Big|<p\sum_{l,m}[f_l r(l,m)+c(l,m)]$$

so (**??**) becomes

$$\xi_2(t,p,S)<2NBp\cdot\Big(\sum_{l,m}[f_l r(l,m)+c(l,m)]\Big) \tag{45}$$

Substituting (**??**) and (**??**) into (**??**) yields (**??**). ■

Given Lemma **??**, we show that for a time t within any number K of τ-updated cycles, the following quantity is bounded:

$$\begin{aligned}
&\sum_{t=1}^{K\tau}\mathbb{E}\Big\{|X(t+1)|^2-|X(t)|^2|X(t)\Big\}\\
&=\sum_{t=1}^{K-1}\sum_{p=0}^{\tau-1}\mathbb{E}\Big\{|X(t+p+1)|^2-|X(t+p)|^2|X(t+p)\Big\}\\
&<\sum_{t=1}^{K-1}\sum_{p=0}^{\tau-1}(Y+h(p)-2\varepsilon\eta|X(t+p)|)\\
&<-2\varepsilon\eta\sum_{t=1}^{K\tau}|X(t)|+(K-1)\Big(\tau Y+\sum_{p=0}^{\tau-1}h(p)\Big) \qquad (46)
\end{aligned}$$

which, when taking the expectation, yields

$$\begin{aligned}
&\mathbb{E}\Big\{|X(K\tau+1)|^2-|X(1)|^2\Big\}\\
&<-2\varepsilon\eta\sum_{t=1}^{K\tau}\mathbb{E}\Big\{|X(t)|\Big\}+(K-1)(\tau Y+\sum_{p=0}^{\tau-1}h(p)) \qquad (47)
\end{aligned}$$

Rearranging gives

$$\begin{aligned}
\frac{1}{K\tau}\sum_{t=1}^{K\tau}\mathbb{E}\Big\{|X(t)|\Big\}&<\frac{1}{2\varepsilon\eta K\tau}\mathbb{E}\Big\{|X(1)|^2-|X(K\tau+1)|^2\Big\}\\
&\quad+\frac{\tau-1}{2\varepsilon\eta K\tau}\Big(\sum_{p=0}^{\tau-1}h(p)+\tau Y\Big)\Big] \qquad (48)\\
&<\frac{1}{2\varepsilon\eta\ K\tau}\mathbb{E}\Big\{|X(1)|^2\Big\}+\frac{1}{2\varepsilon\eta\tau}\left(\sum_{p=0}^{\tau-1}h(p)+\tau Y\right) \qquad (49)
\end{aligned}$$

By (**??**), the bound

$$\frac{1}{K\tau}\sum_{t=1}^{K\tau}\mathbb{E}\Big\{|X(t)|\Big\} < \frac{1}{2\varepsilon\eta\tau}\left[\tfrac{1}{K}\mathbb{E}\Big\{|X(1)|^2\Big\} + \sum_{p=0}^{\tau-1}h(p) + \tau Y\right]$$

establishes that the cycle-based max pressure controller $u^{c*}(X(t))$ defined in (**??**) will stabilize a vertical queueing network with dynamics $X(t)$ as in (**??**)-(**??**). However, the maximum expected queue lengths will be higher in the case of a cycle-based max pressure than in the standard frequent update setting. This relative increase can be shown to be linear in cycle length τ.

VI. Numerical Implementation

A cycle-based max pressure controller was implemented on a network of 11 signalized intersections modeled in Aimsun, a micro-simulation platform commonly used by practitioners. The calibrated model was generated as part of the I-15 Integrated Corridor Management project undertaken by the San Diego Association of Governments in San Diego, CA. The cycle-based controller was set to run with a cycle length of 90 seconds and minimum time constraints of 10 seconds for each of the 3-4 available standard signal control phases.

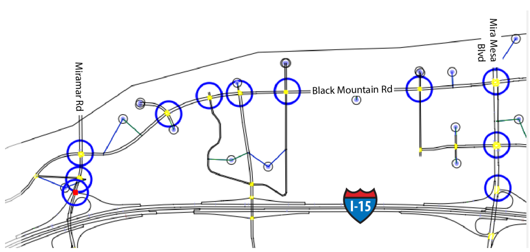

Fig. 1. The chosen network was calibrated to represent realistic demands and physical parameters observed on a stretch of Black Mountain Road near the I-15 freeway in San Diego, California.

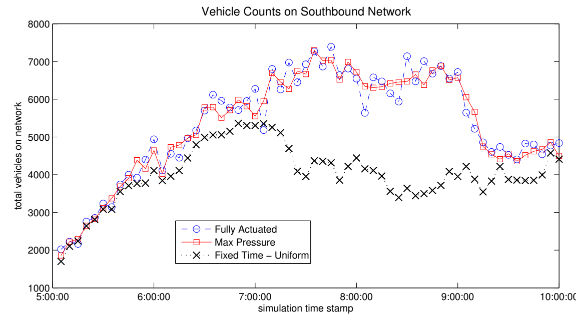

Fig. 2. During congestion, cycle-based max pressure demonstrated service rates that are higher than the uniformly allocated fixed-time controller and

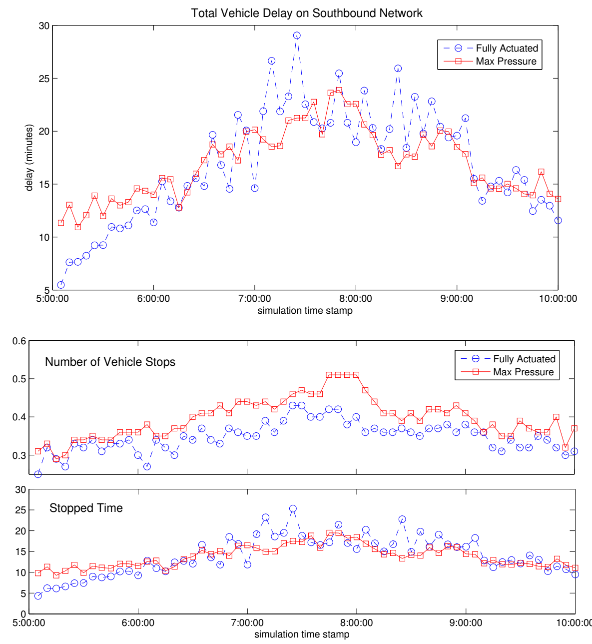

Fig. 3. Cycle-based max pressure outperforms the fully actuated system during periods of high demand. While max pressure caused more vehicle stop events, stoppage times were similar to those observed using the standard fully-actuated controller.

Various performance metrics were compared between model runs using the cycle-based max pressure controller and two alternative controllers: a fixed-time control plan that divides each signal cycle equally between all available phases, and a "fully-actuated" control system such as that which is currently operational on the real road network represented by the model. The fully actuated-controller is essentially a flexible fixed-time plan in which green times can be shortened or extended in real time to promote continuity of flows in response to instantaneous link demand measurements.

The comparison of network vehicle counts in Figure **??** suggests that the uniformly-allocated fixed time controller caused significantly fewer vehicles to be served than the other two controllers, which were comparable to each other in vehicle service rates. This difference made it difficult to fairly compare other performance metrics between the uniform fixed-time controller and the other two options.

Differences between the fully-actuated and cycle-based max pressure controllers were observed in measurements of delays and queue lengths. While the fully-actuated system appeared to produce less delay and shorter queues than max pressure during periods of relatively low network demand, max pressure was equally as effective or even more effective at reducing delays and queue lengths given larger demands. Figure **??** shows how delays were reduced and had less variance over time when the max pressure controller was applied than with the fully-actuated controller. However, cycle-based max pressure consistently induced more stops during a vehicle's journey across the network, which is expected given the design objectives of the fully-actuated system. Total stoppage times were higher with max pressure given low demand, but improved over the existing controller during peak demand.

VII. Conclusion

In this work we have defined an extension of the max pressure controller required for application on a real network of signalized traffic intersections. Given only the constraint that the network demands are serviceable in average, we have proven that updating a max pressure controller which allocates a fixed minimum proportion of service to each permissible phase at a slower rate than that which governs traffic flow will not destroy the stabilizing properties of the controller.

Max pressure provides theoretical guarantees on network-wide performance that are lacking in other existing signal control algorithms. Our implementation in the referenced micro-simulation platform furthermore demonstrates how it could comply with the hardware and communications constraints commonly encountered in existing roadway infrastructure. These numerical simulations reveal that an implementation of cycle-based max pressure competes with the existing standard feedback controller in many performance metrics, especially during periods of high congestion. It also appears to provide less variance in delays than the existing alternative. Future work will involve further analysis of the effect of cycle length on the performance of cycle-based max pressure, as well as performance improvements that can be achieved via the addition of common signal coordination practices in a network running cycle-based max pressure.

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APPENDIX I
FEASIBLE FLOWS WITH A τ-UPDATED CONTROLLER

***Lemma** 9:* All flows which satisfy Property **??** given a controller u updated at every model time step will also satisfy Property **??** with a τ-updated controller for some τ.

Proof: Given the set admissible phases U, define:

- $\mathcal{U}$ is the set of control sequences with distinct elements $\{S(1), S(2) \dots S(t) \dots | S(\cdot) \in U\}$,
- $\mathcal{U}_\tau$ is the set of τ-updated control sequences $\{S(1), S(1), \dots, S(\tau+1), S(\tau+1), \dots, S(n\tau+1), S(n\tau+1), \dots | S(\cdot) \in U\}$,

Also define the following sets of *long-term control proportion matrices*, which are similar to the formulation in (**??**):

$$M_{\mathcal{U}} = \Big\{ \liminf_T \frac{1}{T} \sum_{t=1}^{T} S(t) \Big| \{S(1), S(2), \dots, S(t), \dots\} \in \mathcal{U} \Big\}$$

$$M_{\mathcal{U}_\tau} = \Big\{ \liminf_T \frac{1}{T} \sum_{t=1}^{T} S(t) \cdot \Big| \{S(1), S(1), \dots, S(\tau+1), S(\tau+1), \dots\} \in \mathcal{U}_\tau \Big\}$$

By Property **??**, a demand d is only feasible if there exists a control sequence $\overline{S}$ such that the corresponding long-term control proportion matrix $M_{\overline{S}}$ satisfies (**??**). Here we show $M_{\mathcal{U}} = M_{\mathcal{U}_\tau}$, and therefore any flows that are admissible given an unrestricted controller in $\mathcal{U}$ can also be accommodated using a τ-updated controller in $\mathcal{U}_\tau$.

Obviously, $M_{\mathcal{U}_\tau} \subset M_{\mathcal{U}}$. To show equality, we must also demonstrate that $M_{\mathcal{U}} \subset M_{\mathcal{U}_\tau}$. Suppose there exists a control sequence $\hat{S} = \{S(1), S(2), \dots\} \in \mathcal{U}$. By definition,

$$M_{\hat{S}} = \liminf_T \frac{1}{T} \sum_{t=1}^{T} S(t) = \liminf_T \frac{1}{\tau T} \sum_{t=1}^{\tau T} \tilde{S}(t)$$

$$\text{where } \tilde{S} = \{S(1), S(1), \dots, S(t), S(t), \dots\}$$

$$= \liminf_T \frac{1}{T} \sum_{t=1}^{T} \tilde{S}(t) \in M_{U_\tau} \quad \implies M_{\mathcal{U}} \subset M_{\mathcal{U}_\tau}$$

■