Problem Set 3

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Problem 3-1.

Draw unit vectors u and v that are *not* orthogonal. Show that $w = v - u(u^T v)$ is orthogonal to u (and add w to your picture).

Solution:

The easiest way to testify orthogonality is by performing a dot product.

$$\begin{aligned} u^T w &= u^T(v - u(u^T v)) && (1)\\ &= u^T v - u^T u(u^T v) && (2)\\ &= u^T v - 1 \cdot u^T v && (3)\\ &= 0 && (4) \end{aligned}$$

Problem 3-2.

Key property of every orthogonal matrix: $||Qx||^2 = ||x||^2$ for every vector x. More than this, show that $(Qx)^T(Qy) = x^T y$ for every vector x and y. So *lengths and angles are not changed by* Q. **Computations with Q never overflow!**

Solution:

If the transformation matrix Q is orthogonal or orthonormal, which means $Q^T Q = I$, then it's like the new basis of the column space is still like the original cartesian coordinate system, so the lengths and angles are not changing.

The math proof is below:

$$\begin{aligned} (Qx)^T(Qy) &= x^T Q^T Q y && (5)\\ &= x^T(Q^T Q)y && (6)\\ &= x^T y && (7) \end{aligned}$$

Problem 3-3.

A **permutation matrix** has the same columns as the identity matrix (in some order). *Explain why this permutation matrix and every permutation matrix is orthogonal*: $P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ has orthonormal columns so $\underline{P^T P = I}$ and $\underline{P^{-1} = P^T}$.

When a matrix is symmetric or orthogonal, **it will have orthogonal eigenvectors**. This is the most important source of orthogonal vectors in applied mathematics.