Some of the theoretical contents are taken from the instructor manual of the book „Digital Image Processing (4th Edition)“ by Rafael C. Gonzalez and Richard E. Woods. Therefore, the confidentiality class of this document is „internally extended“. The data can be used by OST members as of September 1, 2020, but must not be passed on to third parties.

Image Processing and Computer Vision 1

Chapter 3 – Spatial Filtering – week 5

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1 Book

1.1 Book by Gonzalez and Woods, 3.31 (a)

Show that the Gaussian kernel, $G(s,t)$, in Eq. (3-45) is separable.
Hint: read the paragraph *separable filter kernels* on Page 161 in Section 3.4. Alternatively, and in alignment with the definition (2-57) use the commutativity of filtering to swap image $f(x, y)$ and kernel $w(x, y)$ in Eq. (3-31) so that the filtering i.e. correlation is expressed in the general form (2-55). Finally, show that the kernel satisfies Eq. (2-57).

Book section 3.5 *Smoothing Spatial Filters*

$$w(s,t) = G(s,t) = Ke^{-\frac{s^2+t^2}{2\sigma^2}} \tag{3-45}$$

Book section3.4 *The Mechanics of Linear Spatial Filtering*

$$g(x, y) = \sum_{s=-a}^{a} \sum_{t=-b}^{b} w(s, t)\, f(x+s, y+t) \tag{3-31}$$

Book section 2.6 *Image Transforms*
The kernel $r(x, y, u, v)$ of the image transformation

$$T(u,v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y)\, r(x, y, u, v) \tag{2-55}$$

is separable if

$$r(x, y, u, v) = r_1(x, u)\, r_2(y, v). \tag{2-57}$$

Solution according to the hint. *As stated at the beginning of the subsection entitled Separable Filter Kernels in section 3.4, a function $G(x,y)$ is separable if it can be written as $G(x,y) = G_1(x)G_2(y)$. The 2-D Gaussian function can be written as*

$$G(x,y) = e^{-\frac{x^2+y^2}{2\sigma^2}} = e^{-\frac{x^2}{2\sigma^2}} e^{-\frac{y^2}{2\sigma^2}} = G_1(x)G_2(y)$$

Hence, the Gaussian kernel is separable.

Alternative solution. *The Alternative derivation makes use of the original definition of separability (2-57). Commutativity of the correlation operation implies*

$$g(x,y) = \sum_{s=-a}^{a} \sum_{t=-b}^{b} f(s,t) w(x+s, y+t).$$

For this equation to comply with (2-55) we can set

$$r(x,y,u,v) = w(x+u, y+v).$$

Substituting the arguments $s \to x+u$ and $t \to y+v$ into the kernel definition (3-45) yields

$$\begin{aligned} w(x+s, t+s) = G(x+1, t+s) &= Ke^{-\frac{(x+u)^2+(y+v)^2}{2\sigma^2}} \\ &= \underbrace{Ke^{-\frac{(x+u)^2}{2\sigma^2}}}_{G_1(x,u)} \underbrace{e^{-\frac{(y+v)^2}{2\sigma^2}}}_{w_2(y,v)}, \end{aligned}$$

which proves the separability property of the 2D Gaussian filter kernel.

1.2 Book by Gonzalez and Woods, 3.39*

An image is filtered with a kernel whose coefficients sum to 1. Show that the sum of the pixel values in the original and filtered images is the same.

We assume full convolution, as explained in connection with Eqs. (3-36) and (3-37). The kernel $w(s,t)$ hast Koefficients sum up to 1, is according Let the coefficients of a general kernel be denoted by $c_1, c_2, c_3, \ldots, c_K$. We know that in full convolution each coefficient in the kernel multiplies each pixel in the input image one time. The only operation performed is a sum of products. Therefore, if we were to expand the convolution summation, add all the terms , and collect the coefficients multiplying every pixel, $f(i,j)$, of the original image, we would find that they appear as follows:

$$f(i,j)[c_1 + c_2 + c_3 + \cdots + c_K]$$

But the sum of the coefficients is assumed to be 1. So, if we carry out this operation for all location (i,j) in the image and sum the results, we conclude that the sum of the pixels in a filtered image will be equal to the sum of the pixels in the original image.

Proof by Benjamin Bucheli a student of this course in 2022: *Assume the image $f(x,y)$ contains only a single pixel with a grayvalue that is nonzero. The convolution of $f(x,y)$ with the kernel h will show a replication of the kernel on the output image $g(x,y)$ weighted with the nonzero grayvalue. Thus, the sum of the grayvalues of $g(x,y)$ equals the sum of grayvalues of $h(x,y)$. The proof can be generalized to arbitrary images, by considering the linearity of the convolution operation.*

1.3 Book by Gonzalez and Woods, 3.49

Show that the Laplacian defined in Eq. (3-59) is isotropic (invariant to rotation of the directions of derivation). Assume continuous quantities. From Table 2.3, coordinate rotation by an angle θ is given by

$$x = x' \cos\theta - y' \sin\theta \text{ and } y = x' \sin\theta + y' \cos\theta$$

where (x, y) are the unrotated and (x', y') are the rotated coordinates, respectively.

The Laplacian operator is defined as

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = f_{xx} + f_{yy}$$

for the unrotated coordinates, and

$$\nabla'^2 f = \frac{\partial^2 f}{\partial x'^2} + \frac{\partial^2 f}{\partial y'^2}$$

for the rotated coordinates. It is given that

$$x = x' \cos\theta - y' \sin\theta \textit{ and } y = x' \sin\theta + y' \cos\theta$$

where θ is the angle of rotation. We want to show that the right sides of the first two equations are equal. We start with

$$\begin{aligned} \frac{\partial f}{\partial x'} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial x'} \\ &= f_x\, c + f_y\, s \end{aligned} \tag{1}$$

Taking the partial derivative of this expression again with respect to x' yields

$$\begin{aligned} \frac{\partial^2 f}{\partial x'^2} &= \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x'}\right]\frac{\partial x}{\partial x'} + \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x'}\right]\frac{\partial y}{\partial x'} \\ &= \frac{\partial}{\partial x}\left[f_x\, c + f_y\, s\right] c + \frac{\partial}{\partial y}\left[f_x\, c + f_y\, s\right] s \\ &= f_{xx}\, c^2 + f_{yx}\, s\, c + f_{xy}\, s\, c + f_{yy}\, s^2 \end{aligned}$$

Next, we compute

$$\begin{aligned} \frac{\partial f}{\partial y'} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial y'} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial y'} \\ &= -f_x\, s + f_y\, c \end{aligned}$$

Taking the derivative of this expression again with respect to y gives

$$\begin{aligned} \frac{\partial^2 f}{\partial y'^2} &= \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y'}\right]\frac{\partial x}{\partial y'} + \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y'}\right]\frac{\partial y}{\partial y'} \\ &= \frac{\partial}{\partial x}\left[f_x\,(-s) + f_y\, c\right](-s) + \frac{\partial}{\partial y}\left[f_x\,(-s) + f_y\, c\right] c \\ &= f_{xx}\, s^2 - f_{yx}\, s\, c - f_{xy}\, s\, c + f_{yy}\, c^2 \end{aligned}$$

Finally, adding the two expressions for the second derivatives gives the result:

$$\begin{aligned} \frac{\partial^2 f}{\partial x'^2} + \frac{\partial^2 f}{\partial y'^2} &= f_{xx}[s^2 + c^2] + f_{yy}[s^2 + c^2] \\ &= f_{xx} + f_{yy} \\ &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y'^2} \end{aligned}$$

which proves that the Laplacian operator is independent of rotation of the directions of derivation. s

1.4 Book by Gonzalez and Woods, 3.53

Show that subtracting the Laplacian from an image gives a result that is proportional to the unsharp mask in Eq. (3-64). Use the definition for the Laplacian given in Eq. (3-62).
Hint: Rearrange Eq. (3.62) so that part of it has the form of an average filter.

Consider the following equation:

$$\begin{aligned} f(x,y) - \nabla^2 f(x,y) &= f(x,y) - [f(x+1,y) + f(x-1,y) + f(x,y+1) + f(x,y-1) - 4f(x,y)] \\ &= 6f(x,y) - [f(x+1,y) + f(x-1,y) + f(x,y+1) + f(x,y-1) + f(x,y)] \end{aligned}$$

We can write this expression as

$$\begin{aligned} f(x,y) - \nabla^2 f(x,y) &= 5\left\{1.2f(x,y) - \frac{1}{5}[f(x+1,y) + f(x-1,y) + f(x,y+1) + f(x,y-1) + f(x,y)]\right\} \\ &= 5\left[1.2f(x,y) - \bar{f}(x,y)\right] \end{aligned}$$

where $\bar{f}(x,y)$ *denotes the average of* $f(x,y)$ *in a neighborhood centered at* (x,y) *and including the center pixel and its four immediate neighbors. Treating the constants in the last line of the above equation as proportionality factors, we may write*

$$f(x,y) - \nabla^2 f(x,y) \sim f(x,y) - \bar{f}(x,y)$$

The right side of this equation is recognized within the just-mentioned proportionality factors to be of the same form as the definition of unsharp mask given in Eq. (3-64).