1 Exercises from the Book

1.1 Book by Gonzalez and Woods, 2.22

In Chapter 3 we will deal with operators whose function is to compute the sum of pixel values in a small sub image area, S_{xy}, as in Eq. (2-43), where the denominator mn equals the number of pixels in the area S_{xy}:

$$g(x,\, y) = \frac{1}{mn} \sum_{(r,c)\in S_{xy}} f(r,\, c) \tag{2-43}$$

Show that these are linear operators.

Note by the lecture: *The original solution is complicated, formaly incorrect and thus missleading. Specifically, it is not necessary to specialize the images to binary images. Furthermore this specialization is formally complicated and even treaded wrongly in the original solution.*

Instead we can choose the most general and thus the simpler defined arbitrary images $f_1(x, y)$ and $f_2(x, y)$. According to Eq. (2-43) we have

$$g(x) = H[f(x, y)] := \frac{1}{mn} \sum_{(r,c)\in S_{xy}} f(r, c).$$

Inserting into the Eq. (2.23) we get

$$\begin{aligned} g_s(x, y) := H\left\{a_1 f_1(x, y) + a_2 f_2(x, y)\right\} &= \frac{1}{mn} \sum_{(r,c)\in S_{xy}} a_1 f_1(r, c) + a_2 f_2(r, c) \\ &= \frac{a_1}{mn} \sum_{(r,c)\in S_{xy}} f_1(r, c) + \frac{a_2}{mn} \sum_{(r,c)\in S_{xy}} f_2(r, c) \\ &= a_1 H\{f_1(x, y)\} + a_2 H\{f_2(x, y)\}, \end{aligned}$$

what head to be proved.

Original solution by the authors of the book: *With reference to Eqs (2-22) and (2-23), let H denote the sum operator, let S_1 and S_2 denote two different small subimage areas of the same size, and let $S_1 + S_2$ denote the corresponding elementwise sum of the elements in S_1 and S_2, as explained in Section 2.6. The operator H computes the sum of pixel values in a neighborhood, and thus yields a scalar for a given neighborhood. Then, $H(aS_1 + bS_2)$ means: (1) multiply the pixels in each of the subimage areas by the constants shown, (2) add the pixel-by-pixel values from aS_1 and bS_2 (which produces a single subimage area), and (3) compute the sum of the values of all the pixels in that single subimage area, which produces a scalar. Let ap_1 and bp_2 denote two arbitrary (but corresponding) pixels from $aS_1 + bS_2$. Then we can write*

$$\begin{aligned} H(aS_1 + bS_2) &= \sum_{p_1\in S_1 \text{ and } p_2\in S_2} ap_1 + bp_2 = \sum_{p_1\in S_1} ap_1 + \sum_{p_2\in S_2} bp_2 = a \sum_{p_1\in S_1} p_1 + b \sum_{p_2\in S_2} p_2 \\ &= aH(S_1) + bH(S_2) \end{aligned}$$

1.2 Book by Gonzalez and Woods, 2.26

Averaging of noisy images for noise reduction. Prove the validity of

(a)

$$\mathbb{E}\left\{\bar{g}(x, y)\right\} = f(x, y) \tag{2-27}$$

(b)

$$\sigma^2_{\bar{g}(x,y} = \frac{1}{K} \sigma^2_{\eta(x,y)} \tag{2-28}$$

For part (b) you will need the following facts from probability: (1) the variance of a constant times a random variable is equal to the constant squared times the variance of the random variable. (2) The variance of the sum of uncorrelated random variables is equal to the sum of the variances of the individual random variables.

(a) From Eq. (2-26), at any point (x, y),

$$\bar{g}(x,y) = \frac{1}{K}\sum_{i=1}^{K} g_i(x,y) = \frac{1}{K}\sum_{i=1}^{K} f_i(x,y) + \frac{1}{K}\sum_{i=1}^{K} \eta_i(x,y)$$

Because this equation is applicable at any coordinates (x, y), we can simplify the notation by dropping the coordinates. Then

$$\mathbb{E}\{\bar{g}\} = \frac{1}{K}\sum_{i=1}^{K} \mathbb{E}\{f_i\} + \frac{1}{K}\sum_{i=1}^{K} \mathbb{E}\{\eta_i\}$$

But all the f_i are the same, so $\mathbb{E}\{f_i\} = f$. In other words, the noisy images are formed by adding noise to the same image. The noise changes from image to image, but f remains the same. Also, it is given that the noise has zero mean, so $\mathbb{E}\{\eta_i\} = 0$. Thus, it follow that $\mathbb{E}\{\bar{g}\} = f$, or $\mathbb{E}\{\bar{g}(x,y)\} = f(x,y)$, which proves the validity of Eq. (2-27).

(b) To prove the validity of Eq. (2-28), consider the preceding equation again:

$$\bar{g} = \frac{1}{K}\sum_{i=1}^{K} g_i = \frac{1}{K}\sum_{i=1}^{K} f_i + \frac{1}{K}\sum_{i=1}^{K} \eta_i$$

As mentioned in the problem statement, the variance ot the sum of uncorrelated random variables is equal to the sum of the variances of the individual variables. Because all the f_i are the same (remember, the noisy images are formed by adding noise to a noiseless image f), the variance of their sum is zero. So, we are left with

$$\mathrm{Var}[\bar{g}] = \mathrm{Var}\left[\frac{1}{K}\sum_{i=1}^{K} \eta_i\right]$$

We are given in the problem statement that the variance of a sum of uncorrelated random variables is equal to the sum of the individual variables. We are given also that the variance of a constant times a random variable is the constant squared times the variance of the random variable. Using these facts results in the following expression.

$$\mathrm{Var}[\bar{g}] = \frac{1}{K^2}\mathrm{Var}\left[\sum_{i=1}^{K} \eta_i\right] = \frac{1}{K^2}\left\{\mathrm{Var}[\eta_1] + \mathrm{Var}[\eta_2] + \ldots + \mathrm{Var}[\eta_\mathrm{K}]\right\}$$

Remember that this expression is for any pair of coordinates (x, y). Because the characteristics of the noise is the same at any location, all the variances on the right side of the preceding equation are the same: $\mathrm{Var}[\eta]$. Using this fact yields,

$$\mathrm{Var}[\bar{g}] = \frac{1}{K}\mathrm{Var}[\eta] = \frac{1}{K}\sigma^2_{\eta(x,y)} = \sigma^2_{\bar{g}(x,y)}$$

which proves the validity of Eq. (2-28).

Note by the lecturer: *The principle of averaging is the more effective to reduce noise, the less the random noise samples η_i are correlated. It is not necessary but simplifying the procedure of denoising, when the mean value also called expected value of the noise samples is zero, i.e., $E\{\eta_i\} = 0$. Given uncorrelated and zero mean noise samples, we can show that summing up noise* *samples yields a new random variable, whose variance is the sum of the variances of the noise samples:*

$$Var\left\{\sum_{i=1}^{K}\eta_i\right\} = \sum_{i=1}^{K} Var\{\eta_i\}$$

Using the defintionion for variance $Var\{\eta_i\} := E\left\{(\eta_i - E\{\eta_i\})^2\right\}$ *and making use of the assumption* $E\{\eta_i\} = 0$ *we get*

$$\begin{aligned}
Var\left\{\sum_{i=1}^{K}\eta_i\right\} &= E\left\{\left(\sum_{i=1}^{K}\eta_i - E\left\{\sum_{i=1}^{K}\eta_i\right\}\right)^2\right\} \\
&= E\left\{\left(\sum_{i=1}^{K}\eta_i - \sum_{i=1}^{K}\underbrace{E\{\eta_i\}}_{0}\right)^2\right\} \\
&= E\left\{\left(\sum_{i=1}^{K}\eta_i\right)^2\right\} \\
&= E\left\{\left(\sum_{i=1}^{K}\eta_i\right)\left(\sum_{j=1}^{K}\eta_j\right)\right\} \\
&= E\left\{\sum_{i=1}^{K}\sum_{j=1}^{K}\eta_i\eta_j\right\} \\
&= \sum_{i=1}^{K}\sum_{j=1}^{K}E\{\eta_i\eta_j\} \\
&= \sum_{i=1}^{K}E\{\eta_i^2\} \\
&= \sum_{i=1}^{K}\sigma_{\eta_i}^2 \\
&= \sum_{i=1}^{K}Var\{\eta_i\} \\
&= K\sigma_\eta^2.
\end{aligned}$$

The step from line 6 to line 7 makes use of the uncorrelated noise sample assumption, $E\{\eta_i\eta_j\} = 0$ *for* $i \neq j$. *The last step holds if* $\sigma_{\eta_i}^2 = \sigma_\eta^2$ *is independent of* i, *what can be assumed to be true in our application.*

1.3 Book by Gonzalez and Woods, 2.32

Give expressions (in terms of sets A, B, and C) for the sets shown shaded in the following figures. The shaded areas in each figure constitute one set, so give only one expression for each of the four figures.

Write the expressions using the set operators $\cup$, $\cap$ and ^{c} for union, intersection and the compliment, respectively.

(a) $[(A\cap B)\cup(A\cap C)] - A\cap B\cap C$. *A shorter form is:* $(A\cap B^c)\cup(B\cap A^c)$, *forming an EXOR (exclusive or) operation.*

(b) $[(B\cap C) - A\cap B\cap C]\cup[A - A\cap(B\cup C))]$. *A shorter form is:* $(B\cap C\cap A^c)\cup(A\cap B^c)\cup C^c$

(c) $[B-(A\cap B)]\cup C$ *A shorter form is:* $B\cap A^c\cap C$

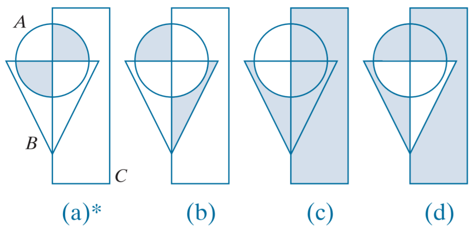

Figure belonging to exercise 2.34

(d) $[C - (B \cap C)] \cup [A - (A \cap B)] \cup [B - B \cap (A \cup C)]$. *A shorter form is:* $[B \cap (A \cup C)]^c \cap (A \cup B \cup C)$

1.4 Book by Gonzalez and Woods, 2.42

Show that 2-D transforms with separable, symmetric kernels can be computed by

(a) computing 1-D transforms along the individual rows (columns) of the input, followed by

(b) computing 1-D transforms along the columns (rows) of the result from step (a)

From Eq. (2-59) and the definition of separable kernels

$$T(u,v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) r(x,y,u,v) = \sum_{x=0}^{M-1} r_1(x,u) \sum_{y=0}^{N-1} f(x,y) r_2(y,v) = \sum_{x=0}^{M-1} T(x,v) r_1(x,u)$$

where

$$T(x,v) = \sum_{y=0}^{N-1} f(x,y) r_2(y,v)$$

For a fixed value of x, this equation is recognized as the 1-D transform along one row of $f(x,y)$. By letting x vary from 0 to $M-1$ we compute the entire array $T(x,v)$. Then, by substituting this array into the last line of the previous equation we have the 1-D transform along the columns $T(x,v)$. In other words, when a kernel is separable, we can compute the 1-D transform along the rows of the image. Then we compute the 1-D transform along the columns of this intermediate result to obtain the final 2-D transform, $T(u,v)$. We obtain the same result by computing the 1-D transform along the columns of $f(x,y)$ followed by the 1-D transform along the rows of the intermediate result.

Note by the lecturer: *The general transform with the kernel $r(x,y,u,v)$ requires the computation of $M \cdot N$ products. Instead, separability of the kernel allows to conduct a column-wise transform followed by a raw-wise transform, requiring the computation of only $M+N$ products! This is a remarkable saving which lets practitioners prefer separable kernels over non-separable kernels whenever possible.*