Matrix Methods In Data Analysis, Signal Processing, And Machine Learning: 18.065
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Problem Set 6

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Problem 6-1.

A symmetric matrix $S = S^T$ has orthonormal eigenvectors v_1 to v_n. Then any vector x can be written as a combination $x = c_1 v_1 + \cdots + c_n v_n$. Explain these two formulas:

$$x^T x = c_1^2 + \cdots + c_n^2 \qquad x^T S x = \lambda_1 c_1^2 + \cdots + \lambda_n c_n^2.$$

Solution:

Write x as

$$x = VC, \quad V = \begin{bmatrix} \vdots & \vdots & \vdots \\ v_1 & \cdots & v_n \\ \vdots & \vdots & \vdots \end{bmatrix}, \quad C = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix},$$

now $x^T x$ can be written as

$$x^T x = (VC)^T VC = C^T (V^T V) C = C^T C.$$

Also, S can be factorized into a diagonal matrix and an orthonormal matrix, i.e.,

$$S = V \Lambda V^T, \quad \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix},$$

so $x^T S x$ can be written as

$$x^T S x = (VC)^T (V \Lambda V^T)(VC) = C^T (V^T V) \Lambda (V^T V) C = C^T \Lambda C,$$

which will give us the final answer.

Problem 6-2.

Find the σ's and v's and u's in the SVD for $A = \begin{bmatrix} 3 & 4 \\ 0 & 5 \end{bmatrix}$. Use equation (12).

Solution:

Use the SVD, by computing $A^T A$ to get the eigenvectors V and eigenvalues Σ, then compute AA^T to get U.

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad \Sigma = \frac{1}{\sqrt{20}} \begin{bmatrix} 30 & 0 \\ 0 & 10 \end{bmatrix}, \quad V = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix},$$

so that $A = U \Sigma V^T$.