Stability of Modified Max Pressure Controller with Application to Signalized Traffic Networks

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Abstract—This work describes a type of distributed feedback control algorithm that acts on a vertical queueing network where flow dynamics may greatly outpace the rate of feedback and actuation. The modeled network has a known, finite set of feasible actuations for the binary controllers located at each network node. It also has known expected demands, split ratios, and maximum service rates. Previous work proposed the application of a max pressure controller to maximize throughput on such a network without the need for centralized computation of a control policy. Here, we extend the max pressure controller to a practical scenario. Specifically, we extend the controller so as to satisfy some practical requirements. The set of allowable controllers in this setting is extended to any convex combination of available signal phases to account for signal changes within a single signal "cycle". Moreover the controller cannot be updated every time step. We show that this proposed extended max pressure controllers stabilize the network (queue lengths remain bounded in expectation) given slight restrictions on admissible network demand flows. This work is motivated by the application of controlling traffic signals on arterial road networks. Max pressure provides an intriguing alternative to existing feedback control systems due to its theoretical guarantees, but cannot be directly applied as originally formulated due to hardware and safety constraints. We ultimately apply our extension of max pressure to a simulation of an existing arterial roadway and provide comparison to the control policy that is currently deployed on this site.

Keywords—max pressure, vertical queueing network, network stability, adaptive signal control, arterial modeling

I. Introduction

This article investigates the design and stability of decentralized controller for vertical queueing networks. In a *vertical queueing network*, agents traveling across the network are stored in "point queues" which do not inhabit a "horizontal" position along the length of a network edge, but instead are considered to be "vertical" stacks at the front end of each edge. In the present work, we examine a vertical queueing network that can be controlled at junctions. Such networks are good models for urban traffic, but are inherited from other fields such as supply chain management or internet routing. The flows passing through each node of this network are constrained by a controller, similar to how a traffic light dictates allowable flows across a traffic intersection: only a finite set of non-conflicting turning movements or *phases* can be permitted to flow simultaneously across each node. At some regular interval, each of the the node controllers selects one element from its set of feasible phases to actuate. Hence flows through the network are dictated largely by the control policy applied at the nodes.

A signal control policy is said to be *stabilizing* if it ensures that the mean length of the queues waiting at each of the nodes remains bounded for all time. One such stabilizing control policy that is known to maximize network throughput is the *max pressure* controller. Here, we show that such network stabilization is still achieved given various extensions of the original max pressure controller. These extensions are motivated by the practical physical constraints imposed on realistic traffic signal controllers, and they present new mathematical problems in the analysis of queue dynamics which are investigated in this article.

The general problem of a stabilizing controller in this context was first considered from the point of view of multihop radio networks in Tassiulas et Ephremides [**?**]. The control of a network is a broad subject with many applications such as wireless networks [**?**][**?**], oscilatory networks [**?**], or queuing networks [**?**][**?**]. In the particular context of arterial traffic control, little work has been made on the approximation equilibrium of the network, at the exception of Osorio and Bierlaire [**?**][**?**]. Most of the literature focuses on the design of feedback responsive control. Comprehensive on this topic are provided in Baras and Levine [**?**][**?**], Mirchandani and Head [**?**], Papageorgiou et al. [**?**], Osorio and Bierlaire [**?**], and Xie et al. [**?**].

The max pressure controller is a specific distributed network signal control policy derived from the concept of a "back pressure controller", which was first studied in the context of routing packets through a communications network [**?**]. The idea has been applied to road traffic management more recently by Varaiya [**?**], as well as in the work of Wongpiromsarn et al. [**?**]. The concept of max pressure control is intuitive: at each intersection, priority is given to the phase which will be able to service the most traffic given knowledge of both available upstream demand and the subsequent feasibility of downstream queues. It is a particularly attractive concept for control of a signalized urban traffic network because ... TODO...

- Local (distributed)
- theoretical guarantees

Variaya's original formulation of this controller in [**?**], however, does not fully consider the practical limitations on the rate of queue measurement and signal actuation in practical traffic networks. For example, a standard max pressure controller has no bound on the rate of signal switches which may

occur relative to the rate of modeled queue formation and dissipation in the network. In implementation, a traffic signal incurs a penalty upon every change in actuation in the form of capacity loss due to "intersection clearance time": a 2-3 second period where all movements are given a red light in order to allow traffic from the previously actuated direction to clear the intersection before possibly conflicting movements can be permitted to enter. Traffic managers also place high value on synchronization between adjacent signals in a network to promote continuity of flow and limit vehicle stops on a preferred throughway. This is typically achieved by constraining the onset of the coupled phases to fixed relative offsets (to account for transit time between successive signals in the prioritized route). Furthermore, a standard max pressure implementation provides no explicit upper bound on the service time of queues on minor approaches where demand may be very small related to the main direction. Traffic controllers can overcome this inequity by operating on a cycle in which each allowable phase must be actuated for some minimum time within a fixed period.

These constraints motivate a new extension of the max pressure control algorithm which bounds signal switches and can maintain timed cyclical behaviors for signal coordination and queue service equity. While a similar concept was suggested in [**?**], this work further extends the concept of a simple proportional cycle controller to allow model dynamics to explicitly act at a faster rate than the controller update period. Using Variaya's stability proof as a starting point, we prove that our proposed extension of a max pressure controller still provides the desired guarantee of queue stability in a vertical queueing network. We then discuss the penalty to the theoretical bound on queue lengths (relative to the standard max pressure controller) due to this constrained controller formulation.

TODO: re-do this organization description.

The remainder of this article is organized as follows: first, Section II describes our modeling framework and Section III gives a mathematical definition of controllers and the concept of network stability.

Section IV describes the practical problem the max pressure encounters in practice. Section V describes . Finally, Section VI outlines

numerical results provided by this controller using a micro-simulation running in the Aimsun platform of a real arterial traffic network within the I-15 corridor which was part of a pilot program operated by the San Diego Association of Governments in San Diego, CA.

II. MODEL FRAMEWORK

Definitions

We consider a network of arterial roads with infinite storage capacity, modeled topologically as a graph with road links being edges and intersections being vertices. An individual link $l \in \mathcal{L}$ can be either at the entry of the network ($l \in \mathcal{L}_{\text{ent}}$) or in the interior of the network ($l \in \mathcal{L} \backslash \mathcal{L}_{\text{ent}}$). The inflow on entry links is a defined entirely by a random demand d_l, while the input flows of all other links depend on queues on upstream links and the relevant set of physical flow constraints

are defined within the network. We require that each link has an exit path, that is, a continuous set of subsequent links on which vehicles can travel from the link to eventually exit the network. Each link in the network model can have multiple *queues* corresponding to individual *movements*: all vehicles in a given queue on any link are intending to advance onto the same subsequent link (though not necessarily the same subsequent queue).

We describe the dynamics of these queues as a discrete time dynamical model using the following notation:

- A *movement* (l, m) distinguishes an intention to travel from link l to link m, (in that case, say that $m \in Out(l)$ where $Out(l)$ is the set of links immediately downstream l);
- A *queue* $x(l, m)(t)$ is the number of vehicles on link l waiting to enter link m at timestep t, and $X(t)$ is the set (vector or matrix) of all the queue lengths on the network at timestep t;
- A *saturation flow* $c(l, m)$ is the expected number of vehicles that can travel from link l to link m per time step given maximum demand for the queue $x(l, m)$, and $C(l, m)(t)$ is the *realized saturation flow* at time t;
- The *turn ratio* $r(l, m)$ is the expected proportion of vehicles that are leaving l which are intending to enter m, and $R(l, m)(t)$ is the *realized turn ratio* at time t;
- The *demand vector* d of dimension $|\mathcal{L}_{\text{ent}}|$ specifies demands at network entry links;
- The *flow vector* f of dimension $|\mathcal{L}|$ denotes flows on all links of the network such that f_l is the flow within link l.

Note that there is necessarily a linear relationship between the expected link flow f and the boundary demand d:

$$f = dP \tag{1}$$

where the matrix P depends only on expected routing proportions within the network. A proof of this is given in [**?**]. ??

Controller

A road intersection is modeled as a node in our framework. Controllers (traffic signals) are placed at every node to limit the set of queues permitted to discharge at any given time. A set of movements that can be simultaneously actuated without flow conflicts is called a *phase*. Each permissible phase for a given intersection can be represented as a binary control matrix S that is defined as follows:

$$S(l, m) = \begin{cases} 1 & \text{if movement } (l, m) \text{ is activated} \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

We denote U_n the known finite set of permissible control matrices for node n. Note that in this article, we often drop the subscript n for ease of notation.

Practically, only one phase can be actuated at any point in time: at each model time step t, a single control matrix $S(t)$ encodes which set of queues approaching the intersection are permitted to discharge during that time step. However, in this

work we consider a *relaxed controller* which operates on a contiguous set of modeled time steps. This relaxed controller is defined as a matrix S^r with each element $S^r(l, m)$ representing the fraction of the operational time steps that are allocated to the movement $\{l, m\}$:

$$S^r(l, m) = \lambda_{l,m} \in [0, 1] \tag{3}$$

Such a relaxed controller can be seen as a convex combination of all possible control matrices,

$$S^r = \sum_{S \in U} \lambda_S S \tag{4}$$

In general, the selection of such a controller can be based on feedback representing the state of the network queues at a previous time step.

Queue Dynamics

The evolution of queue state $X(t)$ can be seen as a Markov chain: the state of the network at time $(t+1)$ is a function of only the network state at time t and external demand vector d,

$$X(t + 1) = F(X(t), d) \tag{5}$$

Define $[a \wedge b] := \min\{a, b\}$. To describe queue dynamics explicitly, we must make a distinction between entry links and internal links: if $l \in \mathcal{L}_{\text{ent}}$,

$$x(l, m)(t + 1) = x(l, m)(t) + d_l(t + 1) \tag{6}$$
$$- [C(l, m)(t + 1)S(l, m)(t + 1) \wedge x(l, m)(t)]$$

and if $l \in \mathcal{L} \backslash \mathcal{L}_{\text{ent}}$,

$$x(l, m)(t + 1) = x(l, m)(t) + \tag{7}$$
$$\sum_k [C(k, l)(t + 1)S(k, l)(t + 1) \wedge x(k, l)(t)]R(l, m)(t + 1)$$
$$- [C(l, m)(t + 1)S(l, m)(t + 1) \wedge x(l, m)(t)]$$

Demand Feasibility

We focus on networks for which the boundary inflow demands $d = (d_l)_{(l \in \mathcal{L}_e)}$ are *feasible*—that is, the network is servicing a distribution of inflows for which it is possible to find a controller that allows *in average* more departures than arrivals at each link.

Define $conv(U)$ to be the convex hull of the set of permissible control matrices U. The following properties are then shown in [**?**]:

***Property** 1:* A matrix M is in $conv(U)$ if and only if $\exists$ a sequence of control matrices $\overline{S} = \{S(1), S(2), ..., S(t), ...|S(\cdot) \in U\}$ such that $\forall(l, m)$

$$M(l, m) = \liminf_T \frac{1}{T} \sum_{t=1}^{T} S(l, m)(t) \tag{8}$$

The element $M(l, m)$ in (8) can be interpreted as the long-term average proportion of intersection capacity given to movement (l, m). Hence define $M_{\overline{S}}$ to be the long-term control proportion matrix constructed as in (8) using a specific control sequence $\overline{S} = \{S(1), S(2), ..., S(t), ...\}$.

***Property** 2:* A demand d is *feasible* if and only if $\exists M_{\overline{S}} \in conv(U)$ and $\varepsilon > 0$ such that

$$c(l, m)M_{\overline{S}}(l, m) > f_l r(l, m) + \varepsilon. \tag{9}$$

where $f = dP$ as in (1).

Define D^0 to be the set of all average demand vectors $d = \{d_l\}$ that satisfy (9) and are therefore feasible.

Network Stability

A network is *stable* if the following quantity is bounded:

$$\frac{1}{T} \sum_{t=1}^{T} \mathbb{E}\left\{|X(t)|_1\right\} \tag{10}$$

where $|X|_1 = \sum_{l,m} |x(l, m)|$ and the network state evolves according to dynamics under state dynamics (6-7).

III. STANDARD MAX PRESSURE CONTROLLER

Consider a weight assigned to each queue (l, m) as a function of all network queue lengths X:

$$w(l, m)(X(t)) = x(l, m)(t) - \sum_{p \in Out(m)} r(m, p)x(m, p)(t) \tag{11}$$

where $Out(m)$ is the set of all links receiving flow from link m. The *pressure* $\gamma(S)$ that is potentially alleviated by a control action S at time step t is defined as follows:

$$\gamma(S)(X(t)) = \sum_{l,m} c(l, m)w(l, m)(X(t))S(l, m)(t) \tag{12}$$

$$= \sum_{l,m:S(l,m)(t)=1} c(l, m)w(l, m)(X(t)) \tag{13}$$

At each time step t, the standard max pressure controller $u^*(X(t))$ explicitly choses the phase $S^* \in U$ that maximizes $\gamma(S)(X(t))$:

$$S^*(t) = u^*(X(t)) = \arg\max\{\gamma(S)(X(t))|S \in U\} \tag{14}$$

Varaiya [**?**] shows the following stability result for the standard max pressure controller on a vertical queueing network:

***Theorem** 3:* The max pressure control u^* is stabilizing whenever the average demand vector $d = \{d_l\}$ is within the set of feasible demands D^0. There is no stabilizing control when $d \notin D^0$.

IV. A CYCLE-BASED MAX PRESSURE CONTROLLER

The attractive qualities of max pressure for controlling vehicular traffic in urban road networks are presented in [**?**]. Yet as previously discussed, this original formulation of the controller is not practical for application on a signalized traffic network for three reasons:

a) it does not account for capacity reductions (lost time) due to excessive signal switching,

b) it cannot enforce coordination between subsequent intersections for purposes of maximizing flow continuity, and

c) it does not provide guarantees that low-demand queues will be served within a finite time period.

These limitations motivate our extension of the standard immediate feedback max pressure control algorithm. In the following section, we define a new *cycle-based* max pressure controller which bounds the number of signal switches per fixed time period, provides capacity for standard signal coordination methods, and can easily guarantee a minimum service rate for all intersection phases. We then show that the application of this controller yields a similar stability guarantee to that shown by Varaiya for the standard controller given slightly weaker conditions on demand flow. The structure of this proof is as follows:

i. First, we formalize a calculation of the *lost time* incurred by signal switching actions.

ii. Then we introduce a formulation of the cycle-based max pressure algorithm and briefly describe how it inherently rectifies issues a), b), and c) above.

iii. Next, we introduce the concept of a τ-*updated* controller that can only be updated once every τ model time steps (or once per *cycle*), and we show that this does not impact the feasibility of demands.

iv. We finally show that queue stability holds with a cycle-based max pressure controller consisting of τ-*updated sequences of relaxed control matrices with minimum proportion constraints*. However, network stability requires *slightly stricter conditions on demand flows* than assumed for stability using the original max pressure formulation.

Lost time due to switching

For safety reasons, an intersection controller cannot switch signal phase actuation immediately. Instead, it must incorporate a pause of $R \approx 2.5$ seconds in which all signal phases have a red light. This *clearance time* allows all vehicles in the previously actuated phase to clear the intersection before a conflicting phase can be permitted to use the intersection.

In the standard formulation of max pressure, the controller chooses an appropriate action based feedback received at every time step of the modeled dynamics. To accurately capture queuing behaviors observed on arterial roadways, a model would need to operate with a time discretization of $\Delta t < 10$ seconds. A signal switch at every time step could therefore result in more than 25% loss of intersection service capacity, which is not considered in the theoretical examination presented in [**?**].

In this work, we explicitly fix the number of signal switches that can occur in a fixed number of model time steps using the familiar concept of a *signal cycle*. As typical with modern traffic signals, the *cycle-based* max pressure controller rotates through all available signal movements within a fixed time period. We define *cycle time* T as a fixed number of model time steps and require that each controller phase S must be green for some proportion $\lambda_S \geq \kappa_S$ of the T steps, where the *minimum green splits* $\kappa_S \in (0, 1) \ \forall \ S \in U$ are parameters selected by a network manager to enforce equity in movement actuation.

The selection of a cycle time T intuitively affects intersection capacity. Our proof of network stability in the following sections relies on the fact that road links are *undersaturated*:

that is, the expected demand is served (on average) within a signal cycle. To avoid link saturation, we pose the following convex optimization problem (extended from that in [**?**]) to determine minimum constrained feasible actuation time Λ^*:

$$\Lambda^* = \min_{\lambda = \{\lambda_S\}} \sum_{S \in U} \lambda_S$$
$$\text{subject to} \quad \lambda_S \geq \kappa_S \ \forall \ S \in U \tag{15}$$
$$f_l r(l, m) < \sum_S \lambda_S c(l, m) S(l, m)$$

where $\kappa_S \in [0, 1] \ \forall S \in U$, and $\sum_S \kappa_S < 1$. If $\Lambda^* > 1$, the demand is not feasible under the set of control constraints $\{\kappa_S\}$ for any cycle length. If $\Lambda^* < 1$, then we can define a cycle length for which flow is admissible without link saturation. However, this cycle length T must be significantly greater than Λ^* to account for clearance times. If we define $L = \text{ceil}(\frac{R}{\Delta t} \cdot |U|)$ to be total *lost time steps* per cycle, a feasible cycle length T must satisfy the following condition:

$$T > \frac{L}{1 - \Lambda^*} \tag{16}$$

Cycle-based controller formulation

The cycle-based max pressure controller is a relaxed control matrix that is constructed as follows:

$$S^{r*}(t) = u^{c*}(X(t)) = \sum_{S \in U} \lambda_S^* S, \quad \text{where} \tag{17}$$

$$\{\lambda_S^*\} = \arg\max_{\lambda_1, \ldots, \lambda_{|U|}} \sum_{S \in U} \lambda_S \gamma(S)(X(t)) \tag{18}$$

$$\text{subject to} \quad \lambda_S \geq \kappa_s$$
$$\sum_S \lambda_S \leq 1 - \frac{L}{T}$$

At time step $t = nT$ for integer n, the controller u^{c*} uses feedback measurements $x(t)$ to select a relaxed control matrix S^{r*} with components λ_S^* that satisfy (18). This relaxed controller is then applied for the subsequent T time steps $\{t, t+1, \ldots, t+T-1\}$ before the controller is updated.

Note that this controller is modeled such that all phases in an intersection are simultaneously actuated at some proportion of their maximum flow capacity. This is not possible in practice, as many phases will have to make conflicting use of the same intersection resources. Hence individual phases S will have to be actuated in series, with each having a duration corresponding to a number of "time units" that are equal to cycle proportions $(\lambda_S T \cdot \Delta t)$. Feedback measurements will then be a measure of "average" cycle queue length acquired over a set of measurements spanning the previous cycle.

Because cycle-based max pressure can be implemented such that phases occur in a predictable order, a controller running cycle-based max pressure can be synchronized with neighboring controllers to enforce a "green-wave progression" designed to minimize individual vehicle stoppage and delay for a prioritized movement. This is standard practice in existing traffic signal control design.

V. STABILITY OF CYCLE-BASED MAX PRESSURE

Cycle-based max pressure is fundamentally different from the standard max pressure formulation of Variaya [?] in two ways: first, it only updates the controller once every signal cycle (or T model time steps); second, it applies a relaxed phase actuation (which is some convex combination of standard phase actuations). The following two sections will separately address how each of these modifications individually affects the stability of the resulting controlled networks.

Properties of a τ-updated controller

Suppose that we are given the model dynamics $X(t)$ as in (6)-(7) and apply the standard max pressure controller (14). However, we further impose that the control actuation $S^*(t)$ can only be updated every τ model time steps. A resulting τ-*updated control sequence* is composed of a single (one-phase) control matrix repeated for τ time steps of the model dynamics. Hence for any integer n, the controller that maximizes the pressure at time step $n\tau + 1$ is continuously applied until time step $(n+1)\tau$:

$$S(n\tau + 1) = S(n\tau + 2) = \ldots = S((n+1)\tau) \qquad (19)$$
$$= S^*(n\tau + 1) = \arg\max_{S \in U}\{\gamma(S)(X(n\tau + 1))\}$$

We first prove that the set of demands that can be accommodated using τ-updated control sequences is the same set of feasible flows as in (9). Given the set admissible phases U, define the following sets:

- $\mathcal{U}$ is the set of control sequences with distinct elements $\{S(1), S(2)\ldots S(t)\ldots | S(\cdot) \in U\}$,
- $\mathcal{U}_\tau$ is the set of τ-updated control sequences $\{S(1), S(1), \ldots, S(\tau+1), S(\tau+1), \ldots, S(n\tau+1), S(n\tau+1), \ldots | S(\cdot) \in U\}$,

Also define the following sets of *long-term control proportion matrices*, which are similar to the formulation in (8):

$$M_{\mathcal{U}} = \Big\{ \liminf_T \frac{1}{T} \sum_{t=1}^T S(t) \Big| \{S(1), S(2), \ldots, S(t), \ldots\} \in \mathcal{U} \Big\}$$

$$M_{\mathcal{U}_\tau} = \Big\{ \liminf_T \frac{1}{T} \sum_{t=1}^T S(t)$$
$$\Big| \{S(1), S(1), \ldots, S(\tau+1), S(\tau+1), \ldots\} \in \mathcal{U}_\tau \Big\}$$

By Property 2, a demand d is only feasible if there exists a control sequence $\overline{S}$ such that the corresponding long-term control proportion matrix $M_{\overline{S}}$ satisfies (9). Here we show $M_{\mathcal{U}} = M_{\mathcal{U}_\tau}$, and therefore any flows that are admissible given an unrestricted controller in $\mathcal{U}$ can also be accommodated using a τ-updated controller in $\mathcal{U}_\tau$.

Obviously, $M_{\mathcal{U}_\tau} \subset M_{\mathcal{U}}$. To show equality, we can also demonstrate that $M_{\mathcal{U}} \subset M_{\mathcal{U}_\tau}$. Suppose there exists a control sequence $\hat{S} = \{S(1), S(2), \ldots\} \in \mathcal{U}$. By definition,

$$M_{\hat{S}} = \liminf_T \frac{1}{T} \sum_{t=1}^T S(t)$$
$$= \liminf_T \frac{1}{\tau T} \sum_{t=1}^{\tau T} \tilde{S}(t)$$
$$\text{where } \tilde{S} = \{S(1), S(1), \ldots, S(t), S(t), \ldots\}$$
$$= \liminf_T \frac{1}{T} \sum_{t=1}^T \tilde{S}(t)$$
$$\in M_{\mathcal{U}_\tau} \qquad \implies M_{\mathcal{U}} \subset M_{\mathcal{U}_\tau}$$

This result implies that a τ-updated control sequence can accommodate the same set of flows as a control sequence updated at every time step. The equivalence becomes intuitive when one considers that our definition of feasible flows considers only the long-term (more precisely, infinite-term) average of both demand and service rates, and any infinite control sequence with limited admissible phases can be re-arranged to form a τ-updated sequence for some τ.

As we will show in the following sections, the only additional impact of occasional updating will be an increased bound on queue lengths relative to the standard max pressure setting.

Stability of cycle-based max pressure

Here we examine the stability of the cycle-based max pressure controller using T-updated sequences of relaxed controllers with fixed minimum phase proportion constraints, as formulated in (17)-(18).

Define $conv_\kappa$ as the set of convex combinations of control matrices with coefficients larger than κ:

$$conv_\kappa = \Big\{ \sum_S \lambda_S S | \lambda_S > \kappa_S \ \forall S \in U \Big\} \qquad (20)$$

Also define a set of *undersaturated* admissible demands D_κ with elements d such that $f = dP$ and

$$f_l r(l,m) < c(l,m) S^r(l,m) \qquad (21)$$

As in (15), this condition ensures that a demand $d \in D_\kappa$ can in average be served *within a single cycle* by a relaxed control matrix that maintains a specified minimum time allocation for each phase.

Theorem 4: The cycle-based max pressure controller defined in (17)-(18) stabilizes a network whenever the demand is within a set of feasible undersaturated demands D^κ.

Proof: Consider the expectation of the following function of queue state with perturbation

$$\delta(t) = X(t+1) - X(t) \qquad (22)$$

conditioned on the past queue state:

$$|X(t+1)|^2 - |X(t)|^2 = |X(t) + \delta(t)|^2 - |X(t)|^2 \qquad (23)$$
$$= 2X(t)^T \delta(t) + |\delta(t)|^2$$
$$= 2\alpha(t) + \beta(t)$$

$$\text{with} \quad \alpha(t) = X(t)^T \delta(t) \quad \text{and} \quad \beta(t) = |\delta(t)|^2 \qquad (24)$$

We continue by addressing bounds on β and α separately.

Bound on $\beta(t) = |\delta(t)|^2$

If we define B as the maximum of all of the quantities $\{\overline{C}(l,m), \sum_k \overline{C}(k,l), \overline{d}(l,m)\}$ and N as the number of queues in the network, we can derive a bound for β which depends on only B and N:

$$\beta(t) = |\delta(t)|^2 \leq NB^2 \tag{25}$$

The proof of this bound is exactly the same as in [**?**], and will therefore not be repeated here. However, note that because these bounds hold for any $S(l,m)(t) \in [0,1]$, this bound on β is trivially extended to any convex combination of control matrices; hence it is still valid in our relaxed controllers (as will be shown later in this article).

Bound on $\alpha(t) = X(t)^T \delta(t)$

Again following (**??**), we define the following sub-terms:

$$\mathbb{E}\{\alpha(t)|X(t)\} = \sum_{l \in \mathcal{L}, m} w(l,m)(t) \cdot \tag{26}$$

$$\left[f_l r(l,m) - \mathbb{E}\left\{ \left[C(l,m)(t+1)S(l,m)(t) \wedge x(l,m)(t) \right] \big| X(t) \right\} \right]$$
$$= \alpha_1(t) + \alpha_2(t)$$

with

$$\alpha_1(t) = \sum_{l \in \mathcal{L}, m} \left[f_l r(l,m) - c(l,m)S(l,m)(t) \right] w(l,m)(t) \tag{27}$$

and

$$\alpha_2(t) = \sum_{l \in \mathcal{L}, m} S(l,m)(t)w(l,m)(t) \cdot$$
$$\left[c(l,m) - \mathbb{E}\left\{ \left[C(l,m)(t+1) \wedge x(l,m)(t) \right] \big| X(t) \right\} \right] \tag{28}$$

Lemma 5: For all l, m, t,

$$\alpha_2(t) \leq \sum_{l \in \mathcal{L}, m} c(l,m)\overline{C}(l,m) \tag{29}$$

where $\overline{C}(l,m)$ is the maximum value of the random service rate $C(l,m)(t)$.

The proof of Lemma 5 again follows [**?**] directly, as an extension from a binary controller S to an $S \in [0,1]$ is trivial. In fact, the modifications made in the cycle-based max pressure controller only affect the $\alpha_1(t)$ term, as we see in the proof of the following lemma.

Lemma 6: If a relaxed max pressure control policy u^{c*} is applied at time step t and the demand d is in the set of feasible undersaturated demands D^κ, then there exists an $\varepsilon > 0$, $\eta > 0$ such that

$$\alpha_1(t) \leq -\varepsilon\eta|X(t)| \tag{30}$$

Proof of Lemma 6:
Consider the relaxed control matrix S^{r*} specified by (18). By construction, it must be true that $\forall S^r \in conv_\kappa$,

$$\sum_{l,m} c(l,m)w(l,m)(X(t))S^r(l,m)$$
$$\leq \sum_{l,m} c(l,m)w(l,m)(X(t))S^{r*}(l,m) \tag{31}$$

with equality only if $S^r = S^{r*}$. Therefore $\forall S^r \neq S^{r*}$,

$$\sum_{l,m} [f_l r(l,m) - c(l,m)S^{r*}(l,m)(t)]w(l,m)(X(t))$$
$$< \sum_{l,m} [f_l r(l,m) - c(l,m)S^r(l,m)]w(l,m)(X(t)) \tag{32}$$

If the demand flow is admissible according to (**??**), then $\exists \Sigma \in conv_\kappa$ such that

$$c(l,m)\Sigma(l,m) = \begin{cases} f_l r(l,m) + \varepsilon & \text{if } w(l,m)(X(t)) > 0 \\ 0 & \text{otherwise} \end{cases}$$

Hence following the same logic as in (**??**),

$$\sum_{l,m} [f_l r(l,m) - c(l,m)S^{r*}(l,m)(t)]w(l,m)(X(t))$$
$$< -\varepsilon \sum_{l \in \mathcal{L}, m} \max\{w(l,m)(X(t)), 0\}$$
$$+ \sum_{l \in \mathcal{L}, m} f_l r(l,m) \min\{w(l,m)(X(t)), 0\} \tag{33}$$

We assume that by our choice of $\sigma(l,m)$, $f_l r(l,m) > \varepsilon$ (omitting the cases where $r(l,m) = 0$). Therefore:

$$\sum_{l,m} [f_l r(l,m) - c(l,m)S^{r*}(l,m)(t)]w(l,m)(X(t))$$
$$< -\varepsilon \sum_{l,m} x(l,m)(t) \tag{34}$$

Explicit bound on queues, immediate feedback MP

Combining (**??**) and (25), we obtain

$$\mathbb{E}\left\{ |X(t+1)|^2 - |X(t)|^2 | X(t) \right\} = \mathbb{E}\left\{ 2\alpha(t) + \beta(t) \right\}$$
$$< -2\varepsilon\eta |X(t)| + 2\sum_{l \in \mathcal{L}, m} [c(l,m)\overline{C}(l,m)] + NB^2 \tag{35}$$

where N is the number of links in the network and $B = \max\{\overline{C}(l,m), \sum_k \overline{C}(k,l), \overline{d}(l,m)\}$. For simplicity, we combine all constant additive terms to define a new constant K:

$$K = 2\sum_{l,m} c(l,m)\overline{C}(l,m) \tag{36}$$

$$+ N\sum_{l,m} \max\left\{ \overline{C}(l,m), \sum_k \overline{C}(k,l), \overline{d}(l,m) \right\}^2$$

So, when the control is updated at time t:

$$\mathbb{E}\left\{ |X(t+1)|^2 - |X(t)|^2 | X(t) \right\} < -2\varepsilon\eta |X(t)| + K \tag{37}$$

We then establish a bound on the incremental queue differences within a cycle of length τ, following the form of (**??**):

Lemma 7: For a given cycle (i.e. when the controller is not updated) consisting of time steps $\{t, t+1, \ldots, t+\tau\}$, $\forall p \in [0, \tau - 1]$,

$$\mathbb{E}\left\{ |X(t+p+1)|^2 - |X(t+p)|^2 | X(t) \ldots X(t+p-1) \right\}$$
$$< -2\varepsilon\eta |X(t+p)| + B(p) + K \tag{38}$$

where

$$K = 2 \sum_{l,m} c(l,m)\overline{C}(l,m) \tag{39}$$

$$+ N \sum_{l,m} \max\{\overline{C}(l,m), \sum_k \overline{C}(k,l), \overline{d}(l,m)\}^2$$

and

$$B(p) = p \left(2\varepsilon\eta \sum_{l,m} + 2(\sum_{l,m}[f_l r(l,m) + c(l,m)]) \right)$$

$$\cdot \max\left\{\overline{C}(l,m), \sum_k \overline{C}(k,l), \overline{d}(l,m)\right\} \tag{40}$$

<u>Proof of Lemma 7:</u>
As above, we have:

$$|X(t+p+1)|^2 - |X(t+p)|^2 \tag{41}$$
$$= 2(\alpha_1(t+p) + \alpha_2(t+p)) + \beta(t+p)$$

where β, α_1 and α_2 are quantities that depend on the controller applied at time step $t+p$, as defined in Section III:

$$\beta(t+p) = |X(t+p+1) - X(t+p)|^2$$
$$\alpha_1(t+p) = \sum_{l,m} \left(f_l r(l,m) - c(l,m)S(l,m)(t) \right)$$
$$\cdot w(l,m)(X(t+p))$$
$$\alpha_2(t+p) = \sum_{l,m} \Big(c(l,m)S(l,m)(t)$$
$$-\mathbb{E}\Big\{ [C(l,m)(t+p+1) \wedge x(l,m)(t+p)]|X(t+p)\Big\}\Big)$$
$$\cdot w(l,m)(X(t+p))$$

As previously derived, the following bounds on $\beta(\cdot)$ and $\alpha_2(\cdot)$ will hold for any binary control matrix:

$$\alpha_2(\cdot) < \sum_{l\in\mathcal{L},m} c(l,m)\overline{C}(l,m) \tag{42}$$

$$\beta(\cdot) < N \sum_{l,m} \max\left\{\overline{C}(l,m), \sum_k \overline{C}(k,l), \overline{d}(l,m)\right\}^2 \tag{43}$$

These two terms form the constant K from (36), which also appears in (38). To complete the bound in (38) we are only left with the α_1 term, which is directly dependent on the explicit form of the binary controller S:

$$\mathbb{E}\Big\{|X(t+p+1)|^2 - |X(t+p)|^2|X(t)\dots X(t+p)\Big\}$$
$$= \mathbb{E}\Big\{2\alpha_1(t+p) + 2\alpha_2(t+p) + \beta(t+p)|X(t)\dots X(t+p)\Big\}$$
$$< \mathbb{E}\Big\{2\alpha_1(t+p)|X(t)\dots X(t+p)\Big\} + K$$
$$= 2\sum_{l,m}[f_l r(l,m) - c(l,m)S(l,m)(t)]w(l,m)(X(t+p)) + K$$
$$\tag{44}$$

Examine the remaining term, $\alpha_1(t)$:

$$2\sum_{l,m}[f_l r(l,m) - c(l,m)S(l,m)(t)]w(l,m)(X(t+p))$$
$$= 2\sum_{l,m}[f_l r(l,m) - c(l,m)S(l,m)(t)]w(l,m)(X(t))$$
$$+ 2\sum_{l,m}[f_l r(l,m) - c(l,m)S(l,m)(t)]$$
$$\cdot (w(l,m)(X(t+p) - X(t)))$$
$$= 2\xi_1 + 2\xi_2$$

With

$$\xi_1(t,S) = \sum_{l,m}[f_l r(l,m) - c(l,m)S(l,m)(t)] \, w(l,m)(X(t))$$
$$\tag{45}$$

and

$$\xi_2(t,p,S) = \sum_{l,m}[f_l r(l,m) - c(l,m)S(l,m)(t)]$$
$$\cdot (w(l,m)(X(t+p) - X(t))) \tag{46}$$

<u>Bound on ξ_1</u>
By Lemma 6 we know that

$$2\sum_{l,m}[f_l r(l,m) - c(l,m)S(l,m)(t)]w(l,m)(X(t)) < -2\varepsilon\eta|X(t)|$$

Then noting that

$$|X(t)| = |X(t+p) - (X(t+p) - X(t))|$$
$$> ||X(t+p)| - |X(t+p) - X(t)||$$
$$> |X(t+p)| - |X(t+p) - X(t)|$$

we are left with

$$2\xi_1(t,S) < -2\varepsilon\eta(|X(t+p)| - |X(t+p) - X(t)|)$$
$$< -2\varepsilon\eta|X(t+p)|$$
$$+ 2\varepsilon\eta\sum_{i=1}^p |X(t+i) - X(t+i-1)|$$
$$= -2\varepsilon\eta|X(t+p)| + 2\varepsilon\eta\sum_{i=1}^p |\delta(t+i-1)| \tag{47}$$

So by (47) and (25),

$$2\sum_{l,m}[f_l r(l,m) - c(l,m)S(l,m)(t)]w(l,m)(X(t)) \tag{48}$$
$$< -2\varepsilon\eta|X(t+p)|$$
$$+ 2\varepsilon\eta p\sum_{l,m} \max\left\{\overline{C}(l,m), \sum_k \overline{C}(k,l), \overline{d}(l,m)\right\}$$

Plugging (48) into (44), we have

$$
\mathbb{E}\Big\{|X(t+p+1)|^2 - |X(t+p)|^2 | X(t), \ldots, X(t+p)\Big\}
$$
$$
< K - 2\varepsilon\eta|X(t+p)|
$$
$$
+ 2\varepsilon\eta p \sum_{l,m} \max\Big\{\overline{C}(l,m), \sum_k \overline{C}(k,l), \overline{d}(l,m)\Big\}
$$
$$
+ 2\sum_{l,m} [f_l r(l,m) - c(l,m)S(l,m)(t)]
$$
$$
\cdot \Big(w(l,m)(X(t+p)) - w(l,m)(X(t))\Big) \quad (49)
$$

Bound on ξ_2

We now have to bound the term

$$
2\xi_2(t,p,S) = 2\sum_{l,m} [f_l r(l,m) - c(l,m)S(l,m)(t)]
$$
$$
\cdot \Big(w(l,m)X((t+p)) - w(l,m)(X(t))\Big) \quad (50)
$$

For that purpose we study the term

$$
w(l,m)(X(t+p)) - w(l,m)(X(t))
$$
$$
= \sum_{n=1}^{p} w(l,m)(X(t+n)) - w(l,m)(X(t+n-1))
$$
$$
= \sum_{n=1}^{p} \Big\{x(l,m)(t+n) - x(l,m)(t+n-1)
$$
$$
- \sum_{s \in Out(m)} [x(m,s)(t+n) - x(m,s)(t+n-1)]r(m,s)\Big\}
$$
$$
= \sum_{n=1}^{p} w(l,m)(\delta(t+n-1)) \quad (51)
$$

By **(??)** and the fact that $w(\cdot)$ is linear,

$$
|w(l,m)(\delta(t+n-1))|
$$
$$
< \sum_{u,v} \max\Big\{\overline{C}(u,v), \sum_k \overline{C}(k,u), \overline{d}(u,v)\Big\} \quad (52)
$$

Therefore plugging (52) back into (50), we get

$$
2\xi_2(t,p,S) = 2\Big(\sum_{l,m}([f_l r(l,m) - c(l,m)S(l,m)]
$$
$$
\cdot \sum_{n=1}^{p} w(l,m)(\delta(t+n-1)))\Big)
$$
$$
< 2\sum_{n=1}^{p}\sum_{l,m} [f_l r(l,m) - c(l,m)S(l,m)]
$$
$$
\cdot \sum_{u,v} \max\Big\{\overline{C}(u,v), \sum_k \overline{C}(k,u), \overline{d}(u,v)\Big\}
$$
$$
(53)
$$

Also note that

$$
|\sum_{n=1}^{p}\sum_{l,m}[f_l r(l,m) - c(l,m)S(l,m)]| < p\sum_{l,m}[f_l r(l,m) + c(l,m)]
$$
$$
(54)
$$

so (53) becomes

$$
2\xi_2(t,p,S) < 2p\left(\sum_{l,m}[f_l r(l,m) + c(l,m)]\right) \quad (55)
$$
$$
\cdot\left(\sum_{l,m} \max\Big\{\overline{C}(l,m), \sum_k \overline{C}(k,l), \overline{d}(l,m)\Big\}\right)
$$

Substituting (55) into (49) yields the final bound expressed in (38):

$$
\mathbb{E}\Big\{|X(t+p+1)|^2 - |X(t+p)|^2 | X(t), \ldots, X(t+p)\Big\}
$$
$$
< K - 2\varepsilon\eta|X(t+p)|
$$
$$
+ 2\varepsilon\eta p \sum_{l,m} \max\Big\{\overline{C}(l,m), \sum_k \overline{C}(k,l), \overline{d}(l,m)\Big\}
$$
$$
+ 2p\left(\sum_{l,m}[f_l r(l,m) + c(l,m)]\right)
$$
$$
\cdot\left(\sum_{l,m} \max\Big\{\overline{C}(l,m), \sum_k \overline{C}(k,l), \overline{d}(l,m)\Big\}\right)
$$
$$
= K - 2\varepsilon\eta|X(t+p)| + B(p) \quad (56)
$$

with K and $B(p)$ given by (39) and (40), respectively.

Once we establish Lemma 7, we can show that for a time step t within any number of cycles T, the following quantity is bounded:

$$
\sum_{t=1}^{\tau T} \mathbb{E}\Big\{|X(t+1)|^2 - |X(t)|^2 | X(t)\Big\}
$$
$$
= \sum_{t=1}^{T-1}\sum_{p=0}^{\tau-1} \mathbb{E}\Big\{|X(t+p+1)|^2 - |X(t+p)|^2 | X(t+p)\Big\}
$$
$$
< \sum_{t=1}^{T-1}\sum_{p=0}^{\tau-1} (-2\varepsilon\eta|X(t+p)| + B(p) + K)
$$
$$
< -2\varepsilon\eta\sum_{t=1}^{\tau T}|X(t)| + (T-1)\left(\sum_{p=0}^{\tau-1} B(p) + \tau K\right) \quad (57)
$$

Which, when taking the expectation, yields

$$
\mathbb{E}\Big\{|X(\tau T+1)|^2 - |X(1)|^2\Big\}
$$
$$
< -2\varepsilon\eta\sum_{t=1}^{\tau T}\mathbb{E}\Big\{|X(t)|\Big\} + (T-1)(\sum_{p=0}^{\tau-1} B(p) + \tau K) \quad (58)
$$

Or, rearranging we obtain

$$
\frac{1}{\tau T}\sum_{t=1}^{\tau T}\mathbb{E}\Big\{|X(t)|\Big\} < \frac{1}{2\varepsilon\eta\tau T}\mathbb{E}\Big\{|X(1)|^2 - |X(\tau T+1)|^2\Big\}
$$
$$
+ \frac{T-1}{2\varepsilon\eta\tau T}\left(\sum_{p=0}^{\tau-1} B(p) + \tau K\right)
$$

$$< \frac{1}{2\varepsilon\eta\tau T}\mathbb{E}\Big\{|X(1)|^2\Big\} + \frac{1}{2\varepsilon\eta\tau}\left(\sum_{p=0}^{\tau-1} B(p) + \tau K\right) \tag{59}$$

This establishes stability: the quantity $\frac{1}{\tau T}\sum_{t=1}^{\tau T}\mathbb{E}\Big\{|X(t)|\Big\}$ must then be bounded.

$$\frac{1}{\tau T}\sum_{t=1}^{\tau T}\mathbb{E}\Big\{|X(t)\Big\} < \frac{1}{2\varepsilon\eta\tau T}\mathbb{E}\Big\{|X(1)|^2\Big\} \tag{60}$$
$$+ \frac{1}{2\varepsilon\eta\tau}\sum_{p=0}^{\tau-1}(B(p) + \tau K)$$

where ε, η and K are constants which depend on network topology and demand.

∎

The proof above shows that the system is stable.

Increased bound on queue length

Max pressure controllers that are only updated every τ model time steps will stabilize a network; however the resulting bound on the queues will be higher than in the immediate feedback setting. Comparing (**??**) and (60), note that the constants are increased by a factor of

$$\frac{1}{\varepsilon\tau}\sum_{p=1}^{\tau-1} B(p) = \frac{1}{\varepsilon\tau}\frac{\tau(\tau-1)}{2}\cdot(\text{constant}) = \frac{\tau-1}{2\varepsilon}\cdot(\text{constant}) \tag{61}$$

The relative cost to queue bounds is therefore linear in τ.

VI. NUMERICAL IMPLEMENTATION

A cycle-allocated max pressure controller was implemented on a network of 11 signalized intersections modeled in AIMSUN, a mico-simulation platform commonly used by practitioners. The present simulation was generated as part of the I-15 Integrated Corridor Management project, led by SANDAG in San Diego, CA. The controller was set to run with a cycle length of 90 seconds and minimum time constraints of 10 seconds for each of the 3-4 available standard signal control stages.
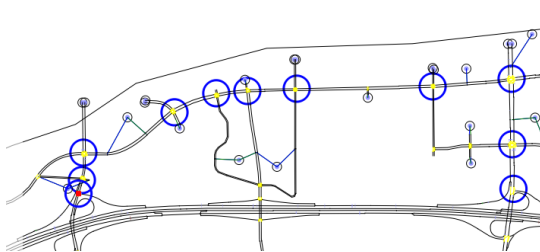

Fig. 1. The chosen network was calibrated to represent realistic demands and physical parameters observed on a stretch of Black Mountain Road near the I-15 freeway in San Diego, California.

The rest of this section provides a comparison of the performance of the max pressure algorithm with two other algorithms described below. While it makes sense to compare the numerical performance of these algorithms, it should be noted that the max pressure algorithm is the only algorithm, to the best of our knowledge, which provides theoretical guarantees: while the other algorithms seem to eprform well in simulation, they come with no proof or guarantees, as they are heuristics commonly used in the practioners field. Various performance metrics were compared between model runs using the max pressure controller and two alternative controllers: a fixed-time control plan that divides each signal cycle equally between all available phases, and a "fully-actuated" control system such as that which is currently operational on the real road network represented by the model. The fully actuated-controller is essentially a flexible fixed-time plan in which green times can be shortened or extended in real time to promote continuity of flows in response to instantaneous link demand measurements. The comparison of network vehicle counts in Figure 2 suggests that the uniformly-allocated fixed time controller caused significantly fewer vehicles to be served than the other two controllers, which were comparable to each other in vehicle service rates. This difference made it difficult to fairly compare other performance metrics between the uniform fixed-time controller and the other two options.
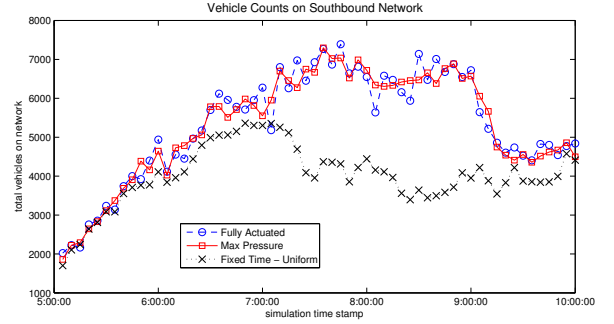

Fig. 2. During congestion, cycle-allocated max pressure demonstrated service rates that are higher than the uniformly allocated fixed-time controller and consistent with a fully-actuated control system.
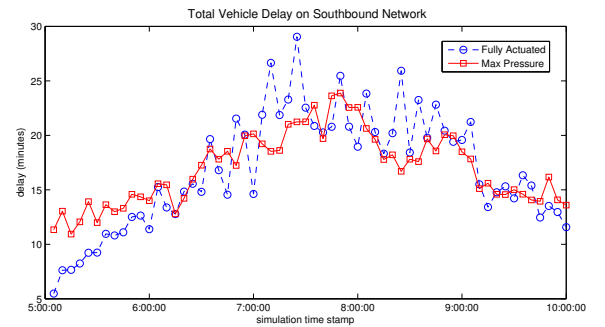

Fig. 3. Max pressure outperforms the fully actuated system during periods of high demand.

Differences between the fully-actuated and max pressure controllers were observed in measurements of delays and queue lengths: while the fully-actuated system appeared to

produce less delay and shorter queues than max pressure during periods of relatively low network demand, max pressure was equally as effective or even more effective at reducing delays and queue lengths given larger demands. Figure 3 shows how delays were reduced and had less variance over time when the max pressure controller was applied than with the fully-actuated controller.
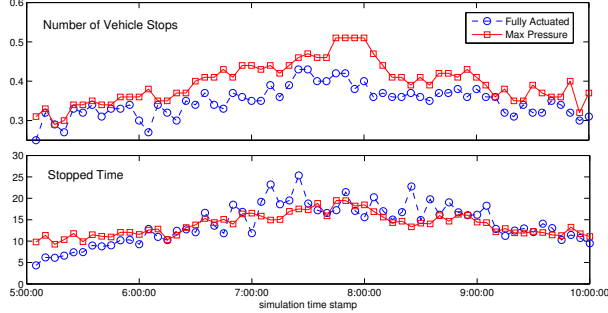

Fig. 4.   While max pressure caused more vehicle stop events, stoppage times were similar to those observed using the standard fully-actuated controller.

Max pressure consistently induced more stops during a vehicle's journey across the network, which is expected given the design objectives of the fully-actuated system. As illustrated in Figure 4, total stoppage times were higher with max pressure given low demand, but improved over the existing controller during peak demand.

VII.   CONCLUSION

In this work we have defined an extension of the max pressure controller required in order to be applied to a real network of signalized traffic intersections. Given only the constraint that the network demands are (in average) serviceable, we have proven that updating a max pressure controller which allocates a fixed minimum proportion of service to each permissible phase at a slower rate than that which governs traffic flow will not destroy the stabilizing properties of the controller.

While there are additional modifications to the control model that must be addressed before max pressure can be considered a feasible algorithm for real-time operation of traffic signal controllers, our implementation in the reference micro-simulation platform demonstrates how it could comply with the hardware and communications constraints commonly encountered in existing roadway infrastructure. These numerical simulations furthermore reveal that an implementation of cycle-allocated max pressure competes with the existing standard feedback controller in many performance metrics, especially during periods of high congestion. This comes on top of theoretical proofs it provides, unlike algoriithms used in practice. It also appears to provide less variance in delays than the existing alternative.