Some of the theoretical contents are taken from the instructor manual of the book „Digital Image Processing (4th Edition)“ by Rafael C. Gonzalez and Richard E. Woods. Therefore, the confidentiality class of this document is „internally extended“. The data can be used by OST members as of September 1, 2020, but must not be passed on to third parties.

Image Processing and Computer Vision 1

Chapter 4 – Filter Design in the Frequency Domain – week 10

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1 Book

1.1 Book by Gonzalez and Woods, 4.99

Without MATLAB, Python or calculator (exam example) calculate the 2-D Discrete Fourier Transform of the images:

$$\text{(a)} \quad f_a = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{(b)} \quad f_b = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{(c)} \quad f_c = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

(a)

$$F_a[u,v] = \mathcal{F}(f_a) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f_a[x,y] e^{-2j\pi(ux/M+vy/N)} = f_a[0,0] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

(b)

$$F_b[u,v] = \mathcal{F}(f_b) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f_b[x,y] e^{-2j\pi(ux/M+vy/N)} = f_b[1,1] e^{-2j\pi(u/3+v/3)}$$

$$= \cos(2\pi \frac{u+v}{3}) - j\sin(2\pi \frac{u+v}{3}) = \begin{bmatrix} 1 & -\frac{1}{2} - j\frac{\sqrt{3}}{2} & -\frac{1}{2} + j\frac{\sqrt{3}}{2} \\ -\frac{1}{2} - j\frac{\sqrt{3}}{2} & -\frac{1}{2} + j\frac{\sqrt{3}}{2} & 1 \\ -\frac{1}{2} + j\frac{\sqrt{3}}{2} & 1 & -\frac{1}{2} - j\frac{\sqrt{3}}{2} \end{bmatrix}$$

(c) First lets rewrite f_c as:

$$f_c \overset{!}{=} \mathbf{1} - f_b$$

Therefore we only need to compute the Fourier Transform of $\mathbf{1}$, which is just the weighted DC-term:

$$\mathcal{F}(\mathbf{1})[u,v] = \begin{cases} MN, & \text{if } i = 0 \text{ and } j = 0 \\ 0, & \text{else} \end{cases}$$

Finally calculate f_c

$$\mathcal{F}(f_c) = \mathcal{F}(\mathbf{1}) - \mathcal{F}(f_b) = \begin{bmatrix} 8 & \frac{1}{2} + j\frac{\sqrt{3}}{2} & \frac{1}{2} - j\frac{\sqrt{3}}{2} \\ \frac{1}{2} + j\frac{\sqrt{3}}{2} & \frac{1}{2} - j\frac{\sqrt{3}}{2} & -1 \\ \frac{1}{2} - j\frac{\sqrt{3}}{2} & -1 & \frac{1}{2} + j\frac{\sqrt{3}}{2} \end{bmatrix}$$

1.2 Book by Gonzalez and Woods, 4.53

Given an image of size $M \times N$, you are asked to perform an experiment that consists of repeatedly lowpass filtering the image using a Gaussian lowpass filter with a given cutoff frequency, D_0. You may ignore computational round-off errors.

(a) Let K denote the number of applications of the filter. Can you predict (without doing the experiment) what the result (image) will be for a sufficiently large value of K? If so, what is that result?

One application of the filter gives

$$G(u,v) = H(u,v)F(u,v) = e^{-D^2(u,v)/(2D_0^2)}F(u,v)$$

Similarly, K applications of the filter would give

$$G_K(u,v) = e^{-KD^2(u,v)/(2D_0^2)}F(u,v)$$

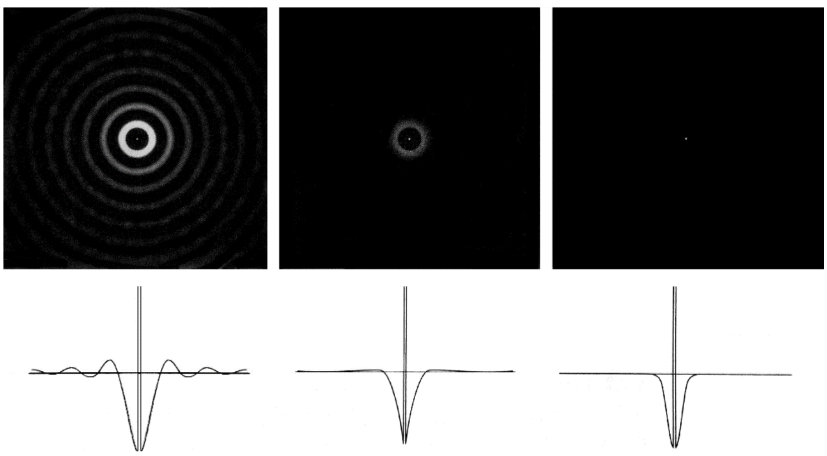
The inverse DFT of $G_K(u,v)$ would give the image resulting from K passes of the Gaussian filter. If K is „large enough “, the Gaussian LPF will become a notch pass filter, passing only $F(0,0)$. We know that this term is equal to the average value of the image. So, there is a value of K after which the result of repeated lowpass filtering will simply produce a constant image. The value of all pixels in this image will be equal to the average value of the original image. Note that the answer applies even as K approaches infinity. In this case the filter will approach an impulse at the origin, and this would still give us $F(0,0)$ as the result of filtering.

Note by the lecturer: The following equality shows that the cutoff frequency of the effectively applied Gaussian filter $H_K(u,v)$ is proportional to $D_0/\sqrt{K}$, i.e., it approaches zero for large enough K:

$$H_K(u,v) := (H(u,v))^K = e^{-\frac{KD^2(u,v)}{2D_0^2}} = e^{-\frac{D^2(u,v)}{2\left(\frac{D_0}{\sqrt{K}}\right)^2}}$$

1.3 Book by Gonzalez and Woods, 4.59

Each spatial highpass kernel in Fig. 4.52 has a strong spike in it the center. Explain the source of these spikes.



a b c

FIGURE 4.52 Spatial representation of typical (a) ideal, (b) Butterworth, and (c) Gaussian frequency domain highpass filters, and corresponding intensity profiles through their centers.

Book figure 4.52 *Spatial representation of typical highpass filters*

With reference to Eq. (4-118),

$$H_{HP}(u,v) = 1 - H_{LP}(u,v) \tag{4-118}$$

the transfer functions of a highpass filter can be expressed as 1 minus the transfer function of a lowpass filter. The highpass spatial kernels in Fig 4.52 were obtained by taking the IDFT of Eq.(4-118) using the appropriate lowpass transfer function. The spikes are unit impulses resulting from taking the IDFT of 1.