CFD WI4011

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Estimating the global error (I)

The *global* error is directly related to the *local truncation* error:

$$L_h e = \tau$$

Try to find a *Barrier function* E such that:

$$L_h E \geq |\tau|$$

We will show that this implies $|e| \leq E$.

Estimating the global error (II)

$$L_h(\pm e - E) \leq 0 \, .$$

We want to use the discrete maximum principle:
Therefore the scheme should be of positive type or $Pe < 2$.
The maximum occurs at Dirichlet boundary $\Rightarrow$

$$\pm e_j - E_j \leq \pm e_1 - E_1, \quad j = 2, \cdots, J \, . \tag{2.58}$$

Assume locally uniform grid: $h_1 = h_2 = h_{\frac{3}{2}}$:

$$L\varphi_1 = \left(\frac{u}{2} + \frac{3\varepsilon}{h_1}\right)\varphi_1 + \left(\frac{u}{2} - \frac{\varepsilon}{h_1}\right)\varphi_2 = a\varphi_1 - b\varphi_2 \leq 0$$

$$0 < b < a$$

Estimating the global error (III)

This means:

$$b\,(\pm e_2 - E_2) \leq b\,(\pm e_1 - E_1) \quad \text{Using max principle}$$

$$a\,(\pm e_1 - E_1) \leq b\,(\pm e_2 - E_2) \quad \text{Using } L_h \leq 0$$

Because $0 < b < a$ this implies:

$$\pm e_1 - E_1 \leq 0 \Rightarrow |e_1| \leq E_1$$

And using the discrete maximum principle:

$$|e_j| \leq E_j, \quad j = 1, \cdots, J.$$

Choosing/Finding a suitable $E(x)$ is an art.

Estimating the global error (IV)

Analysis on a uniform grid:

$$E_j = M\psi(x_j), \quad \psi(x) \equiv 1 + 3x - x^2$$

$$\begin{aligned}
L_h\psi(x_1) &= \ldots > 2\varepsilon/h \\
L_h\psi(x_j) &= uh(3 - 2x_j) + 2\varepsilon h > 2\varepsilon h, \quad j = 2, \ldots, J-1 \\
L_h\psi(x_J) &= \ldots > \varepsilon/h
\end{aligned}$$

Estimating the global error (V)

We have already shown that

$$\begin{aligned} \tau_1 &< M_1 h \\ \tau_j &< M_2 h^3, \quad j = 2, \ldots, J-1 \\ \tau_J &< M_3 h^2 \end{aligned}$$

Hence, choose:

$$M = \frac{h^2}{\varepsilon} \max\{M_1/2, M_2/2, M_3\}$$

Then $L_h E \geq |\tau|$ and $|e| < E = \mathcal{O}(h^2)$

The low-order *local truncation error* at the boundaries does not effect the accuracy(=*global error*) of the solution!

Estimating the global error (VI)

Some remarks:

- Discretisation on a nonuniform grid:
  - Similar procedure, more complicated barrier function(s)
  - Similar result:

$$e_j = \mathcal{O}(\Delta^2), \quad \Delta = \max(h_j)$$

- Discretisation on a vertex centered grid
  - Similar procedure
  - Similar result:

$$e_j = \mathcal{O}(\Delta^2), \quad \Delta = \max(h_j)$$

The fact that the local truncation error of the *vertex-centered* case is smaller than for the *cell-centered* case does not increase the global error!

Summary of the properties of the stationary convection-diffusion equation in 1D(I)

Analytical properties:

- Conservative
- Maximum principle
- Well-posed problem for $Pe \longrightarrow \infty$ with the correct boundary conditions.
- Constant coefficient case can be solved exactly.

Summary of the properties of the stationary convection-diffusion equation in 1D(II)

Finite volume discretisation of the stationary convection-diffusion equation in 1D:

- Central scheme I:accurate solution for $Pe_h < 2$
- Upwind scheme add unwanted artificial diffusion.
- Solution: use local grid refinement in the region of strong gradient, of known location and thickness: Discretisation becomes *uniform* in *Pe*
- The *Roughness* introduced in the grid does <u>not</u> reduce the accuracy of the solution

The stationary convection-diffusion equation in 2D (I)

$$(u_\alpha \varphi)_{,\alpha} - (\varepsilon \phi_{,\alpha})_{,\alpha} = q(\mathbf{x}), \quad (\mathbf{x}) \in \Omega \equiv [0,1]^\alpha$$

with boundary conditions

$$\begin{aligned} \varphi &= f(\mathbf{x}) \quad \mathit{on} \quad \partial\Omega_i \quad \text{Dirichlet} \\ \varphi &= f(\mathbf{x}) \quad \mathit{on} \quad \partial\Omega_o \quad \text{Dirichlet or} \\ \hat{n}_{,\alpha}\varphi_{,\alpha} &= g(\mathbf{x}) \quad \mathit{on} \quad \partial\Omega_o \quad \text{Neumann} \end{aligned}$$

The stationary convection-diffusion equation in 2D (II)

Analytical aspects:

- Conservation form
- Maximum principle
- Well-posed problem for $Pe \longrightarrow \infty$ with the correct boundary conditions.

The stationary convection-diffusion equation in 2D (III)

For efficient numerical discretisation we want to solve the same strategy of *local grid refinement*, but where are those regions of strong gradients and how thick are they going to be?
These regions are called *boundary layers* and there thickness and location can be derived using *singular perturbation theory*

The stationary convection equation in 2D (IV)

Assume when $\varepsilon \ll 1$ we can discard diffusion effects:

$$u_\alpha \varphi_{,\alpha} - = \tilde{q}(x, y), \quad (x, y) \in \Omega \equiv [0, 1]^\alpha$$
$$\tilde{q} = q(x, y) - \varphi u_{\alpha,\alpha}$$

with boundary conditions

$$\varphi = f(x, y) \quad on \quad \partial\Omega_i \quad \text{Dirichlet}$$

The stationary convection equation in 2D (V)

This equation is *hyperbolic*.
Parameterize a curve in Ω

$$\mathbf{r}(s) = \hat{\mathbf{e}}_\alpha x_\alpha(s)$$
$$(x_\alpha)_{,s} = u_\alpha$$

Then

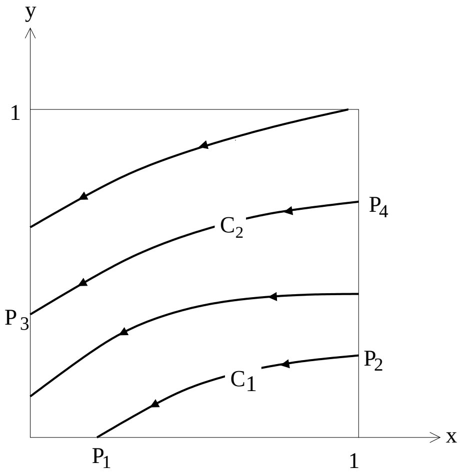
$$\varphi_{,s} = u_\alpha \varphi_{,\alpha} = \tilde{q} \quad \textit{using the PDE}$$

$\varepsilon = 0$, no source $(q = 0,\ u_{\alpha,\alpha} = 0))$
the solution is constant along $\mathbf{r}(s)$: **characteristics**

$\varepsilon \ll 1$, no source $(q = 0,\ u_{\alpha,\alpha} = 0))$:
define **subcharacteristics**, similar to the **characteristics** of the case $\varepsilon = 0$

The stationary convection-diffusion equation in 2D (VI)

A major problem:

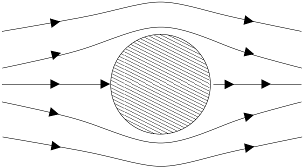


What is the value of $\varphi(P_1)$???

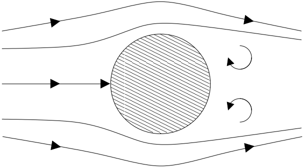
The stationary convection-diffusion equation in 2D (VII)

The convection-diffusion equation is a **singular perturbation problem**

This is related to the **Paradox of d'Alembert** for high Re flow around <u>bluff</u> bodies:



$\varepsilon = 0$

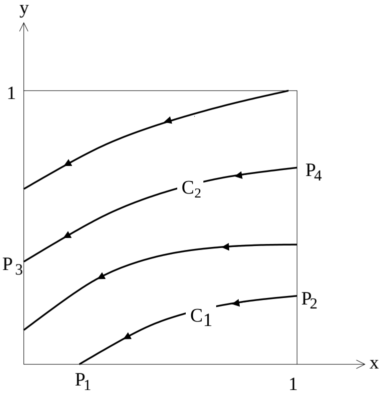


$0 < \varepsilon << 1$

The stationary convection-diffusion equation in 2D (IIX)

A singular perturbation problem

- The classification of the equation changes when $\varepsilon \neq 0$
- We can not neglect diffusion



$\Rightarrow$ the product $\epsilon\varphi_{,\alpha\alpha}$ is not negligible everywhere!! Let's go back to 1-D and find out...

The stationary convection-diffusion equation in 2D (IX)

Back to the one dimensional case:

$$u\varphi_{,x}, -\varepsilon\varphi_{,xx} = 0$$
$$\varphi(0) = a, \quad \varphi(1) = b$$

We assume that in a small region near $x = 1$:

$$\frac{\partial^m \varphi}{\partial x^m} = \mathcal{O}(\delta^{-m}),$$

with the **boundary layer thickness** $\delta = \mathcal{O}(\epsilon^{\alpha})$

Introduce locally a stretched coordinate

$$\tilde{x} = (1 - x)\varepsilon^{-\alpha},$$

α to be defined

which leads to the **boundary layer equation**

The stationary convection-diffusion equation in 2D (X)

$$-\varepsilon^{-\alpha} u \varphi_{,\tilde{x}} - \varepsilon^{1-2\alpha} \varphi_{,\tilde{x}\tilde{x}} = 0$$

Requirements on the solution of the boundary layer equation:

- The **matching principle**

$$\lim_{\tilde{x} \to \infty} \varphi_{\text{inner}}(\tilde{x}) = \lim_{x \uparrow 1} \varphi_{\text{outer}}(x)$$

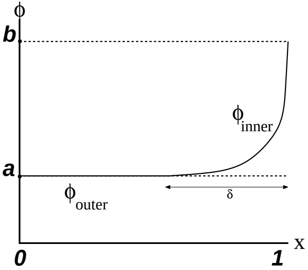
The *inner solution*
is the solution to the boundary layer equation
The *outer solution*
is the solution to the *inviscid* equation ($\varepsilon = 0$)

The stationary convection-diffusion equation in 2D (XI)

- The boundary condition on $\tilde{x} = 0$

$$\varphi_{\text{inner}}(0) = b$$

Lets find out if there is an α that fulfills these conditions...



The stationary convection-diffusion equation in 2D (XII)

- $\alpha < 1$: no solution
- $\alpha = 1$: solution possible ☺
- $\alpha > 1$: no solution

We refer to $\alpha = 1$ as the *distinguished* limit.

The stationary convection-diffusion equation in 2D (XIII)

Inner solution in the stretched variable for $\varepsilon \downarrow 0$, $\tilde{x} \in [0, \infty\rangle$:

$$\varphi(\tilde{x}) = a + (b - a)e^{-u\tilde{x}} \tag{1}$$

Inner solution in the unstretched variable for $\varepsilon \downarrow 0$:

$$\varphi(x) = a + (b - a)e^{-u\frac{1-x}{\varepsilon}} \tag{2}$$

Total solution (also)

$$\varphi(x) = a + (b - a)e^{-u\frac{1-x}{\varepsilon}} \tag{3}$$

We assumed that the boundary layer occurred at $x = 1$ why not at $x = 0$? Lets check...