2^{nd} Homework

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Exercise 3

Our given model is an instance of the parametric set of 2^{nd} degree polynomials of 3 variables, thus we have an input of 3 dimensions $x = [x_1, \quad x_2, \quad x_3]$ $f_\theta: R^3 \to R$.

A suitable transformation would be:

$$\varphi(\chi) = \begin{pmatrix} \varphi_1(\chi) \\ \varphi_2(\chi) \\ \varphi_3(\chi) \\ \varphi_4(\chi) \\ \varphi_5(\chi) \\ \varphi_6(\chi) \\ \varphi_7(\chi) \\ \varphi_8(\chi) \\ \varphi_9(\chi) \end{pmatrix} = \begin{pmatrix} x_1^2 \\ x_2^2 \\ x_3^2 \\ x_1x_2 \\ x_1x_3 \\ x_2x_3 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

The relation could be re-written as a linear combination of the transformed values as:

$$y = 3\varphi_1(\chi) + 4\varphi_2(\chi) + 5\varphi_3(\chi) + 7\varphi_4(\chi) + \varphi_5(\chi) + 4\varphi_6(\chi) - 2\varphi_7(\chi) - 3\varphi_8(\chi) - 5\varphi_9(\chi) + \eta$$

However, that is an instance of the parametric set of 1^{st} degree polynomials of 9 variables, thus we have an input of 9 dimensions

$$x' = [\varphi_1(\chi) \quad \varphi_2(\chi) \quad \varphi_3(\chi) \quad \varphi_4(\chi) \quad \varphi_5(\chi) \quad \varphi_6(\chi) \quad \varphi_7(\chi) \quad \varphi_8(\chi) \quad \varphi_9(\chi)] \; f'_\theta: R^9 \to R$$

Exercise 4

Our given model is an instance of the parametric set of 2^{nd} degree polynomials of 3 variables, thus we have an input of 3 dimensions $x = [x_1, \quad x_2, \quad x_3]$ $f_\theta: R^3 \to R$.

A suitable transformation would be:

$$\varphi(\chi) = \begin{pmatrix} \varphi_1(\chi) \\ \varphi_2(\chi) \\ \varphi_3(\chi) \\ \varphi_4(\chi) \\ \varphi_5(\chi) \end{pmatrix} = \begin{pmatrix} x_1^2 \\ x_2^2 \\ x_3^2 \\ x_1x_2 \\ x_2x_3 \end{pmatrix}$$

The relation could be re-written as a linear combination of the transformed values as:

$$y = \varphi_1(\chi) + 3\varphi_2(\chi) + 6\varphi_3(\chi) + \varphi_4(\chi) + \varphi_5(\chi) + \eta$$

However, that is an instance of the parametric set of 1st degree polynomials of 5 variables, thus we have an input of 5 dimensions

$$x' = [\varphi_1(\chi), \quad \varphi_2(\chi), \quad \varphi_3(\chi), \quad \varphi_4(\chi), \quad \varphi_5(\chi)]$$

Exercise 5

Our data is 2D, $x \in R^2$ and a class set of (+1, -1). We have 5 points in our data that will create the following X and y matrices:

$$X = \begin{pmatrix} x_1^T \\ x_2^T \\ x_3^T \\ x_4^T \\ x_5^T \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & 0 & 0.5 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \end{pmatrix}$$

and

$$y = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$$

We will solve this problem by assuming a linear model such as:

$$0 = \theta_0 + \theta_1 x_1 + \theta_2 x_2$$

where $\theta = (\theta_0 \quad \theta_1 \quad \theta_2)^T \in R^3$ is the parametric vector.

In order to solve this with least squares we need to calculate first the following matrices:

$$X^T = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & -1 & -1 \\ 1 & -1 & 0.5 & 1 & -1 \end{pmatrix}$$

$$X^T X = \begin{pmatrix} 5 & 0 & 2.5 \\ 0 & 4 & 0 \\ 0.5 & 0 & 4.25 \end{pmatrix}$$

$$(X^T\, X)^{-1} = \begin{pmatrix} 0.2024 & 0 & -0.023 \\ 0 & 0.25 & 0 \\ -0.023 & 0 & 0.2381 \end{pmatrix}$$

Then according to the least square criterion:

$$\theta = (X^T X)^{-1} X^T Y = (-0.1905 \quad 1 \quad 0.0952)^T$$

That means that the hyperplane is $0 = -0.1905 + x_1 + 0.0952 x_2$

Exercise 6

Let's define X and Y two random variables with $X = \{x_1, x_2, .., x_n\}$ and $Y = \{y_1, y_2, .., y_m\}$

For each $x \in X$, the events {X=x, Y=y_i} are mutually exclusive and the union is the event {X=x}:

$$P(X = x) = P(\cup_i \{X = x, Y = y_i\}) = \sum_j P(X = x, Y = y_i) = \sum_y P(X = x, Y = y)$$

From the concept of conditional probability, we have:

$$P(x|y) = \frac{P(x, y)}{P(y)}$$

Re-write the above we have:

$$P(x, y) = P(x|y)P(y)$$

The Bayes rule can be verified using the product rule:

$P(x, y) = P(x|y)P(y)$ and $P(x, y) = P(y|x)P(x)$, we have $P(y|x)P(x) = P(x|y)P(y)$

$$P(y|x) = \frac{P(x|y)P(y)}{P(x)}$$