Image Processing and Computer Vision 1

Chapter 5 – Image restauration – week 12

Martin Weisenhorn

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1 Book

1.1 Book by Gonzalez and Woods, 5.34

Using the transfer function in Problem 5.33, give the expression $W(u,v)$ for a Wiener filter transfer function, assuming that the ratio of power spectra of the noise and undegraded signal is a constant.

Transfer function of Problem 5.33:

Spatial domain	$h(x,y) = \frac{x^2+y^2-2\sigma^2}{\sigma^4} \cdot e^{-\frac{x^2+y^2}{2\sigma^2}}$
Frequency domain	$H(u,v) = -8\pi^4\sigma^2 \cdot (u^2+v^2) \cdot e^{-2\pi^2\sigma^2(u^2+v^2)}$

This is a simple plug in problem. Its purpose is to gain familiarity with the various terms of the Wiener filter. From Eq. (5-81),

$$\begin{aligned} W(u,v) &= \left[\frac{1}{H(u,v)}\frac{|H(u,v)|^2}{|H(u,v)|^2+K}\right] \\ &= \left[\frac{1}{H(u,v)}\frac{H(u,v)H(u,v)}{|H(u,v)|^2+K}\right] \\ &= \frac{H(u,v)}{|H(u,v)|^2+K} \end{aligned}$$

because $H(u,v)$ is real. Then, using the results from Problem 5.33,

$$\begin{aligned} |H(u,v)|^2 &= H^*(u,v)H(u,v) \\ &= H^2(u,v) \\ &= 64\pi^8\sigma^4(u^2+v^2)^2 e^{-4\pi^2\sigma^2(u^2+v^2)} \end{aligned}$$

and

$$W(u,v) = \frac{8\pi^4\sigma^2(u^2+v^2)e^{-2\pi^2\sigma^2(u^2+v^2)}}{64\pi^8\sigma^4(u^2+v^2)^2 e^{-4\pi^2\sigma^2(u^2+v^2)}+K}$$

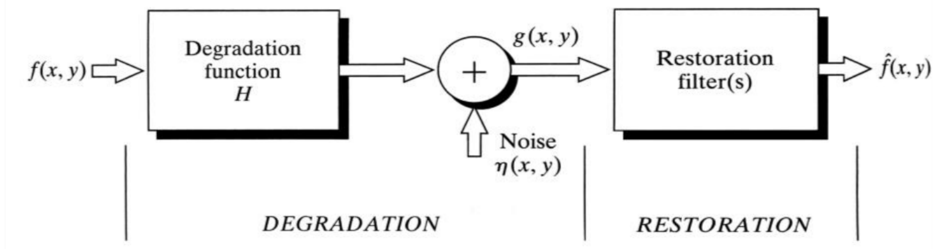
1.2 Book by Gonzalez and Woods, 5.38

Assume that the model in Fig. 5.1 is linear and position invariant and that the noise and image are uncorrelated. Show that the power spectrum of the output is

$$|G(u,v)|^2 = |H(u,v)|^2 \cdot |F(u,v)|^2 + |N(u,v)|^2$$

Refer to Eq. (5-65) and Eq. (4-89).

FIGURE 5.1 A model of the image degradation/ restoration process



Book figure 5.1 *A model of the image degradation / restoration process*

$$G(u,v) = H(u,v)F(u,v) + N(u,v) \tag{5-65}$$

$$P(u,v) = |F(u,v)|^2 = R^2(u,v) + I^2(u,v) \tag{4-89}$$

$P(u,v)$ is the power spectrum, $R(u,v)$ and $I(u,v)$ are the real an imaginary parts of the Digital Fourier Transform $F(u,v)$.

Because the system is assumed linear and position invariat, it follow that Eq. (5-63) holds. Furthermore, we can use superposition (additivity, as discussed in Section 2.6) and obtain the response of the system first to $F(u,v)$ and then to $N(u,v)$ because we kknow that the image and noise are uncorrelated. The sum of the two individual responses then gives the complete response. First, using only $F(u,v)$,

$$G_1(u,v) = H(u,v)F(u,v)$$

and

$$|G_1(u,v)|^2 = |H(u,v)|^2|F(u,v)|^2$$

Then, using only $N(u,v)$,

$$G_2(u,v) = N(u,v)$$

and

$$|G_2(u,v)|^2 = |N(u,v)|^2$$

so that

$$\begin{aligned} |G(u,v)|^2 &= |G_1(u,v)|^2 + |G_2(u,v)|^2 \\ &= |H(u,v)|^2|F(u,v)|^2 + |N(u,v)|^2 \end{aligned}$$

Note by the lecturer: To show independency formally, we need to use a more precise notion of power spectrum. The proper definition is called the cross power spectrum of $A(u,v)$ and $B(u,v)$

$$S_{a,b}(u,v) := \mathrm{E}\left\{A(u,v)B^*(u,v)\right\},$$

and the auto power spectrum of $A(u,v)$

$$S_a(u,v) := \mathrm{E}\left\{A(u,v)A^*(u,v)\right\}.$$

The statistical dependence of the noise spectrum $N(u,v)$ and the image $F(u,v)$ is expressed by their cross spectrum

$$S_{f,n}(u,v) = \mathrm{E}\left\{F(u,v)N^*(u,v)\right\}.$$

This will be zero, i.e $S_{f,n}(u,v) = 0$ if $N(u,v)$ is the spectrum of a zero mean random noise image $n(x,y)$ that is statistically independent from the spectrum $F(u,v)$ of the image. Here the $F(u,v)$ is not random but a constant. Hence, as $N(u,v)$ is assumed to be zero mean, the expectation $\{F(u,v)N^*(u,v)\}$ is zero, obviously. Equipped with these definitions and knowing that independence of noise and image lets their cross power spectrum become zero, allows us to prove the stated equation

$$|G(u,v)|^2 = |H(u,v)|^2|F(u,v)|^2 + \mathrm{E}\left\{|N(u,v)|^2\right\}$$

which – when formulated more precisely and according to our new definition – is expressed by

$$S_g(u,v) = S_h(u,v)S_f(u,v) + S_n(u,v).$$

1.3 Book by Gonzalez and Woods, 5.39

Cannon [1974] suggested a restoration filter $R(u,v)$ satisfying the condition

$$|\hat{F}(u,v)|^2 = |R(u,v)|^2|G(u,v)|^2$$

The restoration filter is based on the premise of forcing the power spectrum of the restored image, $|\hat{F}(u,v)|^2$, to equal the spectrum of the original image, $|F(u,v)|^2$. Assume that the image and noise are uncorrelated.

(a) Find $R(u,v)$ in terms of $|F(u,v)|^2$ and $|N(u,v)|^2$.
Hint: Refer to Fig. 5.1, Eq.(5.65), and Problem 5.38.

(b) Use your result in (a) to state a result in the form of Eq. (5.81).

$$\hat{F} = \left[\frac{H^* \cdot S_f}{S_f \cdot |H|^2 + S_\eta}\right] \cdot G = \left[\frac{H^*}{|H|^2 + \dfrac{S_\eta}{S_f}}\right] \cdot G = \left[\frac{1}{H} \cdot \frac{|H|^2}{|H|^2 + \dfrac{S_\eta}{S_f}}\right] \cdot G \qquad (5.81)$$

We show $\hat{F}$, H, G, S_f and S_η instead of $\hat{F}(u,v)$, $H(u,v)$, $G(u,v)$, $S_f(u,v)$ and $S_\eta(u,v)$ to simplify the notation.

(a) It is given that

$$|\hat{F}(u,v)|^2 = |R(u,v)|^2|G(u,v)|^2$$

Because the image and noise are uncorrelated, it follow from Problem 5.38 that

$$|\hat{F}(u,v)|^2 = |R(u,v)|^2 \left[|H(u,v)|^2|F(u,v)|^2 + |N(u,v)|^2\right]$$

Forcing $|\hat{F}(u,v)|^2$ to equal $|F(u,v)|^2$ gives

$$R(u,v) = \left[\frac{|F(u,v)|^2}{|H(u,v)|^2|F(u,v)|^2 + |N(u,v)|^2}\right]^{1/2}$$

(b)

$$\begin{aligned}\hat{F}(u,v) &= R(u,v)G(u,v) \\ &= \left[\frac{|F(u,v)|^2}{|H(u,v)|^2|F(u,v)|^2 + |N(u,v)|^2}\right]^{1/2} G(u,v) \\ &= \left[\frac{1}{|H(u,v)|^2 + \frac{|N(u,v)|^2}{|F(u,v)|^2}}\right]^{1/2} G(u,v)\end{aligned}$$

and, because $|\hat{F}(u,v)|^2 = S_f(u,v)$ and $|N(u,v)|^2 = S_\eta(u,v)$,

$$\hat{F}(u,v) = \left[\frac{1}{|H(u,v)|^2 + \frac{S_\eta(u,v)}{S_f(u,v)}}\right]^{1/2} G(u,v)$$

Note by the lecturer: The derived reconstruction function can only reconstruct the magnitude of the image spectrum but not the phase. This can be shown by a simplified analysis where the noise is *set to zero, i.e $N(u,v) = 0$:*

$$\begin{aligned}\hat{F}(u,v) &= R(u,v)G(u,v)\\ &= R(u,v)H(u,v)F(u,v)\\ &= \left[\frac{1}{|H(u,v)|^2 + \frac{|N(u,v)|^2}{|F(u,v)|^2}}\right]^{1/2} H(u,v)F(u,v)\\ &= \frac{1}{|H(u,v)|}H(u,v)F(u,v)\end{aligned}$$

Hence, the reconstruction yields $\hat{F}(u,v) = F(u,v)$ only if $H(u,v) = |H(u,v)|$, which is true if $H(u,v)$ is real. This is the case if the point spreading function $h(x,y)$ is symmetric. An example for a symmetric point spreading function is given in the above exercise 5.34. Note that the Wiener filter and the constraint least squares filter in the book also work for nonsymmetric $h(x,y)$.