Mesh Field Theory – Lecture 10: Mesh Hidden Subgroup Problem

Mirroring CMU Quantum Computation Lecture 10

Introduction

This lecture mirrors the structure of CMU Lecture 10 from the 15-859BB Quantum Computation course at Carnegie Mellon University, but reconstructs the Hidden Subgroup Problem (HSP) using Mesh Field Theory.

In Mesh, the HSP is expressed as causal coherence redundancy across a group domain, with hidden subgroup symmetries manifesting through causal interference and divergence-triggered collapse, without tensor products or unitary group actions.

1 Hidden Subgroup Problem in Mesh

Let G be a finite group and $H \leq G$ a hidden subgroup. A function $f : G \to S$ satisfies:

$$f(x) = f(y) \quad \text{iff} \quad xH = yH$$

In Mesh, this means:

- Elements $x \in G$ are mapped to spatial coherence regions R_x,
- Coherence fields $\vec{C}_x(x,t)$ satisfy:

$$\vec{C}_f(x) = \vec{C}_f(y) \quad \text{iff } xH = yH$$

This creates real coherence redundancy over left cosets.

2 Initialization: Coherence Superposition Over Group Elements

Mesh initializes real causal coherence fields for each group element:

$$\vec{C}_{\text{total}}(x) = \sum_{x \in G} \vec{C}_x(x,t)$$

All fields initialized uniformly, preserving causal balance.

3 Oracle Structure: Coherence Coset Mapping

The Mesh oracle identifies coset structures by imposing causal redundancy:

$$\vec{C}_x(x) = \vec{C}_x(xh) \quad \text{for all } h \in H$$

- This encodes the coset condition causally.
- Coherence fields over the same coset evolve identically.

4 Causal Interference and Subgroup Detection

Mesh applies the causal Mesh Fourier transform over the group:

$$\phi_k(g) = \frac{2\pi k \cdot g}{|G|} \quad \Rightarrow \quad \vec{C}_k(g) = \nabla \phi_k(g) \cdot \chi(g)$$

Then:

$$\mathcal{F}(k) = \sum_{g \in G} \vec{C}_{\text{input}}(g,t) \cdot \vec{C}_k(g,t)$$

- Interference patterns localize on characters orthogonal to H.
- Collapse occurs at regions with maximal coherence alignment.

5 Collapse and Subgroup Reconstruction

As in prior Mesh algorithms:

$$\Gamma(g,t) = \nabla \cdot \vec{C}(g,t) \quad \Rightarrow \quad \text{collapse when } \Gamma > \Gamma_{\text{crit}}$$

Each measurement yields information about the subgroup:

$$\chi(g) = 1 \quad \text{only if } g \cdot h = 0 \quad \forall h \in H$$

This provides constraints on the hidden subgroup H, recovered by classical post-processing.

6 Worked Example: Mesh HSP Over $\mathbb{Z}_8$

Let $G = \mathbb{Z}_8$, $H = \langle 2 \rangle = \{0, 2, 4, 6\}$

Oracle mapping:

$$\vec{C}_0 = \vec{C}_2 = \vec{C}_4 = \vec{C}_6 \quad \text{(coset 0)}$$
$$\vec{C}_1 = \vec{C}_3 = \vec{C}_5 = \vec{C}_7 \quad \text{(coset 1)}$$

Apply Mesh Fourier transform over $\mathbb{Z}_8$.
Interference localizes at frequencies orthogonal to H, i.e. multiples of 4.
Collapse occurs at $k = 4$, from which the subgroup is recovered.

7 Summary

In this Mesh mirror of CMU Lecture 10, we established:

- Hidden subgroup structure manifests as coherence redundancy over group cosets.
- Oracle operation is implemented as causal coherence equivalence.
- Fourier interference over the group domain reveals subgroup orthogonality.
- Collapse deterministically extracts subgroup constraints.

Mesh Field Theory thus reconstructs the Hidden Subgroup Problem causally, without Hilbert space vectors, group unitaries, or quantum probability postulates.