Quantitative Portfolio Management: Theory and Practice

Chapter 6: Factor-based Portfolios: Parametric Portfolio Policies

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The big picture: Plan for the entire course

Part A: Preliminaries

Chapter 1: Managing financial data

Chapter 2: Performance measurement (especially out of sample)

Part B: Modern portfolio management

Chapter 3: Mean-variance portfolios that *ignore* estimation error

Chapter 4: Mean-variance portfolios that *adjust for* estimation error

Part C: Post-Modern Portfolio Management

Chapter 5: CAPM-based portfolios: Black-Litterman model

Chapter 6: Factor-based portfolios: Parametric portfolio policies

Chapter 7: Volatility-timed factor portfolios

Chapter 8: Portfolios exploiting systematic risk factors *and* unsystematic risk

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What do we want to do in Chapter 6

In this chapter, we study how asset-pricing factor models can be used to construct optimal portfolios.

We study the use of factor models for estimating both expected returns and the covariance matrix.

We conclude by studying the "parametric portfolio policies" of Brandt, Santa-Clara, and Valkanov.

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Motivation for the material in this chapter

- We start by explaining why asset-pricing factor models may be important.
- First we explain the problems in implementing Markowitz mean-variance portfolios
  - by now, you should understand this perfectly.
- Then, we explain how factor models can help resolve the problems
  - for estimating expected returns
  - for obtaining well-conditioned covariance matrices.
- which allows us to construct portfolios that perform well out-of-sample.

Distinction between factor models and portfolio construction

- Throughout this chapter, remember there is a difference between:
  - Factor models (relation between risk and return)
  - Portfolio choice (weights that exploit optimally risk-return tradeoff)

Error in estimating $\mathbb{E}[R]$ and $\mathbb{V}[R]$ from sample moments

- Mean-variance-optimal portfolio weights depend on estimates of return means and covariances:

$$w = \frac{1}{\gamma}(\mathbb{V}[R])^{-1}(\mathbb{E}[R] - R_f 1_N).$$

- In earlier chapters, we saw that
  - Sample-based estimates of $\mathbb{E}[R]$ are imprecise;
  - Sample-based estimates of $\mathbb{V}[R]$ are ill-conditioned;
  - Consequently, mean-variance portfolios perform poorly out of sample.

First shrinkage method: Using only sample moments

- Having identified the problems in sample-based estimates of $\mathbb{E}[R]$ and $\mathbb{V}[R]$, we study shrinkage methods to solve these problems.
- The first kind of shrinkage methods we studied relied only on sample moments but did not take advantage of asset-pricing theory.
- For example, Bayesian shrinkage of expected returns relies on shrinking sample estimate of $\mathbb{E}[R]$ toward a "grand mean," which is
  - either the average of all mean returns
  - or the expected return on the GMV portfolio.
- Similarly, the Ledoit and Wolf methods relies on shrinking the sample estimate of $\mathbb{V}[R]$ toward
  - either a diagonal matrix with the average variance on its diagonal
  - or a matrix where all the cross-asset correlations are replaced by the average correlation.

Empirical performance of sample-based models . . . I

- When we evaluated the empirical performance of models relying only on sample moments of returns, we saw that
  - models shrinking sample estimates of expected returns fail to outperform the simple $1/N$ benchmark;
  - models that ignore expected returns altogether and choose weights based on minimization of portfolio variance perform better,
    - especially when shrinking the covariance matrix of returns using either a short-sale constraint or the Ledoit and Wolf approach,
- However, even these models fail to consistently outperform the $1/N$ portfolio (DeMiguel, Garlappi, and Uppal 2009; Jacobs, Müller, and Weber 2014).

Second shrinkage method: Using asset-pricing models . . . I

- In the last class, we studied how to take advantage of asset-pricing theory to construct portfolios.
- The Black-Litterman model showed how to
  1. use the Capital Asset Pricing Model of Sharpe (1964) to obtain estimates of expected returns from market-portfolio weights;
  2. combine these CAPM-implied expected returns with the views of the investor to obtain the posterior distribution of asset returns, which are then used to construct mean-variance optimal weights.

Second shrinkage method: Using asset-pricing models . . . II

- In this class, we study other asset-pricing models, besides the CAPM, to see how they can be used for portfolio construction.
- In particular, we will look at the asset-pricing factor models proposed by Fama and French (1992, 1993, 2012, 2015, 2018).
- We will then study parametric portfolio policies
  - developed by Brandt, Sant-Clara, and Valkanov (2009),
  - who find a very clever way to build portfolios using the factors identified by Fama and French and other researchers.

Importance of factor models

- Factor models are the core of modern
  - empirical asset pricing and
  - portfolio construction.
- For asset pricing, factor models provide the link between
  - covariances (risk) and
  - expected returns.
- For portfolio construction, factor models reduce the task of
  - searching among thousands of assets
  - to the more tractable problem of finding
  - the optimal risk-return tradeoff among a small number of factors.

Portfolio construction with factor models

- Recall that in the Markowitz model, the expression for the optimal portfolio is given by

$$w = \frac{1}{\gamma}(\mathbb{V}[R])^{-1}(\mathbb{E}[R] - R_f 1_N),$$

  where we estimate $\mathbb{E}[R]$ and $\mathbb{V}[R]$ from their sample moments.

- When using a factor model to construct the optimal portfolio,
  - we still use the expression for w in the equation above,
  - but estimate $\mathbb{E}[R]$ and $\mathbb{V}[R]$ using the factor model.

- As we show below, using a K-factor model substantially reduces the number of parameters we need to estimate, from $N(N+3)/2$ to $2N + 2K + (N \times K)$.

- After that, we will see if it is possible to reduce the number of parameters to be estimated to only K, no matter how large is N.

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Timeline: Quantitative portfolio management ideas . . . I

- We can see how ideas about investment have progressed over time.
- 4th century: $1/N$
  - "One should always divide his wealth into three parts: a third in land, a third in merchandise, and a third ready to hand [cash].
    Rabbi Issac bar Aha, Babylonian Talmud: Tractate Baba Mezi'a, folio 42a
  - "My ventures are not in one bottom trusted"
    "Merchant of Venice, "Shakespeare ((1564–1616) on the importance of diversification in investing
  - Do not put all your eggs in one basket

Timeline: Quantitative portfolio management ideas . . . II

- 1950s: Mean-variance optimization
  (Markowitz 1952, 1959)

- 1964: CAPM
  (Sharpe 1964)

- 1970–2000s: Bayesian shrinkage
  (Klein and Bawa 1976; Bawa, Brown, and Klein 1979; Jorion 1985; Jorion 1988; Jorion 1992; Pástor and Stambaugh 2000)

- 1990s: Black-Litterman model
  (Black and Litterman 1990, 1991a, 1991b, 1992; He and Litterman 1999; Litterman 2003)

. . . . *This is the point where we are in the course* . . . .

Timeline: Quantitative portfolio management ideas . . . III

. . . . *Today's class* . . . .

- 1970s: Factor models
  (Ross 1976, 1977)
- 1980s Macro factor models
  (Chen, Roll, and Ross 1986)
- 1990–2020s: Fundamental (firm-characteristic-based) factor models
  (Fama and French 1992, 1993, 2012, 2015, 2018).
- 2015–2020: Principal-component-based factor models
  (Kozak, Nagel, and Santosh 2018, 2020; Lettau and Pelger 2018, 2020).
- 2009-2025: Parametric portfolio policies
  (Brandt, Santa-Clara, and Valkanov 2009; DeMiguel, Martín-Utrera, Nogales, and Uppal 2020).

Timeline: Quantitative portfolio management ideas . . . IV

. . . . *Next two classes* . . . .

- 2017-2023: Volatility-timing of factors
  (Moreira and Muir 2017, 2019; Cederburg, O'Doherty, Wang, and Yan 2020; Barroso and Detzel 2021; DeMiguel, Martín-Utrera, and Uppal 2024).

- 2023-2024: Portfolio construction: Beyond systematic risk
  (Raponi, Uppal, and Zaffaroni 2023; Dello-Preite, Uppal, Zaffaroni, and Zviadadze 2024).

- For a more detailed history of the development of ideas about investment, see the book by Rubinstein (2006).

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An overview of factor models

- The theory of factor models was developed
  - by Ross (1976, 1977),
  - with important advances by Huberman (1982), Chamberlain (1983), Chamberlain and Rothschild (1983), and Ingersoll (1984).
- The theory is completely silent about
  - how many factors to include in the model;
  - which factors to include in the model.
- Thus, in practice, we need to determine the
  - number (K) of factors and
  - identity of these factors.

- The objective is to identify factors that explain well the variation in stock returns.

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Main idea: Shrinking the "asset space" . . . I

- In quantitative portfolio management, factor investing aims to simplify the problem of portfolio construction by reducing the dimension of the space to search in.
- If a model has $K < N$ factors, then one can view these models as shrinking the asset space from N to K.
  - Usually, $N \approx 100$; and
  - Usually, $K \approx 5$ and rarely larger than 7.

Main idea: Shrinking the "asset space" . . . II

- Below, we will see how the Fama-French-type factor models reduce the number of parameters that need to be estimated.
- The Brandt, Sant-Clara, and Valkanov (2009) parametric portfolios "shrink" further the number of parameters to be estimated.
  - This approach leads to substantial performance gains.

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A single-factor model

- To understand factor models, we start with a single-factor model.
- Usually, the single factor is assumed to be the return on the market portfolio, $R_{\text{mkt}} = R_m$.
  - Therefore, the single-factor model is sometimes called a single index model, where the index is the market index.
- It is important to understand the difference between
  - CAPM: an equilibrium model of asset returns
  - Market model: a statistical model of asset returns
- The CAPM implies the market model, but the reverse is not true.

Implications of single-factor model for returns . . . I

- In a single-factor model, the (excess) return on asset n is

$$R_n - R_f = \alpha_n + \beta_n(R_m - R_f) + e_n, \quad \mathbb{E}[e_n] = 0, \quad \text{where}$$

  - α_n is the component of security n's return that is independent of the market's return;
  - β_n measures the expected change in R_n given a change in R_m;
  - e_n is uncorrelated with R_m, $\text{Cov}[e_n, R_m] = 0$; i.e., how well the model describes the return of security n is independent of R_m.

Implications of single-factor model for returns . . . II

- If (excess) returns are given by a single-index model,

$$R_n - R_f = \alpha_n + \beta_n(R_m - R_f) + e_n, \quad \mathbb{E}[e_n] = 0, \quad \text{then}$$

- The expected excess return on asset n is:

$$\begin{aligned} \mathbb{E}[R_n] - R_f &= \alpha_n + \beta_n(\mathbb{E}[R_m] - R_f) \\ &= \alpha_n + \beta_n\lambda_m. \qquad \ldots N \text{ alphas, } N \text{ betas, one } \lambda_m \end{aligned}$$

- Note that $\lambda_m = \mathbb{E}[R_m] - R_f$ in the above equation is the price of risk, i.e., the compensation for bearing beta risk.

Implications of single-factor model for returns . . . III

- Important not to confuse this λ_m with the Lagrange multipliers on the constraints in the Markowitz mean-variance optimization that
  - the weights sum to 1, λ_w;
  - the expected portfolio return has to equal a target mean return, λ_R;
  - the weights cannot take short positions, λ_{ss}.
- We are using the same Greek letter, λ, but the subscripts tell you:
  - in this chapter λ_m is the price of market risk,
  - while in the earlier chapters, λ_w, λ_R, and λ_{ss} were Lagrange multipliers.

Implications of single-factor model for returns . . . IV

- Rewriting that (excess) returns are given by a single-index model,

$$R_n - R_f = \alpha_n + \beta_n(R_m - R_f) + e_n, \quad \mathbb{E}[e_n] = 0, \quad \text{then}$$

- The variance of a security's return is:

$$\sigma_n^2 = \beta_n^2 \sigma_m^2 + \sigma_{e_n}^2. \qquad \ldots N \text{ betas, } N \text{ terms of } \sigma_{e_n}^2, \text{ one } \sigma_m^2$$

- The covariance of returns between securities i and j is:

$$\sigma_{i,j} = \beta_i \beta_j \sigma_m^2. \qquad \ldots \text{no new quantity to be estimated!}$$

Key restriction (assumption) of the single-index model

- The key assumption of the single-index model is that e_i is independent of e_j; i.e.,

$$\mathbb{E}[e_i e_j] = 0, \quad \text{for all pairs } i \neq j.$$

- This means that the only reason for assets to vary together, i.e., systematically, is because of their common comovement with the market return.
- So, this model rules out any effects beyond the market (e.g., industry effects) that account for comovement between securities.

Diversification & portfolio risk in a single-factor model . . . I

- The last item we discuss in the context of the single-factor model is the variance of a portfolio p.

- Let a portfolio be defined by its weights in the N available assets:

$$w = \begin{bmatrix} w_1 \\ \vdots \\ w_N \end{bmatrix}$$

- Then, the alpha of the portfolio and the beta of the portfolio are the weighted averages of the asset alphas and betas:

$$\alpha_p = \sum_{n=1}^{N} w_n \alpha_n; \quad \text{and} \quad \beta_p = \sum_{n=1}^{N} w_n \beta_n.$$

Diversification & portfolio risk in a single-factor model . . . II

- Recall that for the single-index model

$$R_n - R_f = \alpha_n + \beta_n(R_m - R_f) + e_n, \qquad \mathbb{E}[e_n] = 0.$$

- The portfolio variance can then be written as

$$\begin{aligned}
\sigma_p^2 &= \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \beta_i \beta_j \sigma_m^2 + \sum_{i=1}^{N} w_i^2 \sigma_{e_i}^2 && \text{. . . from market-model returns} \\
&= \left(\sum_{i=1}^{N} w_i \beta_i\right)\left(\sum_{j=1}^{N} w_j \beta_j\right)\sigma_m^2 + \sum_{i=1}^{N} w_i^2 \sigma_{e_i}^2 && \text{. . . grouping terms} \\
&= \beta_p \beta_p \sigma_m^2 + \sum_{i=1}^{N} w_i^2 \sigma_{e_i}^2 && \text{. . . definition of average beta} \\
&= \beta_p^2 \sigma_m^2 + \sum_{i=1}^{N} w_i^2 \sigma_{e_i}^2. && \text{. . . collecting the } \beta_p \text{ terms}
\end{aligned}$$

Diversification & portfolio risk in a single-factor model . . . III

- Now consider a portfolio with equal weights, $w_n = 1/N$.
- The portfolio variance in this case can be written as

$$
\begin{aligned}
\sigma_p^2 &= \beta_p^2 \sigma_m^2 + \sum_{i=1}^{N} w_i^2 \sigma_{e_i}^2 && \text{. . . from previous page} \\
&= \beta_p^2 \sigma_m^2 + \sum_{i=1}^{N} \left(\frac{1}{N}\right)^2 \sigma_{e_i}^2 && \text{. . . replace } w_i \text{ by } 1/N \\
&= \beta_p^2 \sigma_m^2 + \left(\frac{1}{N}\right) \left(\sum_{i=1}^{N} \frac{1}{N} \sigma_{e_i}^2\right) && \text{. . . grouping terms} \\
&= \beta_p^2 \sigma_m^2 + \left(\frac{1}{N}\right) [\text{average residual risk}] && \text{. . . defn. avg. resi. risk} \\
\lim_{N \to \infty} \sigma_p^2 &= \beta_p^2 \sigma_m^2 && \text{. . . taking the limit}
\end{aligned}
$$

- Thus, a diversified portfolio has only beta (systematic) risk.

Comparing number of parameters to be estimated . . . I

- In the absence of a single-factor model, we needed to estimate:
  - N mean returns
  - N variances
  - $(N^2 - N)/2$ covariances
  - for a total of $\frac{N(N+3)}{2}$, which, for $N = 100$, is 5150.
- In the presence of a single-factor model, we needed to estimate:
  - N alphas
  - N betas
  - N asset-specific volatilities, $\sigma^2_{e_n}$
  - mean and volatility of the market excess return, i.e., λ_m and σ_m
  - for a total of only $(3N + 2)$, which, for $N = 100$, is 302.

Comparing number of parameters to be estimated . . . II

- So, in terms of the number of parameters to be estimated, we have made considerable progress:
  - In the Markowitz sample-based model,

$$[\text{Number of parameters}] = \frac{1}{2}N(N+3) \approx \frac{1}{2}N^2.$$

  - In the single-factor model,

$$[\text{Number of parameters}] = 3N + 2 \approx 3N$$

- Thus, as N increases, the number of parameters to be estimated will increase at much slower for the single-factor model.

Comparing number of parameters to be estimated . . . III

- For an investor holding about 11,000 assets (e.g., Norges Bank)
  - The sample-based approach would need to estimate

$$[\text{Number of parameters}] = \frac{1}{2}N(N+3) = 60,516,500.$$

  - The single-factor model would need to estimate

$$[\text{Number of parameters}] = 3N+2 = 33,002.$$

How to reduce the number of parameters even further?

- Can you think of a way to reduce the number of parameters to be estimated even further?
- Is it possible to de-link entirely
  - the number of parameters to be estimated
  - from the number of assets, N?
- Is it possible to formulate the portfolio problem in such a way that
  - the number of parameters to be estimated is small and constant
  - even when the number of assets, N, is large and increasing?
- Is it possible that, even if $N = 11,000$, we need to estimate only $K = 5$ parameters?
  - We will answer this question later today.

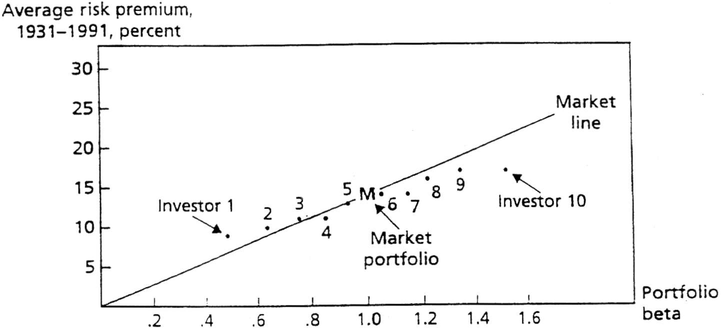
Performance of the market model

- We now examine the empirical evidence on the single-factor model, with the factor being the return on the market portfolio.
- A single-factor model that uses the market return as the factor performs poorly at explaining the cross-section of stock returns.
- As early as the 1990's, Fischer Black had found that the market model did not explain differences in stock returns very well:
  - low-beta stocks had returns higher than those predicted by the market-model;
  - high-beta stocks had returns lower than those predicted by the market-model;

Empirical evidence on performance of market model . . . I

- Actual average risk premiums from portfolios with different betas do not line up with the Security Market Line.

1931–1991
(full sample)

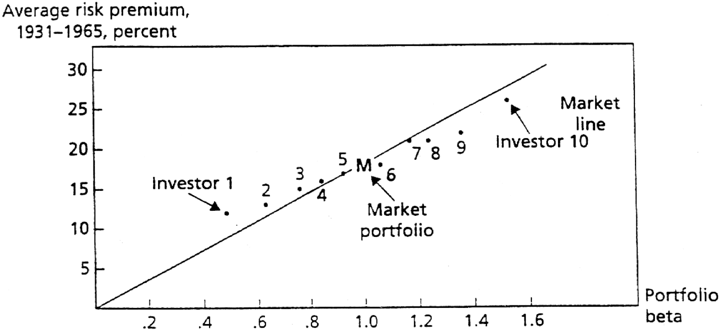


(Source: “Beta and Return” Black (1993))

Empirical evidence on performance of market model . . . II

- CAPM works moderately well over some periods of time

1931–1965
(first half of sample)

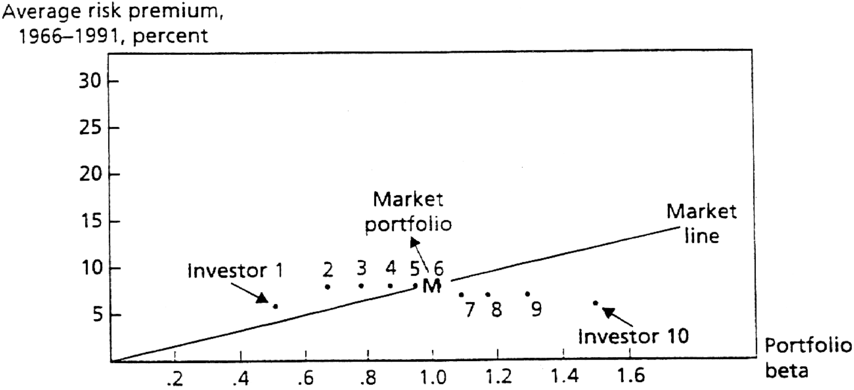


(Source: "Beta and Return" Black (1993))

Empirical evidence on performance of market model . . . III

- CAPM does not work well over significant periods of time

1961–1991
(second half of sample)



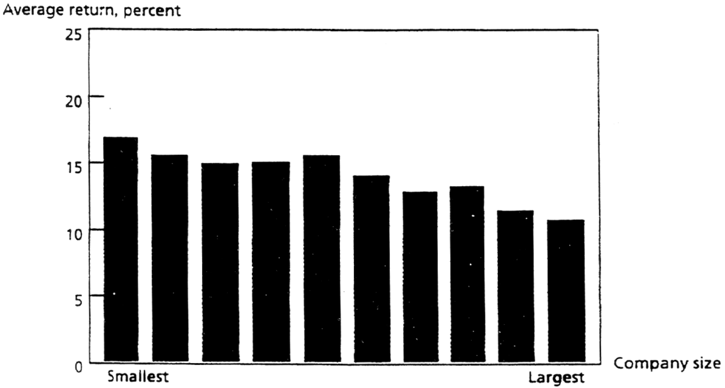
(Source: "Beta and Return" Black (1993))

Empirical evidence on performance of market model . . . IV

- Fama and French showed that factors other than beta seem important in pricing assets; these factors include
  - Size
  - Value (ratio of market value to book value)
- Following their finding, follow-up work has identified hundreds of other factors that seem to be related to returns.

Empirical evidence on performance of market model . . . V

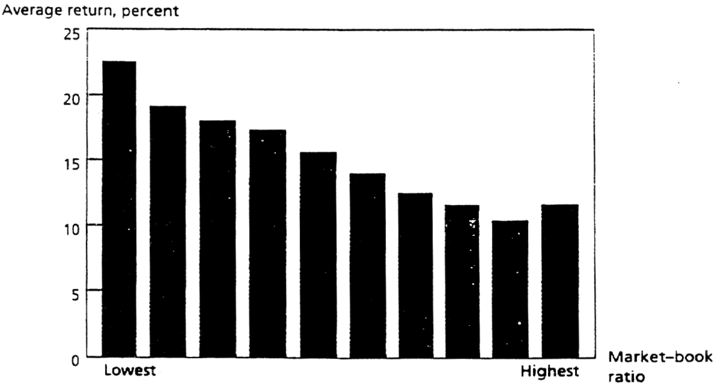
- Since the 1960s, small stocks have outperformed large stocks



Source: "The Cross-Section of Expected Stock Returns" Fama and French (1992)

Empirical evidence on performance of market model . . . VI

- Since mid-1960s, stocks with low ratios of market-to-book value, have outperformed stocks with high ratios.



Source: "The Cross-Section of Expected Stock Returns" Fama and French (1992)

Overall assessment of CAPM (and market model)

- CAPM is a good theoretical model
  - It is simple and sensible.
  - It is built on modern portfolio theory
  - It distinguishes diversifiable risk and non-diversifiable risk
  - It provides a simple pricing model.
  - It is relatively easy to implement in practice.
- But, empirical evidence supporting the market model is weak.

- Recall that our objective was to identify factors that explain well the variation in stock returns.
- The market model is not very successful at explaining the variation in stock returns.

Underlying assumptions of CAPM

- Assumptions of the CAPM:
  - Investors care about returns over the next short horizon
  - Financial market is perfect:
    - all assets are traded
    - no frictions such as trading costs and taxes
    - all investors have perfect information
  - Investors hold fully diversified mean-variance frontier portfolios.
- These assumptions are often violated in practice.
  - E.g., investors do not hold fully diversified portfolios.
- Deviations from these assumptions lead to alternative models.
  - E.g., the model in Merton (1987), in which investors do not hold fully diversified portfolios.

From single-factor to multifactor models

- The poor empirical performance of the single-factor market model motivates the development of multifactor models.
- The hope is that by including multiple factors, we will be able to do a better job of explaining the variation in stock returns.

Start of focus

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A multifactor model in terms of K factors

- We now look at a model with K factors.

$$R_n - R_f = \alpha_n + \beta_{n,1} F_1 + \beta_{n,2} F_2 + \ldots + \beta_{n,K} F_K + e_n, \quad \mathbb{E}[e_n] = 0,$$

  - where $\beta_{n,k}$ is the nth row of a $N \times K$ matrix of betas with respect to the K factors, and
  - $F_1, F_2, \ldots, F_K$ are the K factors,
  - $\text{Cov}[F_k, F_\ell] = 0$; that is, the factors are assumed to be orthogonal to one another.
  - If the original factors are not orthogonal, they can always be transformed so that they become orthogonal, without affecting any implications for asset returns.

A multifactor model in terms of K factor returns

- If the factors are tradable (as opposed to being, e.g., macro factors), then the model can be written in terms of factor returns.
- The K-factor model in terms of factor returns R_{F_k} is:

  $$R_n - R_f = \alpha_n + \beta_{n,1} R_{F_1} + \beta_{n,2} R_{F_2} + \ldots + \beta_{n,K} R_{F_K} + e_n, \quad \mathbb{E}[e_n] = 0,$$

  which implies that expected (excess) returns, are

  $$\begin{aligned} \mathbb{E}[R_n] - R_f &= \alpha_n + \beta_{n,1}\lambda_1 + \beta_{n,2}\lambda_2 + \ldots + \beta_{n,K}\lambda_K \\ &= \alpha_n + \beta_n \lambda, \quad \text{where} \end{aligned}$$

  - $\lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_K\}$ is the $K \times 1$ vector of factor prices of risk.

Return variance and covariance in a multifactor model

- The K-factor model in terms of factor returns is

$$R_n - R_f = \alpha_n + \beta_{n,1} R_{F_1} + \beta_{n,2} R_{F_2} + \ldots + \beta_{n,K} R_{F_K} + e_n, \quad \mathbb{E}[e_n] = 0,$$

which implies that the return variance is

$$\sigma_n^2 = \beta_{n,1}^2 \sigma_{F_1}^2 + \beta_{n,2}^2 \sigma_{F_2}^2 + \ldots + \beta_{n,K}^2 \sigma_{F_K}^2 + \sigma_{e_n}^2, \quad \text{where}$$

$\sigma_{F_k}^2$ is the variance of the return on the kth factor.

- The K-factor model implies that the return covariance between assets i and j is a

$$\sigma_{i,j}^2 = \beta_{i,1}\beta_{j,1}\sigma_{F_1}^2 + \beta_{i,2}\beta_{j,2}\sigma_{F_2}^2 \ldots + \beta_{i,K}\beta_{j,K}\sigma_{F_K}^2$$

where we have assumed that $\mathrm{Cov}(e_i, e_j) = 0$ and also the factor returns are orthogonal.

Number of parameters to be estimated for portfolio construction

- For portfolio construction, we need estimates of expected returns and risk (variances and covariances).
- To estimate the mean return and risk for N assets in a K-factor model, we need to estimate the following parameters:
  - α_n for each of the N stocks ... N;
  - $\sigma^2_{e_n}$ for each of the N stocks ... N;
  - $\beta_{n,k}$ for each of the N stocks for each of the K factors; ... $N \times K$;
  - λ_k and $\sigma^2_{F_k}$ for the K factors. ... $2K$;
- This is a total of: $2N + 2K + (N \times K)$.
  - For $N = 100$ and $K = 5$, we need to estimate 710 parameters;
  - For $N = 11,000$ and $K = 5$, we need to estimate 77,010 parameters.

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Three types of factors in asset-pricing factor models

- To improve the market model's performance for explaining the cross-section of stock returns, these models add to it new factors.
- Depending on the type of factors we add, multifactor models of security market returns can be divided into three types.
  1. Macroeconomic factor models use observable economic time series, such as inflation and interest rates, as measures of pervasive shocks to security returns, i.e., as factors.
  2. Fundamental factor models use the returns on portfolios associated with observed security characteristics such as dividend yield, the book-to-market ratio, size, value, and industry identifiers.
  3. Statistical factor models use factors from the principal-components analysis (PCA) of the dataset of security returns.

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First type: Macroeconomic factor models

- The first empirical application of a macroeconomic factor model was a paper by Chen, Roll, and Ross (1986).
  - Cochrane (2017) provides a review of the factors that emerge from macro-economic theories.
- Some of the macroeconomic variables typically used as factors are
  - inflation,
  - the percentage change in industrial production,
  - the excess return to long-term government bonds, and
  - the realized return premium of low-grade corporate bonds relative to high-grade bonds.
- To estimate the risk-premia (λ_k) for macroeconomic factors, we estimate the risk-premia on factor-mimicking portfolios.
- Macroeconomic factor models are not very good at explaining stock returns, compared to the other types of factors models.

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Second type: Fundamentals-based factor models . . . I

- The literature on fundamentals-based factor models started with the
  - three-factor model of Fama and French (1993),
  - extended to five factors by Fama and French (2015), and then
  - extended to six factors by Fama and French (2018).
- Python code to access Fama-French data is available from:
  - PyAnomaly (efficient data download from WRDS using asynco).
  - pyassetpricing.
  - getFamaFrenchFactors.
- Hou, Xue, and Zhang (2015, 2017) propose new fundamentals and evaluate how well different factor models explain many "anomalies."

Second type: Fundamentals-based factor models . . . II

- The finance literature has "discovered" hundreds of fundamentals or firm characteristics that drive returns.
  - An excellent website with data on 331 predictors is Open Source Asset Pricing; for details, see Chen and Zimmermann (2022).
- In the last few years, machine-learning techniques have been applied to factor investing; for a review, see Giglio, Kelly, and Xiu (2022).
  - Cazalet and Roncalli (2014) and Clarke, Silva, and Thorley (2016) discuss the practical issues in using machine-learning factor models.

How to construct a fundamental factor . . . I

- Define the company-specific characteristic:
  - This could be a fundamental metric like earnings, dividend yield, or a financial ratio, or it could be based on non-financial attributes like ESG criteria.
- Data collection:
  - Collect historical data for the selected company-specific characteristics for a sample of companies.
- Selection criteria:
  - Establish criteria for selecting companies to include in the portfolio.
  - In academic research, we use all companies listed on NYSE, NASDAQ, and Amex exchanges.
  - But, you could use other criteria to select firms.

How to construct a fundamental factor . . . II

- Characteristic portfolio formation
  - Create "two" sub-portfolios: one for companies with high exposure to the chosen factor (High Factor Portfolio) and another for companies with low exposure (Low Factor Portfolio).
  - But, you could also form 4 quartile portfolios and consider only the top and bottom quartile.
  - Or, you could form 10 decile portfolios and consider only the top and bottom decile.
  - Or you could rank all firms and consider all of them.

How to construct a fundamental factor . . . III

- Equal-weighted or value-weighted characteristic portfolio
  - Decide whether the characteristic will be equally weighted (each stock has the same weight) or value-weighted (weights based on market capitalization).
  - In most cases, it is better to use value weights so that bigger firms receive more weight (see Plyakha, Uppal, and Vilkov 2021).
- Rebalancing:
  - Determine the rebalancing frequency of the portfolio.
  - Rebalancing ensures that the portfolio continues to represent the chosen factor over time.
  - In academics, we usually rebalance the portfolio annually (to limit turnover).
  - But, you could rebalance quarterly or monthly.

How to construct a fundamental factor . . . IV

- Benchmark selection:
  - Identify a benchmark index or portfolio against which you will compare the performance of your characteristic portfolio.
- Performance measurement:
  - Calculate performance metrics for both the “High” and “Low” portfolios. (Or, for all quartile/decile portfolios.)
  - Common performance metrics include annualized returns, cumulative returns, volatility, and the Sharpe ratio.
- Sensitivity analysis:
  - Perform sensitivity analyses by varying parameters such as portfolio construction criteria, rebalancing frequency, and benchmark selection to assess the robustness of your findings.

Portfolio choice with the Fama-French factor model

- Suppose that the N stock returns are driven by the three-factor ($K = 3$) Fama-French model,
- where the three factors are long-short portfolios
  - returns on the market (mkt) minus the risk-free asset,
  - size (smb), and
  - value (hml).

$$R_j - R_f = \alpha_j + \beta_{j,\mathsf{mkt}}(R_{\mathsf{mkt}} - R_f) + \beta_{j,\mathsf{smb}} R_{\mathsf{smb}} + \beta_{j,\mathsf{hml}} R_{\mathsf{hml}} + e_j,$$

  - where $\mathbb{E}[e_j] = 0, \mathbb{V}[e_j] = \sigma^2_{e_j}, \mathrm{Cov}[e_i, e_j] = 0$, and
  - σ^2_{mkt}, σ^2_{smb}, and σ^2_{hml} are the return variances of the three factors, and
  - the factors are assumed to be orthogonal to one another.

Moments of asset returns

- Returns are described by

$$R_j - R_f = \alpha_j + \beta_{j,\mathsf{mkt}}(R_{\mathsf{mkt}} - R_f) + \beta_{j,\mathsf{smb}} R_{\mathsf{smb}} + \beta_{j,\mathsf{hml}} R_{\mathsf{hml}} + e_j,$$

- From the returns above, we can compute:
  - Expected value (mean) of the returns of asset j is given by:

$$\mathbb{E}[R_j] - R_f = \alpha_j + \beta_{j,\mathsf{mkt}}(\mathbb{E}[R_{\mathsf{mkt}}] - R_f) + \beta_{j,\mathsf{smb}}\mathbb{E}[R_{\mathsf{smb}}] + \beta_{j,\mathsf{hml}}\mathbb{E}[R_{\mathsf{hml}}]$$

  - Variance of the returns of asset j is given by:

$$\mathbb{V}[R_j] = \beta^2_{j,\mathsf{mkt}}\sigma^2_{\mathsf{mkt}} + \beta^2_{j,\mathsf{smb}}\sigma^2_{\mathsf{smb}} + \beta^2_{j,\mathsf{smb}}\sigma^2_{\mathsf{smb}} + \sigma^2_{e_j}.$$

  - Covariance between the returns on asset i and j is given by:

$$\mathbb{C}[R_i, R_j] = \beta_{i,\mathsf{mkt}}\beta_{j,\mathsf{mkt}}\sigma^2_{\mathsf{mkt}} + \beta_{i,\mathsf{smb}}\beta_{j,\mathsf{smb}}\sigma^2_{\mathsf{smb}} + \beta_{i,\mathsf{smb}}\beta_{j,\mathsf{smb}}\sigma^2_{\mathsf{smb}}.$$

Optimal portfolio weights based on Fama-French model

- To find the optimal portfolio weights when returns are give by the Fama-French model, we need to use the expression

$$w = \frac{1}{\gamma}(\mathbb{V}[R])^{-1}(\mathbb{E}[R] - R_f 1_N),$$

but where the vector of expected returns, $\mathbb{E}[R]$, and the return covariance matrix, $\mathbb{V}[R]$, are obtained as shown on the previous slide.

Performance of firm-characteristic based factor models

- Recall that our objective was to identify factors that explain well the variation in stock returns.
- Factor models based on firm characteristics
  - perform better than the market model
  - but still have limited success in explaining variation in stock returns.
- In our final class, we will see how to construct portfolios that deliver much higher Sharpe ratios than those based on firm characteristics.

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Third type: Statistical factor models (based on principal components) . . . I

- Recall that our objective is to identify factors that explain well the variation in stock returns.
- Principal-components analysis (PCA) is a statistical technique designed to do exactly that.
- PCA identifies the principal components (PCs) that explain the largest variation in stock returns.

Third type: Statistical factor models (based on principal components) . . . II

- PCA is a statistical technique for factor construction and dimensionality reduction.
- In the context of factor-based investing,
  - PCA helps identify underlying factors in a dataset by transforming the original variables
  - into a set of linearly uncorrelated variables, called PCs.
- Python is very very convenient for doing PCA.
  - See the "PCA" function in the "sklearn.decomposition" library.
- For an introduction to PCA, see
  - this excellent article, or
  - this article for an even simpler exposition.

Steps for using PCA for factor construction . . . I

- Note: factor construction is different from portfolio construction.
  - In factor construction, we construct the factors using the PCs.
  - In portfolio construction, we construct portfolio weights using the factors selected from the PCs.
- In this section, we discuss factor construction.
- In the next section, we discuss portfolio construction.

Steps for using PCA for factor construction . . . II

1. Data preparation for factor construction
    - Collect relevant financial data, such as stock returns for a set of assets (e.g., stocks).
        - One can also apply PCA to other quantities, such as financial ratios, or other firm characteristics.
    - Decide whether to standardize the data to have a mean of zero and a standard deviation of one.
2. Covariance matrix calculation
    - Calculate the covariance matrix of the (standardized) return data.

Steps for using PCA for factor construction . . . III

3. Eigen-system decomposition

- Perform eigenvalue decomposition on the (standardized) return covariance matrix, i.e., decompose it into a set of eigenvectors and eigenvalues:

$$\mathbb{V}[R] = E \Lambda E^\top,$$

where

$$\begin{aligned} \mathbb{V}[R] &= \text{original } N \times N \text{ return covariance matrix} \\ E &= (E_1, E_2, \dots E_N) = \text{matrix of } N \text{ eigenvectors} \\ \Lambda &= \operatorname{diag}(\Lambda_1^2, \Lambda_2^2, \dots \Lambda_N^2) = \text{diagonal matrix with } N \text{ eigenvalues} \end{aligned}$$

and, its inverse is given by

$$(\mathbb{V}[R])^{-1} = E \Lambda^{-1} E^\top.$$

Steps for using PCA for factor construction . . . IV

4. Interpretation of the eigen-system
   - Eigenvectors represent the directions (principal components = PCs) of maximum variance;
   - Eigenvalues represent the magnitude of the variance in those directions.
5. Principal-components extraction
   - Arrange the eigenvectors in descending order of their corresponding eigenvalues.
   - The first PC explains the most variance in the data, the second PC explains the second most, and so on.

Steps for using PCA for factor construction . . . V

6. PC loadings
    - The PCs obtained from PCA are linear combinations of the original variables.
    - These linear combinations are represented by PC loadings, which indicate the contribution of each original variable to the PC.
    - The PC loadings provide insights into which variables are driving the variation captured by each PC.
7. "Factor" construction
    - The first few PCs that capture significant variance are considered potential "factors" (use cross-validation to decide how many PCs to retain).
    - These "factors" represent the dominant sources of variation in the dataset.

Steps for using PCA for factor construction . . . VI

8. PC interpretation
    - Interpret the PC based on the characteristics of the original variables with high factor loadings.
    - For example, if the first PC has high loadings on the market return, it might be interpreted as a "market factor."
    - Interpreting factors is crucial for understanding their economic significance.

We have finished our discussion of
using PCA for building factor models.

Now we study
how to use PCA to construction portfolios.

Steps for using PCA for portfolio construction . . . I

1. Download historical stock data: Use the yfinance library to download historical stock data for a set of companies.
   - This data should be organized in a matrix, where rows represent time periods and columns represent different assets.
2. Calculate returns: Calculate monthly returns from the adjusted closing prices.
3. Standardize returns: Standardize the returns by subtracting the mean and dividing by the standard deviation for each asset, so that each variable has a mean of zero and a standard deviation of one.
4. Split data into training and testing sets: Split the data into a training set (used for constructing portfolios) and a testing set (used for evaluating out-of-sample performance).

Steps for using PCA for portfolio construction . . . II

5. Perform PCA: Use the training set to perform PCA to decompose the covariance matrix into its principal components; i.e., find the eigenvalues and eigenvectors of the covariance matrix.

6. Select principal components: Determine the number of principal components to retain based on the explained variance.
   - You may choose a certain percentage of total variance to retain (e.g., 95%).

7. Construct principal components portfolio: Construct portfolios using the retained principal components.
   - Each portfolio will be a linear combination of the original assets,
   - with the weights given by the corresponding eigenvector of the retained PCs.

Steps for using PCA for portfolio construction . . . III

8. Determine portfolio weights for each asset: Determine the weights *for each asset* in the portfolio by combining the weights across the retained principal components.

9. Normalize weights: Normalize the portfolio weights to ensure that they sum to one, making the portfolio fully invested.

10. Apply weights to testing data:
    - Apply the weights to the testing set to evaluate the performance of the PCA portfolio using a metric such as the Sharpe ratio.
    - Choose the set of weights that perform best out-of-sample.

GMV with PCA-based dimensionality reduction

- Let us denote the return-covariance matrix by $V = \mathbb{V}[R]$.
- We know that the global minimum-variance portfolio (GMV) is

$$w_{\text{GMV}} = \frac{V^{-1} 1_N}{1_N^\top V^{-1} 1_N}.$$

- We also know that the return-covariance matrix can be written as

$$V = E \Lambda E^\top, \quad \text{with its inverse given by:} \quad V^{-1} = E \Lambda^{-1} E^\top,$$

  where Λ contains the N principal components (eigenvectors).

- Putting the two together,

$$w_{\text{GMV}} = \frac{(E \Lambda^{-1} E^\top) 1_N}{1_N^\top (E \Lambda^{-1} E^\top) 1_N},$$

  which would just be another way of getting exactly the old result.

Sample code for PCA-based portfolios . . . I

- The code below shows that, exactly as one would expect, you get the same results if you use
  - the standard $N \times N$ covariance matrix
  - all N PCs from the PCA decomposition of the covariance matrix
- The objective of this code is to
  - illustrate this (obvious) result, and
  - set the stage for the case where we consider $K < N$ PCs.

Sample code for PCA-based portfolios . . . II

```
import numpy as np

# Simulate returns for 10 Assets
np.random.seed(0)
T, N, K = 100, 10, 3
F = np.random.normal(0, 1, size=(T, K))      # Latent factors
B = np.random.uniform(-1, 1, size=(N, K))    # Factor loadings
E = np.random.normal(0, 0.05, size=(T, N))   # Idiosyncratic noise
R = F @ B.T + E                              # Simulated returns

#Sample mean vector and covariance matrix of returns
mu = R.mean(axis=0)                   # Sample mean returns
Sigma_full = np.cov(R, rowvar=False)  # Sample covariance matrix
inv_Sigma = np.linalg.inv(Sigma_full) # Inverse of cov. matrix

# Minimum variance portfolio (no expected return constraint)
ones = np.ones(N)
w_full = inv_Sigma @ ones
w_full /= ones @ inv_Sigma @ ones
```

Sample code for PCA-based portfolios . . . III

```
# PCA-based portfolio choice
from sklearn.decomposition import PCA

pca = PCA(n_components=N)  # No truncation: use all N components
pca.fit(R)
V_N = pca.components_.T
Lambda_N = np.diag(pca.explained_variance_)

# Untruncated covariance and its inverse
Sigma_pca = V_N @ Lambda_N @ V_N.T
Sigma_pca_inv = V_N @ np.linalg.inv(Lambda_N) @ V_N.T

# PCA-based weights
w_pca = Sigma_pca_inv @ ones
w_pca /= ones @ Sigma_pca_inv @ ones
```

Sample code for PCA-based portfolios . . . IV

```
print("\nWeights of the GMV portfolio using the standard (full)
    covariance matrix = ")
print(w_full)

print("\nWeights of the GMV portfolio using the PCA (full) covariance
    matrix = ")
print(w_pca)

Weights of GMV portfolio using standard (full) covariance matrix =
[ 0.08867118  0.23587618  0.06426487  0.06258271  0.08986071
0.25729893  0.15513514  0.08365772  0.01134367 -0.0486911 ]

Weights of GMV portfolio using PCA (full) covariance matrix =
[ 0.08867118  0.23587618  0.06426487  0.06258271  0.08986071
0.25729893  0.15513514  0.08365772  0.01134367 -0.0486911 ]
```

Truncated PCA covariance estimation

- To improve out-of-sample performance, we reduce noise by retaining only the top $K < N$ principal components.
  - You can choose K using cross-validation.
- Let $E_K \in \mathbb{R}^{N \times K}$ and $\Lambda_K \in \mathbb{R}^{K \times K}$, so the reduced-dimension matrix is

$$\hat{V}_K = E_K \Lambda_K E_K^\top.$$

- Then the (pseudo) inverse of $\hat{V}_K$ is given by:

$$\hat{V}_K^{-1} = E_K \Lambda_K^{-1} E_K^\top.$$

- The GMV weights, based on the reduced-dimension matrix, are

$$w_{\text{GMV}} = \frac{\hat{V}_K^{-1} 1_N}{1_N^\top \hat{V}_K^{-1} 1_N}.$$

Sample code for *truncated*-PCA GMV portfolios . . . I

- We now look at the case where we do not use all the PCs.
- Instead, we truncate the number of PCs, keeping only K PCs.
- The code below shows the results for the GMV portfolio if you truncate the PCA to keep only $K = 3$ PCs.

Sample code for *truncated*-PCA GMV portfolios . . . II

```
import numpy as np
# Simulate returns for 10 Assets
np.random.seed(0)
T, N, K = 100, 10, 3
F = np.random.normal(0, 1, size=(T, K))      # factor returns
B = np.random.uniform(-1, 1, size=(N, K))    # betas
Err = np.random.normal(0, 0.1, size=(T, N))  # residual errors
R = F @ B.T + Err  # simulated excess returns using factor model
ones = np.ones(N)  # vector of ones

# Sample moments
mu = R.mean(axis=0)                        # Sample mean returns
Sigma_full = np.cov(R, rowvar=False)   # Sample return covariance matrix
inv_Sigma = np.linalg.inv(Sigma_full)  # Inverse of covariance matrix
print("\nSample mean returns = ")
print(mu)
print("\nSample covariance matrix = ")
print(Sigma_full)

# Minimum variance portfolio (no expected return constraint)
w_full = inv_Sigma @ ones
w_full /= ones @ inv_Sigma @ ones
```

Sample code for *truncated*-PCA GMV portfolios . . . III

```
# PCA-based portfolio choice
from sklearn.decomposition import PCA

# PCA with only K = 3 elements included (i.e., with truncation)
pca = PCA(n_components=K)
pca.fit(R)   # fit the simulated returns keeping only K components
E_K = pca.components_.T   # eigenvalues
Lambda_K = np.diag(pca.explained_variance_)   # eigenvectors
print("\nEigenvectors when K = ", K)
print(E_K)
print("\nEigenvalues when K = ", K)
print(Lambda_K)

# Reconstruct rank-K covariance matrix
V_K = E_K @ Lambda_K @ E_K.T
V_K_inv = E_K @ np.linalg.inv(Lambda_K) @ E_K.T

print("\nSample covariance matrix reconstructed from truncated PCA = ")
print(V_K)

# PCA-based weights
w_pca = V_K_inv @ ones
w_pca /= ones @ V_K_inv @ ones
```

Sample code for *truncated*-PCA GMV portfolios . . . IV

```
print("\nWeights of the GMV portfolio using the standard (full)
    covariance matrix = ")
print(w_full)

print("\nWeights of GMV portfolio with K = 3 truncated PCA covariance
    matrix = ")
print(w_pca)

# Output of the print statements

Sample mean returns =
[ 0.12708671 -0.16183823  0.04197566 -0.05191858  0.16836941  0.16672355
 -0.1176492  -0.17905506 -0.08783624 -0.0241036 ]
```

Sample code for *truncated*-PCA GMV portfolios . . . V

```
Sample covariance matrix =
[[ 0.78095393 -0.49041375 -0.12489251  0.6365252   0.42636192
0.09346918  -0.14552655  0.11854582  0.37169097  0.82012935]
 [-0.49041375  0.70871751 -0.56471794  0.0583193  -0.9199419
 -0.46095861   0.4286234   0.41375973 -0.07788141 -0.26635618]
 [-0.12489251 -0.56471794  2.02442023 -0.34594391  1.36384133
 -0.23111022  -0.21837588  0.16318862  0.69051056  0.24033844]
 [ 0.6365252   0.0583193  -0.34594391  1.37999485 -0.22606286
 -0.90015408   0.46308246  1.24660673  1.0640816   1.42801229]
 [ 0.42636192 -0.9199419   1.36384133 -0.22606286  1.46660552
 0.38634751  -0.52256429 -0.37653416  0.32575217  0.3299647 ]
 [ 0.09346918 -0.46095861 -0.23111022 -0.90015408  0.38634751
 1.27531849  -0.66979766 -1.44249204 -1.08167927 -0.92001436]
 [-0.14552655  0.4286234  -0.21837588  0.46308246 -0.52256429
 -0.66979766   0.44018513  0.74632079  0.41538559  0.34165468]
 [ 0.11854582  0.41375973  0.16318862  1.24660673 -0.37653416
 -1.44249204   0.74632079  1.72328278  1.34745599  1.3035032 ]
 [ 0.37169097 -0.07788141  0.69051056  1.0640816   0.32575217
 -1.08167927   0.41538559  1.34745599  1.35972644  1.36714473]
 [ 0.82012935 -0.26635618  0.24033844  1.42801229  0.3299647
 -0.92001436   0.34165468  1.3035032   1.36714473  1.72964436]]
```

Sample code for *truncated*-PCA GMV portfolios . . . VI

```
Eigenvectors when K = 3
[[ 0.12668987  0.16152746 -0.5232372 ]
 [ 0.04005712 -0.36740922  0.26813076]
 [ 0.06822867  0.5931274   0.52954895]
 [ 0.41130711 -0.08183566 -0.31543207]
 [-0.02205572  0.6012429  -0.05156336]
 [-0.38439408  0.12290759 -0.31998217]
 [ 0.18044483 -0.19896489  0.15881839]
 [ 0.48314812 -0.1173505   0.20341851]
 [ 0.42571837  0.16574378  0.09315237]
 [ 0.46156536  0.15163668 -0.30672813]]

Eigenvalues when K = 3
[[6.74909868 0.         0.        ]
 [0.         4.01792971 0.        ]
 [0.         0.         2.05425031]]
```

Sample code for *truncated*-PCA GMV portfolios ...VII

```
Sample covariance matrix reconstructed from truncated PCA =
[[ 0.77556432 -0.49240332 -0.12590944  0.63761857  0.42677506
0.09503061  -0.14554898  0.11830449  0.37145019  0.82276092]
 [-0.49240332  0.70089637 -0.56546393  0.05826203 -0.9219337
 -0.46160855   0.42997837  0.41589898 -0.07827325 -0.26801405]
 [-0.12590944 -0.56546393  2.02098337 -0.34876189  1.36660032
 -0.23218501  -0.21830363  0.16410182  0.69236057  0.24024777]
 [ 0.63761857  0.05826203 -0.34876189  1.37306986 -0.22550843
 -0.90013196   0.46341728  1.24797173  1.06691515  1.43017637]
 [ 0.42677506 -0.9219337   1.36660032 -0.22550843  1.46119851
 0.38802748  -0.52433279 -0.3769562   0.32715791  0.33009957]
 [ 0.09503061 -0.46160855 -0.23218501 -0.90013196  0.38802748
1.26826651  -0.67078115 -1.44510115 -1.08382827 -0.92094232]
 [-0.14554898  0.42997837 -0.21830363  0.46341728 -0.52433279
 -0.67078115   0.43062574  0.74857606  0.41634801  0.34081939]
 [ 0.11830449  0.41589898  0.16410182  1.24797173 -0.3769562
 -1.44510115   0.74857606  1.71579077  1.34896518  1.30540805]
 [ 0.37145019 -0.07827325  0.69236057  1.06691515  0.32715791
 -1.08382827   0.41634801  1.34896518  1.35138257  1.36846366]
 [ 0.82276092 -0.26801405  0.24024777  1.43017637  0.33009957
 -0.92094232   0.34081939  1.30540805  1.36846366  1.72350068]]
```

Sample code for *truncated*-PCA GMV portfolios . . . VIII

```
Weights of the GMV portfolio using the standard (full) covariance matrix
     =
[ 0.0903513   0.23522936  0.06315848  0.06223911  0.09010167  0.25788985
  0.15465218  0.08350277  0.01110026 -0.04822498]

Weights of GMV portfolio with K = 3 truncated PCA covariance matrix =
[ 0.18394367 -0.15265068  0.13217576  0.16635709  0.20041182 -0.03796389
 -0.03046346  0.09304727  0.18555295  0.25958947]
```

Mean-variance portfolio using truncated PCA . . . I

- As before, we start by doing a PCA of returns, and retain only the top $K < N$ principal components.
  - You can choose K using cross-validation.
- Then, as before, write the factor model for returns on the N assets,

  $$R = B\, R_{\text{PCA}} + \epsilon,$$

  but now we are using matrix notation, so
  - $R \in \mathbb{R}^N$,
  - $B \in \mathbb{R}^{N \times K}$ is the matrix of betas,
  - $R_{\text{PCA}} \in \mathbb{R}^K$ is the vector of K statistical-factor returns, and
  - $\epsilon \in \mathbb{R}^N$:

Mean-variance portfolio using truncated PCA . . . II

- Using the eigenvalues $E_K \in \mathbb{R}^{N \times K}$, the return on the K statistical factors, denoted R_{PCA}, is:

$$\hat{R}_{\text{PCA}} = E_K^\top R.$$

- Use the returns on the statistical factors to estimate expected returns on the N assets:

$$\mathbb{E}[R] = \hat{B}\, \mathbb{E}[\hat{R}_{\text{PCA}}].$$

- Similarly, estimate the covariance matrix of N asset returns:

$$\mathbb{V}[R] = \hat{B}\, \mathbb{V}[\hat{R}_{\text{PCA}}]\, \hat{B}^\top + D \quad \ldots D \text{ is diagonal matrix of idio (residual) variances}$$

- Then, the mean-variance optimal portfolio based on PCA is:

$$w_{\text{MV}} = \frac{1}{\gamma}(\mathbb{V}[R])^{-1}(\mathbb{E}[R] - R_f 1_N).$$

Python Code: Expected Returns and Optimal Weights

```
# Estimate factor returns and loadings
R_PCA = R @ E_K  # R, E_K are from earlier code for PCA decomposition
B_hat = E_K
mu_R_PCA = R_PCA.mean(axis=0)   # mean returns for statistical factors
mu = B_hat @ mu_R_PCA   # mean returns for N risky assets

# Factor covariance and idiosyncratic variance
cov_R_PCA = np.cov(R_PCA, rowvar=False)
D = np.diag(np.var(R - R_PCA @ B_hat.T, axis=0))

# Total covariance of returns for the $N$ assets
Sigma = B_hat @ cov_R_PCA @ B_hat.T + D

# Optimal mean-variance weights (gamma = 1 for simplicity)
inv_Sigma = np.linalg.inv(Sigma)
w_MV = inv_Sigma @ mu
w_MV /= np.sum(w_MV)  # normalize to sum to 1
```

Papers and Python code for portfolio construction using principal components

- Papers explaining portfolio construction using PCA:
  - Partovi and Caputo (2004).
  - Meucci (2009).
  - Pasini (2017).
- Python code to construct portfolios using PCA
  - "Eigen-portfolio construction using PCA".
  - "A principal component analysis of portfolio risk shift-pandemic period".

Advantages of PCA for factor construction . . . I

- Balance:
  - PCA balances structure (via statistical factor models) and robustness (via dimension reduction).
- Dimensionality reduction:
  - PCA allows for the reduction of the dimensionality of the dataset while retaining most of the important information.
  - This is particularly useful when dealing with a large number of variables.
- Uncorrelated factors:
  - The principal components obtained through PCA are uncorrelated, which can simplify subsequent analyses.
  - This is valuable when constructing factors that are intended to be independent of each other.

Advantages of PCA for factor construction . . . II

- Capturing commonalities:
  - PCA effectively captures commonalities across variables and identifies the dominant sources of variation in the data.
  - This is particularly relevant in finance when seeking to identify common factors affecting asset returns.
  - Expected returns can be incorporated through latent factor structures estimated by PCA.
- Quantitative approach:
  - PCA provides a quantitative and data-driven approach to factor construction.
  - It does not require a priori assumptions about the nature of the factors, making it a flexible method.

Limitations of PCA for factor construction . . . I

- Interpretability:
  - While PCA identifies statistical factors, the economic interpretation of these factors may not always be straightforward.
  - Researchers often need to rely on additional information to interpret and name the factors meaningfully.
- Assumption of linearity:
  - PCA assumes a linear relationship between variables.
  - If the relationship is highly nonlinear, PCA may not capture the underlying factors accurately.

Limitations of PCA for factor construction . . . II

- Data quality:
  - The effectiveness of PCA depends on the quality and relevance of the input data.
  - Noisy or irrelevant variables may lead to less meaningful factors.
- Factor stability:
  - PCA assumes that the factors derived are stable over time.
  - In dynamic market conditions, factors may change, and ongoing monitoring and adjustment may be necessary.
  - Recent papers show how to do conditional PCA; see Kelly, Pruitt, and Su (2020). Python code for this paper is available from Pruitt's homepage.

Evaluation of truncated-PCA portfolios

- Full mean-variance uses all eigen-directions, including noisy ones.
- Truncated-PCA approach filters out small-eigenvalue directions—usually dominated by estimation error.
  - Choice of truncation parameter depends on trade-off between *fit* and *stability*.
- PCA portfolios tend to be more stable, diversified, and robust in practice.

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2. Motivation for the material in this chapter
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4. Three types of factors in asset-pricing factor models
4.1 First type: Macroeconomic factor models
4.2 Second type: Fundamentals-based factor models
4.3 Third type: Statistical factor models (based on PCs)
4.4 Comparison of the three types of factor models
5. Implementing factor models
6. Portfolio construction using a factor model
7. Parametric portfolio policies
8. Factor models for other asset classes besides equities
9. To do for next class: Readings and assignment
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Comparison of the three types of factor models . . . I

- Macroeconomic and statistical factor models both estimate a firm's factor-beta by time-series regression.
    - Given the nature of security returns data, this limitation is substantial.
    - Time-series regression requires a long and stable history of returns to estimate the factor betas accurately.

Comparison of the three types of factor models ...II

- Fundamental factor models do not require time series regression; it uses observed company attributes as factor betas.
  - These models rely on the empirical finding that company attributes such as firm size, dividend yield, book-to-market ratio, and industry classification explain a substantial proportion of common return.
  - The factors in a fundamental factor model are the realized returns to a set of mimicking portfolios designed to capture the marginal returns associated with a unit exposure to each attribute.
    - For example, the dividend yield factor is the realized return per extra unit of dividend yield, holding other attributes constant.
  - The betas are exogenously determined, firm-specific attributes rather than estimated sensitivities to random factors, and

Comparison of the three types of factor models . . . III

- Portfolios relying on PCA perform well empirically.
- With improving computing capabilities, in the last 10 years there have been substantial advances in
  - the use of PCA-based methods along with
  - the use of techniques from machine learning.
- In our last class, we will look at a PCA-based model in detail.

Comparison of the three types of factor models . . . IV

- In theory, i.e., in the absence of estimation error and with no limits on data availability, the three types of models are fully consistent with one another;
- That is, the three models are simply restatements or (to use a technical term from factor modeling) rotations of one another.
- In this sense, the three types of factor models can all hold simultaneously.
- In practice, as we have discussed above, their performance will vary.

End of focus

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How are factors identified?

- Factors are identified in various ways – depending on type of factors.

1. Factor models could be based on macroeconomic equilibrium asset pricing model.
   - E.g., long-run consumption growth, past consumption habits, and macroeconomic uncertainty.

2. The Fama-French factors are based on economic intuition for firm characteristics that may be indicators of expected returns.
   - E.g., size, value, profitability, investment.

   Other factors originate as a trading strategy that is discovered to produce abnormal returns:
   - E.g., momentum, price reversal, low beta, and idiosyncratic volatility.

3. A purely statistical approach starts by estimating the principal components from a panel of returns.

Challenges in implementing factor models

- While conceptually elegant, the empirical identification of factors is a difficult problem.
- One problem is that there is a very large number of factors.
  - Cochrane (2011) coined the term "factor zoo" referring to the literally hundreds of factors discovered so far.
- Dello-Preite, Uppal, Zaffaroni, and Zviadadze (2024) give a list of
  - 457 firm-specific characteristics in the finance literature;
  - 103 macro factors.

Questions when using factor models

- How many true factors are there?
  - Given the large number of factors "discovered" so far, how should we "tame the factor zoo" (Feng, Giglio, and Xiu 2020).
- Which factors are priced and so should be included?
  - For portfolio management, there is no point taking factor risk for which there is no reward (Daniel, Mota, Rottke, and Santos 2020).
- How stable is the factor structure?
  - A single factor model with time-varying beta's is often equivalent to a multifactor model, and vice versa.
- What is the economic interpretation of the factors?
  - Why does a particular factor drive returns?

Estimating how many factors are needed

- Suppose that one wishes to test whether K factors are sufficient to explain all the pervasive movements in security returns.
- Then, one could estimate two models on the same data set.
  - One model with K factors;
  - The other model with $K+1$ factors.
- Comparing the unexplained residual cross-sectional variance by the two models will then tell us whether we need the $K+1$ factors.
- If $K+1$ factors are necessary, then the difference in the average asset-specific variance should be strictly positive.
- Thus, the increase in explanatory power from adding a factor is the basis for testing whether the additional factor is needed.

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Portfolio construction using a factor model . . . I

- Having selected a factor model, the next question is how to use it to construct a portfolio.
- To use factors to construct a portfolio, we need to compute the returns associated with the factor.
  - That is, each factor needs to be translated into a trading strategy so that we can compute its returns.
- For some factors, this is straightforward because the factor is itself defined as the trading strategy that generates (abnormal) returns.
- However, for macro factors, such as inflation, a trading strategy that replicates (mimics) the factor needs to be constructed.

Portfolio construction using a factor model . . . II

- Lamont (2001) explains how to use regression models to construct factor-mimicking portfolios.
- But when the number of assets or number of trading strategies is large, standard regression models do not work well.
- Implementing regressions in such a setting requires regularization (shrinkage) techniques, such as Lasso, Boosting, or Bayesian priors.

Portfolio construction using a factor model . . . III

- In the last ten years, machine-learning techniques have been used to regularize large-scale portfolio problems.
  - One example is Fan, Zhang, and Yu (2012), who use the LARS (Least Angle Regression) algorithm (explained in the appendix to these slides).
  - Heaton, Polson, and Witte (2016) apply "deep learning" techniques to portfolio optimization; Goodfellow, Bengio, and Courville (2016) is a textbook on deep learning.
  - DeMiguel, Martin-Utrera, Nogales, and Uppal (2017) use the lasso.
- Other approaches to regularization of portfolio weights have been motivated by model uncertainty
  (see Goldfarb and Iyengar 2003; Garlappi, Uppal, and Wang 2007).

Portfolio construction using a factor model . . . IV

- Once you have selected the K factors, there are two steps:
  1. Estimate the factor model and compute the means and covariance matrix for these factor returns;
  2. Use mean-variance portfolio optimization to select how much weight to put on each factor.
- After choosing the optimal weight on the K factors, then identify what that implies for weights on the underlying N risky assets.

Start of focus

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Parametric portfolio policies: The big picture

- Parametric portfolio policies (PPP) were developed by Brandt, Santa-Clara, and Valkanov (2009) and rely on a very clever insight.
- Their insight is that,
  - instead of having two steps:
    1. first estimate a factor model for returns and
    2. then, in a second step, use the factor model to identify the optimal weights,
  - to specify, directly in one step,
    1. a factor model for the portfolio *weights*.

Parametric portfolio policies: The details . . . I

- That is, instead of using a factor model for *asset returns*, where we use two steps to find the optimal portfolio weights:
  1. Start with a K-factor model for the N asset returns
     $$\mathbb{E}[R - R_f 1_N] = \alpha + \beta\lambda;$$
     and find the asset return means and covariances in terms of the parameters of the factor model
     - N alphas for the assets, α
     - N residual variances for the assets, $\sigma^2_{e_n}$
     - $N \times K$ asset betas for the K factors, $\beta_{n,k}$
     - K means and K variances for the factors, λ_k and $\sigma^2_{F_k}$;
  2. Then, in a second step, use these in a mean-variance optimization to find the optimal portfolio weights.

Parametric portfolio policies: The details . . . II

- Brandt, Santa-Clara, and Valkanov (2009) propose, in a single step,
- specifying a parametric (factor) model for the $N_t \times 1$ vector of portfolio weights, $w_t(\theta)$

$$w_t(\theta) = w_{b,t} + (F_{1,t}\theta_1 + F_{2,t}\theta_2 + \ldots + F_{K,t}\theta_K)/N_t, \quad \text{where}$$

  - $w_{b,t}$ is the $N_t \times 1$ vector of benchmark portfolio weights at time t,
  - $F_{k,t}$ is the $N_t \times 1$ long-short characteristic portfolio obtained by standardizing the kth firm-specific characteristic at time t,
  - θ_k is the (scalar) weight on the kth characteristic portfolio in the parametric portfolio,
  - N_t is the number of firms at time t.
- Note θ does not depend on time, but firm characteristics, $F_{k,t}$, do.

Parametric portfolio policies: The details . . . III

- In the expression for parametric portfolio weight

$$w_t(\theta) = w_{b,t} + (F_{1,t}\theta_1 + F_{2,t}\theta_2 + \ldots + F_{K,t}\theta_K)/N_t, \text{ where}$$

  - $w_{b,t}$ is determined by the client (could be market portfolio);
  - $F_{k,t}$ is determined by the data on firm characteristics (e.g., size, value, momentum, etc.);
  - N_t is determined by how many firms we have data for;
  - $\theta = \{\theta_1, \ldots, \theta_K\}$ is a $K \times 1$ vector to be chosen by the investor.
- So, the only unknown is the $K \times 1$ vector $\theta = \{\theta_1, \ldots, \theta_K\}$.
- So, even if $N = 11,000$, we have to estimate only K parameters.
  - Whether $K = 3, 4, 5, \ldots$ will depend on your choice of factor model; but K is usually a small number (less than 10).

Parametric portfolio policies: The details . . . IV

- The weights of the characteristics in the parametric portfolio are scaled by the number of stocks N_t
  - so that they are meaningful for the case where the number of stocks is varying over time;
  - otherwise, increasing the number of stocks while keeping the weights fixed would result in more aggressive portfolio allocations.
- On the next few slides we explain (and illustrate) the design of parametric portfolio policies.

Standardizing characteristic portfolios . . . I

- Brandt, Santa-Clara, and Valkanov (2009) suggest that you standardize each characteristic so that
  - it has a cross-sectional mean of zero, and
  - a cross-sectional standard deviation of one.
- The resultant standardized characteristic is a long-short portfolio that is
  - long stocks whose characteristic is above the cross-sectional average;
  - short stocks whose characteristic is below cross-sectional average.

Standardizing characteristic portfolios . . . II

- Each row represents a particular stock.
- Each number represents the value of the characteristic for that stock.

Firm-specific characteristics

	Value long-short	**Momentum long-short**	**Size long-short**
Stock 1:	0.01%	-0.02%	0.03%
Stock 2:	-0.02%	+0.01%	-0.01%
Stock 3:	-0.03%	+0.12%	0.13%
⋮	.	.	.
	.	.	.
	.	.	.
Stock N:	0.03%	-0.02%	-0.01%

Illustration of parametric portfolios

	Parametric portfolio		Benchmark portfolio		Value long-short		Momentum long-short		Size long-short
Stock 1:	0.05%		0.09%		0.01%		-0.02%		0.03%
Stock 2:	0.05%		0.05%		-0.02%		+0.01%		-0.01%
Stock 3:	0.17%	=	0.21%	$+\,\Theta_1$	-0.03%	$+\,\Theta_2$	+0.12%	$+\,\Theta_3$	0.13%
⋮	⋮		⋮		⋮		⋮		⋮
Stock N:	0.17%		0.15%		0.03%		-0.02%		-0.01%

Full-invested in risky assets

- Brandt, Santa-Clara, and Valkanov (2009) consider a portfolio that is fully invested in risky assets.
- Thus, the parametric portfolio weights on the N_t stocks sum to one.
- Because the weights of the stocks in each characteristic long-short portfolio sum to zero,
- it implies that the parametric weight on the benchmark portfolio must be equal to one.

Returns on the parametric portfolio ... I

- The parametric portfolio can be written in compact matrix notation.
- Define F_t, the $N_t \times K$ matrix whose kth column is $F_{k,t}$.
- Then,

$$w_t(\theta) = w_{b,t} + F_t\theta/N_t, \quad \text{where} \tag{1}$$

- θ is the $K \times 1$ parameter vector, whose kth component is the weight of the kth characteristic θ_k, and
- $F_t\theta/N_t$ is the characteristic portfolio at time t.

Returns on the parametric portfolio . . . II

- The return of the parametric portfolio at time $t+1$, $r_{p,t+1}(\theta)$, is

$$\begin{aligned} r_{p,t+1}(\theta) &= w_{b,t}^{\top} r_{t+1} + \theta^{\top} F_t^{\top} r_{t+1} / N_t \\ &= \; r_{b,t+1} + \theta^{\top} r_{c,t+1}, \quad \text{where} \end{aligned} \tag{2}$$

  - r_{t+1} is the $N_t \times 1$ return vector at time $t+1$,
  - $r_{b,t+1} = w_{b,t}^{\top} r_{t+1}$ is the benchmark portfolio return at time $t+1$,
  - $r_{c,t+1} = F_t^{\top} r_{t+1} / N_t$ is the characteristic return vector at time $t+1$, which contains the returns of the long-short portfolios corresponding to the K characteristics scaled by the number of firms N_t.

- Equation (2) shows that the parametric-portfolio return is
  - the benchmark-portfolio return
  - plus the return of the characteristic portfolio.

Mean and variance of parametric-portfolio returns

- Once you have the expression we derived above for the return of a parametric portfolio

$$r_{p,t+1}(\theta) = r_{b,t+1} + \theta^\top r_{c,t+1}, \tag{2}$$

- we can find the mean and variance of the portfolio returns using standard methods for finding the moments of portfolio returns.
- Once we have the mean and variance of the portfolio return, we can write the optimization problem of a mean-variance investor.

Determining the weights on the characteristic portfolios

- Assume that the investor choses the weights θ by maximizing mean-variance utility:

$$\max_{\theta} \quad \mathbb{E}_t[r_{p,t+1}(\theta)] - \frac{\gamma}{2}\mathbb{V}_t[r_{p,t+1}(\theta)], \quad \text{where} \tag{3}$$

  - γ is the investor's risk-aversion parameter
  - $\mathbb{E}_t[r_{p,t+1}(\theta)]$ is the mean of the parametric-portfolio return.
  - $\mathbb{V}_t[r_{p,t+1}(\theta)]$ is the variance of the parametric-portfolio return.
  - Note, again, that θ does not depend on time.

- If you want, you can
  - add nonnegativity constraints on θ;
  - apply shrinkage to the covariance matrix;
  - adjust for transaction costs.

Regularized parametric portfolios

- To deal with the large number of characteristics in Finance, DeMiguel, Martín-Utrera, Nogales, and Uppal (2020) develop a new class of parametric portfolios.
- These are called regularized parametric portfolios.
- These portfolios are obtained by imposing a lasso (least absolute shrinkage and selection operator).
- The lasso constraint
  - reduces the impact of estimation error and
  - acts as a variable-selection method that helps to reduce problem dimensionality.

Interpretation of parametric portfolio policies

- Parametric portfolios use firm-specific characteristics to tilt the benchmark portfolio toward stocks that improve performance.
  - Parametric portfolios are similar to Black-Litterman portfolios in the sense that both portfolios are a tilt of a benchmark portfolio.
- Parametric portfolios are obtained by
  - adding to the benchmark portfolio
  - a linear combination of long-short characteristic portfolios, obtained by standardizing K firm-specific characteristics cross-sectionally.
- Black-Litterman portfolios are obtained by
  - adding to the benchmark (market) portfolio
  - the views of the investor.

Python code for parametric portfolio policies

- Python code for parametric portfolio policies is available from:
  - The Tidy Finance website (in the section on "Portfolio optimization").
  - But, to run exactly what they run, you may also need access to WRDS.
  - Alternatively, you can use the data file I provided for the assignment.
  - Note that the Tidy Finance website shows optimization for a power utility function, which is what Brandt, Sant-Clara, and Valkanov (2009) use, while we use a mean-variance utility function.

End of focus

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Factor models for other asset classes

- Factor models are relevant not just for stocks, but also other asset classes.
- The challenge is to identify a single set of factors that explain returns for stocks and also other asset classes.
  - Note that stocks, corporate bonds, and equity options have the same underlying: the assets of the firm.
- Some of the current research is focused on the search for a factor model that works across many asset classes:
  - equities
  - bonds
  - options
  - currencies
  - commodities
- See, e.g., Bali, T. G., H. Beckmeyer, and A. Goyal. 2024. A joint factor model for bonds, stocks, and options. Available at SSRN 4589282, from this link.

Road map

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2. Motivation for the material in this chapter
3. Factor models
4. Three types of factors in asset-pricing factor models
5. Implementing factor models
6. Portfolio construction using a factor model
7. Parametric portfolio policies
8. Factor models for other asset classes besides equities
9. To do for next class: Readings and assignment
10. Appendix: Least Angle Regression (LARS) algorithm
11. Bibliography

What we plan to do in the next chapter

What's next?

In the next chapter, we will study if it is possible to time the factors we have found to be important for explaining the cross-section of stock returns.

Specifically, we will study if we can use the volatility of a factor to time our investment in that factor.

Then we will examine what this implies for the risk-return trade-off; i.e., is risk related to returns?

To do for next class: Readings

- Readings
  - A good reference for "factor investing" is the book by Ang (2014).
    - Ang was Head of Factors, Sustainable and Solutions for BlackRock.
    - You can read more about him on this website.
  - For parametric portfolio policies, see the Tidy Finance website (in the section on "Portfolio optimization").
    - This website includes Python code for parametric portfolios.

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Least Angle Regression (LARS) algorithm . . . I

- The Least Angle Regression (LARS) algorithm is a method
  - used for linear regression,
  - particularly in the context of high-dimensional data
  - that was introduced by Efron, Hastie, Johnstone, and Tibshirani (2004) and can be downloaded from this link.
- The main goal of LARS is to find a sparse linear regression model
  - when the number of predictors (features) is much larger than the number of observations.
- It is particularly useful in situations where the number of predictors is comparable to or even exceeds the number of observations.

Key concepts underlying LARS . . . I

- Forward Selection: LARS builds the regression model by adding predictors to the model one at a time. At each step, it identifies the predictor most correlated with the response variable.
- Equal Correlation (Least Angle): LARS moves towards the predictor with the highest correlation with the residual, but it does so in a way that maintains all predictors with equal absolute correlations with the residual.
  - This is where the name "Least Angle" comes from; the algorithm takes the least angle necessary to include a new predictor.

Key concepts underlying LARS . . . II

- Regularization: LARS involves a regularization parameter that controls the shrinkage of coefficients.
  - As the algorithm progresses, it shrinks the coefficients of variables not in the model toward zero.
- Multiple Coefficient Paths: LARS can trace the entire path of the solution for different values of the regularization parameter.
  - This allows for a comprehensive understanding of how the coefficients change as the regularization strength varies.

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Bibliography . . . I

Ang, A. 2014. *Asset management: A systematic approach to factor investing.* Oxford University Press. (Cited on page 148).

Bali, T. G., H. Beckmeyer, and A. Goyal. 2024. A joint factor model for bonds, stocks, and options. Available at SSRN 4589282. (Cited on page 145).

Barroso, P., and A. L. Detzel. 2021. Do limits to arbitrage explain the benefits of volatility-managed portfolios? *Journal of Financial Economics* 140 (3): 744–767. (Cited on page 22).

Bawa, V. S., S. Brown, and R. Klein. 1979. *Estimation risk and optimal portfolio choice.* North Holland, Amsterdam. (Cited on page 20).

Black, F. 1993. Beta and return. *Journal of Portfolio Management* 20 (1): 8–18. (Cited on pages 43–45).

Black, F., and R. Litterman. 1990. Asset allocation: Combining investor views with market equilibrium. Goldman, Sachs & Co. (Cited on page 20).

———. 1991a. Combining investor views with market equilibrium. *Journal of Fixed Income* 1 (2): 7–18. (Cited on page 20).

———. 1991b. Global asset allocation with equities, bonds, and currencies. *Fixed Income Research* 2 (15-28): 1–44. (Cited on page 20).

Bibliography . . . II

Black, F., and R. Litterman. 1992. Global portfolio optimization. *Financial Analysts Journal* 48:28–43. (Cited on page 20).

Brandt, M. W., P. Sant-Clara, and R. Valkanov. 2009. Parametric portfolio policies: Exploiting characteristics in the cross section of equity returns. *Review of Financial Studies:* Forthcoming. (Cited on pages 14, 27, 142).

Brandt, M. W., P. Santa-Clara, and R. Valkanov. 2009. Parametric portfolio policies: Exploiting characteristics in the cross-section of equity returns. *Review of Financial Studies* 22 (9): 3411–3447. (Cited on pages 21, 127, 129, 132, 135).

Cazalet, Z., and T. Roncalli. 2014. Facts and fantasies about factor investing. Available at SSRN 2524547. (Cited on page 64).

Cederburg, S., M. S. O'Doherty, F. Wang, and X. Yan. 2020. On the performance of volatility-managed portfolios. *Journal of Financial Economics* 138 (1): 95–117. (Cited on page 22).

Chamberlain, G. 1983. Funds, factors and diversification in arbitrage pricing models. *Econometrica* 51:1305–1324. (Cited on page 24).

Chamberlain, G., and M. Rothschild. 1983. Arbitrage, factor structure and mean-variance analysis on large asset markets. *Econometrica* 51:1281–1304. (Cited on page 24).

Bibliography . . . III

Chen, A. Y., and T. Zimmermann. 2022. Open source cross-sectional asset pricing. *Critical Finance Review* 11 (2): 207–264. (Cited on page 64).

Chen, N.-F., R. Roll, and S. Ross. 1986. Economic forces and the stock market. *Journal of Business* 59:383–403. (Cited on pages 21, 61).

Clarke, R., H. de Silva, and S. Thorley. 2016. Fundamentals of efficient factor investing. *Financial Analysts Journal* 72:9–26. (Cited on page 64).

Cochrane, J. 2017. Macro-finance. *Review of Finance* 21:1–42. (Cited on page 61).

Daniel, K., L. Mota, S. Rottke, and T. Santos. 2020. The cross-section of risk and returns. *Review of Financial Studies* 33 (5): 1927–1979. (Cited on page 118).

Dello-Preite, M., R. Uppal, P. Zaffaroni, and I. Zviadadze. 2024. Cross-sectional asset pricing with unsystematic risk. Available at SSRN 4135146. (Cited on pages 22, 117).

DeMiguel, V., L. Garlappi, and R. Uppal. 2009. Optimal versus naive diversification: How inefficient is the $1/N$ portfolio strategy? *Review of Financial Studies* 22 (5): 1915–1953. (Cited on page 12).

DeMiguel, V., A. Martin-Utrera, F. J. Nogales, and R. Uppal. 2017. A portfolio perspective on the multitude of firm characteristics. Working Paper, London Business School. (Cited on page 123).

Bibliography . . . IV

DeMiguel, V., A. Martín-Utrera, F. J. Nogales, and R. Uppal. 2020. A Transaction-Cost Perspective on the Multitude of Firm Characteristics. *Review of Financial Studies* 33, no. 5 (April): 2180–2222. (Cited on pages 21, 140).

DeMiguel, V., A. Martín-Utrera, and R. Uppal. 2024. A multifactor perspective on volatility-managed portfolios. *Journal of Finance* 79 (6): 3859–3891. (Cited on page 22).

Efron, B., T. Hastie, I. Johnstone, and R. Tibshirani. 2004. Least angle regression. *Annals of Statisitcs* 32 (2): 407–499. (Cited on page 150).

Fama, E. F., and K. R. French. 1992. The cross-section of expected stock returns. *Journal of Finance* 47, no. 2 (June): 427–465. (Cited on pages 14, 21, 47, 48).

———. 1993. Common risk factors in the returns on stocks and bonds. *Journal of Financial Economics* 33 (1): 3–56. (Cited on pages 14, 21, 63).

———. 2012. Size, value, and momentum in international stock returns. *Journal of Financial Economics* 105 (3): 457–472. (Cited on pages 14, 21).

———. 2015. A five-factor asset pricing model. *Journal of Financial Economics* 116 (1): 1–22. (Cited on pages 14, 21, 63).

———. 2018. Choosing factors. *Journal of Financial Economics* 128 (2): 234–252. (Cited on pages 14, 21, 63).

Bibliography . . . V

Fan, J., J. Zhang, and K. Yu. 2012. Vast portfolio selection with gross-exposure constraints. *Journal of the American Statistical Association* 107:592–606. (Cited on page 123).

Feng, G., S. Giglio, and D. Xiu. 2020. Taming the factor zoo: a test of new factors. *Journal of Finance* 75 (3): 1327–1370. (Cited on page 118).

Garlappi, L., R. Uppal, and T. Wang. 2007. Portfolio selection with parameter and model uncertainty: A multi-prior approach. *The Review of Financial Studies* 20:41–81. (Cited on page 123).

Giglio, S., B. Kelly, and D. Xiu. 2022. Factor models, machine learning, and asset pricing. *Annual Review of Financial Economics* 14:337–368. (Cited on page 64).

Goldfarb, D., and G. Iyengar. 2003. Robust portfolio selection problems. *Mathematics of Operations Research* 28 (1): 1–38. (Cited on page 123).

Goodfellow, I., Y. Bengio, and A. Courville. 2016. *Deep learning.* MIT Press. (Cited on page 123).

He, G., and R. Litterman. 1999. The intuition behind Black-Litterman model portfolios. *Investment Management Research (Goldman, Sachs & Company).* (Cited on page 20).

Bibliography . . . VI

Heaton, J., N. Polson, and J. Witte. 2016. Deep learning for finance: deep portfolios. *Applied Stochastic Models in Business and Industry* 33:3–12. (Cited on page 123).

Hou, K., C. Xue, and L. Zhang. 2015. Digesting anomalies: An investment approach. *Review of Financial Studies* 28 (3): 650–705. (Cited on page 63).

———. 2017. Replicating anomalies. *The Review of Financial Studies.* (Cited on page 63).

Huberman, G. 1982. A simple approach to the Arbitrage Pricing Theory. *Journal of Economic Theory* 28:183–191. (Cited on page 24).

Ingersoll, J. E., Jr. 1984. Some results in the theory of arbitrage pricing. *Journal of Finance* 39:1021–1039. (Cited on page 24).

Jacobs, H., S. Müller, and M. Weber. 2014. How should individual investors diversify? An empirical evaluation of alternative asset allocation policies. *Journal of Financial Markets* 19:62–85. (Cited on page 12).

Jorion, P. 1988. Bayes-stein estimation for portfolio analysis. *Journal of Financial and Quantitative Analysis* 21 (3): 279–292. (Cited on page 20).

Jorion, P. 1985. International portfolio diversification with estimation risk [in English]. *Journal of Business* 58 (3): pp. 259–278. (Cited on page 20).

Bibliography . . . VII

Jorion, P. 1992. Portfolio optimization in practice. *Financial Analysts Journal* 48 (1): 68–74. (Cited on page 20).

Kelly, B. T., S. Pruitt, and Y. Su. 2020. Instrumented principal component analysis. *Available at SSRN 2983919.* (Cited on page 107).

Klein, R. W., and V. S. Bawa. 1976. The effect of estimation risk on optimal portfolio choice. *Journal of Financial Economics* 3 (3): 215–31. (Cited on page 20).

Kozak, S., S. Nagel, and S. Santosh. 2018. Interpreting factor models. *Journal of Finance* 73 (3): 1183–1223. (Cited on page 21).

———. 2020. Shrinking the cross-section. *Journal of Financial Economics* 135 (2): 271–292. (Cited on page 21).

Lamont, O. 2001. Economic tracking portfolios. *Journal of Econometrics* 105:161–184. (Cited on page 122).

Lettau, M., and M. Pelger. 2018. *Estimating latent asset-pricing factors.* Technical report. National Bureau of Economic Research. (Cited on page 21).

———. 2020. Factors that fit the time-series and cross-section of stock returns. *Review of Financial Studies* 33:2274–2325. (Cited on page 21).

Litterman, R. 2003. *Modern investment management: An equilibrium approach.* New York: Wiley. (Cited on page 20).

Bibliography . . . VIII

Markowitz, H. M. 1952. Portfolio selection. *Journal of Finance* 7 (1): 77–91. (Cited on page 20).

———. 1959. *Portfolio selection: Efficient diversification of investments.* New York: Wiley. (Cited on page 20).

Merton, R. C. 1987. A simple model of capital market equilibrium with incomplete information [in English]. *Journal of Finance* 42 (3): 483–510. (Cited on page 50).

Meucci, A. 2009. Managing diversification. *Risk:* 74–79. (Cited on page 103).

Moreira, A., and T. Muir. 2017. Volatility-managed portfolios. *Journal of Finance* 72 (4): 1611–1644. (Cited on page 22).

———. 2019. Should long-term investors time volatility? *Journal of Financial Economics* 131 (3): 507–527. (Cited on page 22).

Partovi, M. H., and M. Caputo. 2004. Principal portfolios: Recasting the efficient frontier. *Economics Bulletin* 7 (3): 1–10. (Cited on page 103).

Pasini, G. 2017. Principal component analysis for stock portfolio management. *International Journal of Pure and Applied Mathematics* 115 (1): 153–167. (Cited on page 103).

Pástor, Ľ., and R. F. Stambaugh. 2000. Comparing asset pricing models: an investment perspective. *Journal of Financial Economics* 56 (3): 335–381. (Cited on page 20).

Bibliography . . . IX

Plyakha, Y., R. Uppal, and G. Vilkov. 2021. Equal or value weighting? Implications for asset-pricing tests. In *Risk management and modeling,* edited by R. Benkraiem, I. Kalaitzoglou, and C. Zopounidis. Springer. (Cited on page 67).

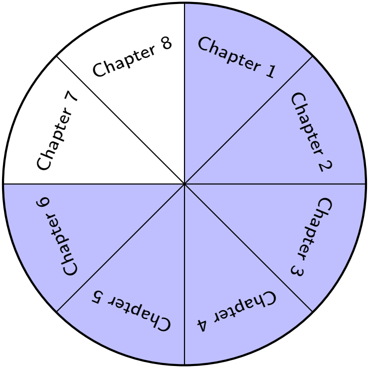
Raponi, V., R. Uppal, and P. Zaffaroni. 2023. Robust portfolio choice. Working Paper, SSRN eLibrary. (Cited on page 22).

Ross, S. A. 1976. The arbitrage theory of capital asset pricing. *Journal of Economic Theory* 13:341–360. (Cited on pages 21, 24).

———. 1977. Return, risk, and arbitrage. In *Risk and return in finance,* edited by I. Friend and J. Bicksler. Cambridge, MA: Ballinger. (Cited on pages 21, 24).

Rubinstein, M. 2006. *A history of the theory of investments: My annotated bibliography.* Hoboken, NJ: John Wiley & Sons. (Cited on page 22).

Sharpe, W. 1964. Capital asset prices: A theory of market equilibrium under conditions of risk. *Journal of Finance* 1919 (3): 425–442. (Cited on pages 13, 20).



End of Chapter 6