Image Processing and Computer Vision 1

Chapter 4 – One Dimensional Fourier Transform – week 6

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1 Book

1.1 Book by Gonzalez and Woods, 4.9 (a)

Show that the following expression is true:

$$\mathscr{F}\left\{\sin(2\pi\mu_0 t)\right\} = \frac{1}{2j}\left[\delta(\mu-\mu_0) - \delta(\mu+\mu_0)\right]$$

Hint. Make use of the Euler formula $e^{jx} = \cos(x) + j\sin(x)$ and the Fourier transfrom pair $\mathscr{F}\left\{e^{j2\pi\mu_0 t}\right\} = \delta(\mu-\mu_0)$.

1.2 Book by Gonzalez and Woods, 4.10

Consider the function $f(t) = \sin(2\pi n t)$, where n is an integer. Its Fourier transform, $F(\mu)$, is purely imaginary (see Problem 4.12). Because the transform, $\tilde{F}(\mu)$, of sampled data consists of periodic copies of $F(\mu)$, it follows that $\tilde{F}(\mu)$ will also be purely imaginary. Draw a diagram similar to Fig. 4.6, and answer the following questions based on your diagram (assume that sampling starts at $t = 0$).

(a) What is the period of $f(t)$?

(b) What is the frequency of $f(t)$?

(c) What would the sampled function and its Fourier transform look like in general if $f(t)$ is sampled at a rate higher than the Nyquist rate?

(d) What would the sampled function look like in general if $f(t)$ is sampled at a rate lower than the Nyquist rate?

(e) What would the sampled function look like if $f(t)$ is sampled at the Nyquist rate with samples taken at $t = 0, \Delta T, 2\Delta T, \ldots$?

1.3 Book by Gonzalez and Woods, 4.11

Prove the validity of the convolution theorem of one continuous variable, as given in Eqs. (4-25) and (4-26).

Book section 4.2.5 *Convolution*

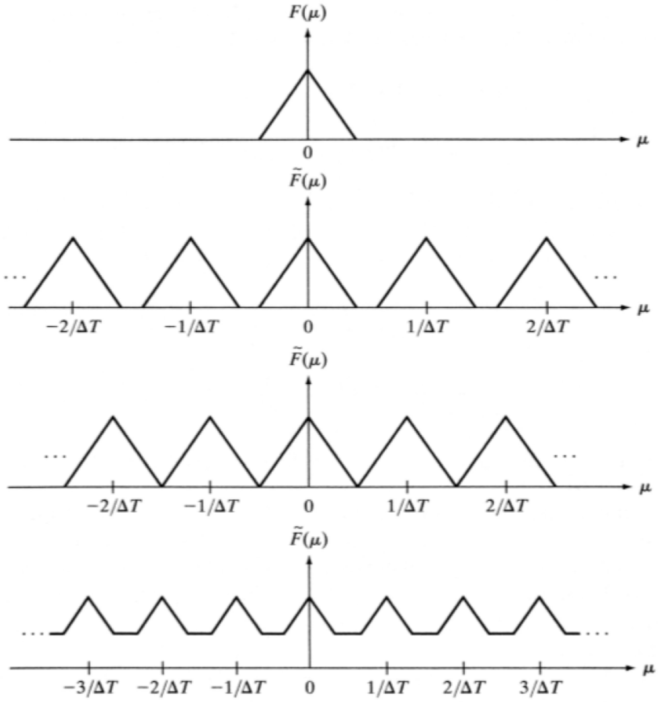
$$(f \star h)(t) \Leftrightarrow (F \cdot H)(\mu) \tag{4.25}$$
$$(f \cdot h)(t) \Leftrightarrow (F \star H)(\mu) \tag{4.26}$$

1.4 Book by Gonzalez and Woods, 4.14

Show that $\tilde{F}(\mu)$ in Eq. (4-40) is infinitely periodic in both directions, with period $1/\Delta T$.

a
b
c
d

FIGURE 4.6
(a) Fourier transform of a band-limited function.
(b) - (d) Transforms of the corresponding sampled function under the conditions of over-sampling, critically-sampling, and under-sampling, respectively.



Book figure 4.6 *Fourier Transform of band-limited function*

Eq. 4-40

$$\tilde{F}(\mu) = \sum_{n=-\infty}^{\infty} f_n e^{-j2\pi\mu n\Delta T}$$

1.5 Book by Gonzalez and Woods, 4.15

Do the following:

(a) Show that Eqs (4-42) and (4-43) constitute a Fourier transform pair: $f_n \Leftrightarrow F_m$.

(b) Show that Eqs. (4-44) and (4-45) also are a Fourier transform pair: $f(x) \Leftrightarrow F(u)$.

You will need the following orthogonality property of exponentials in both parts of this problem:

$$\sum_{x=0}^{M-1} e^{j2\pi rx/M} \cdot e^{-j2\pi ux/M} = \begin{cases} M, & \text{if } r = u \\ 0, & \text{otherwise} \end{cases}$$

Book from Section 4.4 *The Discrete Fourier Transform of One Variable*

$$F_m = \sum_{n=0}^{M-1} f_n e^{-j2\pi mn/M} \qquad m = 0, 1, 2, \ldots, M-1 \tag{4-42}$$

$$f_n = \frac{1}{M} \sum_{m=0}^{M-1} F_m e^{j2\pi mn/M} \qquad n = 0, 1, 2, \ldots, M-1 \tag{4-43}$$

$$F(u) = \sum_{x=0}^{M-1} f(x) e^{-j2\pi ux/M} \qquad u = 0, 1, 2, \ldots, M-1 \tag{4-44}$$

$$f(x) = \sum_{u=0}^{M-1} F(u) e^{j2\pi ux/M} \qquad x = 0, 1, 2, \ldots, M-1 \tag{4-45}$$

1.6 Book by Gonzalez and Woods, 4.18

Show that the 1-D convolution theorem given in Eqs. (4-25) and (4-26) also holds for discrete variables, but with the right side of Eq. (4-26) multiplied by $1/M$. That is, show that

(a) $(f \star h)(x) \Leftrightarrow (F \cdot H)(u)$, and

(b) $(f \cdot h)(x) \Leftrightarrow \frac{1}{M}(F \star H)(u)$

2 Practical Exercise

Write a program, which implements the linear convolution between two finite sequences (vectors) h and f. Now use the FFT (a fast version of the DFT) to calculate the circular convolution between h and f by multiplying their DFTs. To be able to do that, add zeros to the end of the shorter vector, so that both are of the same length. This is called „Zero Padding“. Compare the results for different length of h and f, what do you observe? How would you have to zero pad the vectors, so that the circular convolution is identical to the linear one?