Some of the theoretical contents are taken from the instructor manual of the book „Digital Image Processing (4th Edition)“ by Rafael C. Gonzalez and Richard E. Woods. Therefore, the confidentiality class of this document is „internally extended“. The data can be used by OST members as of September 1, 2020, but must not be passed on to third parties.

Image Processing and Computer Vision 1

Chapter 4 – Two Dimensional Fourier Transform – week 7

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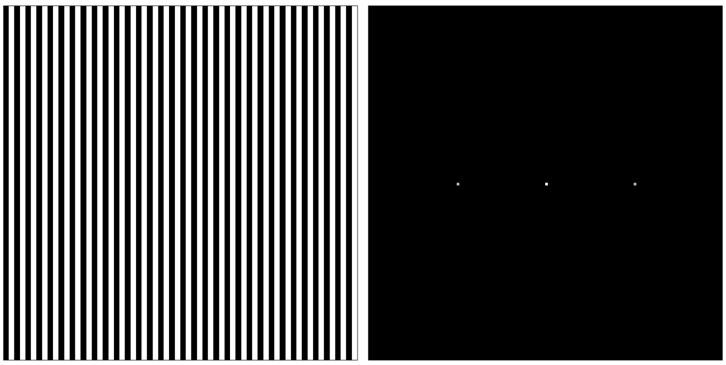
1 Book

1.1 Book by Gonzalez and Woods, 4.25

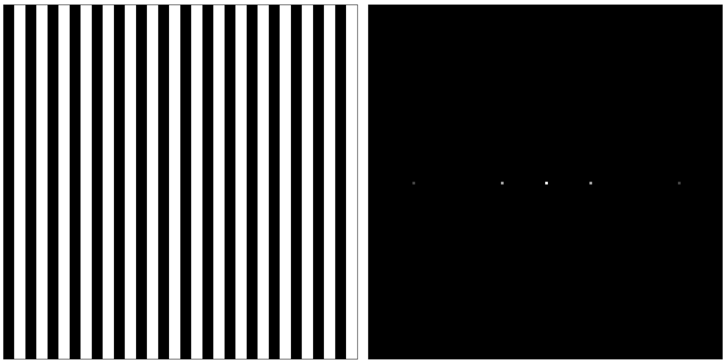
The image on the left in Fig. 4.25 consists of alternating stripes of black/white, each stripe being two pixels wide. The image on the right is the Fourier spectrum of the image on the left, showing the DC term and the frequency terms corresponding to the stripes. (Remember, the spectrum is symmetric so all components, other than the DC term, appear in two symmetric locations.)

(a) Suppose that the stripes of an image of the same size are four pixels wide. Sketch what the spectrum of the image would look like, including only the DC term and the two highest-value frequency terms, which correspond to the two spikes in the spectrum of Fig. 4.25.



Exercise Figure 4.25

(a) Because all rows of the image are identical, we can focus attention on one row, which is a 1-D square wave with Period, P, of four pixels. Therefore, the frequency of this signal is $f = 1/4 = 0.25$ cycles/pixel. If the stripes are now four pixels wide, then the period is eight pixels, and the frequency of the signal is $f = 1/8 = 0.125$ cycles/pixel, which is one-half the frequency of the original signal. The center peak in the spectrum shown in the problem statement is the DC term, and the other tow dominant peaks appear on the horizontal axis of the spectrum, exactly half-way between the center and ends of the horizontal axis of the spectrum. The corresponding peaks in the new spectrum have half the frequency, so they will appear midway between original peaks and the center of the spectrum. That is, one-quarter of the axis length on either side of center. (The spectrum contains other harmonic frequency components that are of lower amplitude and are not shown).



Solution of exercise 4.25 (a)

1.2 Book by Gonzalez and Woods, 4.28

With reference to the discussion on linearity in Section 2.5, demonstrate that

(a) The 2-D continuous Fourier transform is a linear operator.

(b) The 2-D DFT is a linear operator also.

(a) From Eq. (4-59)

$$F(\mu,\nu) = \mathscr{F}\left[f(t,z)\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,z)e^{-j2\pi(\mu t+\nu z)}dtdz$$

Substituting Eq. (2-23) into the definition of the Fourier transform yields

$$\begin{aligned}\mathscr{F}\left[af_1(t,z)+bf_2(t,z)\right] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[af_1(t,z)+bf_2(t,z)\right]e^{-j2\pi(\mu t+\nu z)}dtdz\\ &= a\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_1(t,z)e^{-2j\pi(\mu t+\nu z)}dtdz + b\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_2(t,z)e^{-2j\pi(\mu t+\nu z)}dtdz\\ &= a\mathscr{F}\left[f_1(t,z)\right]+b\mathscr{F}\left[f_2(t,z)\right]\end{aligned}$$

where the second step follows from the distributive property of the integral. The linearity of the inverse transform is proved in exactly the same way.

(b) The linearity of the discrete case is demonstrated in the same way:

$$\begin{aligned}\mathscr{F}\left[af_1(t,z)+bf_2(t,z)\right] &= \sum_{x=0}^{M-1}\sum_{y=0}^{N-1}\left[af_1(x,y)+bf_2(x,y)\right]e^{-j2\pi(ux/M+vy/N)}\\ &= a\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f_1(x,y)e^{-j2\pi(ux/M+vy/N)} + b\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f_2(x,y)e^{-j2\pi(ux/M+vy/N)}\\ &= a\mathscr{F}\left[f_1(x,y)\right]+b\mathscr{F}\left[f_2(x,y)\right]\end{aligned}$$

The linearity of the inverse transform is proved in exactly the same way.

1.3 Book by Gonzalez and Woods, 4.29

With reference to Eqs. (4-71) and (4-72), demonstrate the validity of the following translation (shifting) properties of 2-D, discrete Fourier transform pairs from Table 4.4. (Hint: Study the solutions to Problem 4.17.) (*Hint:* Study the solutions to Problem 4.11)

(a) $f(x,y)\cdot e^{j2\pi(u_0x/M+v_0y/N)} \Leftrightarrow F(u-u_0,v-v_0)$

(a) We solve the problem by direct substitution into the forward Fouriert transform, Eq. (4-59):

$$\begin{aligned}\mathscr{F}\left[f(x,y)e^{j2\pi(\mu_0 t+\nu_0 z)}\right] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{j2\pi(\mu_0 t+\nu_0 z)}e^{-j2\pi(\mu t+\nu z)}dtdz \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi([\mu-\mu_0]t+[\nu-\nu_0]z)}dtdz \\ &= F(\mu-\mu_0,\nu-\nu_0)\end{aligned}$$

Because we used direct substitution into Eq. (4-59 and this equation and Eq. (4-60) are a Fourier transform pair, it must follow that the left side of the double arrow is the inverse transform of the right, that is $\mathscr{F}^{-1}\left[F(\mu-\mu_0,\nu-\nu_0)\right] = f(x,y)e^{j2\pi(\mu_0 t+\nu_0 z)}$

Note by the lecturer: The appearance of discrete variables (x,y) *and* (u,v) *directs us to consider the DFT instead of to the FT as applied above. Following the straight forward approach analogously to the above, we get*

$$\begin{aligned}DFT\left\{f(x,y)\cdot e^{j2\pi(u_0x/M+v_0y/M)}\right\} &= \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)\cdot e^{j2\pi(u_0x/M+v_0y/M)}\cdot e^{-j2\pi(\frac{u}{M}x+\frac{v}{N}y)} \\ &= \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)\cdot e^{-j2\pi[(u-u_0)x/M+(v-v_0)y/M]} \\ &= F(u-u_0,v-v_0)\end{aligned}$$

with $F(u,v)$ *being the DFT of* $f(x,y)$.

1.4 Book by Gonzalez and Woods, 4.39

The following problems are related to the properties in Table 4.1.

(a) Demonstrate the validity of property 2.

(b) Demonstrate the validity of property 4.

(c) Demonstrate the validity of property 5.

(d) Demonstrate the validity of property 7.

(a) Property 2: If $f(x,y)$ *is imaginary,* $f(x,y) \Leftrightarrow F^*(-u,-v) = -F(u,v)$. *Because f(x,y) is imaginary, we can express it as* $jg(x,y)$, *where* $g(x,y)$ *is a real function. Then the proof is as follows:*

$$\begin{aligned}F^*(-u,-v) &= \left[\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} jg(x,y)e^{j2\pi(ux/M+vy/N)}\right]^* \\ &= \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} -jg(x,y)e^{-j2\pi(ux/M+vy/N)} \\ &= -\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}\left[jg(x,y)\right]e^{-j2\pi(ux/M+vy/N)} \\ &= -\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux/M+vy/N)} \\ &= -F(u,v)\end{aligned}$$

(b) Property 4: If $f(x,y)$ *is imaginary, then* $R(u,v)$ *is odd and* $I(u,v)$ *is even.*
Proof: F is complex, so it can be expressed as

$$\begin{aligned}F(u,v) &= \Re\left[F(u,v)\right] + j\Im\left[F(u,v)\right] \\ &= R(u,v) + jI(u,v).\end{aligned}$$

Then, $-F(u,v) = -R(u,v) - jI(u,v)$, *and* $F^*(-u,-v) = R(-u,-v) - jI(-u,-v)$. *But, because* $f(x,y)$ *is imaginary,* $F^*(-u,-v) = -F(u,v)$ *(see Property 2). It then follows from the previous two equations that* $R(u,v) = -R(-u,-v)$ *(i.e., R is odd) and* $I(-u,-v)$ *(I is even).*

(c) *Property 5:* $f(-x,-y) \Leftrightarrow F^*(u,v)$*. That is, if* $f(x,y)$ *is real and its transform is* $F(u,v)$*, then the transform of* $f(-x,-y)$ *is* $F^*(u,v)$*. And conversely.*
Proof: From Example 4.12,

$$\mathscr{F}\left[f(-x,-y)\right] = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m,n)e^{j2\pi(um/M+vn/N)}$$

To see what happens when $f(x,y)$ *is real, we write the right side ode this equation as*

$$\begin{aligned}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m,n)e^{j2\pi(um/M+vn/N)} &= \left[\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f^*(m,n)e^{-j2\pi(um/M+vn/N)}\right]^* \\ &= \left[\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m,n)e^{-j2\pi(um/M+vn/N)}\right]^* \\ &= F^*(u,v)\end{aligned}$$

where the second step follows from the fact that $f(x,y)$ *is real. Thus, we have shown that*

$$\mathscr{F}\left[f(-x,-y)\right] = F^*(u,v)$$

(d) *Property 7: When* $f(x,y)$ *is complex,* $f^*(x,y) \Leftrightarrow F^*(-u,-v)$*.*
Proof:

$$\begin{aligned}\mathscr{F}\left[f^*(x,y)\right] &= \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f^*(x,y)e^{-j2\pi(ux/M+vy/N)} \\ &= \left[\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} = f(x,y)e^{j2\pi(ux/M+vy/N)}\right]^* \\ &= F^*(-u,-v).\end{aligned}$$