Problem Set 7

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Problem 7-1.

Find a closest rank-1 approximation to these matrices (L^2 or Frobenius norm):

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad A = \begin{bmatrix} 0 & 3 \\ 2 & 0 \end{bmatrix} \quad A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Solution:

Use Singular Value Decomposition (SVD) first, then find the largest value in Σ.

(a)

$$A = U\Sigma V^T, \quad U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad V = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$A_1 = u_1\sigma_1 v_1^T$ to get the closest rank-1 matrix, which is

$$A_1 = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(b)

$$A = U\Sigma V^T, \quad U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, \quad V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$A_1 = u_1\sigma_1 v_1^T$ to get the closest rank-1 matrix, which is

$$A_1 = \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}$$

(c)

$$A = U\Sigma V^T, \quad U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}, \quad V = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$A_1 = u_1\sigma_1 v_1^T$ to get the closest rank-1 matrix, which is

$$A_1 = \frac{3}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Problem 7-2.

If A is a 2 by 2 matrix with $\sigma_1 \geq \sigma_2 > 0$, find $||A^{-1}||_2$ and $||A^{-1}||_F^2$.

Solution:

$$\begin{aligned} & A = U\Sigma V^T && (1) \\ \Longrightarrow & A^{-1} = (U\Sigma V^T)^{-1} && (2) \\ \Longrightarrow & A^{-1} = (V^T)^{-1}\Sigma^{-1}U^{-1} = V\Sigma^{-1}U^T, && (3) \end{aligned}$$

Since the norm of a matrix depends on its singular values, we get

$$||A^{-1}||_2 = 1/\sigma_2, \quad ||A^{-1}||_F^2 = (1/\sigma_1)^2 + (1/\sigma_2)^2.$$