Problem Set 5

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Problem 5-1.

For which numbers b and C are these matrices positive definite?

$$A = \begin{bmatrix} 1 & b \\ b & 9 \end{bmatrix} \quad A = \begin{bmatrix} 2 & 4 \\ 4 & c \end{bmatrix} \quad A = \begin{bmatrix} c & b \\ b & c \end{bmatrix}.$$

With the pivots in D and multiplier in L, factor each A into LDL^T.

Solution:

(a)Let's compute the leading determinant, $D_1 = 1 > 0$, $D_2 = 9 - b^2 > 0 \longrightarrow -3 < b < 3$. Then, we can write down the pivots, which are 1 and $9 - b^2$.

$$A = \begin{bmatrix} 1 & b \\ b & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 9 - b^2 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}.$$

(b)$D_1 = 2 > 0$ and $D_2 = 2c - 16 > 0 \longrightarrow c > 8$. We got pivots 2 and $c - 8$.

$$A = \begin{bmatrix} 2 & 4 \\ 4 & c \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & c - 8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

(c)$D_1 = c > 0$, $D_2 = c^2 - b^2 > 0 \longrightarrow -c < b < c$. We got pivot c and $(c^2 - b^2)/c$.

$$A = \begin{bmatrix} c & b \\ b & c \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ b/c & 1 \end{bmatrix} \begin{bmatrix} c & 0 \\ 0 & (c^2 - b^2)/c \end{bmatrix} \begin{bmatrix} 1 & b/c \\ 0 & 1 \end{bmatrix}.$$

Problem 5-2.

Find the 3 by 3 matrix S and its pivots, rank, eigenvalues, and determinant:

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} & & \\ & S & \\ & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 4(x_1 - x_2 + 2x_3)^2$$

Solution:

$$S = 4 \begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & -2 \\ 2 & -2 & 4 \end{bmatrix}.$$

It has only one pivot, which is 4, has rank 1, eigenvalues $24, 0, 0$, and $\det(S) = 0$.

Problem 5-3.

Compute the three upper left determinants of S to establish positive definiteness. Verify that their ratios give the second and third pivots.

$$\textbf{Pivots=ratios of determinants} \quad S = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 3 \\ 0 & 3 & 8 \end{bmatrix}.$$

Solution:

$D_1 = 2 > 0, D_2 = 2 \times 5 - 2 \times 2 = 6 > 0, D_3 = 30 > 0 \Longrightarrow S$ is positive definite.