WI4011
Computational Fluid Dynamics
Assignment 1.2 2016-2017

The completion of Assignment 1.2 is required to pass the exam for WI4011. Provide clear and motivated answers to the questions. Try to use the information available in the book of Prof. Wesseling and the lecture notes. Assignment 1.1 should be handed in (by email or in hard copy) before October 23, 23:59. You are advised but not obliged to work in pairs of two students. If you work in a pair, please state both your names clearly on your work. Preferably use LaTeX. Only well-organised hand-written reports are accepted.

The Method of Manufactured Solutions

We study discretization schemes for the following boundary value problem for the one-dimensional constant coefficient stationary convection diffusion equation defined as:

$$\begin{aligned} \phi_{,1}(x_1) - \epsilon\phi_{,11}(x_1) = q(x_1), \quad x_1 \in \Omega = \langle 0, 1\rangle, \\ \phi(0) = a, \phi(1) = b, \end{aligned} \tag{1}$$

where $0 < \epsilon \ll 1$ and $0 < a, b \in \mathbb{R}$.

- Assume $q(x_1) \equiv 0$. Write a small program for the discretization of (1). Your discretization should be based on a cell-centered finite volume discretization on a uniform mesh with a central approximation of the convective and diffusive fluxes. Use the exact solution of (1) to confirm the numerical solution ϕ_j converges with error $e_{L_2} = \mathcal{O}(h^2)$ to $\phi(x_j)$ for $j = 1 \ldots J$, where $h = J^{-1}$ and J is the number of control volumes. The error e_{L_2} is defined as:

$$e_{L_2} = J^{-\frac{1}{2}} \sqrt{\sum_{j=1}^{J} (\phi_j - \phi(x_j))^2}. \tag{2}$$

  You can use the sample MATLAB™ program available on Blackboard™ for inspiration/starter.

- In many cases analytical solutions are only available for very simple test cases. Luckily, there is the *Method of Manufactured Solutions* [1] for verification of your program. In this approach you will *define your solution* and approximate this self-defined solution numerically.

  Suppose we want to verify the implementation of a numerical approximation L_h of the second order differential operator L. Furthermore, we have theoretically determined the order p of the global error of numerical approximations to solutions to $L\phi = 0$ obtained with L_h.

  We can proceed as follows: First choose a (simple) function $\phi_{\text{man}}(x,t) \in C^2$ and define $q(x,t) := L\phi_{\text{man}}(\boldsymbol{x},t)$. Naturally, $u(x,t) = \phi_{\text{man}}(x,t)$ will be a solution of the differential equation $Lu(x,t) = q(x,t)$. The discrete approximation of the latter differential equation is given by $L_h u_i = q_i$ and $u_i = \phi_{\text{man}}(x_i) + Ch^p$ if we make sure that the approximation $q_i = q(x_i)$ is sufficiently accurate to not be decisive for the order of the global error.

  If the solution of (1) is given by:

$$\phi_{\text{man}}(x_1) = \alpha \cos\left(\frac{x_1\pi}{2}\right) + \beta(1 - e^{x_1}), \tag{3}$$

  - How should α and β be chosen to make $\phi_{\text{man}}(x_1)$ comply with the boundary conditions of (1)?
  - How should $q(x_1)$ be defined to make $\phi_{\text{man}}(x_1)$ a solution of (1)?

- Incorporate the nonzero $q(x_1)$ in your program and establish experimentally that the numerical solution converges to (3) with $e_{L_2} = \mathcal{O}(h^2)$. Make sure you choose $q(x_1)$ in relation to the value of a and b in a way it has a significant effect on the solution!

Overlapping grids

In the lecture we have discussed a Pécèt uniform method based on local grid refinement. Another commonly used approach is to use overlapping grids: a boundary conforming grid with uniform mesh width h and a background grid with uniform mesh width H, with $h \ll H$. We will refer to the mesh

[1] It is good practice to apply this technique for verification of *every* numerical scheme you code, e.g. in your MSc thesis project

with mesh width h as the *fine* grid and the mesh with mesh width H as the *coarse* grid. Both grids do not share a common face, but overlap each other. The solution on one grid is used as a boundary condition for the other grid and vice versa. To make the solution available at the necessary location (on the other grid), it is interpolated from the cell center values. The lay-out of the problem is shown in Figs. 1 and 2. We want to solve the following boundary value problem using this overlapping-grids or *Chimera* method:

$$\begin{aligned} -\phi_{,11}(x_1) &= \sin(x_1\pi), \quad x_1 \in \Omega = \langle 0, 1\rangle, \\ \phi(0) &= a, \phi(1) = b, \end{aligned} \tag{4}$$

where $0 < a, b \in \mathbb{R}$.

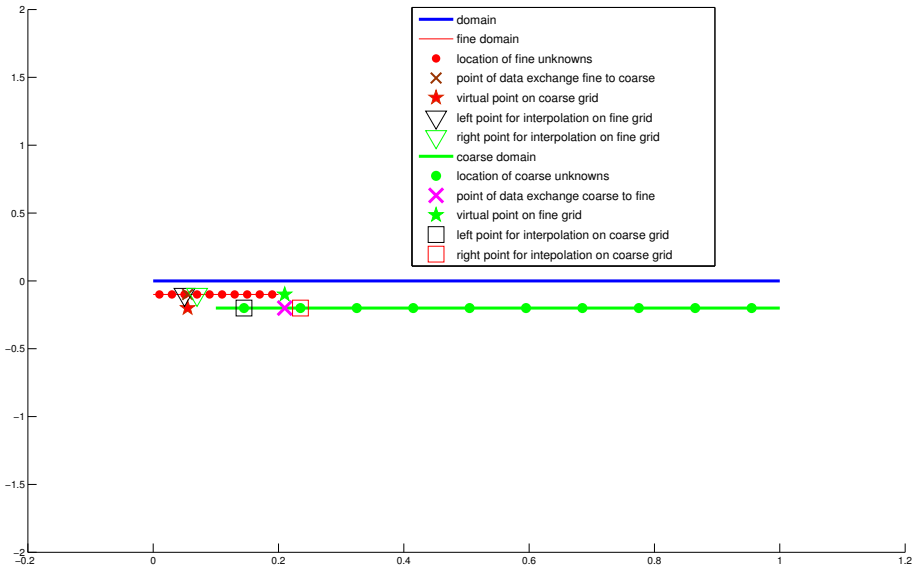

Figure 1: Layout of the overlapping grids.

- We use a cell centered finite volume discretisation with a central approximation to the (diffusive) fluxes. Assume the number of control volumes in the fine grid is J and in the coarse grid is K, and the grids show sufficient overlap to make the exchange of the solution between the two grids possible. The numerical approximations on the fine grid and coarse grid are given by $\phi_j = \phi(x_j) + \mathcal{O}(h^p)$, $j = 1 \ldots J$ and

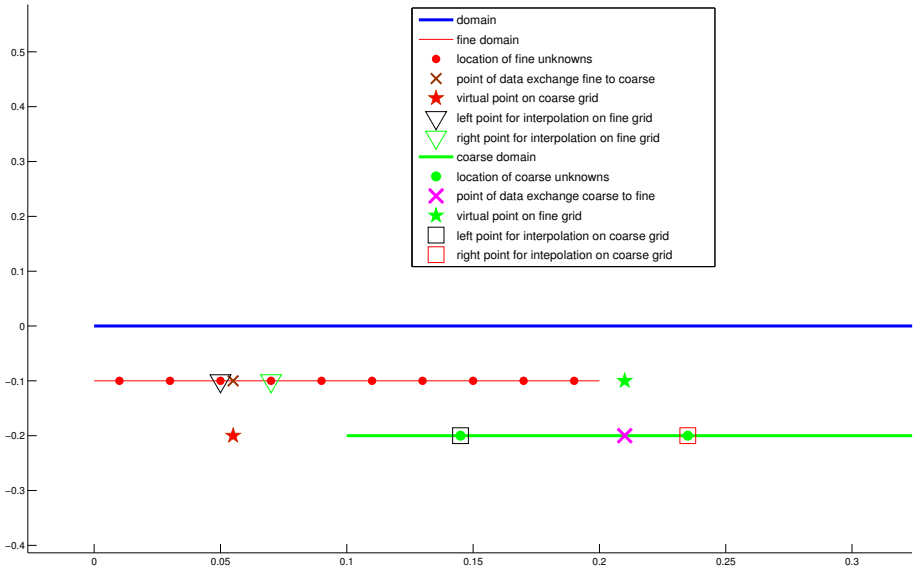

Figure 2: Zoom of the layout of the overlapping grids.

$\Phi_j = \phi(X_j) + \mathcal{O}(H^p),\ j = 1 \ldots K$, respectively. The values ϕ_{J+1} and Φ_0 are exchanged between the two grids, using linear interpolation. Assume interpolation from the fine to the coarse grid and vice versa can be symbolically prescribed as:

$$\begin{aligned} \phi(x_j) = \beta_0^C \Phi(X_j) + \beta_1^C \Phi(X_j + 1) = \\ \Phi(x_j) + \mathcal{O}(H^2),\ x_j \in [X_j, X_{j+1}]\,, \\ \Phi(X_j) = \beta_0^F \phi(x_j) + \beta_1^F \phi(x_{j+1}) = \\ \phi(X_j) + \mathcal{O}(h^2),\ X_j \in [x_j, x_{j+1}]\,, \end{aligned} \tag{5}$$

where $\beta_i^{C/F} \in [0, 1]$. Determine the local truncation error in the numerical scheme for $\phi_j = \phi(x_j),\ j = 1 \ldots J$ and $\Phi_j,\ j = 1 \ldots K$ (6 cases). Refer to the results in the lecture notes if possible.

- After discretisation the problem can be described by the following linear system:

$$A\boldsymbol{\Phi} = \boldsymbol{r},\ \boldsymbol{\Phi} = [\phi_1, \phi_2, ..\phi_J, \Phi_1, \Phi_2, ..\Phi_K]^T. \tag{6}$$

The matrix $A \in \mathbb{R}^{(J+K)\times(J+K)}$ has the following block structure:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \tag{7}$$
$$A_{11} \in \mathbb{R}^{J\times J},\ A_{12}, A_{21}^T \in \mathbb{R}^{J\times K},\ A_{22} \in \mathbb{R}^{K\times K}.$$

Give a complete description of the blocks of the matrix A and the vector $\boldsymbol{r}$.

- Write a small program to compute an approximation to the solution of (4) using the overlapping grid method. Investigate different configurations of the overlapping meshes and show that in the case of *minimal* overlap the order of the global error of the solution is only $\mathcal{O}(h)$.
- There seem to be two possible options to consider for raising the accuracy back to second order:
  - Raise the order of the interpolation,
  - interpolate the approximation to the flux $\phi_{,1}$ instead of the approximation to the solution.

  Discuss what you can expect of either approach. Bonus points for experimental evidence that supports your claims!