Stability of Modified Max Pressure Controller with Application to Signalized Traffic Networks

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***Abstract*—This work describes a type of distributed feedback control algorithm that acts on a vertical queueing network where flow dynamics may greatly outpace the rate of feedback and actuation. The modeled network has a known, finite set of feasible actuations for the binary controllers located at each network node. It also has known expected demands, split ratios, and maximum service rates. Previous work proposed the application of a max pressure controller to maximize throughput on such a network without the need for centralized computation of a control policy. Here, we extend the max pressure controller to a practical scenario . Specifically, we extend the controller so as to satisfy some practical requirements. The set of allowable controllers in this setting is extended to any convex combination of available signal phases to account for signal changes within a single signal "cycle". Moreover the controller cannot be updated every time step. We show that this proposed extended max pressure controllers stabilize the network (queue lengths remain bounded in expectation) given slight restrictions on admissible network demand flows. This work is motivated by the application of controlling traffic signals on arterial road networks. Max pressure provides an intriguing alternative to existing feedback control systems due to its theoretical guarantees, but cannot be directly applied as originally formulated due to hardware and safety constraints. We ultimately apply our extension of max pressure to a simulation of an existing arterial roadway and provide comparison to the control policy that is currently deployed on this site.**

***Keywords*—max pressure, vertical queueing network, network stability, adaptive signal control, arterial modeling**

I. Introduction

This article investigates the design and stability of decentralized controller for vertical queueing networks. In a *vertical queueing network*, agents traveling across the network are stored in "point queues" which do not inhabit a "horizontal" position along the length of a network edge, but instead are considered to be "vertical" stacks at the front end of each edge. In the present work, we examine a vertical queueing network that can be controlled at junctions. Such networks are good models for urban traffic, but are inherited from other fields such as supply chain management or internet routing. The flows passing through each node of this network are constrained by a controller, similar to how a traffic light dictates allowable flows across a traffic intersection: only a finite set of non-conflicting turning movements or *phases* can be permitted to flow simultaneously across each node. At some regular interval, each of the the node controllers selects one element from its set of feasible phases to actuate. Hence flows through the network are dictated largely by the control policy applied at the nodes. A signal control policy is said to be *stabilizing* if it ensures that the mean length of the queues waiting at each of the nodes remains bounded for all time. One such stabilizing control policy that is known to maximize network throughput is the *max pressure* controller. Here, we show that such network stabilization is still achieved given various extensions of the original max pressure controller. These extensions are motivated by the practical physical constraints imposed on realistic traffic signal controllers, and they present new mathematical problems in the analysis of queue dynamics which are investigated in this article.

The general problem of a stabilizing controller in this context was first considered from the point of view of multihop radio networks in Tassiulas et Ephremides [?]. The control of a network is a broad subject with many applications such as wireless networks [?][?], oscilatory networks [?], or queuing networks [?][?]. In the particular context of arterial traffic control, little work has been made on the approximation equilibrium of the network, at the exception of Osorio and Bierlaire [?][?]. Most of the literature focuses on the design of feedback responsive control. Comprehensive on this topic are provided in Baras and Levine [?][?], Mirchandani and Head [?], Papageorgiou et al. [?], Osorio and Bierlaire [?], and Xie et al. [?].

The max pressure traffic controller [?] is a specific distributed network signal control policy derived from the concept of a "back pressure controller", which was first studied in the context of routing packets through a communications network (see for example the pioneering work of Tassiulas and Ephremides [?]). The ideas has been applied to road traffic management more recently by Varaiya [?], as well as in the work of Wongpiromsarn et al. [?].

The concept of max pressure control is intuitive: at each intersection, give priority to the phase which will be able to service the most traffic while taking into account the subsequent feasibility of downstream demand queues. Variaya's original formulation of this controller in [?], however, does not fully consider the practical limitations on the rate of queue measurement and signal actuation in practical traffic networks. Using Variaya's stability proof as a starting point, we prove that two extended max pressure controllers that account for these constraints will still provide queue stability. Specifically, we address the fact that a practical control policy cannot be updated at the same time scale as that at which queues form: due to hardware limitations and safety constraints, most

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existing signal controllers must cycle through all allowable sets of phases while adhering to a fixed cycle length and strict limitations on the minimum and maximum allowable actuation time for each phase within the cycle. During the course of one signal cycle, however, vehicle queues may aggregate (or dissipate) significantly. Therefore, we first show that a "non-updated" max pressure controller, or one which only received occasional feedback upon which to update its actuation policy, will still stabilize network queues. We then show that under slightly stronger conditions on admissible input flows, a "relaxed" max pressure controller that allocates an entire actuation cycle between each signal phase given occasional feedback will also stabilize the network. We also examine the effect that each of these limitations on control will have on the ultimate bound on queue length relative to the chosen cycle time and network parameters.

The remainder of this article is organized as follows: first, Section II describes our modeling framework and Section III gives a mathematical definition of controllers and the concept of network stability. Section IV describes the problem the max pressure encounters in practice. Sections V describe the new controller we propose to solve those problems and expose the proof of stability for this new controller. Finally, Section VI outlines numerical results provided by this controller using a micro-simulation running in the Aimsum platform of a real arterial traffic network within the I-15 corridor which was part of a SANDAG pilot program ran in San Diego, CA.

II. Model Framework

Definitions

We consider a network of arterial roads with infinite storage capacity, modeled topologically as a graph with road links being edges and intersections being vertices. An individual link $l \in \mathcal{L}$ can be either at the entry of the network ($l \in \mathcal{L}_{\text{ent}}$) or in the interior of the network ($l \in \mathcal{L} \backslash \mathcal{L}_{\text{ent}}$). The inflow on entry links is a defined entirely by a random demand d_l, while the input flows of all other links depend on queues on upstream links and the relevant set of physical flow constraints are defined within the network. We require that each link has an exit path, that is, a continuous set of subsequent links on which vehicles can travel from the link to eventually exit the network. Each link in the network model can have multiple *queues* corresponding to individual *movements*: all vehicles in a given queue on any link are intending to advance onto the same subsequent link (though not necessarily the same subsequent queue).

We describe the dynamics of these queues as a discrete time dynamical model using the following notation:

- A *movement* (l, m) distinguishes an intention to travel from link l to link m, (in that case, say that $m \in Out(l)$ where $Out(l)$ is the set of links immediately downstream l);
- A *queue* $x(l,m)(t)$ is the number of vehicles on link l waiting to enter link m at timestep t, and $X(t)$ is the set (vector or matrix) of all the queue lengths on the network at timestep t;
- A *saturation flow* $c(l, m)$ is the expected number of vehicles that can travel from link l to link m per time step given maximum demand for the queue $x(l, m)$, and $C(l, m)(t)$ is the *realized saturation flow* at time t;
- The *turn ratio* $r(l, m)$ is the expected proportion of vehicles that are leaving l which are intending to enter m, and $R(l, m)(t)$ is the *realized turn ratio* at time t;
- The *demand vector* d of dimension $|\mathcal{L}_{\text{ent}}|$ specifies demands at network entry links;
- The *flow vector* f of dimension $|\mathcal{L}|$ denotes flows on all links of the network such that f_l is the flow within link l.

Note that there is necessarily a linear relationship between the flow vector f and the demand vector d: $f = dP$ where the matrix P depends only on routing proportions.

Controller

A road intersection is modeled as a node in our framework. Controllers (traffic signals) are placed at every node to limit the set of queues permitted to discharge at any given time. A set of movements that can be simultaneously actuated without flow conflicts is called a *phase*. Each permissible phase for a given intersection can be represented as a binary control matrix S defined as follows:

$$S(l, m) = \begin{cases} 1 & \text{if movement } (l, m) \text{ is activated} \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

We denote U_n the known finite set of permissible control matrices for node n. Note that in this article, we often drop the subscript n for ease of notation.

While more than one phase cannot realistically be simultaneously actuated, a "relaxed" controller can also be defined as a matrix representing the fraction of each time step allocated to each phase:

$$S^r(l, m) = \lambda_{l,m} \in [0, 1] \tag{2}$$

Such a relaxed controller can be seen as a convex combination of all possible control matrices,

$$S^r = \sum_{S \in U} \lambda_S S \tag{3}$$

We suppose that at each model time step t, a controller selects a single control matrix $S(t)$ that encodes which set of queues approaching the intersection are permitted to discharge during that time step. In general, the determination of such a controller is based on the state of the network at a previous time step.

Queue Dynamics

The evolution of queue state $X(t)$ can be seen as a Markov chain: the state of the network at time $t + 1$ is a function of only the network state at time t and external demand vector d,

$$X(t + 1) = F(X(t), d) \tag{4}$$

Define $[\,a \wedge b\,] := \min\{a, b\}$. To describe queue dynamics explicitly, we must make a distinction between entry links and internal links: if $l \in \mathcal{L}_{\text{ent}}$,

$$\begin{aligned} x(l,m)(t+1) = x(l,m)(t) + d_l(t+1) \\ - [C(l,m)(t+1)S(l,m)(t+1) \wedge x(l,m)(t)] \end{aligned} \tag{5}$$

and if $l \in \mathcal{L} \backslash \mathcal{L}_{\text{ent}}$,

$$\begin{aligned} &x(l,m)(t+1) = x(l,m)(t) + \\ &\sum_k [C(k,l)(t+1)S(k,l)(t+1) \wedge x(k,l)(t)]R(l,m)(t+1) \\ &\quad - [C(l,m)(t+1)S(l,m)(t+1) \wedge x(l,m)(t)] \end{aligned} \tag{6}$$

Stability conditions

We focus on networks for which the boundary inflow demands $d = (d_l)_{(l \in \mathcal{L}_e)}$ are *feasible*—that is, the network is servicing a distribution of inflows for which it is possible to find a controller that allows *in average* more departures than arrivals at each link. For a specific sequence of control matrices $\overline{S} = \{S(1), S(2), ..., S(t), ...\}$, we define the *long-term control proportion* matrix $M_{\overline{S}}$ as follows:

$$M_{\overline{S}}(l,m) = \liminf_T \frac{1}{T} \sum_{t=1}^{T} S(l,m)(t) \tag{7}$$

We define $co(U)$ as the convex hull of the set of permissible control matrices U. The following property can be shown:

***Property** 1:* As shown in [**?**], $M \in co(U)$ if and only if $\exists$ a sequence of control matrices $\overline{S} = \{S(1), S(2), ..., S(t), ...|S(\cdot) \in U\}$ such that $\forall (l,m)$

$$M(l,m) = \liminf_T \frac{1}{T} \sum_{t=1}^{T} S(l,m)(t) \tag{8}$$

***Property** 2:* As defined in [**?**], a demand is feasible if and only if $\exists\, M_{\overline{S}} \in co(U)$ and $\varepsilon > 0$ such that

$$c(l,m)M_{\overline{S}}(l,m) > f_l r(l,m) + \varepsilon. \tag{9}$$

Furthermore, we say that a network is *stable* if the following quantity is bounded:

$$\frac{1}{T} \sum_{t=1}^{T} \mathbb{E}\Big\{|X(t)|_1\Big\} \tag{10}$$

where $|X|_1 = \sum_{l,m} |x(l,m)|$. If the demand on a network is feasible, then it is possible to find a controller that stabilizes this network as shown by Varaiya [**?**].

III. Standard Max Pressure Controller: Immediate Feedback

This section summarizes results from [**?**] and [**?**] used later in this article.

Principle

We consider a weight assigned to each queue (l,m) as a function of all network queue lengths X:

$$w(l,m)(X(t)) = x(l,m)(t) - \sum_{p \in Out(m)} r(m,p)x(m,p)(t) \tag{11}$$

where $Out(m)$ is the set of all links receiving flow from link m. The original *immediate feedback* max pressure controller actuates the phase $S^* \in U$ which alleviates the most *pressure* at the intersection. The pressure $\gamma(S)$ that is potentially alleviated by a control action S is defined as follows:

$$\gamma(S)(X(t)) = \sum_{l,m} c(l,m)w(l,m)(X(t))S(l,m)(t) \tag{12}$$

$$= \sum_{l,m:S(l,m)(t)=1} c(l,m)w(l,m)(X(t)) \tag{13}$$

Explicitly, $u^*(X)$ therefore simply choses the phase $S^* \in U$ that maximizes $\gamma(S)(X(t))$ at each time step:

$$S^*(t) = u^*(X(t)) = \arg\max\{\gamma(S)(X(t))|S \in U\} \tag{14}$$

First Stability Result

Varaiya [**?**] shows that the following stability result holds for the immediate feedback max pressure controller:

***Theorem** 3:* The max pressure control u^* is stabilizing whenever the average demand vector $d = \{d_l\}$ is within the set of feasible demands, called D^0. There is no stabilizing control when d is not within the set of feasible demands D^0.

IV. Practical Constraints on the Controller

Varaiya [] assumes that the controller can be updated every time step. In real traffic systems, this is not possible. As a matter of fact, for hardware/technical reasons, one must keep the same control every τ time steps. This can be seen as the fact that the dynamics are updated quicker than the control.

Also, Varaiya suppose that only one control matrix is chosen during the optimisation procedure, i.e. only one of the possible non conflicting phases is chosen. This is not realistic in practice either, since each set of phases must be allocated a minimal (non nul) time. In that case, instead of choosing the controller over a finite set, a convex combination of control matrices is chosen. The coefficient λ_S chosen for each phase S represents the fraction (between 0 and 1) of the physical time allocated. To ensure that each phase is guaranteed a minimum time, we want to constraint the coefficients of the convex combination to be all larger than κ. (It is pretty straightforward to notice that κ must satisfy the following:

$$\kappa|\text{set of non conflicting phases }| < 1$$

We therefore reformulate the controller so that instead of a single phase being chosen for actuation at every model step, all feasible phases are actuated in parallel during the time step at a proportionally reduced rate. The control output will no longer be a binary matrix, but rather will be a *relaxed* control matrix

$S^r(t)$ which is a convex combination of all elements in the set of feasible control phases U. Each element $S^r(t)(i,j) \in [0,1]$ will represent fraction of the flow capacity at time step t that is allocated to movement (i,j).

V. Relaxation of the Max Pressure Controller. Stability of the new Controller

We reformulate the controller so as to fullfill those guarantees. Then we show that a similar stability result as the one shown by Varaiya holds (under some *slightly* weaker conditions). To fulfil the first condition, we allow the control to be a convex combination of the different possible control matrices. For the second condition we consider a control that is not updated every time steps. The final control that we choose is a combination of the two.

Minimum cycle time

Consider that the length T of the controller update time period (signal cycle) is not pre-defined. However, there is a known amount of *lost time* L per cycle during which no phases can be actuated to account for physical clearance of the intersection between actuated phases. Hence it must be the case that $T > L$. We furthermore impose that each feasible phase has an associated minimum proportion constraint: phase S must be actuated for at least $\lambda_S T$ seconds during any one cycle. We then want to determine the minimum control period or *cycle length* for which a fixed demand flow can be served while the stated phase proportion constraints are satisfied.

Unlike the previous case where we constrained our analysis to flows which could be served in average over an arbitrary long-term time horizon, here our proof of stability depends on the assumption that the average demand is served *within a single cycle*. As suggested in [**?**], we then pose the selection of a cycle length as a convex optimization problem with constraints applied to enforce the desired minimum phase proportions κ:

$$\begin{aligned} \underset{\lambda=(\lambda_S)_S}{\text{minimize}} \quad & \sum_{S\in U} \lambda_S \\ \text{subject to} \quad & \lambda_S \geq \kappa_S \\ & f_l r(l,m) < \sum_S \lambda_S c(l,m) S(l,m) \end{aligned} \tag{15}$$

where $\kappa_S \in [0,1]\ \forall S \in U$, and $\sum_S \kappa_S < 1$.

Let us denote Λ^* to be the optimum of (15). If $\Lambda^* > 1$, the demand is not feasible under the set of control constraints $\{\kappa_S\}$ for any length time step. If $\Lambda^* < 1$, then the flow is admissible for a time step of length

$$T > \frac{L}{1-\Lambda^*} \tag{16}$$

We can now use this lower bound to select an appropriate cycle length.

Cycle max pressure

Cycle max pressure enables the control to act at a slower time scale than the queue dynamics, as would be the case in a practical traffic application. Suppose that we are given the model dynamics $X(t)$ as in (5)-(6), but the controller $S^*(t)$ can only be updated every τ model time steps (or once per *cycle*). The "τ-non updated" control sequence is therefore composed of control matrices repeated for at least τ model time steps: for a fixed cycle size τ and integer n,

$$\begin{aligned} S(n\tau+1) &= S(n\tau+2) = \ldots = S((n+1)\tau) = S^*(n\tau+1) \\ &= \arg\max\{\gamma(S)(X(n\tau+1))|S \in U\} \end{aligned} \tag{17}$$

Physically, the controller that maximizes the pressure at time step $n\tau+1$ is continuously applied until time step $(n+1)\tau$.

Relaxed Max Pressure Controller

Let $\tau \in \mathbb{N}$ be fixed, the number of time steps between two actuation of the controller $\forall n \in \mathbb{N}$,

$$\begin{aligned} S(n\tau+1) &= \ldots = S((n+1)\tau) = S^*(n\tau+1) \\ &= \arg\max\{\gamma(S)(X(n\tau+1))|S \in U\} \quad (18) \\ &= \underset{\lambda_1,\ldots,\lambda_{|U|}}{\arg\max} \sum_{S\in\mathcal{U}} \lambda_S \Big(\sum_{l,m} w(l,m)(t)c(l,m)S(l,m)\Big) \quad (19) \\ &\quad \text{subject to} \\ &\quad \lambda_S \geq \kappa_s \\ &\quad \sum_S \lambda_S \leq 1 - \tfrac{L}{T} \quad (20) \\ &= S^{r*} \quad (21) \\ &= \sum_{S\in\mathcal{U}} \lambda^*_S S \quad (22) \end{aligned}$$

Phase distribution within a cycle

We next investigate how the max pressure controller must be altered to allow the minimum proportion constraints on each phase in the set of allowable phases $\mathcal{U}$.

Let T be the cycle length satisfying (16) and κ_S be the minimum proportion of the cycle which must be allocated to each $S \in U$. The allocated max pressure controller selects a set of λ_S that maximizes alleviated pressure under the following constraints:

- $\lambda_S \geq \kappa_S$
- $f_l r_{l,m} < \sum_S \lambda_S c(l,m) S(l,m)$
- $\sum_S \lambda_S = 1 - \frac{L}{T}$

The desired controller is therefore determined by the solution to the following linear program:

$$\begin{aligned} \{\lambda^*_S\} = \underset{\lambda_1,\ldots,\lambda_{|U|}}{\arg\max} \quad & \sum_{S\in\mathcal{U}} \lambda_S \Big(\sum_{l,m} w(l,m)(t)c(l,m)S(l,m)\Big) \\ \text{subject to} \quad & \lambda_S \geq \kappa_s \\ & \sum_S \lambda_S \leq 1 - \tfrac{L}{T} \end{aligned} \tag{23}$$

The solution to (23) selects coefficients $\{\lambda_S^*\}$ which form a corresponding relaxed control matrix

$$S^{r*} = \sum_{S \in \mathcal{U}} \lambda_S^* S \tag{24}$$

τ-admissible flows

We first prove that this set of τ-admissible flows (demands that can be accommodated using τ-non updated sequences) is in fact the same set of flows that is admissible under typical updated control sequences, defined in equation (9).
Define the following sets:

- U is the set of admissible control matrices as in (9),
- $U_{\mathbb{N}}$ is the set of control sequences $\{S(1), S(2) \dots S(t) \dots | S(\cdot) \in U\}$ where elements are applied at consecutive time steps,
- $U_{\tau\mathbb{N}}$ is the set of control sequences $\{S(1), S(1), \dots, S(\tau+1), S(\tau+1), \dots, S(n\tau+1), S(n\tau+1), \dots | S(\cdot) \in U\}$ where controls are updated only once every τ steps,
- $conv(U) = \Big\{ \liminf_T \dfrac{1}{T} \sum_{t=1}^T S(t) | \{S(1), S(2), \dots, S(t), \dots\} \in U_{\mathbb{N}} \Big\}$
- $conv(U_\tau) = \Big\{ \liminf_T \dfrac{1}{T} \sum_{t=1}^T S(t) | \{S(1), S(1), \dots, S(\tau+1), S(\tau+1), \dots\} \in U_{\tau\mathbb{N}} \Big\}$

Obviously, $conv(U_\tau) \subset conv(U)$. But we can also show that $conv(U) \subset conv(U_\tau)$:
Suppose $M \in conv(U)$, so $\exists \{S(1) \dots S(t) \dots\}$ such that $M = \liminf_T \dfrac{1}{T} \sum_{t=1}^T S(t)$.

$$\begin{aligned} M &= \liminf_T \frac{1}{T} \sum_{t=1}^{T} S(t) = \liminf_T \frac{1}{\tau T} \sum_{t=1}^{\tau T} \tilde{S}(t) \\ &\quad \text{with } \tilde{S} = \{S(1), \dots, S(1), \dots, S(t), \dots, S(t), \dots\} \\ &= \liminf_T \frac{1}{T} \sum_{t=1}^{T} \tilde{S}(t) \end{aligned}$$

Trivially, $\tilde{S} \in conv(U_\tau)$. Because $conv(U_\tau) \subset conv(U)$ and $cont(U) \subset conv(U_\tau)$, it must hold that $conv(U) = conv(U_\tau)$. This establishes Property 4:

***Property** 4:*

$$conv(U) = conv(U_\tau)$$

This property implies that a τ-non updated control sequence can accommodate the same set of flows as a control sequence updated at every time step. The equivalence becomes intuitive when one considers that our definition of feasible flows considers only the long-term average of demand and service rates: note that a τ control matrix in $\tilde{S}$ is simply the average of the corresponding τ matrices in $\overline{S}$, such that $M_{\tilde{S}} = M_{\overline{S}}$. Hence,

$$\begin{aligned} f_l r(l,m) < c(l,m) M_{\overline{S}}(l,m) \implies \\ f_l r(l,m) < c(l,m) M_{\tilde{S}}(l,m). \end{aligned} \tag{25}$$

However, as we will show in the following sections, the bound on queue lengths when the cycle controller is applied will be larger than in the immediate feedback setting.

Stability of the new controller

Here we extend the previous proof of stability of an immediate feedback max pressure controller to the case of our relaxed max pressure controller:

***Theorem** 5:* The relaxed max pressure controller, defined above, updated every τ iterations stabilizes the network whenever the demand is within a set of feasible demands D^κ.

Proof:

Define $conv_\kappa$ as the set of convex combinations of control matrices with coefficients larger than κ:

$$conv_\kappa = \Big\{ \sum_S \lambda_S S | \, \lambda_S > \kappa_S \; \forall S \in U \Big\} \tag{26}$$

Also define a set of reduced admissible demands D_κ which can in average be served in a single cycle with a relaxed control matrix that maintains the minimum time allocation for a given cycle time (as in (15)):

$$\begin{aligned} &d \in D_\kappa \text{ iff } \exists\, S^r \in conv_\kappa \\ &\qquad \text{such that } f_l r(l,m) < c(l,m) S^r(l,m) \end{aligned} \tag{27}$$

Immediate feedback Max Pressure: proof of Theorem 3: Consider the expectation of the following function of queue state with perturbation

$$\delta(t) = X(t+1) - X(t) \tag{28}$$

conditioned on the past queue state:

$$\begin{aligned} |X(t+1)|^2 - |X(t)|^2 &= |X(t) + \delta(t)|^2 - |X(t)|^2 \\ &= 2X(t)^T \delta(t) + |\delta(t)|^2 \\ &= 2\alpha(t) + \beta(t) \end{aligned} \tag{29}$$

with

$$\alpha(t) = X(t)^T \delta(t) \quad \text{and} \quad \beta(t) = |\delta(t)|^2 \tag{30}$$

We continue by addressing bounds on β and α separately.

Bound on $\beta(t) = |\delta(t)|^2$

Define $\overline{C}$ as the maximum realized saturation flow and $\overline{d}$ as the maximum possible value of the demand vector. If $l \in \mathcal{L}_{\text{ent}}$ and $m \in Out(l)$,

$$\begin{aligned} |\delta(l,m)(t)| = \Big| &- [C(l,m)(t+1) S(l,m)(t) \wedge x(l,m)(t)] \\ &+ D(l,m)(t+1) \Big| \\ &\le \max\{\overline{C}(l,m), \overline{d}(l,m)\} \end{aligned} \tag{31}$$

where $D(l,m)(t) = D(l)(t) R(l,m)(t)$ with $D(l)(t)$ defined as the realized demand on link l at time t. This is because we know that both $C(l,m)(t+1) S(l,m)(t) \wedge x(l,m)(t)$ and $D(l,m)(t+1)$ are non-negative, so the absolute value of their difference must be less than either of the two quantities individually.

Similarly, if $l \in \mathcal{L}\backslash\mathcal{L}_{\text{ent}}$ and $m \in Out(l)$:

$$|\delta(l,m)(t)| = \Big| -[C(l,m)(t+1)S(l,m)(t) \wedge x(l,m)(t)] \tag{32}$$

$$+ \sum_k [C(k,l)(t+1)S(k,l)(t) \wedge x(k,l)(t)]R(l,m)(t+1)\Big|$$

$$\le \max\left\{\overline{C}(l,m), \sum_k \overline{C}(k,l)\right\} \tag{33}$$

If we define B as the maximum of all of the quantities $\left\{\overline{C}(l,m), \sum_k \overline{C}(k,l), \overline{d}(l,m)\right\}$ and N as the number of queues in the network, we can derive a bound for β which depends on only B and N:

$$\beta(t) = |\delta(t)|^2 \le NB^2 \tag{34}$$

Note that because these bounds hold for any $S(l,m)(t) \in [0,1]$, the bound on β presented here can easily be extended to any convex combination of control matrices; hence it is still valid in our modified controllers, as shown later in this article.

Bound on $\alpha(t) = X(t)^T\delta(t)$

The term α in (30) is explicitly defined in terms of queue state $X(t)$ as follows:

$$\begin{aligned}\alpha(t) &= X(t)^T[X(t+1) - X(t)]\\ &= \sum_{l\in\mathcal{L}\backslash\mathcal{L}_{\text{ent}},m}\sum_k [C(k,l)(t+1)S(k,l)(t) \wedge x(k,l)(t)]\\ &\qquad \cdot R(l,m)(t+1)x(l,m)(t)\\ &\qquad - \sum_{l\in\mathcal{L},m}[C(l,m)(t+1)S(l,m)(t) \wedge x(l,m)(t)]\\ &\qquad - \sum_{l\in\mathcal{L}_{\text{ent}},m} d(l,m)(t+1)x(l,m)(t)\\ &= \sum_{l\in\mathcal{L},m}[C(l,m)(t+1)S(l,m)(t) \wedge x(l,m)(t)]\\ &\qquad \cdot\Big(-x(l,m)(t) + \sum_p R(m,p)(t+1)x(m,p)(t)\Big)\\ &\qquad + \sum_{l\in\mathcal{L}_{\text{ent}},m} d(l,m)(t+1)x(l,m)(t)\end{aligned} \tag{35}$$

Note that only the expectation of these terms appear in equation (**??**), so we are interested in $\mathbb{E}\{\alpha(t)|X(t)\}$. We therefore make the following observation: because $R(m,p)(t+1)$ is independent of $C(l,m)(t+1)$ and $X(t)$,

$$\begin{aligned}&\mathbb{E}\Big\{[C(l,m)(t+1)S(l,m)(t) \wedge x(l,m)(t)]\\ &\qquad \cdot R(m,p)(t+1)x(m,p)(t)|X(t)\Big\}\\ &= \mathbb{E}\Big\{[C(l,m)(t+1)S(l,m)(t) \wedge x(l,m)(t)]|X(t)\Big\}\\ &\qquad \cdot r(m,p)(t+1)x(m,p)(t)\end{aligned}$$

Also, the expectation of demand $d(l,m)$ is equal to the measured demand d_l on link l, times the relevant expected split ratio $r(l,m)$. Hence the desired expectation of (35) can be expressed as

$$\begin{aligned}&\mathbb{E}\{\alpha(t)|X(t)\} =\\ &\sum_{l\in\mathcal{L},m}\mathbb{E}\Big\{[C(l,m)(t+1)S(l,m)(t) \wedge x(l,m)(t)]|X(t)\Big\}\\ &\qquad \cdot\Big(-x(l,m)(t) + \sum_p r(m,p)(t+1)x(m,p)(t)\Big)\\ &\qquad + \sum_{l\in\mathcal{L}_{\text{ent}},m} d_l r(l,m)x(l,m)(t)\\ &= \sum_{l\in\mathcal{L}_{\text{ent}},m} d_l r(l,m)x(l,m)(t)\\ &\quad - \sum_{l\in\mathcal{L},m}\mathbb{E}\Big\{[C(l,m)(t+1)S(l,m)(t) \wedge x(l,m)(t)]|X(t)\Big\}\\ &\qquad \cdot w(l,m)(t)\end{aligned} \tag{36}$$

where $w(l,m)(t) = w(l,m)(X(t))$ is the weight of a link as defined in (11). We also can include the following relation:

$$\begin{aligned}&\sum_{l\in\mathcal{L},m} f_l r(l,m)w(l,m)(t)\\ &= \sum_{l\in\mathcal{L},m} f_l r(l,m)\left[x(l,m) - \sum_p r(m,p)x(m,p)(t)\right]\\ &= \sum_{l\in\mathcal{L},m} f_l r(l,m)x(l,m)(t)\\ &\qquad - \sum_m\left[\sum_{l\in\mathcal{L}} f_l r(l,m)\sum_p r(m,p)x(m,p)(t)\right]\\ &= \sum_{l\in\mathcal{L},m} f_l r(l,m)x(l,m)(t)\\ &\qquad - \sum_{m\in\mathcal{L}\backslash\mathcal{L}_{\text{ent}},p} f_m r(m,p)x(m,p)(t)\\ &= \sum_{l\in\mathcal{L}_{\text{ent}},m} d_l r(l,m)x(l,m)(t)\end{aligned}$$

So (36) is further simplified to:

$$\begin{aligned}\mathbb{E}\{\alpha(t)|X(t)\} &= \sum_{l\in\mathcal{L},m}\Big[f_l r(l,m)\\ &- \mathbb{E}\Big\{[C(l,m)(t+1)S(l,m)(t) \wedge x(l,m)(t)]|X(t)\Big\}\Big]w(l,m)(t)\\ &= \alpha_1(t) + \alpha_2(t)\end{aligned} \tag{37}$$

with

$$\alpha_1(t) = \sum_{l\in\mathcal{L},m}[f_l r(l,m) - c(l,m)S(l,m)(t)]\,w(l,m)(t) \tag{38}$$

and

$$\alpha_2(t) = \sum_{l\in\mathcal{L},m}\Big[c(l,m) - \tag{39}$$

$$\mathbb{E}\Big\{[C(l,m)(t+1)\wedge x(l,m)(t)]|X(t)\Big\}\Big]S(l,m)(t)w(l,m)(t)$$

These $\alpha_{\{1,2\}}$ are derived by first including in (37) the 0-valued term $[c(l,m)S(l,m)(t)w(l,m)(t) - c(l,m)S(l,m)(t)w(l,m)(t)]$, and then assuming that $S(l,m)(t) \in \{0,1\}$ (hence it can be brought outside of the internal min function in α_2 without changing the ultimate result).

***Lemma** 6:* For all l, m, t,

$$\alpha_2(t) \leq \sum_{l\in\mathcal{L},m} c(l,m)\overline{C}(l,m) \tag{40}$$

where $\overline{C}(l,m)$ is the maximum value of the random service rate $C(l,m)(t)$.

Proof of Lemma 6:
By Jensen's inequality,

$$\begin{aligned}\mathbb{E}\Big\{C(l,m)(t+1)\wedge x(l,m)(t)|X(t)\Big\} \\ &\leq \mathbb{E}\Big\{C(l,m)(t+1)|X(t)\Big\}\wedge x(l,m)(t) \\ &= c(l,m)\wedge x(l,m)(t) \\ &\leq c(l,m)\end{aligned}$$

Furthermore, we know that the term $\Big[c(l,m) - \mathbb{E}\Big\{[C(l,m)(t+1)\wedge x(l,m)(t)]|X(t)\Big\}\Big]$ is non-negative, and only equal to 0 when $x(l,m)(t) > \overline{C}(l,m)$. Using these relations and the observations that $w(l,m)(t) \leq x(l,m)(t)$ and $S(l,m)(t) \in \{0,1\}$, the following must hold

$$\begin{aligned}\alpha_2(t) &= \sum_{l\in\mathcal{L},m}\Big[c(l,m) - \mathbb{E}\Big\{[C(l,m)(t+1)\wedge x(l,m)(t)]|X(t)\Big\}\Big] \\ &\qquad \cdot S(l,m)(t)w(l,m)(t) \\ &\leq \sum_{l\in\mathcal{L},m}\Big[c(l,m) - \mathbb{E}\Big\{[C(l,m)(t+1)\wedge x(l,m)(t)]|X(t)\Big\}\Big] \\ &\qquad \cdot S(l,m)(t)x(l,m)(t) \\ &\leq \sum_{l\in\mathcal{L},m} c(l,m)\overline{C}(l,m)\end{aligned}$$

***Lemma** 7:* If the immediate feedback max pressure control policy u^* is applied and the demand d is in the set of feasible demands D^o, then there exists an $\varepsilon > 0$, $\eta > 0$ such that

$$\alpha_1(t) \leq -\varepsilon\eta|X(t)| \tag{41}$$

Proof of Lemma 7:
Applying the definition of immediate feedback max pressure control in (14) as S^*,

$$\begin{aligned}\sum_{l,m} S^*(l,m)(t)c(l,m)w(l,m)(t) \\ &= \max_{S\in U}\sum_{l,m} S(l,m)c(l,m)w(l,m)(t) \\ &= \max_{M\in co(U)}\sum_{l,m} M(l,m)c(l,m)w(l,m)(t)\end{aligned}$$

As in (9), since $d \in D^0$, there exists an $\varepsilon > 0$ and M^+ such that $C(l,m)M^+(l,m) > f_l r(l,m) + \varepsilon\ \forall (l,m)$. Logically, any M' such that $0 \leq M' \leq M^+$ (component-wise) must also be in $co(U)$. Therefore we choose a $M' < M^+$ such that

$$M'(l,m)c(l,m) = \begin{cases} f_l r(l,m) + \varepsilon & \text{if } w(l,m) > 0 \\ 0 & \text{if } w(l,m) \leq 0 \end{cases} \tag{42}$$

Then

$$\begin{aligned}\alpha_1(t) &= \sum_{l\in\mathcal{L},m}[f_l r(l,m) - c(l,m)S^*(l,m)(t)]\,w(l,m)(t) \\ &\leq \sum_{l\in\mathcal{L},m}[f_l r(l,m) - M'(l,m)c(l,m)]\,w(l,m)(t) \\ &= -\varepsilon\sum_{l\in\mathcal{L},m}\max\{w(l,m)(t),0\} \\ &\qquad + \sum_{l\in\mathcal{L},m} f_l r(l,m)\min\{w(l,m)(t),0\} \\ &\leq -\varepsilon\sum_{l\in\mathcal{L},m}|w(l,m)(t)| \end{aligned}\tag{43}$$

Notice that $w(l,m)(X(t))$ is a linear, invertible function of the array $X(t)$, and therefore there exists a $\eta > 0$ such that $\sum_{l,m}|w(l,m)(X(t))| \geq \eta|X(t)|$. Substituting this expression into (43) defines a bound on α_1 (t):

$$\alpha_1(t) \leq -\varepsilon\eta\,|X(t)| \tag{44}$$

Combining the results of Lemmas 6 and 7 generates the desired bound on $\mathbb{E}\Big\{\alpha(t)|X(t)\Big\}$:

$$\mathbb{E}\Big\{\alpha(t)|X(t)\Big\} \leq -\varepsilon\eta\,|X(t)| + \sum_{l\in\mathcal{L},m} c(l,m)\overline{C}(l,m) \tag{45}$$

Explicit bound on queues, immediate feedback MP

Combining (45) and (34), we obtain

$$\begin{aligned}\mathbb{E}\Big\{|X(t+1)|^2 - |X(t)|^2|X(t)\Big\} &= \mathbb{E}\Big\{2\alpha(t)+\beta(t)\Big\} \\ &< -2\varepsilon\eta\,|X(t)| + 2\sum_{l\in\mathcal{L},m}[c(l,m)\overline{C}(l,m)] + NB^2\end{aligned}\tag{46}$$

where N is the number of links in the network and $B = \max\left\{\overline{C}(l,m), \sum_k \overline{C}(k,l), \overline{d}(l,m)\right\}$. For simplicity, we combine all constant additive terms to define a new constant K:

$$\begin{aligned}K &= 2\sum_{l,m} c(l,m)\overline{C}(l,m) \\ &\qquad + N\sum_{l,m}\max\left\{\overline{C}(l,m), \sum_k \overline{C}(k,l), \overline{d}(l,m)\right\}^2\end{aligned}\tag{47}$$

So

$$\mathbb{E}\Big\{|X(t+1)|^2 - |X(t)|^2|X(t)\Big\} < -2\varepsilon\eta\,|X(t)| + K \tag{48}$$

This is sufficient to show stability as previously defined.

Cycle Max Pressure: proof of Theorem 5: First we establish a bound on the incremental queue differences within a cycle of length τ, following the form of (**??**):

***Lemma** 8:* For a given cycle consisting of time steps $\{t, t+1, \ldots, t+\tau\}$, $\forall p \in [0, \tau-1]$,

$$\mathbb{E}\Big\{|X(t+p+1)|^2 - |X(t+p)|^2 | X(t)\ldots X(t+p-1)\Big\} < -2\varepsilon\eta|X(t+p)| + B(p) + K \tag{49}$$

where

$$K = 2\sum_{l,m} c(l,m)\overline{C}(l,m) + N\sum_{l,m}\max\{\overline{C}(l,m), \sum_k \overline{C}(k,l), \overline{d}(l,m)\}^2 \tag{50}$$

and

$$B(p) = p\left(2\varepsilon\eta\sum_{l,m} + 2(\sum_{l,m}[f_l r(l,m) + c(l,m)])\right) \cdot \max\left\{\overline{C}(l,m), \sum_k \overline{C}(k,l), \overline{d}(l,m)\right\} \tag{51}$$

Once we establish Lemma 8, we can show that for a time step t within any number of cycles T, the following quantity is bounded:

$$\begin{aligned}
&\sum_{t=1}^{\tau T}\mathbb{E}\Big\{|X(t+1)|^2 - |X(t)|^2 | X(t)\Big\} \\
&= \sum_{t=1}^{T-1}\sum_{p=0}^{\tau-1}\mathbb{E}\Big\{|X(t+p+1)|^2 - |X(t+p)|^2 | X(t+p)\Big\} \\
&< \sum_{t=1}^{T-1}\sum_{p=0}^{\tau-1}(-2\varepsilon\eta|X(t+p)| + B(p) + K) \\
&< -2\varepsilon\eta\sum_{t=1}^{\tau T}|X(t)| + (T-1)\left(\sum_{p=0}^{\tau-1}B(p) + \tau K\right)
\end{aligned} \tag{52}$$

Which, when taking the expectation, yields

$$\mathbb{E}\Big\{|X(\tau T+1)|^2 - |X(1)|^2\Big\} < -2\varepsilon\eta\sum_{t=1}^{\tau T}\mathbb{E}\Big\{|X(t)|\Big\} + (T-1)(\sum_{p=0}^{\tau-1}B(p) + \tau K) \tag{53}$$

Or, rearranging we obtain

$$\begin{aligned}
\frac{1}{\tau T}\sum_{t=1}^{\tau T}\mathbb{E}\Big\{|X(t)|\Big\} &< \frac{1}{2\varepsilon\eta\tau T}\mathbb{E}\Big\{|X(1)|^2 - |X(\tau T+1)|^2\Big\} \\
&\quad + \frac{T-1}{2\varepsilon\eta\tau T}\left(\sum_{p=0}^{\tau-1}B(p) + \tau K\right) \\
&< \frac{1}{2\varepsilon\eta\tau T}\mathbb{E}\Big\{|X(1)|^2\Big\} + \frac{1}{2\varepsilon\eta\tau}\left(\sum_{p=0}^{\tau-1}B(p) + \tau K\right)
\end{aligned} \tag{54}$$

This establishes stability: the quantity $\frac{1}{\tau T}\sum_{t=1}^{\tau T}\mathbb{E}\Big\{|X(t)|\Big\}$ must then be bounded.

Proof of Lemma 8:

Recall that for immediate feedback MP, we analyzed the following bounds on expected queue behavior per model timestep:

$$|X(t+p+1)|^2 - |X(t+p)|^2 = 2(\alpha_1(t+p) + \alpha_2(t+p)) + \beta(t+p) \tag{55}$$

where β, α_1 and α_2 are quantities that depend on the controller applied at time step $t+p$, as defined in Section V:

$$\begin{aligned}
\beta(t+p) &= |X(t+p+1) - X(t+p)|^2 \\
\alpha_1(t+p) &= \sum_{l,m}\Big(f_l r(l,m) - c(l,m)S(l,m)(t)\Big) \cdot w(l,m)(X(t+p)) \\
\alpha_2(t+p) &= \sum_{l,m}\Big(c(l,m)S(l,m)(t) \\
&\quad - \mathbb{E}\Big\{[C(l,m)(t+p+1) \wedge x(l,m)(t+p)] | X(t+p)\Big\}\Big) \cdot w(l,m)(X(t+p))
\end{aligned}$$

As previously derived, the following bounds on $\beta(\cdot)$ and $\alpha_2(\cdot)$ will hold for any binary control matrix:

$$\alpha_2(\cdot) < \sum_{l\in\mathcal{L},m} c(l,m)\overline{C}(l,m) \tag{56}$$

$$\beta(\cdot) < N\sum_{l,m}\max\left\{\overline{C}(l,m), \sum_k \overline{C}(k,l), \overline{d}(l,m)\right\}^2 \tag{57}$$

These two terms form the constant K from (47), which also appears in (49). To complete the bound in (49) we are only left with the α_1 term, which is directly dependent on the explicit form of the binary controller S:

$$\begin{aligned}
&\mathbb{E}\Big\{|X(t+p+1)|^2 - |X(t+p)|^2 | X(t)\ldots X(t+p)\Big\} \\
&= \mathbb{E}\Big\{2\alpha_1(t+p) + 2\alpha_2(t+p) + \beta(t+p) | X(t)\ldots X(t+p)\Big\} \\
&< \mathbb{E}\Big\{2\alpha_1(t+p) | X(t)\ldots X(t+p)\Big\} + K \\
&= 2\sum_{l,m}[f_l r(l,m) - c(l,m)S(l,m)(t)]w(l,m)(X(t+p)) + K
\end{aligned} \tag{58}$$

Examine the remaining term, $\alpha_1(t)$:

$$\begin{aligned}
&2\sum_{l,m}[f_l r(l,m) - c(l,m)S(l,m)(t)]w(l,m)(X(t+p)) \\
&= 2\sum_{l,m}[f_l r(l,m) - c(l,m)S(l,m)(t)]w(l,m)(X(t)) \\
&\quad + 2\sum_{l,m}[f_l r(l,m) - c(l,m)S(l,m)(t)]
\end{aligned}$$

$$\cdot (w(l,m)(X(t+p)-X(t))) = 2\xi_1 + 2\xi_2$$

With

$$\xi_1(t,S) = \sum_{l,m} [f_l r(l,m) - c(l,m)S(l,m)(t)]\, w(l,m)(X(t)) \tag{59}$$

and

$$\xi_2(t,p,S) = \sum_{l,m} [f_l r(l,m) - c(l,m)S(l,m)(t)] \cdot (w(l,m)(X(t+p)-X(t))) \tag{60}$$

Bound on ξ_1

By Lemma 7 we know that

$$2\sum_{l,m}[f_l r(l,m) - c(l,m)S(l,m)(t)]w(l,m)(X(t)) < -2\varepsilon\eta|X(t)|$$

Then noting that

$$\begin{aligned}|X(t)| &= |X(t+p) - (X(t+p)-X(t))| \\ &> ||X(t+p)| - |X(t+p)-X(t)|| \\ &> |X(t+p)| - |X(t+p)-X(t)|\end{aligned}$$

we are left with

$$\begin{aligned}2\xi_1(t,S) &< -2\varepsilon\eta(|X(t+p)| - |X(t+p)-X(t)|) \\ &< -2\varepsilon\eta|X(t+p)| \\ &\quad + 2\varepsilon\eta\sum_{i=1}^{p}|X(t+i)-X(t+i-1)| \\ &= -2\varepsilon\eta|X(t+p)| + 2\varepsilon\eta\sum_{i=1}^{p}|\delta(t+i-1)|\end{aligned} \tag{61}$$

So by (61) and (34),

$$\begin{aligned}&2\sum_{l,m}[f_l r(l,m) - c(l,m)S(l,m)(t)]w(l,m)(X(t)) \qquad (62)\\ &< -2\varepsilon\eta|X(t+p)| \\ &\quad + 2\varepsilon\eta p\sum_{l,m}\max\left\{\overline{C}(l,m), \sum_k \overline{C}(k,l), \overline{d}(l,m)\right\}\end{aligned}$$

Plugging (62) into (58), we have

$$\begin{aligned}&\mathbb{E}\Big\{|X(t+p+1)|^2 - |X(t+p)|^2 | X(t),\dots,X(t+p)\Big\} \\ &< K - 2\varepsilon\eta|X(t+p)| \\ &\quad + 2\varepsilon\eta p\sum_{l,m}\max\left\{\overline{C}(l,m), \sum_k \overline{C}(k,l), \overline{d}(l,m)\right\} \\ &\quad + 2\sum_{l,m}[f_l r(l,m) - c(l,m)S(l,m)(t)] \\ &\quad\quad \cdot \Big(w(l,m)(X(t+p)) - w(l,m)(X(t))\Big)\end{aligned} \tag{63}$$

Bound on ξ_2

We now have to bound the term

$$\begin{aligned}2\xi_2(t,p,S) = 2\sum_{l,m}&[f_l r(l,m) - c(l,m)S(l,m)(t)] \\ &\cdot \Big(w(l,m)X((t+p)) - w(l,m)(X(t))\Big)\end{aligned} \tag{64}$$

For that purpose we study the term

$$\begin{aligned}&w(l,m)(X(t+p)) - w(l,m)(X(t)) \\ &= \sum_{n=1}^{p} w(l,m)(X(t+n)) - w(l,m)(X(t+n-1)) \\ &= \sum_{n=1}^{p}\Big\{x(l,m)(t+n) - x(l,m)(t+n-1) \\ &\quad - \sum_{s\in Out(m)}[x(m,s)(t+n) - x(m,s)(t+n-1)]r(m,s)\Big\} \\ &= \sum_{n=1}^{p} w(l,m)(\delta(t+n-1))\end{aligned} \tag{65}$$

By (33) and the fact that $w(\cdot)$ is linear,

$$\begin{aligned}&|w(l,m)(\delta(t+n-1))| \\ &\quad < \sum_{u,v}\max\Big\{\overline{C}(u,v), \sum_k \overline{C}(k,u), \overline{d}(u,v)\Big\}\end{aligned} \tag{66}$$

Therefore plugging (66) back into (64), we get

$$\begin{aligned}2\xi_2(t,p,S) &= 2\Bigg(\sum_{l,m}([f_l r(l,m) - c(l,m)S(l,m)] \\ &\qquad \cdot \sum_{n=1}^{p} w(l,m)(\delta(t+n-1))\Bigg) \\ &< 2\sum_{n=1}^{p}\sum_{l,m}[f_l r(l,m) - c(l,m)S(l,m)] \\ &\qquad \cdot \sum_{u,v}\max\Big\{\overline{C}(u,v), \sum_k \overline{C}(k,u), \overline{d}(u,v)\Big\}\end{aligned} \tag{67}$$

Also note that

$$|\sum_{n=1}^{p}\sum_{l,m}[f_l r(l,m) - c(l,m)S(l,m)]| < p\sum_{l,m}[f_l r(l,m) + c(l,m)] \tag{68}$$

so (67) becomes

$$\begin{aligned}2\xi_2(t,p,S) &< 2p\left(\sum_{l,m}[f_l r(l,m) + c(l,m)]\right) \qquad (69)\\ &\qquad \cdot \left(\sum_{l,m}\max\Big\{\overline{C}(l,m), \sum_k \overline{C}(k,l), \overline{d}(l,m)\Big\}\right)\end{aligned}$$

Substituting (69) into (63) yields the final bound expressed in (49):

$$\begin{aligned}
&\mathbb{E}\Big\{|X(t+p+1)|^2 - |X(t+p)|^2 | X(t),\dots,X(t+p)\Big\} \\
&\quad < K - 2\varepsilon\eta|X(t+p)| \\
&\qquad + 2\varepsilon\eta p \sum_{l,m} \max\left\{\overline{C}(l,m), \sum_k \overline{C}(k,l), \overline{d}(l,m)\right\} \\
&\qquad + 2p\left(\sum_{l,m}[f_l r(l,m) + c(l,m)]\right) \\
&\qquad\qquad \cdot \left(\sum_{l,m} \max\left\{\overline{C}(l,m), \sum_k \overline{C}(k,l), \overline{d}(l,m)\right\}\right) \\
&\quad = K - 2\varepsilon\eta|X(t+p)| + B(p)
\end{aligned} \tag{70}$$

with K and $B(p)$ given by (50) and (51), respectively.

Allocated Max Pressure: proof of Theorem 6: Consider the relaxed control matrix S^{r*} specified by (23). By construction, it must be true that $\forall S^r \in conv_\kappa$,

$$\begin{aligned}
&\sum_{l,m} c(l,m)w(l,m)(X(t))S^r(l,m) \\
&\qquad \le \sum_{l,m} c(l,m)w(l,m)(X(t))S^{r*}(l,m)
\end{aligned} \tag{71}$$

with equality only if $S^r = S^{r*}$. Therefore $\forall S^r \neq S^{r*}$,

$$\begin{aligned}
&\sum_{l,m}[f_l r(l,m) - c(l,m)S^{r*}(l,m)(t)]w(l,m)(X(t)) \\
&\qquad < \sum_{l,m}[f_l r(l,m) - c(l,m)S^r(l,m)]w(l,m)(X(t))
\end{aligned} \tag{72}$$

If the demand flow is admissible according to (27), then $\exists \Sigma \in conv_\kappa$ such that

$$c(l,m)\Sigma(l,m) = \begin{cases} f_l r(l,m) + \varepsilon & \text{if } w(l,m)(X(t)) > 0 \\ 0 & \text{otherwise} \end{cases}$$

Hence following the same logic as in (43),

$$\begin{aligned}
&\sum_{l,m}[f_l r(l,m) - c(l,m)S^{r*}(l,m)(t)]w(l,m)(X(t)) \\
&\quad < -\varepsilon \sum_{l\in\mathcal{L},m} \max\{w(l,m)(X(t)), 0\} \\
&\qquad + \sum_{l\in\mathcal{L},m} f_l r(l,m) \min\{w(l,m)(X(t)), 0\}
\end{aligned} \tag{73}$$

We assume that by our choice of $\sigma(l,m)$, $f_l r(l,m) > \varepsilon$ (omitting the cases where $r(l,m) = 0$). Therefore:

$$\begin{aligned}
&\sum_{l,m}[f_l r(l,m) - c(l,m)S^{r*}(l,m)(t)]w(l,m)(X(t)) \\
&\qquad < -\varepsilon \sum_{l,m} x(l,m)(t)
\end{aligned} \tag{74}$$

Notice that besides the modified flow admissibility conditions required due to the non-binary control matrix, the proof presented in this section follows exactly the stability proof for the immediate feedback controller in section V.

$$\begin{aligned}
\frac{1}{\tau T}\sum_{t=1}^{\tau T} \mathbb{E}\Big\{|X(t)\Big\} &< \frac{1}{2\varepsilon\eta\tau T}\mathbb{E}\Big\{|X(1)|^2\Big\} \\
&\quad + \frac{1}{2\varepsilon\eta\tau}\sum_{p=0}^{\tau-1}(B(p) + \tau K)
\end{aligned} \tag{75}$$

where ε, η and K are constants which depend on network topology and demand.

■

The proof above shows that the system is stable.

Stability of allocated max pressure

We now show that the allocated max pressure controller S^{r*} is stable in the sense of (**??**) under conditions slightly modified from those assumed in the stability proofs of immediate feedback and cycle max pressure.

Increased bound on queue length

Max pressure controllers that are only updated every τ model time steps will stabilize a network; however the resulting bound on the queues will be higher than in the immediate feedback setting. Comparing (**??**) and (75), note that the constants are increased by a factor of

$$\frac{1}{\varepsilon\tau}\sum_{p=1}^{\tau-1} B(p) = \frac{1}{\varepsilon\tau}\frac{\tau(\tau-1)}{2}\cdot(\text{constant}) = \frac{\tau-1}{2\varepsilon}\cdot(\text{constant}) \tag{76}$$

The relative cost to queue bounds is therefore linear in τ.

VI. Numerical Implementation

A cycle-allocated max pressure controller was implemented on a network of 11 signalized intersections modeled in AIMSUN, a mico-simulation platform commonly used by practitioners. The present simulation was generated as part of the I-15 Integrated Corridor Management project, led by SANDAG in San Diego, CA. The controller was set to run with a cycle length of 90 seconds and minimum time constraints of 10 seconds for each of the 3-4 available standard signal control stages.

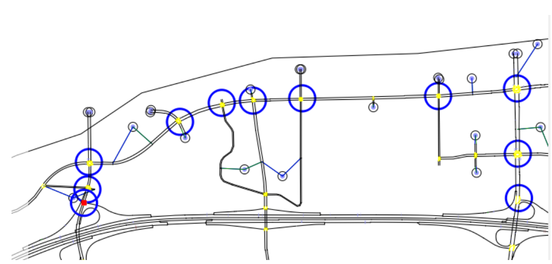

Fig. 1. The chosen network was calibrated to represent realistic demands and physical parameters observed on a stretch of Black Mountain Road near the I-15 freeway in San Diego, California.

The rest of this section provides a comparison of the performance of the max pressure algorithm with two other algorithms described below. While it makes sense to compare the numerical performance of these algorithms, it should be noted that the max pressure algorithm is the only algorithm, to the best of our knowledge, which provides theoretical guarantees: while the other algorithms seem to eprform well in simulation, they come with no proof or guarantees, as they are heuristics commonly used in the practioners field. Various performance metrics were compared between model runs using the max pressure controller and two alternative controllers: a fixed-time control plan that divides each signal cycle equally between all available phases, and a “fully-actuated” control system such as that which is currently operational on the real road network represented by the model. The fully actuated-controller is essentially a flexible fixed-time plan in which green times can be shortened or extended in real time to promote continuity of flows in response to instantaneous link demand measurements. The comparison of network vehicle counts in Figure 2 suggests that the uniformly-allocated fixed time controller caused significantly fewer vehicles to be served than the other two controllers, which were comparable to each other in vehicle service rates. This difference made it difficult to fairly compare other performance metrics between the uniform fixed-time controller and the other two options.

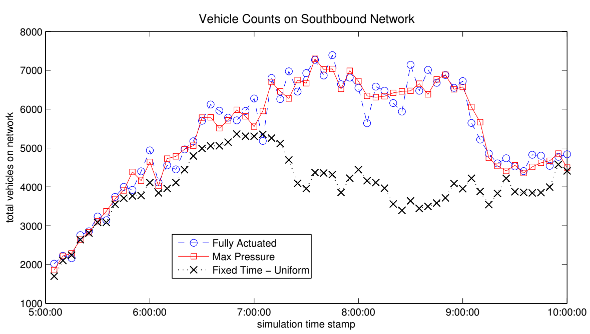

Fig. 2. During congestion, cycle-allocated max pressure demonstrated service rates that are higher than the uniformly allocated fixed-time controller and consistent with a fully-actuated control system.

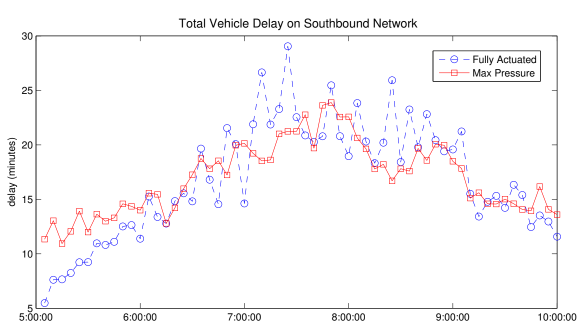

Fig. 3. Max pressure outperforms the fully actuated system during periods of high demand.

Differences between the fully-actuated and max pressure controllers were observed in measurements of delays and queue lengths: while the fully-actuated system appeared to produce less delay and shorter queues than max pressure during periods of relatively low network demand, max pressure was equally as effective or even more effective at reducing delays and queue lengths given larger demands. Figure 3 shows how delays were reduced and had less variance over time when the max pressure controller was applied than with the fully-actuated controller.

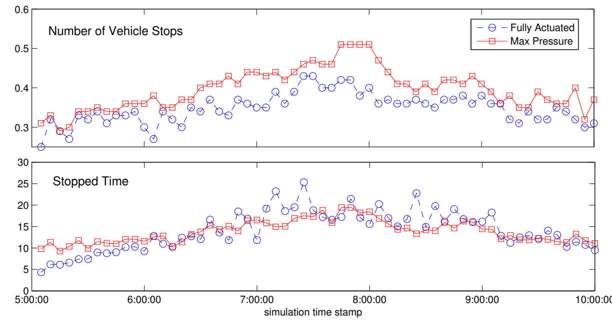

Fig. 4. While max pressure caused more vehicle stop events, stoppage times were similar to those observed using the standard fully-actuated controller.

Max pressure consistently induced more stops during a vehicle’s journey across the network, which is expected given the design objectives of the fully-actuated system. As illustrated in Figure 4, total stoppage times were higher with max pressure given low demand, but improved over the existing controller during peak demand.

VII. Conclusion

In this work we have defined an extension of the max pressure controller required in order to be applied to a real network of signalized traffic intersections. Given only the constraint that the network demands are (in average) serviceable, we have proven that updating a max pressure controller which allocates a fixed minimum proportion of service to each permissible phase at a slower rate than that which governs traffic flow will not destroy the stabilizing properties of the controller.

While there are additional modifications to the control model that must be addressed before max pressure can be considered a feasible algorithm for real-time operation of traffic signal controllers, our implementation in the reference micro-simulation platform demonstrates how it could comply with the hardware and communications constraints commonly encountered in existing roadway infrastructure. These numerical simulations furthermore reveal that an implementation of cycle-allocated max pressure competes with the existing standard feedback controller in many performance metrics, especially during periods of high congestion. This comes on top of theoretical proofs it provides, unlike algoriithms used in practice. It also appears to provide less variance in delays than the existing alternative.

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