Quantitative Portfolio Management

Sample Final Exam

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SOLUTION/CORRECTION

Sketch of solutions – please excuse typos

1 Managing financial data and performance evaluation

Assume that you are an investment manager who wishes to estimate the mean and volatility (standard deviation) of stock market returns over the last 100 years.

Suppose you have data for **daily prices** for the last 100 years; assume that there are 250 trading days each year, so you have a total of 25 000 prices. The data is in an Excel file called "data.xlsx". The first column has the dates in the format yyyy mm dd and the second column has the prices.

Q1.1 (**1 point**) Please write the Python code to calculate the *annualized mean and volatility* (standard deviation) of **daily** log returns over the 100 years. Include code to import any libraries needed for the calculation. Include comments to make it easier to understand the Python code.

```
### To get credit for your answer, the important steps are:
- Import libraries
- Load the data
- Calculate daily log returns
- Calculate mean and volatility of daily returns
- Annualize the daily returns

# Import libraries
import pandas as pd
import numpy as np

# Load the data from the Excel file
df = pd.read_excel("data.xlsx", header=None, names=["Date", "Price"])

# Convert the "Date" column to datetime format
df["Date"] = pd.to_datetime(df["Date"], format="%Y %m %d")

# Calculate daily log returns
df["Log_Returns"] = np.log(df["Price"] / df["Price"].shift(1))

# Calculate the mean and volatility of daily log returns
mean_daily_log_returns = df["Log_Returns"].mean()
volatility = df["Log_Returns"].std()

# Calculate the annualized mean and volatility of daily log returns
# Assuming 250 trading days per year
annualized_mean_log_returns = mean_daily_log_returns * 250
annualized_volatility = volatility * np.sqrt(250)
```

Q1.2 (**1 point**) Please write the Python code to calculate, using the same data, the annualized mean of **monthly** log returns and volatility of log returns over the 100 years. There is no need to import the libraries or data again; i.e., there is no need to repeat any code you have written for the previous question.

```
### To get credit for your answer, the important steps are:
- Resample to go from daily to monthly frequency
- Annualize the monthly log returns

# Resample the data to monthly frequency
df_monthly = df.set_index("Date").resample("M")

# Calculate monthly log returns
df_monthly["Log_Returns"] = np.log(df_monthly["Price"] / df_monthly["Price"].shift(1))

# Calculate the mean and volatility of monthly log returns
mean_monthly_log_returns = df_monthly["Log_Returns"].mean()
volatility_monthly = df_monthly["Log_Returns"].std()

# Calculate the annualized mean and volatility of monthly log returns
annualized_mean_monthly_log_returns = mean_monthly_log_returns * 12
annualized_volatility_monthly = volatility_monthly * np.sqrt(12)
```

Q1.3 (**1 point**) Please **compare** the precision of the **mean** of log returns computed from **daily** data to that computed from **monthly** data. That is, will one be more precise than the other?

The precision of the mean of log returns computed from daily data is the same as that computed from monthly data; in fact, the point estimate of the mean will be exactly the same because the intermediate prices always cancel when computing the mean of log returns so that the estimate of the mean depends only on the last and first observation.

Q1.4 (**1 point**) Please **compare** the precision of the **volatility** of log returns computed from **daily** data to that computed from **monthly** data. That is, will one be more precise than the other?

The volatility estimated from daily returns will be much more precise than that estimated from monthly data. This is because the precision of the return volatility increases with the frequency of prices; that is, there is no canceling out of the intermediate prices when computing the volatility.

Q1.5 (**1 point**) This part is unrelated to the earlier ones.
Consider the setting where Greta works for an asset management company. Greta has a client, Celine Dijon, who has 100 million euros invested in a market index. She now wishes to invest an additional 1 million euros, and she is considering investing in a biotech company that is doing innovative research on the frontier of medical research. The mean returns in excess of the risk-free rate, volatility of excess returns, and Sharpe ratios are given below.

Item	Market	Biotech
Mean of excess returns	0.10	0.15
Volatility of excess returns	0.20	0.50
Sharpe ratio	0.50	0.30

If Greta says that investing in the biotech company is not a good idea because its Sharpe ratio is lower than the market's, what would be your response to her?

Making investment decisions based on the Sharpe ratio is better than making the decision based on expected returns. However, even the Sharpe ratio ignores **correlations**, so it would be incorrect to decide based only on Sharpe ratios. The correct way to evaluate any new investment is to consider the **improvement in Sharpe ratio** as a result of the new investment and then to see if the difference between the Sharpe ratio after making the new investment is significantly higher than the Sharpe ratio before investing. And, in the case under consideration, the correlation of the return on the biotech with the market return is likely to be very low.

END of Q1 ♠

2 Mean-variance portfolio choice (without and with shrinkage)

Suppose that you are a portfolio manager. You have a client who wishes to invest in three risky assets but wants to minimize her risk as much as possible. So, to satisfy the client's desire to minimize risk, you recommend that she invest in the global minimum variance (GMV) portfolio. The client tells you that the three risky assets are **uncorrelated** and have annual return means of μ_1, μ_2, μ_3, and annual return volatilities of σ_1, σ_2, σ_3.

Q2.1 (**1 point**) Write down the **objective function** of the investor and any **constraints** that the optimization problem must satisfy.

Objective function:

$$\min_{w_1,w_2,w_3} \quad \text{Portfolio Variance} = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + w_3^2\sigma_3^2$$

subject to the constraint that the weights sum to 1:

$$w_1 + w_2 + w_3 = 1.$$

Q2.2 (**1 point**) Write the **Python code** to solve the constrained optimization problem above. Include code to import any libraries needed for the optimization.

```
import numpy as np
from scipy.optimize import minimize

# Function to calculate portfolio variance
def portfolio_variance(weights, cov_matrix):
    return np.dot(weights, np.dot(cov_matrix, weights))

# Constraint: weights must sum to 1
def constraint_sum(weights):
    return np.sum(weights) - 1

# Function to find the global minimum variance (GMV) portfolio
def find_gmv_portfolio(returns, cov_matrix):
    num_assets = len(returns)
    initial_weights = np.ones(num_assets) / num_assets  # Equal weights initially

    # Constraints
    constraints = ({'type': 'eq', 'fun': constraint_sum})

    # Minimize the portfolio variance
    result = minimize(portfolio_variance, initial_weights, args=(cov_matrix,),
                      method='SLSQP', constraints=constraints)

    # Extract the optimal weights
    optimal_weights = result.x
    return optimal_weights
```

Q2.3 (**2 points**) **Solve this optimization problem** (by hand; i.e., analytically) to obtain the expression for the optimal weights in the three assets. Your answer should be an expression depending on the return moments, i.e., the annual return means, μ_1, μ_2, μ_3, and the annual return volatilities, σ_1, σ_2, σ_3.

The Lagrangian for this problem is:

$$L(w_1, w_2, w_3, \lambda) = \sum_{i=1}^{3} w_i^2 \sigma_i^2 + \lambda \left(1 - \sum_{i=1}^{3} w_i\right).$$

Taking partial derivatives with respect to w_1, w_2, w_3, and λ and setting them equal to zero gives:

$$\frac{\partial L}{\partial w_1} = 2w_1\sigma_1^2 - \lambda = 0$$
$$\frac{\partial L}{\partial w_2} = 2w_2\sigma_2^2 - \lambda = 0$$
$$\frac{\partial L}{\partial w_3} = 2w_3\sigma_3^2 - \lambda = 0$$
$$\frac{\partial L}{\partial \lambda} = 1 - \sum_{i=1}^{3} w_i = 0$$

Solving these equations gives:

$$w_1 = \frac{\frac{1}{\sigma_1^2}}{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} + \frac{1}{\sigma_3^2}} \quad w_2 = \frac{\frac{1}{\sigma_2^2}}{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} + \frac{1}{\sigma_3^2}} \quad w_3 = \frac{\frac{1}{\sigma_3^2}}{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} + \frac{1}{\sigma_3^2}}$$

These weights represent the optimal allocation of the investment to each asset to achieve the global minimum variance portfolio.

Q2.4 (**1 point**) Would **Ledoit-Wolf shrinkage** applied to the returns covariance matrix improve the out-of-sample performance of the optimal portfolio weights you estimate? Explain why or why not.

The answer is both "yes" and "no." (That is, credit was given for the reasons you gave, instead of whether you said "yes" or "no.")

On the one hand, Ledoit-Wolf shrinkage would reduce sensitivity to outliers in the estimates of volatility, would improve stability of the solution because of regularization, and also through the allowance of prior information.

However, if we are told that the correlations across assets are zero, then the number of parameters to be estimated for the covariance matrix is already small. So, the benefits of using any shrinkage method is also going to be small.

END of Q2 ♠

3 Black-Litterman model

Imagine that you work for an investment bank that uses the Black-Litterman model to construct portfolios for its clients.

Q3.1 (**0.5 points**) Explain to your new client the underlying **logic and motivation** of the two major steps of the Black-Litterman model, which are to:
(i) back-out expected excess returns from the CAPM; and
(ii) blend expected excess returns from the CAPM with the investor's views.

Sample-based mean-variance portfolio weights are plagued by estimation error. The problem is especially severe when estimating expected returns. One way to resolve this is to use reasonable portfolio weights – the weights implied by the CAPM – to infer expected returns.

This is the **first** step of the Black-Litterman approach. Because the market-capitalization weights are not extreme, the expected returns implied by these weights will also not be extreme, and so provide a sensible starting point for constructing portfolios.

The **second** step of the Black-Litterman model is to incorporate any views that the client has about expected returns by blending them to the CAPM-implied expected returns. This "blending" is done by using the standard formula for Bayesian updating of the prior mean and variance; it is also the same formula as one for finding the mean and variance *conditional* on a view.

In summary, this two-step process aims to provide a more robust and realistic estimate of expected returns, resulting in a well-balanced portfolio that incorporates both market equilibrium and investor-specific information.

Q3.2 (**0.5 points**) Explain to the client why the Black-Litterman approach, which is based on the two steps described above, is better than simply **blending the market-portfolio weights** with the views of the investor.

In general, asset returns are correlated. So, there is no direct way of blending the **weights** from the CAPM with the views of an investor about specific asset. For example, if the view is to overweight a particular asset, it is not clear how one should adjust all the other weights.

Similarly, it is not clear how to incorporate the confidence of the investor in her views into the blending of weights directly.

This is why the Black-Litterman approach, which blends expected returns with views, is superior to an approach that blends weights with views.

Q3.3 (**2 points**) Find the market risk aversion, γ_{mkt}, if

$$\gamma_{\text{mkt}} = \frac{\text{SR}_{\text{mkt}}}{\sigma_{\text{mkt}}},$$

and the market consists of only three risky assets, $n = \{1, 2, 3\}$, whose excess returns are **uncorrelated** and their relative stock-market capitalizations (i.e., market weights), mean excess returns, and volatilities (standard deviations) are given below:

Item	Asset 1	Asset 2	Asset 3
Weight in market portfolio	w_1	w_2	w_3
Mean of sample excess returns	μ_1	μ_2	μ_3
Volatility of sample excess returns	σ_1	σ_2	σ_3

Let R_{mkt} represent the **excess** return of the market portfolio.

The expected excess return of the market portfolio is given by the weighted sum of the individual asset excess returns:

$$E[R_{\text{mkt}}] = w_1\mu_1 + w_2\mu_2 + w_3\mu_3$$

The volatility of the market portfolio is given by the weighted sum of the individual asset volatilities, taking into account the correlations (ρ) between the assets:

$$\sigma_{\text{mkt}} = \sqrt{w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + w_3^2\sigma_3^2}$$

and

$$\sigma_{\text{mkt}}^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + w_3^2\sigma_3^2$$

The Sharpe ratio of the market portfolio is the ratio of its expected excess return to its volatility:

$$SR_{\text{mkt}} = \frac{E[R_{\text{mkt}}]}{\sigma_{\text{mkt}}}$$

Then,

$$\gamma_{\text{mkt}} = \frac{\text{SR}_{\text{mkt}}}{\sigma_{\text{mkt}}} = \frac{E[R_{\text{mkt}}]}{\sigma_{\text{mkt}}^2} = \frac{w_1\mu_1 + w_2\mu_2 + w_3\mu_3}{w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + w_3^2\sigma_3^2}.$$

Q3.4 (**1 point**) Now, find the **CAPM-implied expected excess returns**, $\mathbb{E}[R_{\text{CAPM}}]$, if

$$w_{\text{mkt}} = \frac{1}{\gamma_{\text{mkt}}}\mathbb{V}[R_{\text{sample}}]^{-1}\mathbb{E}[R^{\text{capm}}], \quad \text{where:} \quad \gamma_{\text{mkt}} = \frac{\text{SR}_{\text{mkt}}}{\sigma_{\text{mkt}}},$$

which, in expanded form, can be written as

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \frac{1}{\gamma_{\text{mkt}}}\left(\begin{bmatrix} \sigma_1^2 & 0 & 0 \\ 0 & \sigma_2^2 & 0 \\ 0 & 0 & \sigma_3^2 \end{bmatrix}\right)^{-1}\begin{bmatrix} E[R_1^{\text{capm}}] \\ E[R_2^{\text{capm}}] \\ E[R_3^{\text{capm}}] \end{bmatrix}$$

and, **as before**, the market consists of only three risky assets: $n = \{1, 2, 3\}$ whose excess returns are **uncorrelated** and their relative market capitalizations (i.e., market weights), mean excess returns, and volatilities (standard deviations) are:

Item	Asset 1	Asset 2	Asset 3
Weight in market portfolio	w_1	w_2	w_3
Mean of sample excess returns	μ_1	μ_2	μ_3
Volatility of sample excess returns	σ_1	σ_2	σ_3

In the previous part of this question, we have already identified $E[R_{\text{mkt}}]$, σ^2_{mkt}, and γ_{mkt}, which we will use below.

From this part of the question, we are given that

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \frac{1}{\gamma_{\text{mkt}}}\left(\begin{bmatrix} \sigma_1^2 & 0 & 0 \\ 0 & \sigma_2^2 & 0 \\ 0 & 0 & \sigma_3^2 \end{bmatrix}\right)^{-1}\begin{bmatrix} E[R_1^{\text{capm}}] \\ E[R_2^{\text{capm}}] \\ E[R_3^{\text{capm}}] \end{bmatrix}$$

Rearrange this equation so that the object of interest is on the left-hand side,

$$\begin{aligned} \begin{bmatrix} E[R_1^{\text{capm}}] \\ E[R_2^{\text{capm}}] \\ E[R_3^{\text{capm}}] \end{bmatrix} &= \gamma_{\text{mkt}}\begin{bmatrix} \sigma_1^2 & 0 & 0 \\ 0 & \sigma_2^2 & 0 \\ 0 & 0 & \sigma_3^2 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \\ &= \frac{E[R_{\text{mkt}}]}{\sigma^2_{\text{mkt}}}\begin{bmatrix} \sigma_1^2 & 0 & 0 \\ 0 & \sigma_2^2 & 0 \\ 0 & 0 & \sigma_3^2 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \\ &= \frac{E[R_{\text{mkt}}]}{\sigma^2_{\text{mkt}}}\begin{bmatrix} \sigma_1^2 w_1 \\ \sigma_2^2 w_2 \\ \sigma_3^2 w_3 \end{bmatrix}, \end{aligned}$$

or, the expression for the CAPM-implied expected returns for asset n (when returns are uncorrelated), is

$$E[R_n^{\text{capm}}] = \frac{E[R_{\text{mkt}}]}{\sigma^2_{\text{mkt}}}\sigma_n^2 w_n, \quad \text{where } n = \{1, 2, 3\}.$$

Q3.5 (**1 point**) Please write the **pick matrix** P and the **view vector** Q if the investor's beliefs are the following:

- The first and second assets will have equal returns on average;
- The third asset is expected to have a return of 8%.

The pick matrix P is

$$P = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The Q vector of views is

$$Q = \begin{bmatrix} 0.00 \\ 0.08 \end{bmatrix}$$

END of Q3 ♠

4 Factor-based portfolios

Suppose that you work for an asset-management company that believes that the N stock returns are driven by the three-factor ($K = 3$) Fama-French model, where the three factors are the returns on the market (mkt) minus the risk-free asset, size (smb), and value (hml) long-short portfolios. That is, excess returns on assets i and j can be written as:

$$R_i - R_f = \alpha_i + \beta_{i,\text{mkt}}(R_{\text{mkt}} - R_f) + \beta_{i,\text{smb}}R_{\text{smb}} + \beta_{i,\text{hml}}R_{\text{hml}} + e_i, \quad \text{where } \mathbb{E}[e_i] = 0,\ \text{Cov}[e_i, e_j] = 0,$$
$$R_j - R_f = \alpha_j + \beta_{j,\text{mkt}}(R_{\text{mkt}} - R_f) + \beta_{j,\text{smb}}R_{\text{smb}} + \beta_{j,\text{hml}}R_{\text{hml}} + e_j, \quad \text{where } \mathbb{E}[e_j] = 0,\ \text{Cov}[e_i, e_j] = 0,$$

and σ^2_{mkt}, σ^2_{smb}, and σ^2_{hml} are the variances of the returns on the three factors, which are orthogonal to one another; i.e., the covariance between each pair of factors is zero.

Q4.1 (**1 point**) What is the covariance between the returns on Asset i and Asset j?

Note that the question tells us that α_i and α_j are constants, that the three factors are uncorrelated (orthogonal) to one another, and that the residuals e_i and e_j are also uncorrelated. Then, the covariance between the returns on asset i and j are given by:

$$\text{Cov}[R_i, R_j] = \beta_{i,\text{mkt}}\beta_{j,\text{mkt}}\sigma^2_{\text{mkt}} + \beta_{i,\text{smb}}\beta_{j,\text{smb}}\sigma^2_{\text{smb}} + \beta_{i,\text{smb}}\beta_{j,\text{smb}}\sigma^2_{\text{smb}}.$$

Q4.2 (**1 point**) Write down the steps for using this asset-returns factor model for constructing a mean-variance optimal portfolio of N assets; that is, a factor-based portfolio policy (**not** parametric portfolio policy, which is the next question).

Notation:
w = N-vector of weights in the risky assets
$\bar{R}$ = N-vector of individual asset expected returns
V = $N \times N$ covariance matrix of individual asset returns

Short answer: We know that the expression for the optimal mean-variance weights is

$$w = \frac{1}{\gamma}V^{-1}(\bar{R} - R_f 1_N).$$

If asset returns are given by the three-factor Fama-French model, then in the above expression, we need to determine $\bar{R}$ and V from the Fama-French model, as we have already done in the previous part of this question.

A longer, more-detailed, response to Q4.2:

Step 1: Estimate the model
Estimate the expected returns of each asset, the covariances between the asset returns, and the factor exposures (beta values) for each asset in the Fama-French 3-factor model.

Step 2: Calculate expected portfolio return
Calculate the expected return of the portfolio in terms of the N-vector w of weights for each of the N risky assets and their respective expected returns, besides the weight assigned to the risk-free asset.

$$\bar{R}_p = w_f R_f + \sum_{i=1}^{N} w_i \bar{R}_i = (1 - w^\top 1_N) R_f + w^\top \bar{R} = R_f + w^\top (\bar{R} - R_f 1_N)$$

Note that $\bar{R}_i$ will be obtained from the Fama-French factor model.

Step 3: Calculate portfolio variance
Calculate the variance (or standard deviation) of the portfolio returns using the covariance matrix of asset returns and the weights assigned to each asset.

$$\sigma_p^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j \mathrm{Cov}(R_i, R_j) = w^\top V w$$

Note again that σ_p^2 will be obtained from the Fama-French factor model, using the result for $\mathrm{Cov}(R_i, R_j)$ from the previous part of this question.

Step 4: Formulate the optimization problem
Formulate the mean-variance optimization problem by setting up the objective function and adding any constraints, such as the constraint that the sum of weights equals 1.

$$\max_{w_1,\ldots,w_N} \bar{R}_p - \frac{\gamma}{2} \sigma_p^2 \quad \text{subject to:} \quad w_f + \sum_{i=1}^{N} w_i = 1.$$

Step 5: Solve the Optimization Problem
Use optimization techniques (e.g., quadratic programming) to solve the optimization problem and find the optimal weights for each asset in the portfolio.

For the simple problem formulated above, we know that the solution is:

$$w = \frac{1}{\gamma} V^{-1} (\bar{R} - R_f 1_N).$$

Q4.3 (**1 point**) Now we consider **parametric portfolio policies**. What is the logic underlying parametric portfolio policies? What are the strengths of parametric portfolio policies as opposed to using a factor model for asset returns and then choosing portfolio weights based on the factor model?

> The key insight underlying parametric portfolio policies is that it is much better to model portfolio weights directly in terms of a factor model, as opposed to first modeling asset returns using a factor model, and then finding the optimal portfolio weights.
>
> Parametric portfolio policies *parameterize* portfolio weights in terms of the factors, and then optimize over the parametric weights assigned to the factors using mean-variance utility maximization (or expected-utility maximization) as the objective function.
>
> $$w_t(\theta) = w_{b,t} + (\theta_1 F_{1,t} + \theta_2 F_{2,t} + \ldots + \theta_K F_{K,t}),$$
>
> The key advantage of using parametric portfolio policies is that the number of parameters to be estimated is equal to K, where K represents the number of factors. In contrast, the sample-based Markowitz portfolio needs to estimate about $N^2/2$ parameters, while the factor-based approach, where one first models returns in terms of factors and then determines the optimal portfolio, requires the estimation of about $N \times K$ parameters.
>
> Thus, when using parametric portfolio policies, the number of parameters to be estimated is always K, even as the number of assets in the portfolio, N, increases.

Q4.4 (**1 point**) Explain how one can implement **parametric portfolio policies** for the three-factor Fama-French model. In particular, write down the optimization problem of an investor who wishes to maximize **mean-variance utility** to obtain the optimal parametric portfolio. Make sure to write down the **choice variables** over which the investor will be optimizing and also the expressions for the **mean** and **variance** of the returns of the parametric portfolio.

> Assuming that there is no benchmark portfolio, just as before, the investor will be optimizing the same mean-variance objective function, but now the optimization will be over the parametric weights θ_k:
>
> $$\max_{\theta_{\text{mkt}}, \theta_{\text{smb}}, \theta_{\text{hml}}} \quad \mathrm{E}[R_p] - \frac{\gamma}{2}\mathrm{Var}(R_p),$$
>
> where
>
> $$\begin{aligned} R_p &= \theta_{\text{mkt}}(R_{\text{mkt}} - R_f) + \theta_{\text{smb}} R_{\text{smb}} + \theta_{\text{hml}} R_{\text{hml}} \\ \mathrm{E}[R_p] &= \theta_{\text{mkt}}(E[R_{\text{mkt}}] - R_f) + \theta_{\text{smb}} E[R_{\text{smb}}] + \theta_{\text{hml}} E[R_{\text{hml}}] \\ \mathrm{Var}[R_p] &= \theta^2_{\text{mkt}}\sigma^2_{\text{mkt}} + \theta^2_{\text{smb}}\sigma^2_{\text{smb}} + \theta^2_{\text{hml}}\sigma^2_{\text{hml}}. \end{aligned}$$

Q4.5 (**1 point**) Suppose you find empirical evidence that the risk-return relation is not strong; i.e., changes in factor variance do not lead to proportional changes in factor mean returns. Therefore, you would like to extend the parametric portfolios you have described above so that they are **conditional on the volatility of each factor**.

Explain how you will construct a parametric portfolio policy whose weights on each factor are conditional on the volatility of that factor. Make sure that the portfolio you recommend depends only on historical data and does not suffer from look-ahead bias.

The way to construct a parametric portfolio policy that exploits the weak relation between risk and return is to make the parametric portfolio weight θ_k itself a function of the inverse of factor variance or factor volatility. For example, one could specify that

$$\theta_k = a_k + \frac{b_k}{\sigma_k},$$

where σ_k is the realized volatility of the return on factor k and a_k and b_k are constants. In this case, the investor will be optimizing the same mean-variance objective function, but now the optimization will be over a_k and b_k:

$$\max_{\{a_k, b_k\}_{k=1}^{K}} \quad \mathrm{E}[R_p] - \frac{\gamma}{2}\mathrm{Var}(R_p),$$

where

$$\begin{aligned} R_p &= \theta_{\text{mkt}}(R_{\text{mkt}} - R_f) + \theta_{\text{smb}} R_{\text{smb}} + \theta_{\text{hml}} R_{\text{hml}} \\ \mathrm{E}[R_p] &= \theta_{\text{mkt}}(E[R_{\text{mkt}}] - R_f) + \theta_{\text{smb}} E[R_{\text{smb}}] + \theta_{\text{hml}} E[R_{\text{hml}}] \\ \mathrm{Var}[R_p] &= \theta^2_{\text{mkt}}\sigma^2_{\text{mkt}} + \theta^2_{\text{smb}}\sigma^2_{\text{smb}} + \theta^2_{\text{hml}}\sigma^2_{\text{hml}}. \end{aligned}$$

END of Q4 ♠

END of exam ♠