WI4011
Computational Fluid Dynamics
Assignment 2.2 2016-2017

The completion of Assignment 2.2 is required to pass the exam for WI4011. Provide clear and motivated answers to the questions. Try to use the information available in the book of Prof. Wesseling and the lecture notes. Assignment 2.2 should be handed in (by email or in hard copy) before Februari 19, 23:59. Please do not put this off until the last moment, because it will involve some programming. You are advised but not obliged to work in pairs of two students. If you work in a pair, please state both your names clearly on your work. Preferably use LaTeX. Only well-organised hand-written reports are accepted.

Discretisation in general coordinates

We are going to compute the stationary, incompressible, inviscid and irrotational flow around a circular cylinder. For stationary, inviscid, irrotational and incompressible flow, the velocity field $\boldsymbol{u}$ can be expressed as:

$$u_\alpha = \Phi_{,\alpha}, \tag{1}$$

where Φ is the (total) velocity potential function, defined as:

$$\Phi = U_\infty x_1 + \phi, \tag{2}$$

where U_∞ is the free stream velocity in the x_1 direction and ϕ is the *perturbation* potential. The velocity field is solenoidal, i.e. $u_{\alpha,\alpha} = 0$, because of the incompressibility of the flow field. Furthermore, the flow field and (hence) the distribution of the potential functions, are symmetrical with respect to the x_1 axis.

1. The flow is symmetrical with respect to the x_1 axis, so it is sufficient to consider the solution only for $x_2 \geq 0$.

   Using the definition of the velocity field and the solenoidality constraint, derive/define a second order partial differential and proper boundary conditions for the perturbation potential $\phi(\boldsymbol{x})$ in the domain $G = \{(x_1, x_2) : R_1^2 \leq (x_1)^2 + (x_2)^2 \leq R_2^2, \quad x_2 \geq 0\}$ (Fig. 1(a)).

Choose a homogeneous dirichlet boundary condition for the perturbation potential on $\Gamma = \{(x_1, x_2) : (x_1)^2 + (x_2)^2 = R_2^2, \quad x_2 \geq 0\}$.

How does this choice of boundary condition affect the accuracy of the solution? Is the perturbation potential unique with this boundary condition? Are there alternative boundary conditions for this boundary?

2. Using the expression for the Laplacian in general coordinates :

$$\nabla^2 \phi = \frac{1}{\sqrt{g}} \frac{\partial}{\partial \eta^\beta} \left(\sqrt{g} g^{\alpha\beta} \frac{\partial \phi}{\partial \eta^\alpha} \right), \tag{3}$$

show that for the polar coordinate system $\boldsymbol{\eta} = (\eta^1, \eta^2)$, defined as:

$$x_1 = \eta^2 \cos \eta^1, \tag{4}$$

$$x_2 = \eta^2 \sin \eta^1, \tag{5}$$

the expression (3) simplifies to:

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial \eta^2 \partial \eta^2} + \frac{1}{\eta^2} \frac{\partial \phi}{\partial \eta^2} + \frac{1}{(\eta^2)^2} \frac{\partial^2 \phi}{\partial \eta^1 \partial \eta^1}. \tag{6}$$

Why are there no mixed derivatives in (6)?

3. Define a mapping $\boldsymbol{x}(\boldsymbol{\xi})$ that maps the domain $H = \{(\xi^1, \xi^2) : 0 \geq \xi^1, \xi^2 \leq 1\}$ in logical space to the domain G in physical space (Fig. 1).

4. Derive proper boundary conditions on the domain H, in a way these boundary conditions combined with (3) constitute a boundary value problem for the perturbation potential $\phi(\boldsymbol{\xi})$. Define the boundary conditions in terms of the covariant/contravariant basis vectors and the metric tensor. Do <u>not</u> use the *specific* mapping at this stage. *(Hint: use the fact that the boundaries of the domain are of the form $\xi^\alpha(\boldsymbol{x}) =$ constant, and therefore the unit normal vector on the boundary is given by $\pm \boldsymbol{a}^{(\alpha)}/|\boldsymbol{a}^{(\alpha)}|$).*

5. Write a short MATLAB/PYTHON program for the discretisation of your boundary value problem based on (3) and the chosen boundary conditions. Solve the resulting linear system for the unknown perturbation

potential $\phi(\boldsymbol{\xi}_{ij})$. Include a contour plot of $\phi(\boldsymbol{x}(\boldsymbol{\xi}_{ij}))$ in your report. Use the following parameter values:

$$U_\infty = 1, \tag{7}$$
$$R_1 = 1, \tag{8}$$
$$R_2 = 5. \tag{9}$$

I strongly advise you to make a modular set up: Define functions in MATLAB/PYTHON that recursively compute the necessary quantities in each grid point, e.g. the covariant basis vectors from the mapping, the contravariant basis vectors from the covariant basis vectors, the contravariant metric tensor from the contravariant basis vectors etc.
Hints:

- Do *not* expand the derivatives in (3), but discretize (3) directly.
- Before deciding on the location of the unknowns (vertex-centered or cell-centered), consider which quantities need to be extracted from the numerical solution.
- If you only use the definition of the mapping $\boldsymbol{x}(\boldsymbol{\xi})$ in the definition of the covariant basis vectors and compute all other quantities recursively, you can verify that your discretisation leads to the standard discretisation of the Laplacian in Cartesian coordinates if you temporarily set the mapping to $\boldsymbol{x} = \boldsymbol{\xi}$. This is particularly important to check the handling of the corner control volumes.
- The resulting contourlines for the perturbation potential are symmetrical with respect to the x_2 axis. On those boundaries where a homogeneous Neumann boundary condition is imposed, the contourlines should be orthogonal to the boundary (why?).
- It is not a good idea to proceed with the computation of the velocity field in the following questions before you are quite positive the perturbation potential is correct.

6. Formulate the velocity field u_α in terms of the total potential $\Phi(\boldsymbol{\xi})$ and its derivatives. Include a vector (quiver) plot of $\boldsymbol{u}(\boldsymbol{x}(\boldsymbol{\xi}_{ij}))$ in your report.

7. Make a line plot of the magnitude of the computed tangential distribution on the surface of the cylinder and the exact solution[1] for three different grid sizes.

8. Use *Richardson extrapolation* to estimate the order of convergence of the value of the magnitude of the tangential velocity on the top of the cylinder. Make sure you consider the velocity *on* the surface, not in the cell centers of the control volumes adjacent to the surface.

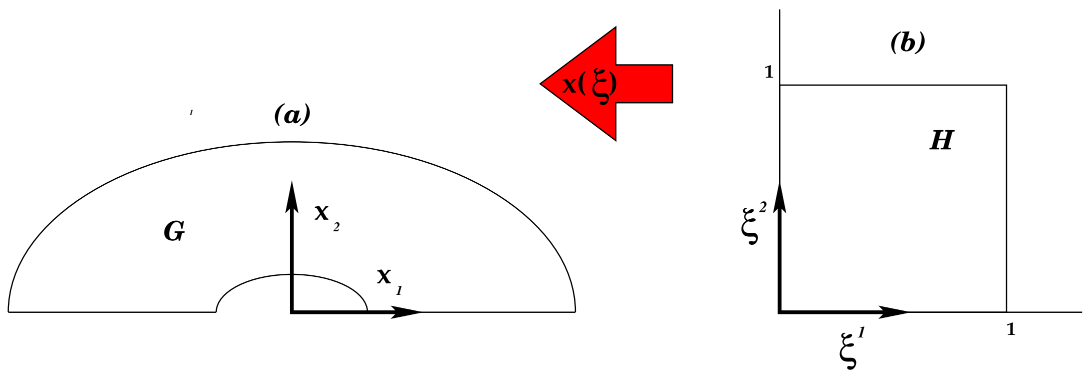

Figure 1: The domain in physical space (a) and in logical space (b).

[1]The exact solution for potential flow around a cylinder *in an unbounded domain* can be found in many reference books or online.