CFD WI4011

Duncan Roy van der Heul
Scientific Computing Group
Delft Institute for Applied Mathematics (DIAM)

September 6, 2011

Course information (I)

- Material
  - Lecture Notes (updated, downloadable from Blackboard, to be available in printed form shortly)
  - Book: P. Wesseling, Principles of Computational Fluid Dynamics, Springer 2001.
  - Book: P.M. Knupp, Fundamentals of Grid Generation, CRC-press, 1993.

Course information (II)

- Examination
  - Practical Assignment I: The convection-diffusion equation
  - Practical Assignment II:
    - Theoretical aspects of the Navier-Stokes equations
    - Transformation to and discretisation in general coordinates
    - Structured grid generation
  - Oral exam/discussion of assignments (by appointment, Januari-Feb. 2011)
- OpenFOAM course/Assignment (most likely optional)

You can work in pairs...

Course information (III)

- Contact: D.R van der Heul
  EWI Room HB 7.030 (by appointment)
  Tel. 82632
  D.R.vanderHeul@tudelft.nl

Focus of the course

WI4011 is focussed on an *in depth* discussion of the basics of Computational Fluid Dynamics. We want you to learn how mathematical tools can help to analyse and design discretisation techniques.

Much more can be analysed than many (engineers) think....

Subjects (I)

- Governing equations of fluid dynamics:
  Navier-Stokes equations for incompressible (laminar) viscous flow
- Convection-diffusion equations as a model equation for the N.-S. equations
  - Analytical aspects
  - Discretisation in space and time
  - Analysing numerical schemes $\Rightarrow$ Accuracy and Stability...
- Discretisation of the incompressible N.-S. equations
  - staggered and colocated schemes
  - time-integration
  - solution methods for the discretised equations *(briefly)*
- Discretisation on unstructured grids
- Discretisation in general coordinates

Subjects (II)

- Grid Generation methods
  - Structured grid generation in 2D
    - Algebraic methods
    - Differential methods
  - Unstructured grid generation in 2D
    - Delaunay triangulation
    - Advancing front triangulation
- Guest Lecture

Introduction

Learning objectives of Chapter 1

- Familiarize yourself with the notation
- Brush up some basic linear algebra
- Master the outline of the derivation of the Reynolds Transport Theorem
- Understand the derivation of the Navier-Stokes equation for laminar incompressible flow, using the Reynolds Transport theorem.
- Appreciate the importance of the Reynolds number to characterize the solution of the Navier-Stokes equations.

Vector analysis (I)

Bold-faced lower case Latin letters denote vectors:

$$\mathbf{x} = (x_1, x_2, .., x_d)$$

Greek letters:scalars, but pressure is denoted by Latin letter:

$$p = p(\mathbf{x})$$

Partial differentiation:

$$\phi_{,\alpha} = \partial\phi/\partial x_\alpha.$$

Vector analysis (II)

Greek subscript:coordinate direction.
Summation convention:

$$\mathbf{u} \cdot \mathbf{v} = u_\alpha v_\alpha = \sum_{\alpha=1}^{d} u_\alpha v_\alpha$$

Laplace operator:

$$\nabla^2 \phi = \phi_{,\alpha\alpha} = \sum_{\alpha=1}^{d} \partial^2 \phi / \partial x_\alpha^2$$

- *Cartesian coordinates*:no difference between sub- and superscripts.
- compact formulations independent of dimensionality of problem.
- Handy for Cartesian coordinates, *essential* for general (curvilinear) coordinates.

Vector analysis (III)

$$u_\alpha + v_\alpha \text{ means } \sum_{\alpha=1}^{d} (u_\alpha + v_\alpha) \text{ (true/false?)}$$

Kronecker delta:

$$\delta_{\alpha\beta} = \begin{cases} 0, & \alpha \neq \beta \\ 1, & \alpha = \beta \end{cases}$$

$$\delta_{\alpha\alpha} = ?$$

Vector analysis (IV)

Divergence theorem:

$$\int_V \phi_{,\alpha} dV = \int_S \phi n_\alpha dS$$

$$\int_V \mathrm{div}\mathbf{u} dV = \int_S \mathbf{u} \cdot \mathbf{n} dS \qquad \text{or...}$$

$$\int_V u_{\alpha,\alpha} dV = \int_S u_\alpha n_\alpha dS$$

Simple flow models (I)

A vectorfield $\mathbf{a}$ for which div$\mathbf{a} = 0$ is called *solenoidal*
If we assume div$\mathbf{u} = 0$, we can define a 2D vectorfield as

$$\mathbf{u} = \nabla \times \boldsymbol{\psi},$$

where

$$\boldsymbol{\psi} = (0, 0, \psi)$$

ψ is the streamfunction
contourlines of ψ are streamlines!

Simple flow models (II)

A vectorfield **a** for which $\nabla \times \mathbf{a} = 0$ is called *irrotational*
If we assume $\nabla \times \mathbf{u} = 0$,

$$\mathbf{u} = \nabla\phi \qquad (\qquad ....u_\alpha = \phi_{,\alpha})$$

ϕ is the *velocity potential*

The Transport Theorem

The total derivative

$$\frac{D\phi}{Dt} \equiv \frac{\partial}{\partial t}\phi[t, \mathbf{x}(t, \mathbf{y})]$$

The Reynolds's transport theorem

$$\frac{d}{dt}\int_{V(t)} \phi dV = \int_{V(t)} \left(\frac{\partial \phi}{\partial t} + \mathrm{div} \phi \mathbf{u} \right) dV$$

Conservation of mass(I)

The mass conservation equation

$$\frac{d}{dt}\int_{V(t)} \rho dV = \int_{V(t)} \sigma dV = .$$

Use the transport theorem:

$$\int_{V(t)} \left(\frac{\partial \rho}{\partial t} + \mathrm{div}\rho\mathbf{u}\right) dV = 0$$

This holds for every $V(t) \Rightarrow$

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\rho\mathbf{u} = 0$$

Conservation of mass(II)

Incompressible flow

The density of each material particle remains the same:

$$\rho[t, \mathbf{x}(t, \mathbf{y})] = \rho(0, \mathbf{y}).$$

$$\frac{D\rho}{Dt} \equiv \frac{\partial \rho}{\partial t} + u_\alpha \rho_{,\alpha} = 0$$

Hence

$$\mathrm{div}\mathbf{u} = 0$$

Near incompressibility is a *flow* property, not a *material* property

Conservation of momentum(I)

$$\frac{d}{dt}\int_{V(t)} \rho u_\alpha dV = \int_{V(t)} \rho f_\alpha^b dV + \int_{S(t)} f_\alpha^s dS.$$

Transport theorem $\Rightarrow$

$$\int_{V(t)} \left[\frac{\partial \rho u_\alpha}{\partial t} + (\rho u_\alpha u_\beta)_{,\beta}\right] dV = \int_{V(t)} \rho f_\alpha^b dV + \int_{S(t)} f_\alpha^s dS.$$

$f_\alpha^s = \tau_{\alpha\beta} n_\beta$: $\qquad \tau_{\alpha\beta}$ is the stress tensor

Conservation of momentum(II)

Use Gauß theorem to convert the boundary integral:

$$\iiint_{V(t)} \left[\frac{\partial \rho u_\alpha}{\partial t} + \left(\rho u_\alpha u_\beta\right)_{,\beta} \right] dV = \iiint_{V(t)} \left(f_\alpha^b + \tau_{\alpha\beta,\beta} \right) dV.$$

$$\frac{\partial \rho u_\alpha}{\partial t} + \left(\rho u_\alpha u_\beta\right)_{,\beta} = f_\alpha^b + \tau_{\alpha\beta,\beta}$$

Constitutive relation

Newtonian fluid:

$$\tau_{\alpha\beta} = -p\delta_{\alpha\beta} + 2\mu \left(e_{\alpha\beta} - \tfrac{1}{3}\Delta\delta_{\alpha\beta} \right)$$

$$e_{\alpha\beta} = \tfrac{1}{2}\left(u_{\alpha,\beta} + u_{\beta,\alpha} \right) : \quad \text{rate of strain tensor}$$

$$\Delta = e_{\alpha\alpha} = \text{div}\mathbf{u}$$

Special case: The incompressible Navier-Stokes equations with μ constant:

$$\frac{Du_\alpha}{Dt} = -p_{,\alpha} + \mu u_{\alpha,\beta\beta} + \rho f^b_\alpha .$$

$$\frac{D\mathbf{u}}{Dt} = -\nabla p + \mu\nabla^2\mathbf{u} + \rho\mathbf{f}$$

Dimensionless Navier-Stokes equations (I)

Four independent physical units: L, U, M, T_r
$M = \rho_r L^3$; regard T as constant (no temperature dependence).
Dimensionless variables:

$$x' = x/L, \quad u' = u/U, \quad \rho' = \rho/\rho_r.$$

$$\frac{L}{U}\frac{\partial \rho' u'_\alpha}{\partial t} + \left(\rho' u'_\alpha u'_\beta\right)_{,\beta} =$$
$$-\frac{1}{\rho_r U^2} p_{,\alpha} + \frac{2}{\rho_r UL}\left\{\mu\left(e'_{\alpha\beta} - \tfrac{1}{3}\Delta' \delta_{\alpha\beta}\right)\right\}_{,\beta} + \frac{L}{U^2}\rho' f^b_\alpha,$$

$$t' = Ut/L, \quad p' = p/(\rho_r U^2), \quad (\mathbf{f}^b)' = L/U^2 \mathbf{f}^b$$

Dimensionless Navier-Stokes equations (II)

$$\frac{Du_\alpha}{Dt} = -p_{,\alpha} + \frac{\mu}{\rho_r UL} u_{\alpha,\beta\beta} + f_\alpha^b =$$

$$\frac{Du_\alpha}{Dt} = -p_{,\alpha} + Re^{-1} u_{\alpha,\beta\beta} + f_\alpha^b,$$

$$Re \equiv \frac{\rho_r UL}{\mu}$$

The Reynolds number (I)

$$Re \equiv \frac{\rho_r UL}{\mu}$$

Inertia term is of order $\rho_r U^2/L$
Viscous term is of order $\mu U/L^2$
Re is a measure of the ratio of the inertial and viscous forces.
$Re \gg 1$ inertia dominates, $Re \ll 1$ viscous effects dominate.
Both effects balanced by pressure gradient.
Typical values:

$$\frac{\mu}{\rho} = 1.5.10^{-5} m^2/s \text{ (air)}, \quad \frac{\mu}{\rho} = 1.1.10^{-6} m^2/s \text{ (water)}$$

The Reynolds number (II)

In most industrial flow problems $Re \gg 1$:
Aircraft wing in flight: $Re = 2.0.10^7$
Model of same wing in windtunnel: $Re = 2.0.10^6$
Yacht:$Re = 2.0.10^7$
Fish: $Re = 2.0.10^4$
In most conditions where $Re \gg 1$ the flow is *turbulent*

A model equation: The convection-diffusion equation (I)

$$\frac{d}{dt}\iiint_{V(t)} \rho\phi dV = \iint_{S(t)} \mathbf{f}\cdot\mathbf{n} dS + \iiint_{V(t)} q dV$$

$\mathbf{f}$ is the *flux* vector. Fick's law:

$$\mathbf{f} = k\nabla\phi,$$

The convection-diffusion equation:

$$\frac{\partial\rho\phi}{\partial t} + \mathrm{div}(\rho\phi\mathbf{u}) = (k_0\phi_{,\alpha})_{,\alpha} + q.$$

A model equation: The convection-diffusion equation (II)

Four independent physical units: L, U, ρ_r
Dimensionless variables:

$$x' = x/L, \quad u' = u/U, \quad \rho' = \rho/\rho_r.$$

The dimensionless convection-diffusion equation :

$$\frac{\partial \rho\phi}{\partial t} + \mathrm{div}(\rho\phi\mathbf{u}) = (\frac{k_0}{\rho_r UL}\phi_{,\alpha})_{,\alpha} + q =$$

$$\frac{\partial \rho\phi}{\partial t} + \mathrm{div}(\rho\phi\mathbf{u}) = (Pe^{-1}\phi_{,\alpha})_{,\alpha} + q,$$

$$Pe \equiv \frac{\rho_r UL}{k_0} \quad \text{the Péclet number}$$