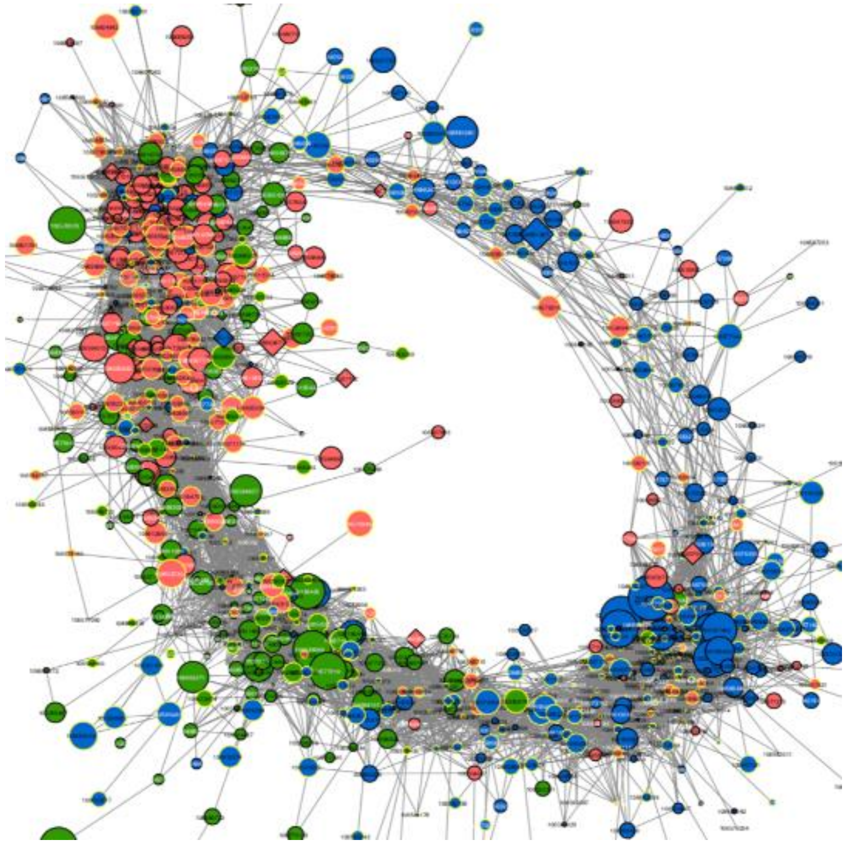
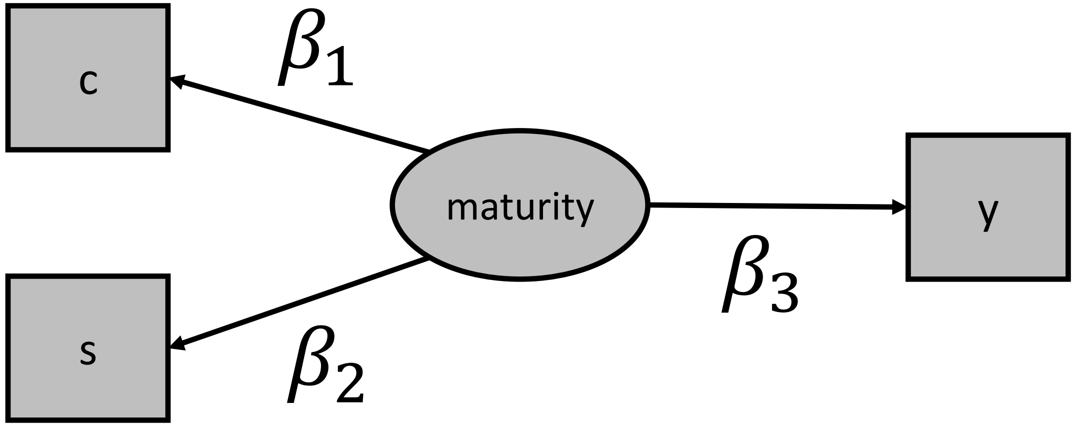
Latent variables





Building blocks

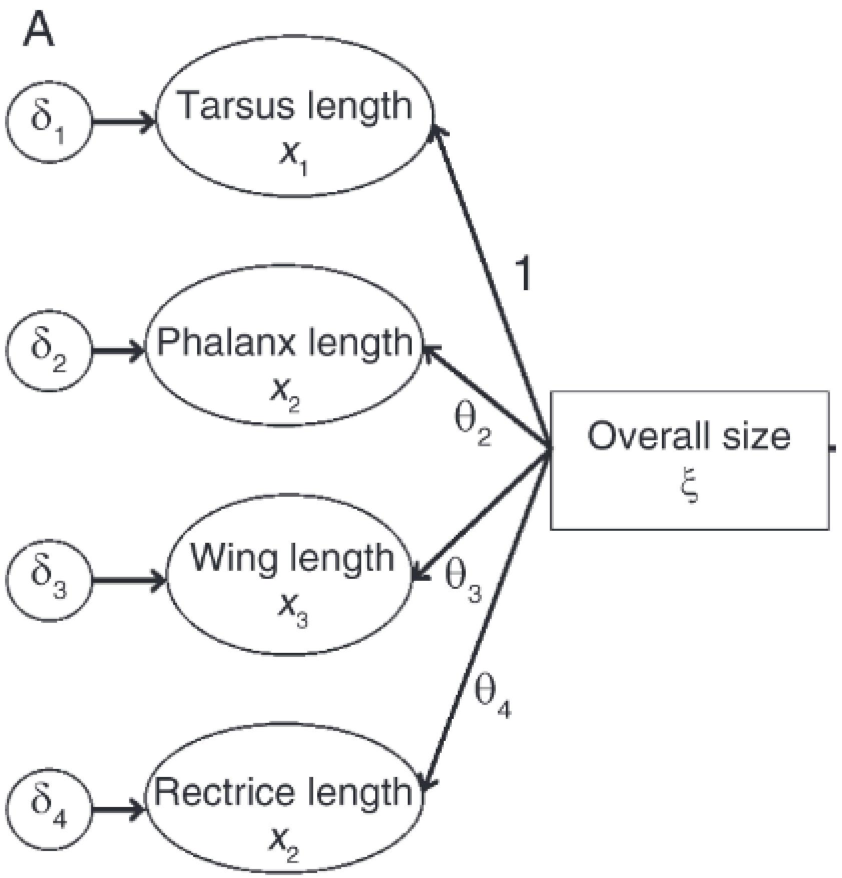
Sarah Cubaynes

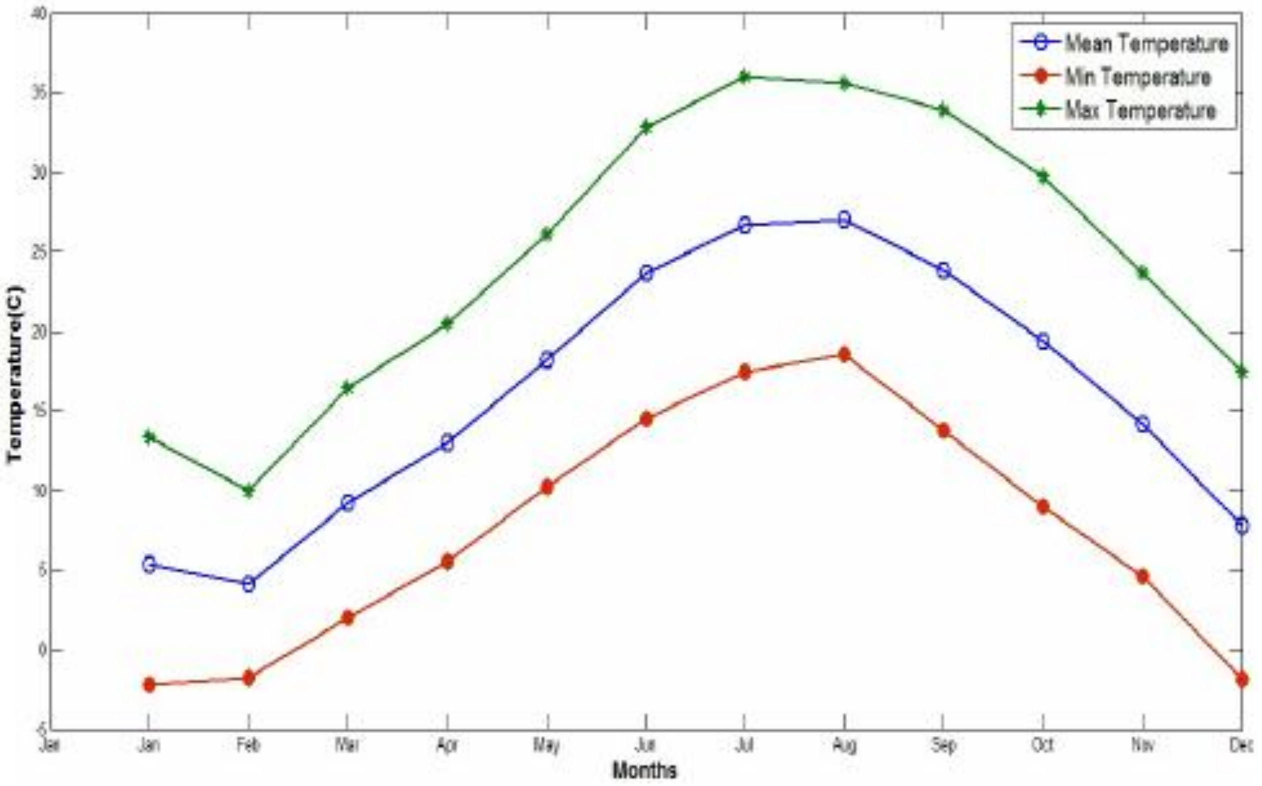
Martijn van de Pol

Cubaynes et al. (**2012**) *Ecology*

van de Pol et al. **(2021)** *Journal of Animal Ecology*

This module’s question: what if we’re kinda measuring the same thing?





What is a latent (or hidden) variable?

A random variable that is unmeasured but not necessarily unmeasureable.
-P Spirtes (2001)

A variable that is hypothesized to exist, but that has not been measured directly
-J Grace (2006)

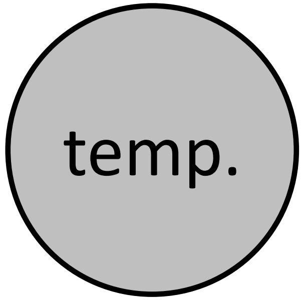
A variable that is not directly observable but is inferred from other variables that can be measured
-Generative AI (yesterday)

Variables that can only be inferred indirectly through a mathematical model from other observable variables
-Wikipedia (also yesterday)

What is a latent (or hidden) variable?

Everything is a latent variable **– LA Dyer**

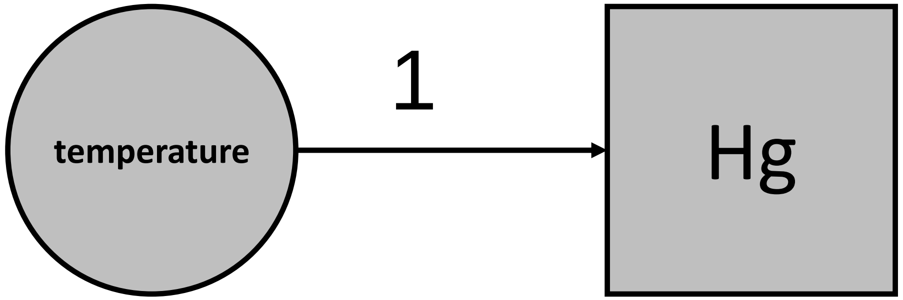
Is temperature a latent variable?



A random variable that is unmeasured but not necessarily unmeasureable.
-P Spirtes (2001)

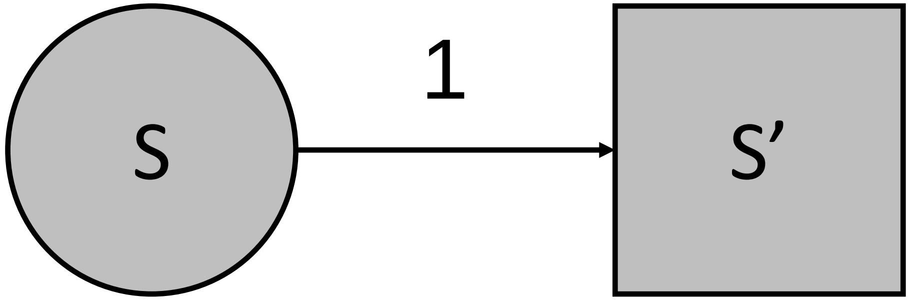
Temperature is the average kinetic energy of particles

Temperature is a latent variable



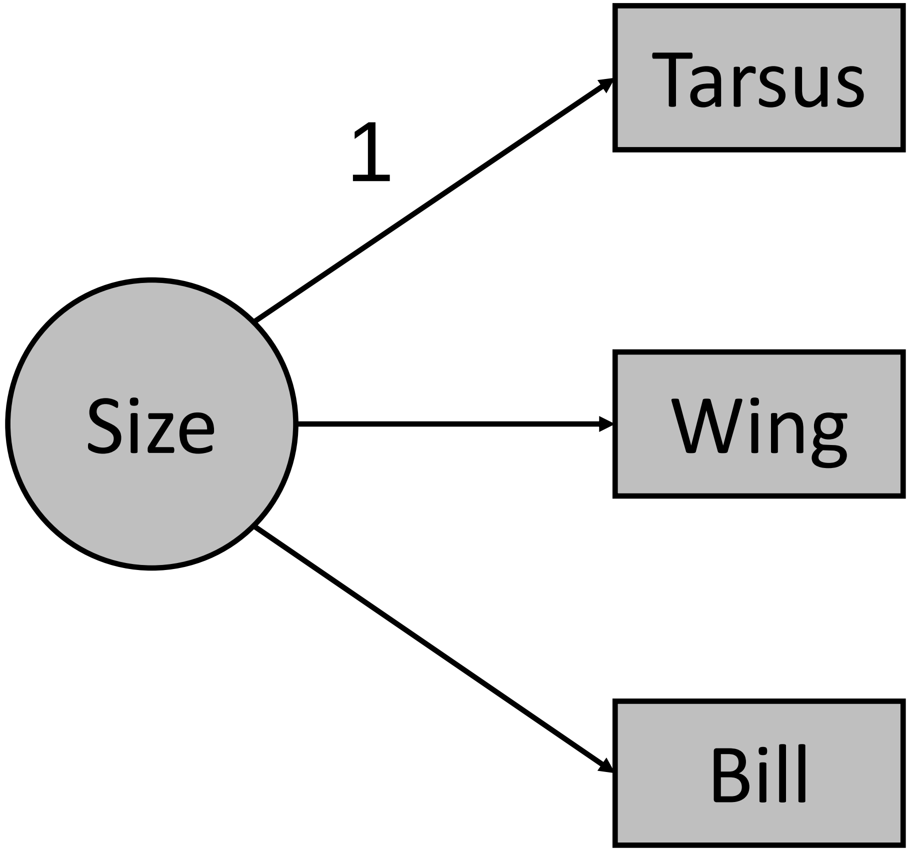
Temperature is the average kinetic energy of particles
We measure it (with error) via the expansion of mercury (or lasers)

Is survival a latent variable?



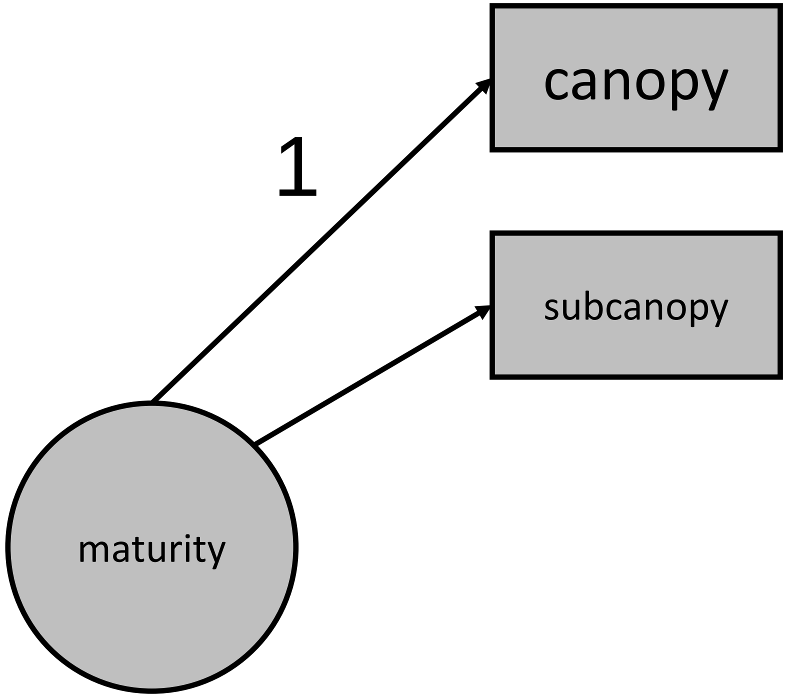
S: survival of a population, S’: survival of a marked sample

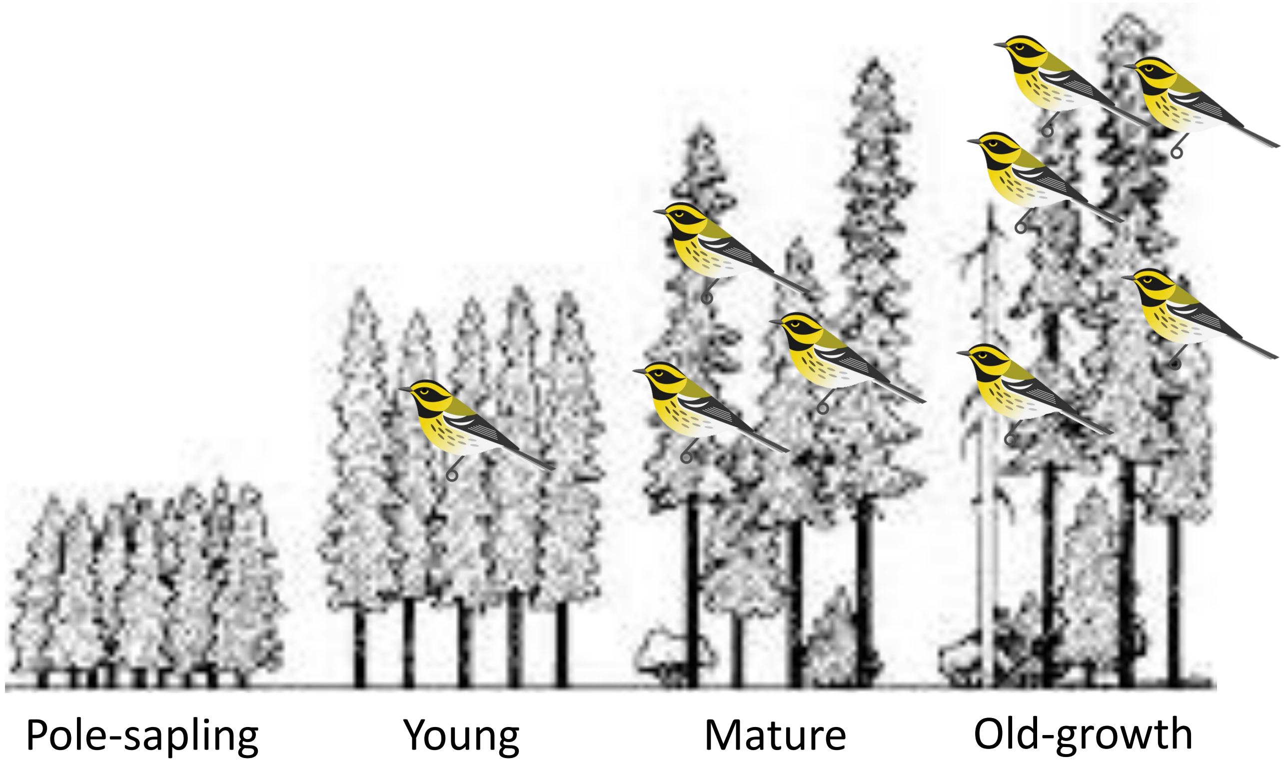
Size



‘Size’ is a human construct (i.e., latent variable)

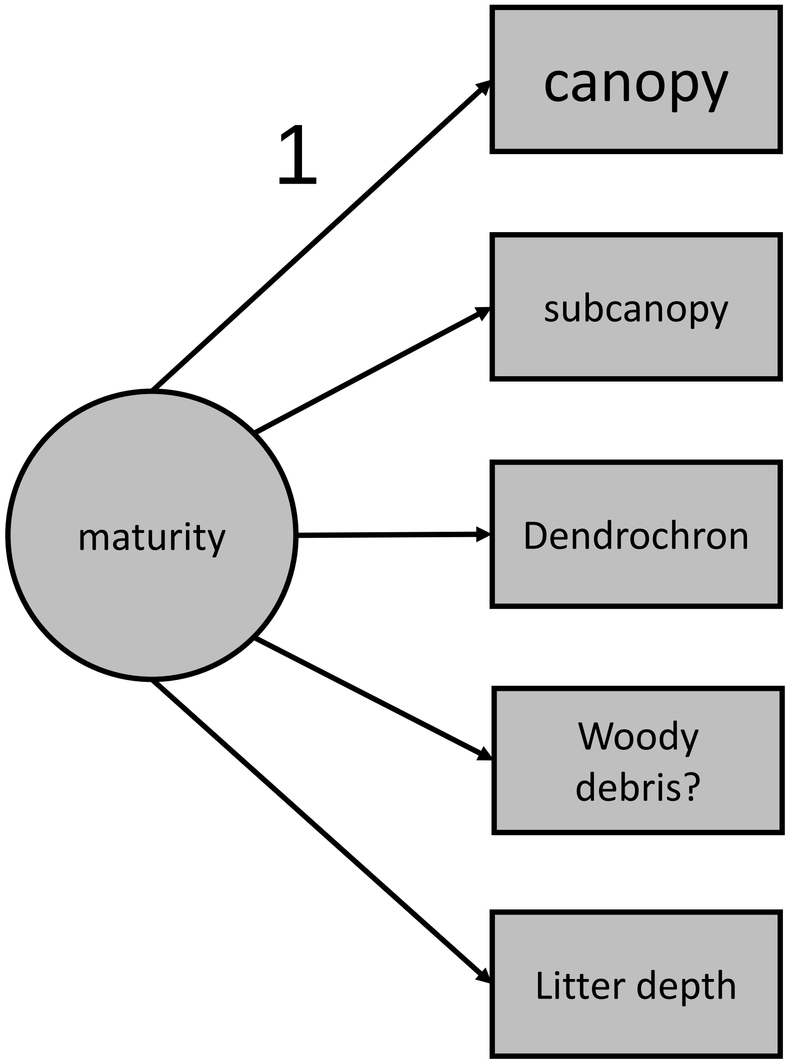
Forest maturity

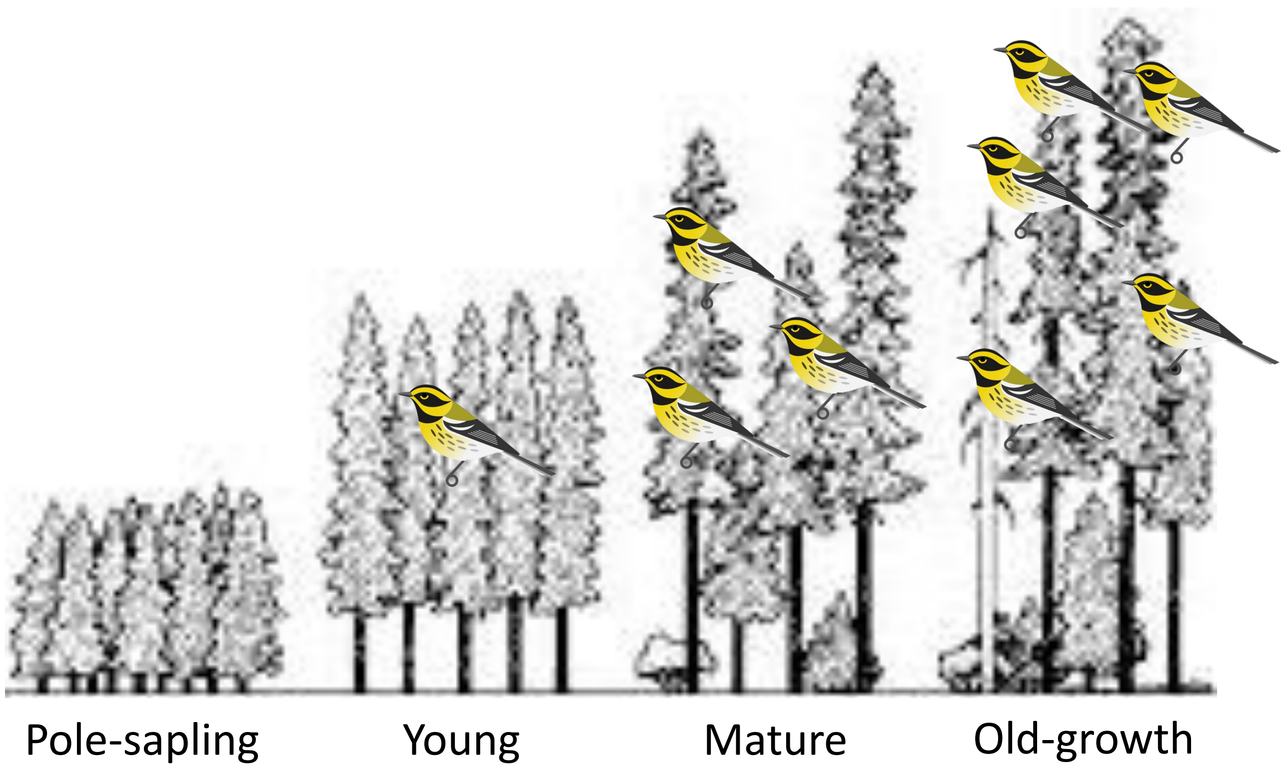




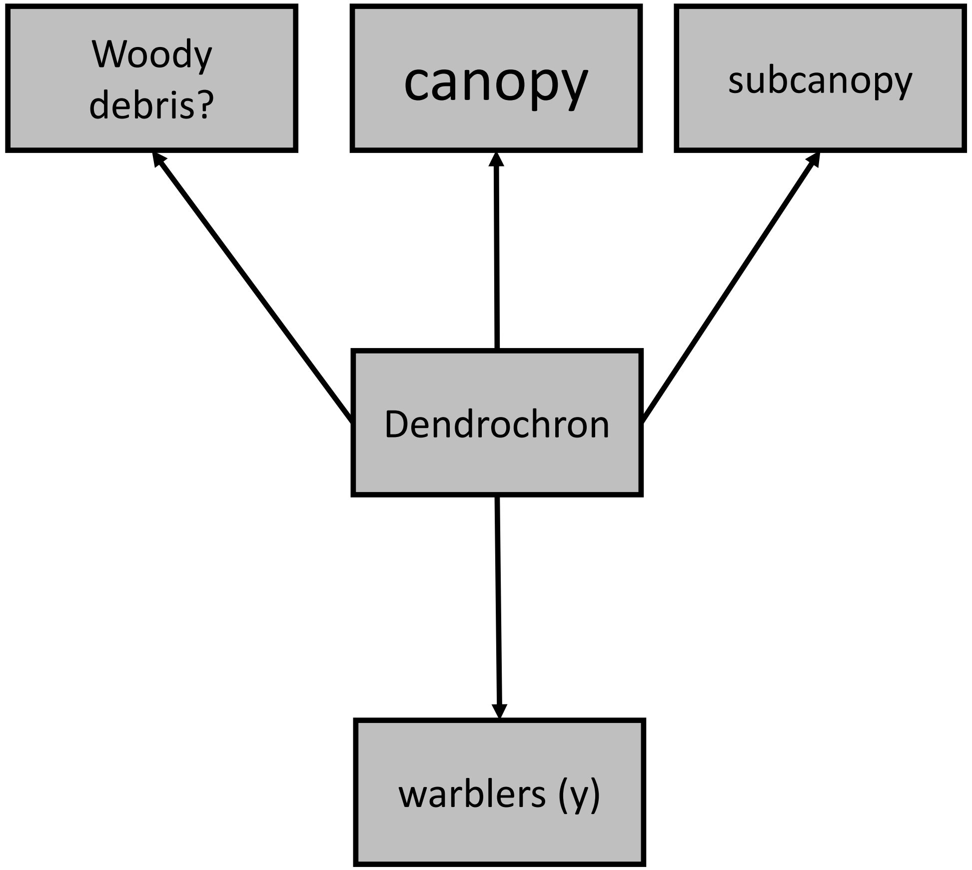
These ‘seral stages’ are human constructs

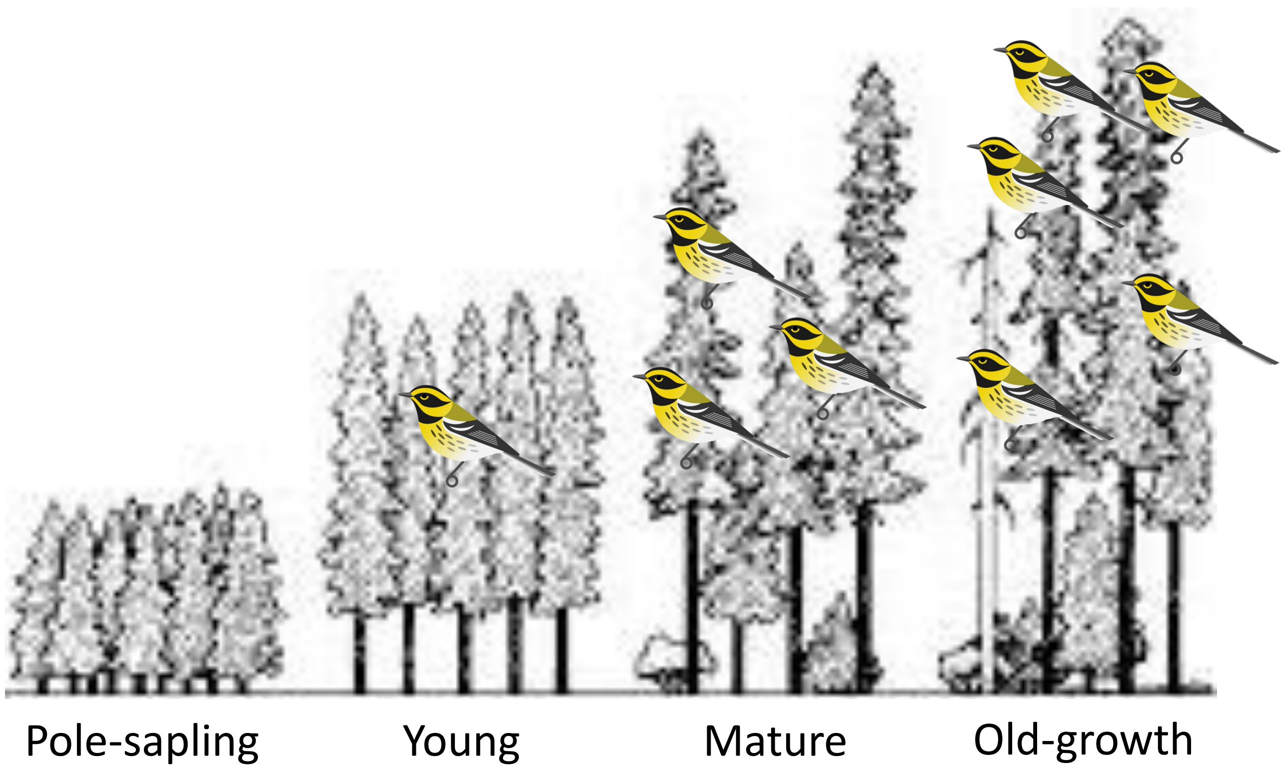
Forest maturity [expanded]





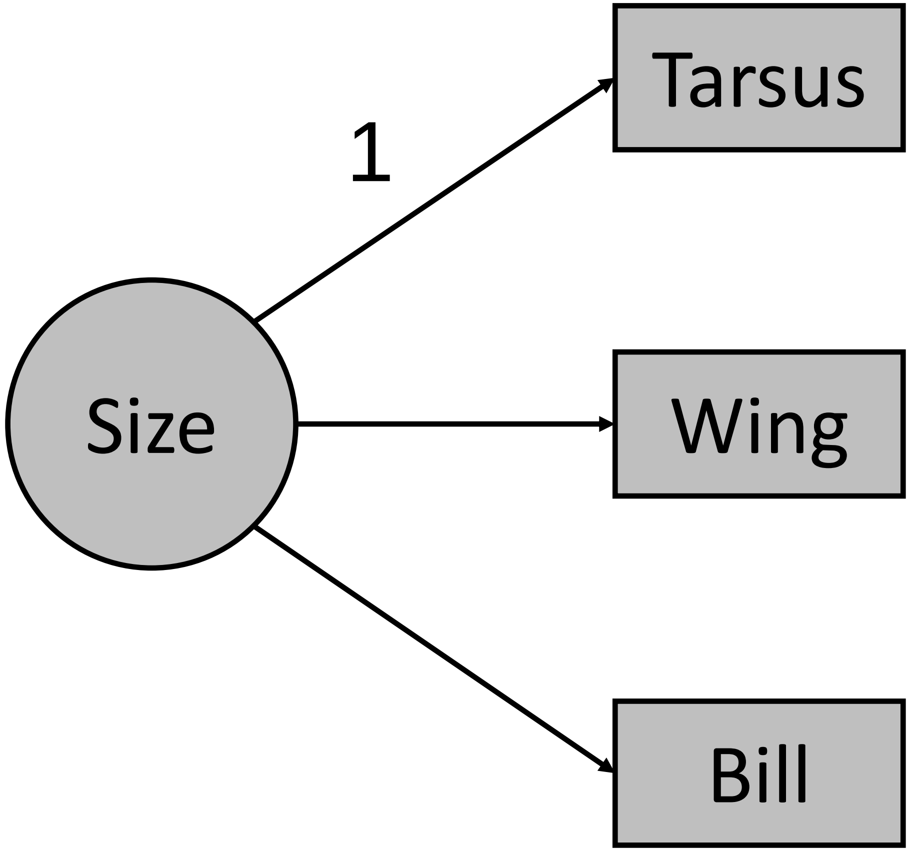
We could structure this differently





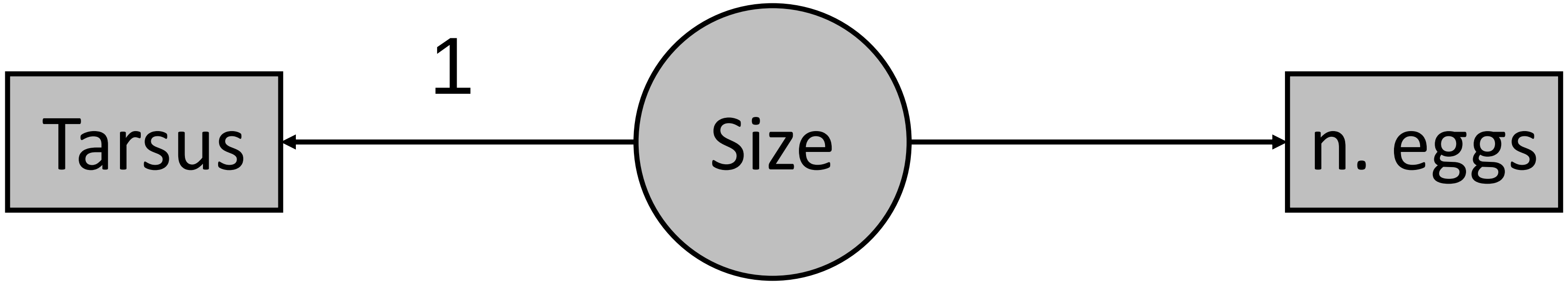
Path analysis: we don't have to use latent variables!

They're just very useful…

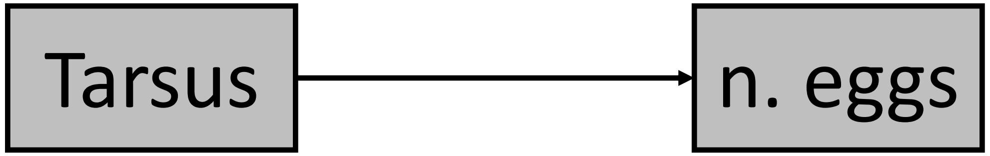


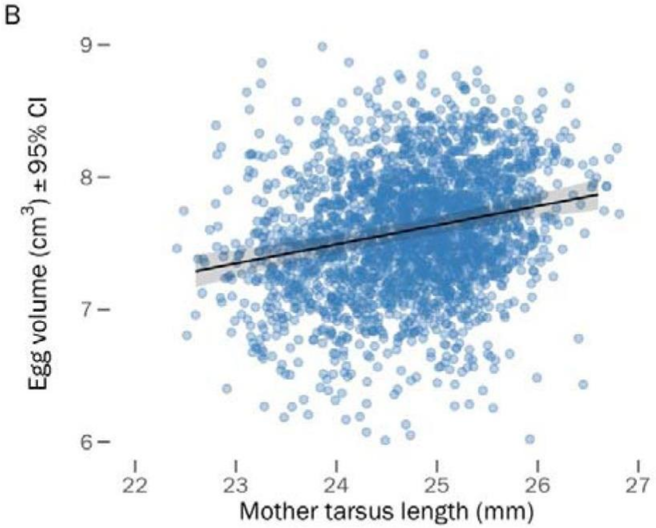
Further, I argue that <u>we use them all the time</u> subconsciously

Subconscious model



Actual model





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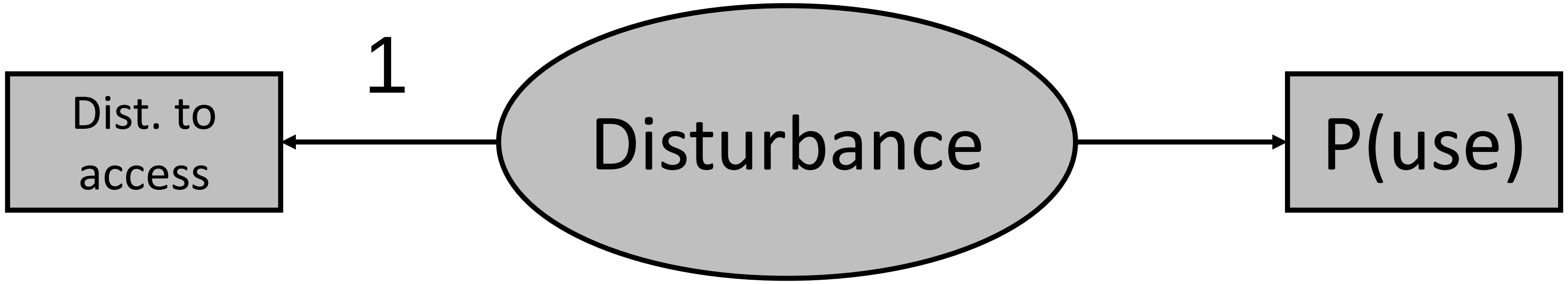
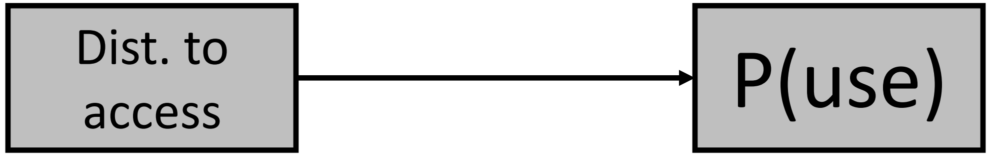
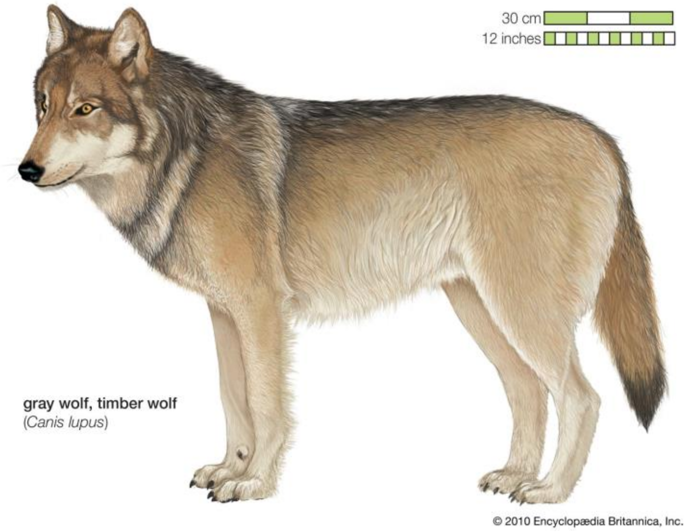


Actual model





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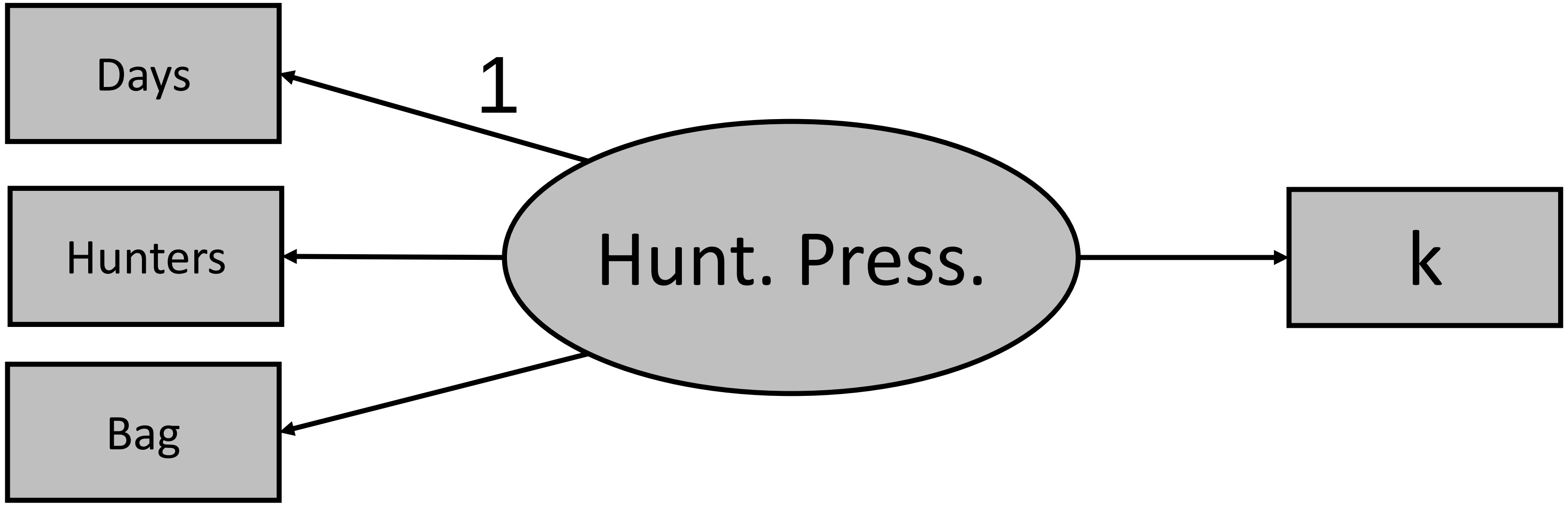
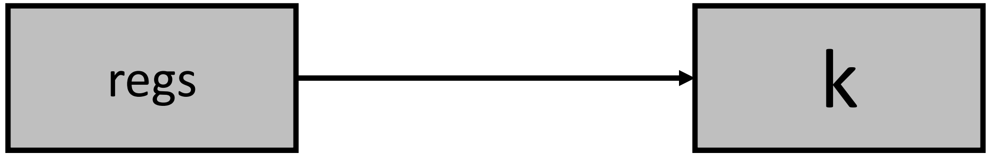
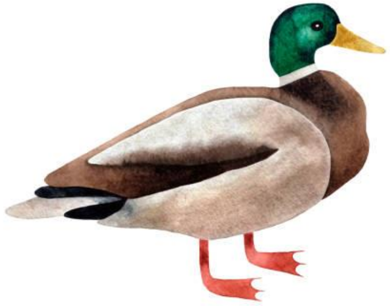
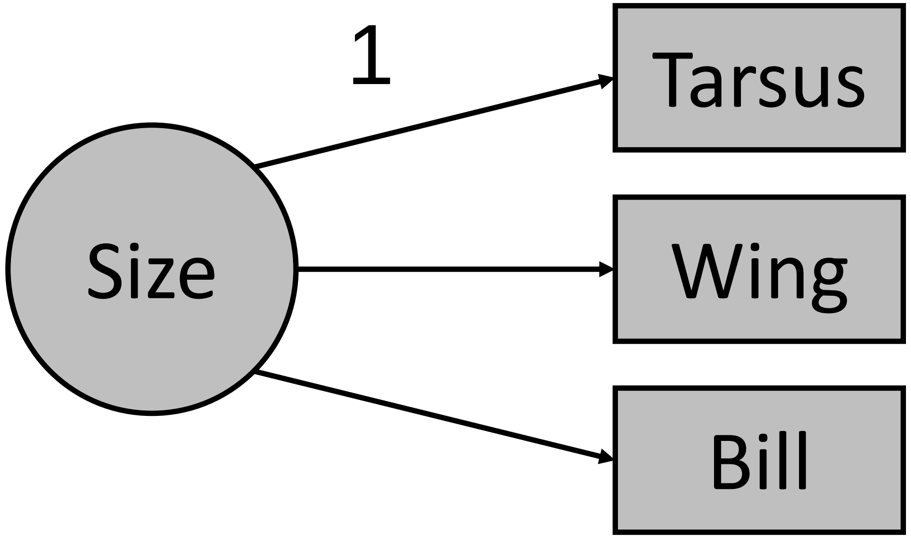


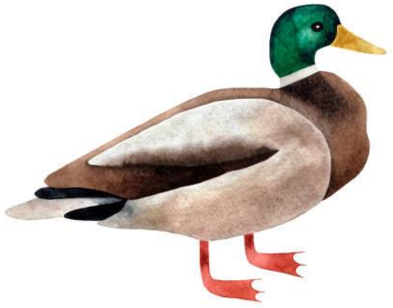
Actual model

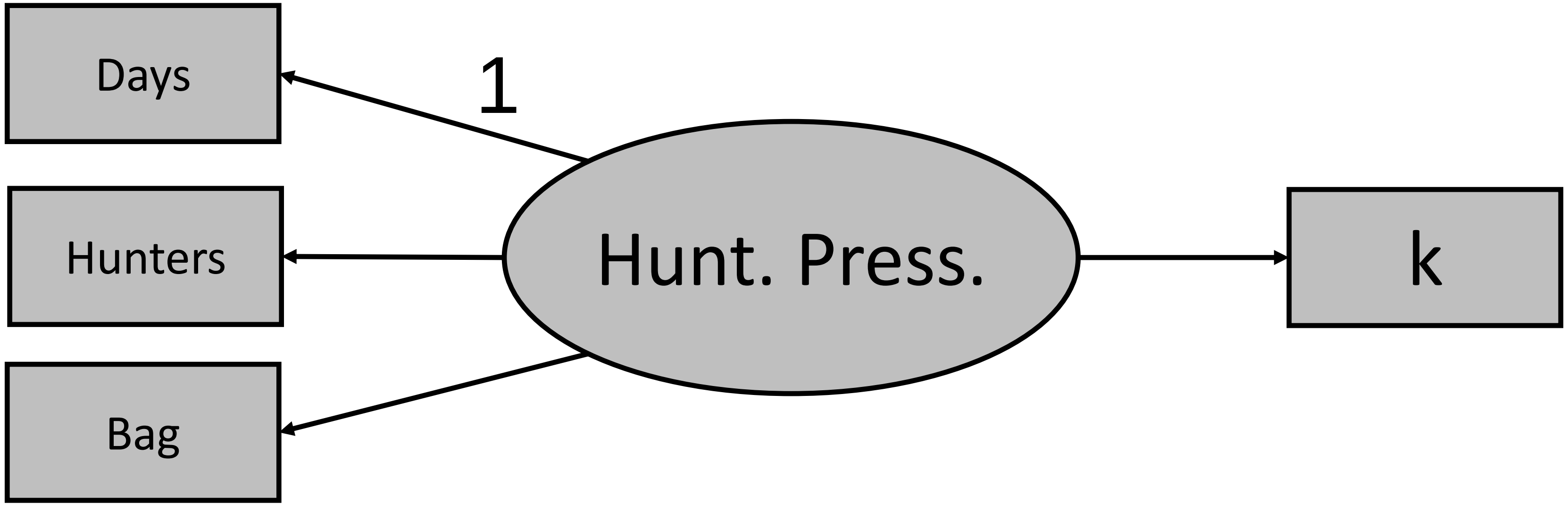




Arrow directionality





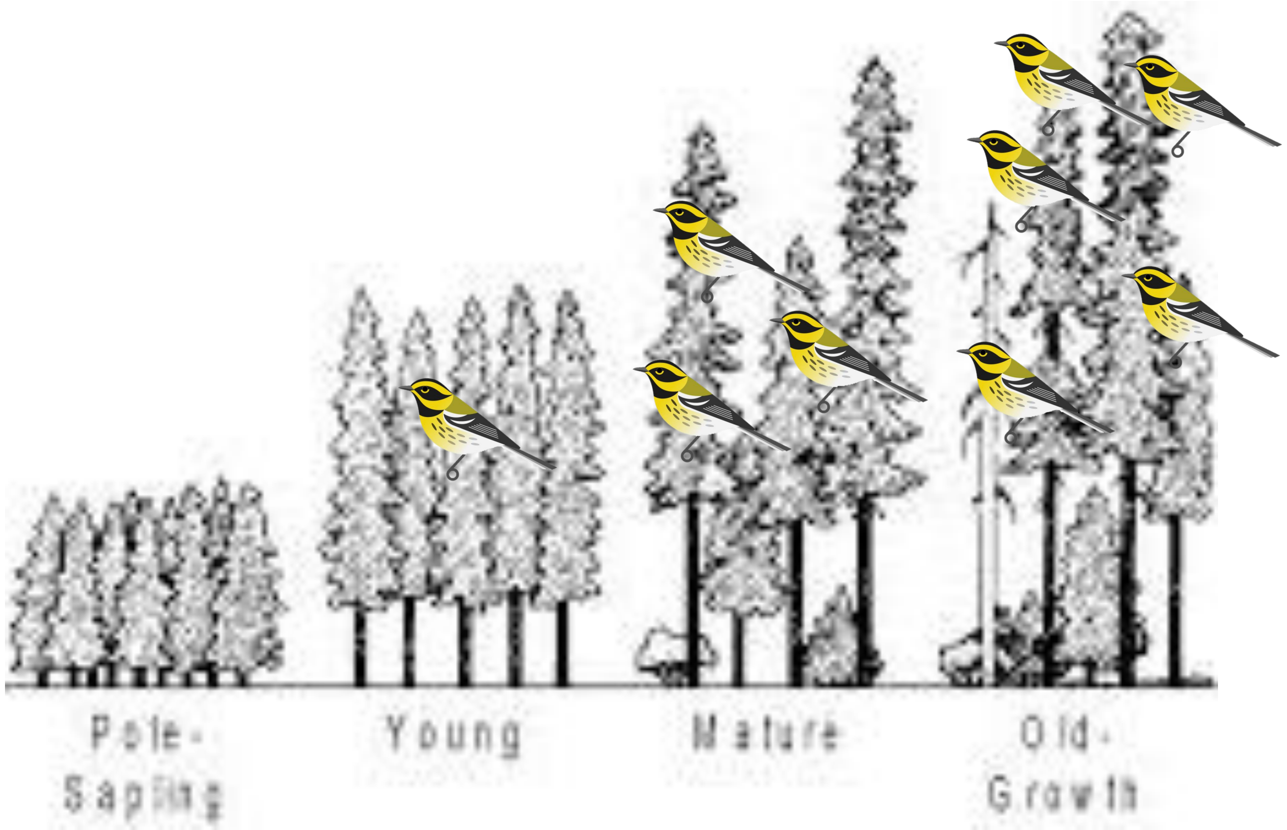


1. **Latent variables are intuitive, and we already informally use them all the time**

2. **They can be used to link multiple measurements of similar processes**

We can make some assumptions about our latent variable(s)

Our first example: forest age as a latent variable



We generally assume they're normally distributed

$$\boldsymbol{m} \sim \text{normal}(\mu, \sigma_m^2)$$

They're really kind of like random effects…

We assume that they are zero-centered b/c they're human constructs

i.e., what should the scale of forest maturity be?

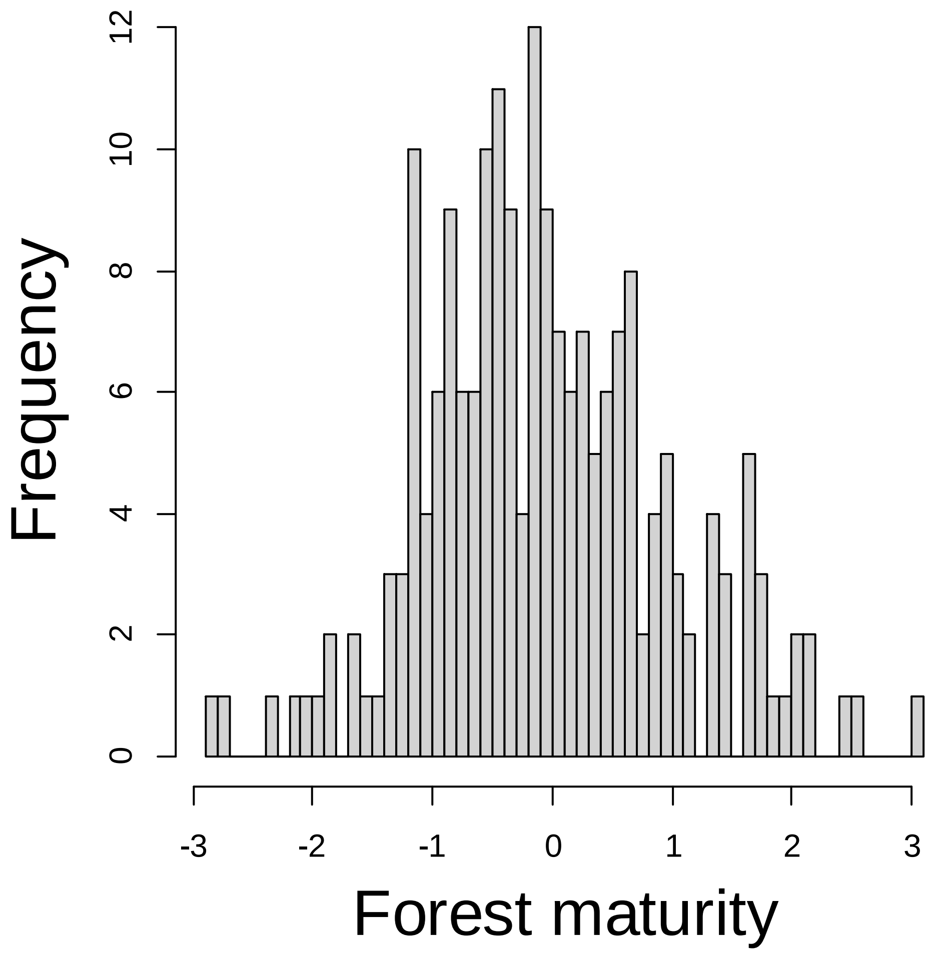
$$\boldsymbol{m} \sim \mathrm{normal}(0, \sigma_m^2)$$

Assigning an intercept would be entirely subjective, plus the math is easier if $\mu = 0$

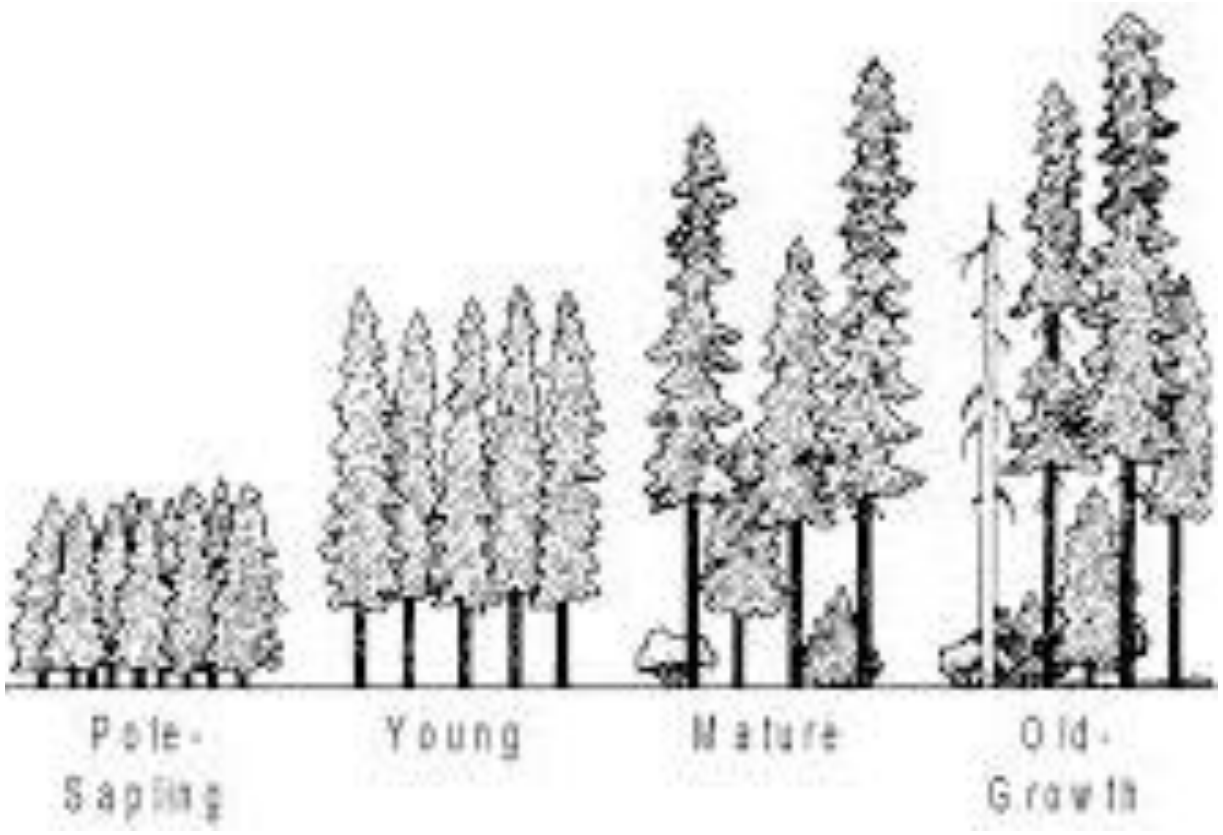
Let’s simulate some data

Data: counts (y) of ‘yellow-footed weeble-wobbles’ at sites with different canopy (c) and sub-canopy (s) heights

Step 1: simulate variation in forest maturity



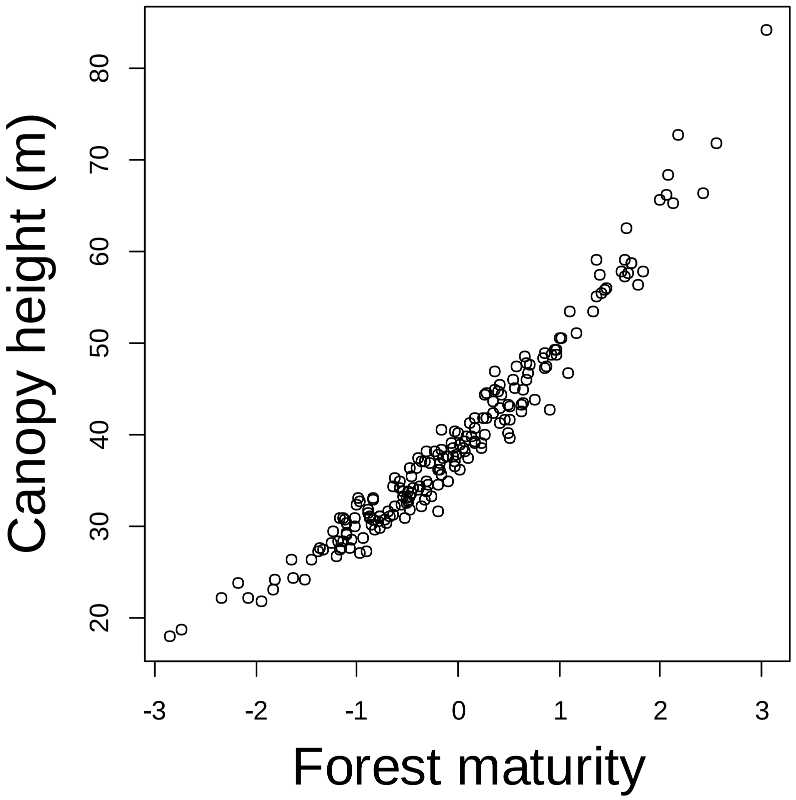
$$\boldsymbol{m} \sim \mathrm{normal}(0, 1)$$

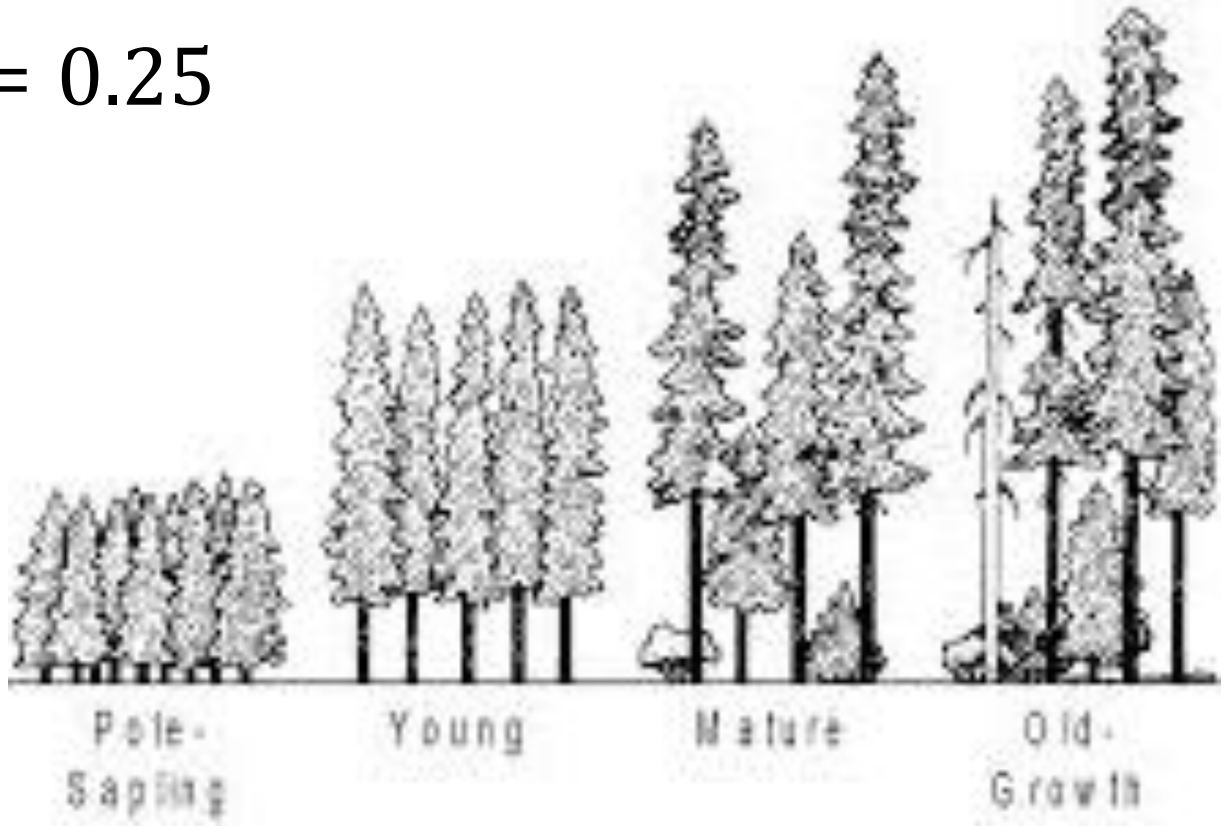


Step 2: simulate variation in canopy height (c)

$$\boldsymbol{c} \sim \text{lognormal}(\alpha_1 + \beta_1 \boldsymbol{m}, \sigma_c = 0.05)$$

$$\alpha_1 = 3.65$$
$$\beta_1 = 0.25$$

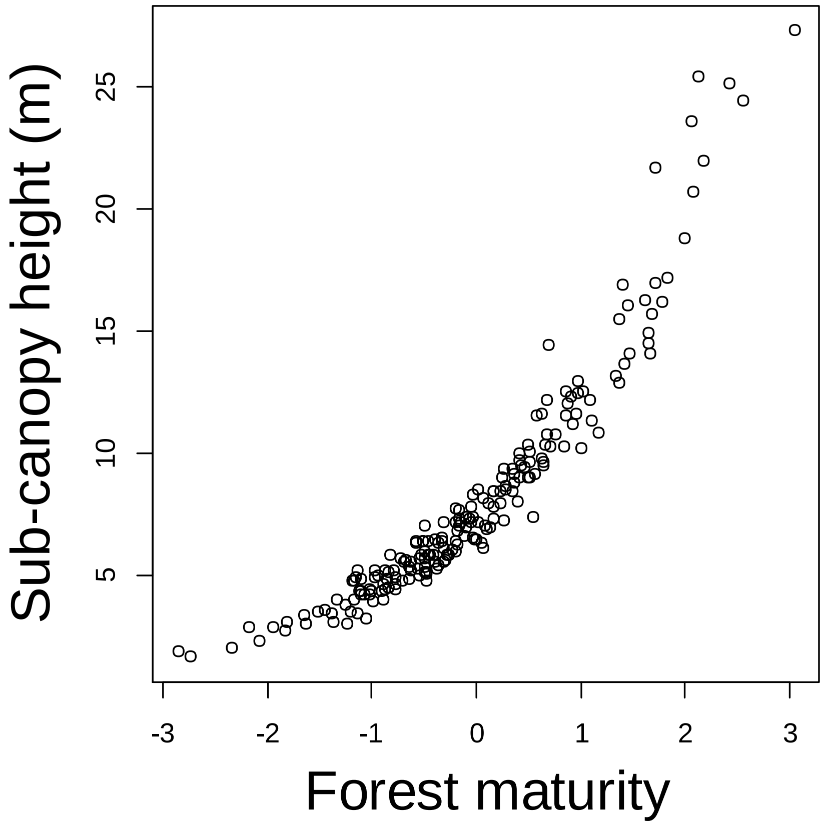


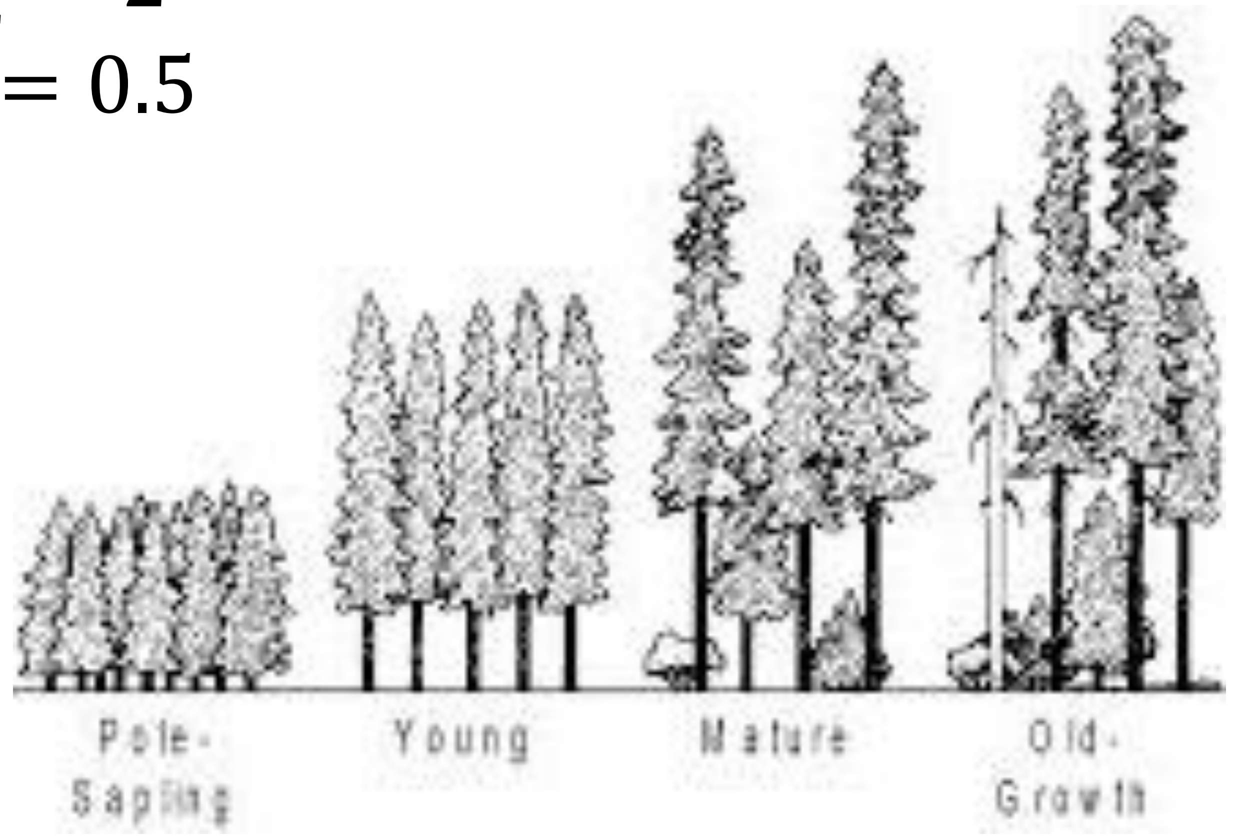


Step 3: simulate variation in sub-canopy height (s)

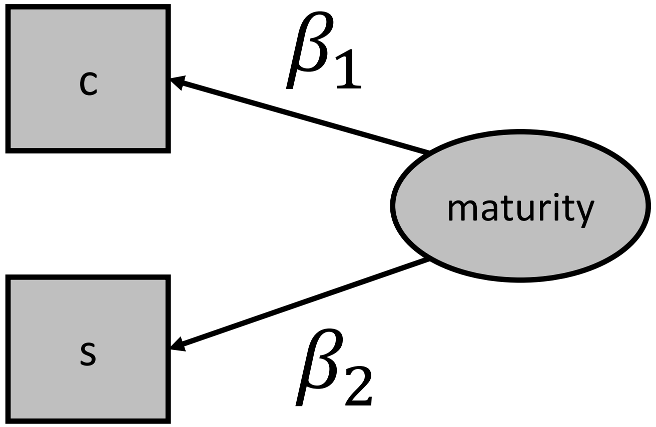
$$\boldsymbol{s} \sim \mathrm{lognormal}(\alpha_2 + \beta_2 \boldsymbol{m}, \sigma_s = 0.05)$$

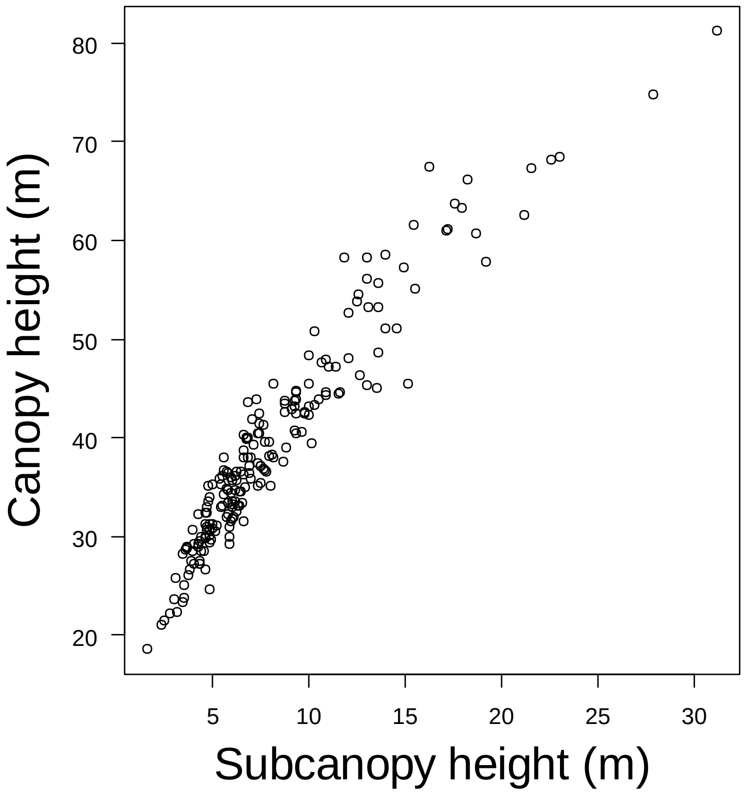
$$\alpha_2 = 2$$
$$\beta_2 = 0.5$$



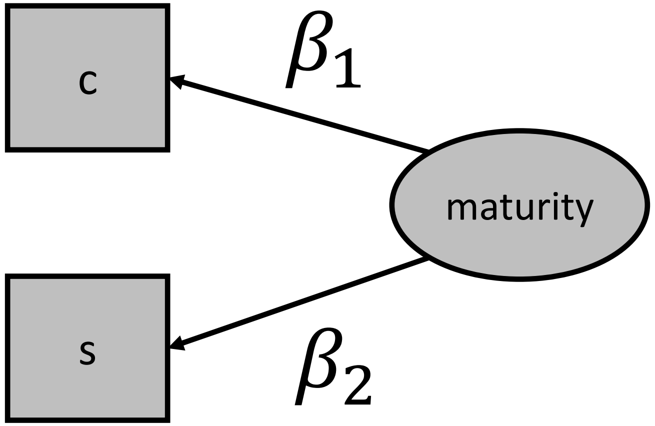


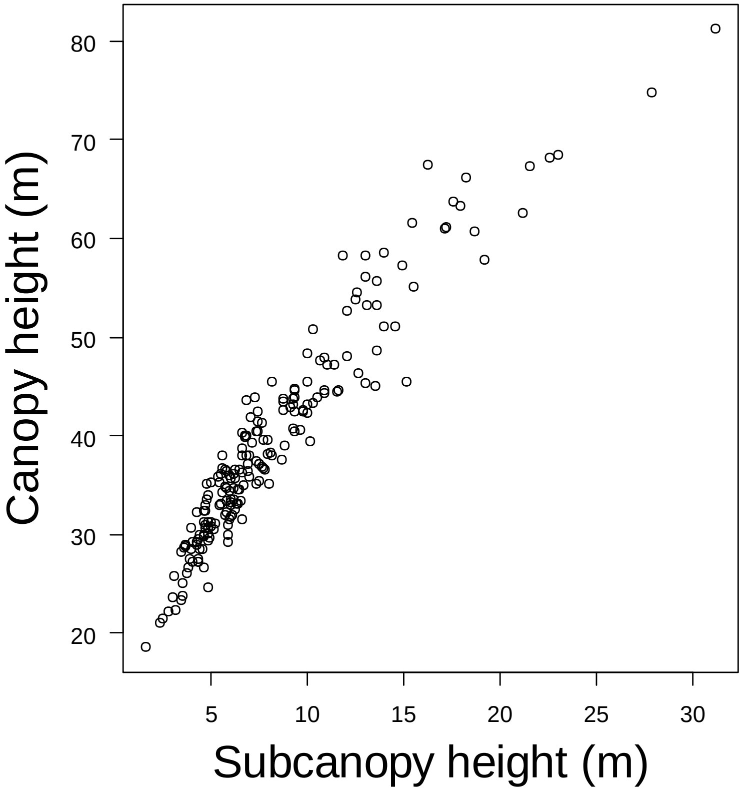
The hypothesis: older forests will have greater canopy heights and greater sub-canopy heights





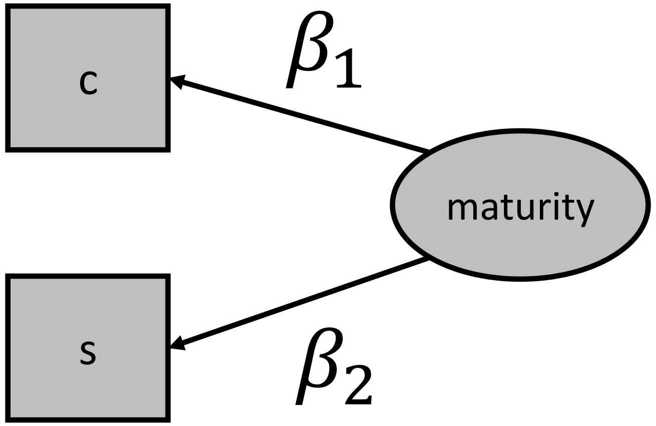
<u>The most important caveat</u>: if things aren't collinear, then you can't assign them to a latent variable





A note on drawing graphs

Squares or rectangles represent measured variables



This oval represents a latent variable

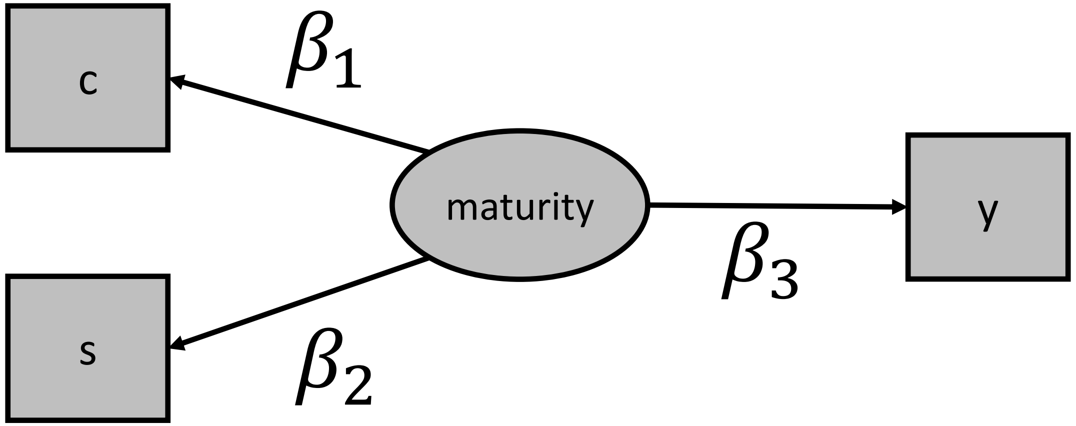
Arrows represent paths (linear models). The direction of the arrow indicates how to parameterize the relationship

Step 4: simulate variation in warbler counts (y)

$$\boldsymbol{y} \sim \text{Poisson}(e^{\alpha_3 + \beta_3 \boldsymbol{m}})$$

$$\alpha_3 = 0.5$$

$$\beta_3 = 0.75$$



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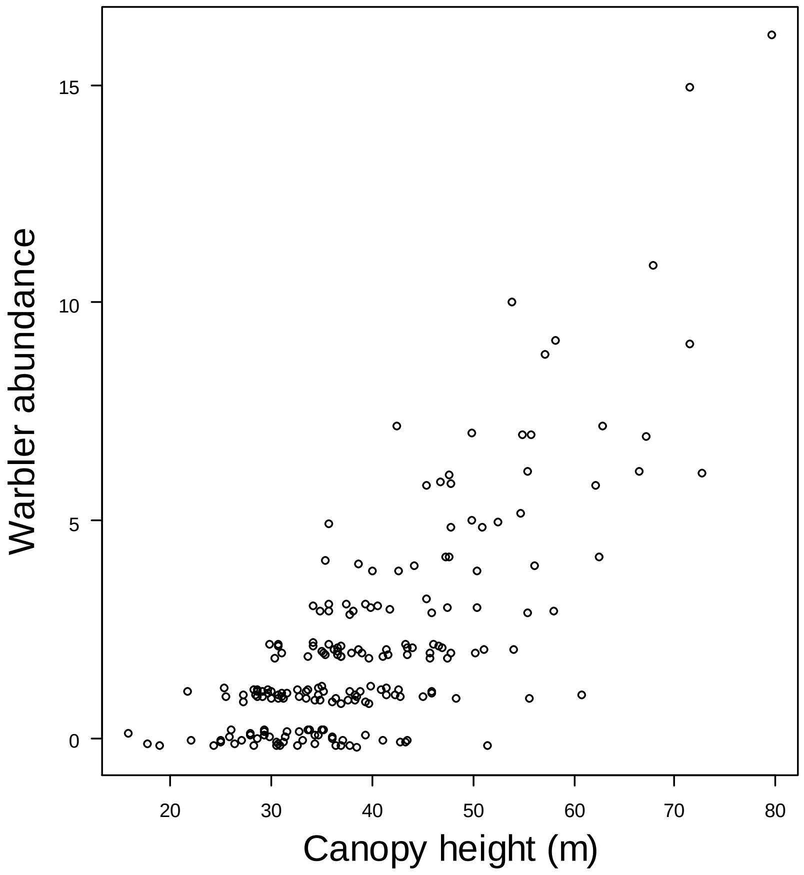
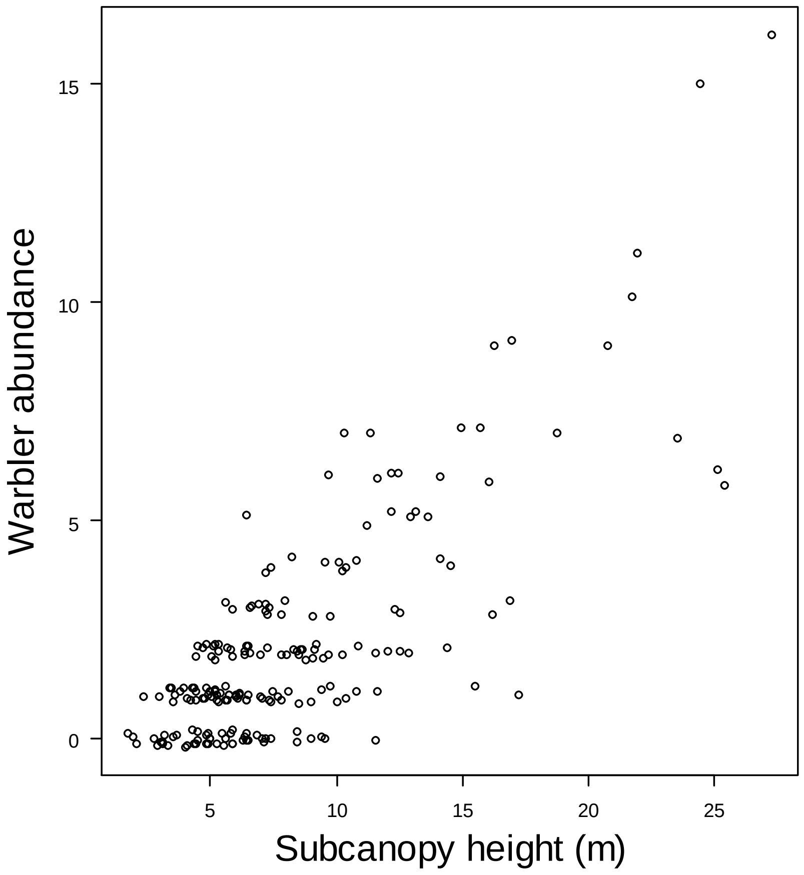
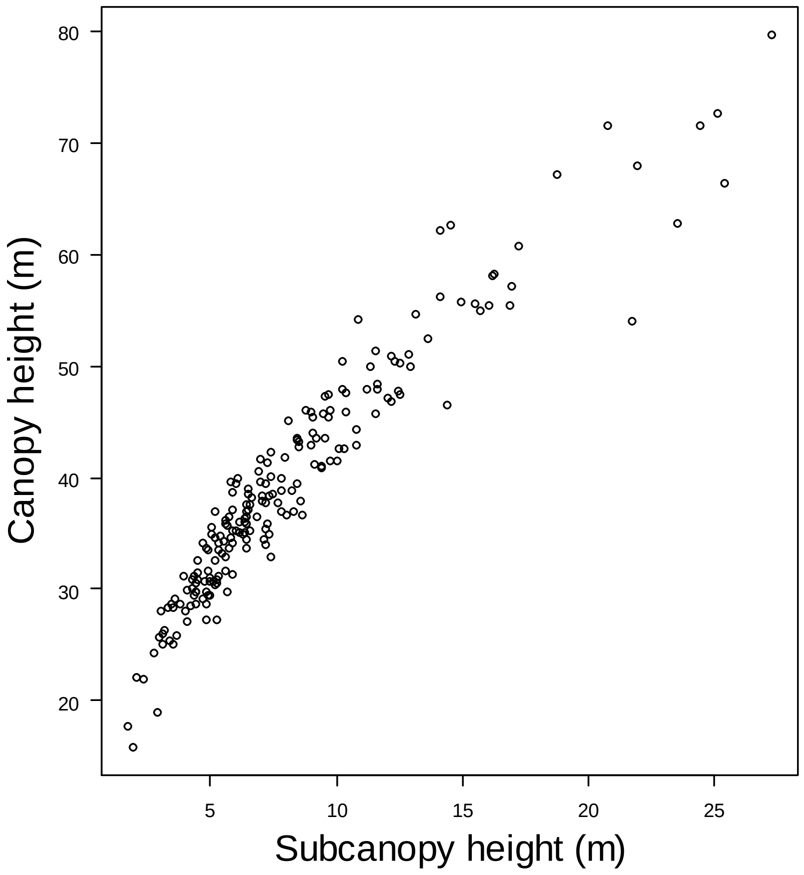






The ecological hypothesis: older forests will have more birds

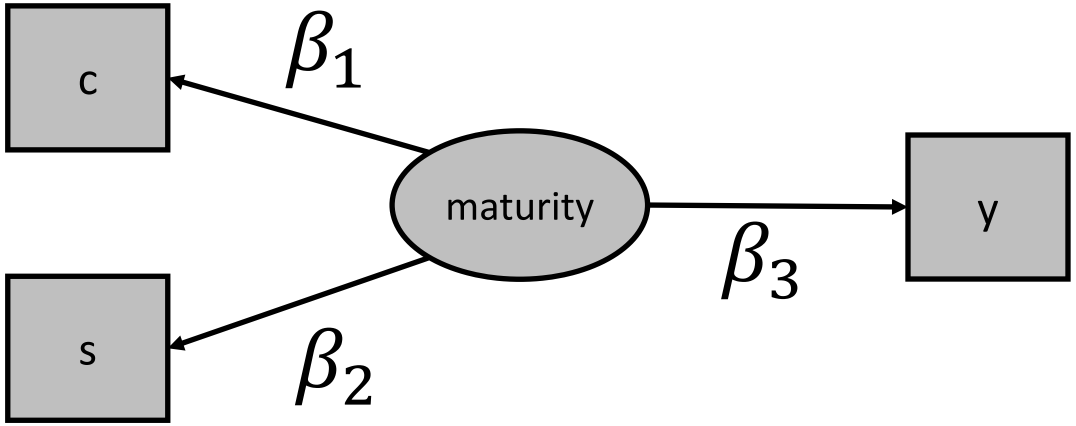
Our (first) model

$$\boldsymbol{m} \sim \mathrm{normal}(0, \sigma_m^2)$$

$$\boldsymbol{c} \sim \mathrm{normal}(\alpha_1 + \beta_1 \boldsymbol{m}, \sigma_c^2)$$

$$\boldsymbol{s} \sim \mathrm{normal}(\alpha_2 + \beta_2 \boldsymbol{m}, \sigma_s^2)$$

$$\boldsymbol{y} \sim \mathrm{normal}(\alpha_3 + \beta_3 \boldsymbol{m}, \sigma_y^2)$$



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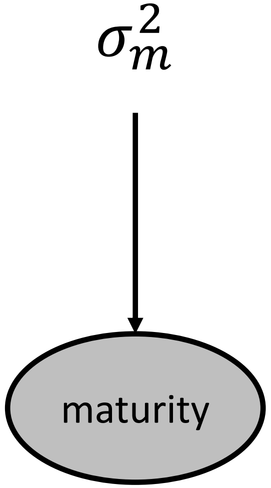

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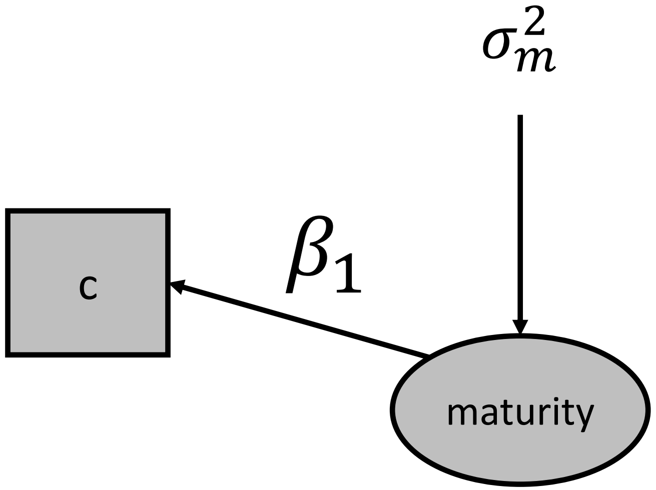

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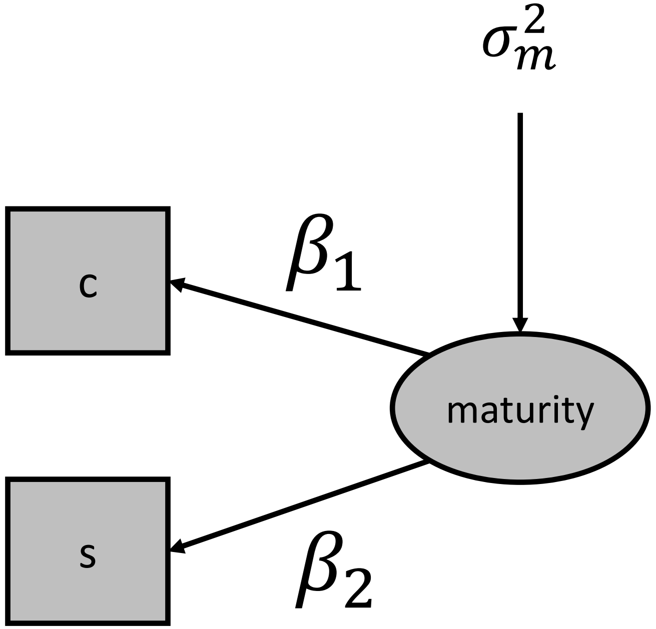

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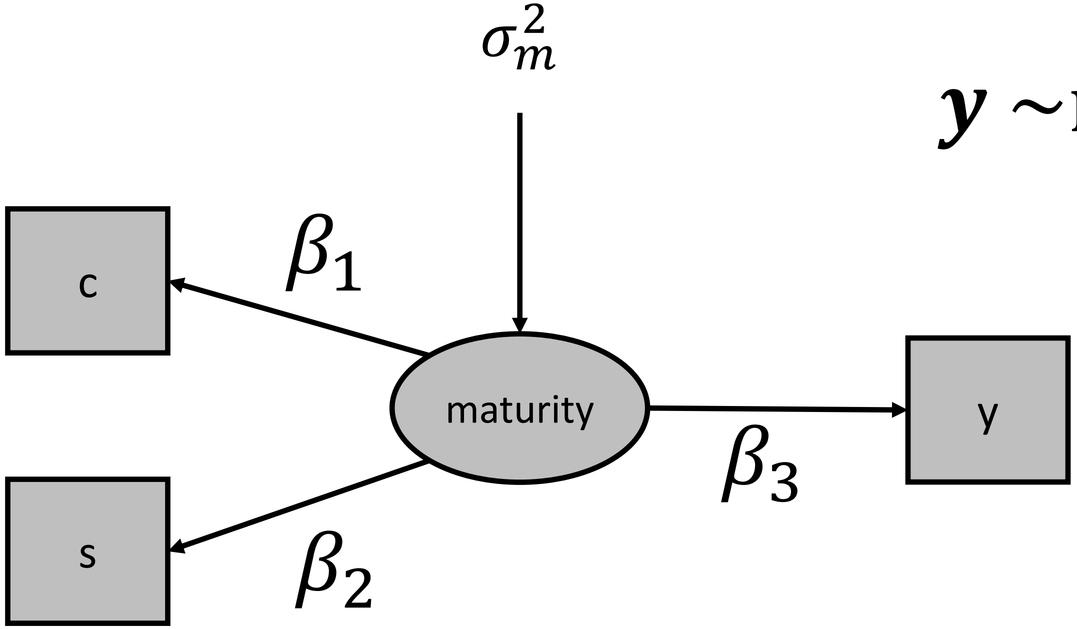
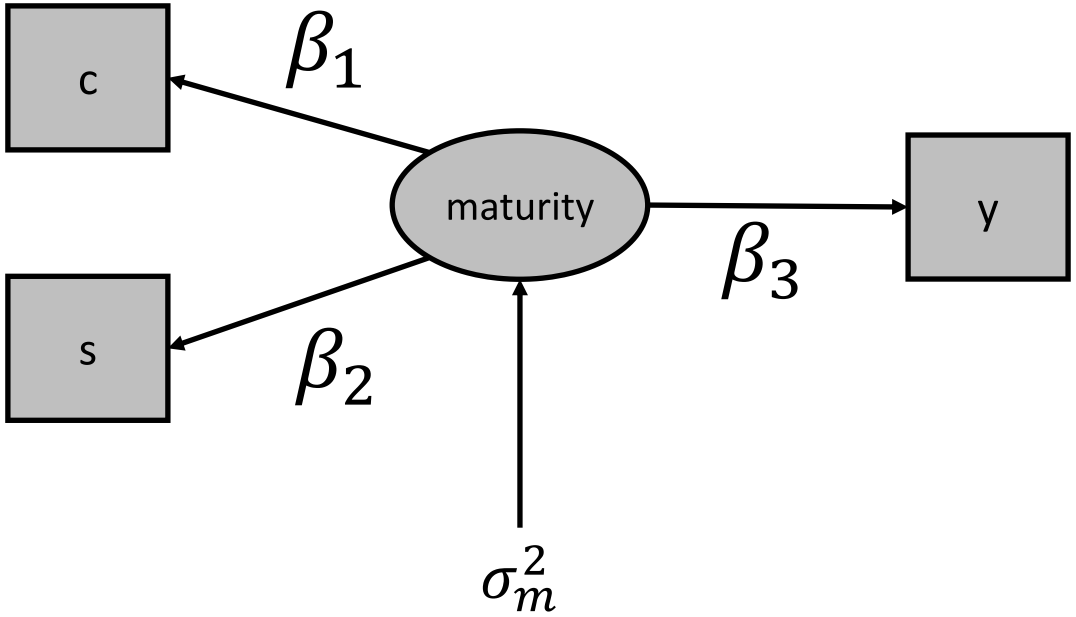
$$\boldsymbol{y} \sim \text{normal}(\alpha_3 + \beta_3 \boldsymbol{m}, \sigma_y^2)$$



There is one very non-intuitive thing to discuss

We must fix a ‘loading’ to 1



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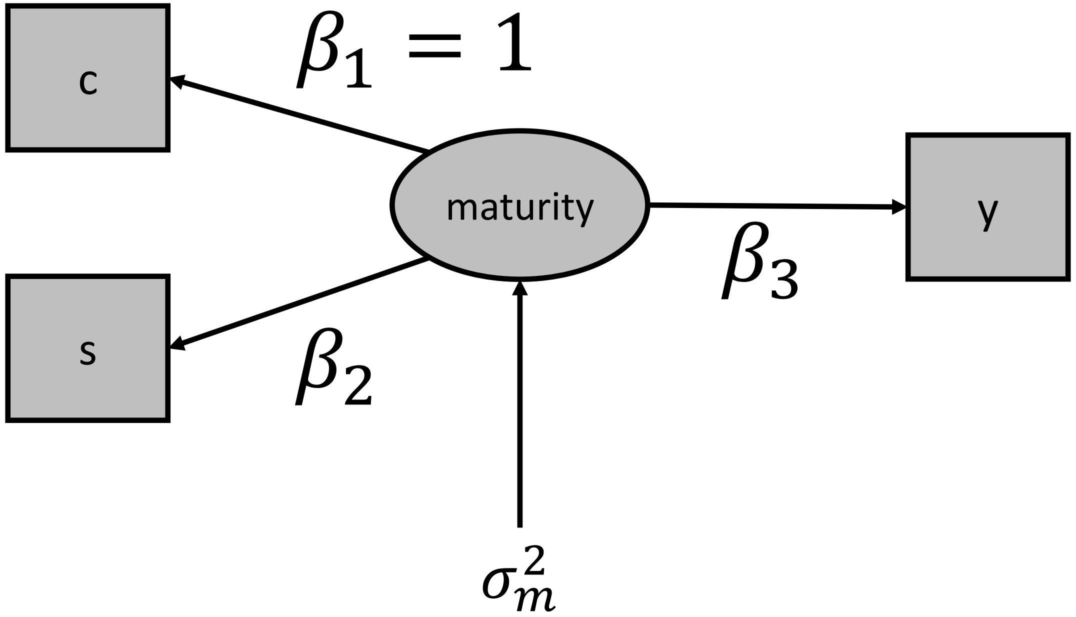

Why?!

Why?!

Well, so the model will be identifiable…

What are the implications of that?

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1. The latent variable will be on the same scale as whatever path we fix = 1

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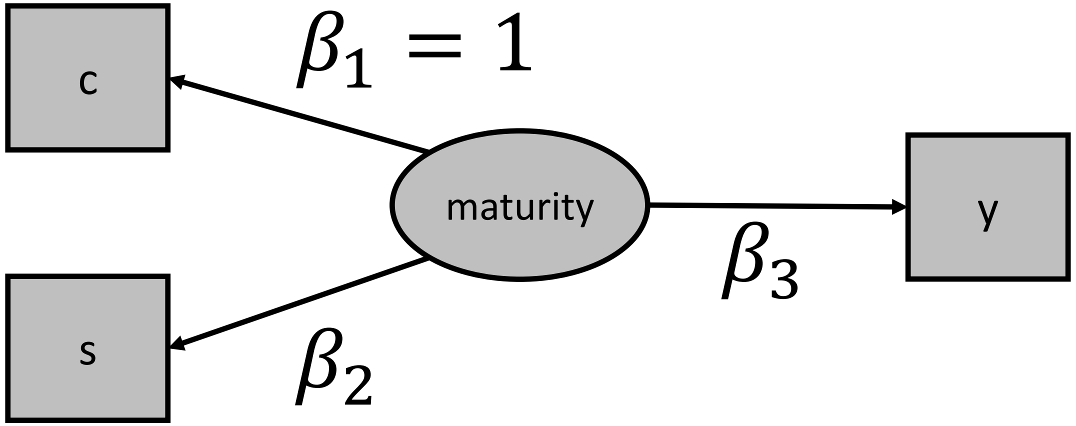
1. The latent variable will be on the same scale as whatever path we fix = 1

2. Our estimates of parameter relationships will be a function of that scale.

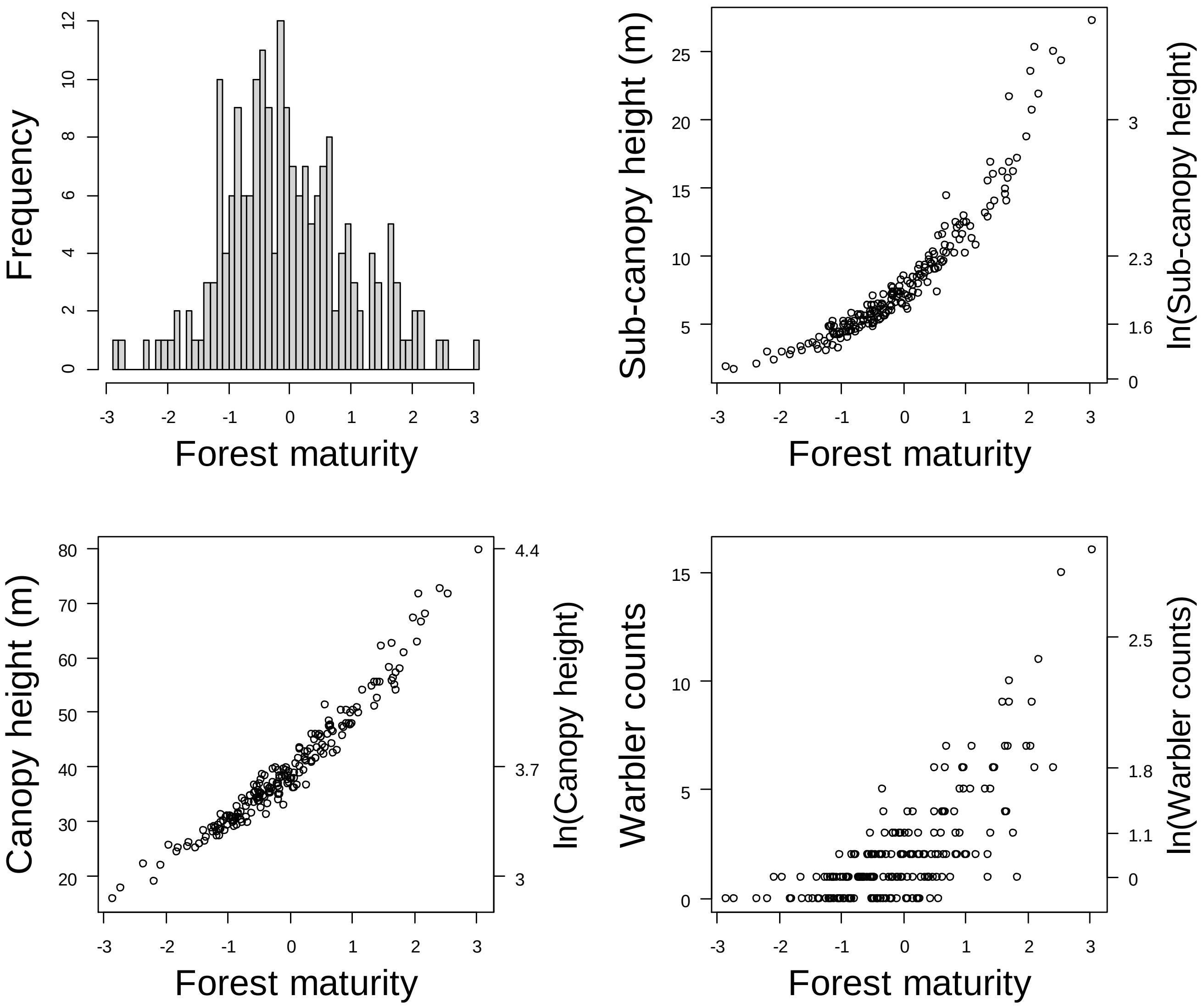
3. That’s it. It won’t change our predictions (i.e., warbler counts)

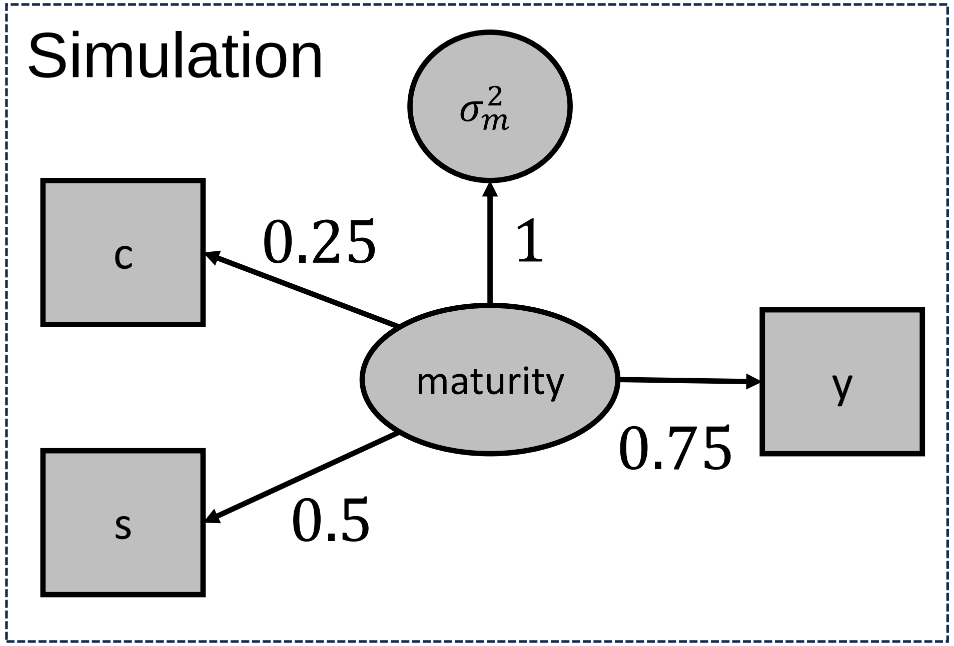
All we're really assuming when we fix that beta is that there is a positive relationship between our latent variable and the measured variable



So, let’s talk about this ‘fixing a loading to 1’ thing

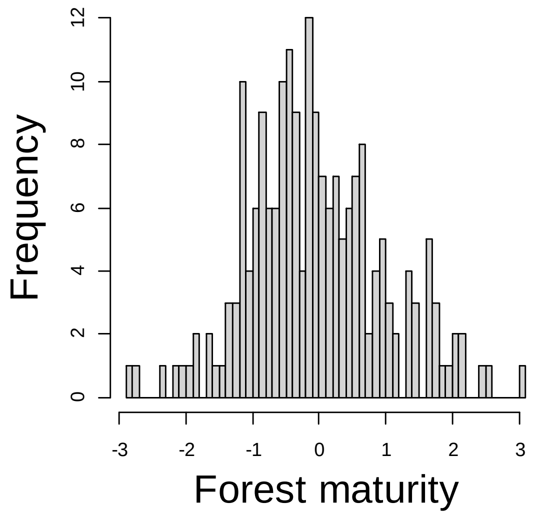
Let’s simulate some data

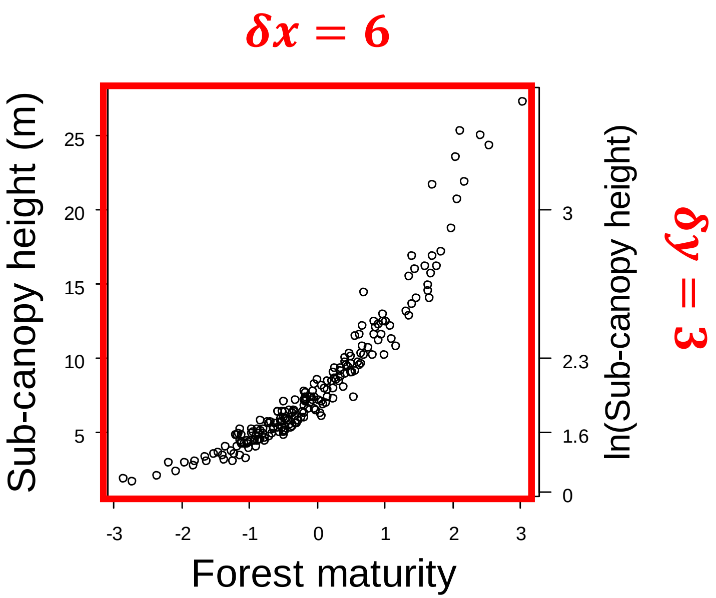


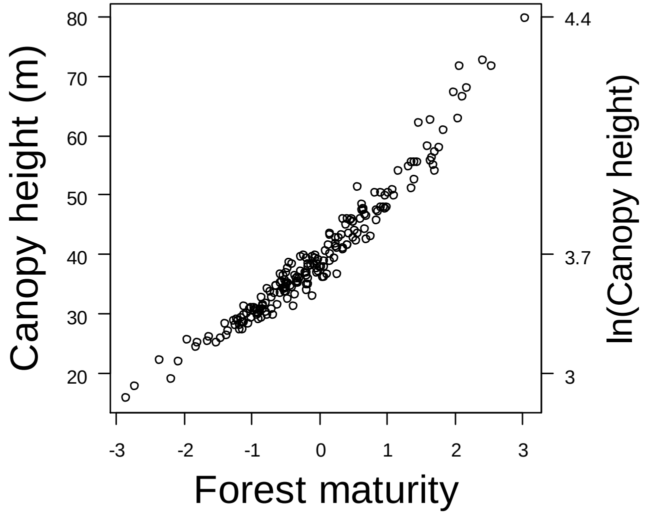


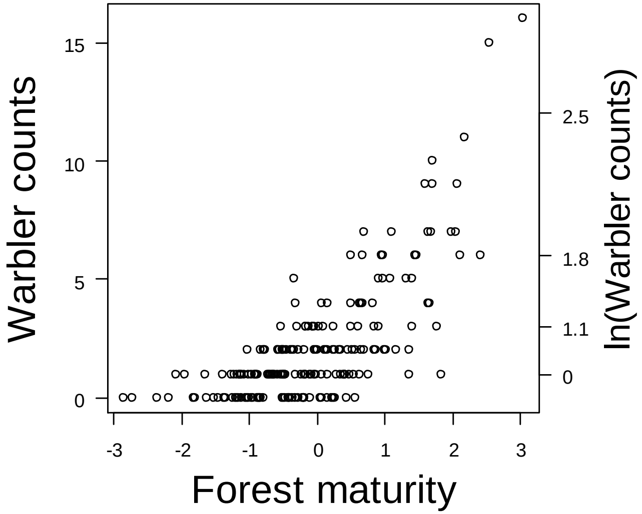
$$\delta y = \delta x \beta$$

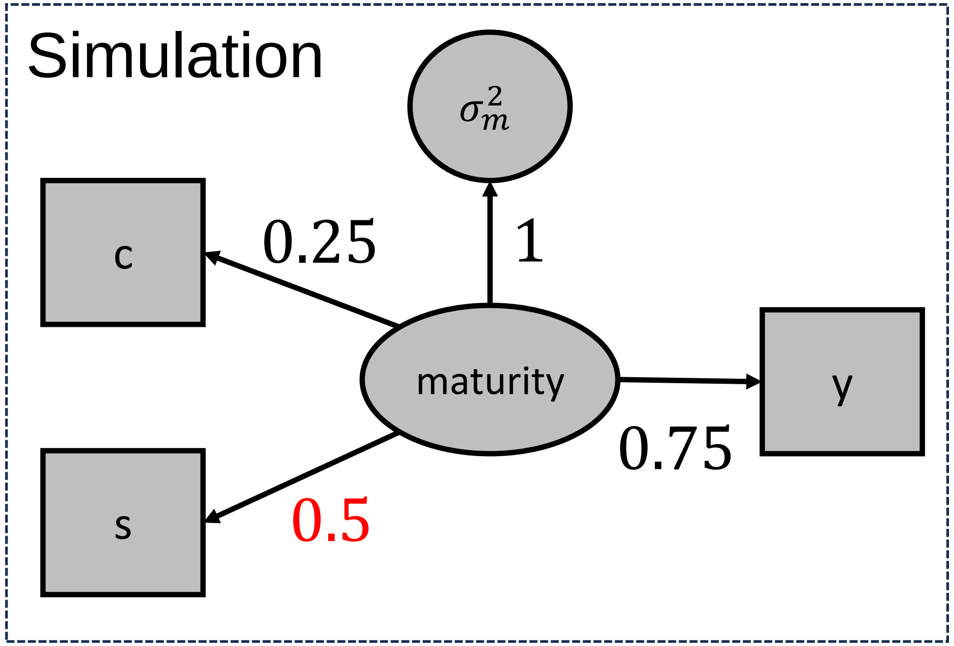
Clarifying a loading…







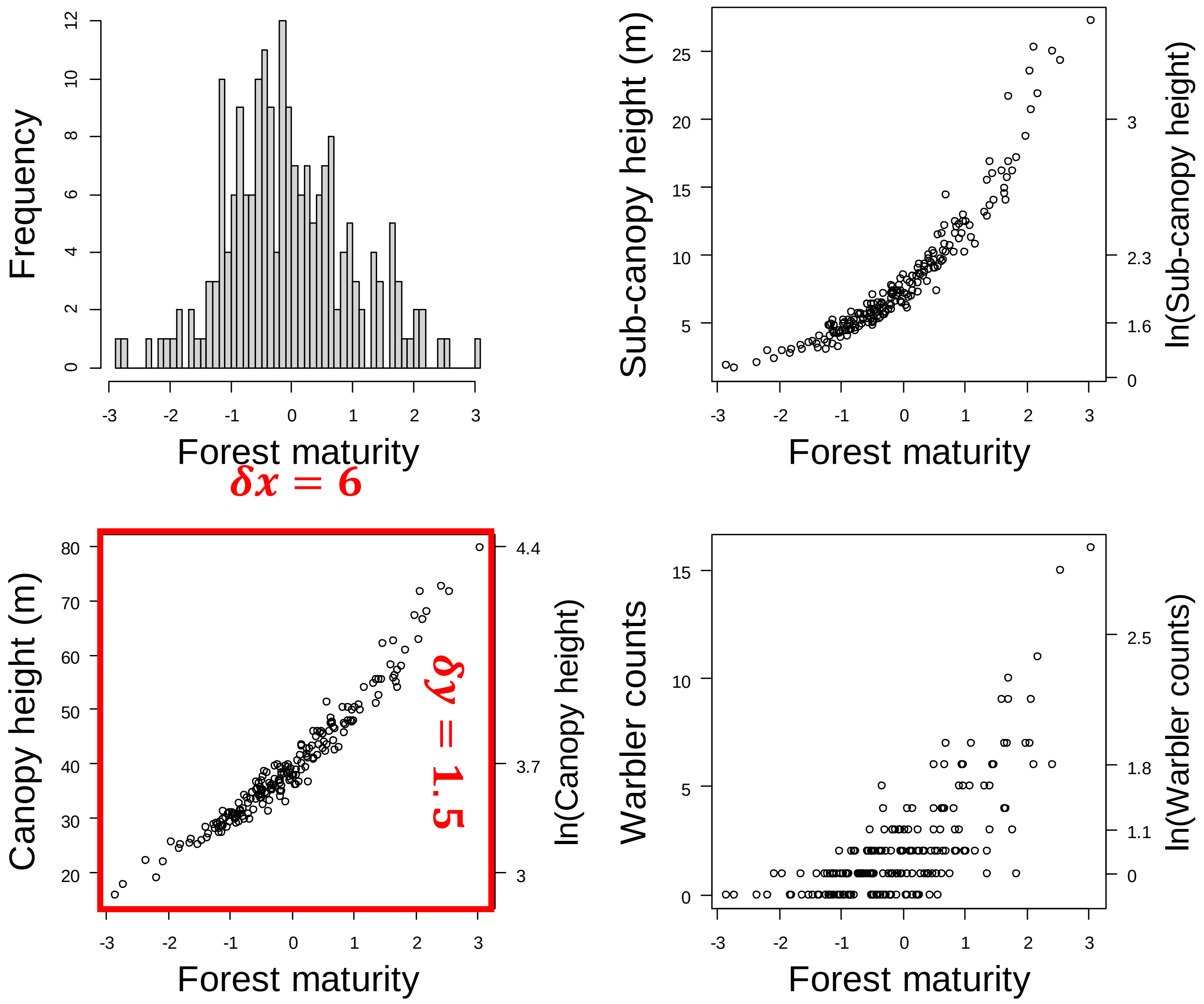


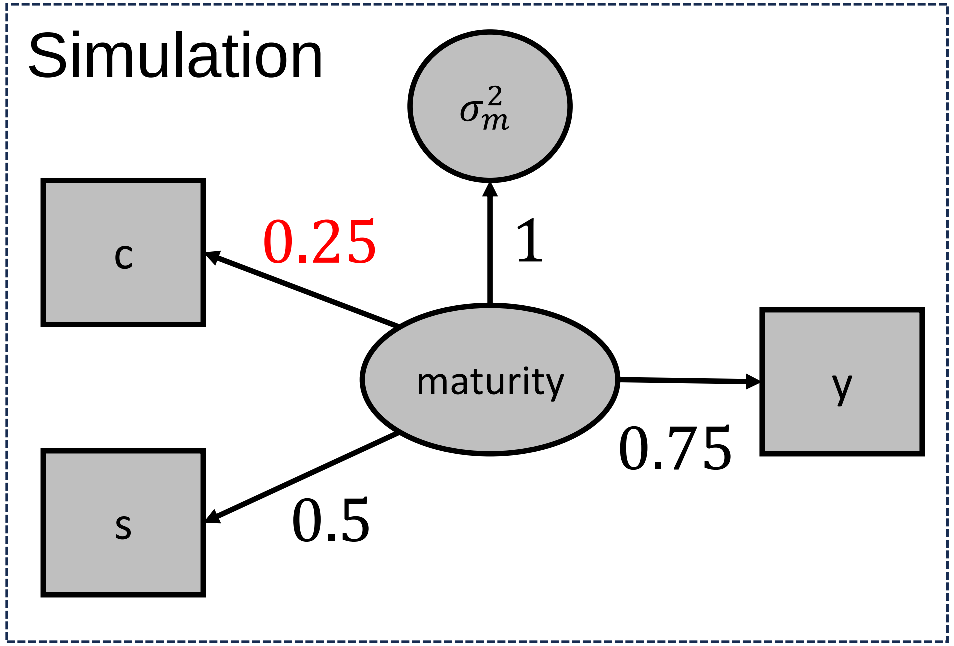


$$\delta y = \delta x \beta$$

$$\beta = 0.5$$

Clarifying a loading…

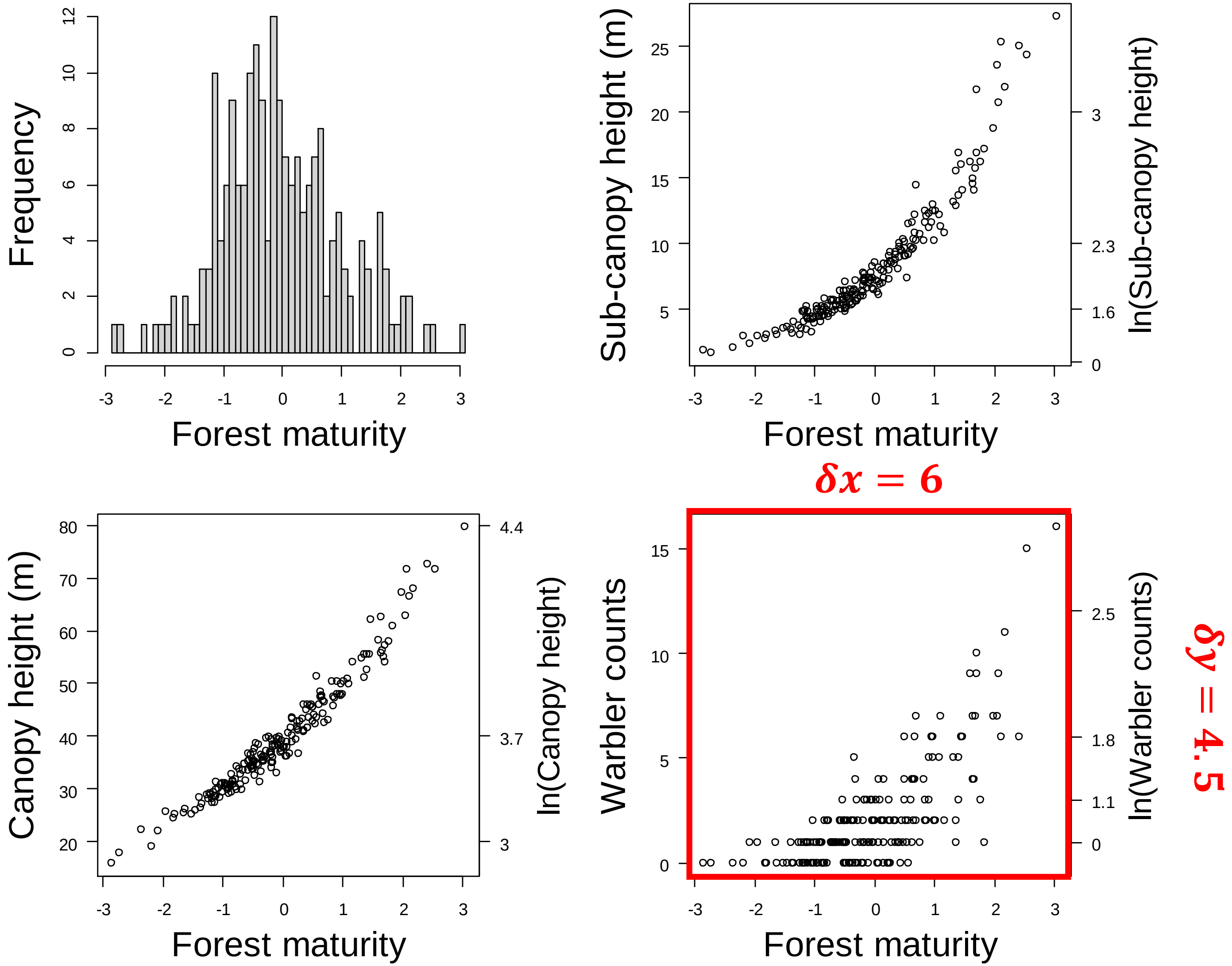


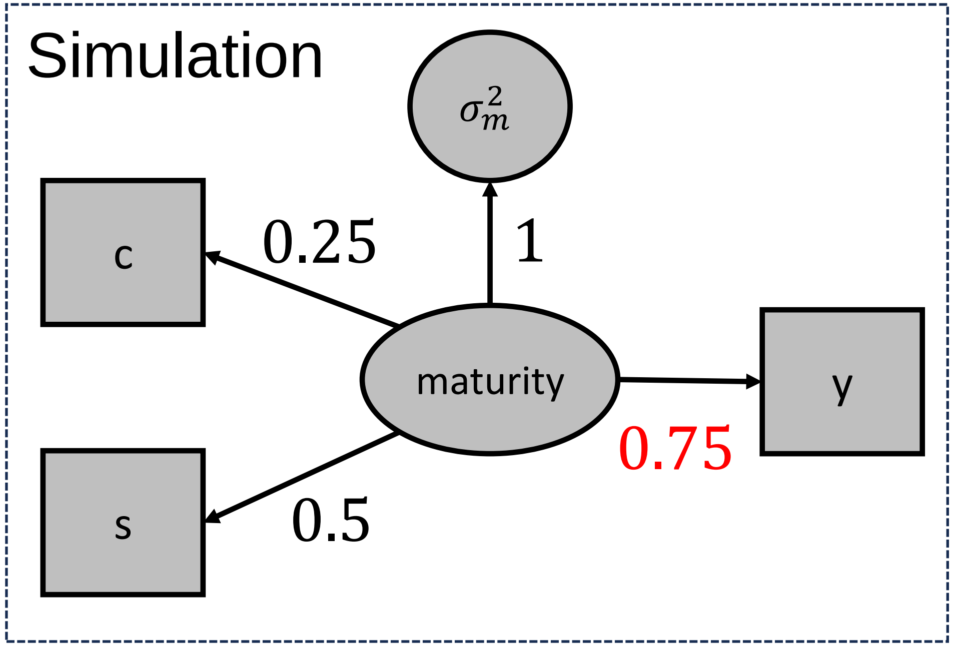


$$\delta y = \delta x \beta$$

$$\beta = 0.25$$

Clarifying a loading…





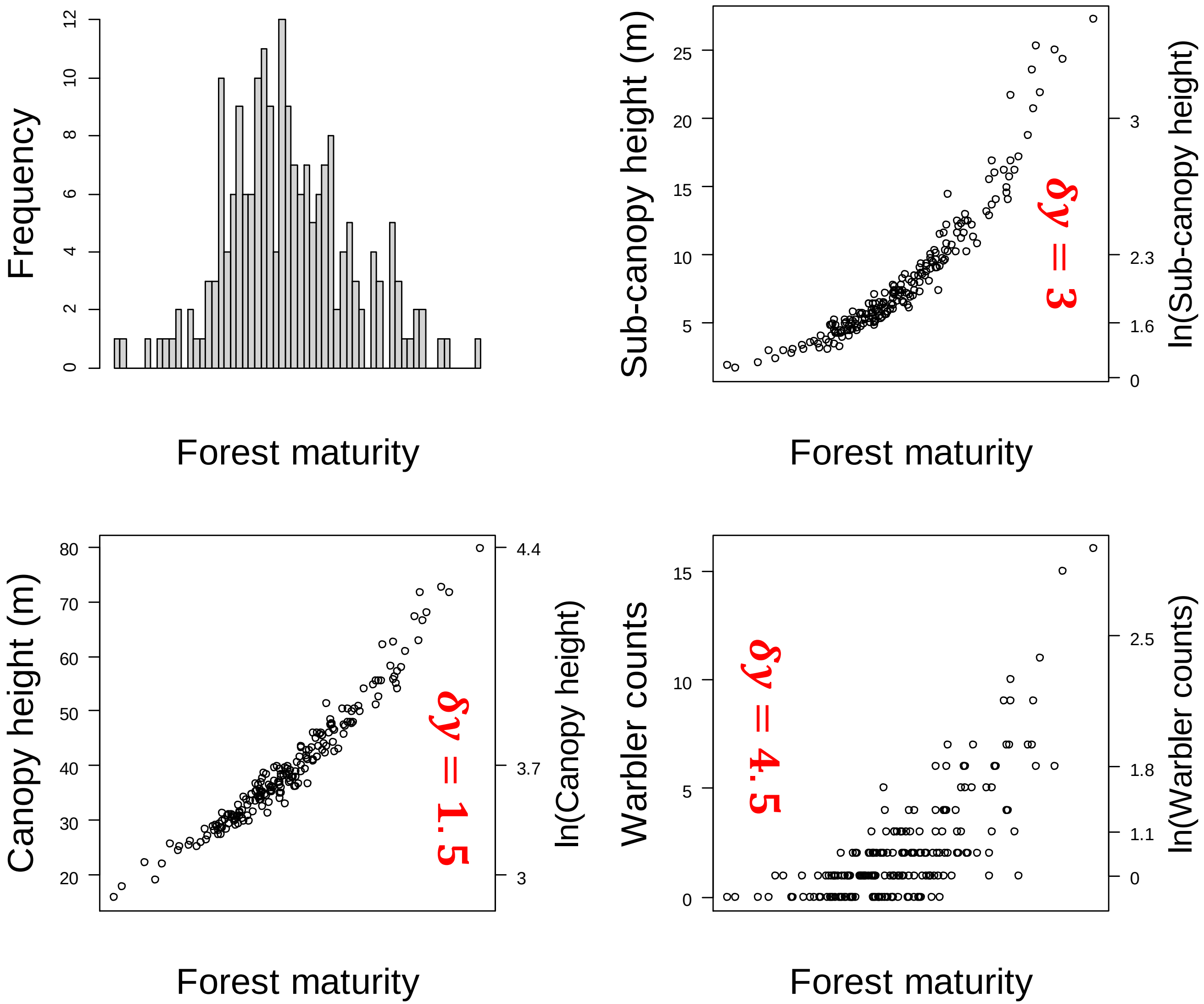
$$\delta y = \delta x \beta$$

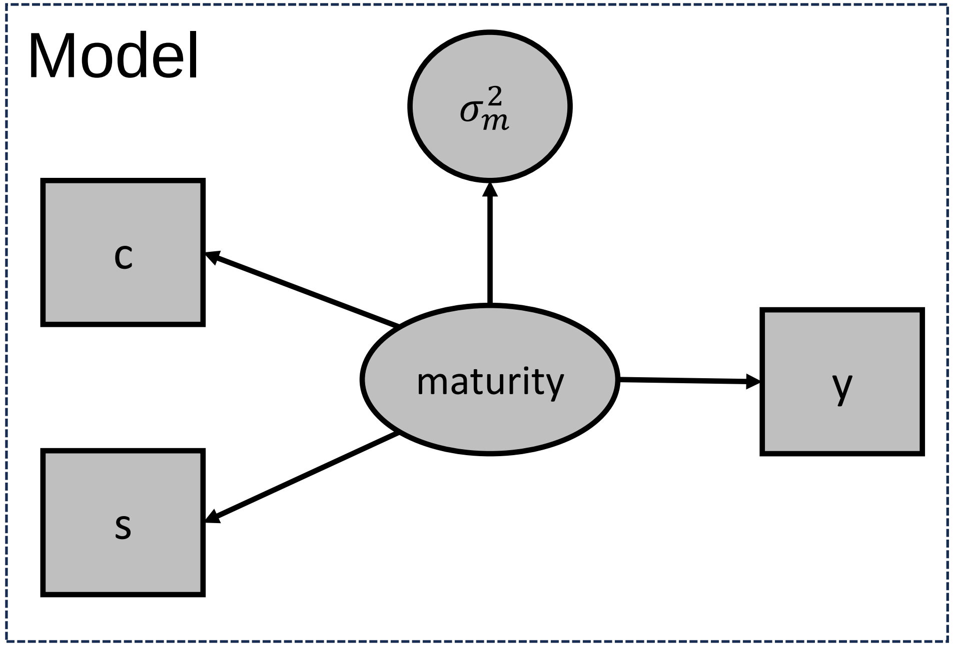
$$\beta = 0.75$$

Our latent variable is unobservable…

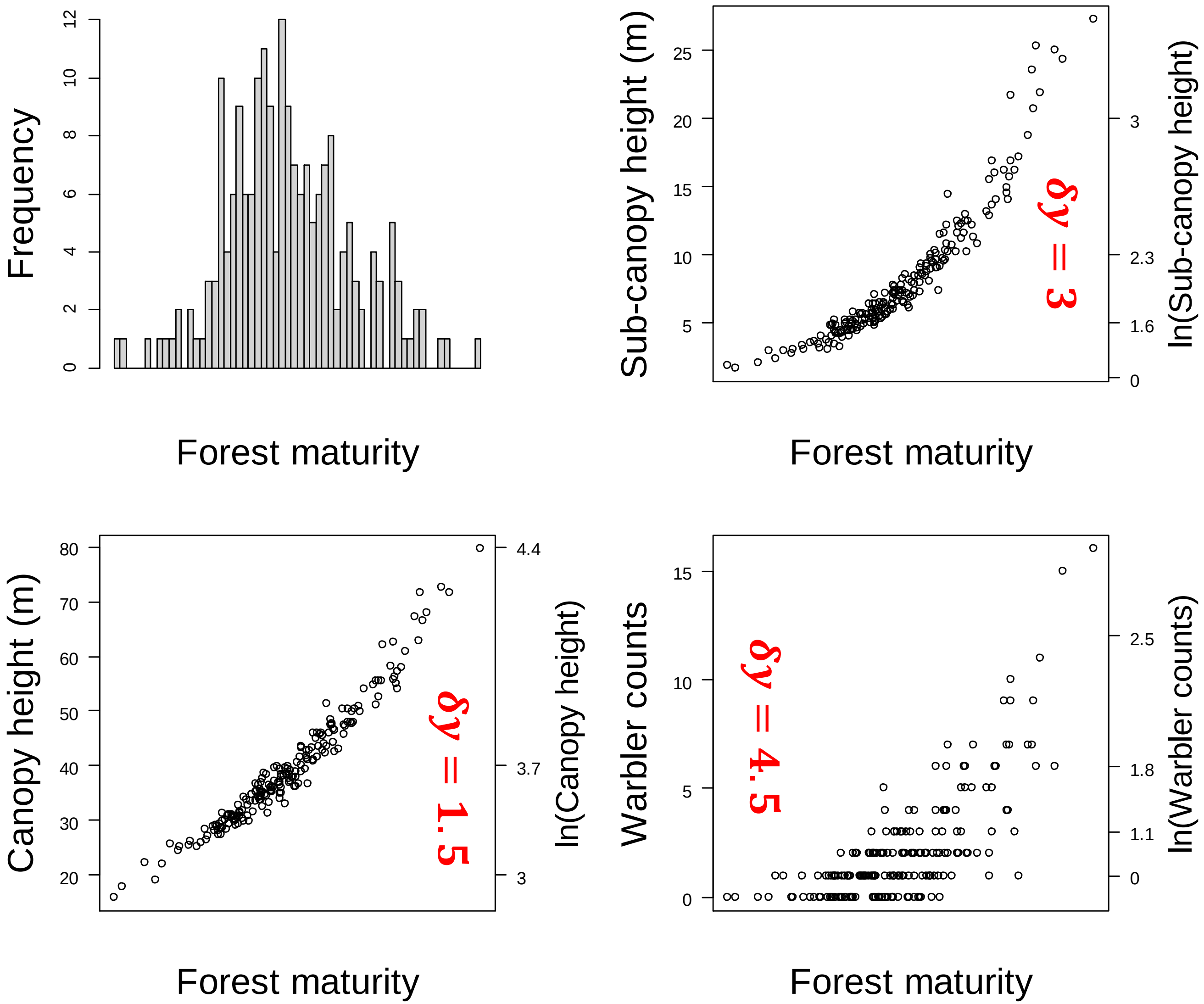
We don't know its scale…

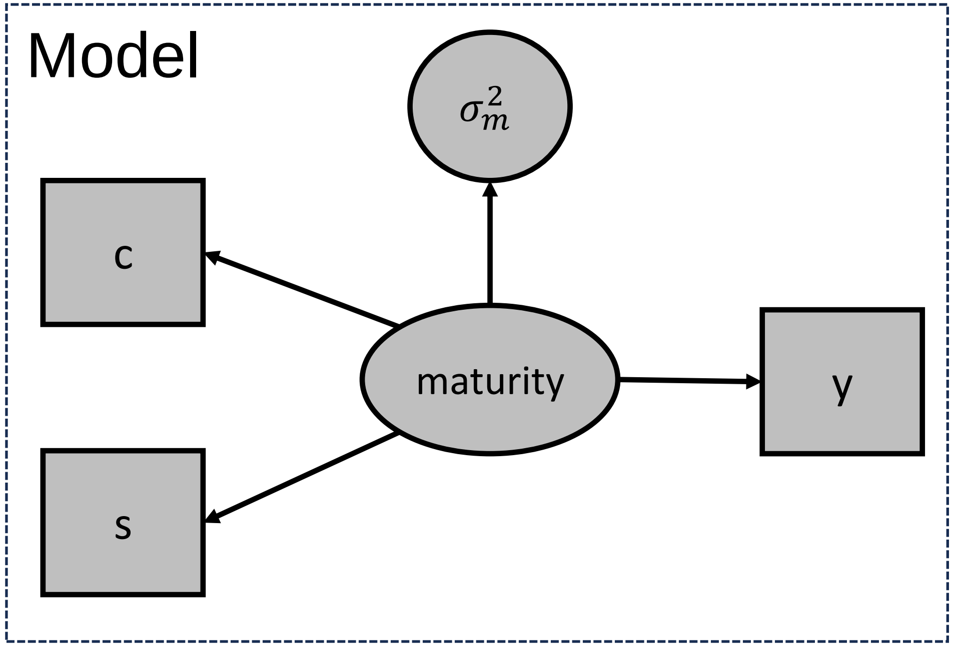
Clarifying a loading…





Shoot…

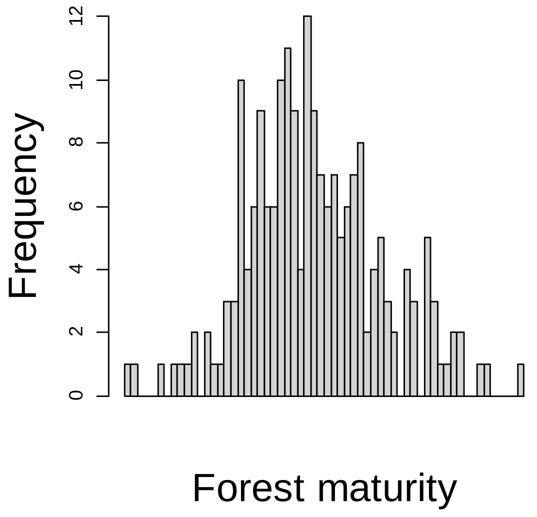


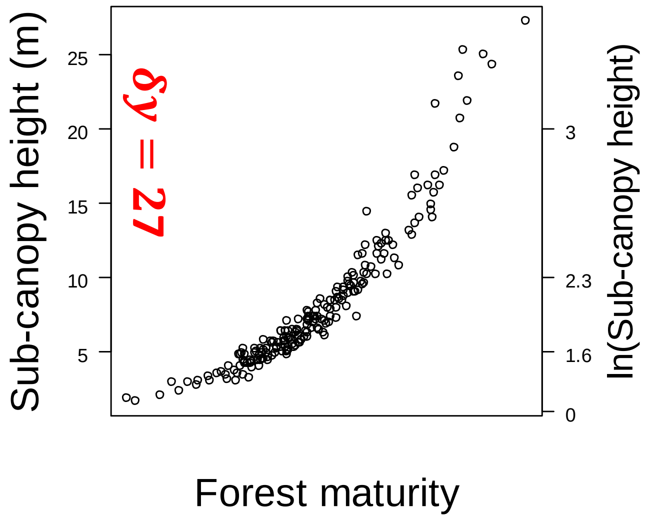


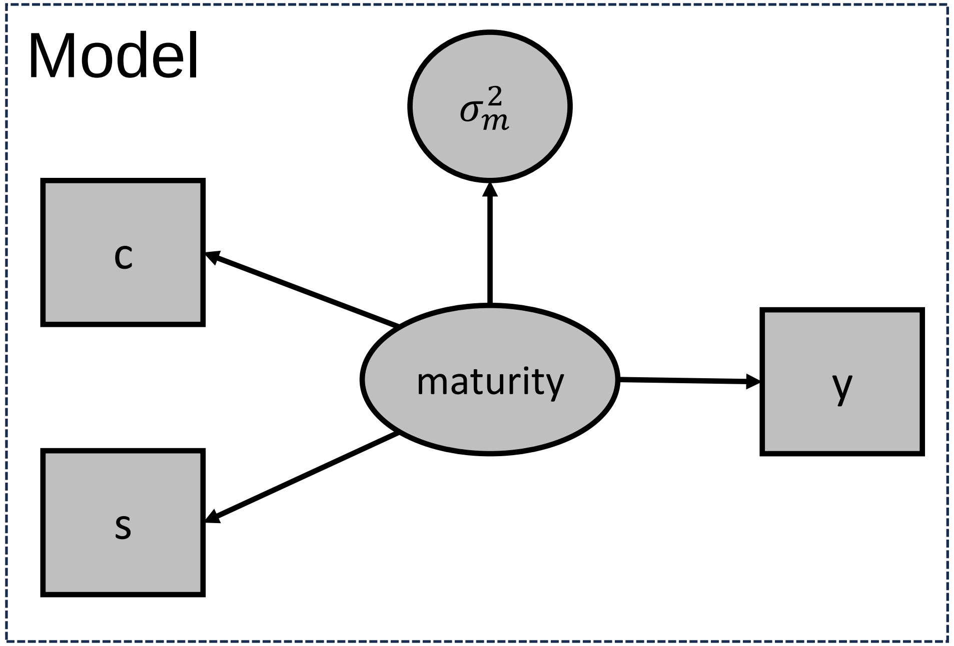
There's a big problem:
We don't know the range of maturity

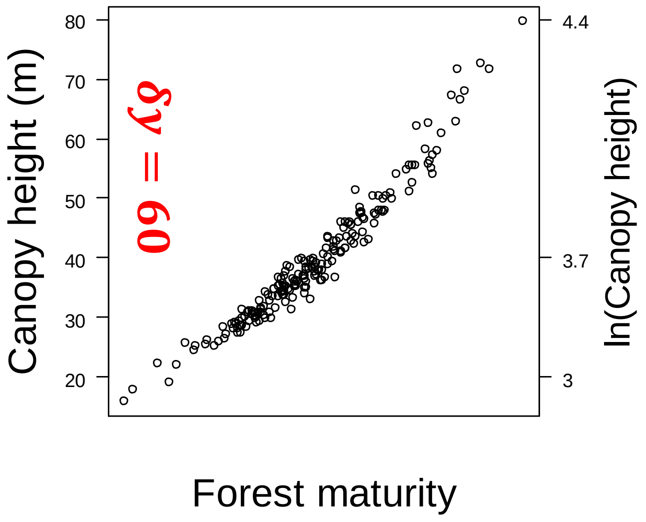
$$\boldsymbol{m} \sim \mathrm{normal}(0, \sigma_m^2)$$

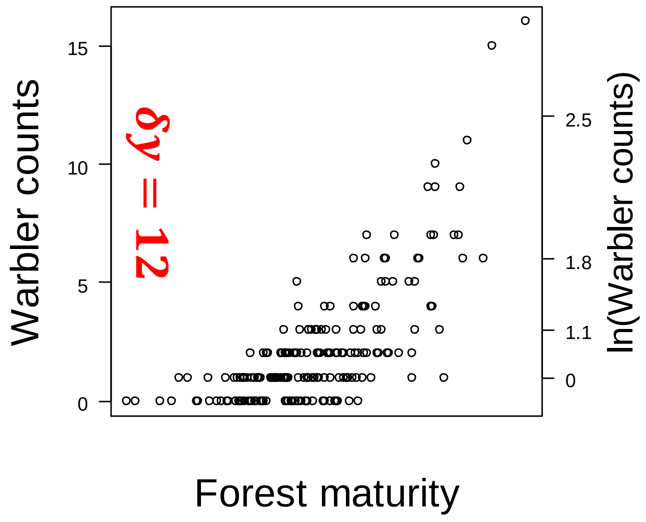
Small groups!







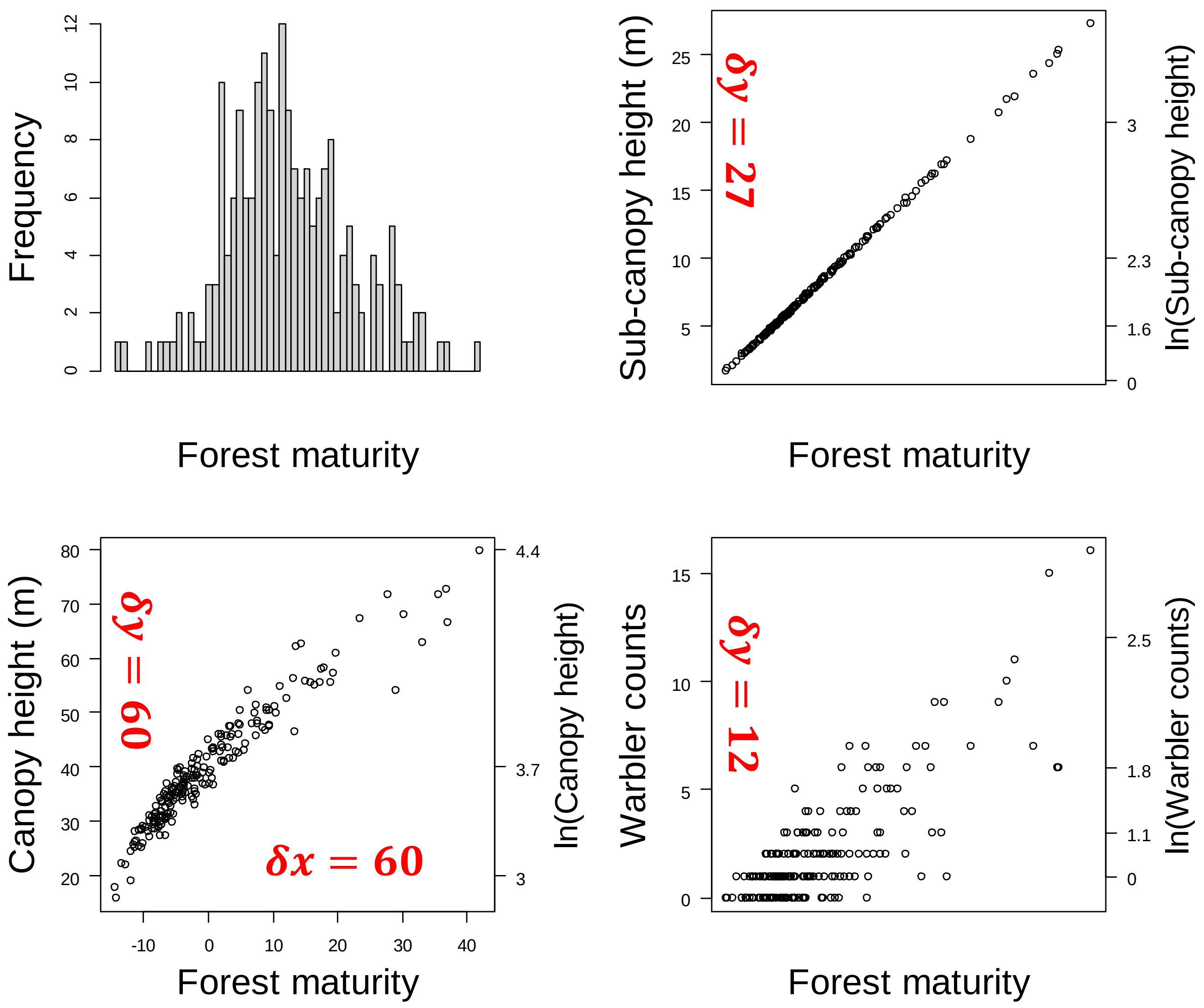


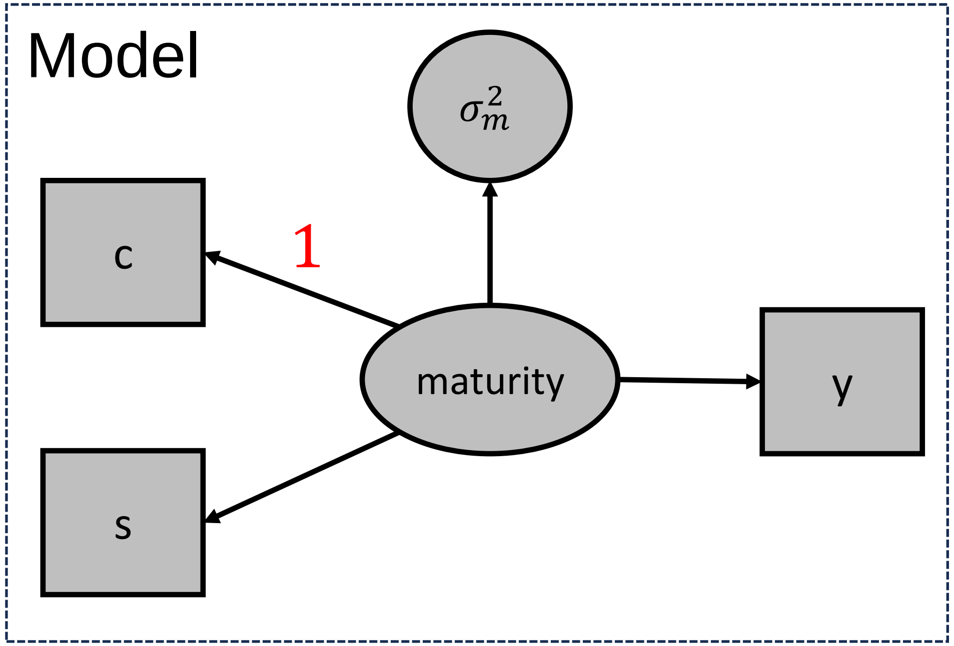


How could we scale maturity, i.e., what should the minimum and maximum values of maturity be?

$$\boldsymbol{m} \sim \text{normal}(0, \sigma_m^2)$$

What if we fix a beta (to give it a scale)?





How could we scale maturity, i.e., what should the minimum and maximum values of maturity be?

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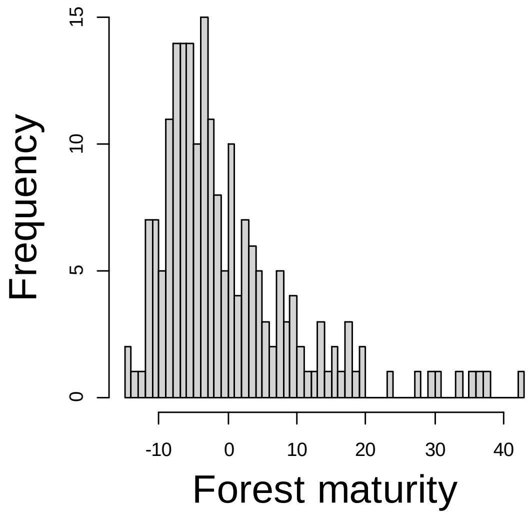
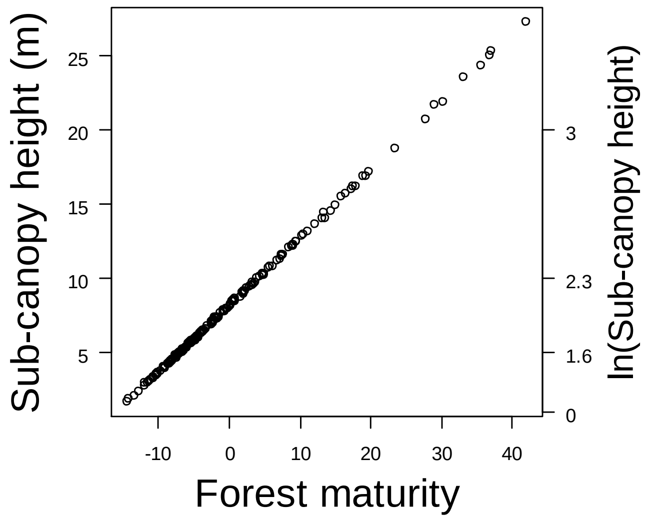
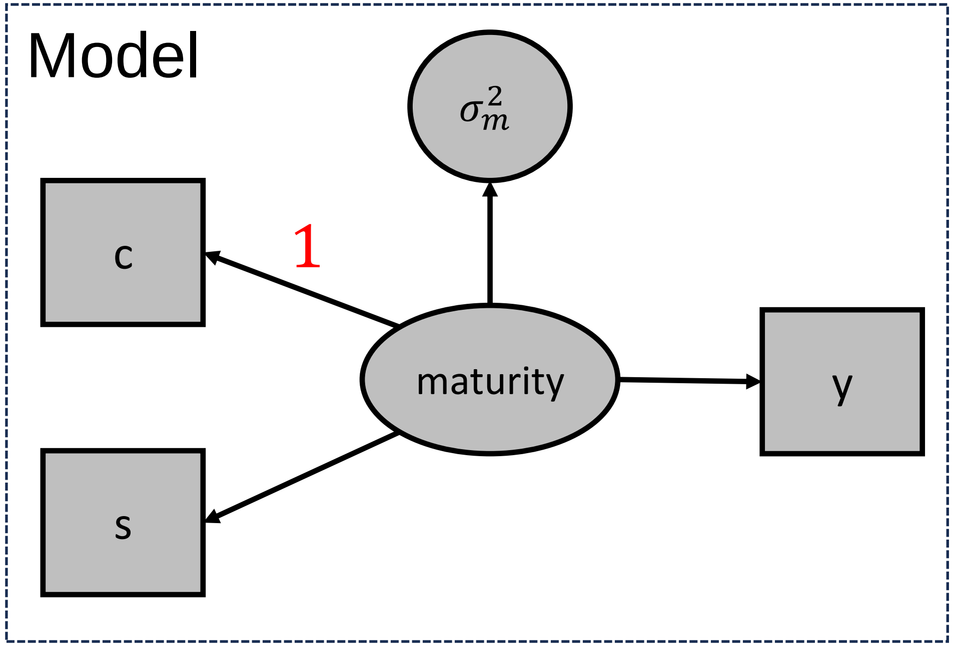
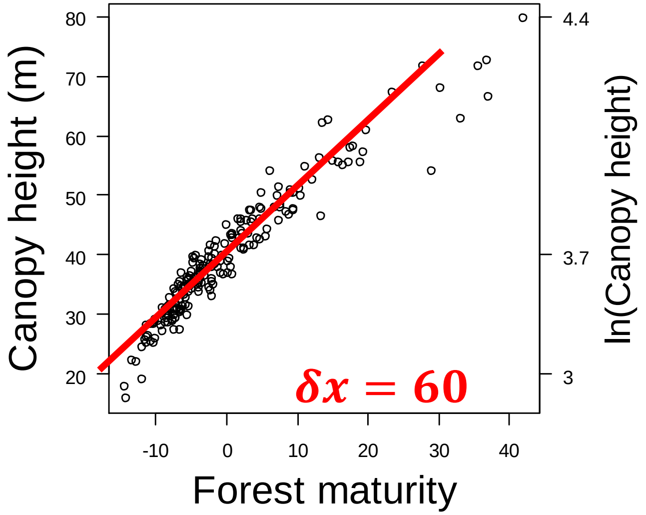
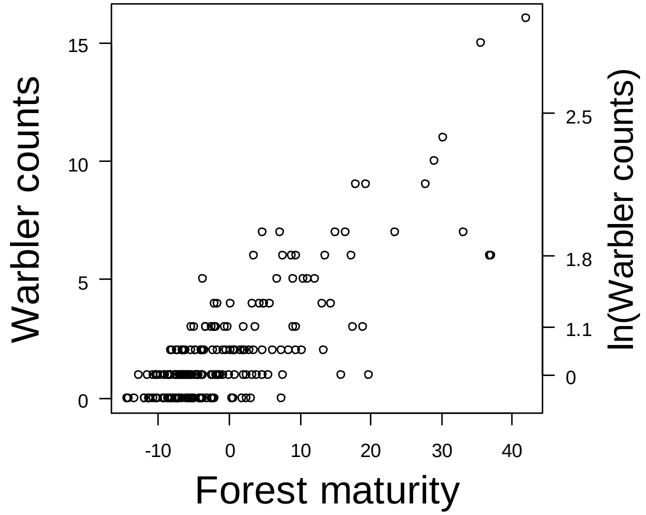
Now we have a scale!!



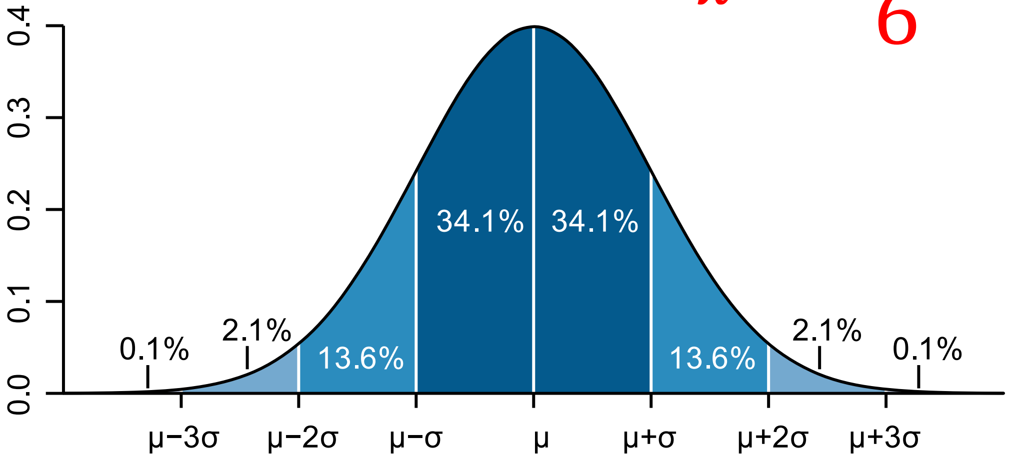




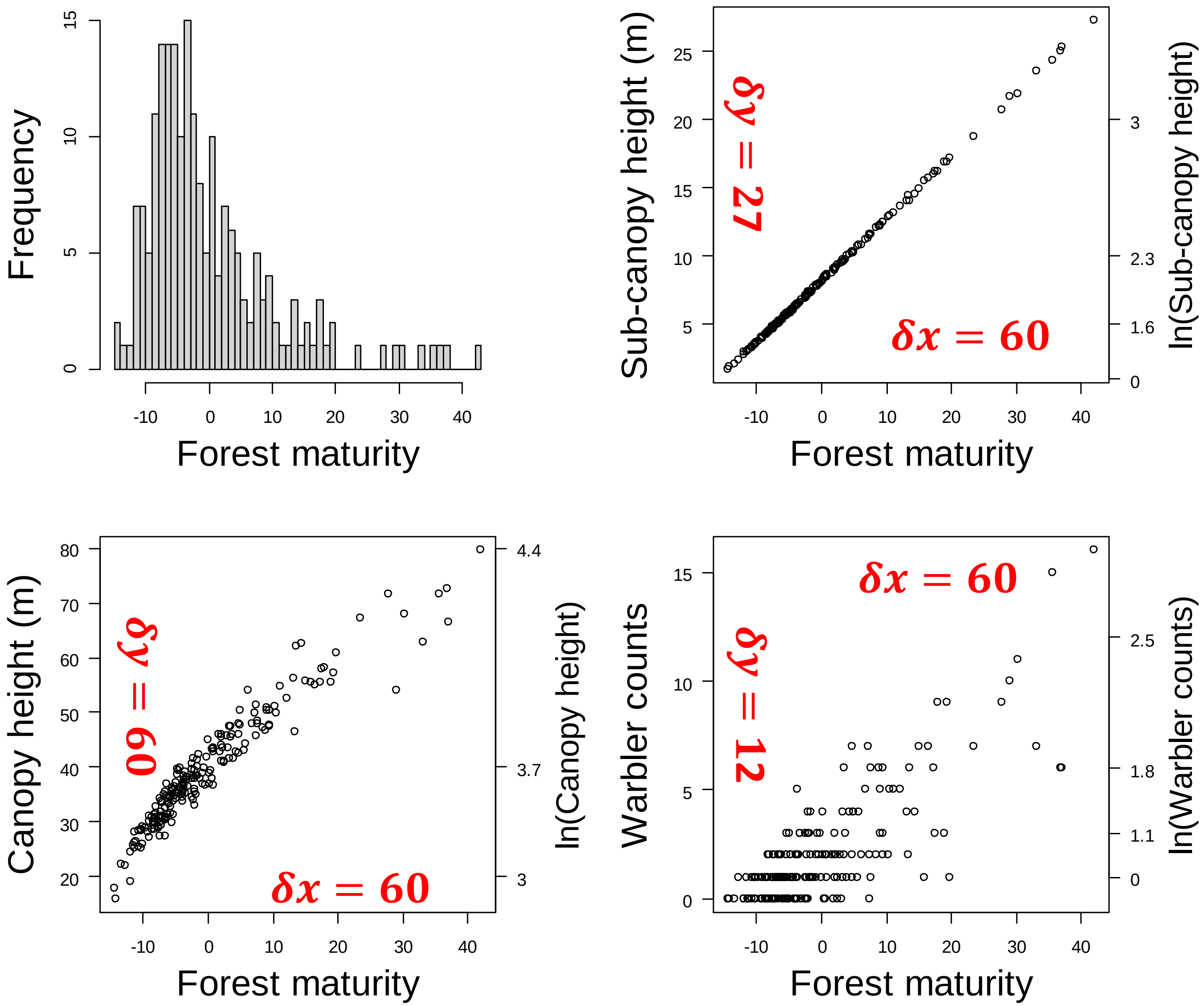


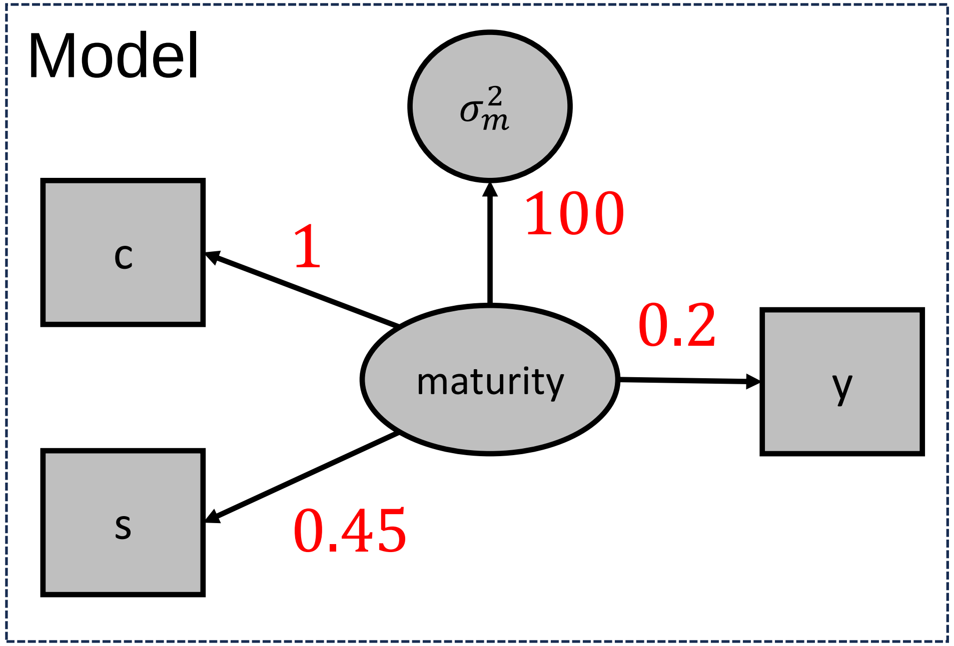


$$\sigma_x \cong \frac{\delta x}{6}$$



We can estimate all the betas

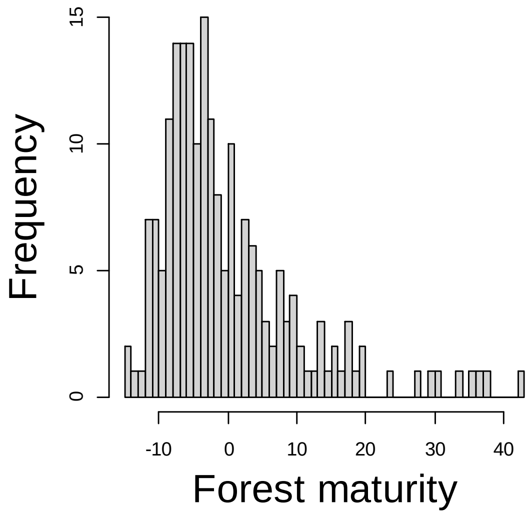


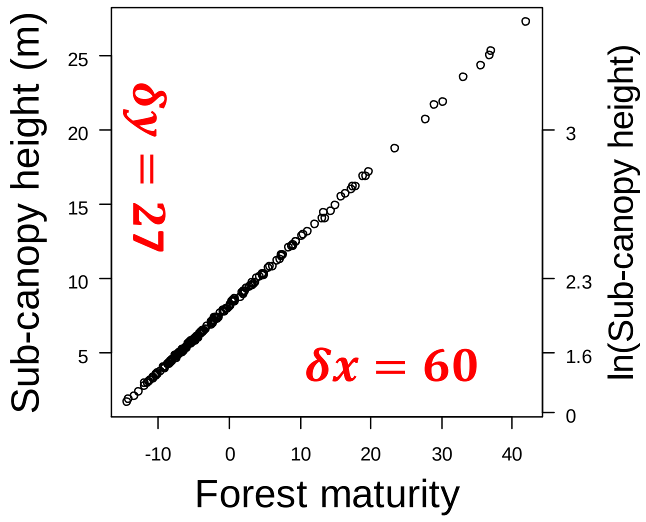


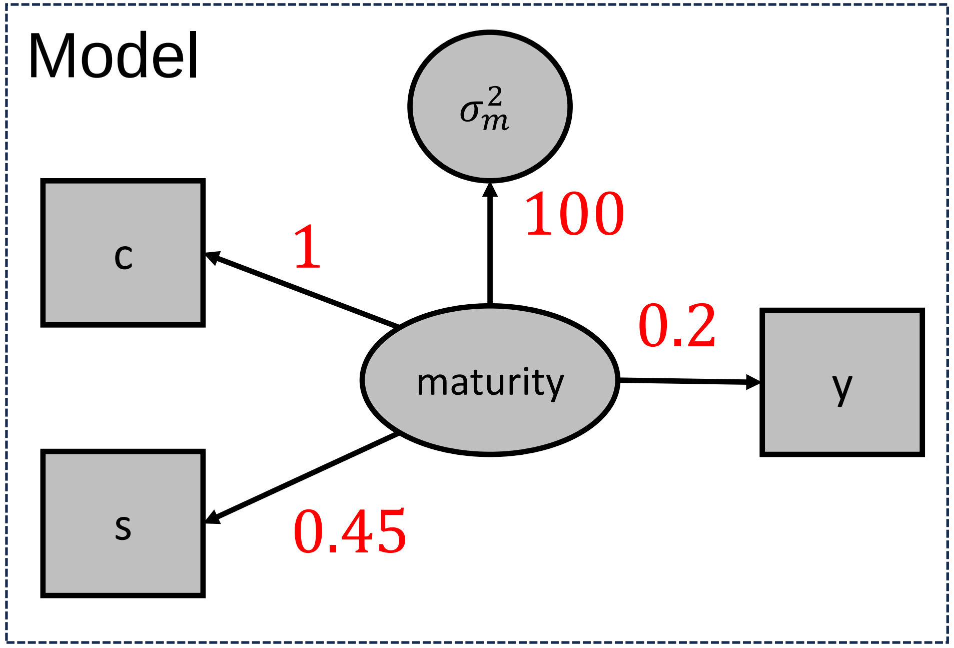
How could we scale maturity, i.e., what should the minimum and maximum values of maturity be?

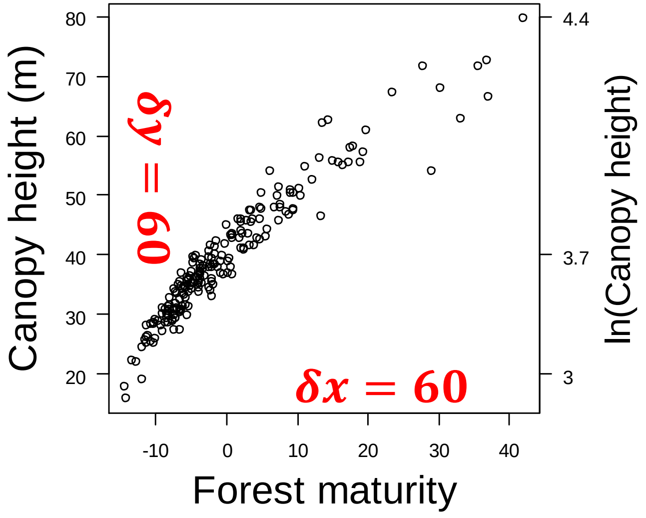
$$\boldsymbol{m} \sim \mathrm{normal}(0, \sigma_m^2)$$

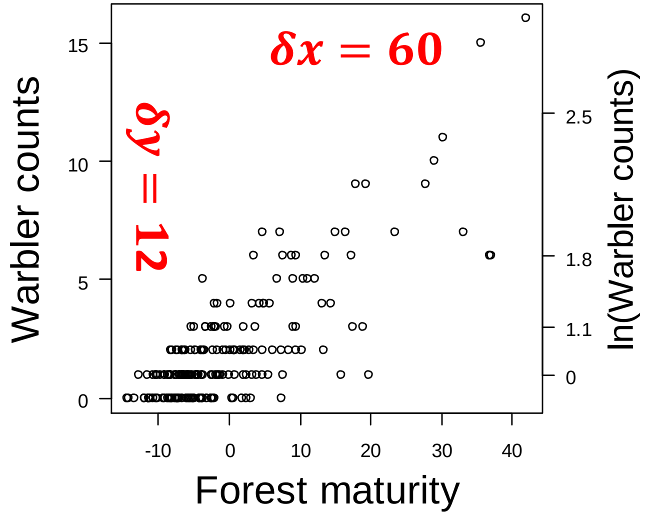
The scale of our latent variable is arbitrary











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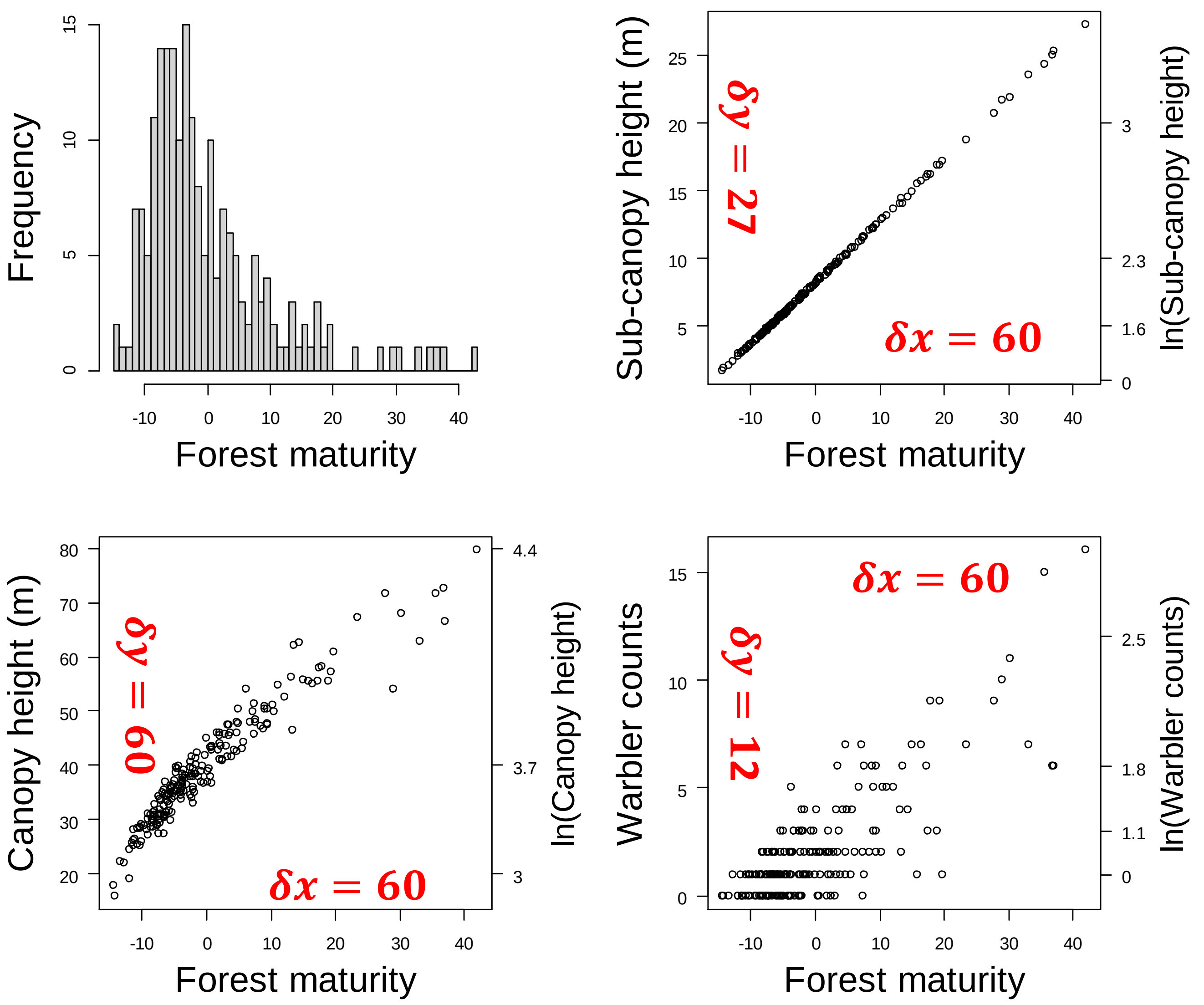
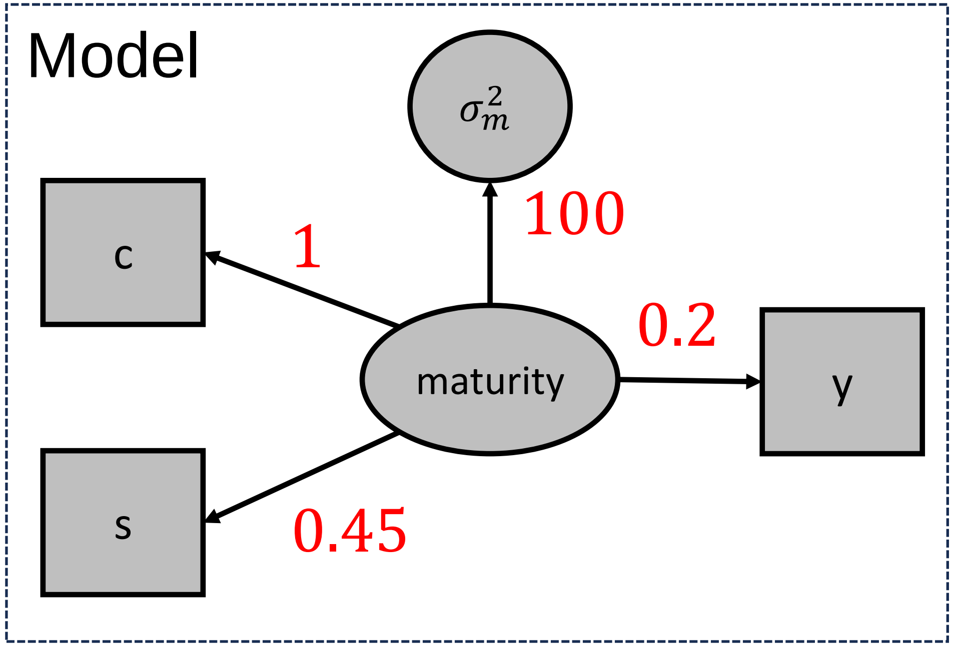
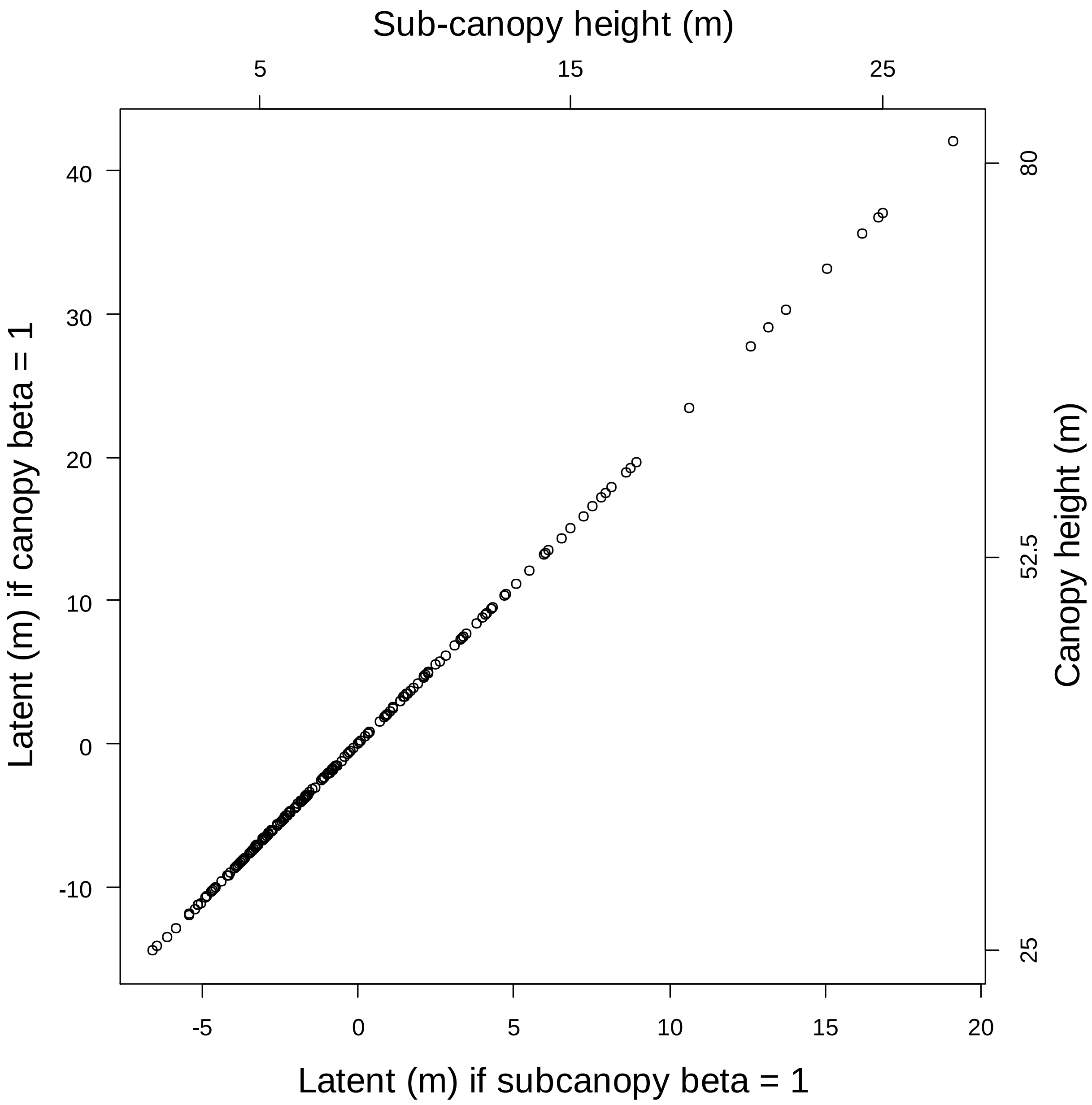


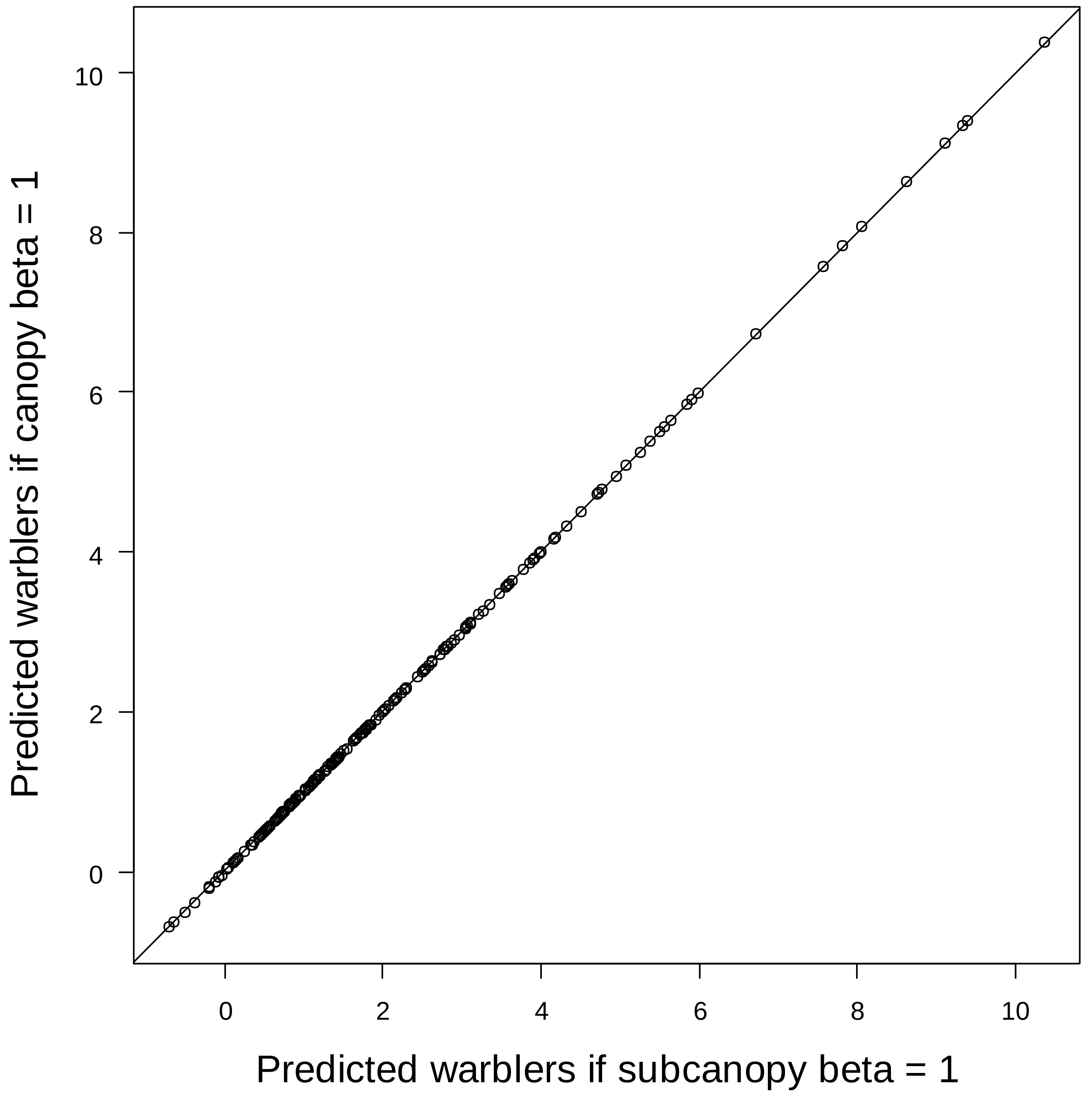


How could we scale maturity, i.e., what should the minimum and maximum values of maturity be?

$$\boldsymbol{m} \sim \text{normal}(0, \sigma_m^2)$$

Scale of latent variable can change, predictions don't



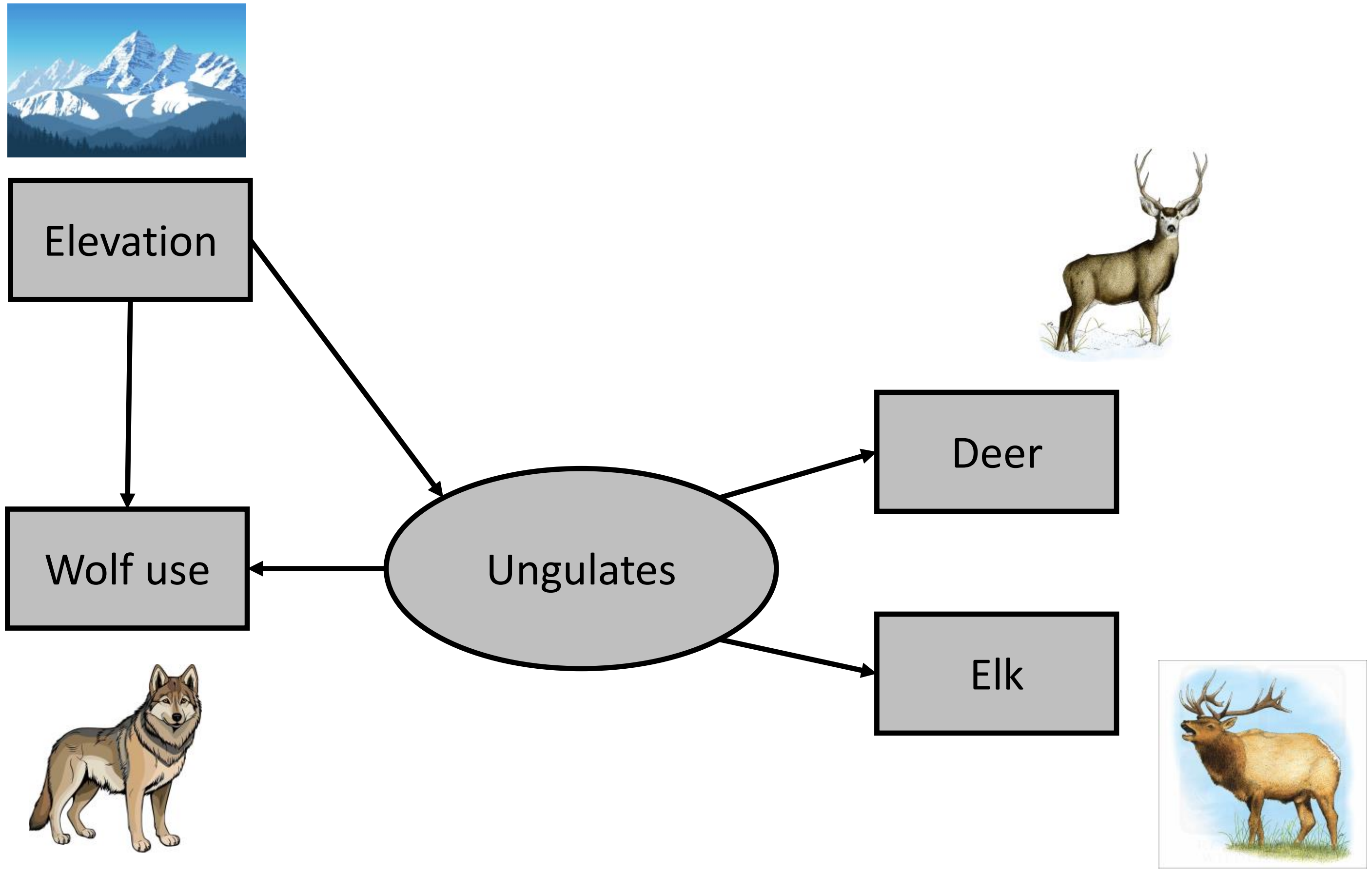


`lavaan` syntax

```
d <- data.frame(c = canopy, s = subcan, y = warblers)
sem1 <- sem('m =~ c + s
            y ~ m
            c ~ 1
            s ~ 1',
            data = d)
summary(sem1)
m.pred <- as.numeric(predict(sem1))
```

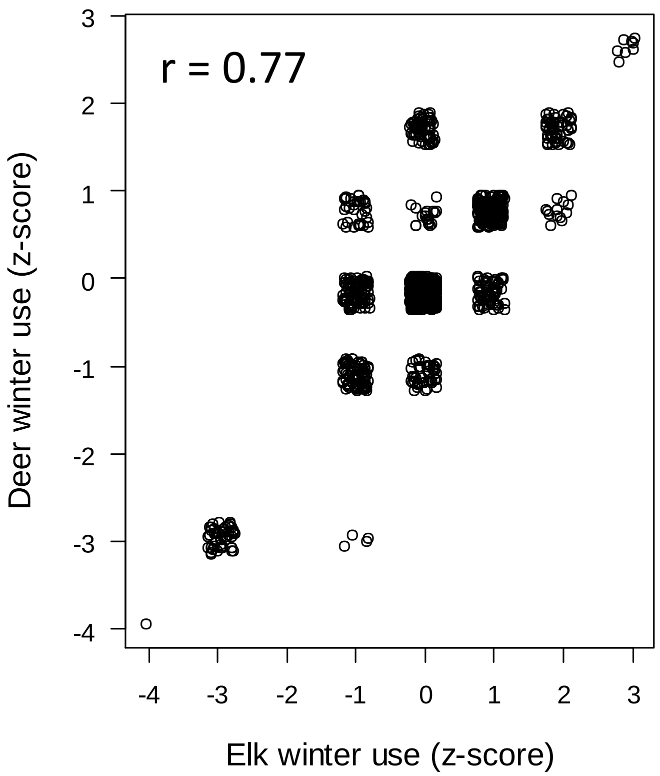
Expanding the Bow Valley wolf analysis with latent variables

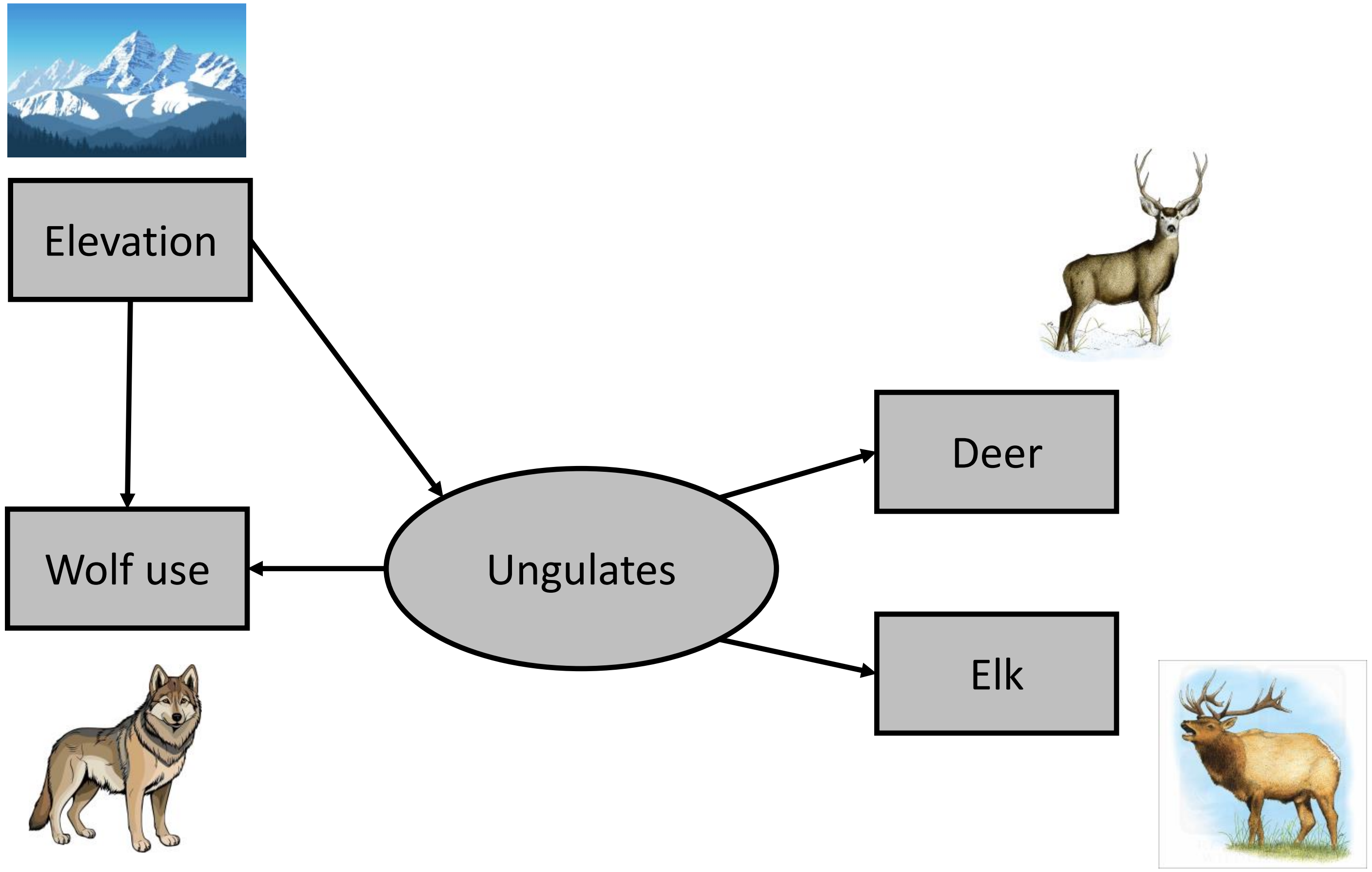
- Imagine that we're not just interested in deer, but in deer and elk abundance



Expanding the Bow Valley wolf analysis with latent variables

- Imagine that we're not just interested in deer, but in modeling deer and elk abundance simultaneously

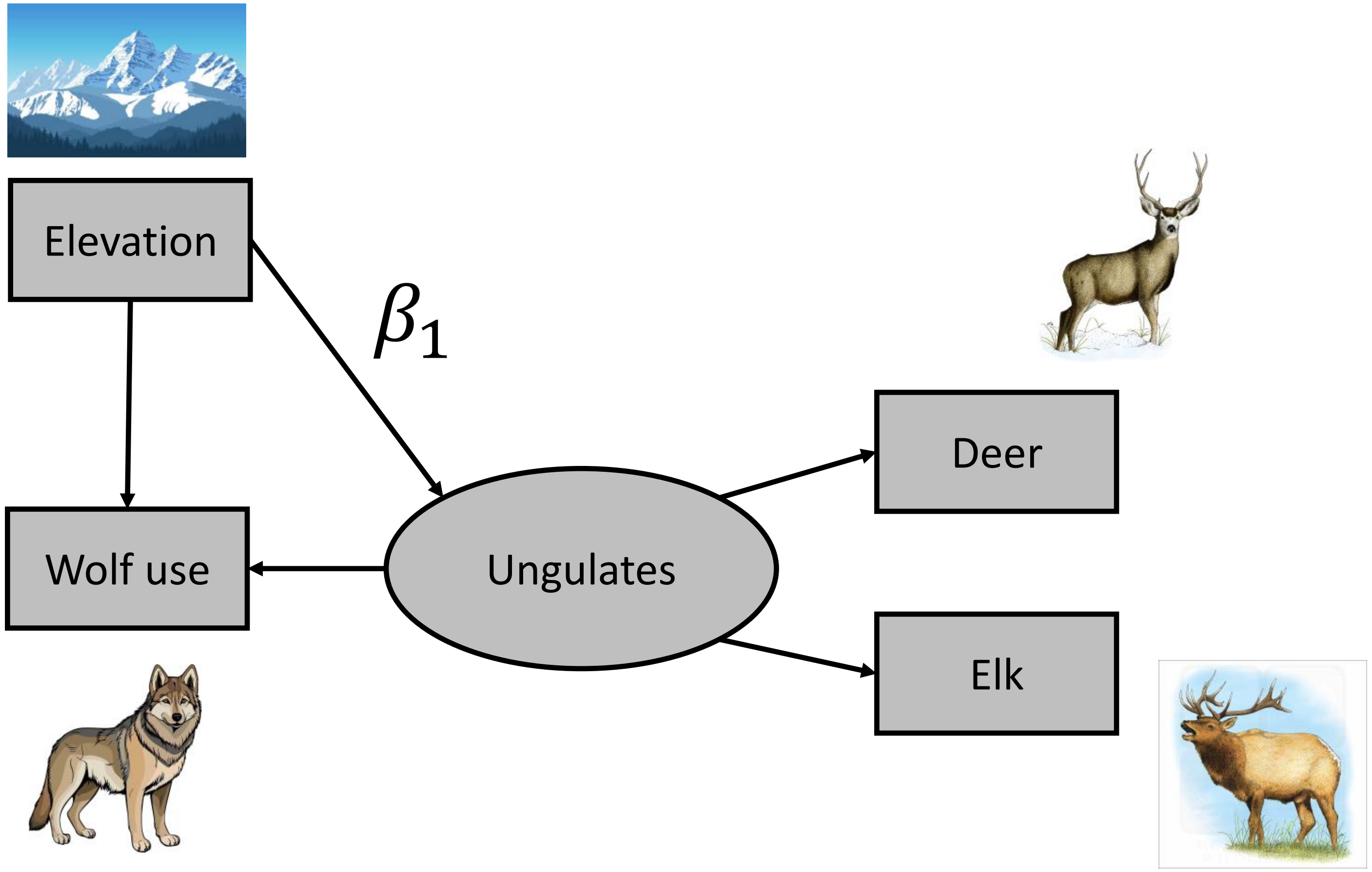




Expanding the Bow Valley wolf analysis with latent variables

- We can construct an ungulate latent variable modeled as a function of elevation (e), and measured via deer and elk winter habitat use…

$$u_i \sim \mathrm{normal}(\beta_1 \times elev_i, \sigma_u^2)$$



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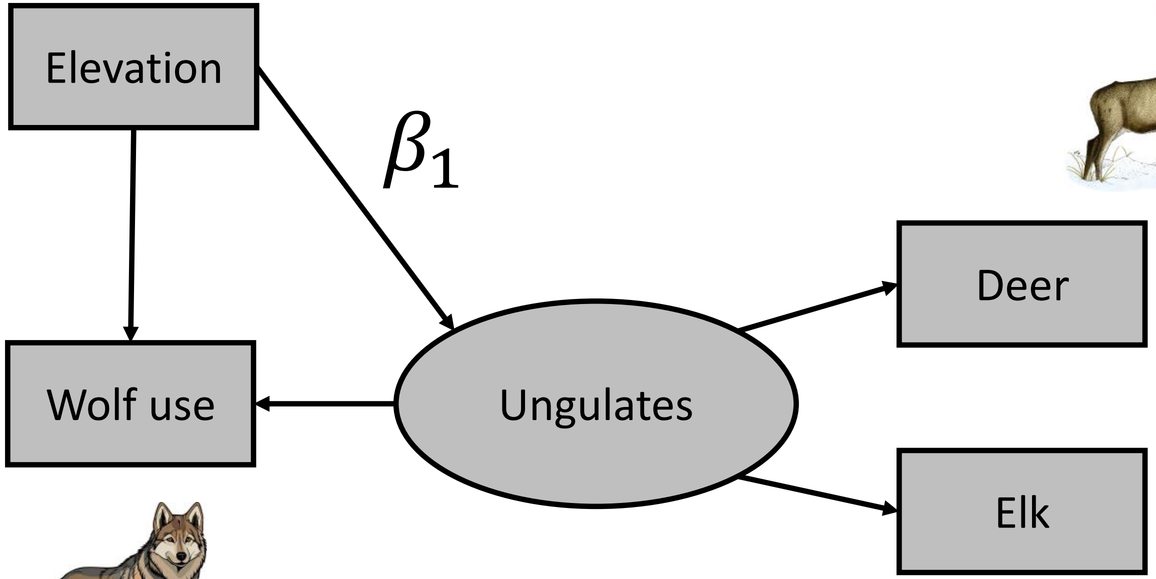
$$\boxed{u_i \sim \text{normal}(\beta_1 \times elev_i, \sigma_u^2)}$$



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model <- "ung =~ z.deer + z.elk
          ung ~ z.elev
          used ~ ung + z.elev
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          z.elk ~ 0"
```

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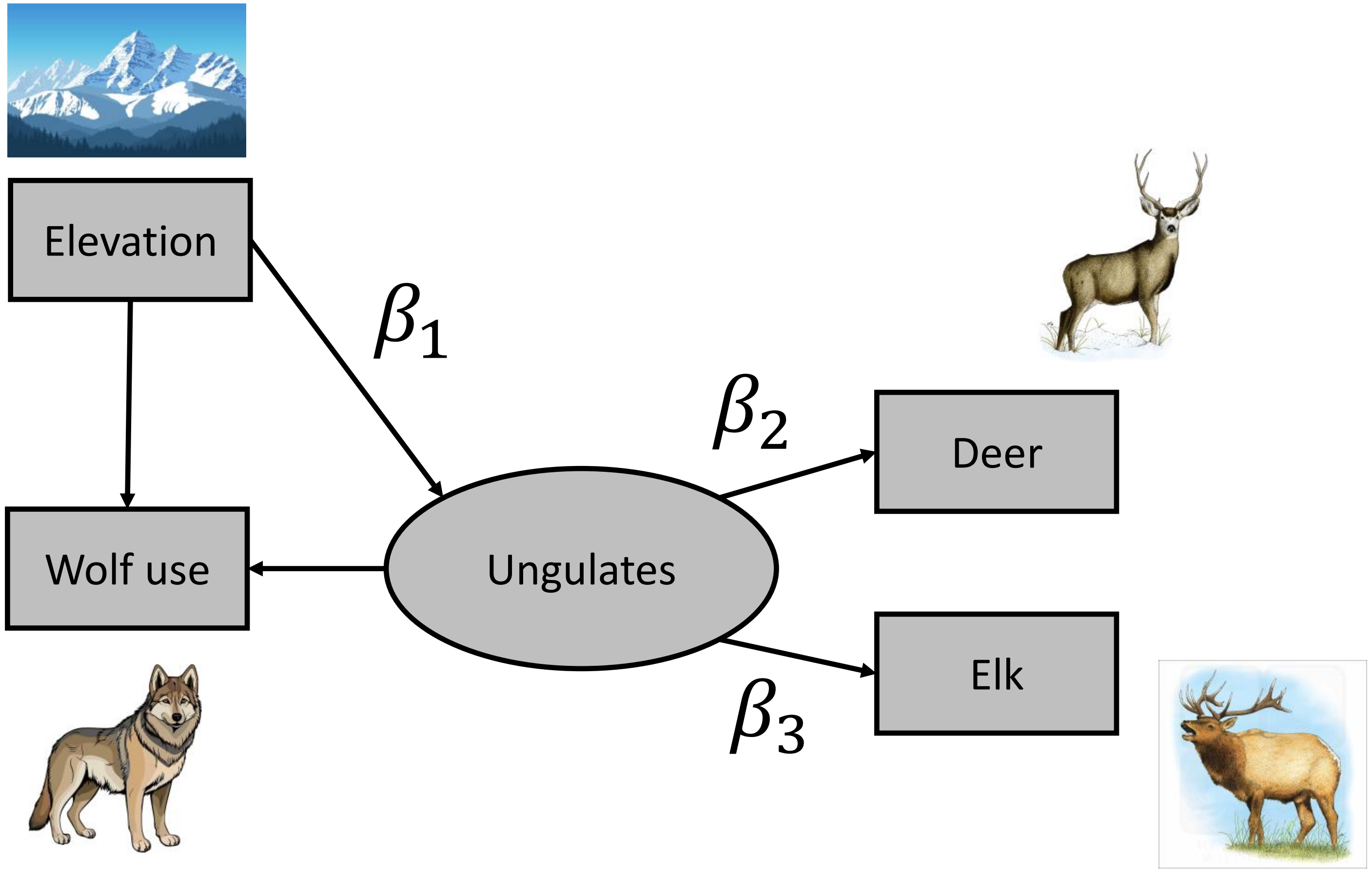
$$deer_i \sim \mathrm{normal}(\beta_2 \times u_i, \sigma_{deer}^2)$$

$$elk_i \sim \mathrm{normal}(\beta_3 \times u_i, \sigma_{elk}^2)$$



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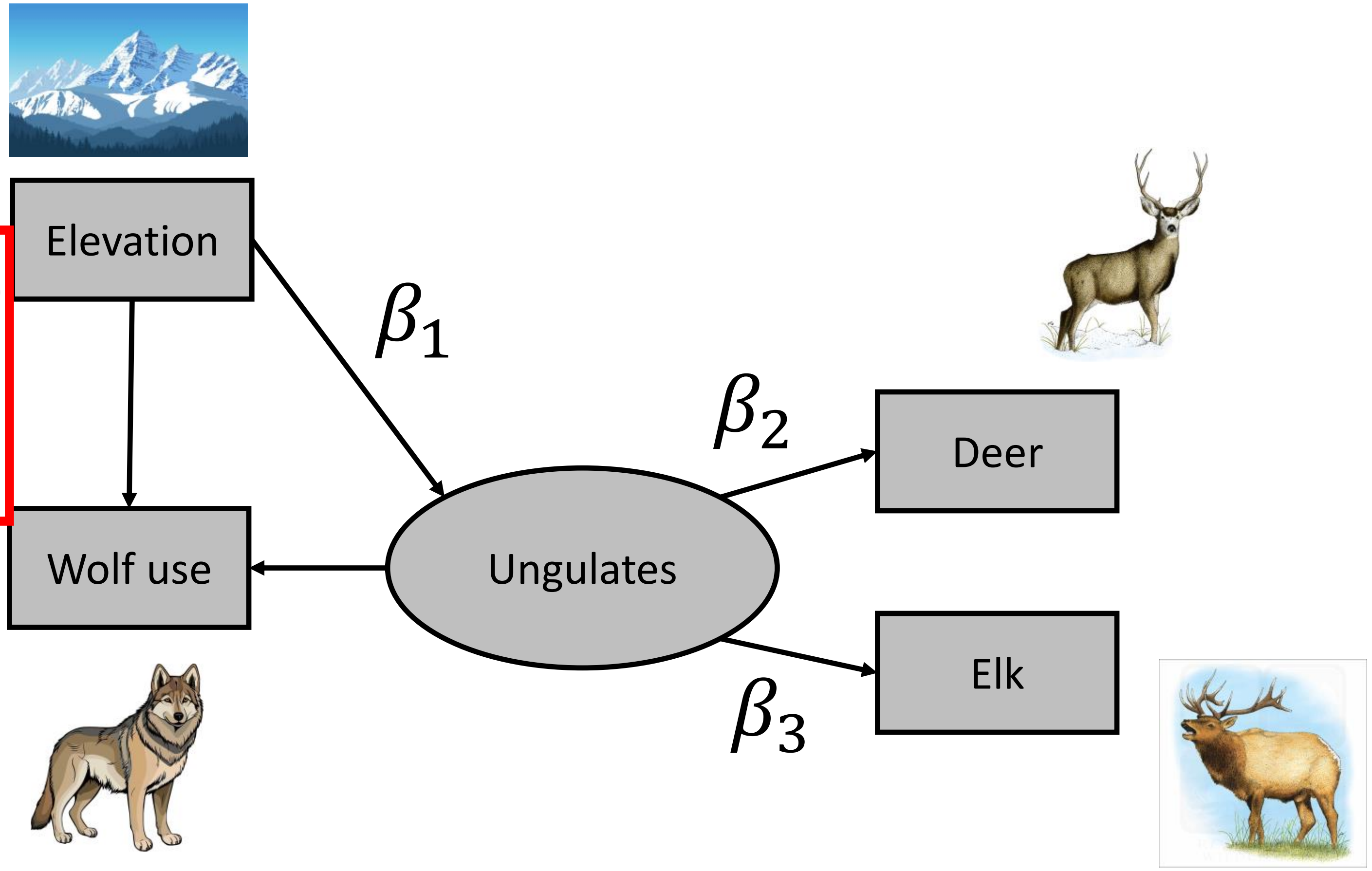

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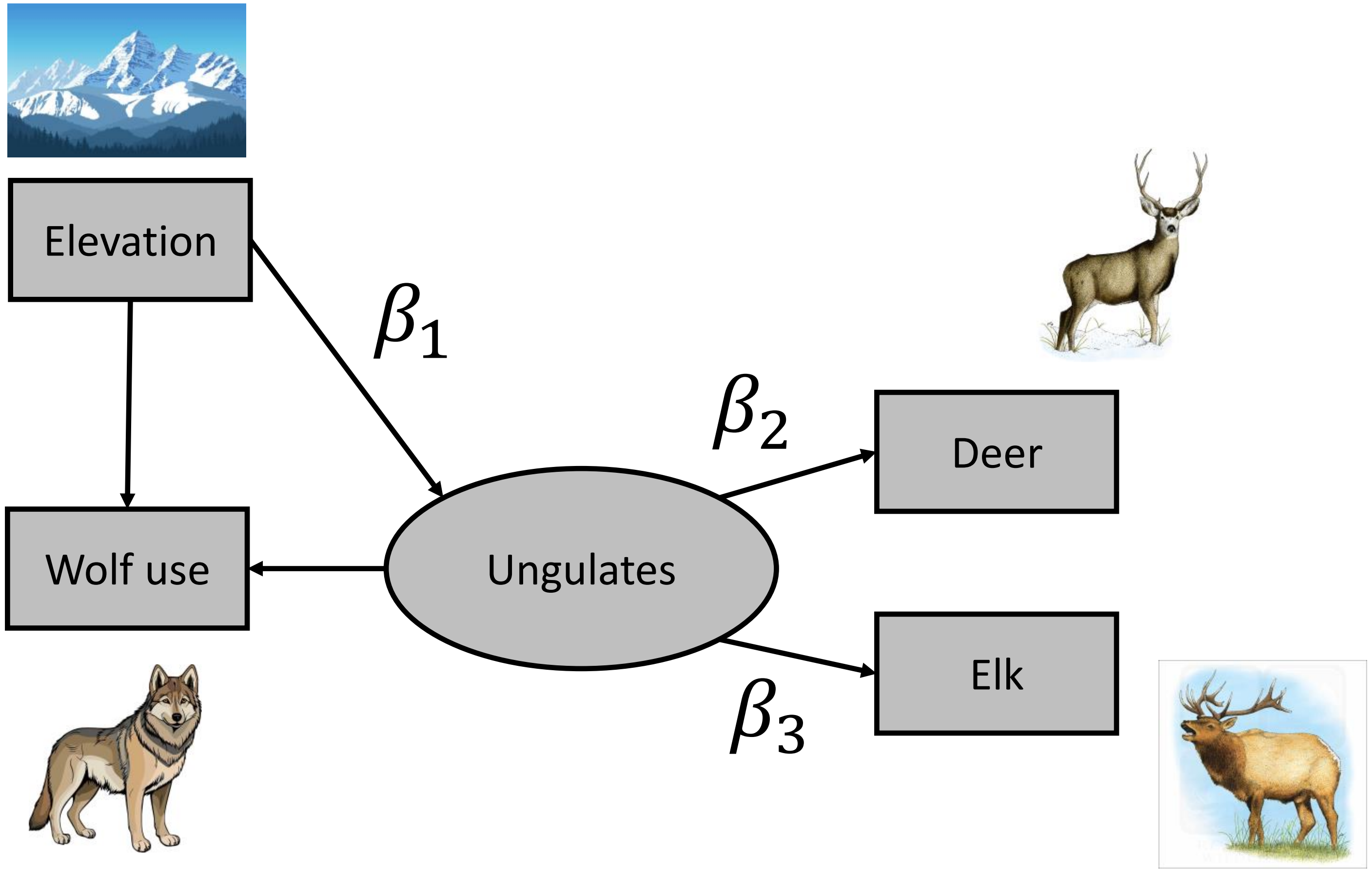
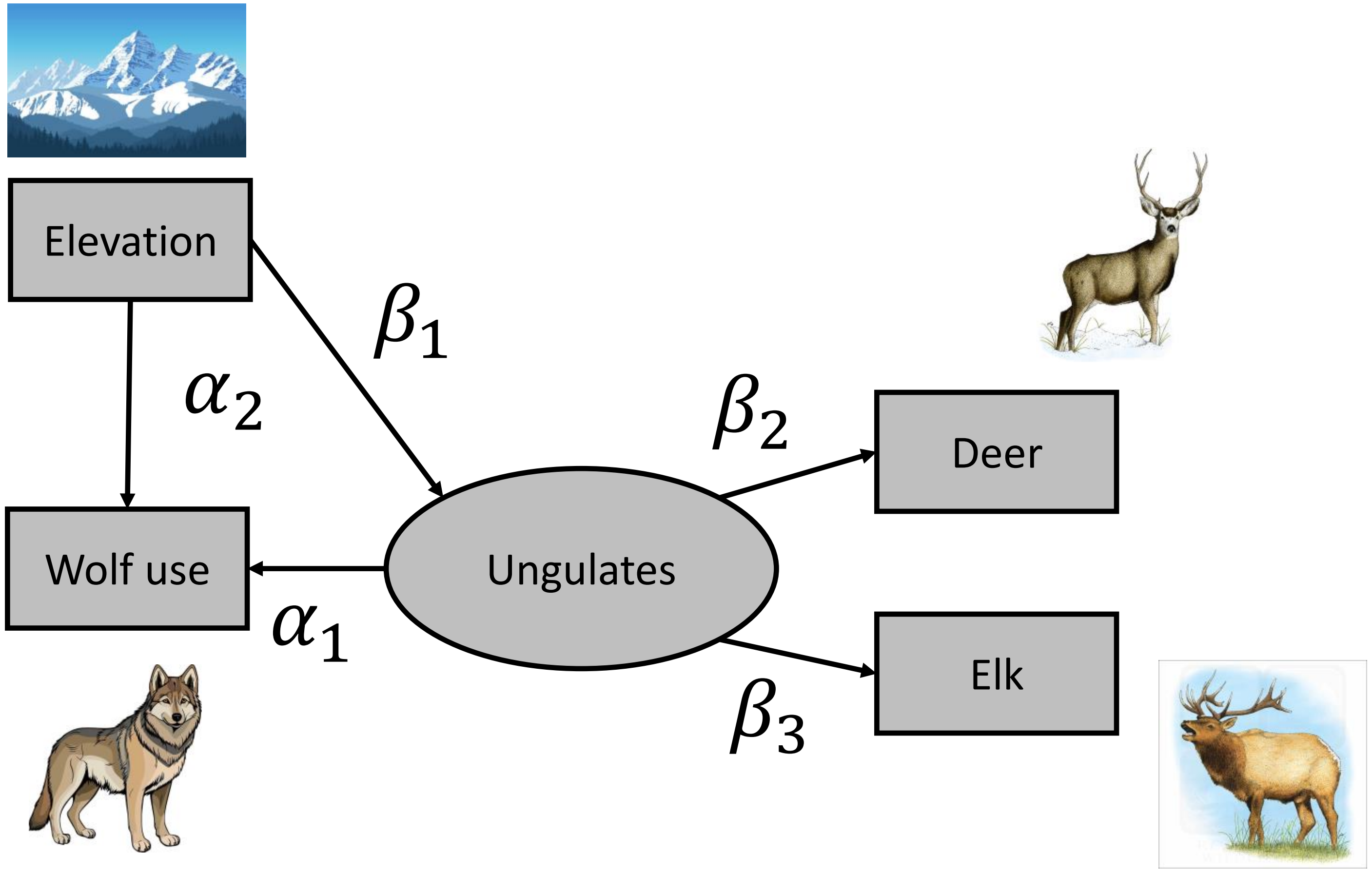


Expanding the Bow Valley wolf analysis with latent variables

- We can construct an ungulate latent variable modeled as a function of elevation (e), and measured via deer and elk winter habitat use…

$$w_i \sim \text{Bernoulli}(\varphi_i)$$

$$\text{logit}(\varphi_i) = \alpha_0 + \alpha_1 u_i + \alpha_2 e_i$$



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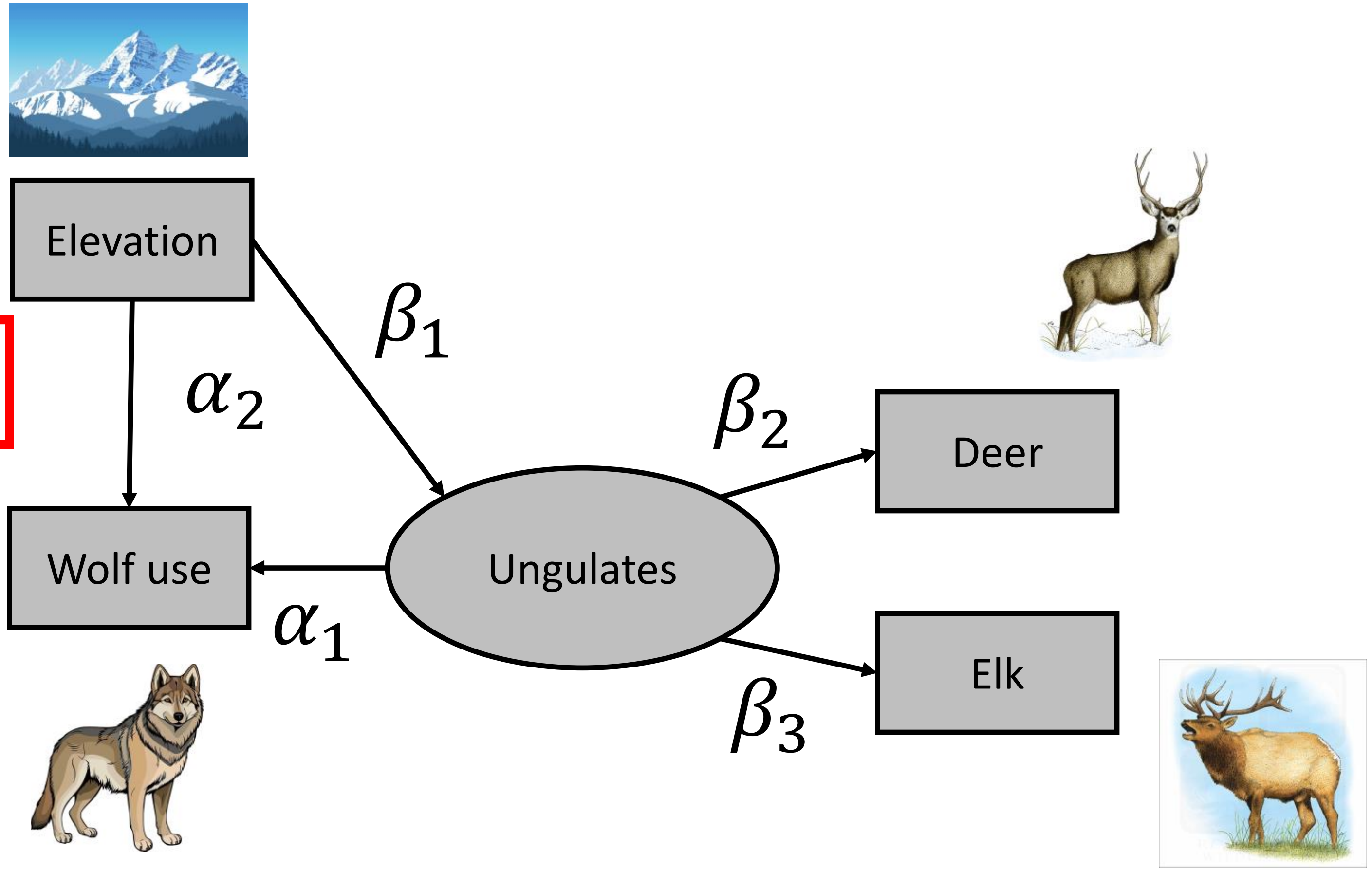
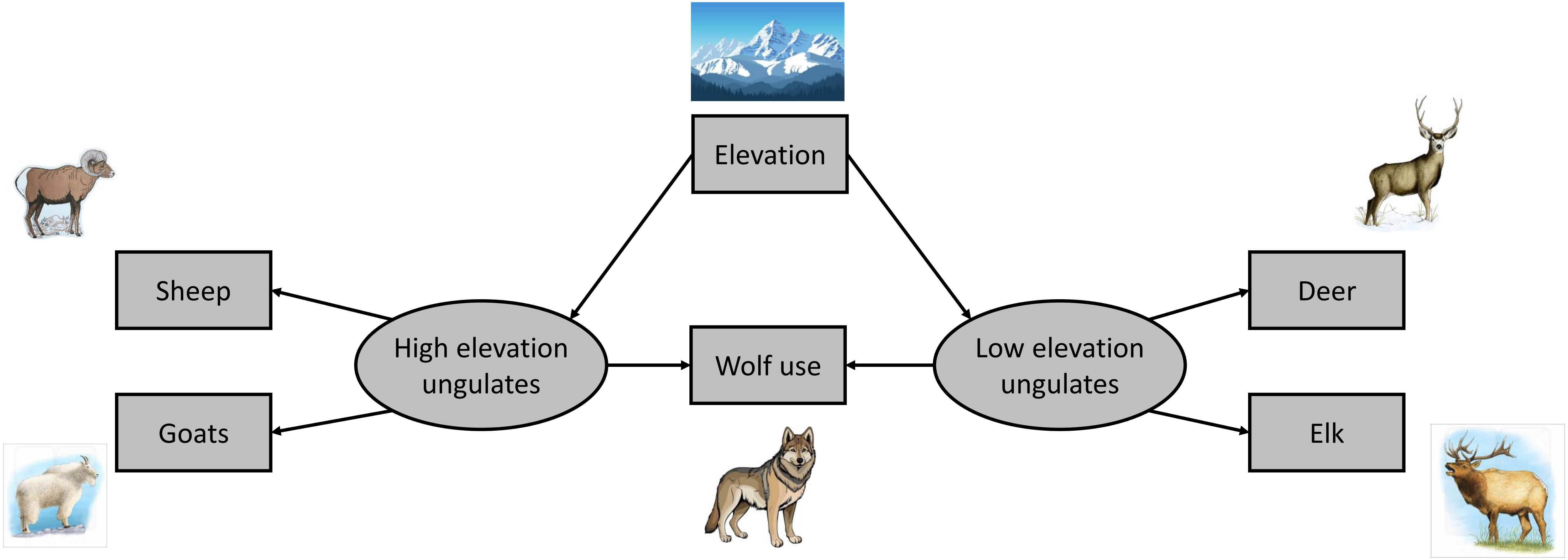


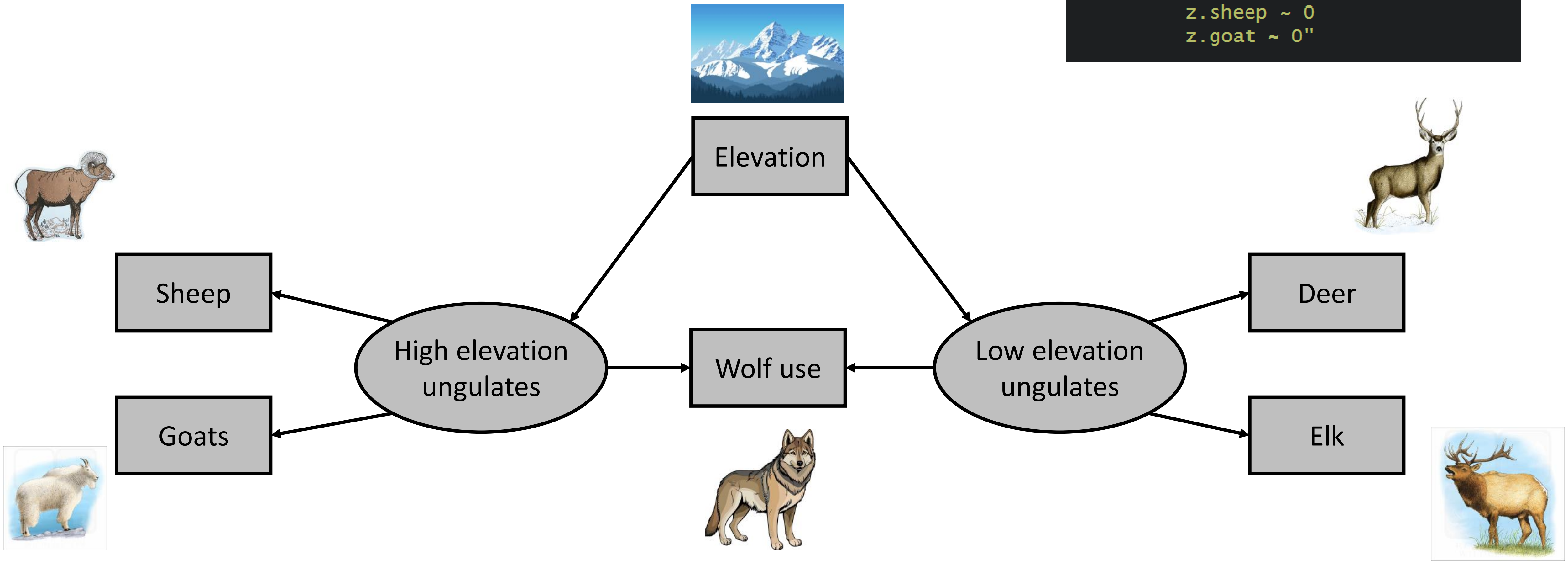
Do wolves prefer high or low elevation ungulates?



Do wolves prefer high or low elevation ungulates?

- Imagine that we're not just interested in wolves, but in how ungulates respond to the landscape, and which ungulates wolves prefer…

```
model <- "lo.ung =~ z.deer + z.elk
          hi.ung =~ z.sheep + z.goat
          lo.ung ~ z.elev
          hi.ung ~ z.elev
          used ~ hi.ung + lo.ung
          z.deer ~ 0
          z.elk ~ 0
          z.sheep ~ 0
          z.goat ~ 0"
```



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          hi.ung ~ z.elev
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          z.elk ~ 0
          z.sheep ~ 0
          z.goat ~ 0"
```