# Stability of Modified Max Pressure Controller with Application to Signalized Traffic Networks

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***Abstract*—This work describes a type of distributed feedback control algorithm that acts on a vertical queueing network where flow dynamics may greatly outpace the rate of feedback and actuation. The modeled network has a known, finite set of feasible actuations for the binary controllers located at each network node. It also has known expected demands, split ratios, and maximum service rates. Previous work proposed the application of a max pressure controller to maximize throughput on such a network without the need for centralized computation of a control policy. Here, we extend the max pressure controller to a practical scenario . Specifically, we extend the controller so as to satisfy some practical requirements. The set of allowable controllers in this setting is extended to any convex combination of available signal phases to account for signal changes within a single signal "cycle". Moreover the controller cannot be updated every time step. We show that this proposed extended max pressure controllers stabilize the network (queue lengths remain bounded in expectation) given slight restrictions on admissible network demand flows. This work is motivated by the application of controlling traffic signals on arterial road networks. Max pressure provides an intriguing alternative to existing feedback control systems due to its theoretical guarantees, but cannot be directly applied as originally formulated due to hardware and safety constraints. We ultimately apply our extension of max pressure to a simulation of an existing arterial roadway and provide comparison to the control policy that is currently deployed on this site.**

***Keywords*—max pressure, vertical queueing network, network stability, adaptive signal control, arterial modeling***

## I. Introduction

This article investigates the design and stability of decentralized controller for vertical queueing networks. In a *vertical queueing network*, agents traveling across the network are stored in "point queues" which do not inhabit a "horizontal" position along the length of a network edge, but instead are considered to be "vertical" stacks at the front end of each edge. In the present work, we examine a vertical queueing network that can be controlled at junctions. Such networks are good models for urban traffic, but are inherited from other fields such as supply chain management or internet routing. The flows passing through each node of this network are constrained by a controller, similar to how a traffic light dictates allowable flows across a traffic intersection: only a finite set of non-conflicting turning movements or *phases* can be permitted to flow simultaneously across each node. At some regular interval, each of the the node controllers selects one element from its set of feasible phases to actuate. Hence flows through the network are dictated largely by the control policy applied at the nodes.

A signal control policy is said to be *stabilizing* if it ensures that the mean length of the queues waiting at each of the nodes remains bounded for all time. One such stabilizing control policy that is known to maximize network throughput is the *max pressure* controller. Here, we show that such network stabilization is still achieved given various extensions of the original max pressure controller. These extensions are motivated by the practical physical constraints imposed on realistic traffic signal controllers, and they present new mathematical problems in the analysis of queue dynamics which are investigated in this article.

The general problem of a stabilizing controller in this context was first considered from the point of view of multihop radio networks in Tassiulas et Ephremides [?]. The control of a network is a broad subject with many applications such as wireless networks [?][?], oscilatory networks [?], or queuing networks [?][?]. In the particular context of arterial traffic control, little work has been made on the approximation equilibrium of the network, at the exception of Osorio and Bierlaire [?][?]. Most of the literature focuses on the design of feedback responsive control. Comprehensive on this topic are provided in Baras and Levine [?][?], Mirchandani and Head [?], Papageorgiou et al. [?], Osorio and Bierlaire [?], and Xie et al. [?].

The max pressure traffic controller [?] is a specific distributed network signal control policy derived from the concept of a "back pressure controller", which was first studied in the context of routing packets through a communications network (see for example the pioneering work of Tassiulas and Ephremides [?]). The ideas has been applied to road traffic management more recently by Varaiya [?], as well as in the work of Wongpiromsarn et al. [?].

The concept of max pressure control is intuitive: at each intersection, priority is given to the phase which will be able to service the most traffic given both available upstream demand and the subsequent feasibility of downstream queues. It is a particularly attractive concept for control of a signalized urban traffic network because ... TODO...

- Local (distributed)
- theoretical guarantees

Variaya's original formulation of this controller in [?], however, does not fully consider the practical limitations on the rate of queue measurement and signal actuation in practical traffic

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This work was sponsored by...

networks. For example, a standard Max Pressure controller has no bound on the rate of signal switches which may occur relative to the rate of modeled queue formation and dissipation in the network. In implementation, a traffic signal incurs a penalty upon every change in actuation in the form of capacity loss due to "intersection clearance time": a 2-3 second period where all movements are given a red light in order to allow traffic from the previously actuated direction to clear the intersection before possibly conflicting movements can be permitted to enter. Traffic managers also place high value on synchronization between adjacent signals in a network to promote continuity of flow and limit vehicle stops on a preferred throughway. This is typically achieved by constraining the onset of the coupled phases to fixed relative offsets (to account for transit time between successive signals in the prioritized route). Furthermore, a standard max pressure implementation provides no explicit upper bound on the service time of queues on minor approaches where demand may be very small related to the main direction. Traffic controllers can overcome this inequity by operating on a cycle in which each allowable phase must be actuated for some minimum time within a fixed period.

These constraints motivate a new extension of the max pressure control algorithm which bounds signal switches and can maintain timed cyclical behaviors for signal coordination and queue service equity. Using Variaya's stability proof as a starting point, we prove that our proposed extension of a max pressure controller still provides the desired guarantee of queue stability in a vertical queueing network. We then discuss the penalty to the theoretical bound on queue lengths (relative to the standard max pressure controller) due to this constrained controller formulation.

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The remainder of this article is organized as follows: first, Section II describes our modeling framework and Section III gives a mathematical definition of controllers and the concept of network stability.

Section IV describes the practical problem the max pressure encounters in practice. Section V describes . Finally, Section VI outlines

numerical results provided by this controller using a micro-simulation running in the Aimsum platform of a real arterial traffic network within the I-15 corridor which was part of a pilot program operated by the San Diego Association of Governments in San Diego, CA.

## II. Model Framework

### *Definitions*

We consider a network of arterial roads with infinite storage capacity, modeled topologically as a graph with road links being edges and intersections being vertices. An individual link $l \in \mathcal{L}$ can be either at the entry of the network ($l \in \mathcal{L}_{\text{ent}}$) or in the interior of the network ($l \in \mathcal{L} \backslash \mathcal{L}_{\text{ent}}$). The inflow on entry links is a defined entirely by a random demand $d_l$, while the input flows of all other links depend on queues on upstream links and the relevant set of physical flow constraints are defined within the network. We require that each link has an exit path, that is, a continuous set of subsequent links on which vehicles can travel from the link to eventually exit the network. Each link in the network model can have multiple *queues* corresponding to individual *movements*: all vehicles in a given queue on any link are intending to advance onto the same subsequent link (though not necessarily the same subsequent queue).

We describe the dynamics of these queues as a discrete time dynamical model using the following notation:

- A *movement* $(l, m)$ distinguishes an intention to travel from link $l$ to link $m$, (in that case, say that $m \in Out(l)$ where $Out(l)$ is the set of links immediately downstream $l$);
- A *queue* $x(l, m)(t)$ is the number of vehicles on link $l$ waiting to enter link $m$ at timestep $t$, and $X(t)$ is the set (vector or matrix) of all the queue lengths on the network at timestep $t$;
- A *saturation flow* $c(l, m)$ is the expected number of vehicles that can travel from link $l$ to link $m$ per time step given maximum demand for the queue $x(l, m)$, and $C(l, m)(t)$ is the *realized saturation flow* at time $t$;
- The *turn ratio* $r(l, m)$ is the expected proportion of vehicles that are leaving $l$ which are intending to enter $m$, and $R(l, m)(t)$ is the *realized turn ratio* at time $t$;
- The *demand vector* $d$ of dimension $|\mathcal{L}_{\text{ent}}|$ specifies demands at network entry links;
- The *flow vector* $f$ of dimension $|\mathcal{L}|$ denotes flows on all links of the network such that $f_l$ is the flow within link $l$.

Note that there is necessarily a linear relationship between the expected link flow vector $f$ and the boundary demand vector $d$: $f = dP$ where the matrix $P$ depends only on expected routing proportions within the network.

### *Controller*

A road intersection is modeled as a node in our framework. Controllers (traffic signals) are placed at every node to limit the set of queues permitted to discharge at any given time. A set of movements that can be simultaneously actuated without flow conflicts is called a *phase*. Each permissible phase for a given intersection can be represented as a binary control matrix $S$ that is defined as follows:

$$S(l, m) = \begin{cases} 1 & \text{if movement } (l, m) \text{ is activated} \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

We denote $U_n$ the known finite set of permissible control matrices for node $n$. Note that in this article, we often drop the subscript $n$ for ease of notation.

Practically, only one phase can be actuated at any point in time: at each model time step $t$, a single control matrix $S(t)$ encodes which set of queues approaching the intersection are permitted to discharge during that time step. However, in this work we consider a *relaxed controller* which operates on a contiguous set of modeled time steps. This relaxed controller is defined as a matrix $S^r$ with each element $S^r(l, m)$ representing the fraction of the operational time steps that are allocated to the movement $\{l, m\}$:

$$S^r(l, m) = \lambda_{l,m} \in [0, 1] \tag{2}$$

Such a relaxed controller can be seen as a convex combination of all possible control matrices,

$$S^r = \sum_{S \in U} \lambda_S S \tag{3}$$

In general, the selection of such a controller can be based on feedback representing the state of the network queues at a previous time step.

*Queue Dynamics*

The evolution of queue state $X(t)$ can be seen as a Markov chain: the state of the network at time $t+1$ is a function of only the network state at time $t$ and external demand vector $d$,

$$X(t+1) = F(X(t), d) \tag{4}$$

Define $[\, a \wedge b \,] := \min\{a, b\}$. To describe queue dynamics explicitly, we must make a distinction between entry links and internal links: if $l \in \mathcal{L}_{\text{ent}}$,

$$\begin{aligned} x(l,m)(t+1) &= x(l,m)(t) + d_l(t+1) \\ &\quad - [C(l,m)(t+1)S(l,m)(t+1) \wedge x(l,m)(t)] \end{aligned} \tag{5}$$

and if $l \in \mathcal{L} \backslash \mathcal{L}_{\text{ent}}$,

$$\begin{aligned} & x(l,m)(t+1) = x(l,m)(t) + \\ & \sum_k [C(k,l)(t+1)S(k,l)(t+1) \wedge x(k,l)(t)] R(l,m)(t+1) \\ & \qquad - [C(l,m)(t+1)S(l,m)(t+1) \wedge x(l,m)(t)] \end{aligned} \tag{6}$$

*Demand Feasibility*

We focus on networks for which the boundary inflow demands $d = (d_l)_{(l \in \mathcal{L}_e)}$ are *feasible*—that is, the network is servicing a distribution of inflows for which it is possible to find a controller that allows *in average* more departures than arrivals at each link.

For a specific sequence of control matrices $\overline{S} = \{S(1), S(2), ..., S(t), ...\}$, define the *long-term control proportion* matrix $M_{\overline{S}}$ as follows:

$$M_{\overline{S}}(l,m) = \liminf_T \frac{1}{T} \sum_{t=1}^{T} S(l,m)(t) \tag{7}$$

Also, define $conv(U)$ as the convex hull of the set of permissible control matrices $U$.

The following properties are shown in [**?**]:

***Property** 1:* $M \in conv(U)$ if and only if $\exists$ a sequence of control matrices $\overline{S} = \{S(1), S(2), ..., S(t), ... | S(\cdot) \in U\}$ such that $\forall (l,m)$

$$M(l,m) = \liminf_T \frac{1}{T} \sum_{t=1}^{T} S(l,m)(t) \tag{8}$$

***Property** 2:* A demand is *feasible* if and only if $\exists\ M_{\overline{S}} \in conv(U)$ and $\varepsilon > 0$ such that

$$c(l,m) M_{\overline{S}}(l,m) > f_l r(l,m) + \varepsilon. \tag{9}$$

Define $D^0$ to be the set of all average demand vectors $\{d_l\}$ that satisfy (9) and are therefore feasible.

*Network Stability*

A network is *stable* if the following quantity is bounded:

$$\frac{1}{T} \sum_{t=1}^{T} \mathbb{E}\Big\{ |X(t)|_1 \Big\} \tag{10}$$

where $|X|_1 = \sum_{l,m} |x(l,m)|$ and the network state evolves according to dynamics under state dynamics (5-6).

## III. Standard Max Pressure Controller

Consider a weight assigned to each queue $(l, m)$ as a function of all network queue lengths $X$:

$$w(l,m)(X(t)) = x(l,m)(t) - \sum_{p \in Out(m)} r(m,p) x(m,p)(t) \tag{11}$$

where $Out(m)$ is the set of all links receiving flow from link $m$. The *pressure* $\gamma(S)$ that is potentially alleviated by a control action $S$ at time step $t$ is defined as follows:

$$\gamma(S)(X(t)) = \sum_{l,m} c(l,m) w(l,m)(X(t)) S(l,m)(t) \tag{12}$$

$$= \sum_{l,m: S(l,m)(t)=1} c(l,m) w(l,m)(X(t)) \tag{13}$$

At each time step $t$, the standard max pressure controller $u^*(X(t))$ explicitly choses the phase $S^* \in U$ that maximizes $\gamma(S)(X(t))$:

$$S^*(t) = u^*(X(t)) = \arg\max\{\gamma(S)(X(t)) | S \in U\} \tag{14}$$

Varaiya [**?**] shows the following stability result for the standard max pressure controller on a vertical queueing network:

***Theorem** 3:* The max pressure control $u^*$ is stabilizing whenever the average demand vector $d = \{d_l\}$ is within the set of feasible demands $D^0$. There is no stabilizing control when $d \notin D^0$.

## IV. Cycle-based Max Pressure Controller

Max pressure is justifiably presented in [**?**] as an attractive option for controlling vehicular traffic in urban road networks. Yet as previously discussed, this original formulation of the max pressure controller is not practical for application on a signalized traffic network for three reasons:

a) it does not account for capacity reductions due to excessive signal switching,
b) it cannot enforce coordination between subsequent intersections for purposes of maximizing flow continuity, and
c) it does not provide guarantees that low-demand queues will be served within a finite time period.

These limitations motivate our extension of the immediate feedback max pressure control algorithm. In the following section, we define a new *cycle-based max pressure* controller which bounds the number of signal switches per fixed time period, provides capacity for standard signal coordination methods, and can easily guarantee a minimum service rate for all interaction phases. We then show that the application of this controller yields a similar stability guarantee to that shown by

Varaiya for the standard immediate feedback controller (given slightly weaker conditions on demand flow). The structure of this proof is as follows:

i. First, we introduce a formulation of the cycle-based max pressure algorithm and briefly describe how it inherently rectifies issues a), b), and c) above.
ii. Next, we introduce the concept of a $\tau$*-non updated* controller that can only be updated once every $\tau$ model time steps (or once per *cycle*), and we show that queue stability holds with such a controller given slightly weaker conditions on demand flows than required for the stability of standard max pressure in [**?**].
iii. Then we show that this result does not change if the $\tau$-non updated controller is a relaxed controller as defined in (2-3).

The last three arguments build to prove that the cycle-based max pressure controller stabilizes a vertical queueing network.

*Phase distribution within a cycle*

We next investigate how the max pressure controller must be altered to allow the minimum proportion constraints on each phase in the set of allowable phases $\mathcal{U}$.

Let $T$ be the cycle length satisfying (22) and $\kappa_S$ be the minimum proportion of the cycle which must be allocated to each $S \in U$. The allocated max pressure controller selects a set of $\lambda_S$ that maximizes alleviated pressure under the following constraints:

- $\lambda_S \geq \kappa_S$
- $f_l r_{l,m} < \sum_S \lambda_S c(l,m) S(l,m)$
- $\sum_S \lambda_S = 1 - \frac{L}{T}$

The desired controller is therefore determined by the solution to the following linear program:

$$\begin{aligned} \{\lambda_S^*\} = \underset{\lambda_1,\ldots,\lambda_{|U|}}{\arg\max} \quad & \sum_{S\in\mathcal{U}} \lambda_S \Big( \sum_{l,m} w(l,m)(t) c(l,m) S(l,m) \Big) \\ \text{subject to} \quad & \lambda_S \geq \kappa_s \\ & \sum_S \lambda_S \leq 1 - \tfrac{L}{T} \end{aligned} \tag{15}$$

The solution to (20) selects coefficients $\{\lambda_S^*\}$ which form a corresponding relaxed control matrix

$$S^{r*} = \sum_{S\in\mathcal{U}} \lambda_S^* S \tag{16}$$

*Cycle max pressure*

Cycle max pressure enables the control to act at a slower time scale than the queue dynamics, as would be the case in a practical traffic application. Suppose that we are given the model dynamics $X(t)$ as in (5)-(6), but the controller $S^*(t)$ can only be updated every $\tau$ model time steps (or once per *cycle*). The "$\tau$-non updated" control sequence is therefore composed of control matrices repeated for at least $\tau$ model time steps: for a fixed cycle size $\tau$ and integer $n$,

$$\begin{aligned} S(n\tau+1) = S(n\tau+2) = \ldots = S((n+1)\tau) = S^*(n\tau+1) \\ = \arg\max\{\gamma(S)(X(n\tau+1))|S \in U\} \end{aligned} \tag{17}$$

Physically, the controller that maximizes the pressure at time step $n\tau+1$ is continuously applied until time step $(n+1)\tau$.

*Relaxed Max Pressure Controller*

Let $\tau \in \mathbb{N}$ be fixed, the number of time steps between two actuation of the controller $\forall n \in \mathbb{N}$,

$$\begin{aligned} S(n\tau+1) = \ldots = S((n+1)\tau) = S^*(n\tau+1) \\ = \arg\max\{\gamma(S)(X(n\tau+1))|S \in U\} \end{aligned} \tag{18}$$

$$= \underset{\lambda_1,\ldots,\lambda_{|U|}}{\arg\max} \sum_{S\in\mathcal{U}} \lambda_S \Big( \sum_{l,m} w(l,m)(t) c(l,m) S(l,m) \Big) \tag{19}$$

subject to

$$\begin{aligned} & \lambda_S \geq \kappa_s \\ & \sum_S \lambda_S = S^{r*} = \sum_{S\in\mathcal{U}} \lambda_S^* S \leq 1 - \tfrac{L}{T} \end{aligned} \tag{20}$$

*Minimum cycle time*

Consider that the length $T$ of the controller update time period (signal cycle) is not pre-defined. However, there is a known amount of *lost time* $L$ per cycle during which no phases can be actuated to account for physical clearance of the intersection between actuated phases. Hence it must be the case that $T > L$. We furthermore impose that each feasible phase has an associated minimum proportion constraint: phase $S$ must be actuated for at least $\lambda_S T$ seconds during any one cycle. We then want to determine the minimum control period or *cycle length* for which a fixed demand flow can be served while the stated phase proportion constraints are satisfied.

Unlike the previous case where we constrained our analysis to flows which could be served in average over an arbitrary long-term time horizon, here our proof of stability depends on the assumption that the average demand is served *within a single cycle*. As suggested in [**?**], we then pose the selection of a cycle length as a convex optimization problem with constraints applied to enforce the desired minimum phase proportions $\kappa$:

$$\begin{aligned} \underset{\lambda=(\lambda_S)_S}{\text{minimize}} \quad & \sum_{S\in U} \lambda_S \\ \text{subject to} \quad & \lambda_S \geq \kappa_S \\ & f_l r(l,m) < \sum_S \lambda_S c(l,m) S(l,m) \end{aligned} \tag{21}$$

where $\kappa_S \in [0,1]\ \forall S \in U$, and $\sum_S \kappa_S < 1$.

Let us denote $\Lambda^*$ to be the optimum of (21). If $\Lambda^* > 1$, the demand is not feasible under the set of control constraints $\{\kappa_S\}$ for any length time step. If $\Lambda^* < 1$, then the flow is admissible for a time step of length

$$T > \frac{L}{1-\Lambda^*} \tag{22}$$

We can now use this lower bound to select an appropriate cycle length.

### $\tau$-admissible flows

We first prove that this set of $\tau$-admissible flows (demands that can be accommodated using $\tau$-non updated sequences) is in fact the same set of flows that is admissible under typical updated control sequences, defined in equation (9).
Define the following sets:

- $U$ is the set of admissible control matrices as in (9),
- $U_{\mathbb{N}}$ is the set of control sequences $\{S(1), S(2) \dots S(t) \dots | S(\cdot) \in U\}$ where elements are applied at consecutive time steps,
- $U_{\tau\mathbb{N}}$ is the set of control sequences $\{S(1), S(1), \dots, S(\tau+1), S(\tau+1), \dots, S(n\tau+1), S(n\tau+1), \dots | S(\cdot) \in U\}$ where controls are updated only once every $\tau$ steps,
- $conv(U) = \Big\{ \liminf_T \frac{1}{T} \sum_{t=1}^{T} S(t) | \{S(1), S(2), \dots, S(t), \dots\} \in U_{\mathbb{N}} \Big\}$
- $conv(U_\tau) = \Big\{ \liminf_T \frac{1}{T} \sum_{t=1}^{T} S(t) | \{S(1), S(1), \dots, S(\tau+1), S(\tau+1), \dots\} \in U_{\tau\mathbb{N}} \Big\}$

Obviously, $conv(U_\tau) \subset conv(U)$. But we can also show that $conv(U) \subset conv(U_\tau)$:
Suppose $M \in conv(U)$, so $\exists \{S(1) \dots S(t) \dots\}$ such that $M = \liminf_T \frac{1}{T} \sum_{t=1}^{T} S(t)$.

$$\begin{aligned} M &= \liminf_T \frac{1}{T} \sum_{t=1}^{T} S(t) = \liminf_T \frac{1}{\tau T} \sum_{t=1}^{\tau T} \tilde{S}(t) \\ &\quad \text{with } \tilde{S} = \{S(1), \dots, S(1), \dots, S(t), \dots, S(t), \dots\} \\ &= \liminf_T \frac{1}{T} \sum_{t=1}^{T} \tilde{S}(t) \end{aligned}$$

Trivially, $\tilde{S} \in conv(U_\tau)$. Because $conv(U_\tau) \subset conv(U)$ and $cont(U) \subset conv(U_\tau)$, it must hold that $conv(U) = conv(U_\tau)$. This establishes Property 4:

***Property** 4:*

$$conv(U) = conv(U_\tau)$$

This property implies that a $\tau$-non updated control sequence can accommodate the same set of flows as a control sequence updated at every time step. The equivalence becomes intuitive when one considers that our definition of feasible flows considers only the long-term average of demand and service rates: note that a $\tau$ control matrix in $\tilde{S}$ is simply the average of the corresponding $\tau$ matrices in $\overline{S}$, such that $M_{\tilde{S}} = M_{\overline{S}}$. Hence,

$$\begin{aligned} f_l r(l,m) < c(l,m) M_{\overline{S}}(l,m) \implies \\ f_l r(l,m) < c(l,m) M_{\tilde{S}}(l,m). \end{aligned} \tag{23}$$

However, as we will show in the following sections, the bound on queue lengths when the cycle controller is applied will be larger than in the immediate feedback setting.

### Stability of the new controller

Here we extend the previous proof of stability of an immediate feedback max pressure controller to the case of our relaxed max pressure controller. We begin by defining two sets we gonna use later.

Define $conv_\kappa$ as the set of convex combinations of control matrices with coefficients larger than $\kappa$:

$$conv_\kappa = \Big\{ \sum_S \lambda_S S | \, \lambda_S > \kappa_S \; \forall S \in U \Big\} \tag{24}$$

Also define a set of reduced admissible demands $D_\kappa$ which can in average be served in a single cycle with a relaxed control matrix that maintains the minimum time allocation for a given cycle time (as in (21)):

$$\begin{aligned} d \in D_\kappa \text{ iff } \exists \, S^r \in conv_\kappa \\ \text{such that } f_l r(l,m) < c(l,m) S^r(l,m) \end{aligned} \tag{25}$$

***Theorem** 5:* The relaxed max pressure controller, defined above, updated every $\tau$ iterations stabilizes the network whenever the demand is within a set of feasible demands $D^\kappa$.

*Proof:*

*Allocated Max Pressure: proof of Theorem 6:* Consider the expectation of the following function of queue state with perturbation

$$\delta(t) = X(t+1) - X(t) \tag{26}$$

conditioned on the past queue state:

$$\begin{aligned} |X(t+1)|^2 - |X(t)|^2 &= |X(t) + \delta(t)|^2 - |X(t)|^2 \\ &= 2X(t)^T \delta(t) + |\delta(t)|^2 \\ &= 2\alpha(t) + \beta(t) \end{aligned} \tag{27}$$

with

$$\alpha(t) = X(t)^T \delta(t) \quad \text{and} \quad \beta(t) = |\delta(t)|^2 \tag{28}$$

We continue by addressing bounds on $\beta$ and $\alpha$ separately.

### Bound on $\beta(t) = |\delta(t)|^2$

Define $\overline{C}$ as the maximum realized saturation flow and $\overline{d}$ as the maximum possible value of the demand vector. If $l \in \mathcal{L}_{\text{ent}}$ and $m \in Out(l)$,

$$\begin{aligned} |\delta(l,m)(t)| &= \Big| - [C(l,m)(t+1) S(l,m)(t) \wedge x(l,m)(t)] \\ &\qquad + D(l,m)(t+1) \Big| \\ &\le \max \big\{ \overline{C}(l,m), \overline{d}(l,m) \big\} \end{aligned} \tag{29}$$

where $D(l,m)(t) = D(l)(t)R(l,m)(t)$ with $D(l)(t)$ defined as the realized demand on link $l$ at time $t$. This is because we know that both $C(l,m)(t+1)S(l,m)(t) \wedge x(l,m)(t)$ and $D(l,m)(t+1)$ are non-negative, so the absolute value of their difference must be less than either of the two quantities individually.

Similarly, if $l \in \mathcal{L} \backslash \mathcal{L}_{\text{ent}}$ and $m \in Out(l)$:

$$|\delta(l,m)(t)| = \Big| - [C(l,m)(t+1)S(l,m)(t) \wedge x(l,m)(t)] \tag{30}$$

$$+ \sum_k [C(k,l)(t+1)S(k,l)(t) \wedge x(k,l)(t)]R(l,m)(t+1)\Big|$$

$$\leq \max\left\{\overline{C}(l,m), \sum_k \overline{C}(k,l)\right\} \tag{31}$$

If we define $B$ as the maximum of all of the quantities $\left\{\overline{C}(l,m), \sum_k \overline{C}(k,l), \overline{d}(l,m)\right\}$ and $N$ as the number of queues in the network, we can derive a bound for $\beta$ which depends on only $B$ and $N$:

$$\beta(t) = |\delta(t)|^2 \leq NB^2 \tag{32}$$

Note that because these bounds hold for any $S(l,m)(t) \in [0,1]$, the bound on $\beta$ presented here can easily be extended to any convex combination of control matrices; hence it is still valid in our modified controllers, as shown later in this article.

*Bound on $\alpha(t) = X(t)^T \delta(t)$*

The term $\alpha$ in (28) is explicitly defined in terms of queue state $X(t)$ as follows:

$$\begin{aligned}
\alpha(t) &= X(t)^T [X(t+1) - X(t)] \\
&= \sum_{l \in \mathcal{L} \backslash \mathcal{L}_{\text{ent}}, m} \sum_k [C(k,l)(t+1)S(k,l)(t) \wedge x(k,l)(t)] \\
&\qquad \cdot R(l,m)(t+1)x(l,m)(t) \\
&\qquad - \sum_{l \in \mathcal{L}, m} [C(l,m)(t+1)S(l,m)(t) \wedge x(l,m)(t)] \\
&\qquad - \sum_{l \in \mathcal{L}_{\text{ent}}, m} d(l,m)(t+1)x(l,m)(t) \\
&= \sum_{l \in \mathcal{L}, m} [C(l,m)(t+1)S(l,m)(t) \wedge x(l,m)(t)] \\
&\qquad \cdot \Big( -x(l,m)(t) + \sum_p R(m,p)(t+1)x(m,p)(t) \Big) \\
&\qquad + \sum_{l \in \mathcal{L}_{\text{ent}}, m} d(l,m)(t+1)x(l,m)(t)
\end{aligned} \tag{33}$$

Note that only the expectation of these terms appear in equation (**??**), so we are interested in $\mathbb{E}\{\alpha(t)|X(t)\}$. We therefore make the following observation: because $R(m,p)(t+1)$ is independent of $C(l,m)(t+1)$ and $X(t)$,

$$\begin{aligned}
&\mathbb{E}\Big\{[C(l,m)(t+1)S(l,m)(t) \wedge x(l,m)(t)] \\
&\qquad \cdot R(m,p)(t+1)x(m,p)(t)|X(t)\Big\} \\
&= \mathbb{E}\Big\{[C(l,m)(t+1)S(l,m)(t) \wedge x(l,m)(t)]|X(t)\Big\} \\
&\qquad \cdot r(m,p)(t+1)x(m,p)(t)
\end{aligned}$$

Also, the expectation of demand $d(l,m)$ is equal to the measured demand $d_l$ on link $l$, times the relevant expected split ratio $r(l,m)$. Hence the desired expectation of (33) can be expressed as

$$\begin{aligned}
&\mathbb{E}\{\alpha(t)|X(t)\} = \\
&\sum_{l \in \mathcal{L}, m} \mathbb{E}\Big\{[C(l,m)(t+1)S(l,m)(t) \wedge x(l,m)(t)]|X(t)\Big\} \\
&\qquad \cdot \Big( -x(l,m)(t) + \sum_p r(m,p)(t+1)x(m,p)(t) \Big) \\
&\qquad + \sum_{l \in \mathcal{L}_{\text{ent}}, m} d_l r(l,m)x(l,m)(t) \\
&= \sum_{l \in \mathcal{L}_{\text{ent}}, m} d_l r(l,m)x(l,m)(t) \\
&\quad - \sum_{l \in \mathcal{L}, m} \mathbb{E}\Big\{[C(l,m)(t+1)S(l,m)(t) \wedge x(l,m)(t)]|X(t)\Big\} \\
&\qquad \cdot w(l,m)(t)
\end{aligned} \tag{34}$$

where $w(l,m)(t) = w(l,m)(X(t))$ is the weight of a link as defined in (11). We also can include the following relation:

$$\begin{aligned}
&\sum_{l \in \mathcal{L}, m} f_l r(l,m)w(l,m)(t) \\
&= \sum_{l \in \mathcal{L}, m} f_l r(l,m) \left[x(l,m) - \sum_p r(m,p)x(m,p)(t)\right] \\
&= \sum_{l \in \mathcal{L}, m} f_l r(l,m)x(l,m)(t) \\
&\qquad - \sum_m \left[\sum_{l \in \mathcal{L}} f_l r(l,m) \sum_p r(m,p)x(m,p)(t)\right] \\
&= \sum_{l \in \mathcal{L}, m} f_l r(l,m)x(l,m)(t) \\
&\qquad - \sum_{m \in \mathcal{L} \backslash \mathcal{L}_{\text{ent}}, p} f_m r(m,p)x(m,p)(t) \\
&= \sum_{l \in \mathcal{L}_{\text{ent}}, m} d_l r(l,m)x(l,m)(t)
\end{aligned}$$

So (34) is further simplified to:

$$\begin{aligned}
\mathbb{E}\{\alpha(t)|X(t)\} &= \sum_{l \in \mathcal{L}, m} \Big[f_l r(l,m) \\
&- \mathbb{E}\Big\{[C(l,m)(t+1)S(l,m)(t) \wedge x(l,m)(t)]|X(t)\Big\}\Big]w(l,m)(t) \\
&= \alpha_1(t) + \alpha_2(t)
\end{aligned} \tag{35}$$

with

$$\alpha_1(t) = \sum_{l \in \mathcal{L}, m} [f_l r(l,m) - c(l,m)S(l,m)(t)]\, w(l,m)(t) \tag{36}$$

and

$$\alpha_2(t) = \sum_{l \in \mathcal{L}, m} \Big[c(l,m) - \tag{37}$$

$$\mathbb{E}\Big\{[C(l,m)(t+1)\wedge x(l,m)(t)]|X(t)\Big\}\Big]S(l,m)(t)w(l,m)(t)$$

***Lemma 6:*** For all $l$, $m$, $t$,

$$\alpha_2(t) \leq \sum_{l\in\mathcal{L},m} c(l,m)\overline{C}(l,m) \tag{38}$$

where $\overline{C}(l,m)$ is the maximum value of the random service rate $C(l,m)(t)$.

<u>Proof of Lemma 6:</u>

By Jensen's inequality,

$$\begin{aligned}\mathbb{E}\Big\{C(l,m)(t+1)\wedge x(l,m)(t)|X(t)\Big\}&\\ \leq \mathbb{E}\Big\{C(l,m)(t+1)|X(t)\Big\}&\wedge x(l,m)(t)\\ = c(l,m)\wedge x(l,m)(t)&\\ \leq c(l,m)&\end{aligned}$$

Furthermore, we know that the term $\Big[c(l,m)-\mathbb{E}\Big\{[C(l,m)(t+1)\wedge x(l,m)(t)]|X(t)\Big\}\Big]$ is non-negative, and only equal to 0 when $x(l,m)(t)>\overline{C}(l,m)$. Using these relations and the observations that $w(l,m)(t) \leq x(l,m)(t)$ and $S(l,m)(t) \in \{0,1\}$, the following must hold

$$\begin{aligned}\alpha_2(t) &= \sum_{l\in\mathcal{L},m}\Big[c(l,m)-\mathbb{E}\Big\{[C(l,m)(t+1)\wedge x(l,m)(t)]|X(t)\Big\}\Big]\\ &\qquad\cdot S(l,m)(t)w(l,m)(t)\\ &\leq \sum_{l\in\mathcal{L},m}\Big[c(l,m)-\mathbb{E}\Big\{[C(l,m)(t+1)\wedge x(l,m)(t)]|X(t)\Big\}\Big]\\ &\qquad\cdot S(l,m)(t)x(l,m)(t)\\ &\leq \sum_{l\in\mathcal{L},m} c(l,m)\overline{C}(l,m)\end{aligned}$$

***Lemma 7:*** If the relaxed max pressure control policy $u^*$ is applied at time step $t$ and the demand $d$ is in the set of feasible demands $D^\kappa$, then there exists an $\varepsilon>0$, $\eta>0$ such that

$$\alpha_1(t) \leq -\varepsilon\eta|X(t)| \tag{39}$$

<u>Proof of Lemma 7:</u>

Consider the relaxed control matrix $S^{r*}$ specified by (20). By construction, it must be true that $\forall S^r \in conv_\kappa$,

$$\sum_{l,m} c(l,m)w(l,m)(X(t))S^r(l,m) \leq \sum_{l,m} c(l,m)w(l,m)(X(t))S^{r*}(l,m) \tag{40}$$

with equality only if $S^r = S^{r*}$. Therefore $\forall S^r \neq S^{r*}$,

$$\begin{aligned}&\sum_{l,m}[f_l r(l,m)-c(l,m)S^{r*}(l,m)(t)]w(l,m)(X(t))\\ &\quad< \sum_{l,m}[f_l r(l,m)-c(l,m)S^{r}(l,m)]w(l,m)(X(t))\end{aligned} \tag{41}$$

If the demand flow is admissible according to (25), then $\exists\Sigma \in conv_\kappa$ such that

$$c(l,m)\Sigma(l,m) = \begin{cases} f_l r(l,m)+\varepsilon & \text{if } w(l,m)(X(t))>0\\ 0 & \text{otherwise}\end{cases}$$

Hence following the same logic as in (**??**),

$$\begin{aligned}&\sum_{l,m}[f_l r(l,m)-c(l,m)S^{r*}(l,m)(t)]w(l,m)(X(t))\\ &\quad< -\varepsilon\sum_{l\in\mathcal{L},m}\max\{w(l,m)(X(t)),0\}\\ &\qquad+\sum_{l\in\mathcal{L},m} f_l r(l,m)\min\{w(l,m)(X(t)),0\}\end{aligned} \tag{42}$$

We assume that by our choice of $\sigma(l,m)$, $f_l r(l,m) > \varepsilon$ (omitting the cases where $r(l,m)=0$). Therefore:

$$\begin{aligned}&\sum_{l,m}[f_l r(l,m)-c(l,m)S^{r*}(l,m)(t)]w(l,m)(X(t))\\ &\quad< -\varepsilon\sum_{l,m} x(l,m)(t)\end{aligned} \tag{43}$$

*Explicit bound on queues, immediate feedback MP*

Combining (**??**) and (32), we obtain

$$\begin{aligned}&\mathbb{E}\Big\{|X(t+1)|^2-|X(t)|^2|X(t)\Big\} = \mathbb{E}\Big\{2\alpha(t)+\beta(t)\Big\}\\ &\quad< -2\varepsilon\eta\,|X(t)| + 2\sum_{l\in\mathcal{L},m}[c(l,m)\overline{C}(l,m)] + NB^2\end{aligned} \tag{44}$$

where $N$ is the number of links in the network and $B = \max\Big\{\overline{C}(l,m), \sum_k \overline{C}(k,l), \overline{d}(l,m)\Big\}$. For simplicity, we combine all constant additive terms to define a new constant $K$:

$$\begin{aligned}K &= 2\sum_{l,m} c(l,m)\overline{C}(l,m)\\ &\quad+ N\sum_{l,m}\max\left\{\overline{C}(l,m),\sum_k\overline{C}(k,l),\overline{d}(l,m)\right\}^2\end{aligned} \tag{45}$$

So, when the control is updated at time $t$:

$$\mathbb{E}\Big\{|X(t+1)|^2-|X(t)|^2|X(t)\Big\} < -2\varepsilon\eta\,|X(t)| + K \tag{46}$$

*Cycle Max Pressure: proof of Theorem 5:* We then establish a bound on the incremental queue differences within a cycle of length $\tau$, following the form of (**??**):

***Lemma 8:*** For a given cycle (i.e. when the controller is not updated) consisting of time steps $\{t, t+1, \ldots, t+\tau\}$, $\forall p \in [0, \tau-1]$,

$$\begin{aligned}&\mathbb{E}\Big\{|X(t+p+1)|^2-|X(t+p)|^2|X(t)\ldots X(t+p-1)\Big\}\\ &< -2\varepsilon\eta|X(t+p)| + B(p) + K\end{aligned} \tag{47}$$

where

$$K = 2\sum_{l,m} c(l,m)\overline{C}(l,m) + N\sum_{l,m}\max\{\overline{C}(l,m),\sum_{k}\overline{C}(k,l),\overline{d}(l,m)\}^2 \tag{48}$$

and

$$B(p) = p\left(2\varepsilon\eta\sum_{l,m} + 2(\sum_{l,m}[f_l r(l,m)+c(l,m)])\right) \cdot \max\left\{\overline{C}(l,m),\sum_{k}\overline{C}(k,l),\overline{d}(l,m)\right\} \tag{49}$$

Proof of Lemma 8:
As above, we have:

$$|X(t+p+1)|^2 - |X(t+p)|^2 = 2(\alpha_1(t+p)+\alpha_2(t+p)) + \beta(t+p) \tag{50}$$

where $\beta$, $\alpha_1$ and $\alpha_2$ are quantities that depend on the controller applied at time step $t+p$, as defined in Section III:

$$\beta(t+p) = |X(t+p+1) - X(t+p)|^2$$
$$\alpha_1(t+p) = \sum_{l,m}\Big(f_l r(l,m) - c(l,m)S(l,m)(t)\Big) \cdot w(l,m)(X(t+p))$$
$$\alpha_2(t+p) = \sum_{l,m}\Bigg(c(l,m)S(l,m)(t) - \mathbb{E}\Big\{[C(l,m)(t+p+1)\wedge x(l,m)(t+p)]|X(t+p)\Big\}\Bigg) \cdot w(l,m)(X(t+p))$$

As previously derived, the following bounds on $\beta(\cdot)$ and $\alpha_2(\cdot)$ will hold for any binary control matrix:

$$\alpha_2(\cdot) < \sum_{l\in\mathcal{L},m} c(l,m)\overline{C}(l,m) \tag{51}$$

$$\beta(\cdot) < N\sum_{l,m}\max\left\{\overline{C}(l,m),\sum_{k}\overline{C}(k,l),\overline{d}(l,m)\right\}^2 \tag{52}$$

These two terms form the constant $K$ from (45), which also appears in (47). To complete the bound in (47) we are only left with the $\alpha_1$ term, which is directly dependent on the explicit form of the binary controller $S$:

$$\begin{aligned}
&\mathbb{E}\Big\{|X(t+p+1)|^2 - |X(t+p)|^2|X(t)\dots X(t+p)\Big\}\\
&= \mathbb{E}\Big\{2\alpha_1(t+p) + 2\alpha_2(t+p) + \beta(t+p)|X(t)\dots X(t+p)\Big\}\\
&< \mathbb{E}\Big\{2\alpha_1(t+p)|X(t)\dots X(t+p)\Big\} + K\\
&= 2\sum_{l,m}[f_l r(l,m) - c(l,m)S(l,m)(t)]w(l,m)(X(t+p)) + K
\end{aligned} \tag{53}$$

Examine the remaining term, $\alpha_1(t)$:

$$\begin{aligned}
&2\sum_{l,m}[f_l r(l,m) - c(l,m)S(l,m)(t)]w(l,m)(X(t+p))\\
&= 2\sum_{l,m}[f_l r(l,m) - c(l,m)S(l,m)(t)]w(l,m)(X(t))\\
&\quad + 2\sum_{l,m}[f_l r(l,m) - c(l,m)S(l,m)(t)]\\
&\quad \cdot (w(l,m)(X(t+p) - X(t)))\\
&= 2\xi_1 + 2\xi_2
\end{aligned}$$

With

$$\xi_1(t,S) = \sum_{l,m}\left[f_l r(l,m) - c(l,m)S(l,m)(t)\right]w(l,m)(X(t)) \tag{54}$$

and

$$\xi_2(t,p,S) = \sum_{l,m}\left[f_l r(l,m) - c(l,m)S(l,m)(t)\right] \cdot (w(l,m)(X(t+p) - X(t))) \tag{55}$$

Bound on $\xi_1$
By Lemma 7 we know that

$$2\sum_{l,m}[f_l r(l,m) - c(l,m)S(l,m)(t)]w(l,m)(X(t)) < -2\varepsilon\eta|X(t)|$$

Then noting that

$$\begin{aligned}
|X(t)| &= |X(t+p) - (X(t+p) - X(t))|\\
&> ||X(t+p)| - |X(t+p) - X(t)||\\
&> |X(t+p)| - |X(t+p) - X(t)|
\end{aligned}$$

we are left with

$$\begin{aligned}
2\xi_1(t,S) &< -2\varepsilon\eta(|X(t+p)| - |X(t+p) - X(t)|)\\
&< -2\varepsilon\eta|X(t+p)| + 2\varepsilon\eta\sum_{i=1}^{p}|X(t+i) - X(t+i-1)|\\
&= -2\varepsilon\eta|X(t+p)| + 2\varepsilon\eta\sum_{i=1}^{p}|\delta(t+i-1)|
\end{aligned} \tag{56}$$

So by (56) and (32),

$$\begin{aligned}
&2\sum_{l,m}[f_l r(l,m) - c(l,m)S(l,m)(t)]w(l,m)(X(t))\\
&\quad < -2\varepsilon\eta|X(t+p)|\\
&\quad + 2\varepsilon\eta p\sum_{l,m}\max\left\{\overline{C}(l,m),\sum_{k}\overline{C}(k,l),\overline{d}(l,m)\right\}
\end{aligned} \tag{57}$$

Plugging (57) into (53), we have

$$
\begin{aligned}
&\mathbb{E}\Big\{|X(t+p+1)|^2 - |X(t+p)|^2 | X(t), \ldots, X(t+p)\Big\} \\
&< K - 2\varepsilon\eta |X(t+p)| \\
&\quad + 2\varepsilon\eta p \sum_{l,m} \max\left\{\overline{C}(l,m), \sum_k \overline{C}(k,l), \overline{d}(l,m)\right\} \\
&\quad + 2\sum_{l,m}[f_l r(l,m) - c(l,m)S(l,m)(t)] \\
&\qquad \cdot \Big(w(l,m)(X(t+p)) - w(l,m)(X(t))\Big)
\end{aligned} \tag{58}
$$

Bound on $\xi_2$

We now have to bound the term

$$
\begin{aligned}
2\xi_2(t,p,S) &= 2\sum_{l,m}[f_l r(l,m) - c(l,m)S(l,m)(t)] \\
&\quad \cdot \Big(w(l,m)X((t+p)) - w(l,m)(X(t))\Big)
\end{aligned} \tag{59}
$$

For that purpose we study the term

$$
\begin{aligned}
&w(l,m)(X(t+p)) - w(l,m)(X(t)) \\
&= \sum_{n=1}^{p} w(l,m)(X(t+n)) - w(l,m)(X(t+n-1)) \\
&= \sum_{n=1}^{p} \Big\{x(l,m)(t+n) - x(l,m)(t+n-1) \\
&\quad - \sum_{s\in Out(m)} [x(m,s)(t+n) - x(m,s)(t+n-1)]r(m,s)\Big\} \\
&= \sum_{n=1}^{p} w(l,m)(\delta(t+n-1))
\end{aligned} \tag{60}
$$

By (31) and the fact that $w(\cdot)$ is linear,

$$
\begin{aligned}
&|w(l,m)(\delta(t+n-1))| \\
&\quad < \sum_{u,v} \max\Big\{\overline{C}(u,v), \sum_k \overline{C}(k,u), \overline{d}(u,v)\Big\}
\end{aligned} \tag{61}
$$

Therefore plugging (61) back into (59), we get

$$
\begin{aligned}
2\xi_2(t,p,S) &= 2\Bigg(\sum_{l,m}([f_l r(l,m) - c(l,m)S(l,m)] \\
&\qquad \cdot \sum_{n=1}^{p} w(l,m)(\delta(t+n-1))\Bigg) \\
&< 2\sum_{n=1}^{p}\sum_{l,m}[f_l r(l,m) - c(l,m)S(l,m)] \\
&\qquad \cdot \sum_{u,v} \max\Big\{\overline{C}(u,v), \sum_k \overline{C}(k,u), \overline{d}(u,v)\Big\}
\end{aligned} \tag{62}
$$

Also note that

$$
|\sum_{n=1}^{p}\sum_{l,m}[f_l r(l,m) - c(l,m)S(l,m)]| < p\sum_{l,m}[f_l r(l,m) + c(l,m)] \tag{63}
$$

so (62) becomes

$$
\begin{aligned}
2\xi_2(t,p,S) &< 2p\left(\sum_{l,m}[f_l r(l,m) + c(l,m)]\right) \\
&\quad \cdot \left(\sum_{l,m} \max\Big\{\overline{C}(l,m), \sum_k \overline{C}(k,l), \overline{d}(l,m)\Big\}\right)
\end{aligned} \tag{64}
$$

Substituting (64) into (58) yields the final bound expressed in (47):

$$
\begin{aligned}
&\mathbb{E}\Big\{|X(t+p+1)|^2 - |X(t+p)|^2 | X(t), \ldots, X(t+p)\Big\} \\
&< K - 2\varepsilon\eta |X(t+p)| \\
&\quad + 2\varepsilon\eta p \sum_{l,m} \max\left\{\overline{C}(l,m), \sum_k \overline{C}(k,l), \overline{d}(l,m)\right\} \\
&\quad + 2p\left(\sum_{l,m}[f_l r(l,m) + c(l,m)]\right) \\
&\qquad \cdot \left(\sum_{l,m} \max\Big\{\overline{C}(l,m), \sum_k \overline{C}(k,l), \overline{d}(l,m)\Big\}\right) \\
&= K - 2\varepsilon\eta |X(t+p)| + B(p)
\end{aligned} \tag{65}
$$

with $K$ and $B(p)$ given by (48) and (49), respectively.

Once we establish Lemma 8, we can show that for a time step $t$ within any number of cycles $T$, the following quantity is bounded:

$$
\begin{aligned}
&\sum_{t=1}^{\tau T} \mathbb{E}\Big\{|X(t+1)|^2 - |X(t)|^2 | X(t)\Big\} \\
&= \sum_{t=1}^{T-1}\sum_{p=0}^{\tau-1} \mathbb{E}\Big\{|X(t+p+1)|^2 - |X(t+p)|^2 | X(t+p)\Big\} \\
&< \sum_{t=1}^{T-1}\sum_{p=0}^{\tau-1} (-2\varepsilon\eta |X(t+p)| + B(p) + K) \\
&< -2\varepsilon\eta \sum_{t=1}^{\tau T} |X(t)| + (T-1)\left(\sum_{p=0}^{\tau-1} B(p) + \tau K\right)
\end{aligned} \tag{66}
$$

Which, when taking the expectation, yields

$$
\begin{aligned}
&\mathbb{E}\Big\{|X(\tau T+1)|^2 - |X(1)|^2\Big\} \\
&< -2\varepsilon\eta \sum_{t=1}^{\tau T} \mathbb{E}\Big\{|X(t)|\Big\} + (T-1)(\sum_{p=0}^{\tau-1} B(p) + \tau K)
\end{aligned} \tag{67}
$$

Or, rearranging we obtain

$$\begin{aligned}\frac{1}{\tau T}\sum_{t=1}^{\tau T}\mathbb{E}\Big\{|X(t)|\Big\} &< \frac{1}{2\varepsilon\eta\tau T}\mathbb{E}\Big\{|X(1)|^2 - |X(\tau T+1)|^2\Big\} \\ &\quad + \frac{T-1}{2\varepsilon\eta\tau T}\left(\sum_{p=0}^{\tau-1}B(p)+\tau K\right) \\ &< \frac{1}{2\varepsilon\eta\tau T}\mathbb{E}\Big\{|X(1)|^2\Big\} + \frac{1}{2\varepsilon\eta\tau}\left(\sum_{p=0}^{\tau-1}B(p)+\tau K\right)\end{aligned} \tag{68}$$

This establishes stability: the quantity $\frac{1}{\tau T}\sum_{t=1}^{\tau T}\mathbb{E}\Big\{|X(t)|\Big\}$ must then be bounded.

$$\begin{aligned}\frac{1}{\tau T}\sum_{t=1}^{\tau T}\mathbb{E}\Big\{|X(t)\Big\} &< \frac{1}{2\varepsilon\eta\tau T}\mathbb{E}\Big\{|X(1)|^2\Big\} \\ &\quad + \frac{1}{2\varepsilon\eta\tau}\sum_{p=0}^{\tau-1}(B(p)+\tau K)\end{aligned} \tag{69}$$

where $\varepsilon$, $\eta$ and $K$ are constants which depend on network topology and demand.

■

The proof above shows that the system is stable.

*Stability of allocated max pressure*

We now show that the allocated max pressure controller $S^{\tau *}$ is stable in the sense of (**??**) under conditions slightly modified from those assumed in the stability proofs of immediate feedback and cycle max pressure.

*Increased bound on queue length*

Max pressure controllers that are only updated every $\tau$ model time steps will stabilize a network; however the resulting bound on the queues will be higher than in the immediate feedback setting. Comparing (**??**) and (69), note that the constants are increased by a factor of

$$\frac{1}{\varepsilon\tau}\sum_{p=1}^{\tau-1}B(p) = \frac{1}{\varepsilon\tau}\frac{\tau(\tau-1)}{2}\cdot(\text{constant}) = \frac{\tau-1}{2\varepsilon}\cdot(\text{constant}) \tag{70}$$

The relative cost to queue bounds is therefore linear in $\tau$.

## V. Numerical Implementation

A cycle-allocated max pressure controller was implemented on a network of 11 signalized intersections modeled in AIMSUN, a mico-simulation platform commonly used by practitioners. The present simulation was generated as part of the I-15 Integrated Corridor Management project, led by SANDAG in San Diego, CA. The controller was set to run with a cycle length of 90 seconds and minimum time constraints of 10 seconds for each of the 3-4 available standard signal control stages.

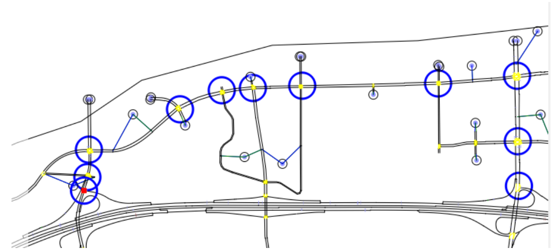

Fig. 1. The chosen network was calibrated to represent realistic demands and physical parameters observed on a stretch of Black Mountain Road near the I-15 freeway in San Diego, California.

The rest of this section provides a comparison of the performance of the max pressure algorithm with two other algorithms described below. While it makes sense to compare the numerical performance of these algorithms, it should be noted that the max pressure algorithm is the only algorithm, to the best of our knowledge, which provides theoretical guarantees: while the other algorithms seem to eprform well in simulation, they come with no proof or guarantees, as they are heuristics commonly used in the practioners field. Various performance metrics were compared between model runs using the max pressure controller and two alternative controllers: a fixed-time control plan that divides each signal cycle equally between all available phases, and a "fully-actuated" control system such as that which is currently operational on the real road network represented by the model. The fully actuated-controller is essentially a flexible fixed-time plan in which green times can be shortened or extended in real time to promote continuity of flows in response to instantaneous link demand measurements. The comparison of network vehicle counts in Figure 2 suggests that the uniformly-allocated fixed time controller caused significantly fewer vehicles to be served than the other two controllers, which were comparable to each other in vehicle service rates. This difference made it difficult to fairly compare other performance metrics between the uniform fixed-time controller and the other two options.

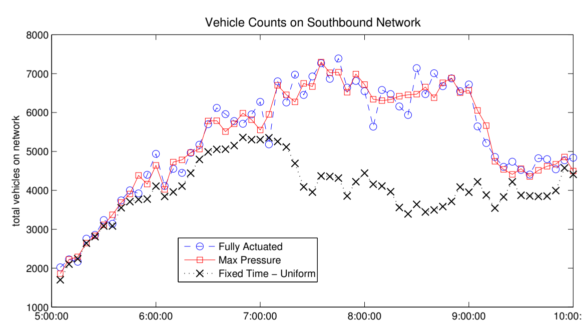


Fig. 2. During congestion, cycle-allocated max pressure demonstrated service rates that are higher than the uniformly allocated fixed-time controller and consistent with a fully-actuated control system.

Differences between the fully-actuated and max pressure controllers were observed in measurements of delays and

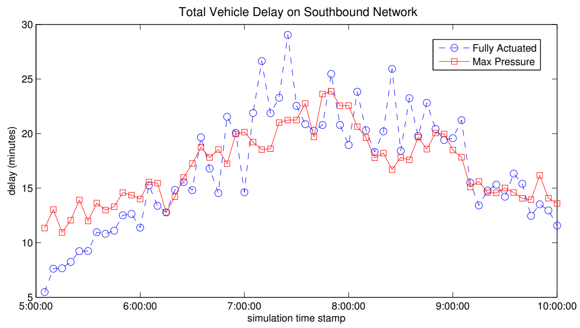


Fig. 3. Max pressure outperforms the fully actuated system during periods of high demand.

queue lengths: while the fully-actuated system appeared to produce less delay and shorter queues than max pressure during periods of relatively low network demand, max pressure was equally as effective or even more effective at reducing delays and queue lengths given larger demands. Figure 3 shows how delays were reduced and had less variance over time when the max pressure controller was applied than with the fully-actuated controller.

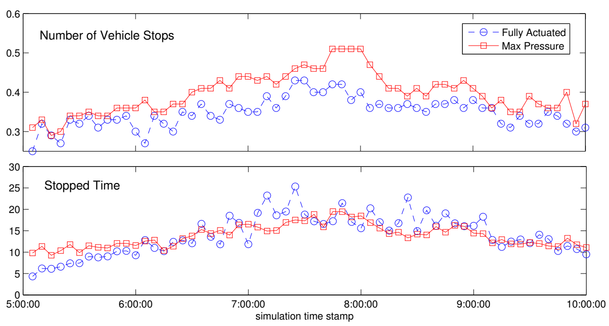


Fig. 4. While max pressure caused more vehicle stop events, stoppage times were similar to those observed using the standard fully-actuated controller.

Max pressure consistently induced more stops during a vehicle's journey across the network, which is expected given the design objectives of the fully-actuated system. As illustrated in Figure 4, total stoppage times were higher with max pressure given low demand, but improved over the existing controller during peak demand.

## VI. Conclusion

In this work we have defined an extension of the max pressure controller required in order to be applied to a real network of signalized traffic intersections. Given only the constraint that the network demands are (in average) serviceable, we have proven that updating a max pressure controller which allocates a fixed minimum proportion of service to each permissible phase at a slower rate than that which governs traffic flow will not destroy the stabilizing properties of the controller.

While there are additional modifications to the control model that must be addressed before max pressure can be considered a feasible algorithm for real-time operation of traffic signal controllers, our implementation in the reference micro-simulation platform demonstrates how it could comply with the hardware and communications constraints commonly encountered in existing roadway infrastructure. These numerical simulations furthermore reveal that an implementation of cycle-allocated max pressure competes with the existing standard feedback controller in many performance metrics, especially during periods of high congestion. This comes on top of theoretical proofs it provides, unlike algoriithms used in practice. It also appears to provide less variance in delays than the existing alternative.

The authors would like to thank Pravin Varaiya for his insights on this work, as well as Brian Peterson and Joe Butler for their support for the implementation.