Quantitative Portfolio Management: Theory and Practice

Chapter 5: CAPM-Based Portfolios: Black-Litterman Model

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The big picture: Plan for the entire course

Part A: Preliminaries

Chapter 1: Managing financial data

Chapter 2: Performance measurement (especially out of sample)

Part B: Modern portfolio management

Chapter 3: Mean-variance portfolios that *ignore* estimation error

Chapter 4: Mean-variance portfolios that *adjust for* estimation error

Part C: Post-Modern Portfolio Management

Chapter 5: CAPM-based portfolios: Black-Litterman model

Chapter 6: Factor-based portfolios: Parametric portfolio policies

Chapter 7: Volatility-timed factor portfolios

Chapter 8: Portfolios exploiting systematic risk factors *and* unsystematic risk

Table of contents

Road map

What do we want to do in Chapter 5

In this chapter and the next two chapters, we study how *asset-pricing factor models* can be used to build portfolios that perform well out of sample.

Today, we study the Black-Litterman model, which is based on the Capital Asset Pricing Model.

In the next two chapters, we will study parametric portfolio policies, which are based on factor asset-pricing models, such as the ones developed by Fama and French.

Road map

Error in estimating $\mathbb{E}[R]$ and $\mathbb{V}[R]$ from sample moments

- In the last-to-last chapter, we saw that mean-variance weights depend on estimates of return means and covariances:

$$w = \frac{1}{\gamma}(\mathbb{V}[R])^{-1}(\mathbb{E}[R] - R_f 1_N).$$

- In the last chapter, we saw that
  - Sample-based estimates of $\mathbb{E}[R]$ are imprecise;
  - Sample-based estimates of $\mathbb{V}[R]$ are ill-conditioned;
  - Consequently, mean-variance portfolios perform poorly out of sample.
- We then studied shrinkage methods to improve the properties of the sample estimates of $\mathbb{E}[R]$ and $\mathbb{V}[R]$.

Shrinkage using only sample moments

- The shrinkage methods we studied relied only on sample moments but did not take advantage of asset-pricing theory.
- For example, Bayesian shrinkage of expected returns relies on shrinking sample estimate of $\mathbb{E}[R]$ toward a "grand mean," which is
  - either the average of all mean returns
  - or the expected return on the GMV portfolio.
- Similarly, the Ledoit and Wolf methods relies on shrinking the sample estimate of $\mathbb{V}[R]$ toward
  - either a diagonal matrix with the average variance on its diagonal
  - or a matrix where all the cross-asset correlations are replaced by the average correlation.

Empirical performance of sample-based models . . . I

- When we evaluated the empirical performance of models relying only on sample moments of returns, we saw that
  - models shrinking sample estimates of expected returns fail to outperform the simple $1/N$ benchmark;
  - models that ignore expected returns altogether and choose weights based on minimization of portfolio variance perform better,
    - especially when shrinking the covariance matrix of returns using either a short-sale constraint or the Ledoit and Wolf approach,
  - however, even these models do not always outperform the simple $1/N$ portfolio.

Empirical performance of sample-based models . . . II

- For the empirical performance of models based on shrinkage of sample moments, we studied the evidence reported in
  - DeMiguel, Garlappi, and Uppal (2009).
- But several other papers confirm that optimizing models based on sample moments performs poorly.
- Some of these papers are described on the next few slides.

Empirical performance of sample-based models . . . III

- Jacobs, Müller, and Weber (2014)
  - Take the perspective of a Euro investor
  - Extend the data period: 1973 to 2013
  - Extend the analysis across countries
  - Extend the analysis across asset classes (stocks, bonds, and commodities)
  - Extend the list of models of optimal portfolio selection studied
  - Extend the performance-evaluation metrics.

Empirical performance of sample-based models . . . IV

- Jacobs, Müller, and Weber (2014) find that:

  "Analyzing more than 5,000 heuristics, our results show that in fact almost any form of well-balanced allocation over asset classes offers similar diversification gains as even recently developed portfolio optimization approaches."

- Jacobs, Müller, and Weber (2014) conclude that:
  - Estimation error leads to poor performance of "optimal" models, relative to fixed-weight portfolios.

Empirical performance of sample-based models . . . V

- However, it is possible to find more sophisticated optimizing portfolios that do outperform the $1/N$ portfolio.
- For example, Ao, Li, and Zheng 2019
  - use machine-learning methods (in particular, lasso constraints) to
  - find shrinkage portfolios that outperform $1/N$ out of sample.
- Another example, which we will study in the last chapter, is the paper by Raponi, Uppal, and Zaffaroni (2023), which
  - shows how to build portfolios that outperform the $1/N$ portfolio
  - by exploiting the compensation for bearing unsystematic risk.

Ideas for Master's projects

- As a Master's project, you could extend the data until 2024 in the paper by Jacobs, Müller, and Weber (2014)
- As a Master's project, you could
  - reproduce the results in Ao, Li, and Zheng (2019), and
  - extend the analysis in Ao, Li, and Zheng (2019) along various dimensions, as in Jacobs, Müller, and Weber (2014).
- As a Master's project, you could
  - reproduce the results in Raponi, Uppal, and Zaffaroni (2023)
  - extend the analysis in Raponi, Uppal, and Zaffaroni (2023) to new asset classes (beyond equities) and along other dimensions, as in Jacobs, Müller, and Weber (2014).

End of discussion of
portfolios based on *sample-moments of returns*

. . . .

Start of discussion of
portfolios based on *asset-pricing models*

Shrinkage using asset-pricing factor models . . . I

- So far, in the shrinkage models we studied, we did not use any information from asset-pricing models.
- We will now study two "shrinkage" methods based on asset-pricing factor models.
  1. The first is based on the Capital Asset Pricing Model of Sharpe (1964) . . . which we will study in this chapter.
  2. The second can be interpreted as being based on the models of Fama and French (1992, 1993, 2012, 2015, 2018) . . . which we will study in the next chapter.
- After that, in the last two chapters of our course, we will see how to go beyond Fama-French models.

Shrinkage using asset-pricing factor models . . . II

1. The first asset-pricing-based model for portfolio choice we study is the famous Black-Litterman model developed at Goldman Sachs, described in a series of papers,
   - Black and Litterman (1990, 1991a, 1991b, 1992); the 1992 paper is available from this link.
   - for the intuition underlying the Black-Litterman model, see He and Litterman (1999), which is available from this link;
   - and, for a historical perspective, see the book by Litterman (2003).
2. The second asset-pricing-based model for portfolio choice we will study (in the next class) is the parametric portfolio policy of Brandt, Sant-Clara, and Valkanov (2009).

- Both models are based on very clever insights, are widely used in industry, and are straightforward to implement using Python.

Road map

Road map

First motivation for Black-Litterman model

- Recall that the solution to the Markowitz problem is given by:

  $$w = \frac{1}{\gamma} \left(\mathbb{V}[R]\right)^{-1} (\mathbb{E}[R] - R_f 1_N),$$

  where $\mathbb{E}[R]$ denotes the $N \times 1$ vector of expected returns and $\mathbb{V}[R]$ is the $N \times N$ the variance-covariance matrix for returns.

- In the Markowitz problem, $\mathbb{E}[R]$ and $\mathbb{V}[R]$ are typically estimated using historical sample data.

- As we have seen, the sample-based Markowitz portfolio performs very poorly out-of-sample.

- This is the first motivation for the Black-Litterman model: how to obtain better estimates of expected returns using an asset-pricing model, the CAPM.

Portfolio choice using the CAPM

- The Capital Asset Pricing Model (CAPM) says that
  - every investor should hold the same portfolio of risky assets,
  - this common portfolio then must be the market portfolio.
- From the CAPM-implied market portfolio weights, we will obtain estimates of expected returns (as shown on the next slide).
  - Fischer Black adored the CAPM model.
  - You can read about this in his biography, "Fischer Black and the Revolutionary Idea of Finance" link to book on Amazon.

Expected returns implied by CAPM weights (without proof)

- Start with the Markowitz mean-variance optimal solution:

$$w_{\text{markowitz}} = \frac{1}{\gamma}\mathbb{V}[R_{\text{sample}}]^{-1}(\mathbb{E}[R_{\text{sample}}] - R_f 1_N) \quad \ldots \text{from our earlier class.}$$

- Apply Markowitz to the Market (mkt) portfolio:

$$w_{\text{mkt}} = \frac{1}{\gamma_{\text{mkt}}}\mathbb{V}[R_{\text{sample}}]^{-1}(\mathbb{E}[R_{\text{capm}}] - R_f 1_N)$$

- Get CAPM-implied expected excess returns from market weights:

$$\underbrace{\mathbb{E}[R_{\text{capm}}]}_{N\times 1} - R_f 1_N = w_{\text{mkt}}\gamma_{\text{mkt}}\,\mathbb{V}[R_{\text{sample}}] \quad \ldots \quad \text{we will see how to eliminate } \gamma_{\text{mkt}}$$

Second motivation for the Black-Litterman model

- Imagine that you work for a quant fund.
- A client walks in and, based on asset-pricing theory (CAPM), you recommend that the client should hold the market portfolio.
- The client says, "but I have views about some of the asset returns."
- Question:
  How exactly should you tilt the client's portfolio away from the market portfolio in response to the client's views?
- It is not obvious how to adjust the weights of assets.
  - Because asset returns are correlated, changing the weight on one asset will also change the other weights.
- The Black-Litterman model tells us how to incorporate the client's views, which is the second motivation for this model.

Deviating from the market portfolio based on "views"

- The client could have absolute views:
  - Apple's return will be 2% below its historical mean.
  - Intel's return will be 4% above its historical mean.
- Or, the client could have relative views:
  - Meta will underperform Google by 5%.
  - NVIDIA will outperform both Apple and Intel by 3%.
- The confidence (inverse of variance) about each absolute and relative view could differ, so we need to account for this also.
  - The more confident the investor is about a particular view, the smaller the variance of that view.

Expressing Absolute and Relative Views

- The investor can have K views, denoted by the $K \times 1$ vector q, about the N expected returns, μ.
- These K views can be expressed as a linear combination of returns, through a "pick" matrix P as follows:

$$P\,\mu = q + \epsilon_q, \quad \text{where:} \quad \epsilon_q \sim \mathcal{N}(0_K, V_{\epsilon_q}), \tag{1}$$

where

  - P is $K \times N$ "pick" matrix;
  - μ is an $N \times 1$ vector of investor's expected returns;
  - q is $K \times 1$ vector of views about future absolute or relative returns;
  - ϵ_q is $K \times 1$ vector of "errors" with multivariate normal distribution,
    - with a mean given by the $K \times 1$ vector of 0_K, and
    - covariances given by the $K \times K$ matrix V_{ϵ_q}.

Example of views expressed via P matrix . . . I

- Suppose that the market has only $N = 5$ securities.
- Suppose that the investor has the following $K = 3$ views:
  1. An absolute view that the return on Security 1 will be 10%, with a view-variance of 9%.
  2. An absolute view that the return on Security 2 will be 8%, with a view-variance of 4%.
  3. A relative view that the return on Security 3 will exceed that on Security 4 by 2%, with a view-variance of 1%.

  - No absolute view about Security 3.
  - No absolute or relative view about Security 5.

Example of views expressed via P matrix . . . II

- To express these $K = 3$ views using Equation (1), which is:

$$P\,\mu = q + \epsilon_q, \quad \text{where:} \quad \epsilon_q \sim \mathcal{N}(0, V_{\epsilon_q}), \tag{1}$$

- we specify the following:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix}}_{\substack{P \\ K \times N}}^{\textit{pick matrix}} \times \underbrace{\begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \\ \mu_5 \end{bmatrix}}_{\substack{\mu \\ N \times 1}}^{\textit{means}} = \underbrace{\begin{bmatrix} 0.10 \\ 0.08 \\ 0.02 \end{bmatrix}}_{\substack{q \\ K \times 1}}^{\textit{views}} + \underbrace{\begin{bmatrix} \epsilon_{q,1} \\ \epsilon_{q,2} \\ \epsilon_{q,3} \end{bmatrix}}_{\substack{\epsilon_q \\ K \times 1}}^{\textit{errors}}$$

- and the $K \times K$ diagonal matrix of "view variances":

$$V_{\epsilon_q} = \begin{bmatrix} 0.09 & 0 & 0 \\ 0 & 0.04 & 0 \\ 0 & 0 & 0.01 \end{bmatrix}.$$

Blend/condition expected returns from CAPM with investor's views (without proof)

- Blend (i.e., condition) the CAPM-implied expected returns with the investor's views to get
  - Expected returns, conditional on views: $\mathbb{E}[R_{\text{capm}}|\text{views}]$, and
  - Return-covariance matrix, conditional on views: $\mathbb{V}[R_{\text{sample}}|\text{views}]$.
- Compute the Black-Litterman (BL) weights using blended views:

$$w_{\text{BL}} = \frac{1}{\gamma}\mathbb{V}[R_{\text{sample}}|\text{views}]^{-1}(\mathbb{E}[R_{\text{capm}}|\text{views}] - R_f 1_N).$$

Road map

Main advantages of Black-Litterman model

- The main advantages of the Black-Litterman model are that
  - You can provide views on only a subset of assets and the model will make the correct adjustments for the covariance with other assets.
  - You can provide confidence (inverse of variance) about your views, which will be reflected in the degree of shrinkage.
  - Using Black-Litterman posterior returns leads to more reasonable portfolios than those from using sample moments of returns.

The importance of the Black-Litterman model

- The He and Litterman (1999, page 13) article says the following:
  - In the Quantitative Strategies group at Goldman Sachs Asset Management, we develop quantitative models to manage portfolios.
  - The Black-Litterman model is the central framework for our modeling process.
  - Our process starts with finding a set of views that are profitable.
    - For example, it is well known that portfolios based on certain value and momentum factors are consistently profitable.

      Note that Mark Carhart, who developed the momentum factor, worked at Goldman Sachs Asset Management; Wikipedia link.
    - We forecast returns for portfolios incorporating these factors and construct a set of views.
  - The Black-Litterman model takes these views and constructs a set of expected returns on each asset.

Relation/link to what we did in the last chapter

- In the last chapter, we dealt with unreasonable sample estimates of asset-return moments (means and covariances)
  - by shrinking them
  - toward a reasonable "value."
- In the Black-Litterman model, we are going to
  - start with a reasonable portfolio, the market portfolio, and
  - tilt away from this portfolio based on an investor's views.

Is the Black-Litterman model a Bayesian model?

- The Black and Litterman (1990, 1992) model is a combination of a model-based (CAPM) and view-based approach.
- Strictly speaking, the Black-Litterman model is not a Bayesian model.
  - A Bayesian model combines a prior with the data;
  - The Black-Litterman model combines the CAPM with a view.
- The similarity with the Bayesian approach is that
  - the equation for combining the CAPM with the investor's views
  - is the same as that for updating the prior with the data in the Bayesian approach.

Road map

The Black-Litterman model in two steps . . . I

- We will study the Black-Litterman model in two steps.
  - Both steps are simple.
  - Only the notation is cumbersome.

1. How to back out CAPM-implied expected returns?
2. From those returns, how to get
   2.1 $\mathbb{E}[R_{\text{capm}}|\text{views}]$, i.e., CAPM-implied expected returns conditional on the investor's views;
   2.2 $\mathbb{V}[R_{\text{sample}}|\text{views}]$, i.e., sample covariance matrix conditional on the investor's views.

The Black-Litterman model in two steps . . . II

- To help bring the theory to life, we will then use an example from a paper by Idzorek (2007), which you can download from this link.
  - Note that there are some minor differences in the theory presented in the Idzorek (2007) paper and what we will do in class.
  - The key difference is how the confidence (variance) about the views is specified.
- Finally, the next assignment will ask you to compute the Black-Litterman portfolio for a given set of views.

Notation used in most Black-Litterman papers/websites

Symbol	What it represents
Returns are random	
$\mu = \mathbb{E}(R)$	$N \times 1$ vector of expected (excess) returns
V_{ϵ_R}	$N \times N$ sample covariance matrix of asset-return residuals
V_R	$N \times N$ sample covariance matrix of asset returns
Means of returns are also random	
$\Pi = \mathbb{E}[R_{\text{capm}}]$	$N \times 1$ vector of prior views of $\mu =$ expected returns from CAPM
$\tau\Sigma = V_\mu$	Variance of mean returns (μ)
τ	(scalar) tuning constant, usually less than 1
Views are also uncertain	
q	$K \times 1$ vector of views, where $K \leq N$
P	$K \times N$ pick matrix which captures views about the N assets
$\Omega = V_{\epsilon_q}$	$K \times K$ uncertainty matrix of views

Explaining our notation . . . I

- When discussing the Black-Litterman model, we will be dealing with several random variables. Thus, we need to be careful with notation.
  - The returns, R, have two components, both of which are random:

    $$R = \mu + \epsilon_R, \quad \text{where}$$

    - The residual is random, with distribution: $\epsilon_R \sim \mathcal{N}(0, V_{\epsilon_R})$.
    - The mean returns, μ, are also random, with covariance matrix V_μ.
    - Thus, $R \sim \mathcal{N}(\mu, V_\mu + V_{\epsilon_R})$.
  - The investor has views that have errors, ϵ_q; the covariance matrix of the errors is denoted by V_{ϵ_q}.
  - The views of the investor are uncertain, with covariance matrix V_q.

Explaining our notation ... II

- We will design notation that is slightly different from what is standard, but one that is easier to understand.

- In particular, all covariance matrices are denoted by $\mathbb{V}[x] = V_x$.

- So
  - if returns are: $R = \mu + \epsilon_R$, with both μ and ϵ_R random, and
  - if views are: $q = P\,\mu - \epsilon_q$, (P is a matrix of constants), then:

Our notation	Equal to (using our notation)	Equal to (using notation used by others)
V_{ϵ_R}	—	Σ
V_μ	$\tau\, V_{\epsilon_R}$	$\tau\Sigma$
V_R	$V_\mu + V_{\epsilon_R}$	$\tau\Sigma + \Sigma = (1+\tau)\Sigma$
V_{ϵ_q}	—	Ω
V_q	$PV_\mu P^\top + V_{\epsilon_q}$	$P(\tau\Sigma)P^\top + \Omega$.

The two steps for deriving the Black-Litterman result

- We now study how to
  1. Back out the CAPM-implied expected returns;
  2. From those returns, get $\mathbb{E}[R_{\text{capm}}|\text{views}]$ and $\mathbb{V}[R_{\text{sample}}|\text{views}]$, and use these to solve the Markowitz mean-variance problem.

The steps that summarize the Black-Litterman model

- The explanation of the Black-Litterman model is divided into the steps below.

Step 1.1 Back out expected returns from the CAPM.

Step 1.2 Obtain aggregate risk aversion of the market, γ_{mkt}.

. . . .

Step 2.1 Start with a prior distribution for expected returns; that is,
- i.e., assume that the vector of mean returns is itself random,
- with the mean of the mean given by expected returns from CAPM.

Step 2.2 Specify subjective views of the investor regarding expected returns.

Step 2.3 Use Bayes' Theorem to condition the prior distribution of expected returns on the investor's views to obtain its posterior distribution.

Step 2.4 From the posterior distribution of *expected returns*, get the posterior mean and variance of returns.

Step 2.5 Use the posterior mean and variance of returns, conditional on the investor's views, to solve for the mean-variance optimal portfolio.

Step 1.1 of Black-Litterman model

Back out expected returns from the CAPM.

Step 1.1: Get expected returns from CAPM

- Recall optimality condition for any mean-variance efficient portfolio:

$$w = \frac{1}{\gamma} V_R^{-1} (\mathbb{E}[R] - R_f 1_N) \qquad \text{. . . condition for optimality;}$$

Then, if the CAPM is true, the above equation for the market-portfolio weights is

$$w_{\text{mkt}} = \frac{1}{\gamma_{\text{mkt}}} V_R^{-1} (\mathbb{E}[R_{\text{capm}}] - R_f 1_N) \qquad \text{. . . under the CAPM;}$$

$$\gamma_{\text{mkt}} V_R w_{\text{mkt}} = (\mu_{\text{capm}} - R_f 1_N) \qquad \text{. . . isolating expected return}$$

$$\mu_{\text{capm}} - R_f 1_N = \gamma_{\text{mkt}} V_R w_{\text{mkt}} \qquad \text{. . . rearranging LHS and RHS}$$

- where $\mu_{\text{capm}} \equiv \mathbb{E}[R_{\text{capm}}]$ is the $N \times 1$ vector with expected returns for the N risky assets according to the CAPM.
- Note that R_f, V_R, and w_{mkt} are observable from market data.
- But γ_{mkt} is not observable, so we need to identify this.

Step 1.2 of Black-Litterman model

Obtain aggregate risk aversion of the market, γ_{mkt}.

Step 1.2: Obtaining aggregate risk aversion of the market

- We start by re-writing the expression from the previous page:

$$\mu_{\text{capm}} = R_f \, 1_N + \gamma_{\text{mkt}} \, V_R \, w_{\text{mkt}} \tag{2}$$

then, multiplying both sides by $w_{\text{mkt}}^\top$, we get

$$\underbrace{w_{\text{mkt}}^\top \mu_{\text{capm}}}_{=\mu_{\text{mkt}}} = R_f \underbrace{w_{\text{mkt}}^\top 1_N}_{=1} + \gamma_{\text{mkt}} \underbrace{w_{\text{mkt}}^\top V_R \, w_{\text{mkt}}}_{=\sigma^2_{\text{mkt}}}$$

$$\mu_{\text{mkt}} = R_f + \gamma_{\text{mkt}} \, \sigma^2_{\text{mkt}} \quad \ldots \quad \text{definition of } \mu_{\text{capm}} \text{ and } \sigma^2_{\text{mkt}}$$

$$\gamma_{\text{mkt}} = \frac{\mu_{\text{mkt}} - R_f}{\sigma^2_{\text{mkt}}} \quad \ldots \quad \text{one expression for } \gamma_{\text{mkt}}$$

$$\gamma_{\text{mkt}} = \underbrace{\left(\frac{\mu_{\text{mkt}} - R_f}{\sigma_{\text{mkt}}} \right)}_{\text{SR}_{\text{mkt}}} \frac{1}{\sigma_{\text{mkt}}} \quad \ldots \quad \text{split denominator into two}$$

$$\gamma_{\text{mkt}} = \frac{\text{SR}_{\text{mkt}}}{\sigma_{\text{mkt}}} \quad \ldots \quad \text{another expression for } \gamma_{\text{mkt}}.$$

CAPM-implied expected returns from Steps 1.1 and 1.2 . . . I

- Putting together our results,

$$\mu_{\text{capm}} - R_f \, 1_N = \gamma_{\text{mkt}} \, V_R \, w_{\text{mkt}} \qquad \ldots \text{Step 1.1}$$

$$\gamma_{\text{mkt}} = \frac{\mu_{\text{mkt}} - R_f}{\sigma^2_{\text{mkt}}} = \frac{\text{SR}_{\text{mkt}}}{\sigma_{\text{mkt}}} \qquad \ldots \text{Step 1.2}$$

leads to the final expression for the CAPM-implied stock returns:

$$\mu_{\text{capm}} - R_f \, 1_N = \frac{\text{SR}_{\text{mkt}}}{\sigma_{\text{mkt}}} V_R \, w_{\text{mkt}} \qquad \ldots \text{in terms of observables.}$$

CAPM-implied expected returns from Steps 1.1 and 1.2 . . . II

- From the last expression on the previous page,

$$\mu_{\text{capm}} - R_f \, 1_N = \left(\frac{\text{SR}_{\text{mkt}}}{\sigma_{\text{mkt}}} \right) V_R \, w_{\text{mkt}},$$

- we see one can get the CAPM-implied expected excess returns, from
  - SR_{mkt}, the market Sharpe ratio;
  - σ_{mkt}, the volatility of the market;
  - V_R, the $N \times N$ sample-covariance matrix of returns;
  - w_{mkt}, the $N \times 1$ vector of market-capitalization weights.

End of Step 1 of the Black-Litterman model

Getting expected returns from CAPM

Step 2 of the Black-Litterman model

Combine expected returns from CAPM with the views of the investor

Step 2.1 of Black-Litterman model

Recognize that the mean returns themselves are random, and identify the distribution of mean returns

Step 2.1: Recognize that mean returns are random . . . I

- Just like the Markowitz model, we start with a vector of random returns;
- We specify the distribution of these returns as follows:

$$R = \mu + \epsilon_R, \quad \text{where}$$
$$\epsilon_R \sim \mathcal{N}\big(0, V_{\epsilon_R} = \Sigma\big);$$
$$R \sim \mathcal{N}\big(\mu, V_R\big)$$

- Black and Litterman assume that expected returns μ themselves are random, where the prior distribution of expected returns is

$$\mu \sim \mathcal{N}\big(\mu_{\text{capm}}, V_\mu\big) \quad \text{. . . i.e., prior mean is given by CAPM.}$$

Step 2.1: Recognize that mean returns are random . . . II

- Therefore, given returns

$$R = \mu + \epsilon_R,$$

- if μ is random, then the distribution of returns, R, is

$$R \sim \mathcal{N}(\mathbb{E}[R], V_R), \tag{3}$$

where

$$\mathbb{E}[R] = \mathbb{E}[\mu_{\mathsf{capm}} + \epsilon_R] = \mu_{\mathsf{capm}} \quad \text{. . . because } \mathbb{E}[\epsilon_R] = 0_N,$$

$$V_R = V_\mu + V_{\epsilon_R}, \quad \text{. . . assuming that } \mu \text{ and } \epsilon_R \text{ are independent.}$$

Step 2.1: Recognize that mean returns are random . . . III

- Black and Litterman suggest that the mean of expected returns should be less variable than the returns themselves; therefore,

$$V_\mu = \tau V_{\epsilon_R}, \quad \text{where} \quad \tau \leq 1.$$

- In classical statistics, $\tau = 1/T^{\text{est}}$, where T^{est} is the number of observations used to estimate sample moments
  - So, if you are using 5 years of monthly observations, $T^{\text{est}} = 60$;
  - implying that τ is a small number, close to 0.
- Thus,

$$\begin{aligned} V_R &= V_\mu + V_{\epsilon_R}, && \ldots \quad \text{assuming that } \mu \text{ and } \epsilon_R \text{ are independent} \\ &= \tau\Sigma + \Sigma \\ &= (1+\tau)\Sigma \\ &\approx \Sigma && \ldots \quad \text{if } \tau = 1/T^{\text{est}} \text{ is small.} \end{aligned}$$

Step 2.2 of Black-Litterman model

Specify subjective views of investor regarding expected returns.

Step 2.2: Views of the investor and their distribution

- In the absence of views that differ from the implied equilibrium return, the investor should hold the market portfolio.
- The views of the investor may be expressed in
  - absolute terms:
    Asset i will have a return of 10%, with a variance of 50%;
  - relative terms:
    Asset n's return will exceed Asset m's, with a variance of 20%.
- The more confident the investor is about a particular view, the smaller the variance of that view.

Expressing Absolute and Relative Views

- The investor can have K views, $q \in \mathbb{R}^K$, about the N returns.
- These K views can be expressed as a linear combination of returns, through a "pick" matrix P as follows:

$$P\,\mu = q + \epsilon_q, \quad \text{where:}$$
$$q \sim \mathcal{N}\big(\mathbb{E}[q], V_q\big) \text{ and } \epsilon_q \sim \mathcal{N}(0, V_{\epsilon_q}),$$

where

  - P is $K \times N$ matrix;
  - μ is $N \times 1$ vector of investor's expected returns;
  - q is $K \times 1$ vector of views about future absolute or relative returns;
  - ϵ_q is $K \times 1$ vector of errors with multivariate normal distribution,
    - with a mean given by the $K \times 1$ vector of 0_K, and
    - covariances given by the $K \times K$ matrix V_{ϵ_q}.

The Distribution of Views

- We wish to determine the distribution of the investor's views, q.
- We can obtain this from Equation (1), which is reproduced below:

$$P\,\mu = q + \epsilon_q, \tag{1}$$

- Re-arranging the terms in the equation above, we get:

$$q = P\,\mu - \epsilon_q, \quad \text{where,}$$
$$\epsilon_q \sim \mathcal{N}(0, V_{\epsilon_q}) \;\; \text{and} \;\; q \sim \mathcal{N}\big(\mathbb{E}[q], V_q\big),$$

- From the expressions above, we get the distribution of views:

$$\mathbb{E}[q] = \mathbb{E}[P\,\mu - \epsilon_q] = P\,\mu \qquad \ldots \quad \mathbb{E}[\epsilon_q] = 0$$
$$V_q = \mathbb{V}[P\,\mu - \epsilon_q] = P\,V_\mu\,P^\top + V_{\epsilon_q} = P(\tau\Sigma)P^\top + \Omega,$$
$$\mathbb{C}[\mu, q] = \mathbb{C}[\mu, P\,\mu - \epsilon_q] = V_\mu\,P^\top \quad \ldots \mathbb{C}[\mu, \epsilon_q] = 0, \text{note: } \mathbb{C} \text{ denotes Cov}$$
$$\mathbb{C}[q, \mu] = \mathbb{C}[P\,\mu - \epsilon_q, \mu,] = P\,V_\mu \quad \ldots \mathbb{C}[\epsilon_q, \mu] = 0.$$

Step 2.3 of Black-Litterman model

Combine expected returns from CAPM with the views of the investor

Use Bayes' Theorem to condition the prior distribution of *expected returns* on the investor's views to obtain its posterior distribution

Step 2.3: Posterior distribution of *expected returns* . . . I

- From standard results about multivariate normal variables, we know
- the expectation of a vector of random variables Y conditional on a particular realization x of the random vector X is:

$$\mathbb{E}[Y|x] = \mathbb{E}[Y] + \underbrace{\mathbb{C}[Y,X]V_X^{-1}}_{\text{beta}}(x - \mathbb{E}[X]),$$

- with the conditional variance of Y given by

$$\mathbb{V}[Y|x] = \mathbb{V}[Y] - \underbrace{\mathbb{C}[Y,X]\,V_X^{-1}}_{\text{beta}}\mathbb{C}[X,Y].$$

Step 2.3: Posterior distribution of *expected returns* ...II

- In our context,
  - $Y = \mu$, and
  - $X = q$.
- Thus, in our context, the conditional expectation of mean returns is

$$\mathbb{E}[\mu|q] = \mathbb{E}[\mu] + \mathbb{C}[\mu, q] V_q^{-1}\big(q - \mathbb{E}[q]\big), \tag{4}$$

with the conditional variance of μ given by

$$\mathbb{V}[\mu|x] = V_\mu - \mathbb{C}[\mu, q]\, V_q^{-1}\, \mathbb{C}[q, \mu]. \tag{5}$$

Step 2.3: Posterior distribution of *expected returns* ...III

- In the context of the Black-Litterman model, $Y = \mu$, with:

$$\mathbb{E}[\mu] = \mu_{\text{capm}} \quad \text{...assumption that prior of mean is given by CAPM,}$$

$$V_\mu = \tau\, V_{\epsilon_R} = \tau\, \Sigma.$$

- Similarly, X corresponds to $q = P\,\mu - \epsilon_q$, with:

$$\mathbb{E}[q] = \mathbb{E}[P\,\mu - \epsilon_q] = P\,\mathbb{E}[\mu] = P\,\mu_{\text{capm}},$$

$$V_q = \mathbb{V}[P\,\mu - \epsilon_q] = P\,V_\mu\,P^\top + V_{\epsilon_q} = P\,(\tau\,\Sigma)\,P^\top + \Omega,$$

$$\mathbb{C}[q,\mu] = \mathbb{C}[P\,\mu - \epsilon_q, \mu] = P\,V_\mu = P\,(\tau\,\Sigma)$$

$$\mathbb{C}[\mu, q] = \mathbb{C}[\mu, P\,\mu - \epsilon_q] = V_\mu\,P^\top = (\tau\,\Sigma)\,P^\top.$$

Step 2.3: Posterior distribution of *expected returns* . . . IV

- Making these substitutions we get the conditional expectation of μ

$$\begin{aligned}
\mathbb{E}[\mu|q] &= \mathbb{E}[\mu_{\text{capm}}|\text{views}] \\
&= \mathbb{E}[\mu] + \mathbb{C}[\mu, q] V_q^{-1} \big(q - \mathbb{E}[q]\big), && (4) \\
&= \mu_{\text{capm}} + \big[(\tau\,\Sigma)\, P^\top \big] \big[P(\tau\,\Sigma)P^\top + \Omega \big]^{-1} \big(q - P\,\mu_{\text{capm}}\big) && (6)
\end{aligned}$$

with the conditional variance of μ

$$\begin{aligned}
\mathbb{V}[\mu|q] &= \mathbb{V}[\mu_{\text{capm}}|\text{views}] \\
&= V_\mu - \mathbb{C}[\mu, q]\, V_q^{-1}\, \mathbb{C}[q, \mu]. && (5) \\
&= \Big[\tau\,\Sigma\Big] - \Big[(\tau\,\Sigma)\, P^\top\Big] \Big[P(\tau\,\Sigma)P^\top + \Omega\Big]^{-1} \Big[P\,(\tau\,\Sigma)\Big]. && (7)
\end{aligned}$$

- **Note** that the results we have in equations (6) and (7) are for
  - the distribution of expected returns, μ,
  - while what we need is the distribution of returns, R.

Step 2.4 of Black-Litterman model

From the posterior distribution of expected returns,
get the posterior distribution of returns.

More precisely, use the posterior distribution of expected returns to obtain the posterior mean and posterior variance of returns (conditional on investor's views).

Step 2.4: Posterior (conditional) distribution of returns . . . I

- In the previous step, we have computed the posterior (conditional) distribution of expected returns.
- We now use that result for expected returns to derive the posterior (conditional) distribution of returns.

Step 2.4: Posterior (conditional) distribution of returns . . . II

- Recall also from Equation (3) on Page 53 that

$$R = \mu + \epsilon_R, \qquad \epsilon_R \sim \mathcal{N}\big(0, V[\epsilon_R]\big), \tag{3}$$

- which implies that conditional on the views q

$$\mathbb{E}[R|q] = \mathbb{E}[\mu|q] + \mathbb{E}[\epsilon_R|q] = \mathbb{E}[\mu|q], \quad \ldots \text{where } \mathbb{E}[\epsilon_R|q] = 0;\ \mathbb{E}[\mu|q] \text{ is in Eqn. (6)}$$

$$= \mu_{\mathbf{capm}} + \left[(\tau\,\Sigma)\,P^\top\right]\left[P(\tau\,\Sigma)P^\top + \Omega\right]^{-1}(q - P\,\mu_{\mathbf{capm}})\,, \tag{8}$$

$$\mathbb{V}[R|q] = V_{\epsilon_R} + \mathbb{V}[\mu|q], \quad \ldots \text{where } \mathbb{V}[\mu|q] \text{ is in (7) and } V_{\epsilon_R} = \Sigma$$

$$= \Sigma + \left[\tau\,\Sigma - \Big((\tau\,\Sigma)\,P^\top\Big)\,\Big(P(\tau\,\Sigma)P^\top + \Omega\Big)^{-1}\,\Big(P\,(\tau\,\Sigma)\Big)\right].$$

$$= (1+\tau)\,\Sigma - \Big((\tau\,\Sigma)\,P^\top\Big)\,\Big(P(\tau\,\Sigma)P^\top + \Omega\Big)^{-1}\,\Big(P\,(\tau\,\Sigma)\Big)\,. \tag{9}$$

Interpreting expression for conditional mean of the return

- The expected return conditional on views can be interpreted as a weighted average of the model-return and the view-return.

$$\mathbb{E}[R|q] = \mathbb{E}[R|\text{views}]$$

$$= \underbrace{\mu_{\text{capm}}}_{\text{model mean}} + \underbrace{\left[(\tau\,\Sigma)\,P^\top\right]\left[P(\tau\,\Sigma)P^\top + \Omega\right]^{-1}}_{\text{weight on your view}} \underbrace{(q - P\,\mu_{\text{capm}})}_{\text{your views vs. model}}, \quad (8)$$

which, after some algebra, can be written in another way:

$$= \left[(\tau\Sigma)^{-1} + P^\top\Omega^{-1}P\right]^{-1}\left[(\tau\Sigma)^{-1}\underbrace{\mathbb{E}[R_{\text{capm}}]}_{\text{model}} + P^\top\Omega^{-1}\underbrace{q}_{\text{views}}\right] \qquad (10)$$

which is the weighted average of the "model" and the "views":

$$= \underbrace{\left[(\tau\Sigma)^{-1} + P^\top\Omega^{-1}P\right]^{-1}(\tau\Sigma)^{-1}}_{\text{weight on CAPM-implied return}}\underbrace{\mathbb{E}[R_{\text{capm}}]}_{\text{model}}$$

$$+ \underbrace{\left[(\tau\Sigma)^{-1} + P^\top\Omega^{-1}P\right]^{-1}P^\top\Omega^{-1}}_{\text{weight on views}}\underbrace{q}_{\text{views}}.$$

Another way of writing the conditional variance of returns

- The expression for the conditional variance can also be simplified (for details of the derivation, see Meucci 2010):

$$\begin{aligned} \mathbb{V}[R|q] &= \mathbb{V}[R|\text{views}] \\ &= (1+\tau)\,\Sigma - \left((\tau\,\Sigma)\,P^\top\right)\left(P(\tau\,\Sigma)P^\top + \Omega\right)^{-1}\left(P\,(\tau\,\Sigma)\right). \qquad (9) \\ &= \Sigma + \left[(\tau\Sigma)^{-1} + P^\top\Omega^{-1}P\right]^{-1}. \qquad (11) \end{aligned}$$

Step 2.5 of Black-Litterman model

Use the posterior mean and variance of returns, conditional on the investor's views, to solve for the Black-Litterman portfolio weights.

Black-Litterman portfolio weights . . . I

- We now know how to obtain the expected returns (and covariance matrix of returns)
  1. implied by the CAPM (Part 1),
  2. conditional on the views of the investor (Part 2).
- To find the optimal portfolio, we proceed just as before, but
  - instead of using the sampling distribution of returns,
  - we use the posterior distribution.

Black-Litterman portfolio weights . . . II

- That is, in contrast to Markowitz portfolio weights,

$$w_{\text{markowitz}} = \frac{1}{\gamma} \left(\mathbb{V}[R_{\text{sample}}]\right)^{-1} \left(\mathbb{E}[R_{\text{sample}}] - R_f 1_N\right),$$

- the Black-Litterman portfolio weights, w_{BL}, are given by:

$$w_{\text{BL}} = \frac{1}{\gamma} \left(\mathbb{V}[R|\text{q}]\right)^{-1} \left(\mathbb{E}[R|\text{q}] - R_f 1_N\right), \tag{12}$$

  with
  - $\mathbb{E}[R|\text{q}]$ defined in Equations (8) or (10), and
  - $\mathbb{V}[R|\text{q}]$ defined in Equations (9) or (11).

Black-Litterman combined with constraints or shrinkage

- Note that the Black-Litterman approach can be combined with the methods we studied in the last class. For example:
  - When choosing the portfolio weights using mean-variance optimization with the Black-Litterman μ_{BL} and Σ_{BL}, one can impose portfolio constraints;
  - The covariance matrix used can be Σ_{BL} with Ledoit-Wolf shrinkage applied to it.

End of the Black-Litterman model with the detailed derivations

Road map

Example of Black-Litterman with one asset, one view . . . I

- To confirm our understanding and gain further intuition, we consider the special case of the Black-Litterman model with
  - one risky asset, and
  - one view about this asset.
- We show that expected returns conditional on views
  - are a weighted average of the returns from the CAPM model and the investors' views about returns;
  - with the weights depending on the relative confidence in the model and in the return views.

Example of Black-Litterman with one asset, one view . . . II

- For the special case of one asset and one view:
  - $V_{\epsilon_R} = \Sigma = \sigma^2_{\epsilon_R}$
  - $P = 1$
  - $V_{\epsilon_q} = \Omega = \sigma^2_{\epsilon_q}$.

Posterior mean of returns ...I

- Recall the general result in Equation (6) that

$$\begin{aligned}\mathbb{E}[R|q] &= \mu_{\textbf{capm}} + \left[(\tau\, V_{\epsilon_R})\, P^\top\right] \left[P(\tau\, V_{\epsilon_R})P^\top + V_{\epsilon_q}\right]^{-1} (q - P\,\mu_{\textbf{capm}})\,. \\ &= \mu_{\textbf{capm}} + \left[(\tau\,\Sigma)\, P^\top\right] \left[P(\tau\,\Sigma)P^\top + \Omega\right]^{-1} (q - P\,\mu_{\textbf{capm}})\,.\end{aligned} \tag{6}$$

- Making the substitutions for
  - $V_{\epsilon_R} = \Sigma = \sigma^2_{\epsilon_R}$
  - $P = 1$
  - $V_{\epsilon_q} = \Omega = \sigma^2_{\epsilon_q}$.

  in Equation (6) leads to the result on the next slide:

Posterior mean of returns . . . II

$$\mathbb{E}[R|q] = \mu_{\text{capm}} + \left(\tau\,\sigma^2_{\epsilon_R}\right)\left[\left(\tau\,\sigma^2_{\epsilon_R}\right) + \sigma^2_{\epsilon_q}\right]^{-1}\left(q - \mu_{\text{capm}}\right)$$

$$= \mu_{\text{capm}} \underbrace{\left[1 - \frac{\tau\,\sigma^2_{\epsilon_R}}{\tau\,\sigma^2_{\epsilon_R} + \sigma^2_{\epsilon_q}}\right]}_{\text{weight on model}} + q \underbrace{\left[\frac{\tau\,\sigma^2_{\epsilon_R}}{\tau\,\sigma^2_{\epsilon_R} + \sigma^2_{\epsilon_q}}\right]}_{\text{weight on view}} \quad \text{. . . collect terms}$$

$$= \mu_{\text{capm}} \underbrace{\left[\frac{\sigma^2_{\epsilon_q}}{\tau\,\sigma^2_{\epsilon_R} + \sigma^2_{\epsilon_q}}\right]}_{\text{weight on model}} + q \underbrace{\left[\frac{\tau\,\sigma^2_{\epsilon_R}}{\tau\,\sigma^2_{\epsilon_R} + \sigma^2_{\epsilon_q}}\right]}_{\text{weight on view}} \quad \text{. . . simplify first fraction}$$

$$= \mu_{\text{capm}} \underbrace{\left[\frac{\frac{1}{\tau\,\sigma^2_{\epsilon_R}}}{\frac{1}{\tau\,\sigma^2_{\epsilon_R}} + \frac{1}{\sigma^2_{\epsilon_q}}}\right]}_{\substack{\text{relative confidence} \\ \text{in model}}} + q \underbrace{\left[\frac{\frac{1}{\sigma^2_{\epsilon_q}}}{\frac{1}{\tau\,\sigma^2_{\epsilon_R}} + \frac{1}{\sigma^2_{\epsilon_q}}}\right]}_{\substack{\text{relative confidence} \\ \text{in view}}} \quad \text{. . . divide by } \tau\,\sigma^2_{\epsilon_R} \times \sigma^2_{\epsilon_q}$$

$$= \mu_{\text{capm}}\,\psi + q\,(1 - \psi) \quad \text{. . . weighted average of view and CAPM}$$

Road map

Example of Black-Litterman model: Numerical

- To check our understanding of the Black-Litterman model, we will now consider a numerical application of the model.
  - We will consider an example from the paper by Idzorek (2007), which you can download from this link.
- The next assignment will give you another opportunity to code the Black-Litterman model in Python.
  - Writing the code for the Black-Litterman model is simple, so I recommend you write the code yourself rather than using a package.
  - In my view, writing the code will be simpler than figuring out how to use a package written by someone else.
  - If you decide to use a package and find one that is simple to use, please let me know (so I can learn about the package).

Inputs needed for implementing Black-Litterman model

- Data from the market:
  - Risk-free rate, R_f.
  - Market-capitalization weights, w_{mkt}.
  - Volatility of the return on the market, $\sqrt{\mathbb{V}[R_{\mathrm{mkt}}]}$.
  - Sharpe ratio of the return on the market, $\frac{\mathbb{E}[R_{\mathbf{mkt}}]-R_f}{\sqrt{\mathbb{V}[R_{\mathbf{mkt}}]}}$.
  - Variance-covariance matrix of sample returns, $V_{\epsilon_R} = \Sigma$.
- Subjective parameters:
  - Parameter to shrink variance-covariance matrix of sample returns, τ.
  - Matrix reflecting absolute and relative views, P.
  - Estimate of returns based on absolute and relative views, q_{views}.
  - Estimates of variance of the error in the views, $V_{\epsilon_q} = \Omega$.

Data provided to us . . . I

- Assume that $\tau = 0.025$ ($\tau \approx= 1/T^{\text{est}}$, where T^{est} is the number of data points used in the estimation of the return moments, so this corresponds to 40 years of annual observations.)
- Number of risky assets = $N = 8$, which are listed on the next page.
- All return data for the assets is in excess of the risk-free rate.
- The market
  - Sharpe ratio is 0.426169, and
  - the market return volatility is 14.0789%.

Data provided to us . . . II

- Historical sample excess mean returns (μ_{sample}) and market-capitalization weights, w_{mkt}.

#	Asset Class	μ_{sample}	w_{mkt}
1	US Bonds	3.15%	0.180409
2	Int'l Bonds	1.75%	0.268921
3	US Large Growth	−6.39%	0.119896
4	US Large Value	−2.86%	0.124435
5	US Small Growth	−6.75%	0.016023
6	US Small Value	−0.54%	0.010849
7	Int'l Dev. Equity	−6.75%	0.243523
8	Int'l Emerg. Equity	−5.26%	0.035942

Data provided to us . . . III

- Covariance matrix of excess returns, Σ is:

Asset Class	US Bonds	Intern Bonds	US Large Growth	US Large Value	US Small Growth	US Small Value	Intern Dev. Equity	Inter Emerg. Equity
US Bonds	0.00100	0.00132	-0.00057	-0.00067	0.00012	0.00012	-0.00044	-0.00043
Intern Bonds	0.00132	0.00727	-0.00130	-0.00061	-0.00223	-0.00098	0.00144	-0.00153
US Large Growth	-0.00057	-0.00130	0.05985	0.02758	0.06349	0.02303	0.03296	0.04803
US Large Value	-0.00067	-0.00061	0.02758	0.02960	0.02657	0.02146	0.02069	0.02985
US Small Growth	0.00012	-0.00223	0.06349	0.02657	0.10248	0.04274	0.03994	0.06599
US Small Value	0.00012	-0.00098	0.02303	0.02146	0.04274	0.03205	0.01988	0.03223
Intern Dev. Equity	-0.00044	0.00144	0.03296	0.02069	0.03994	0.01988	0.02835	0.03506
Inter Emerg. Equity	-0.00043	-0.00153	0.04803	0.02985	0.06599	0.03223	0.03506	0.07995

Data provided to us . . . IV

- The investor has three views, the first absolute, the others relative.
  - View 1: International Developed Equity will have an absolute excess return of 5.25% (with a view variance of = 0.000709).
  - View 2: International Bonds will outperform US Bonds by 25 basis points (view variance = 0.000141).
  - View 3: US Large Growth and US Small Growth will outperform US Large Value and US Small Value by 2% (view variance = 0.000866).
- Thus,

$$q + \epsilon_q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} + \begin{bmatrix} \epsilon_{q_1} \\ \epsilon_{q_2} \\ \epsilon_{q_3} \end{bmatrix} = \begin{bmatrix} 5.25 \\ 0.25 \\ 2.00 \end{bmatrix} + \begin{bmatrix} \epsilon_{q_1} \\ \epsilon_{q_2} \\ \epsilon_{q_3} \end{bmatrix}$$

- And, the $K \times N$ pick matrix corresponding to the $K = 3$ views for the $N = 8$ assets is

$$P = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & -0.5 & 0.5 & -0.5 & 0 & 0 \end{bmatrix}$$

Data provided to us . . . V

- We now need to specify the matrix Ω, which captures the uncertainty in the views.

$$\Omega = \begin{bmatrix} \Omega_{11} & 0 & 0 \\ 0 & \Omega_{22} & 0 \\ 0 & 0 & \Omega_{33} \end{bmatrix} = \begin{bmatrix} 0.000709 & 0 & 0 \\ 0 & 0.000141 & 0 \\ 0 & 0 & 0.000866 \end{bmatrix}$$

- These numbers are obtained from $P_k(\tau\Sigma)P_k^\top$, where $k = \{1, 2, 3\}$ is the k-th row of matrix P.

- I have used one method for specifying the Ω matrix, but there are others (and you may wish to use one of the other methods).

We are ready to start analyzing the data given to us.

Numerical example: Markowitz portfolio weights

- We start by computing the Markowitz portfolio weights (w_{MVU}) for $\gamma_{mkt} = 3.0271189$.

$$w_{\text{MVU}} = \frac{1}{\gamma}\Sigma^{-1}\mu_{\text{sample}}$$

Asset Class	μ_{sample} %	w_{mkt} %	w_{MVU} %
US Bonds	3.15	18.04	1161.24
Int'l Bonds	1.75	26.89	−106.56
US Large Growth	−6.39	11.98	56.00
US Large Value	−2.86	12.44	−5.72
US Small Growth	−6.75	1.60	−61.61
US Small Value	−0.54	1.08	82.81
Int'l Dev. Equity	−6.75	24.35	−105.48
Int'l Emerg. Equity	−5.26	3.59	14.78

- It is clear from the last column that the mean-variance weights based on historical estimates of the sample mean are not reasonable.

Numerical example: Implied CAPM excess returns

- Next, we compute the expected (excess) returns implied by the CAPM, using the expression:

$$w_{mkt} = \frac{1}{\gamma_{mkt}} \Sigma^{-1} \mu_{\text{capm}} \quad \text{which implies} \quad \mu_{\text{capm}} = \gamma_{mkt} \Sigma \, w_{mkt}$$

Asset Class	μ_{sample} %	w_{mkt} %	w_{MVU} %	μ_{capm} %
US Bonds	3.15	18.04	1161.24	0.08
Int'l Bonds	1.75	26.89	−106.56	0.67
US Large Growth	−6.39	11.98	56.00	6.41
US Large Value	−2.86	12.44	−5.724	4.08
US Small Growth	−6.75	1.60	−61.61	7.43
US Small Value	−0.54	1.08	82.81	3.70
Int'l Dev. Equity	−6.75	24.35	−105.48	4.80
Int'l Emerg. Equity	−5.26	3.59	14.78	6.60

- Clearly, the CAPM-implied expected excess returns seem much more reasonable than the sample-based historical mean returns, μ_{sample}.

Numerical example: Posterior mean returns . . . I

- We now combine the CAPM-implied expected excess returns with the investor's views to obtain the posterior expected excess returns:

$$\begin{aligned}\mu_{\mathrm{BL}} &= \mathbb{E}[R_{\mathrm{capm}}|\text{views}] - R_f 1_N \\ &= \left[(\tau\Sigma)^{-1} + P^\top \Omega^{-1} P\right]^{-1} \left[(\tau\Sigma)^{-1} \underbrace{\mu_{\mathrm{capm}}}_{\text{model}} + P^\top \Omega^{-1} \underbrace{q}_{\text{views}}\right]. \qquad (10)\end{aligned}$$

Asset Class	μ_{sample} %	w_{mkt} %	w_{MVU} %	μ_{capm} %	μ_{BL} %
US Bonds	3.15	18.04	1161.24	0.08	0.0621
Int'l Bonds	1.75	26.89	−106.56	0.67	0.5036
US Large Growth	−6.39	11.98	56.00	6.41	6.2827
US Large Value	−2.86	12.44	−5.72	4.08	4.3383
US Small Growth	−6.75	1.60	−61.61	7.43	7.2545
US Small Value	−0.54	1.08	82.81	3.70	3.9105
Int'l Dev. Equity	−6.75	24.35	−105.48	4.80	4.8576
Int'l Emerg. Equity	−5.26	3.59	14.78	6.60	6.6881

Numerical example: Posterior return covariance matrix

- Similarly, the posterior covariance matrix of returns is:

$$\begin{aligned}\Sigma_{\mathrm{BL}} &= \mathbb{V}[R_{\mathrm{sample}}|\mathrm{views}] \\ &= \Sigma + \left[(\tau\Sigma)^{-1} + P^{\top}\Omega^{-1}P\right]^{-1}. \end{aligned} \tag{11}$$

$$\Sigma_{\mathrm{BL}} = \begin{bmatrix} 0.00102 & 0.00135 & -0.00058 & -0.00068 & 0.00013 & 0.00013 & -0.00045 & -0.00044 \\ 0.00135 & 0.00738 & -0.00133 & -0.00063 & -0.00226 & -0.00100 & 0.00144 & -0.00156 \\ -0.00058 & -0.00133 & 0.06061 & 0.02801 & 0.06415 & 0.02331 & 0.03330 & 0.04857 \\ -0.00068 & -0.00063 & 0.02801 & 0.03015 & 0.02692 & 0.02181 & 0.02096 & 0.03030 \\ 0.00013 & -0.00226 & 0.06415 & 0.02692 & 0.10385 & 0.04344 & 0.04034 & 0.06681 \\ 0.00013 & -0.00100 & 0.02331 & 0.02181 & 0.04344 & 0.03267 & 0.02013 & 0.03272 \\ -0.00045 & 0.00144 & 0.03330 & 0.02096 & 0.04034 & 0.02013 & 0.02868 & 0.03546 \\ -0.00044 & -0.00156 & 0.04857 & 0.03030 & 0.06681 & 0.03272 & 0.03546 & 0.08131 \end{bmatrix}$$

Numerical example: Black-Litterman portfolio weights

- We can use our estimates of μBL and Σ_{BL} to compute the Black-Litterman portfolio weights

$$w_{\text{BL}} = \frac{1}{\gamma_{mkt}} \Sigma_{\text{BL}}^{-1} \mu_{\text{BL}}.$$

Asset Class	μ_{sample} %	w_{mkt} %	w_{MVU} %	μ_{capm} %	μ_{BL} %	w_{BL} %
US Bonds	3.15	18.04	1161.24	0.08	0.0621	28.35
Int'l Bonds	1.75	26.89	−106.56	0.67	0.5036	15.48
US Large Growth	−6.39	11.98	56.00	6.41	6.2827	9.07
US Large Value	−2.86	12.44	−5.72	4.08	4.3383	14.76
US Small Growth	−6.75	1.60	−61.61	7.43	7.2545	−1.05
US Small Value	−0.54	1.08	82.81	3.70	3.9105	3.68
Int'l Dev. Equity	−6.75	24.35	−105.48	4.80	4.8576	28.85
Int'l Emerg. Equity	−5.26	3.59	14.78	6.60	6.6881	3.50

Road map

Python code for the Black-Litterman model

- Python code for the Black-Litterman model is available from:
  - PyPortfolioOpt.
  - Luís Fernando Torres.
  - Python for Finance.
  - Robert Martin
  - Cardiel - A portfolio allocation tool based on Black-Litterman with very nice visualization of the portfolio weights.
- My advice to you is to write your own code – you will learn a lot more from writing the code, which is straightforward to do.

Road map

What we plan to do in the next chapter

What's next?

In the next chapter, we will study parametric portfolio policies developed by Brandt, Santa-Clara, and Valkanov (2009).

To do for next class: Readings

- Readings
  - To get just the intuition for the Black-Litterman model, you can read He and Litterman (1999), which is available from this link.
    - The main text of the article offers a non-mathematical discussion;
    - The maths underlying the model is in Appendix B of the article.
  - To read a well-written description of the Black-Litterman model, I recommend Meucci (2010), which provides a careful and detailed analysis. The article can be downloaded from this link.

Road map

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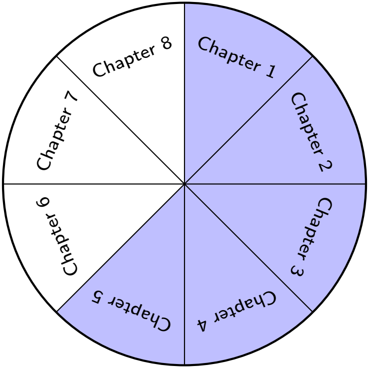

End of Chapter 5