Some of the theoretical contents are taken from the instructor manual of the book „Digital Image Processing (4th Edition)“ by Rafael C. Gonzalez and Richard E. Woods. Therefore, the confidentiality class of this document is „internally extended“. The data can be used by OST members as of September 1, 2020, but must not be passed on to third parties.

Image Processing and Computer Vision 1

Chapter 4 – One Dimensional Fourier Transform – week 6

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1 Book

1.1 Book by Gonzalez and Woods, 4.9 (a)

Show that the following expression is true:

$$\mathscr{F}\left\{\sin(2\pi\mu_0 t)\right\} = \frac{1}{2j}\left[\delta(\mu-\mu_0) - \delta(\mu+\mu_0)\right]$$

Hint. Make use of the Euler formula $e^{jx} = \cos(x)+j\sin(x)$ and the Fourier transfrom pair $\mathscr{F}\left\{e^{j2\pi\mu_0 t}\right\} = \delta(\mu-\mu_0)$.

Using the Euler formula for positive and negative imaginary exponent it follows that

$$\sin(x) = \frac{e^{jx} - e^{-jx}}{2j}.$$

Substituting x by $2\pi\mu_0 t$ yields

$$\sin(2\pi\mu_0 t) = \frac{e^{j2\pi\mu_0 t} - e^{-j2\pi\mu_0 t}}{2j}.$$

Transforming the two sums separately is possible because the Fourier transform is linear, thus we get

$$\begin{aligned}\mathscr{F}\left\{\sin(2\pi\mu_0 t)\right\} &= \frac{\mathscr{F}\left\{e^{j2\pi\mu_0 t}\right\} - \mathscr{F}\left\{e^{-j2\pi\mu_0 t}\right\}}{2j} \\ &= \frac{1}{2j}\left[\delta(\mu-\mu_0) - \delta(\mu+\mu_0)\right].\end{aligned}$$

1.2 Book by Gonzalez and Woods, 4.10

Consider the function $f(t) = \sin(2\pi nt)$, where n is an integer. Its Fourier transform, $F(\mu)$, is purely imaginary (see Problem 4.12). Because the transform, $\tilde{F}(\mu)$, of sampled data consists of periodic copies of $F(\mu)$, it follows that $\tilde{F}(\mu)$ will also be purely imaginary. Draw a diagram similar to Fig. 4.6, and answer the following questions based on your diagram (assume that sampling starts at $t = 0$).

(a) What is the period of $f(t)$?

(b) What is the frequency of $f(t)$?

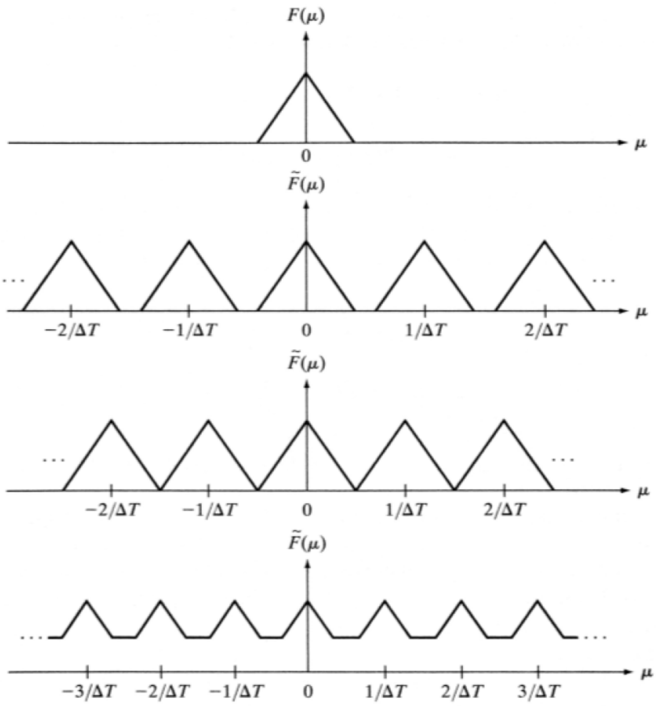
(c) What would the sampled function and its Fourier transform look like in general if $f(t)$ is sampled at a rate higher than the Nyquist rate?

(d) What would the sampled function look like in general if $f(t)$ is sampled at a rate lower than the Nyquist rate?

(e) What would the sampled function look like if $f(t)$ is sampled at the Nyquist rate with samples taken at $t = 0, \Delta T, 2\Delta T, \ldots$?

a
b
c
d

FIGURE 4.6
(a) Fourier transform of a band-limited function.
(b) - (d) Transforms of the corresponding sampled function under the conditions of over-sampling, critically-sampling, and under-sampling, respectively.



Book figure 4.6 *Fourier Transform of band-limited function*

- *The period is such that $2\pi nt = 2\pi$, or $t = 1/n$.*
- *The frequency is 1 divided by the period, or n.*
- *The continuous Fourier transform of the given sine wave looks as in Fig. P4.13(a) (see Problem 4.12(b)), and the transform of the sampled data (showing a few periods) has the general form illustrated in Fig P4.13(b) (The dashed box is an ideal filter that would allow reconstruction if the sine function were sampled, with the sampling theorem being satisfied). The sampled function would look as in Problem 4.9(a) if the sampling theorem were satisfied.*
- *If the function were undersampled it would look like Fig. 4.9 or 4.11 in the book.*
- *The Nyquist sampling rate is exactly twice the highest frequency, or $2n$. That is, $(1/\Delta T) = 2n$, or $\Delta T = 1/2n$. Taking samples at $t = \pm\Delta T, \pm 2\Delta T, \ldots$ would yield the sampled function $\sin(2\pi n \Delta T)$ whose values are all 0s because $\Delta T = 1/2n$ and n is an integer. In terms of Fig. P4.13(b), wee see that when $\Delta T = 1/2n$ all the positive and negative impulses would coincide, thus canceling each other and giving a result of 0 for the sampled data.*

1.3 Book by Gonzalez and Woods, 4.11

Prove the validity of the convolution theorem of one continuous variable, as given in Eqs. (4-25) and (4-26).

Starting from Eq. (4-24),

$$f(t) \star g(t) = \int_{-\infty}^{\infty} f(\tau) g(t - \tau) d\tau$$

(Refer to the solution of Problem 4.5(a) regarding the notation we are using to denote convolution). The Fourier transform of this expression is

$$
\begin{aligned}
\mathscr{F}\left[f(t) \star g(t)\right] &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(\tau) g(t-\tau) d\tau\right] e^{-j2\pi\mu t} dt \\
&= \int_{-\infty}^{\infty} f(\tau) \left[\int_{-\infty}^{\infty} g(t-\tau) e^{-j2\pi\mu t} dt\right] d\tau
\end{aligned}
$$

The term inside the inner bracket is the Fourier transform of $g(t-\tau)$. But, we know from the translation property (Table 4.4) that

$$\mathscr{F}\left[g(t-\tau)\right] = G(\mu) e^{-j2\pi\mu\tau}$$

so,

$$
\begin{aligned}
\mathscr{F}\left[f(t) \star g(t)\right] &= \int_{-\infty}^{\infty} f(\tau) \left[G(\mu) e^{-j2\pi\mu\tau}\right] d\tau \\
&= G(\mu) \int_{-\infty}^{\infty} f(\tau) e^{-2j\pi\mu\tau} d\tau \\
&= G(\mu) F(\mu)
\end{aligned}
$$

This proves that multiplication in the frequency domain is equal to convolution in the spatial domain. The proof that multiplication in the spatial domain is equal to convolution in the frequency domain is done in a similar way.

1.4 Book by Gonzalez and Woods, 4.14

Show that $\tilde{F}(\mu)$ in Eq. (4-40) is infinitely periodic in both directions, with period $1/\Delta T$.

Eq. 4-40

$$\tilde{F}(\mu) = \sum_{n=-\infty}^{\infty} f_n e^{-j2\pi\mu n\Delta T}$$

To prove infinite periodicity in both directions with period $1/\Delta T$ we have to show that $\tilde{F}(\mu + k[1/\Delta T]) = \tilde{F}(\mu)$, for $k = 0, \pm 1, \pm 2, \ldots$. From Eq. (4-40),

$$
\begin{aligned}
\tilde{F}(\mu + k/\Delta T) &= \sum_{n=-\infty}^{\infty} f_n e^{-j2\pi(\mu + k/\Delta T)n\Delta T} \\
&= \sum_{n=-\infty}^{\infty} f_n e^{-j2\pi\mu n\Delta T} e^{-j2\pi kn} \\
&= \sum_{n=-\infty}^{\infty} f_n e^{-j2\pi\mu n\Delta T} \\
&= \tilde{F}(\mu)
\end{aligned}
$$

1.5 Book by Gonzalez and Woods, 4.15

Do the following:

(a) Show that Eqs (4-42) and (4-43) constitute a Fourier transform pair: $f_n \Leftrightarrow F_m$.

(b) Show that Eqs. (4-44) and (4-45) also are a Fourier transform pair: $f(x) \Leftrightarrow F(u)$.

You will need the following orthogonality property of exponentials in both parts of this problem:

$$\sum_{x=0}^{M-1} e^{j2\pi rx/M} \cdot e^{-j2\pi ux/M} = \begin{cases} M, & \text{if } r = u \\ 0, & \text{otherwise} \end{cases}$$

(a) *We solve this problem by direct substitution using orthogonality. First we have to show that F_m the DFT of f_n. Substituting Eq. (4-43) into (4-42) yields*

$$\begin{aligned} F_m &= \sum_{n=0}^{M-1}\left[\frac{1}{M}\sum_{r=0}^{M-1} F_r e^{j2\pi rn/M}\right] e^{-j2\pi mn/M} \\ &= \frac{1}{M}\sum_{r=0}^{M-1} F_r \left[\sum_{n=0}^{M-1} e^{j2\pi rn/M} e^{-2j\pi mn/M}\right] \\ &= \frac{1}{M} F_m M \\ &= F_m \end{aligned}$$

where, because of orthogonality, the third step is 0 unless $r = m$. Next, we have to show that f_n is the inverse DFT of F_m. Substituting Eq. (4-42) into (4-43) and using the same basic procedure yields as above,

$$\begin{aligned} f_n &= \frac{1}{M}\sum_{m=0}^{M-1}\left[\sum_{r=0}^{M-1} f_r e^{j2\pi rm/M}\right] e^{-j2\pi mn/M} \\ &= \frac{1}{M}\sum_{r=0}^{M-1} f_r \left[\sum_{m=0}^{M-1} e^{j2\pi rm/M} e^{-2j\pi mn/M}\right] \\ &= \frac{1}{M} f_n M \\ &= f_n \end{aligned}$$

where, because of orthogonality, the third step is 0 unless $r = n$. By showing that F_m is the DFT of f_n and that f_n is the IDFT of F_m we have established that Eqs. (4-42) and (4-43) constitute a Fourier transform pair.

(b) *We solve this problem as above, by direct substitution and using orthogonality. First we have to show that $F(u)$ the DFT of $f(x)$. Substituting Eq. (4-45) into (4-44) yields*

$$\begin{aligned} F(u) &= \sum_{x=0}^{M-1}\left[\frac{1}{M}\sum_{r=0}^{M-1} F(r) e^{j2\pi rx/M}\right] e^{-j2\pi ux/M} \\ &= \frac{1}{M}\sum_{r=0}^{M-1} F(r) \left[\sum_{n=0}^{M-1} e^{j2\pi rx/M} e^{-2j\pi ux/M}\right] \\ &= \frac{1}{M} F(u) M \\ &= F(u) \end{aligned}$$

where, because of orthogonality, the third step is 0 unless $r = 0$. The prove that $f(x)$ is the inverse DFT of $F(u)$ works similar to the above proofs.

Note by the lecturer: Equations (4-42) and (4-44) describe the DFT with slightly different notations and variable names. The same holds for (4-43)and (4-45). Hence $F(u)$ and $f(x)$ being Fourier transform pairs does not need to be proved once again.

1.6 Book by Gonzalez and Woods, 4.18

Show that the 1-D convolution theorem given in Eqs. (4-25) and (4-26) also holds for discrete variables, but with the right side of Eq. (4-26) multiplied by $1/M$. That is, show that

(a) $(f \star h)(x) \Leftrightarrow (F \cdot H)(u)$, and

(b) $(f \cdot h)(x) \Leftrightarrow \frac{1}{M}(F \star H)(u)$

(a) We have to show first that the Fourier transform of the convolution $f(x) \star h(x)$ is the product $F(u)H(u)$, and vice versa. Using the definition of the 1-D convolution theorem in Eq. (4-24), the DFT in Eq.(4-44), and 1-D discrete convolution in Eq. (4-48), we write

$$\begin{aligned}
\mathscr{F}\left[(f \star h)(x)\right] = \mathscr{F}\left[f(x) \star h(x)\right] &= \sum_{x=0}^{M-1}\left[\sum_{m=0}^{M-1} f(m)h(x-m)\right]e^{-j2\pi ux/M} \\
&= \sum_{m=0}^{M-1} f(m)\left[\sum_{x=0}^{M-1} h(x-m)e^{-j2\pi ux/M}\right] \\
&= \sum_{m=0}^{M-1} f(m)H(u)e^{-j2\pi ux/M} \\
&= H(u)\sum_{m=0}^{M-1} f(m)e^{-j2\pi ux/M} \\
&= H(u)F(u) = (H \cdot F)(u)
\end{aligned}$$

where the third step follow from the translation property of the Fourier transform (see Problem 4.20). Because we showed the preceding properties by substituting into the DFT, and the DFT and IDFT are a transform pair, it must be true that the reverse, i.e., that $f(x) \star h(x)$ is the IDFT of $F(u)H(u)$, is true also (see the discussion on this concept following Eqs. (4-44) and (4-45) in the book).

(b) For the second half of the convolution theorem, we have to show that the DFT of the product $f(x)h(x)$ is the convolution $F(u) \star H(u)$, and vice versa. We only show one of these two properties, and the other follow from the fact that the DFT and IDFT form a Fourier transform pair, as mentioned at the end part (a). In this case, it is much easier to show that the IDFT of $(1/M)F(u) \star H(u)$ is the product $f(x)h(x)$.

$$\begin{aligned}
\mathscr{F}^{-1}\left[\frac{1}{M}F(u) \star H(u)\right] &= \frac{1}{M}\sum_{u=0}^{M-1}\left[\frac{1}{M}\sum_{m=0}^{M-1} F(m)H(u-m)\right]e^{j2\pi ux/M} \\
&= \frac{1}{M}\sum_{m=0}^{M-1} F(m)\left[\frac{1}{M}\sum_{u=0}^{M-1} H(u-m)e^{j2\pi ux/M}\right] \\
&= \frac{1}{M}\sum_{m=0}^{M-1} F(m)h(x)e^{j2\pi xm/M} \\
&= \left[\frac{1}{M}\sum_{m=0}^{M-1} F(m)e^{j2\pi xm/M}\right]h(x) \\
&= f(x)h(x)
\end{aligned}$$

where the third step follows from the translation property of the DFT (see Problem 4.20). Note that we again used the equivalent notation $\mathscr{F}^{-1}\left[(1/M)F(u) \star H(u)\right]$ for $\mathscr{F}^{-1}\left[(1/M)(F \star H)(u)\right]$ and $f(x)h(x)$ for $(f \cdot h)(x)$.