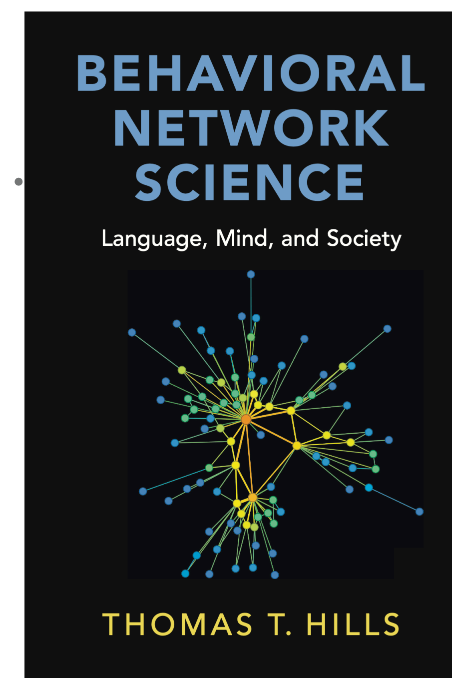
"I always dream of a pen that would be a syringe." — Jacques Derrida

BEHAVIORAL NETWORK SCIENCE
LANGUAGE, MIND, AND SOCIETY



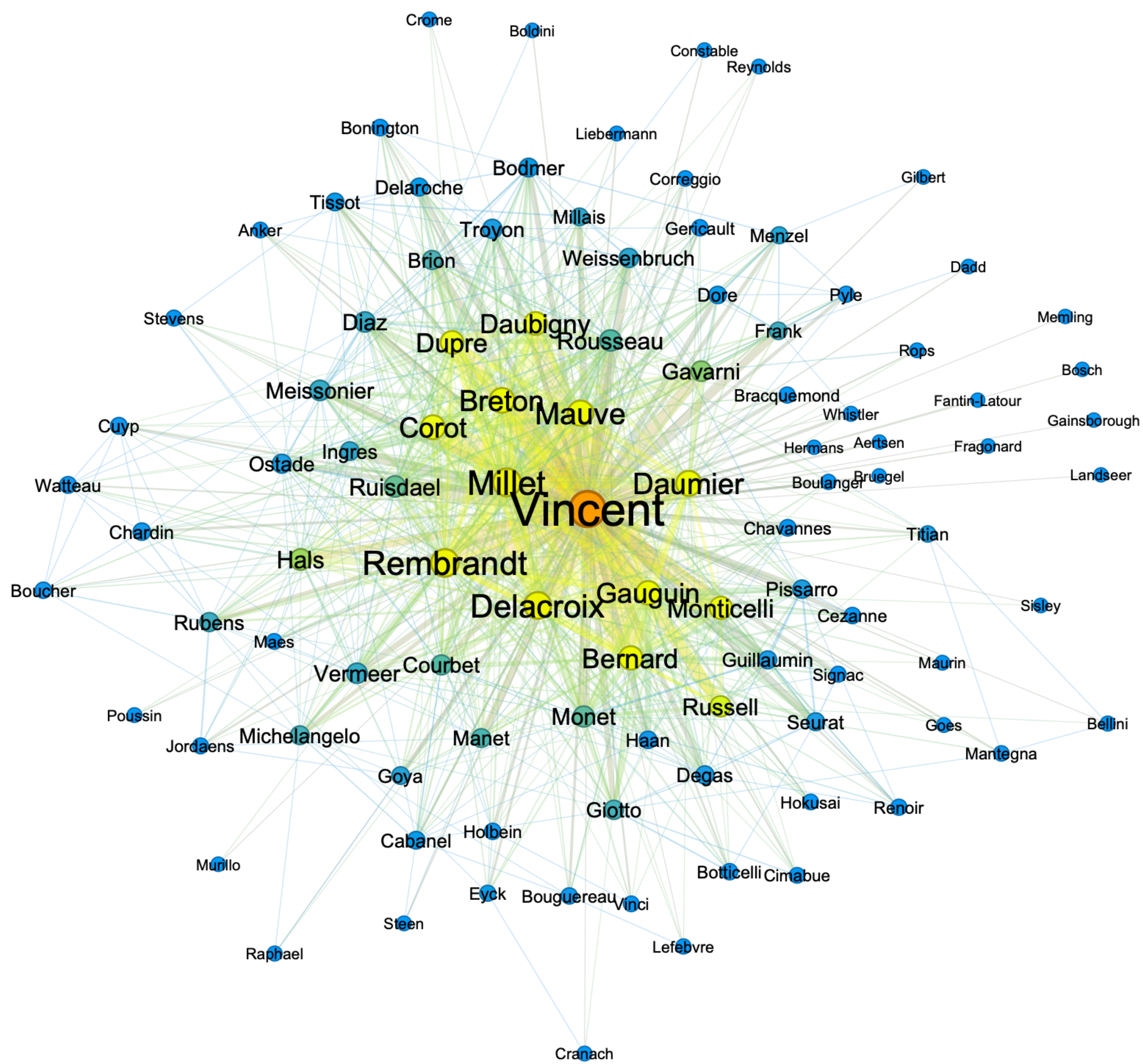
Thomas Hills
University of Warwick

CogSci 2025

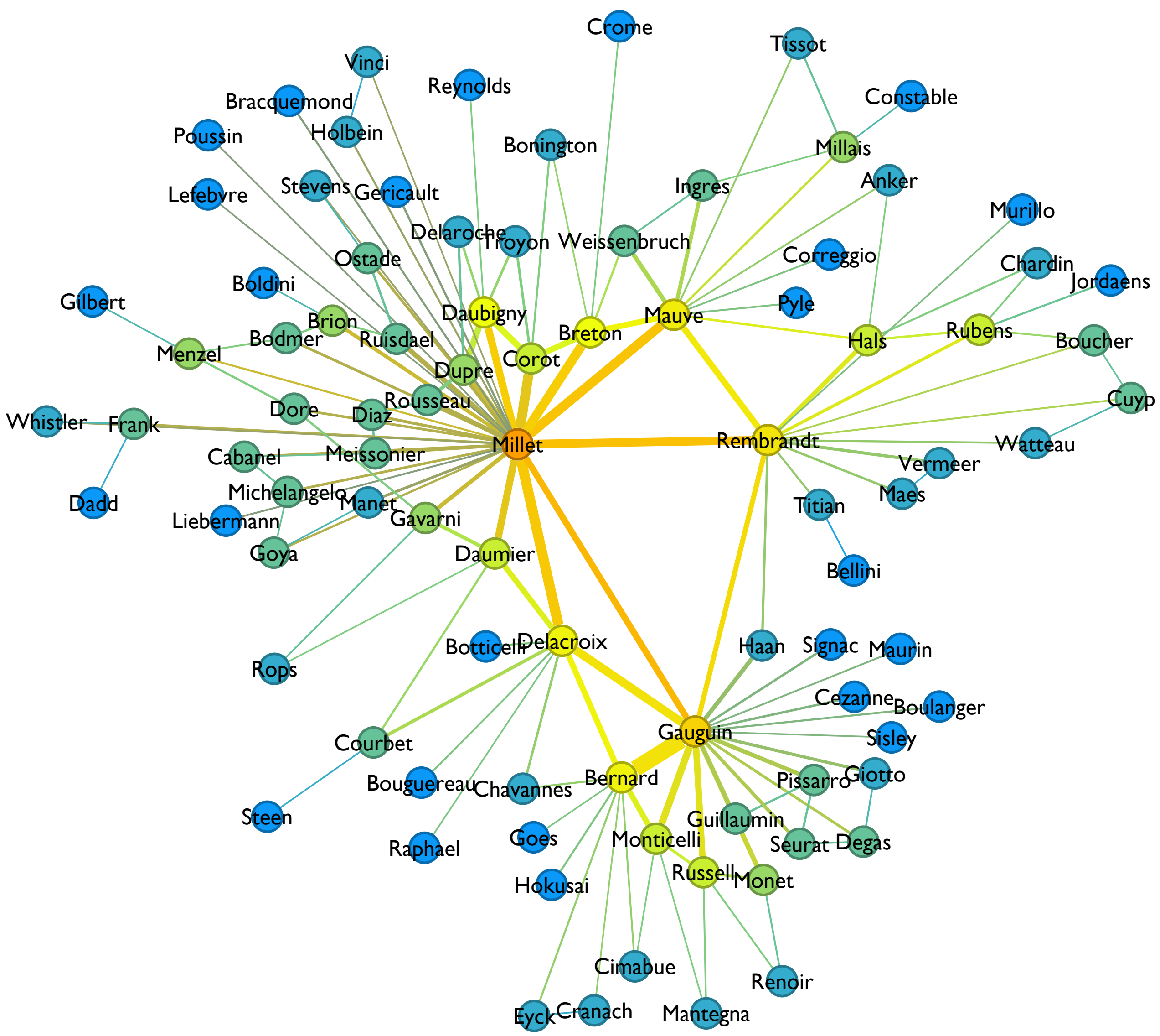
WARWICK
THE UNIVERSITY OF WARWICK

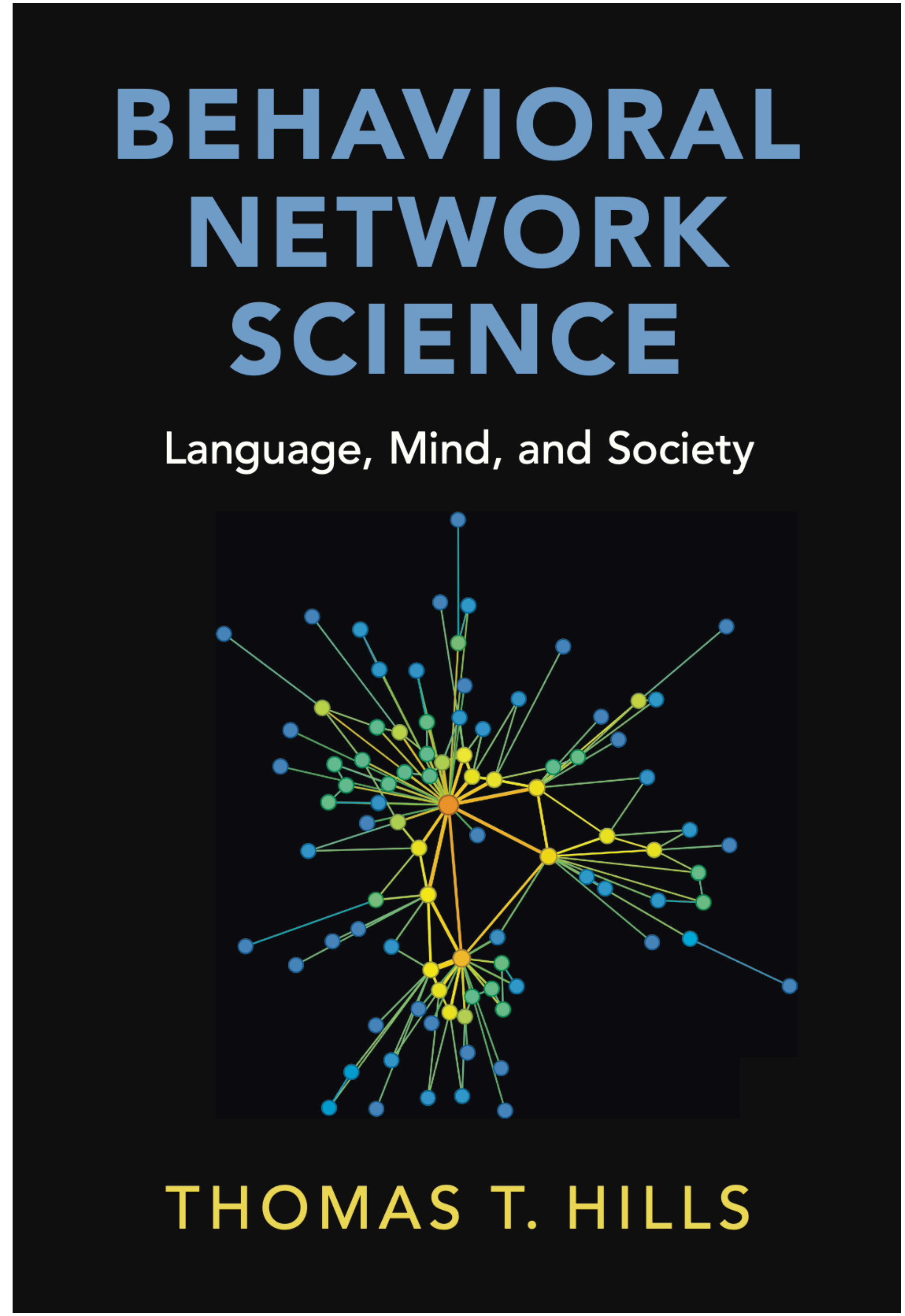
The Alan Turing Institute

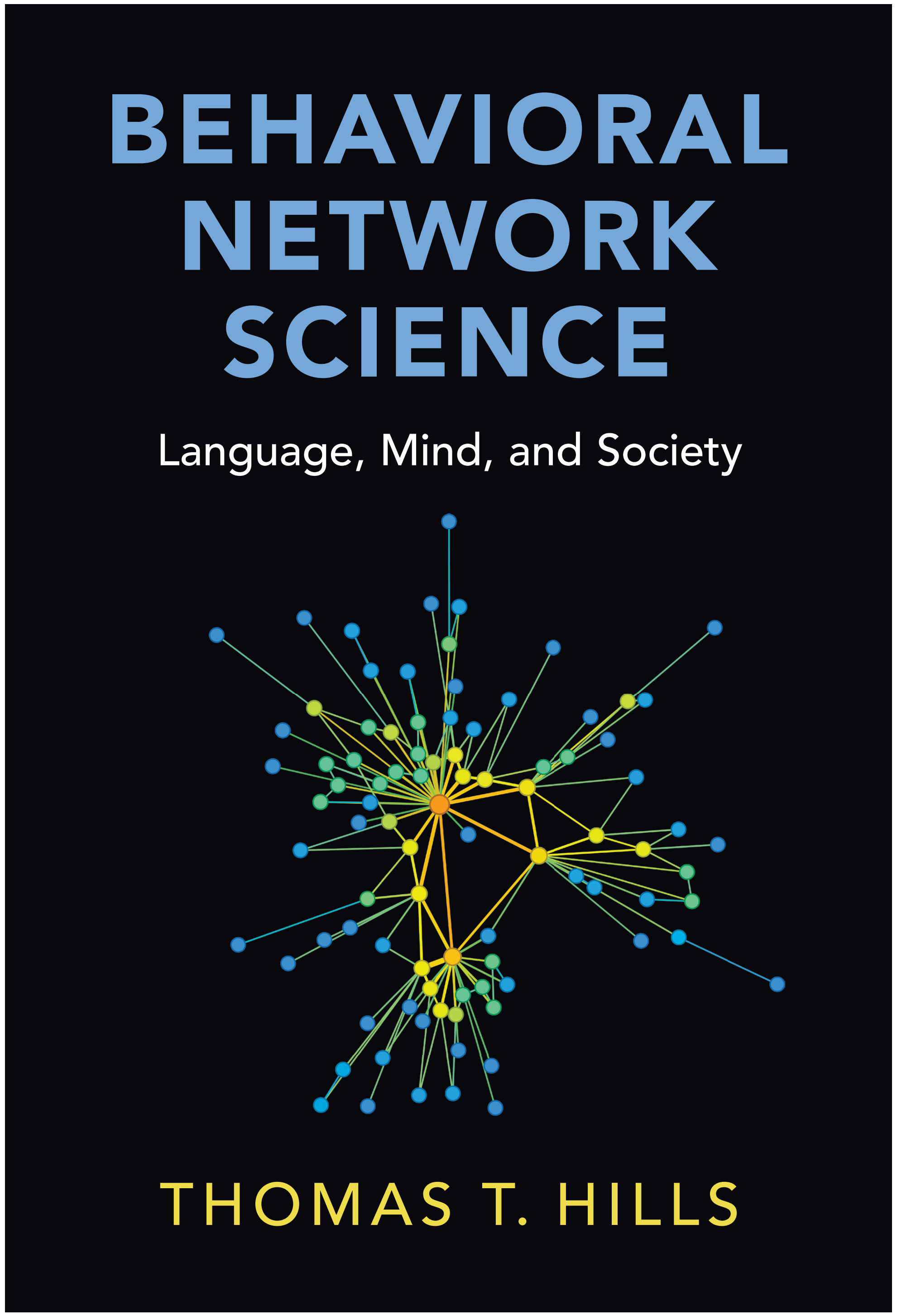
THE ROYAL SOCIETY



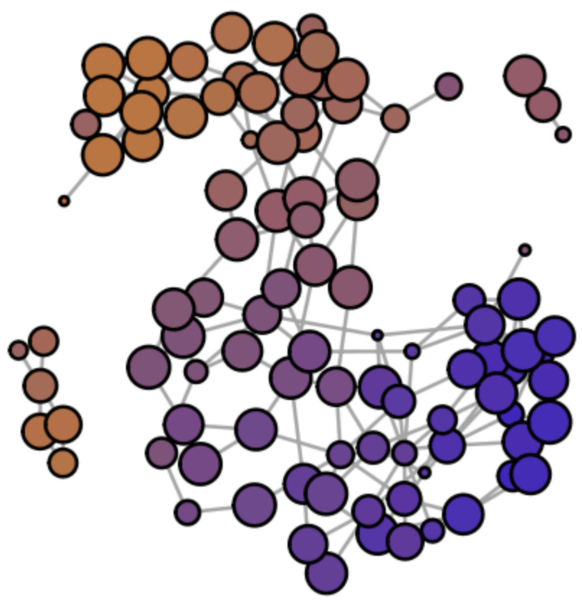
Structure is beautiful


Crome
Tissot
Vinci
Reynolds
Constable
Bracquemond
Holbein
Poussin
Bonington
Millais
Ingres
Anker
Stevens
Gericault
Lefebvre
Murillo
Delaroche
Troyon
Weissenbruch
Correggio
Ostade
Chardin
Jordaens
Boldini
Pyle
Gilbert
Daubigny
Mauve
Brion
Breton
Hals
Rubens
Boucher
Bodmer
Ruisdael
Menzel
Corot
Dupre
Rousseau
Cuyp
Dore
Diaz
Whistler
Frank
Millet
Rembrandt
Watteau
Cabanel
Meissonier
Vermeer
Maes
Michelangelo
Dadd
Manet
Titian
Liebermann
Gavarni
Goya
Daumier
Bellini
Rops
Botticelli
Delacroix
Haan
Signac
Maurin
Cezanne
Boulanger
Gauguin
Sisley
Courbet
Bernard
Pissarro
Giotto
Bouguereau
Chavannes
Guillaumin
Steen
Goes
Monticelli
Seurat
Degas
Raphael
Russell
Monet
Hokusai
Cimabue
Renoir
Eyck
Cranach
Mantegna

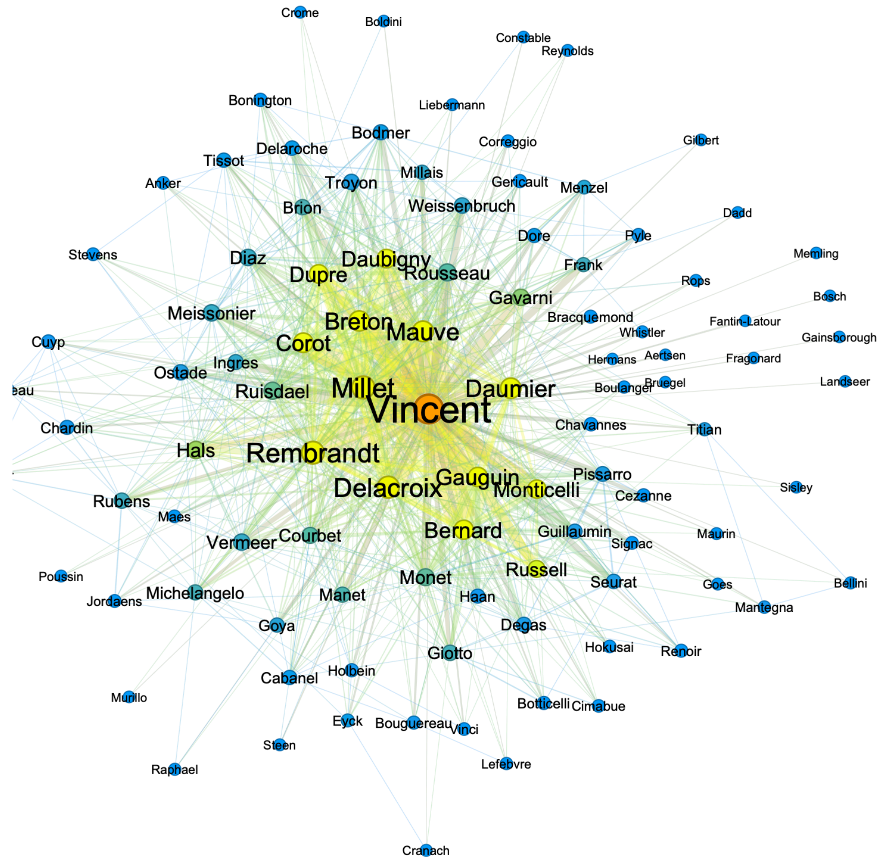

BEHAVIORAL
NETWORK
SCIENCE
Language, Mind, and Society
THOMAS T. HILLS



Language Evolution
Word births
Memory Structure
False memories
Memory Search
Aging
Creativity
Group Problem Solving
Conflict
Network Illusions
Conspiracy beliefs
Agent-based modeling
Turning text into networks

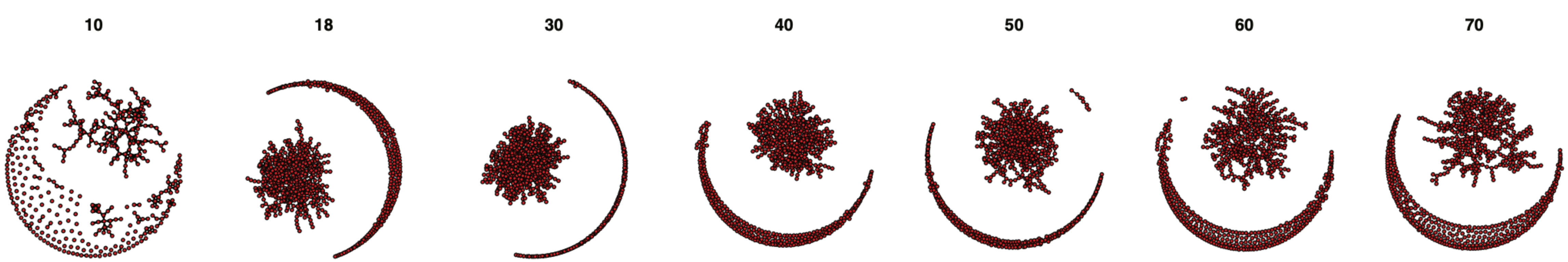


Resource flows
Self-organization
Knowledge development
Conflict
Creativity and Innovation
Language evolution
Social Influence
Economics

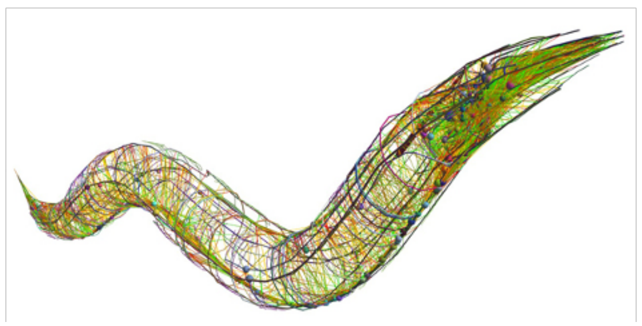


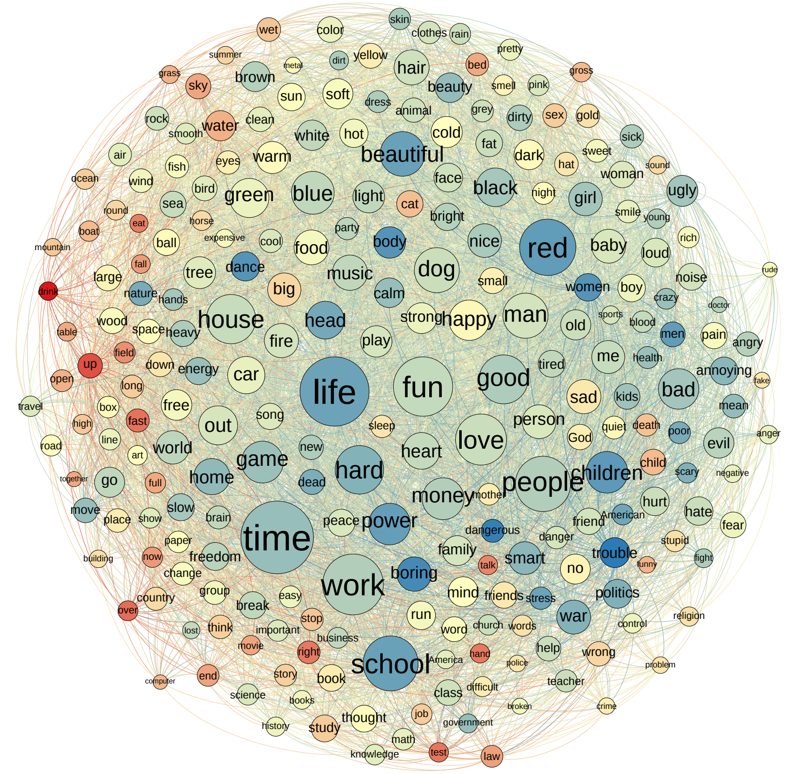
complex systems

Language acquisition
Memory Structure
False memories
Memory Search
Aging
Group Problem Solving
Network Illusions
Conspiracy beliefs

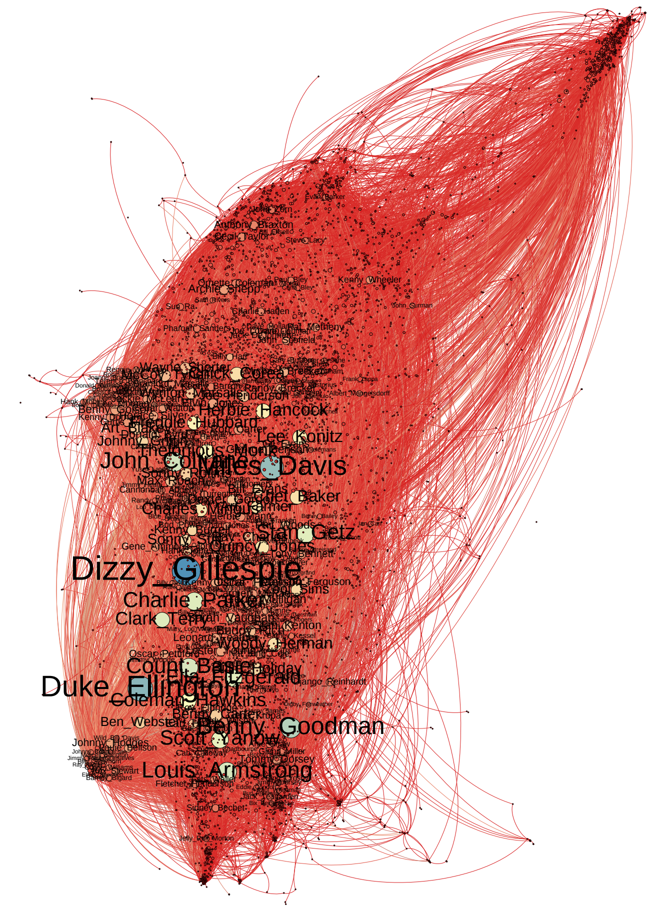


Nematode neurons



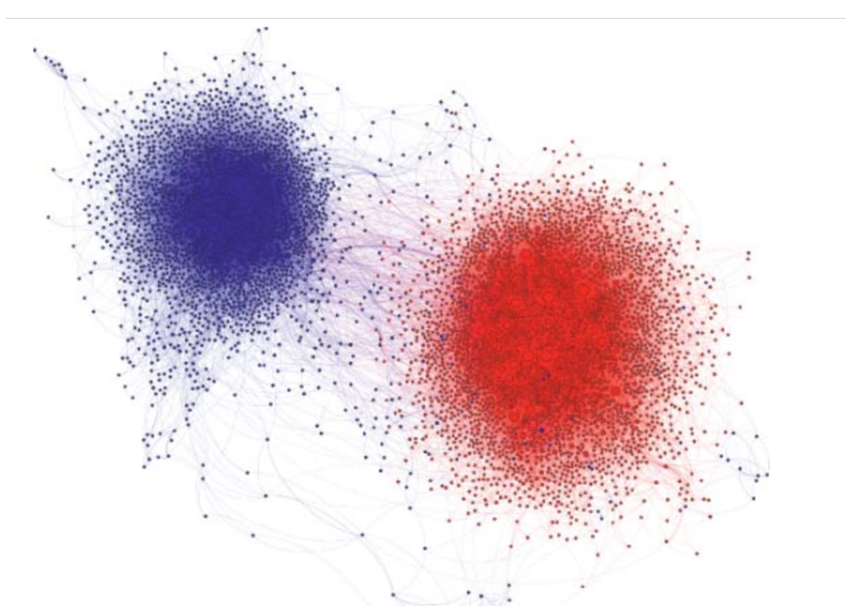


Free associations

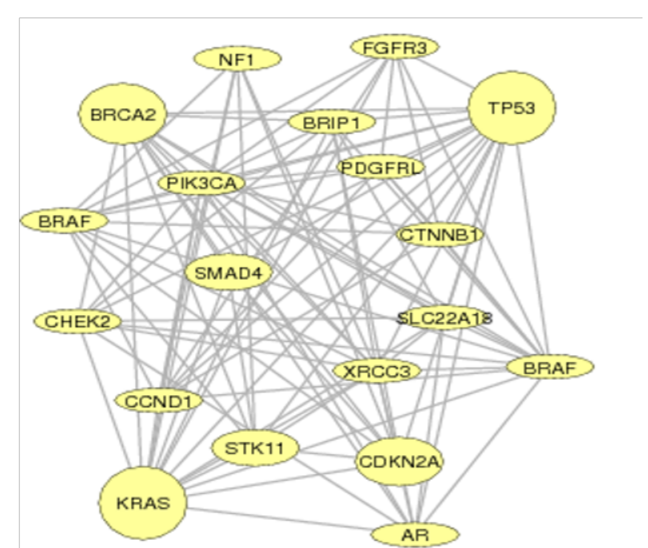


Jazz musicians

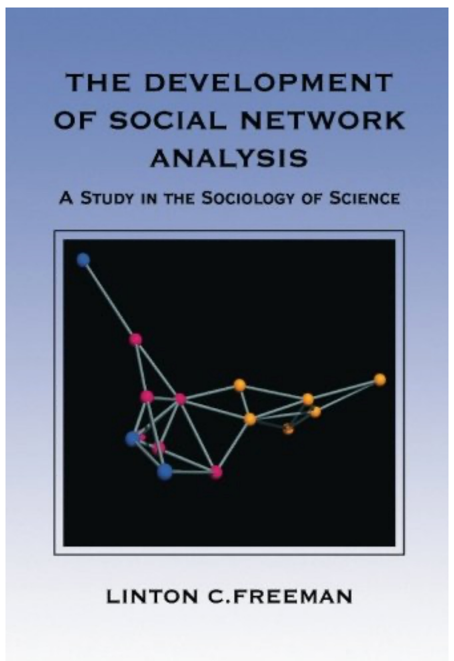
complex systems

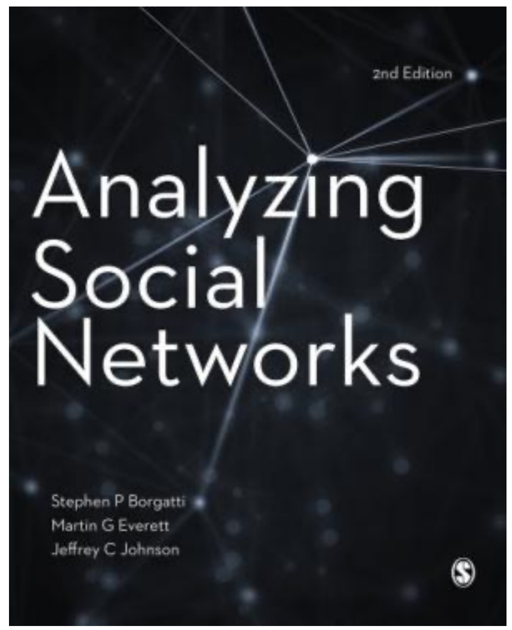


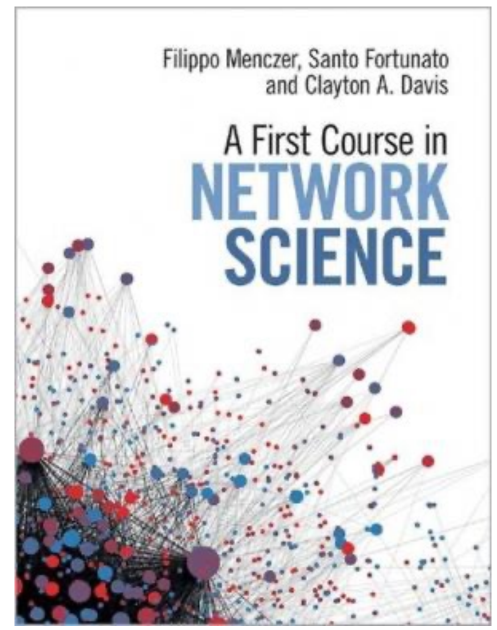
Twitter connections

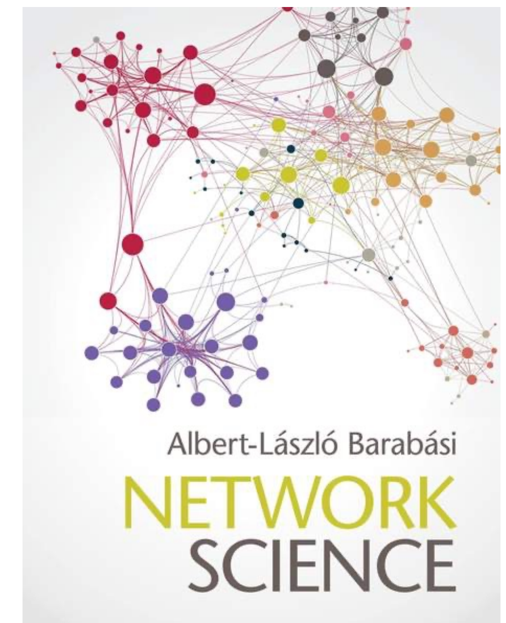


gene networks

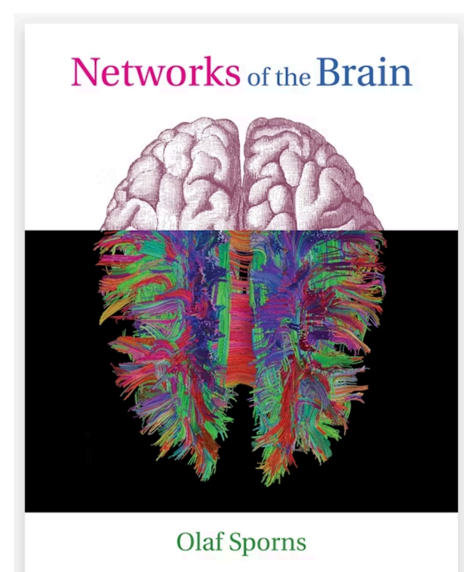








Sociologists **Physicists**



Outline for the workshop

- Part 1 — Introduction, Network basics, representations
- Part 2 — Measuring things on networks
- Part 3 — Generating networks, null hypotheses
- Part 4 — Models and processes on networks

online folder

https://github.com/thomasthills/cogsci2025

This has all the Rcode, slides, and data files.

If there's anything you'd like that you don't have, let me know.

My background

- **Computational Social Sciences (Cognitive Science):** learning, memory, language evolution, aging

- **English (BA) and Biology (BS), Biology (PhD):**
  - mathematical biology and neuroscience of cognitive control
  - process and environment (structure)

- **Methods:**
  - **network analysis** (lexical and cognitive structure)
  - **computational modeling** (explanatory process models)
  - **natural language processing** (derive representations from language)

- Part 1 — Introduction, Network basics, representations

Some examples of why we should care about structure?

Structure can tell us what kinds of systems we're dealing with.

Why is this social network so disconnected?

- Krebs (2002) mapped the network structure of the 9/11 terrorist cells---identifying each individual and the relationships among them.
- What he observed was peculiar for a social network. The 19 hijackers had sparsely interacted. This is rare among social networks. Our friends' friends tend to become our friends. Close-knit groups form from like-minded individuals. And coordinated groups need coordinated communication.
- On the face of it, the structure of the terrorist cells lacked these features. Why?

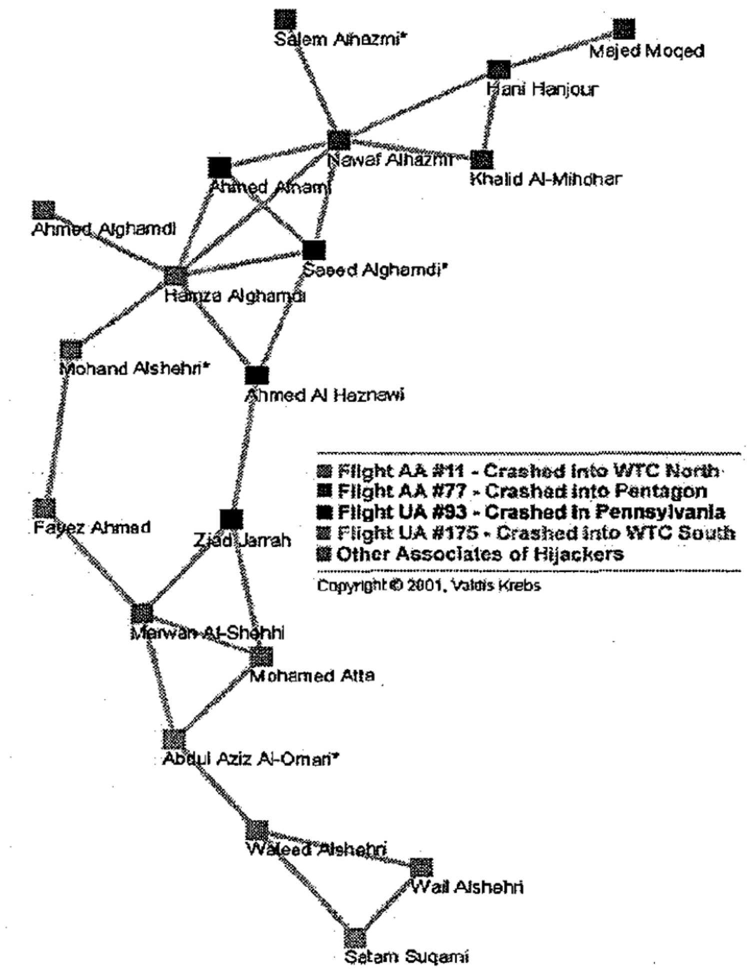


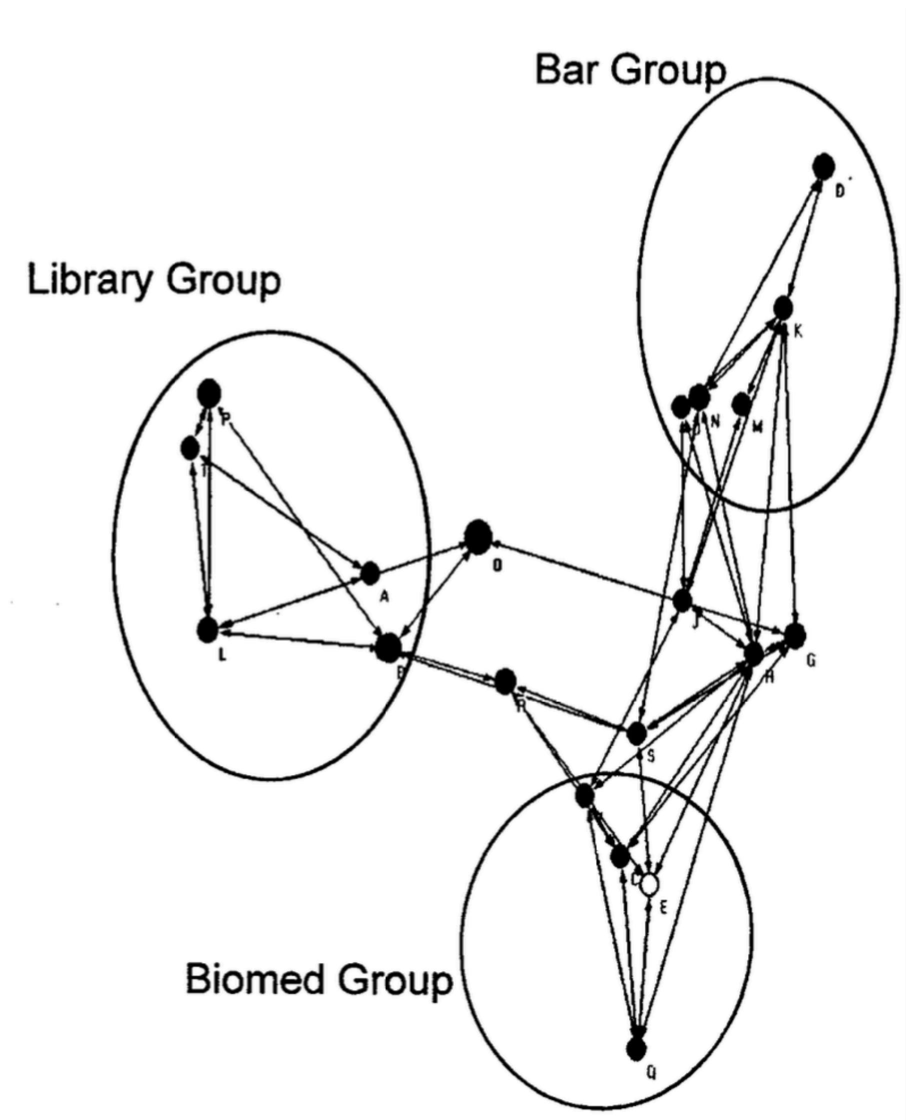
Figure 2 Trusted Prior Contacts

Social networks have a Behavioral Immune System

Structure can help us understand how to make systems better

How to successfully overwinter at the South Pole

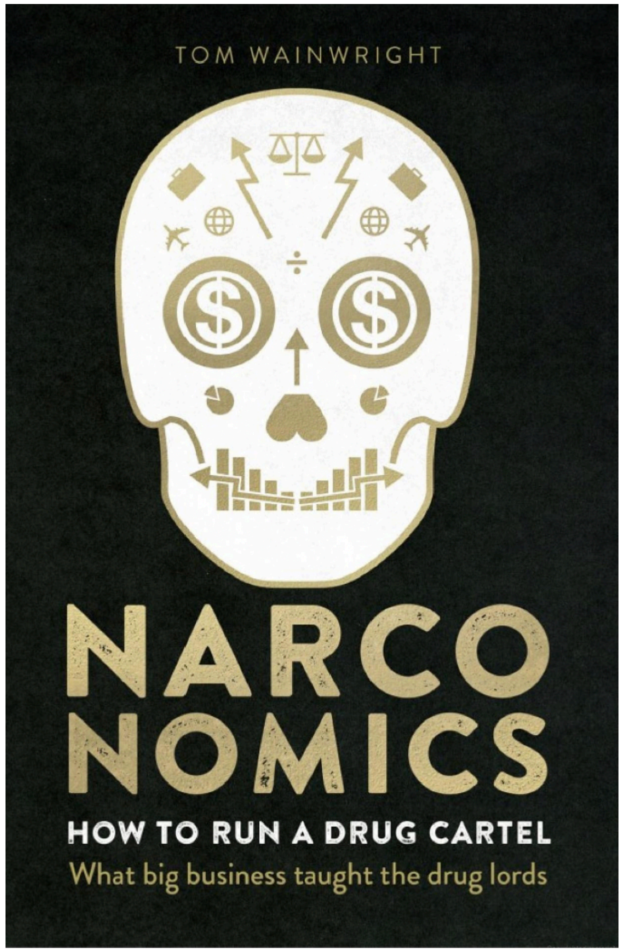
- Johnson et al. (2003)'s pioneering research on overwintering teams in the South Pole.
- These teams stayed 9 months inside small spaces with one another under harsh conditions.
- What holds communities together?
- Expressive leaders—coordinated social interactions to help keep the communities connected.
- Positive deviant—i.e., the clown — violated social boundaries and therefore kept the communities connected.



Unhappy Group C: The lowest coherence and lowest agreed expressive leadership and positive deviants.

Structure can help us predict what’s important and why

What made Juarez the murder capital of the world from 2008-2011?

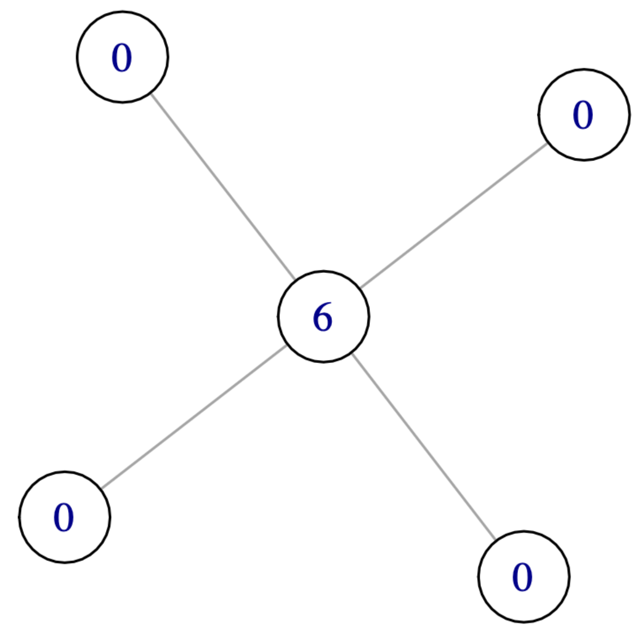


Why is this location important?

Betweenness

Drugs pass through Juarez. The Juarez and Sinaloa cartels began fighting over control of this border because President Calderon led a crackdown on cartels that destabilized the Juarez cartel, which in turn led El Chapo and the Sinaloa to try and take control.

What made Juarez the murder capital of the world from 2008-2011?



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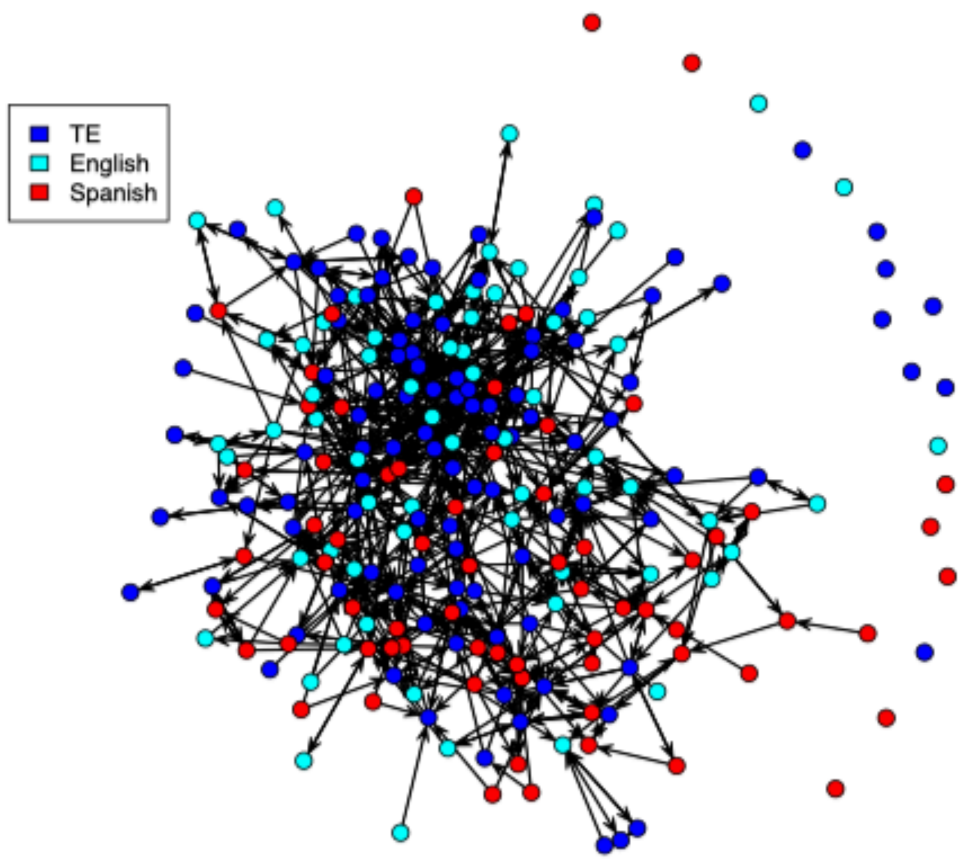
Betweenness

Drugs pass through Juarez. The Juarez and Sinaloa cartels began fighting over control of this border because President Calderon led a crackdown on cartels that destabilized the Juarez cartel, which in turn led El Chapo and the Sinaloa to try and take control.

Structure can help us understand process

How do bilingual first language learners learn languages?



Bilingual first language learners

TE: translational equivalent

Study of 181 bilingual first language learners learning two languages at once.

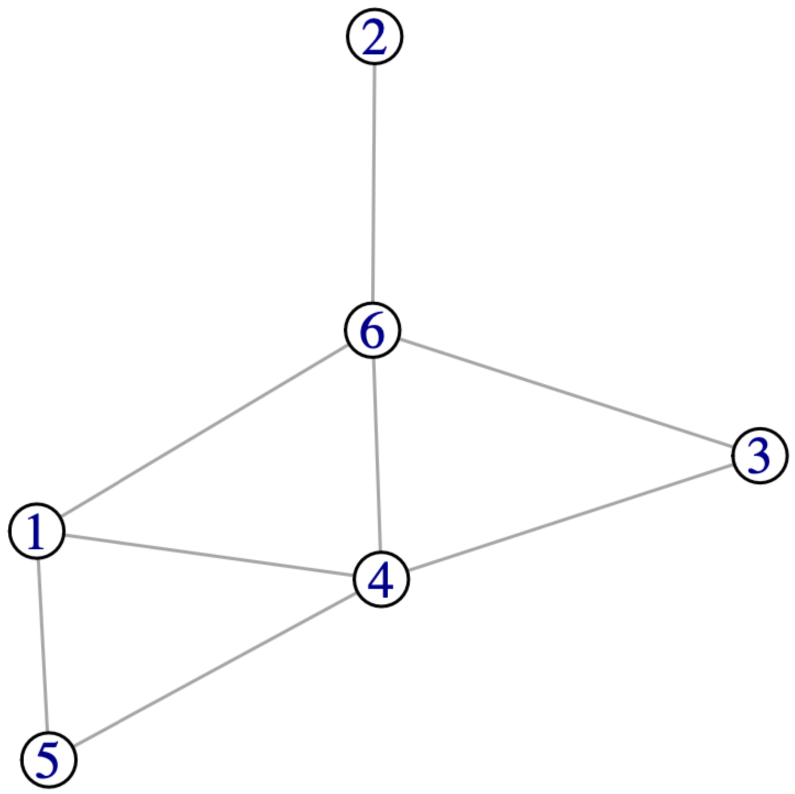
We developed a model to understand how monolingual language learners learned languages.

Then we asked how bilinguals were different from a monolingual learning two independent languages at once.

Result: Semantic facilitation

Bilson et al., 2015

Network Basics



A basic network

Nodes and Edges

- What are nodes/vertices?
- What are edges/links?
- Think about this for a system that's important to you.

Some examples of nodes and edges

TABLE 1: Examples of cognitive networks and their cognitive application.

Cognitive Network	Nodes	Edges	Relevant research areas
Semantic network	Words	Semantic relationships, including free associations, shared features, taxonomic, cooccurrence, semantic roles	Language acquisition; cognitive aging; semantic priming; creativity/insight; cognitive search and navigation; semantic memory
Form similarity network	Words	Phonological or orthographic similarity	Lexical retrieval; production; speech errors; memory recall; word learning
Syntactic network	Words; phrases; sentences	Cooccurrence; parse trees; syntactic dependencies	Language acquisition; language evolution; syntactic learning
Concept network	Concepts; ideas	Cooccurrence; causal; feature similarity	Learning; memory; concept formation
Informational network	Shapes; pictures; any unit of information	Temporal cooccurrence; communication; transmission	Statistical learning of external structure; information transmission
Clinical, personality networks	Symptoms; personality traits; items on a questionnaire	Statistical relationship such as partial correlations; comorbidity	Clinical psychopathology; personality disorders
Social network	People	Friendship; followers on social media; face to face interactions	Collective problem solving; decision making; echo chambers; polarization

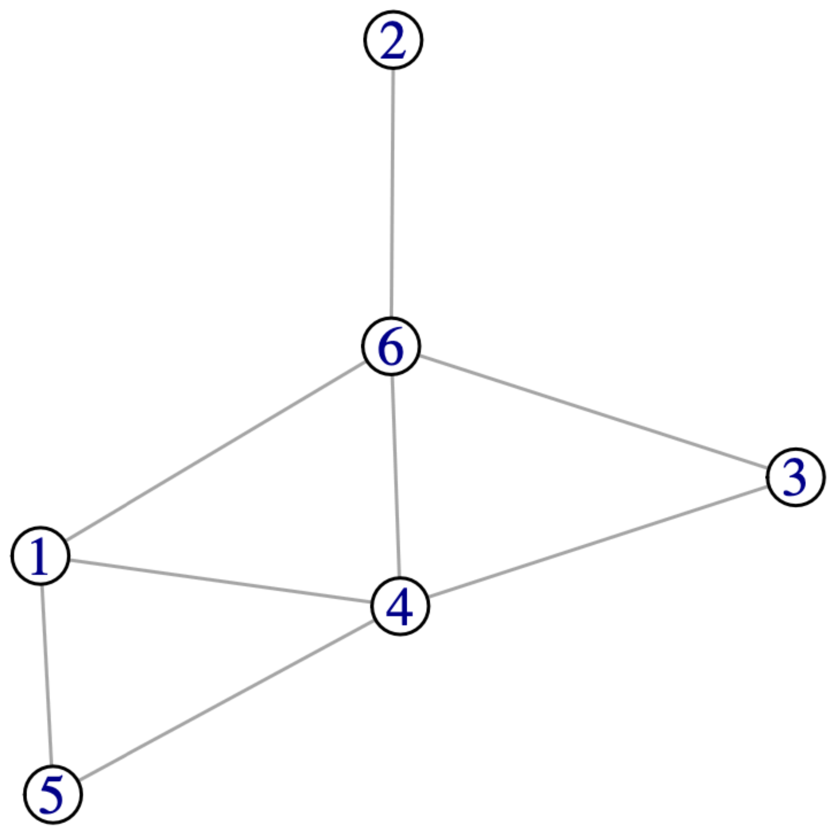
There are many more.

Siew et al., 2019

Representing a simple network

Edge list

V1	V2
1	4
3	4
1	5
4	5
1	6
2	6
3	6
4	6



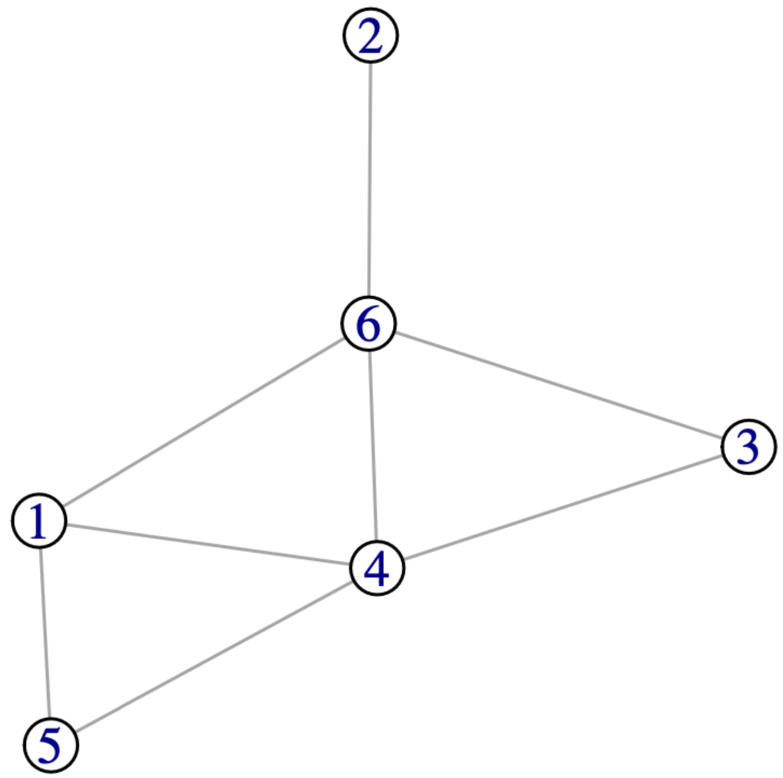
Representing a simple network

Edge list

V1	V2
1	4
3	4
1	5
4	5
1	6
2	6
3	6
4	6

Adjacency Matrix

	1	2	3	4	5	6
1	0	0	0	1	1	1
2	0	0	0	0	0	1
3	0	0	0	1	0	1
4	1	0	1	0	1	1
5	1	0	0	1	0	0
6	1	1	1	1	0	0



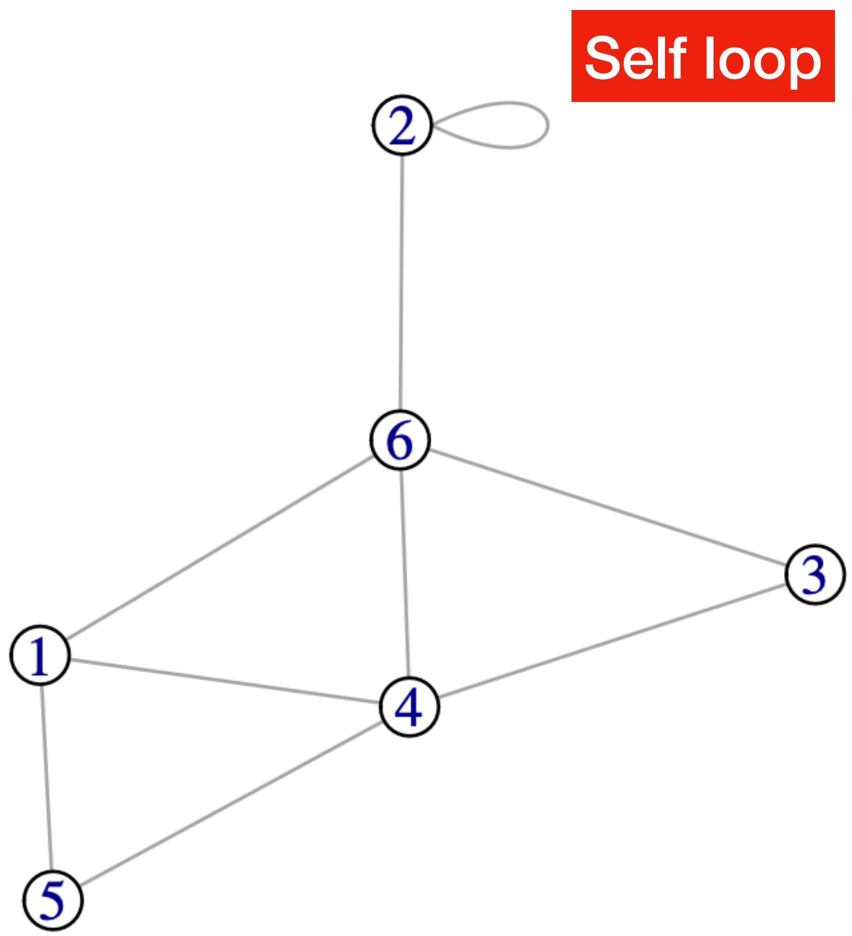
These both contain all the information we need to construct this network.

Why might one be better than another?

Sparse matrices

Self-loops

Self-loops are connections from a node to itself



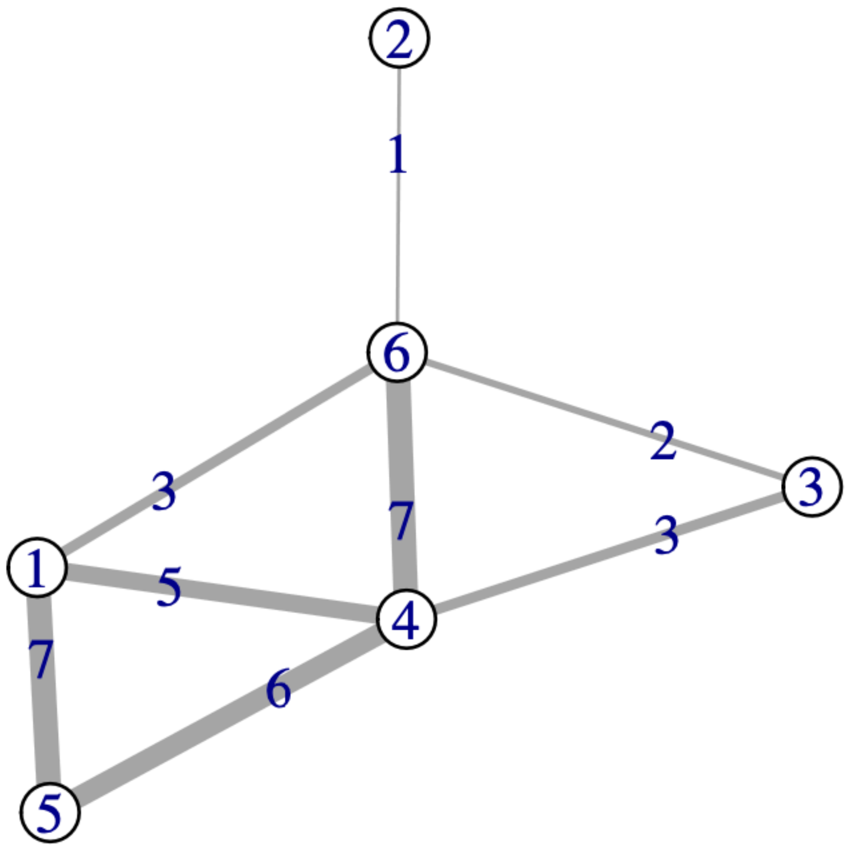
	1	2	3	4	5	6
1	0	0	0	1	1	1
2	0	1	0	0	0	1
3	0	0	0	1	0	1
4	1	0	1	0	1	1
5	1	0	0	1	0	0
6	1	1	1	1	0	0

Diagonal

What might self-loops be good for?
Imagine we all bring cake to share, but some people eat their own cake.

if you plot something with a lot of these loops, you can easily remove them with **igraph::simplify(g)**

Weighted Networks



Adjacency Matrix

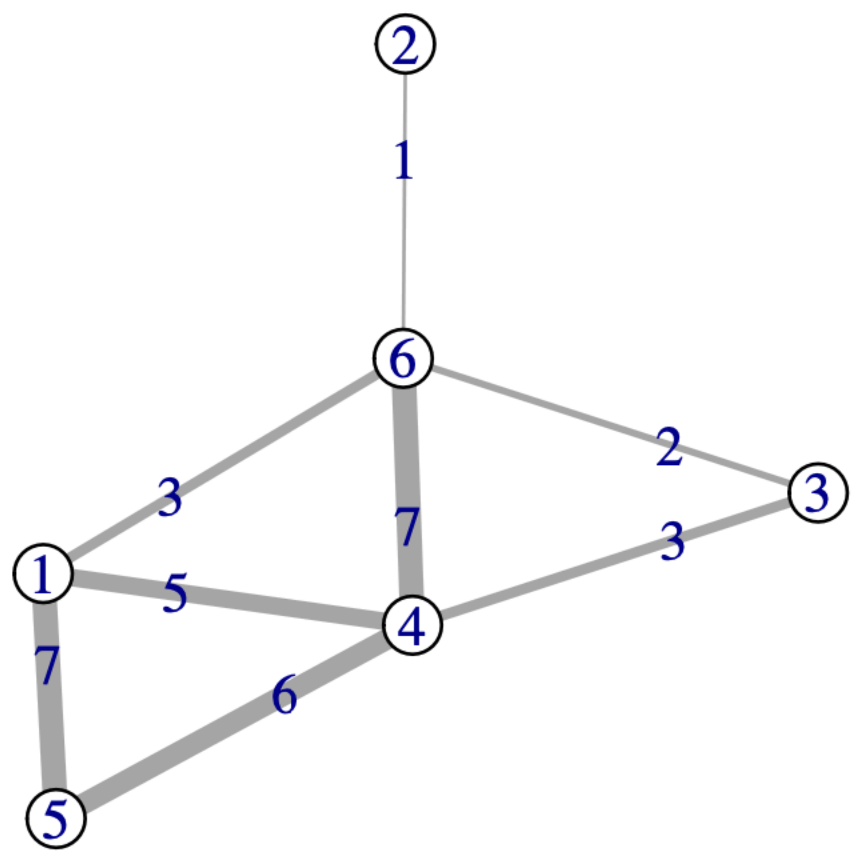
	1	2	3	4	5	6
1	0	0	0	5	7	3
2	0	0	0	0	0	1
3	0	0	0	3	0	2
4	5	0	3	0	6	7
5	7	0	0	6	0	0
6	3	1	2	7	0	0

Edge list

V1	V2	weight
1	4	5
3	4	3
1	5	7
4	5	6
1	6	3
2	6	1
3	6	2
4	6	7

What can weights represent?

Weighted Networks



Adjacency Matrix

	1	2	3	4	5	6
1	0	0	0	5	7	3
2	0	0	0	0	0	1
3	0	0	0	3	0	2
4	5	0	3	0	6	7
5	7	0	0	6	0	0
6	3	1	2	7	0	0

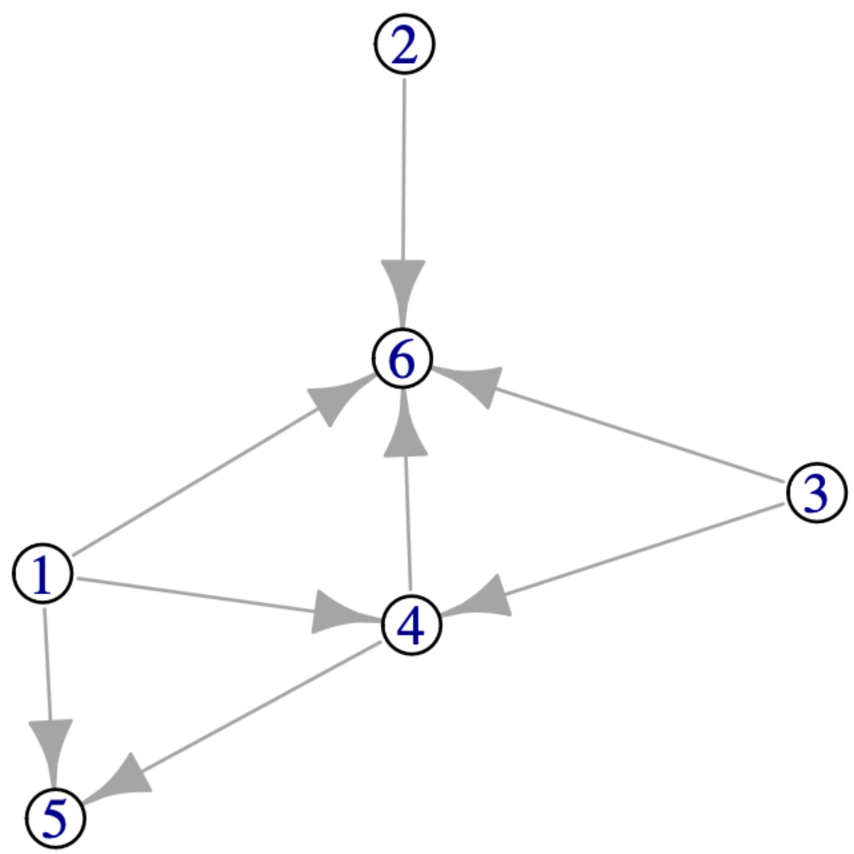
Edge list

V1	V2	weight
1	4	5
3	4	3
1	5	7
4	5	6
1	6	3
2	6	1
3	6	2
4	6	7

What can weights represent?

How similar are two nodes. How old is the relationship. What is the relationship strength.

Directed Networks



Adjacency Matrix

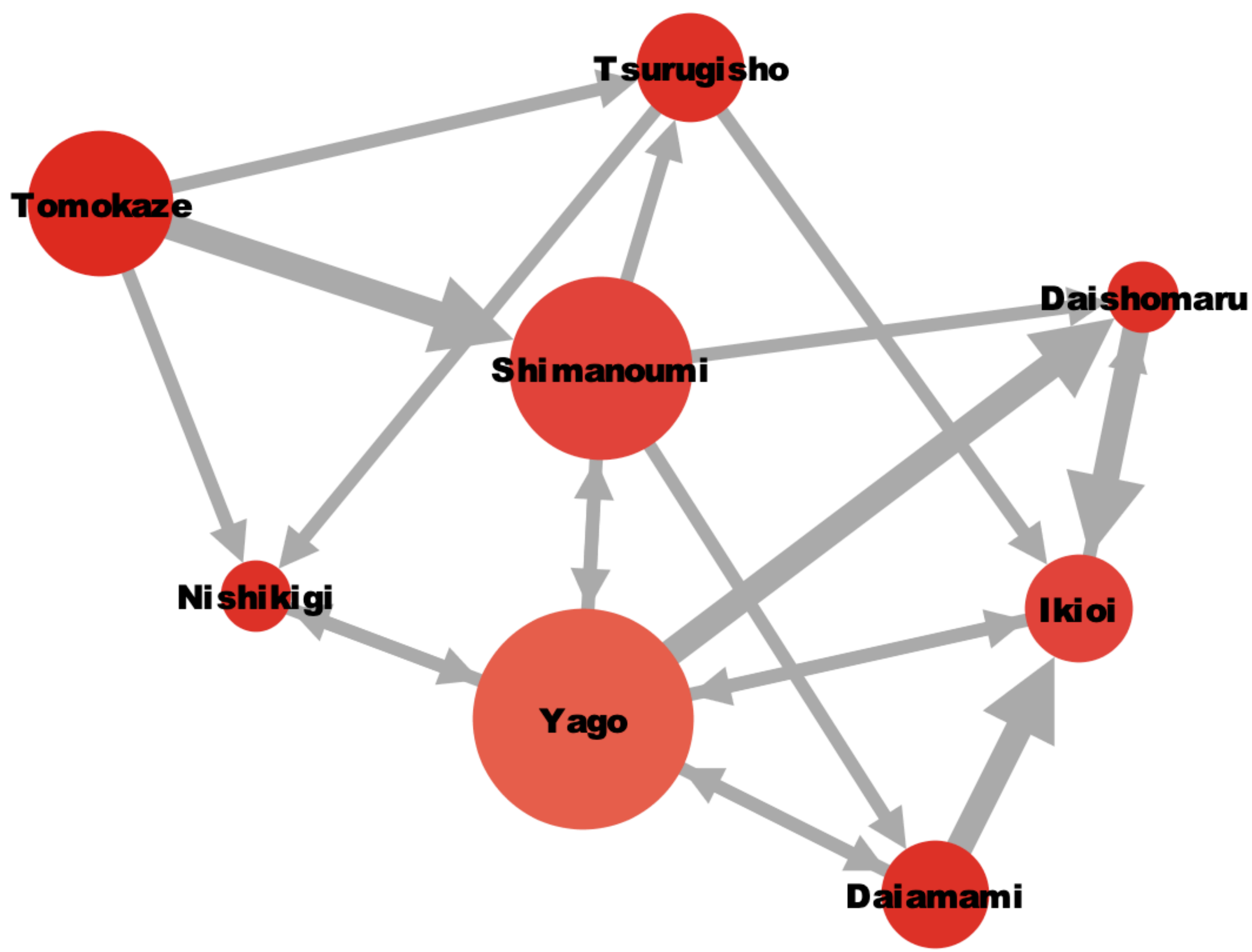
	1	2	3	4	5	6
1	0	0	0	1	1	1
2	0	0	0	0	0	1
3	0	0	0	1	0	1
4	0	0	0	0	1	1
5	0	0	0	0	0	0
6	0	0	0	0	0	0

Edge list

V1	V2
1	4
3	4
1	5
4	5
1	6
2	6
3	6
4	6

Why does the directed edge mean?

Application of directed network

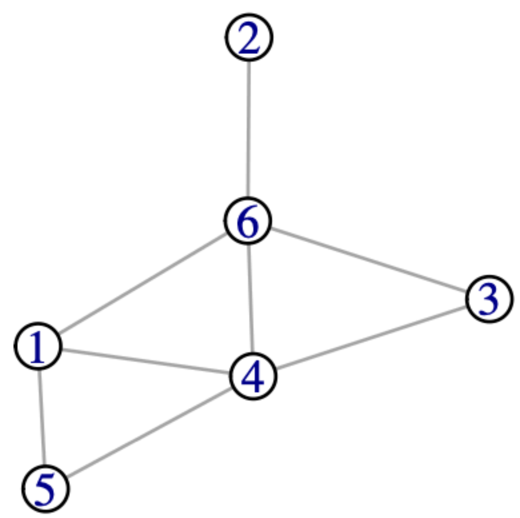


Arrows go from losers to winners

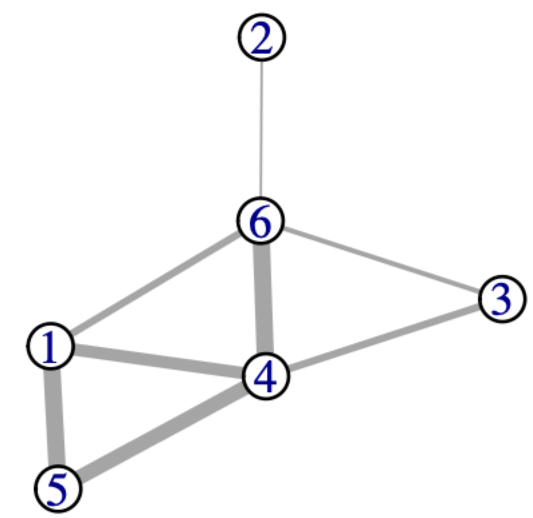
Data from Data.World Sumo Matches 2019

4 kinds of Networks

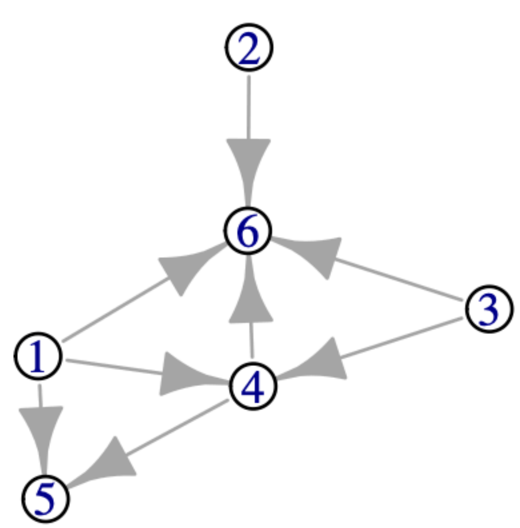
Simple: Unweighted, Undirected



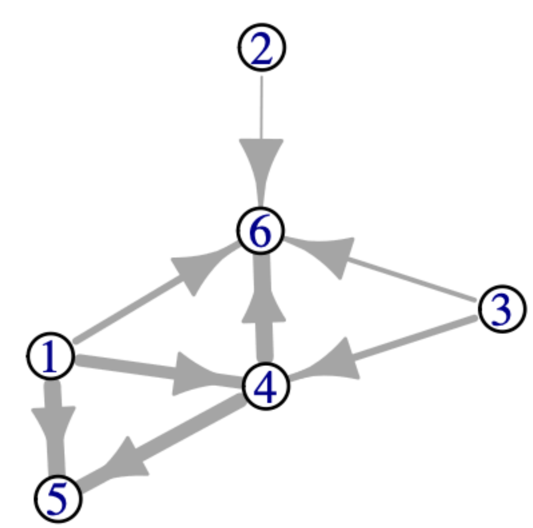
Weighted, Undirected



Directed, Unweighted



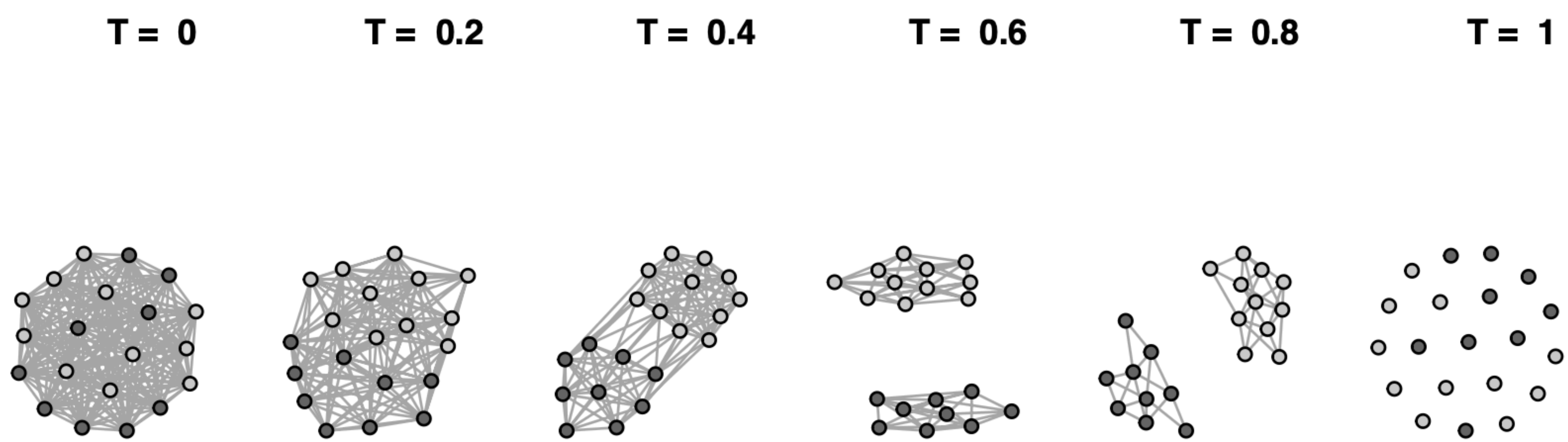
Weighted, Directed



- Often basic (undirected and unweighted) networks are easier to deal with.
- How can we transform a weighted and/or directed network into an unweighted and undirected network?

How to get a basic network from weighted data

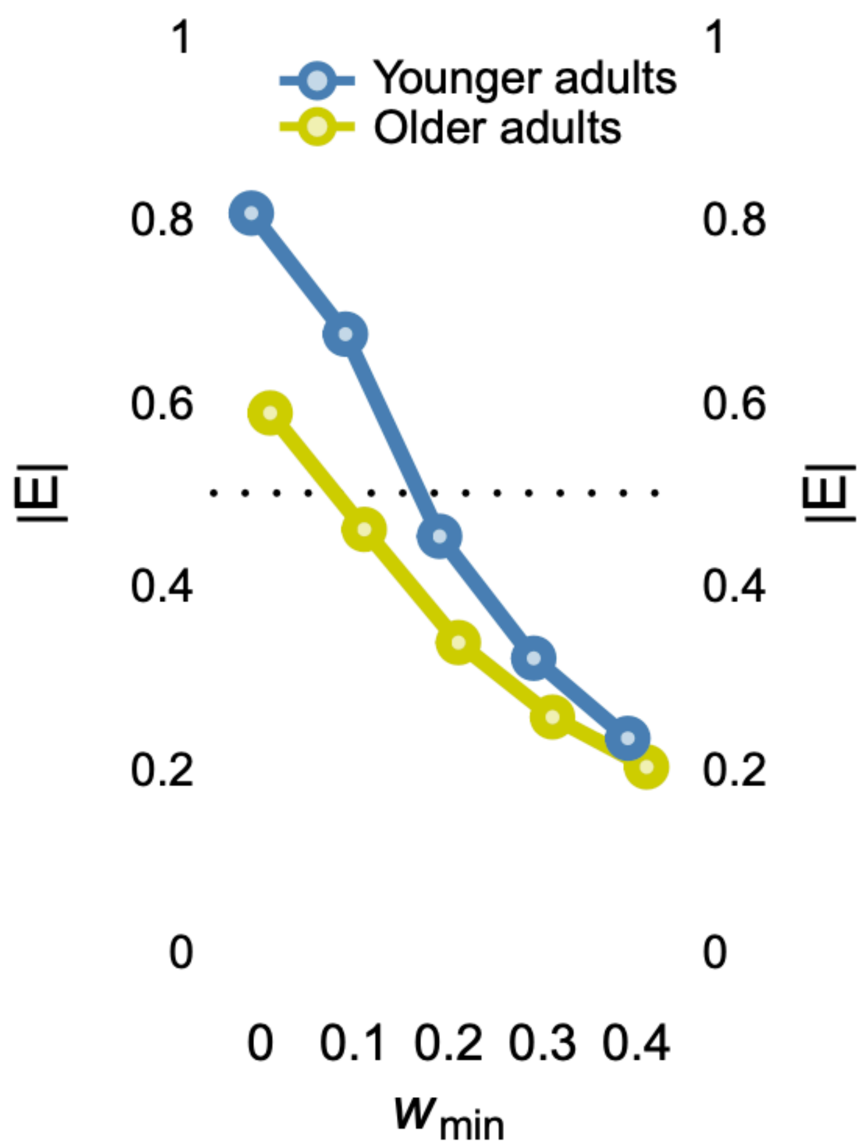
- Apply a moving threshold. Keep edges above threshold.




```
igraph::delete_edges(g, E(g)[E(g)$weight < thresh])
```

Application of moving threshold to aging networks

- Apply a moving threshold. Keep edges above threshold.



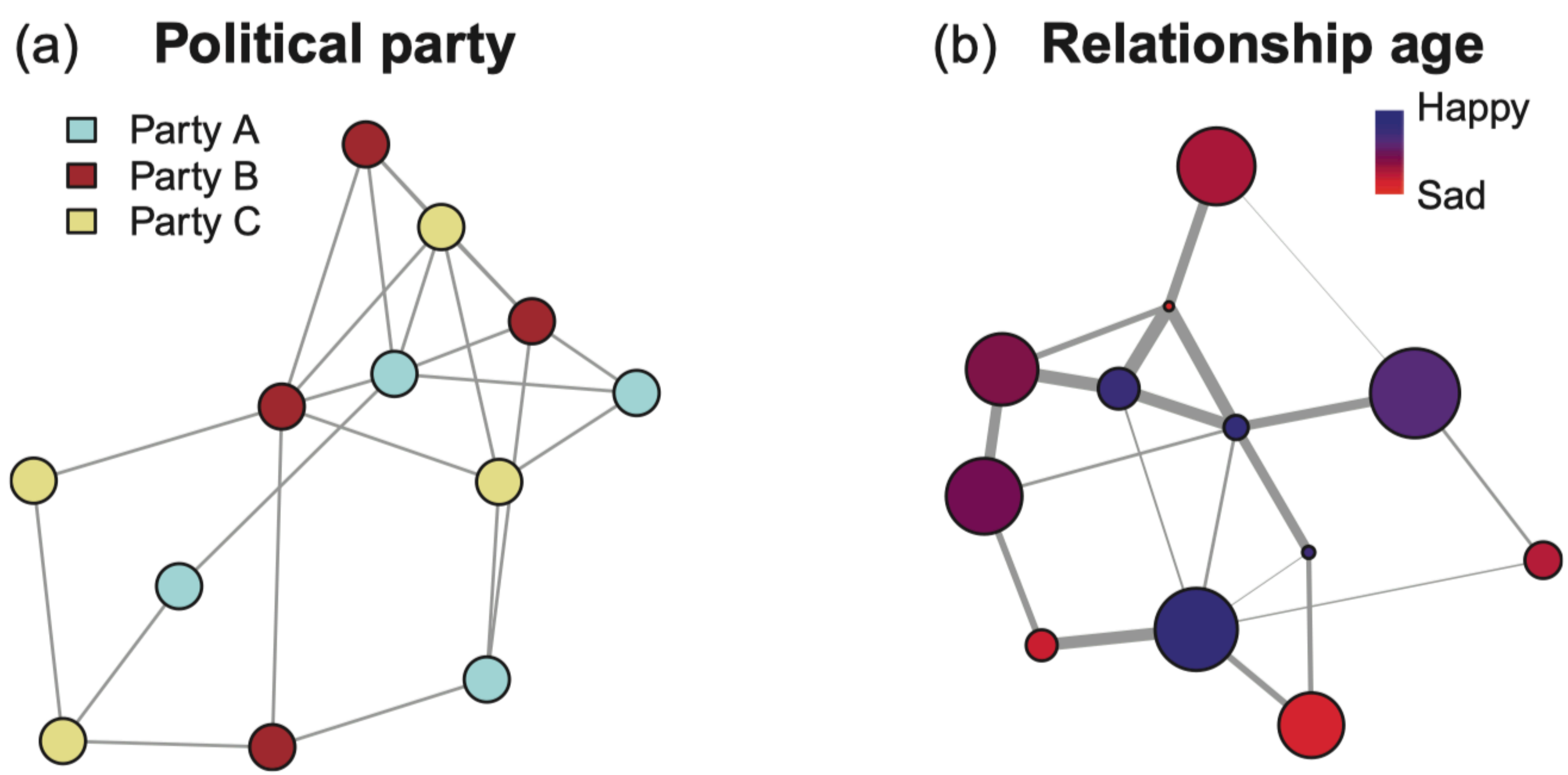
|E| = proportion of edges relative to fully connected graph.

For a given threshold, older adults tend to have sparser networks (up to a point).

This is a general solution, that allows you to be fully transparent. It's hard to pre-register thresholds.

Node and edge attributes

Nodes and edges can have properties of their own



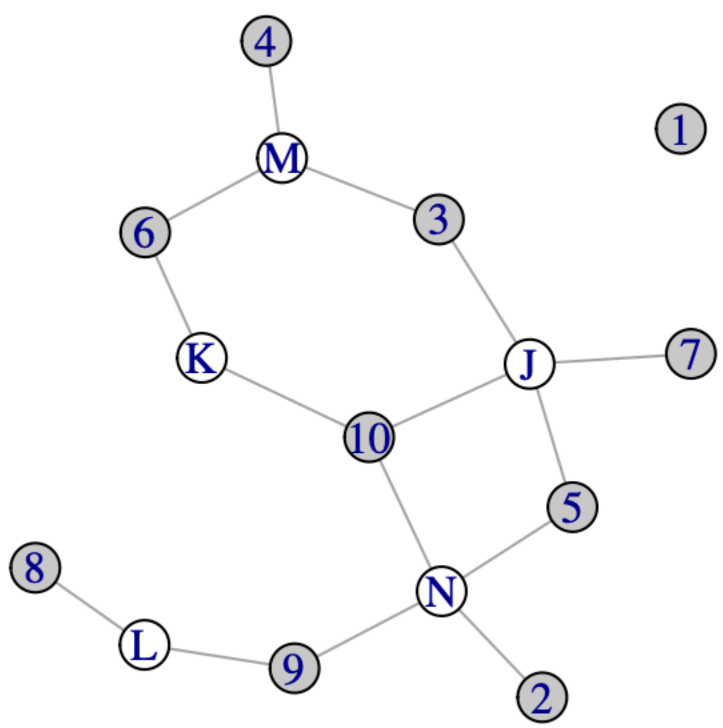
These are often covariates to compare against structure. For example, the frequency of a word in child-directed speech might be included in a model to predict language acquisition based on structure. We can also ask how attributes influence associations (assortativity).

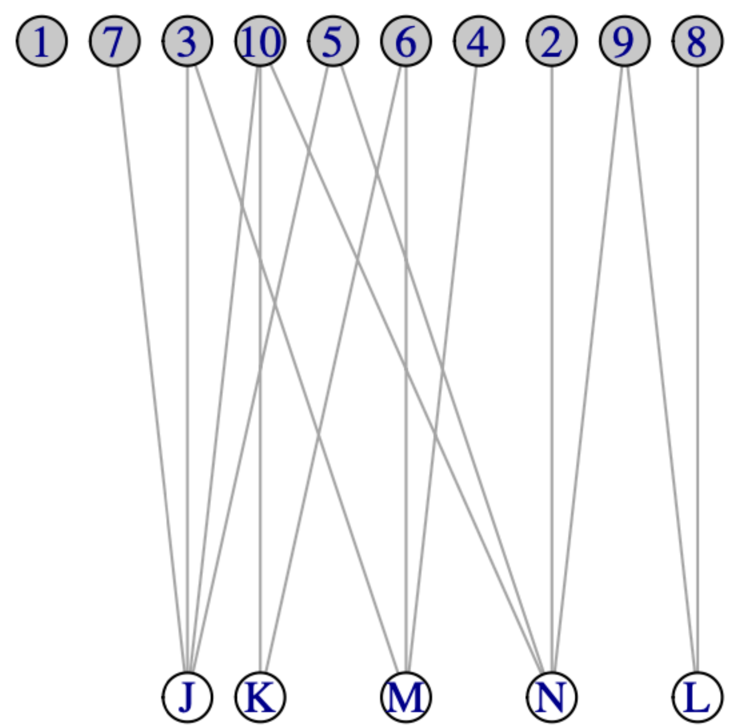
```
V(g)$name_of_new_attribute <- values
```

Bipartite Networks

Bipartite networks have two kinds of nodes and nodes are only connected to nodes of other types.

```
igraph::graph_from_biadjacency_matrix(bam)
```





layout = layout_as_bipartite(g)

	J	K	L	M	N
1	0	0	0	0	0
2	0	0	0	0	1
3	1	0	0	1	0
4	0	0	0	1	0
5	1	0	0	0	1
6	0	1	0	1	0
7	1	0	0	0	0
8	0	0	1	0	0
9	0	0	1	0	1
10	1	1	0	0	1

Bipartite network

These two networks are the same (just visualized differently), and represented by the rectangular 'incidence' matrix on the right.

Bipartite Projections

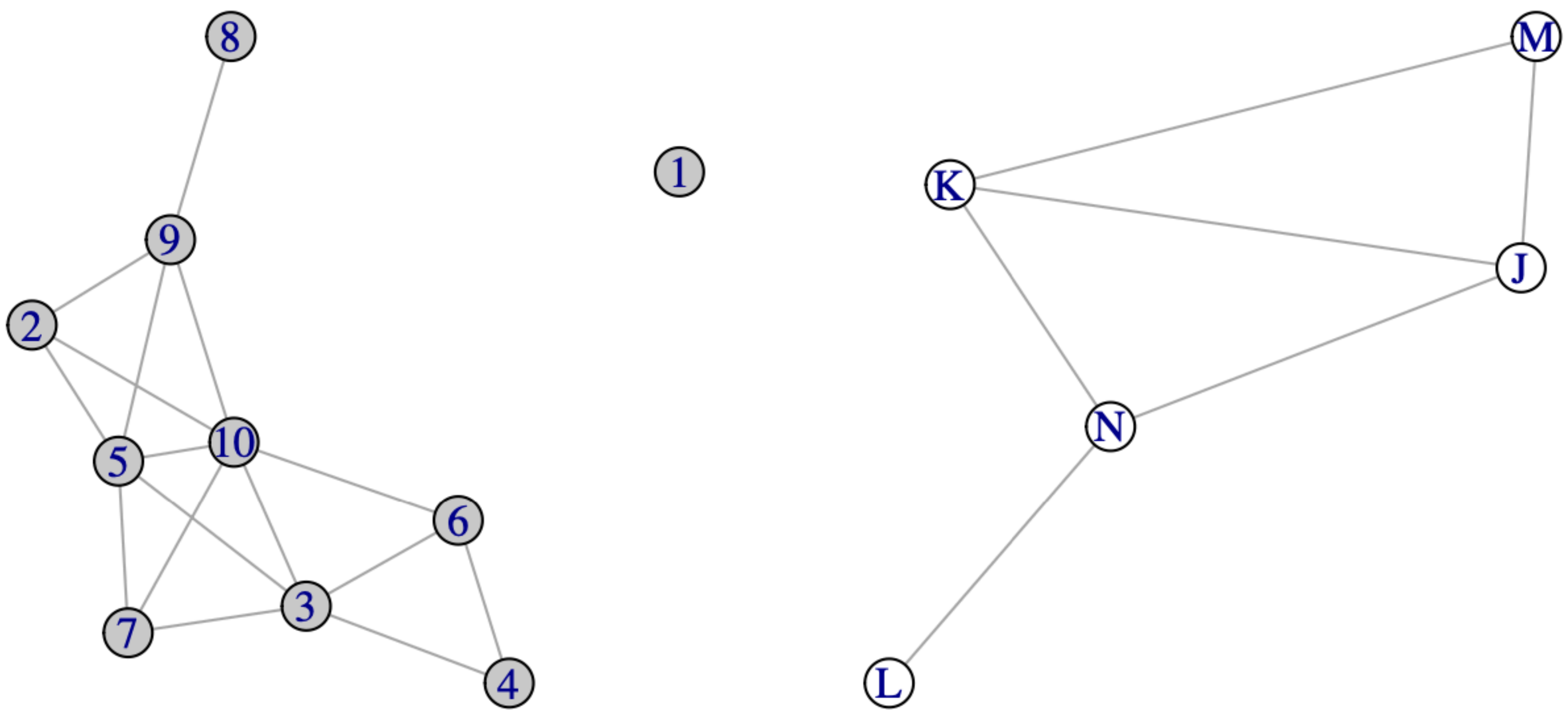
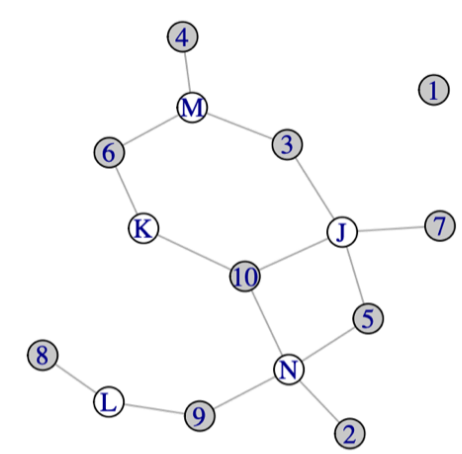


Figure 9: One-mode projections of the two-mode (bipartite) network.



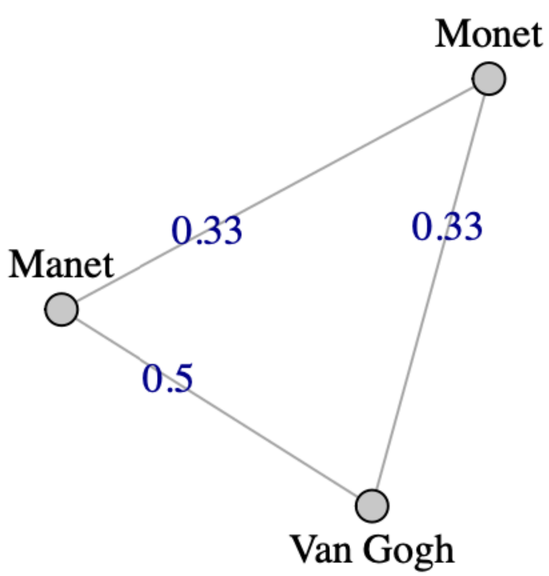

```
g.bp <- igraph::bipartite_projection(g, multiplicity=TRUE)
n1 <- g.bp[[1]] # this is the row projection
n2 <- g.bp[[2]] # this is the column projection
```

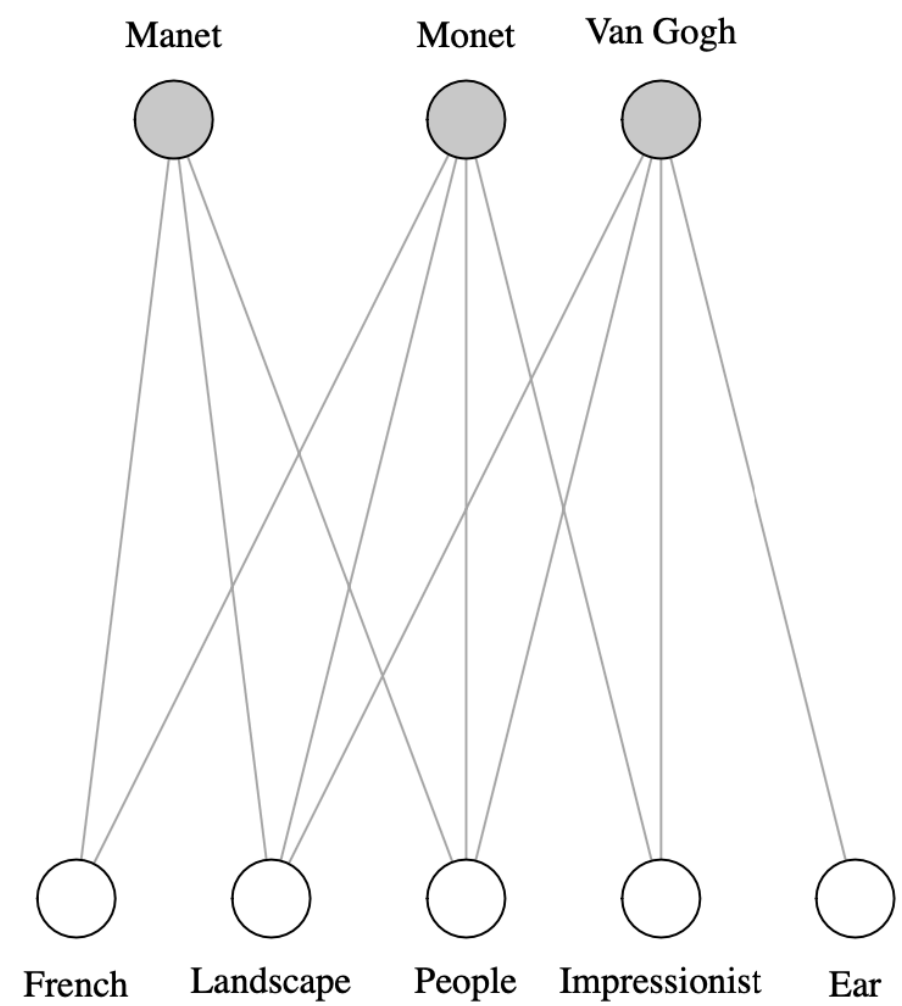
Application of bipartite networks

Figure 1: Paintings from Manet (left), Monet (centre), Van Gogh (right)

Table 1: A bipartite adjacency matrix with two node types: painters and features.

	French	Landscape	People	Ear	Impressionist
Manet	1	0.1	1.0	0	0
Monet	1	1.0	0.1	0	1
Van Gogh	0	1.0	0.1	1	1





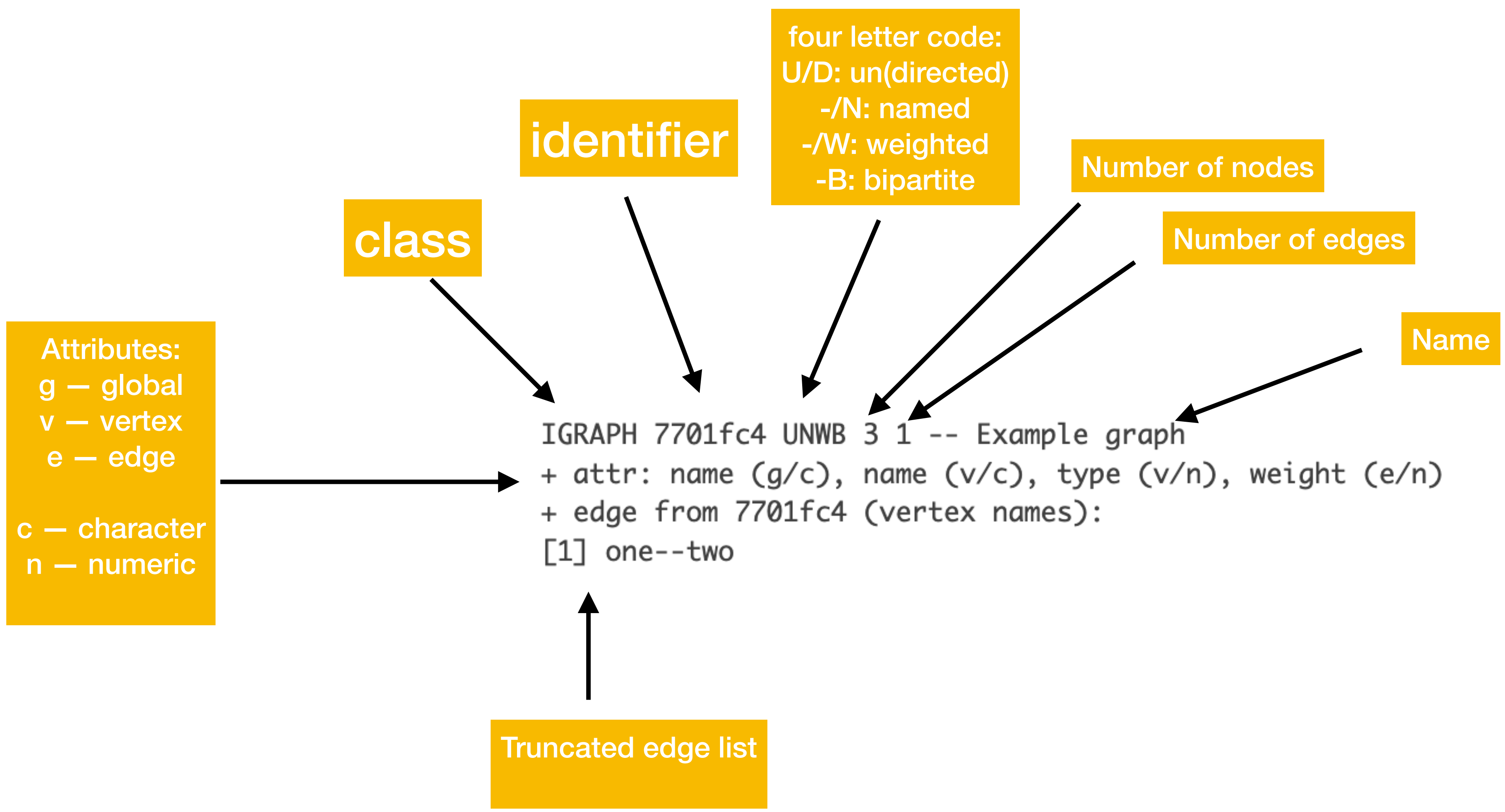
Chapter 6: What is distinctive? Engelthaler and Hills (2016) applied this to shape bias and age of acquisition for words.

Brief code interlude

```
igraph::graph_from_edgelist(el, directed = TRUE)
igraph::graph_from_adjacency_matrix(am)
igraph::graph_from_biadjacency_matrix(bam)
```

Within the functions, you can specify whether the graph should be directed, weighted, etc.

igraph object



to make the above:
library(igraph) # install library
g <- make_graph(c(1,2), directed = FALSE) # simple graph
g

igraph object: test

```
> macaque
IGRAPH f7130f3 DN-- 45 463 --
+ attr: Citation (g/c), Author (g/c), shape (v/c), name (v/c)
+ edges from f7IGRAPH 7701fc4 UNWB 3 1 -- Example graph
 [1] V1 ->V2    + attr: name (g/c), name (v/c), type (v/n), weight (e/n)
 [7] V1 ->PO    + edge from 7701fc4 (vertex names):
[13] V2 ->V4t   [1] one--two
[19] V2 ->PO      V2 ->PIP     V2 ->VIP     V2 ->FST     V2 ->FEF     V3 ->V1
[25] V3 ->V2      V3 ->V3A     V3 ->V4      V3 ->V4t     V3 ->MT      V3 ->MSTd/p
[31] V3 ->PO      V3 ->LIP     V3 ->PIP     V3 ->VIP     V3 ->FST     V3 ->TF
[37] V3 ->FEF     V3A->V1      V3A->V2      V3A->V3      V3A->V4      V3A->VP
[43] V3A->MT      V3A->MSTd/p V3A->MSTl    V3A->PO      V3A->LIP     V3A->DP
+ ... omitted several edges
```

- Part 2 — Measuring things on networks

Network levels of inference

- Macro-scale: what is the structure of an entire network? How does it compare with other networks. What can the network do?
- Micro-scale: What are the properties of the nodes? What are their differences? Who is important?
- Meso-scale: What are the structure of communities? How many communities? How do they they compare with one another?

Network Measures

Variable	Definitions
N	Number of nodes
E	Number of edges
L	Average shortest path length
D	Diameter
C_i	Clustering Coefficient
k, k^{in}, k^{out}	Degree, in-degree, and out-degree
b	Betweeness centrality
c	Closeness centrality
x	Eigenvector centrality
r	Assortativity
Q	Modularity

There are so many more, so it's best to have a theory to help you choose.

G(N,E)

- How many possible edges in an undirected network of N nodes?

	1	2	3	4	5	6
1	0	0	0	1	1	1
2	0	0	0	0	0	1
3	0	0	0	1	0	1
4	1	0	1	0	1	1
5	1	0	0	1	0	0
6	1	1	1	1	0	0

Number of possible edges:
E~N^2

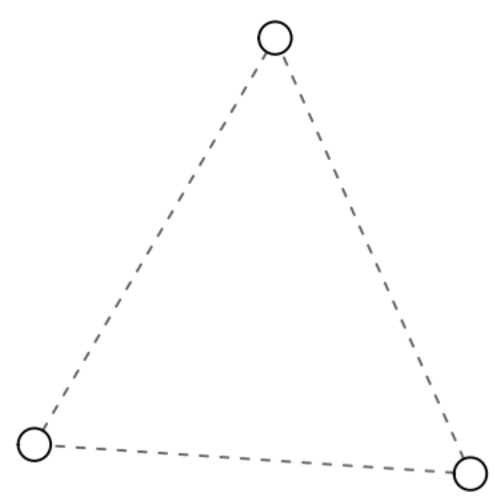
Number of possible without self-loops:
E~N(N-1)

Number of possible without self-loops and undirected:
E~N(N-1)/2

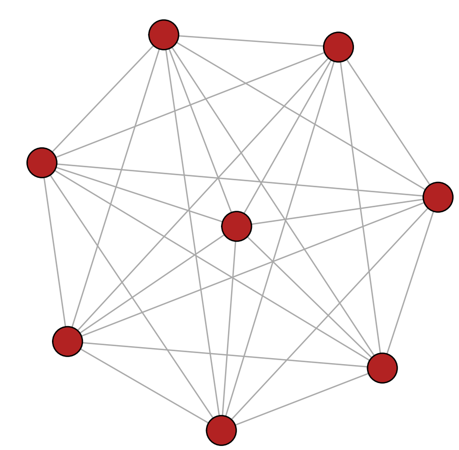
Birthday paradox

What's the probability at least two people in this room share a birthday?

What's the probability for these two people?
1/365



What's the probability for these three people?
1-(364/365)^3

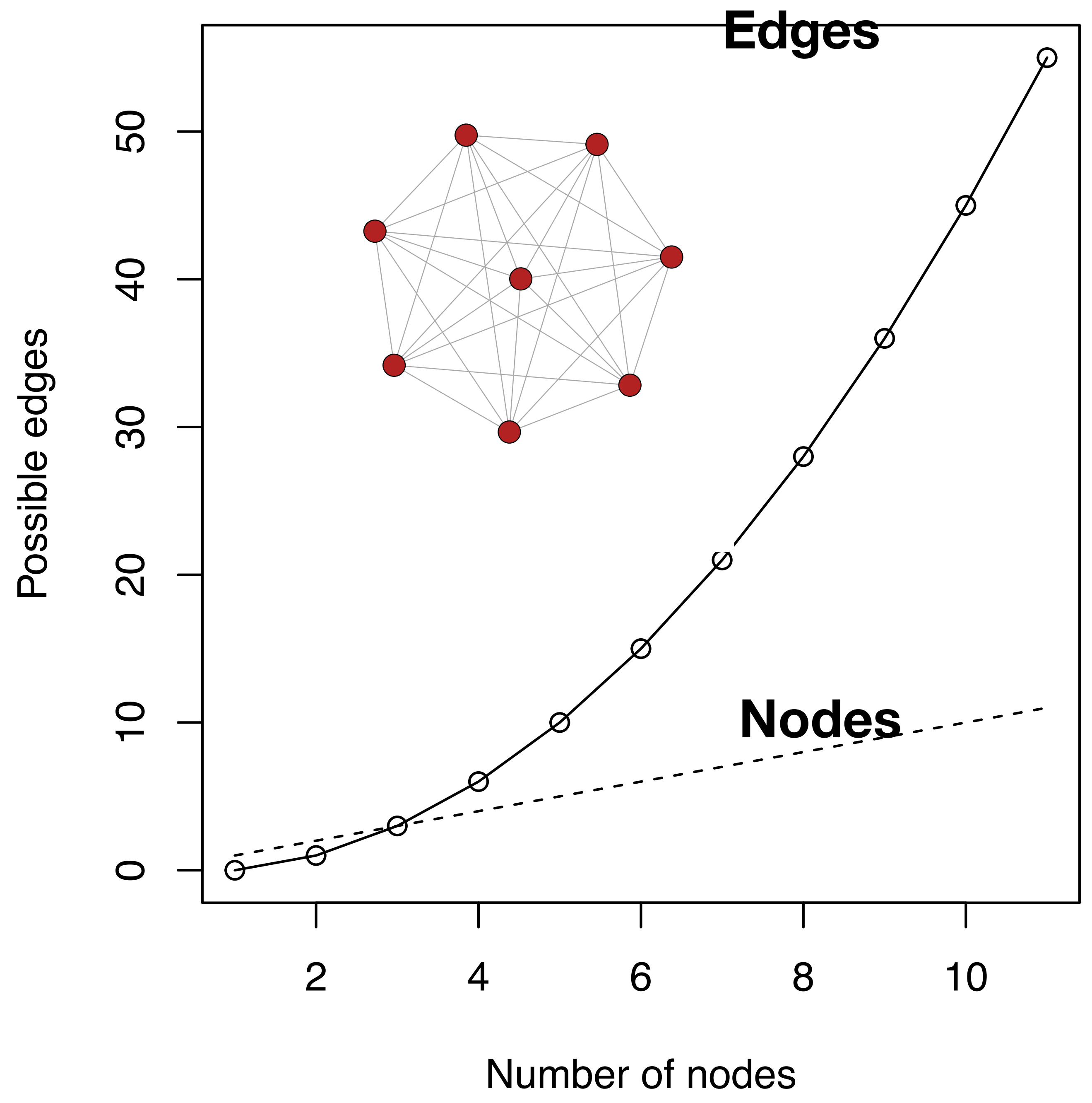


Why is this a network problem?

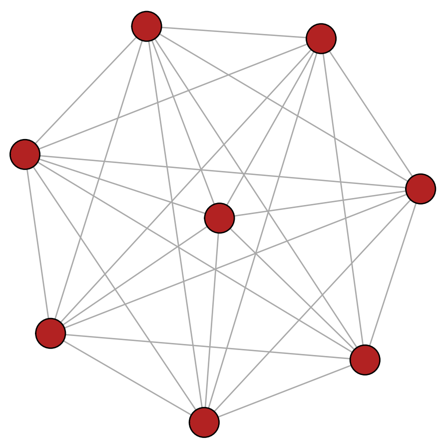
Metcalfe's Law

The 'value' of a network grows as the square of the number of 'users.' Value ~ number of possible connections.

Number of possible edges:
E~N^2



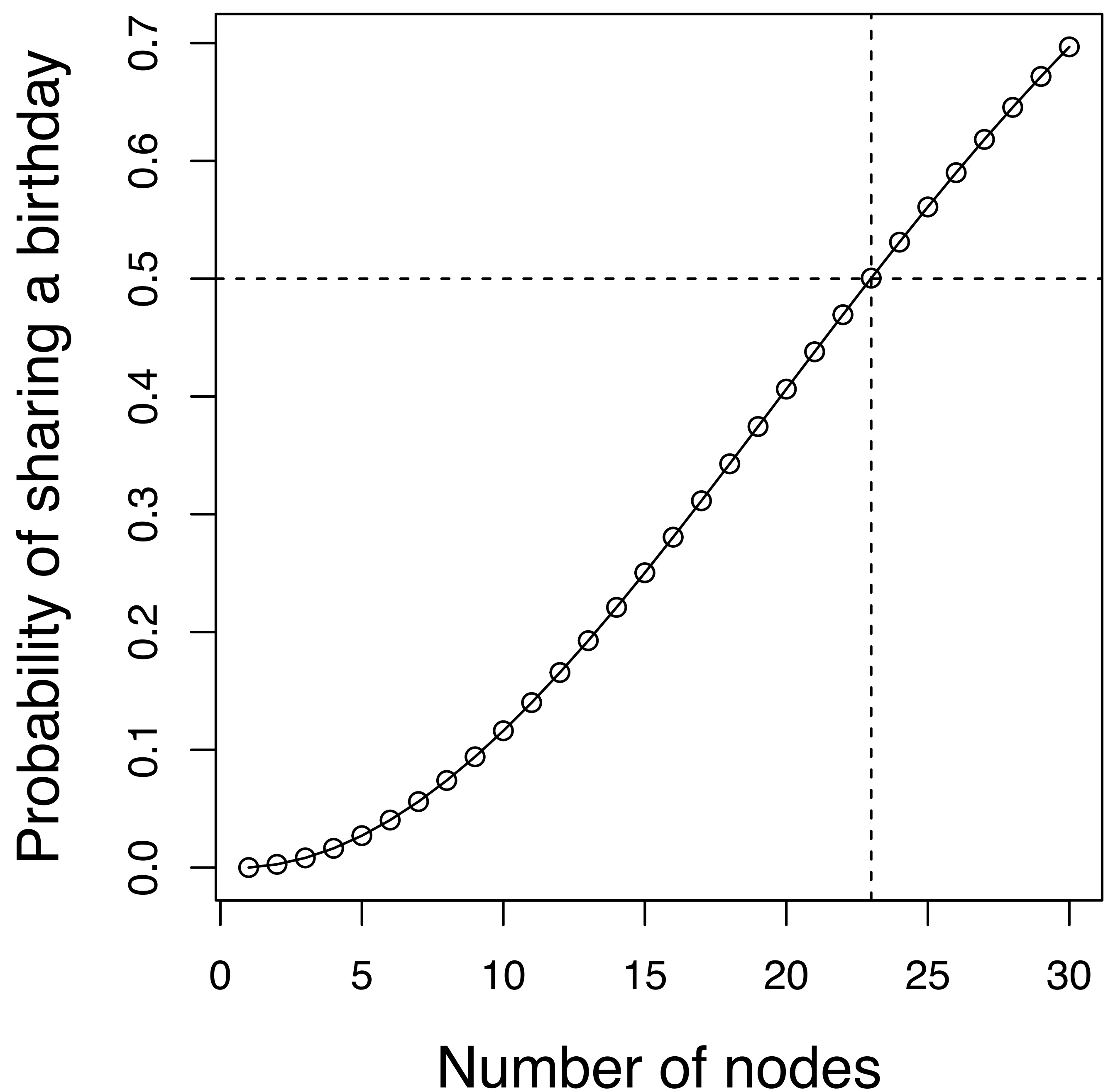
Birthday paradox



Number of possible edges:
E=N(N-1)/2

Probability of not sharing a birthday:
1-(364/365)^E

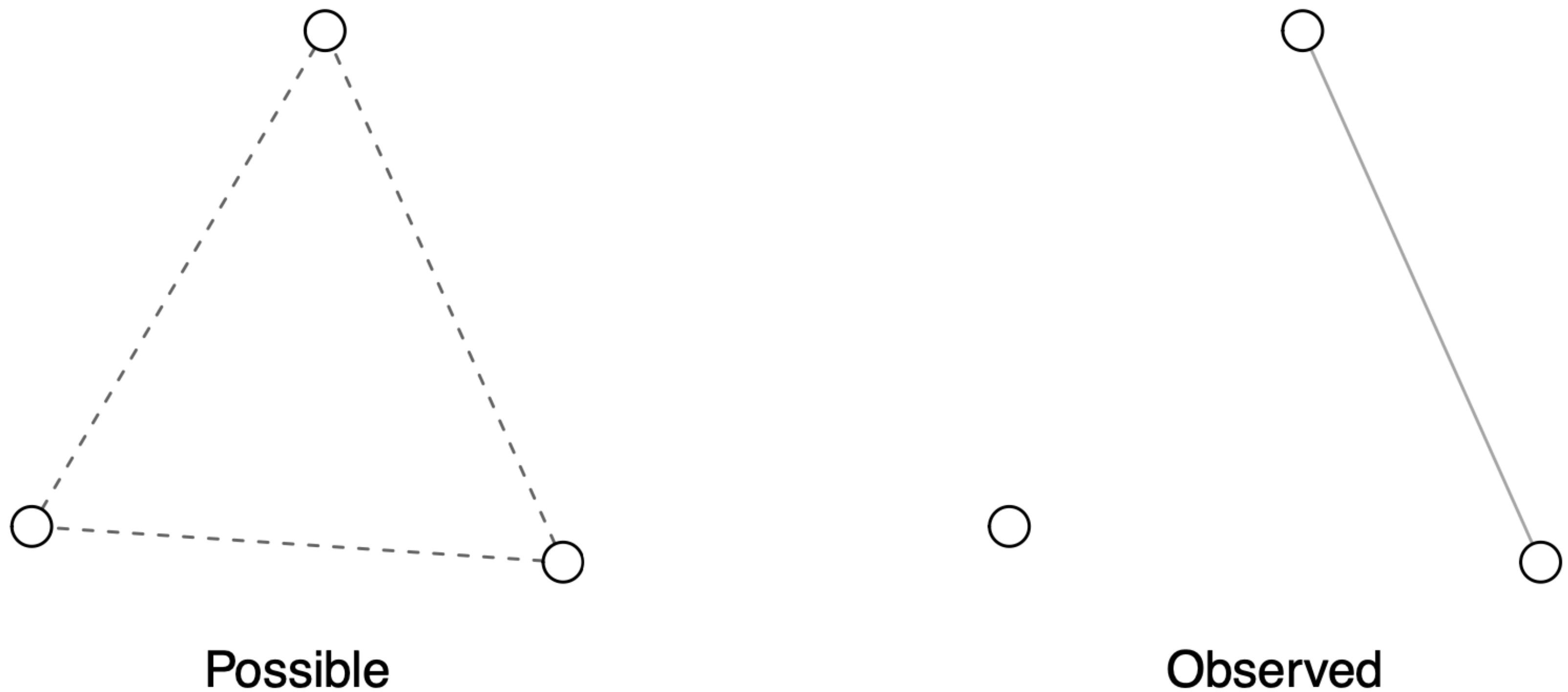
The number of possible edges increases rapidly with N. Therefore the probability that at least two people share such a low probability edge goes up rapidly.



Density

$$\boxed{\rho = \frac{2E}{N(N-1)}}$$

Number of observed edges over number of possible edges.



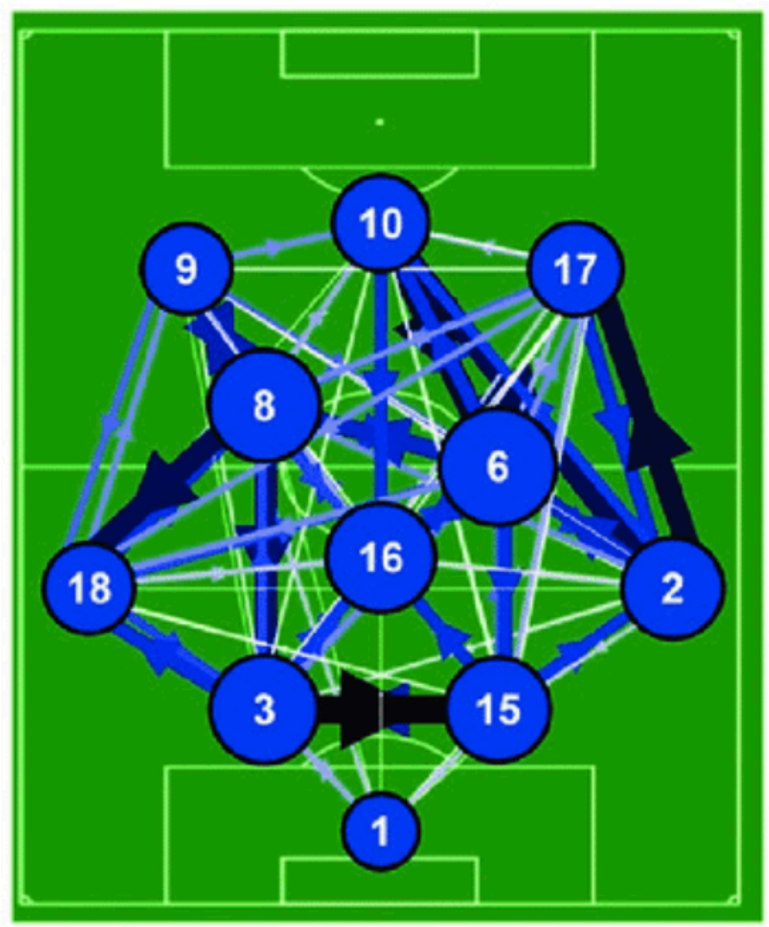
Density = 1/3

`igraph::edge_density()`

What does density tell us?

- Well coordinated teams often have high -density interactions.

- Covert teams often have low density interactions.



9 - Alexis, 10 - Messi, 17 - Pedro,
8 - Iniesta, 16 - Busquets, 6 - Xavi,
18 - Alba, 3 - Piqué, 15 -Bartra, 2 - Alves, 1 - Valdés

FC Barcelona
Agusti et al, 2016

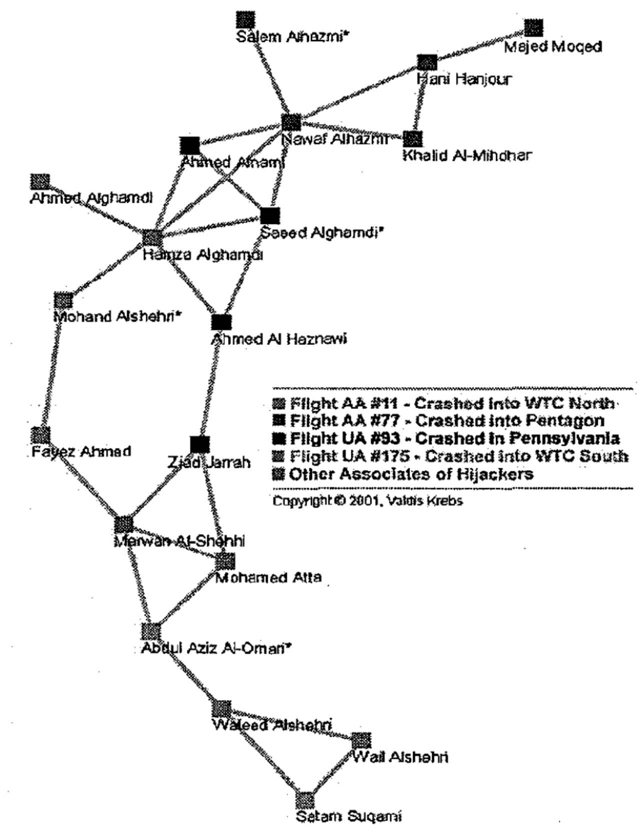
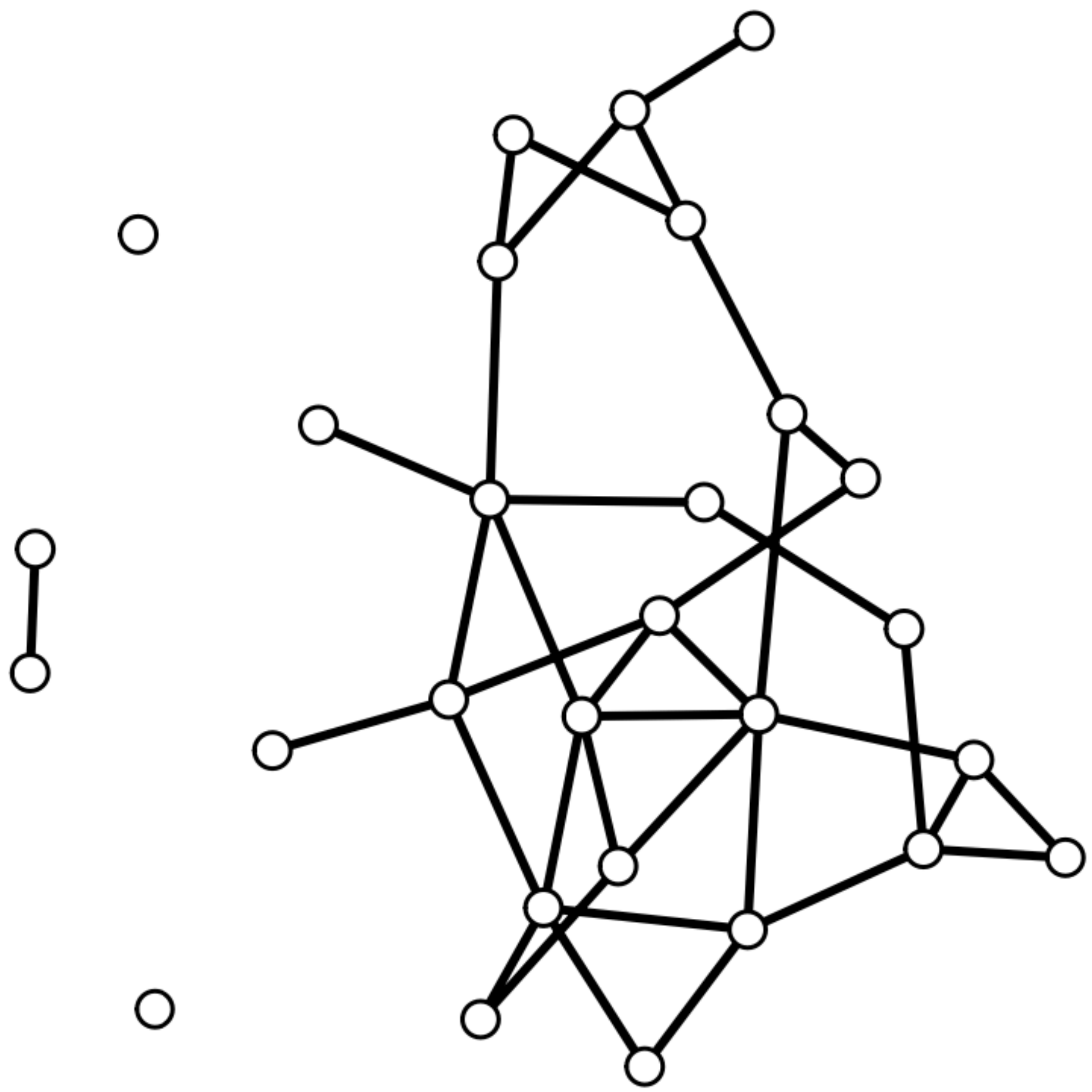


Figure 2 Trusted Prior Contacts

Krebs (2002) Mapping
Terrorist Cells

Components

Collection of nodes that are all ‘reachable’ via a path of edges



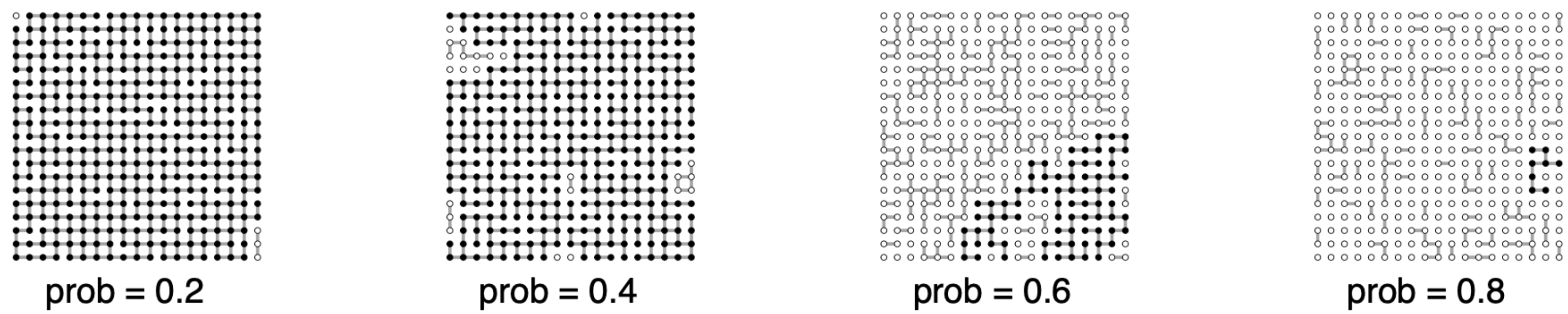
Most network research focuses on the ‘giant’ component.

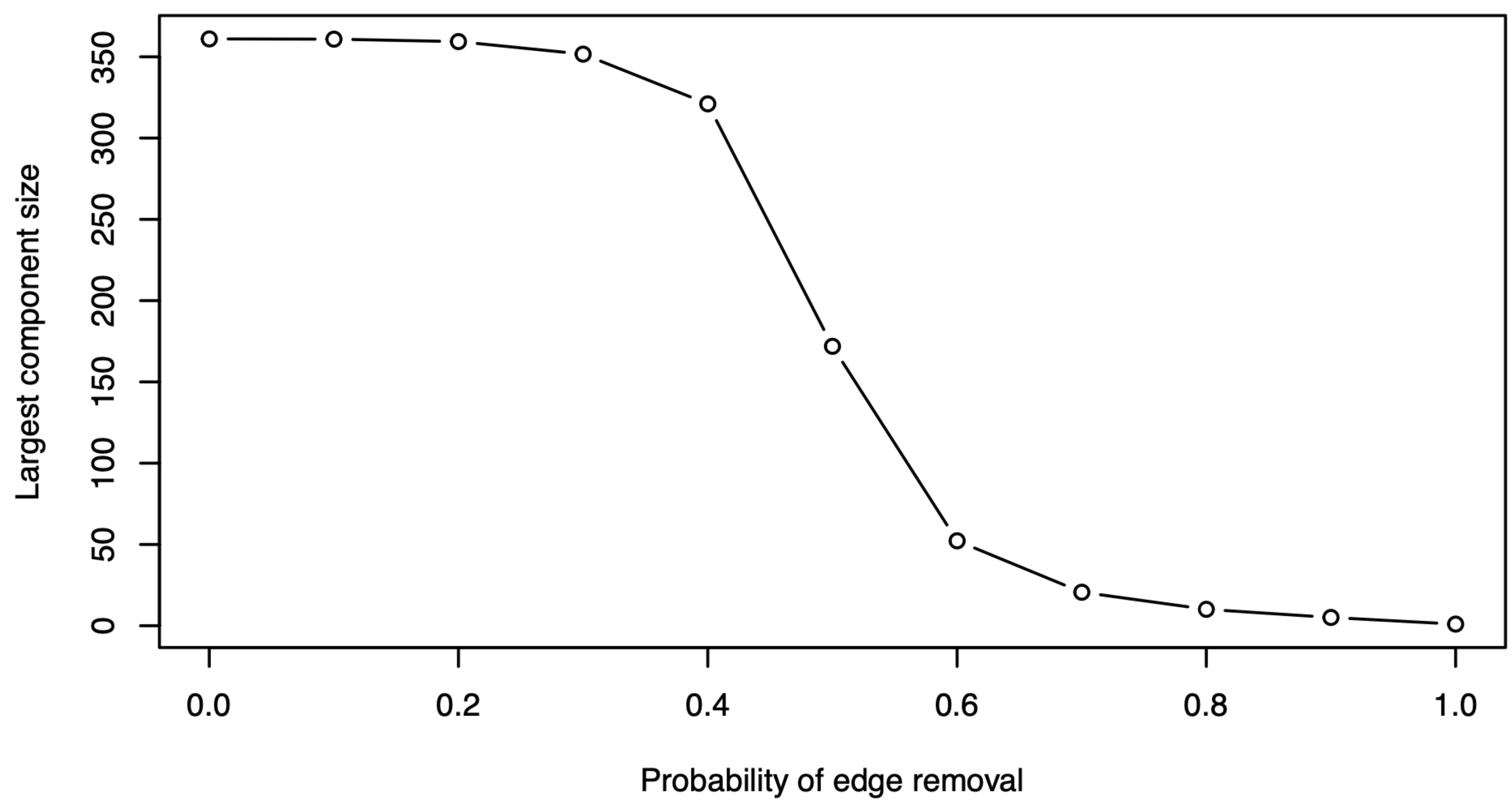
Because many metrics are less meaningful when there is no path between nodes.

`igraph::components(g)`

Percolation analysis

Probability of killing edges in a lattice—how many components?
As we remove edges, the size of the largest component decreases.

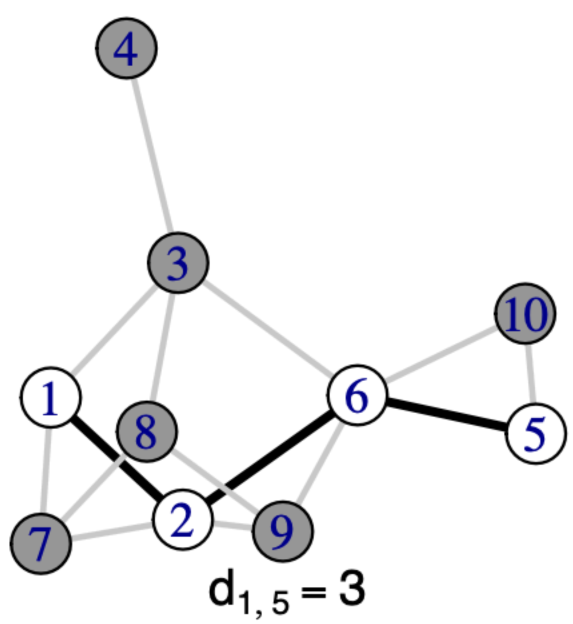


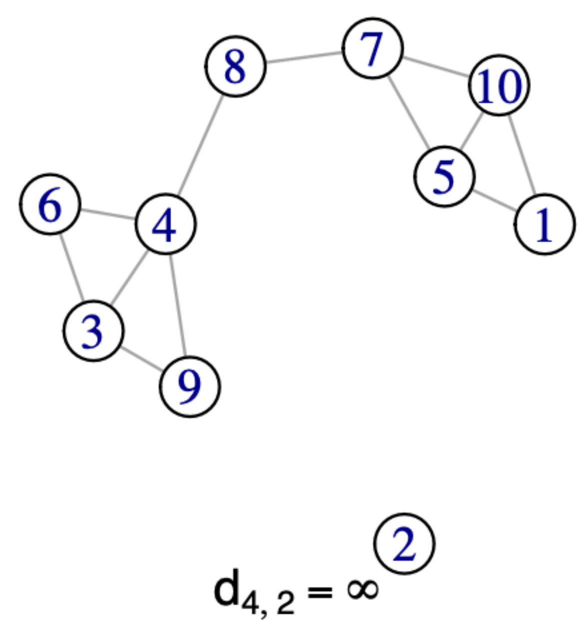


Kenett et al. (2018)

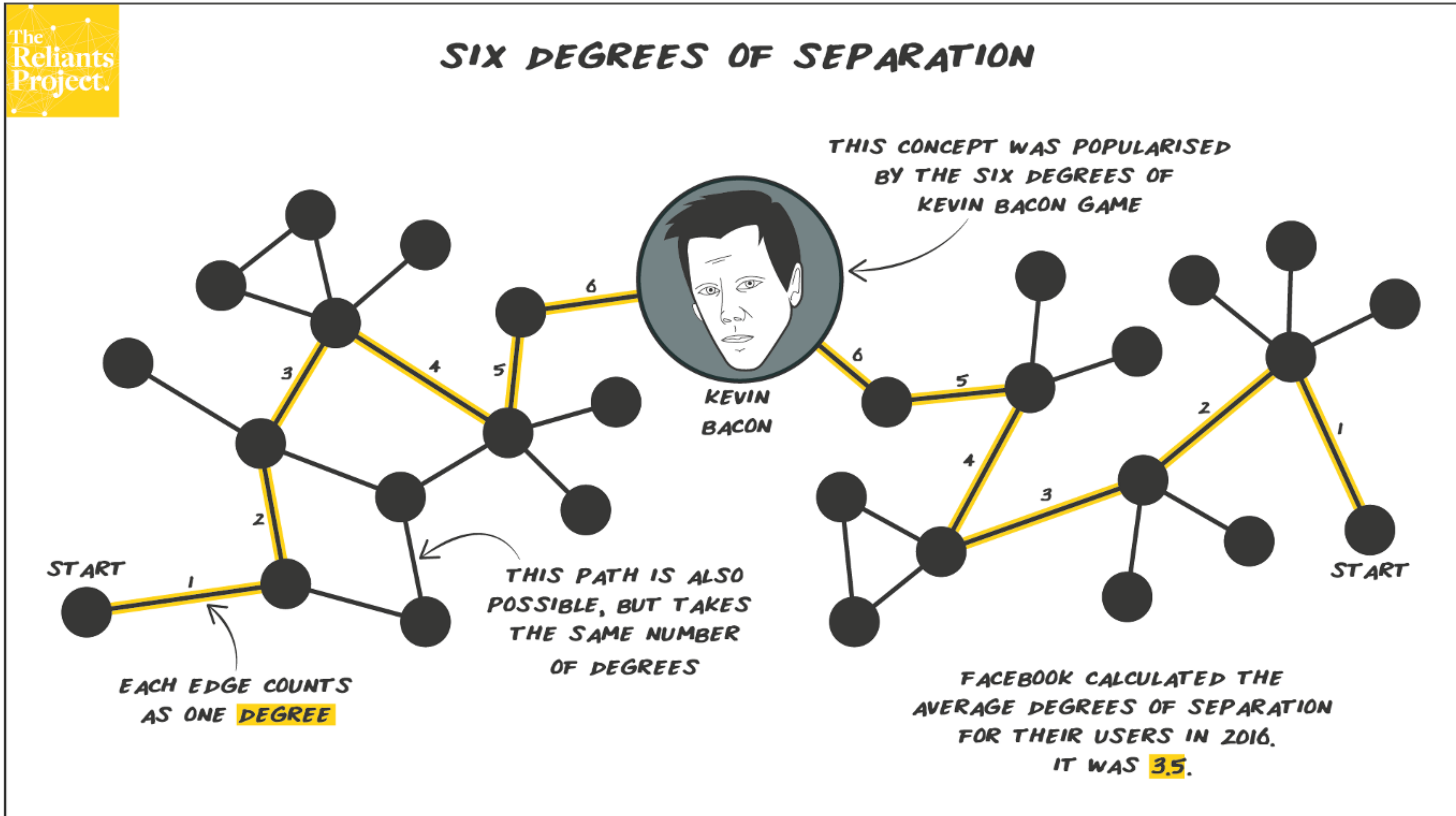
Path Lengths

- Path—a series of contiguous edges.
- Shortest path length (geodesic)
- Diameter—longest shortest path length in a network

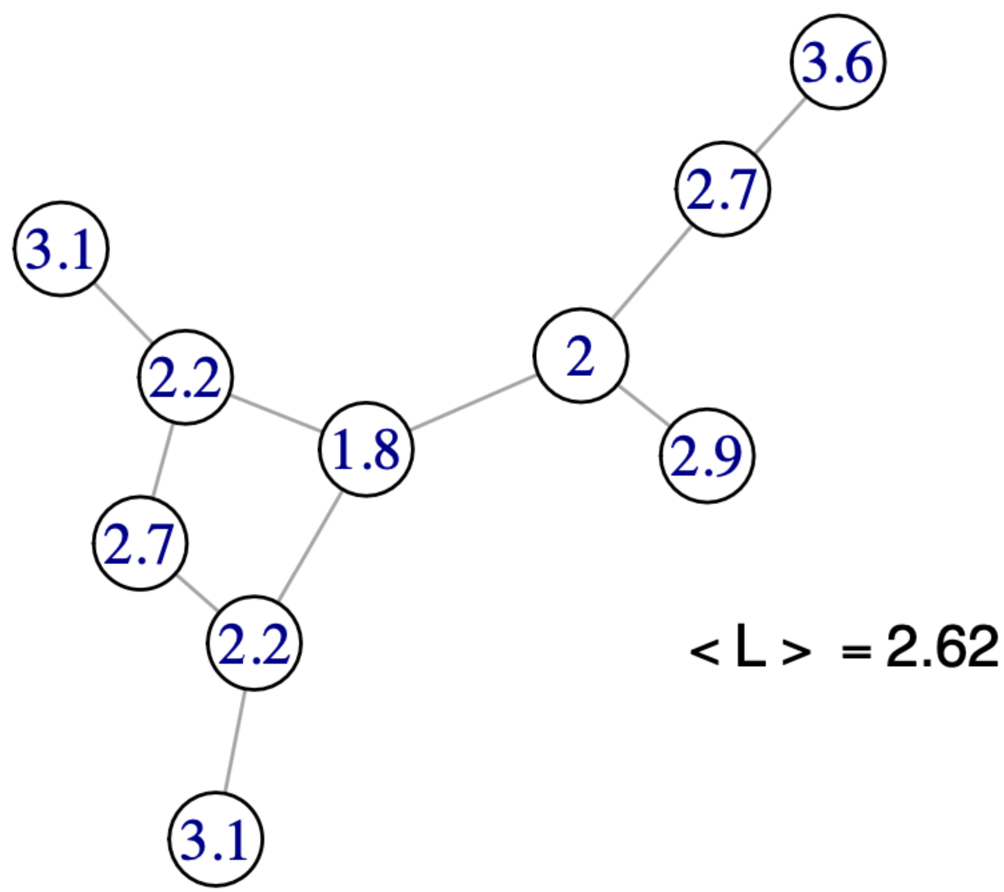




Oracle of Bacon



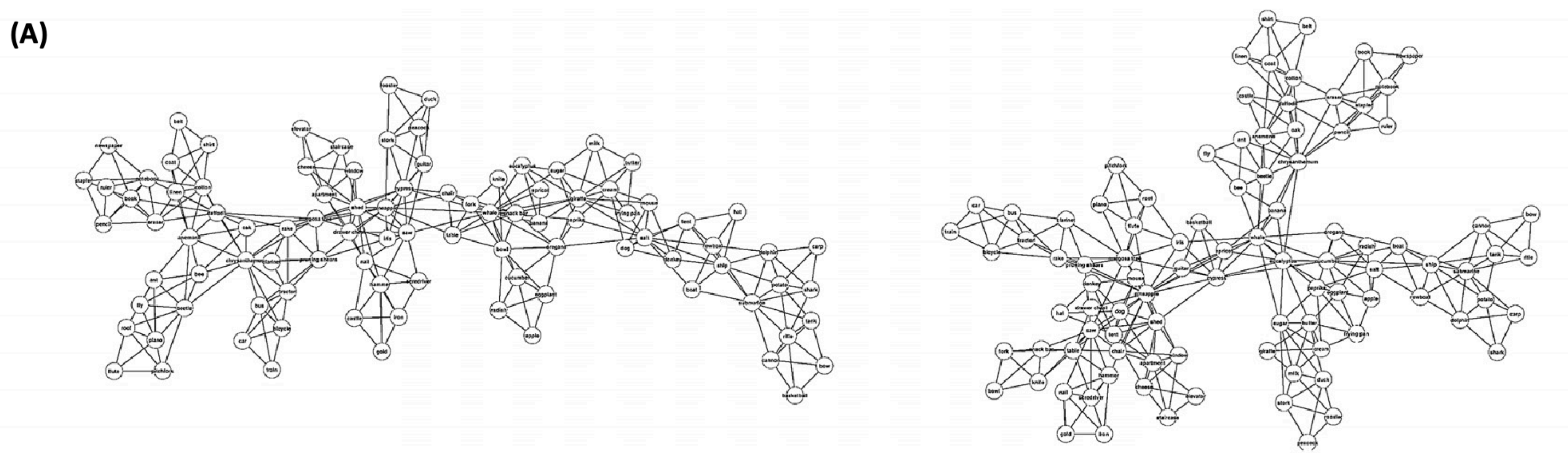
Average shortest path length



Every node is labeled with it's average shortest path distance to all other nodes

```
igraph::distances(g)
igraph::mean_distance(g)
```

Average shortest path lengths are shorter among free associations in more creative people



Low creative

High creative

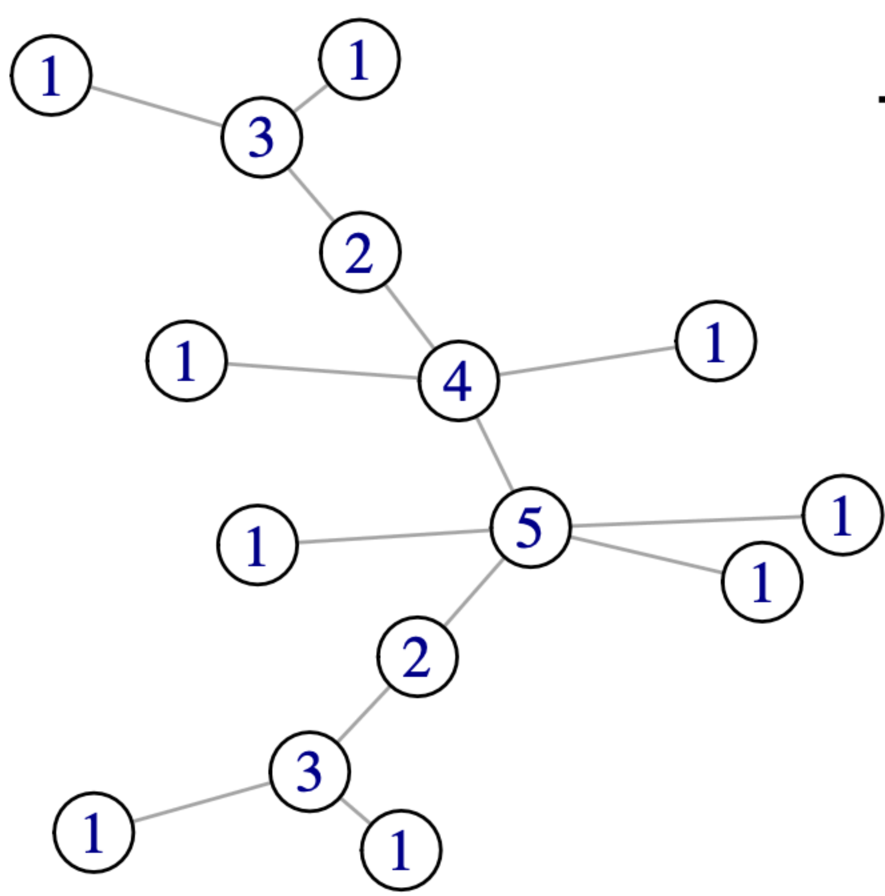
Kenett & Faust (2019)

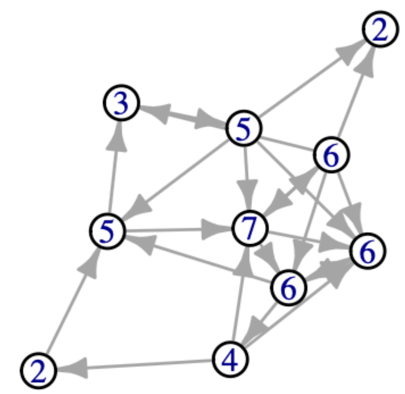
Centrality

- Centrality is a measure of node importance.
- There are many centrality measures:
- Degree, strength
- Clustering coefficient
- Betweenness
- Eigenvector centrality/PageRank
- And many more

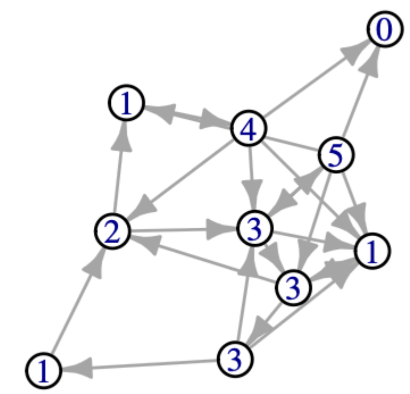
Degree centrality

The number of edges

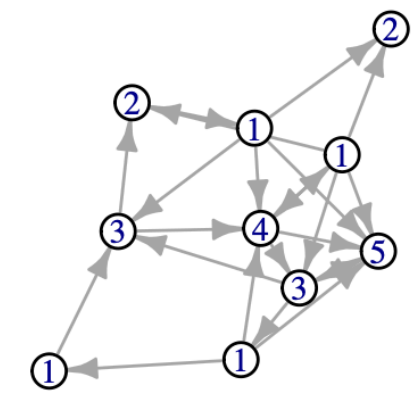




Total



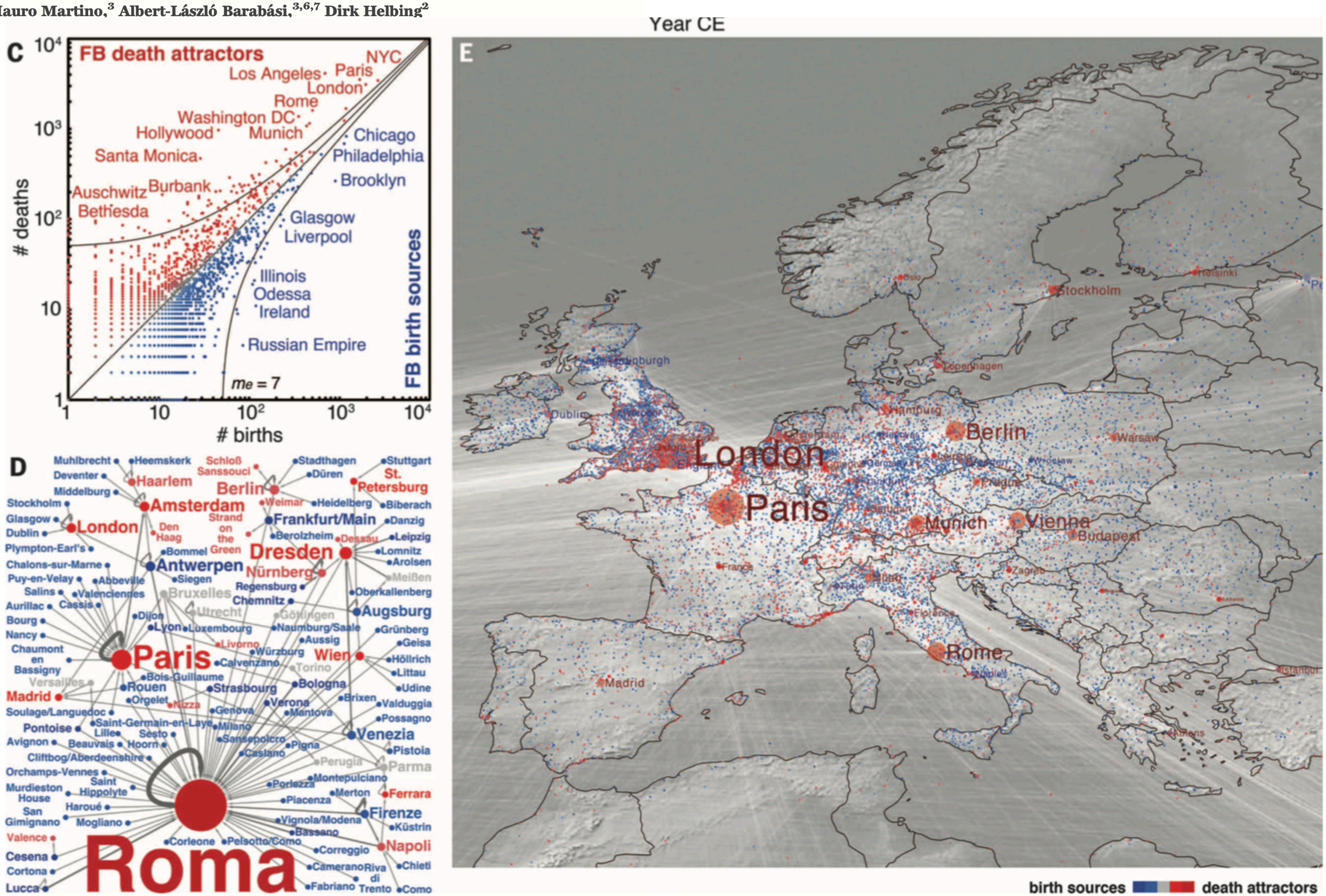
Outdegree



Indegree

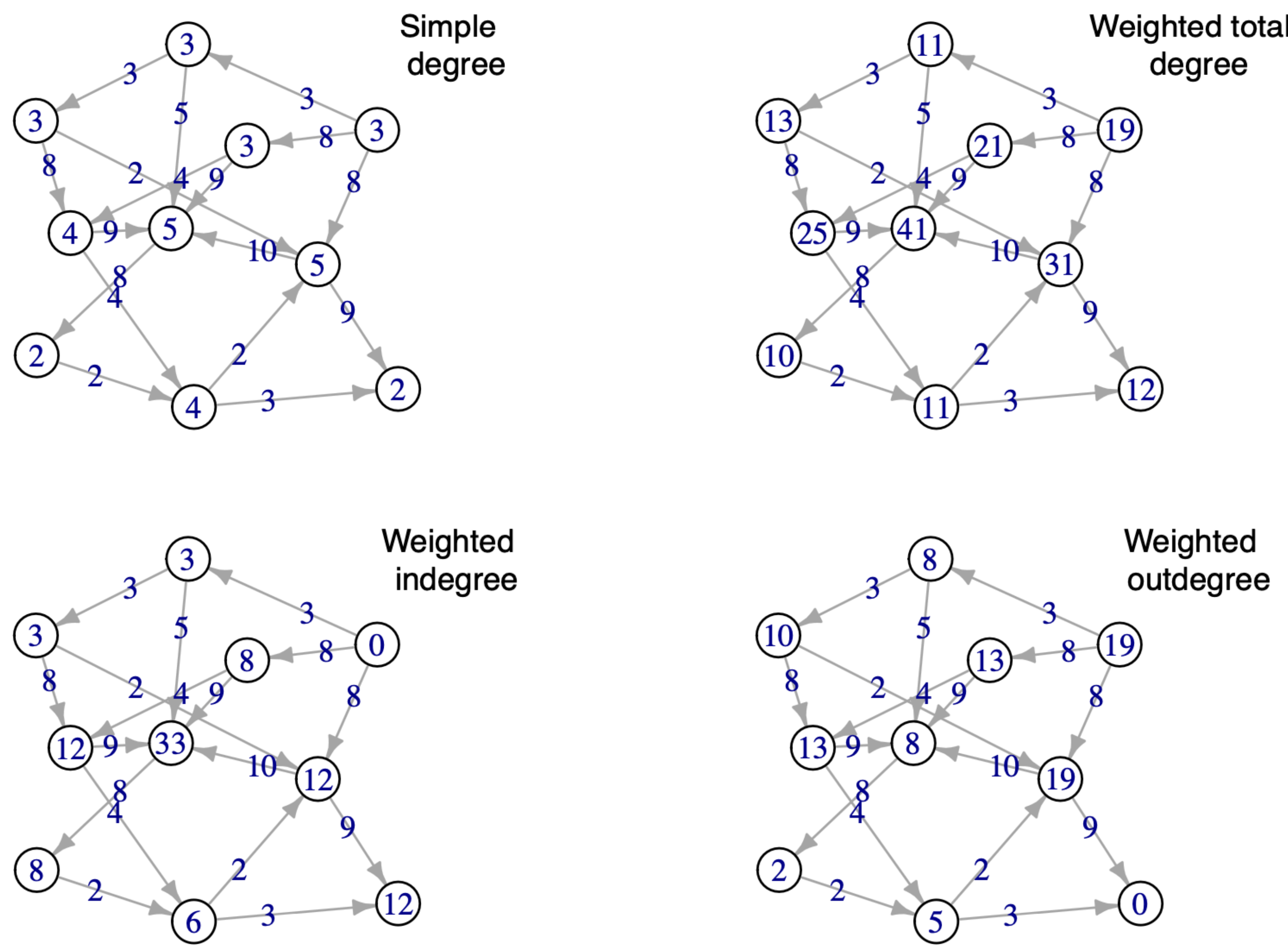
A network framework of cultural history

Maximilian Schich,[1,2,3*] Chaoming Song,[4] Yong-Yeol Ahn,[5] Alexander Mirsky,[2] Mauro Martino,[3] Albert-László Barabási,[3,6,7] Dirk Helbing[2]



Strength centrality

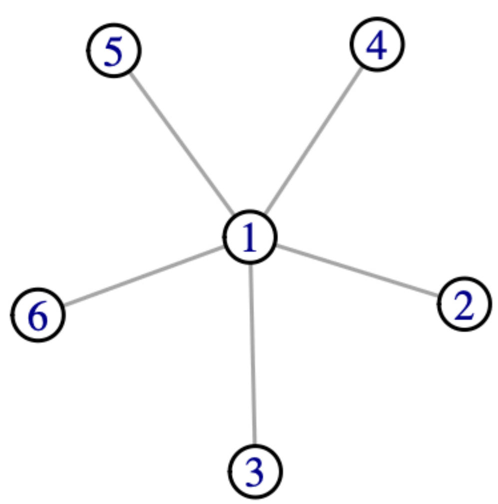
Sum up the weights of the edges



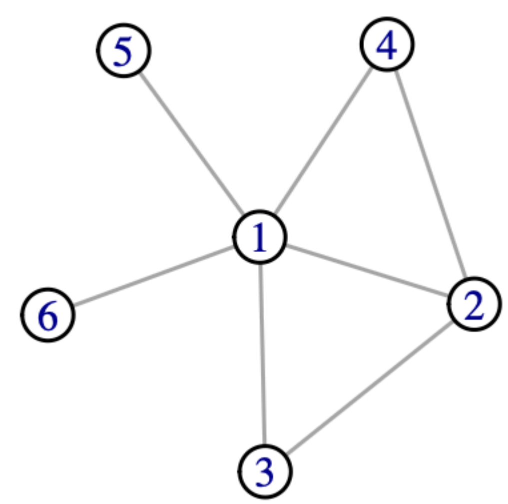
Clustering Coefficient

(Node level)

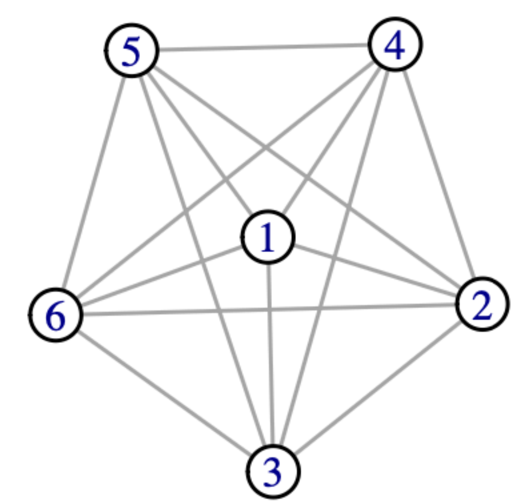
The clustering coefficient has two forms. The first is a node-level or local clustering coefficient. This measures the proportion of a node's neighbors that are connected by an edge.



C=0



C=.2



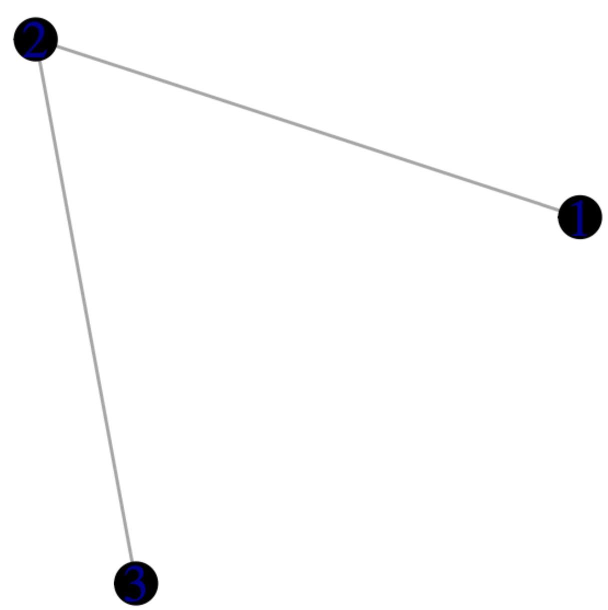
C=1

$$C_i = \frac{2e}{k_i(k_i - 1)}$$

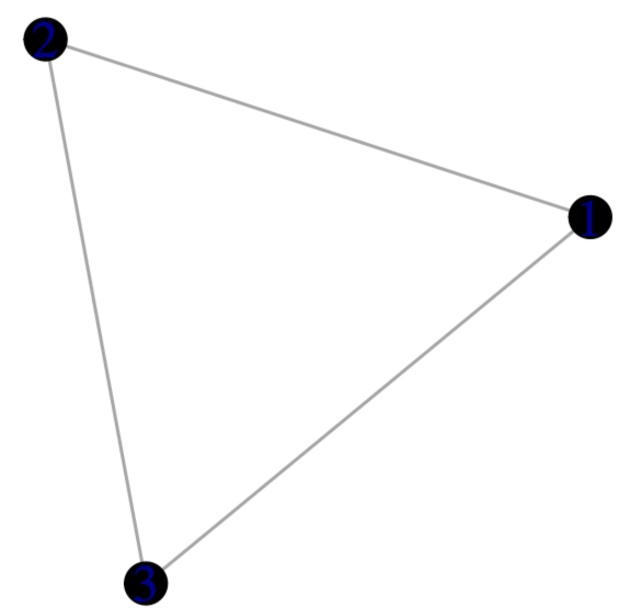
One can compute the average C over all nodes

Clustering Coefficient
(Transitivity: graph level)

Transitivity measures the proportion of triplets in the network that are transitive (i.e. a triangle).



Intransitive triplet

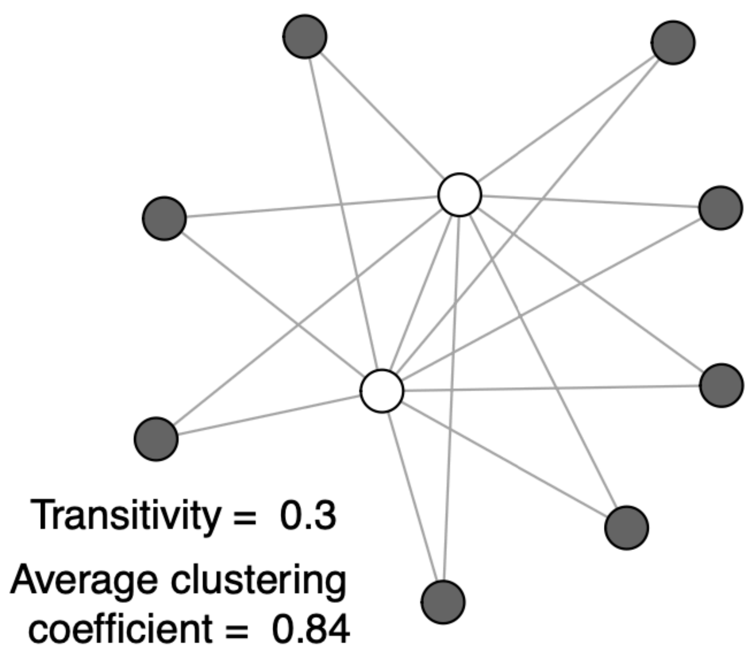


Transitive triplet

$$T = \frac{3\Delta}{\Lambda}$$

Transitivity is a graph level metric

Clustering coefficient and transitivity can diverge (node vs. graph level view)

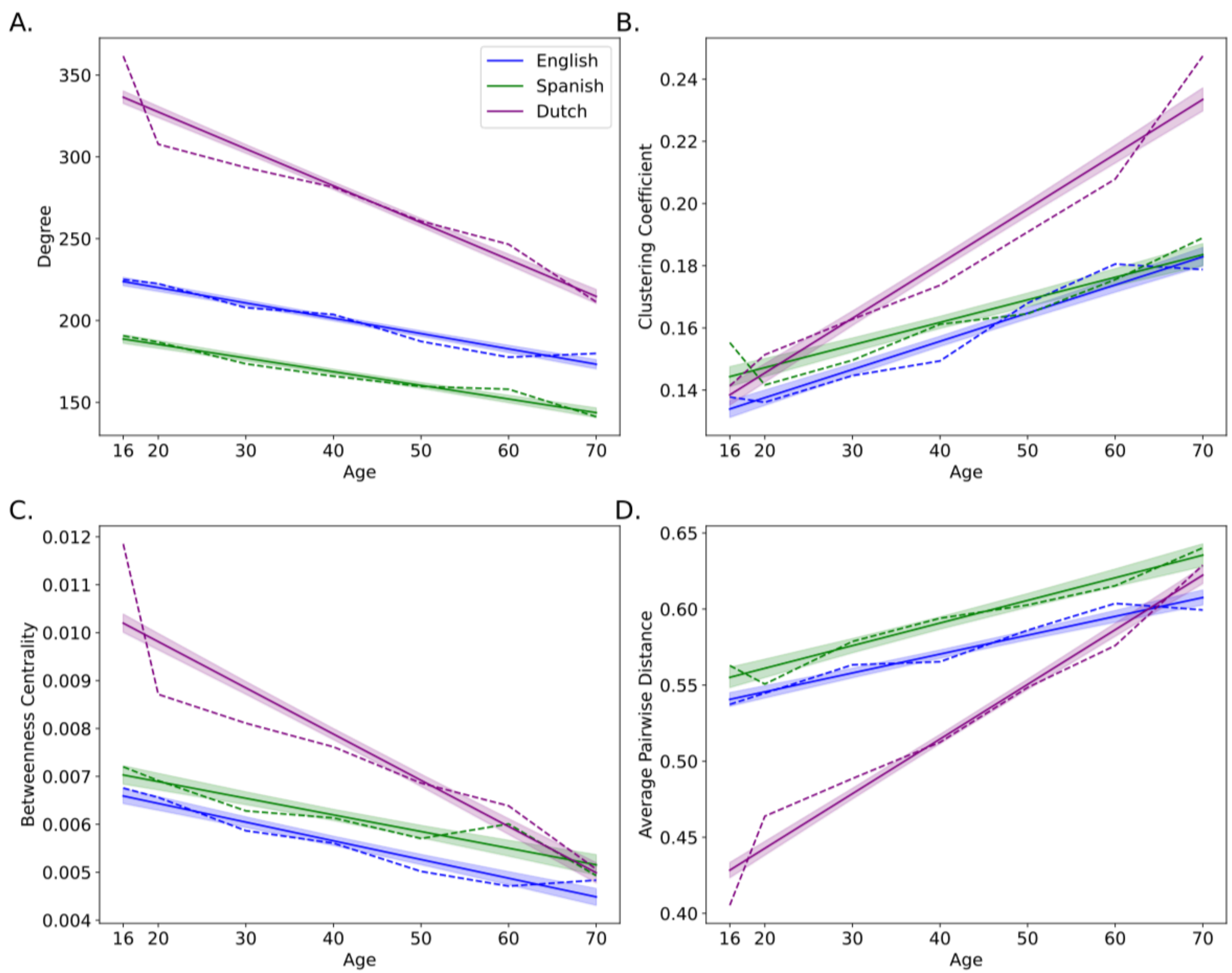


`igraph::transitivity(cnetwork, type =“global”)`

does not produce the same result as

`mean(igraph::transitivity(cnetwork, type =“local"))`

Free associations across the lifespan (what words do people produce)

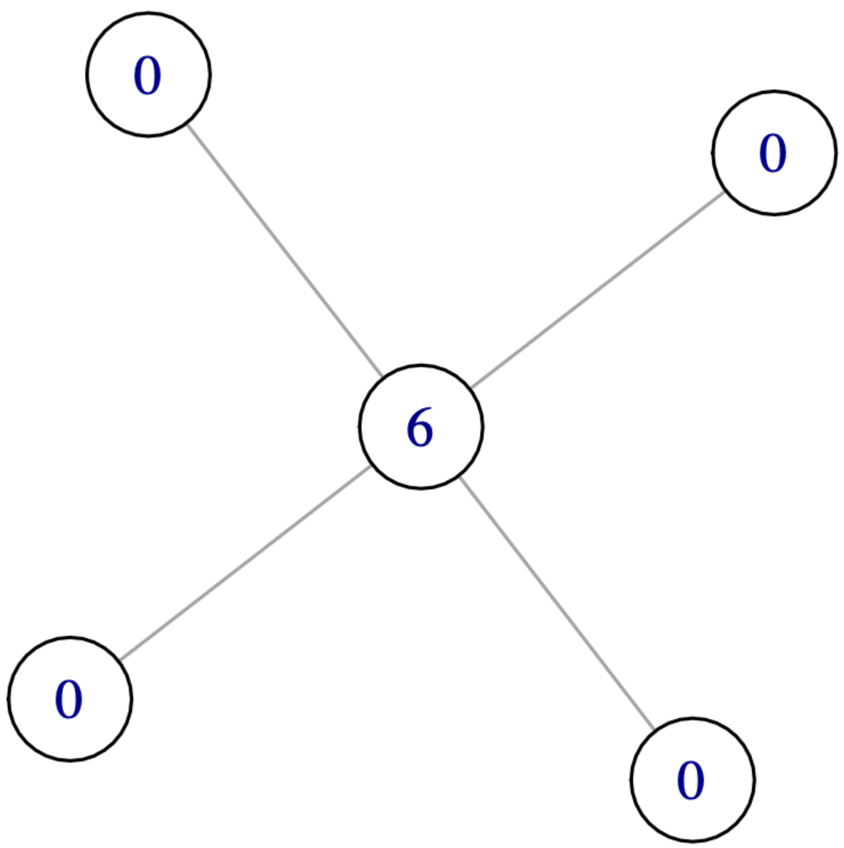


Older adults produce words that are lower degree, higher CC, lower betweenness, and further apart from one another

Jeong & Hills, 2024

Betweenness centrality

The betweenness centrality for a node i is the number of shortest paths between all other pairs of nodes that pass through node i.



$$b_i = \sum_{i \neq j \neq k} \frac{\sigma_{jk}(i)}{\sigma_{jk}}$$

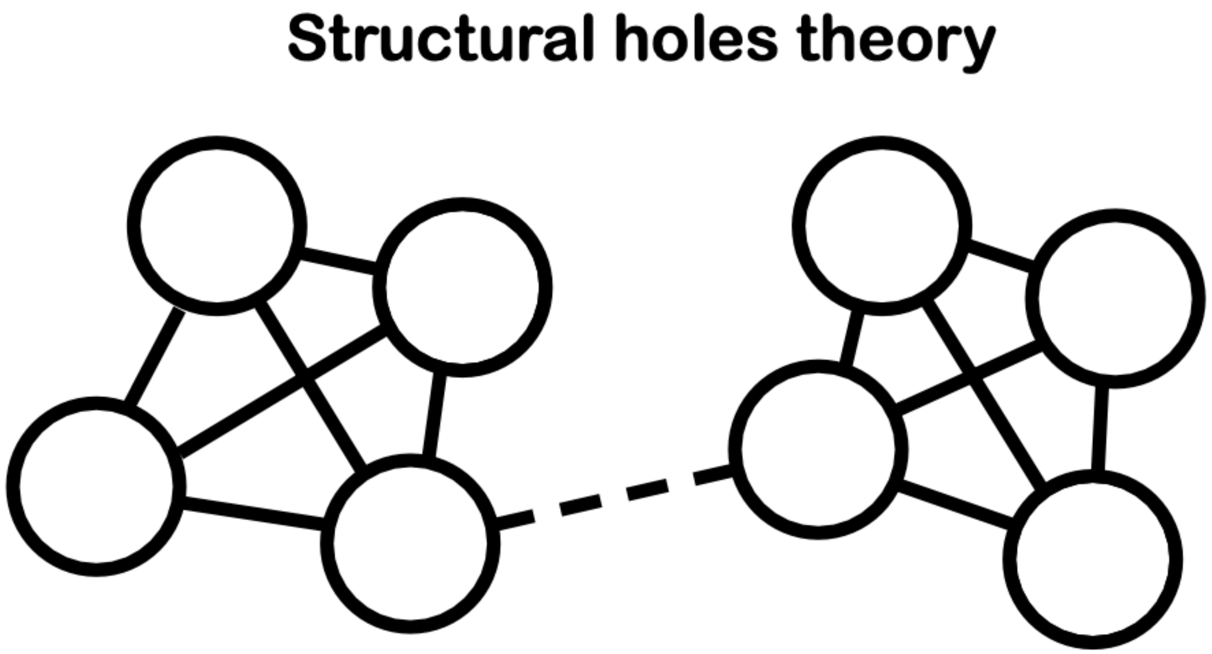
6 possible paths, all pass through the central node.

`igraph::betweenness(g)`

Brokers have high betweenness

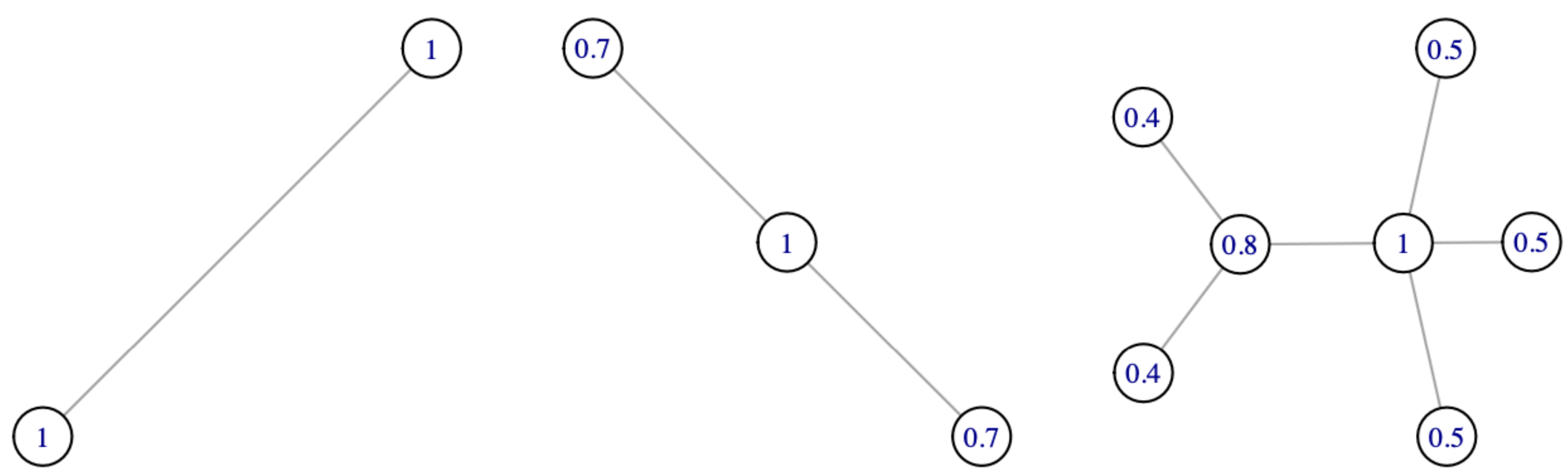
- Burt's *structural holes theory:* brokers between communities add value.

- Note we can also measure betweenness for edges.



Eigenvector centrality

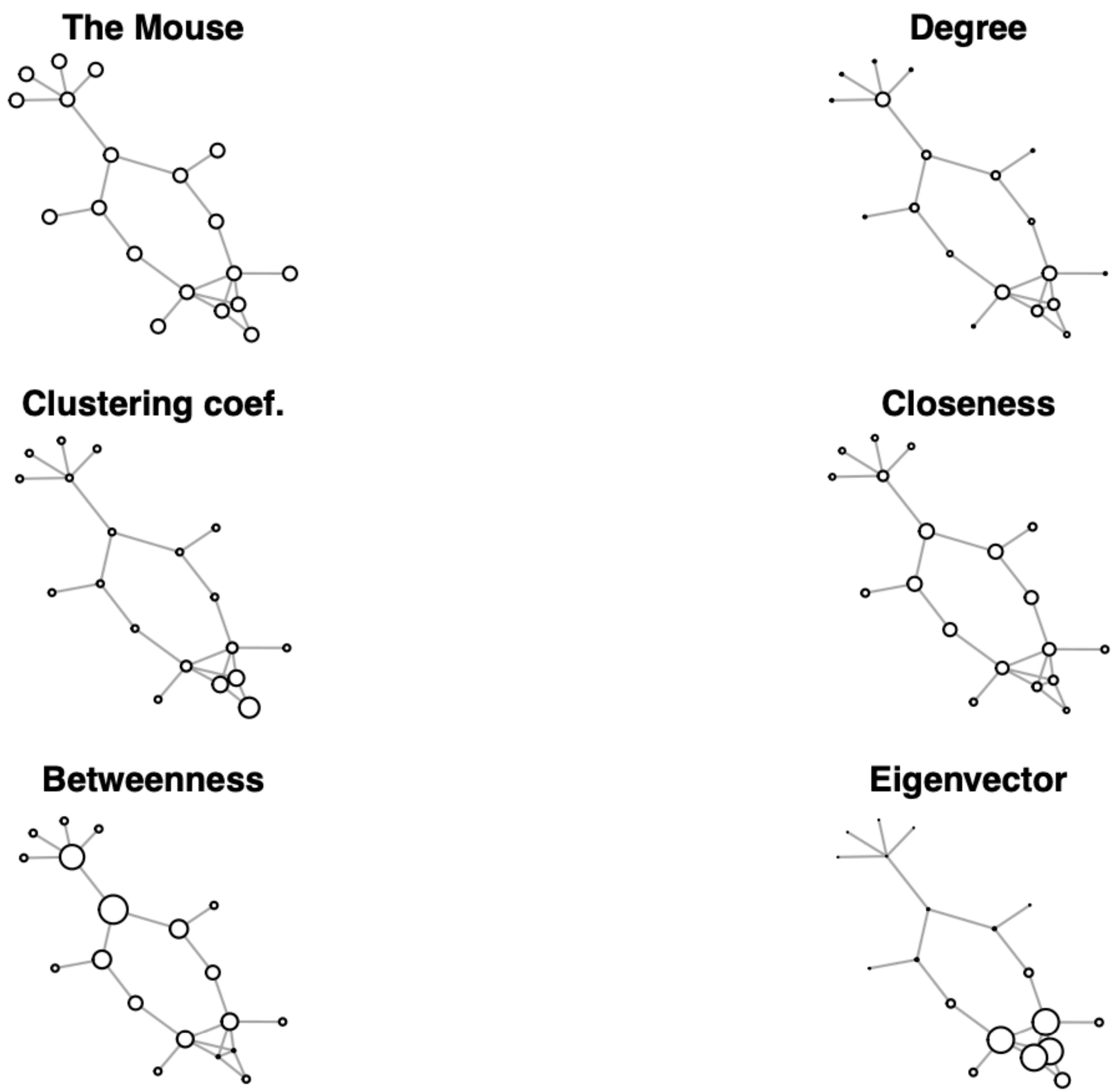
Eigenvector centrality is analogous to prestige. To be prestigious, one must receive prestige from other nodes. The more prestigious the nodes one receives prestige from, the more prestige one receives. The definition is recursive: It requires that we know how prestigious each node is before we can compute the prestige of any node.



This measure is the basis of PageRank and Katz centrality—both look at how nodes recursively give and receive 'value' to their neighbours.

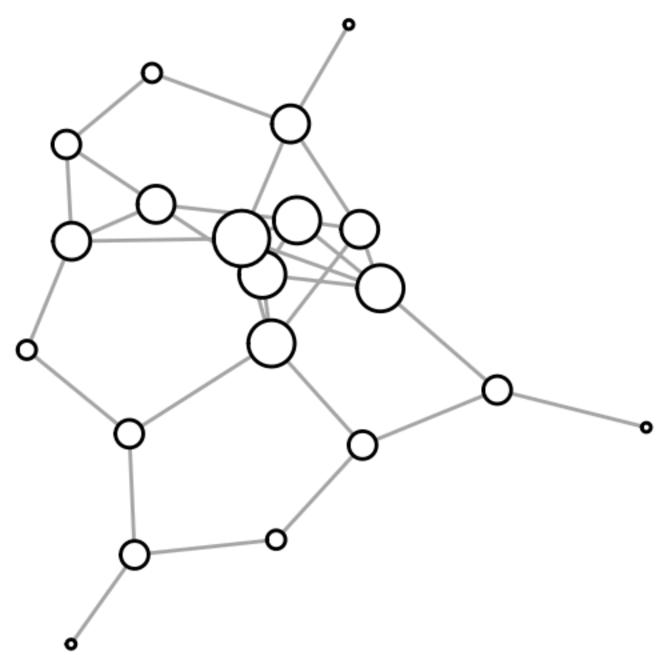
`eigen_centrality(g)$vector`

Measures of centrality

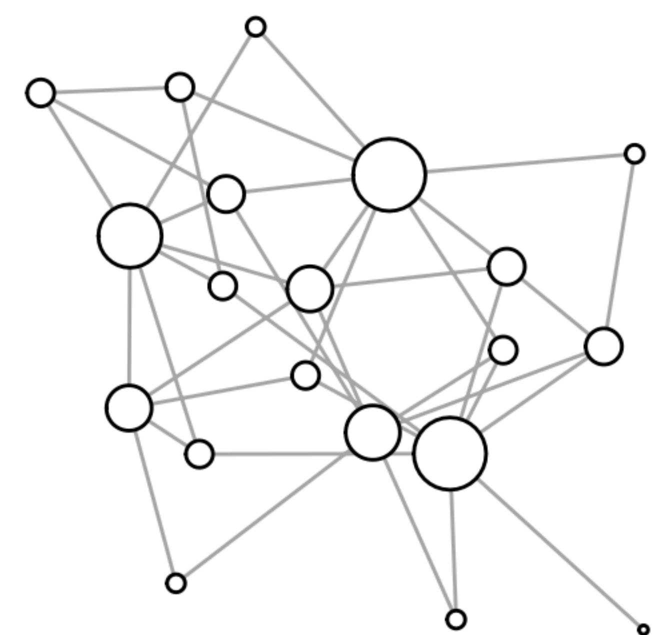


Assortatitivity

Assortativity evaluates the degree to which nodes with similar properties connect with each other. In social networks, this is known as homophily: “birds of a feather flock together.”



r=.45

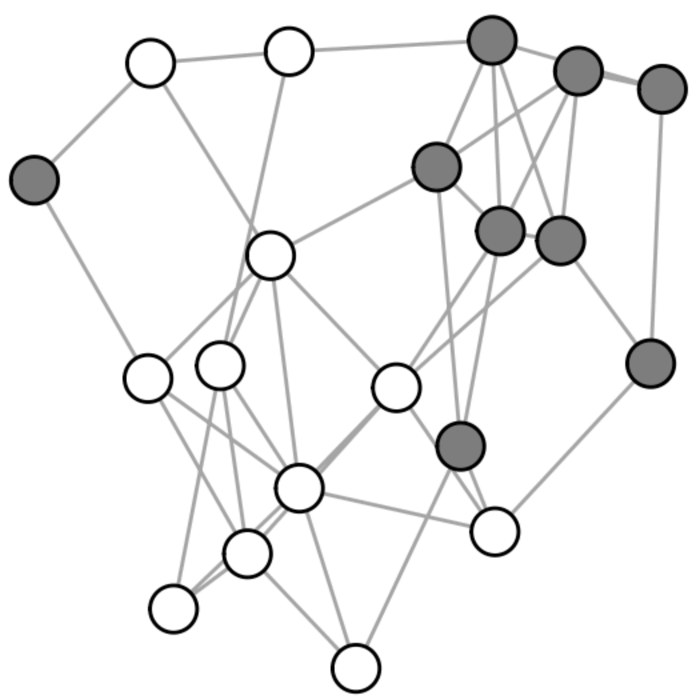


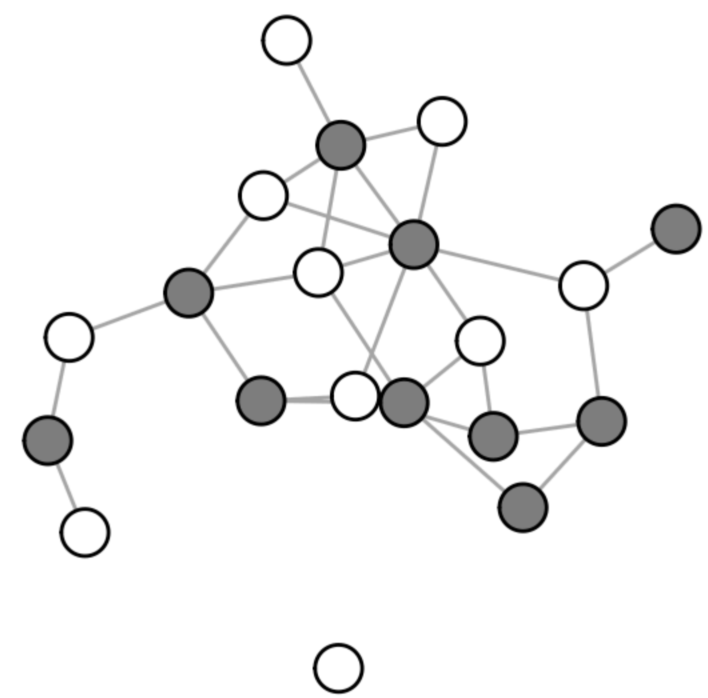
r=-.62

To evaluate assortativity we compute an assortativity coefficient: the Pearson correlation between pairs of connected nodes in the network with respect to the value in question. To do this, generate an edge list from the network, replace the node labels with the value for each node, and take the correlation of the two columns of values.

Assortatitivity

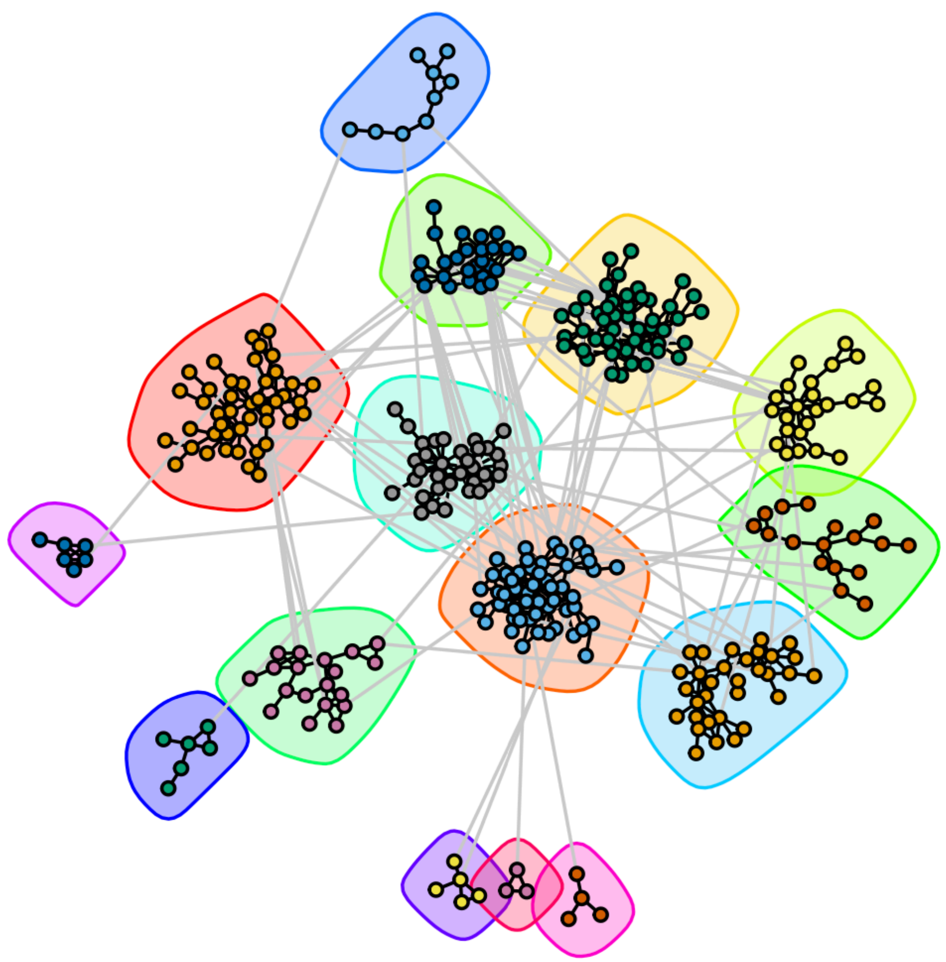
Assortativity by color (a node attribute)





Community Detection

Communities can be detected by identifying clusters of nodes that are more well connected to one another than they are to members of other communities. A division of the network into a set of communities is called a *partition*.

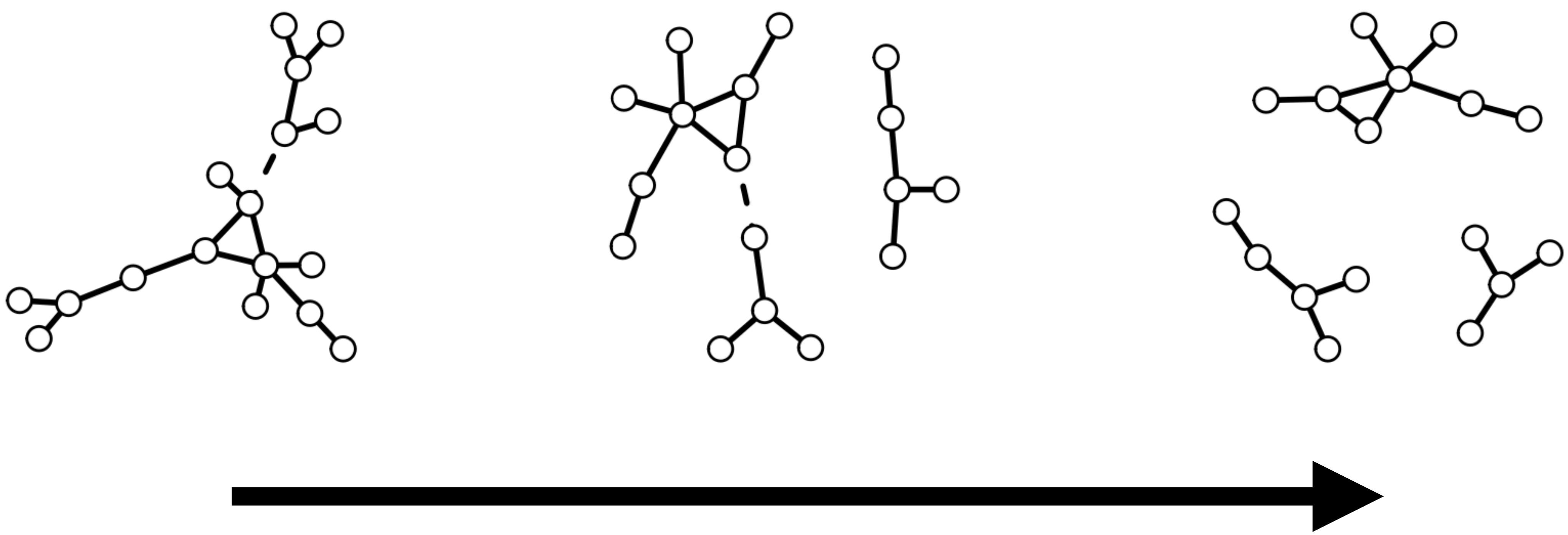


Network science offers a number of methods for identifying structural categories in data.

I'll show you a few, because I want you to understand how they work.

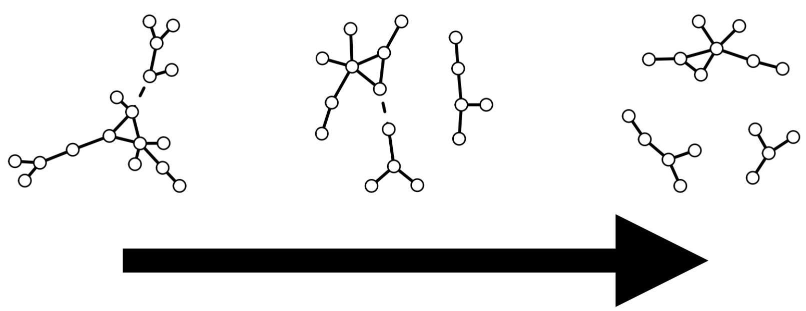
Girvan-Newman Method

The Girvan Newman Method (Girvan and Newman 2002) (or edge betweenness method) is based on the observation that edges connecting separate communities have high edge betweenness: *shortest paths between members of different communities will pass through edges with high edge betweenness.*

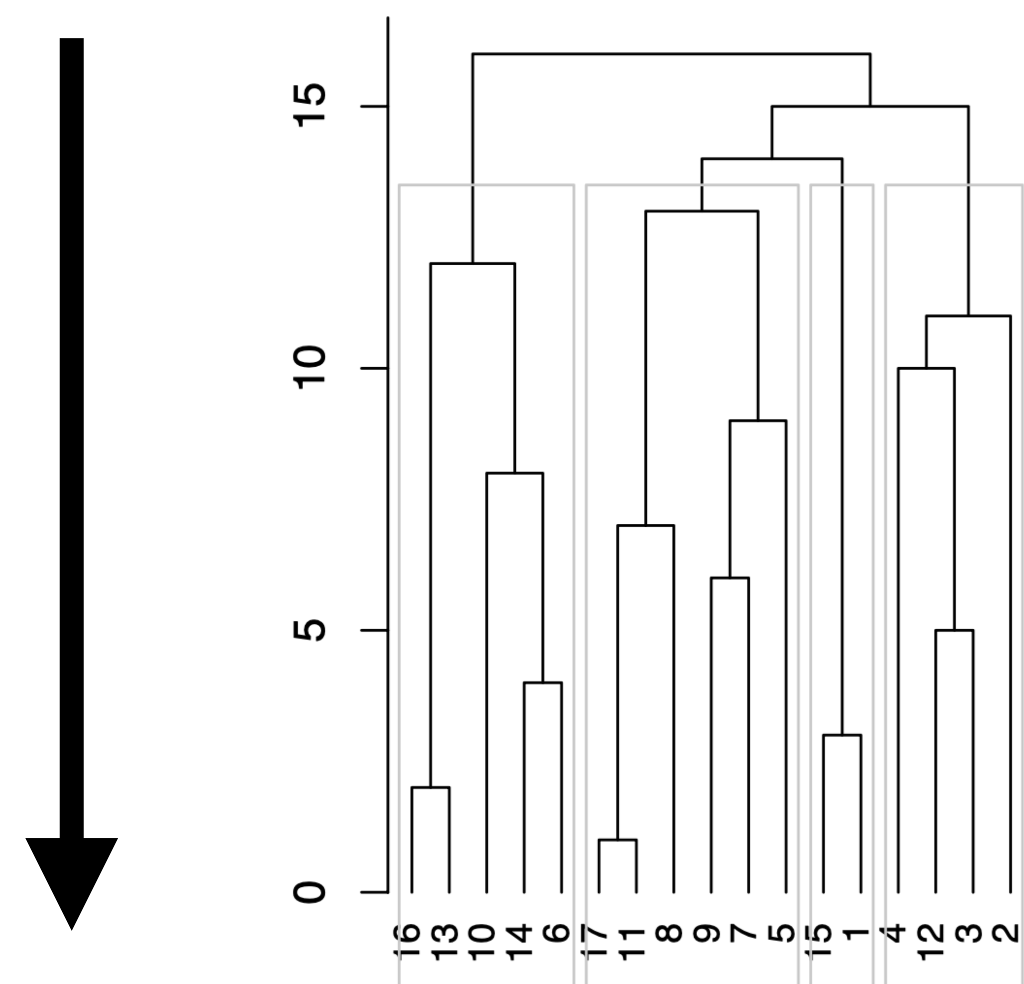


if we cut the edge with the highest betweenness repeatedly, we partition the network into components.

Girvan-Newman Method



Doing the above generates a tree of components, with the number of components growing with additional edge removals.



How to pick the best partition?

Modularity

Modularity, Q, is a measure of the difference between the observed links within communities and the expected links within the same communities if all edges were distributed at random.

$$Q = \frac{1}{2m} \sum_{i.j} [A_{ij} - \frac{k_i * k_j}{2m}] \delta(c_i, c_j)$$

Observed

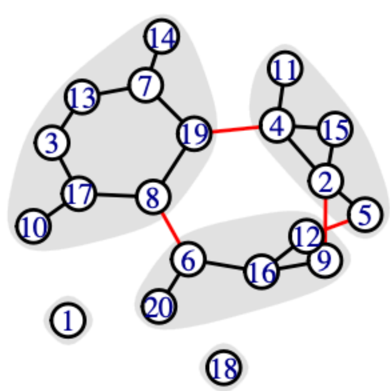
Expected
(approximation)

High modularity means more observed connections between group members than expected.
Choose partition with highest modularity.

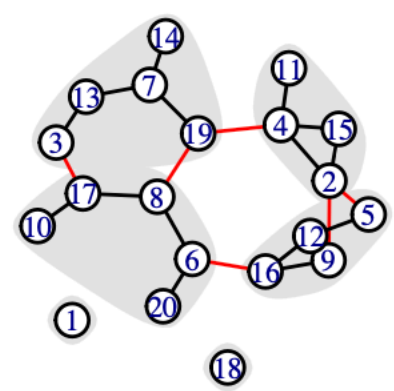
Community detection

(Many methods)

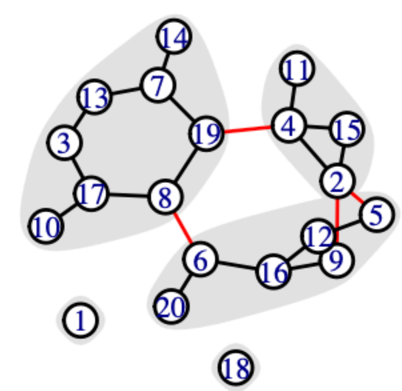
Girvan Newman



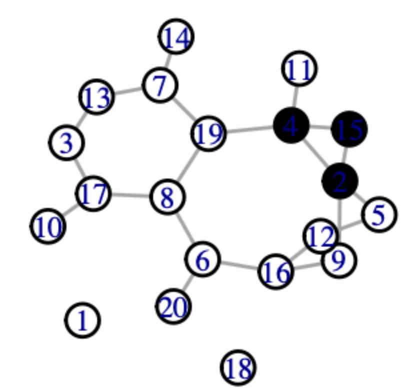
Louvain



Walktrap



Clique Percolation



- Part 3 — Generating networks, null hypotheses

Good network science makes comparisons

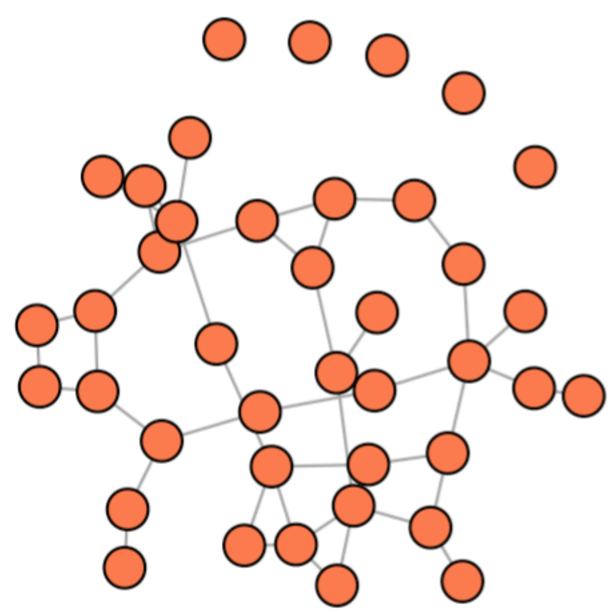
- You can measure structure on networks.
- But this structure needs comparators to be meaningful.
- You could compare:
    - the same network at different times
    - different groups
    - random networks designed to preserve certain features
    - models of network generation
- It's often most meaningful to compare networks of the same size.

`igraph::subgraph(graph, vertices)`

Erdös-Renyi random graphs

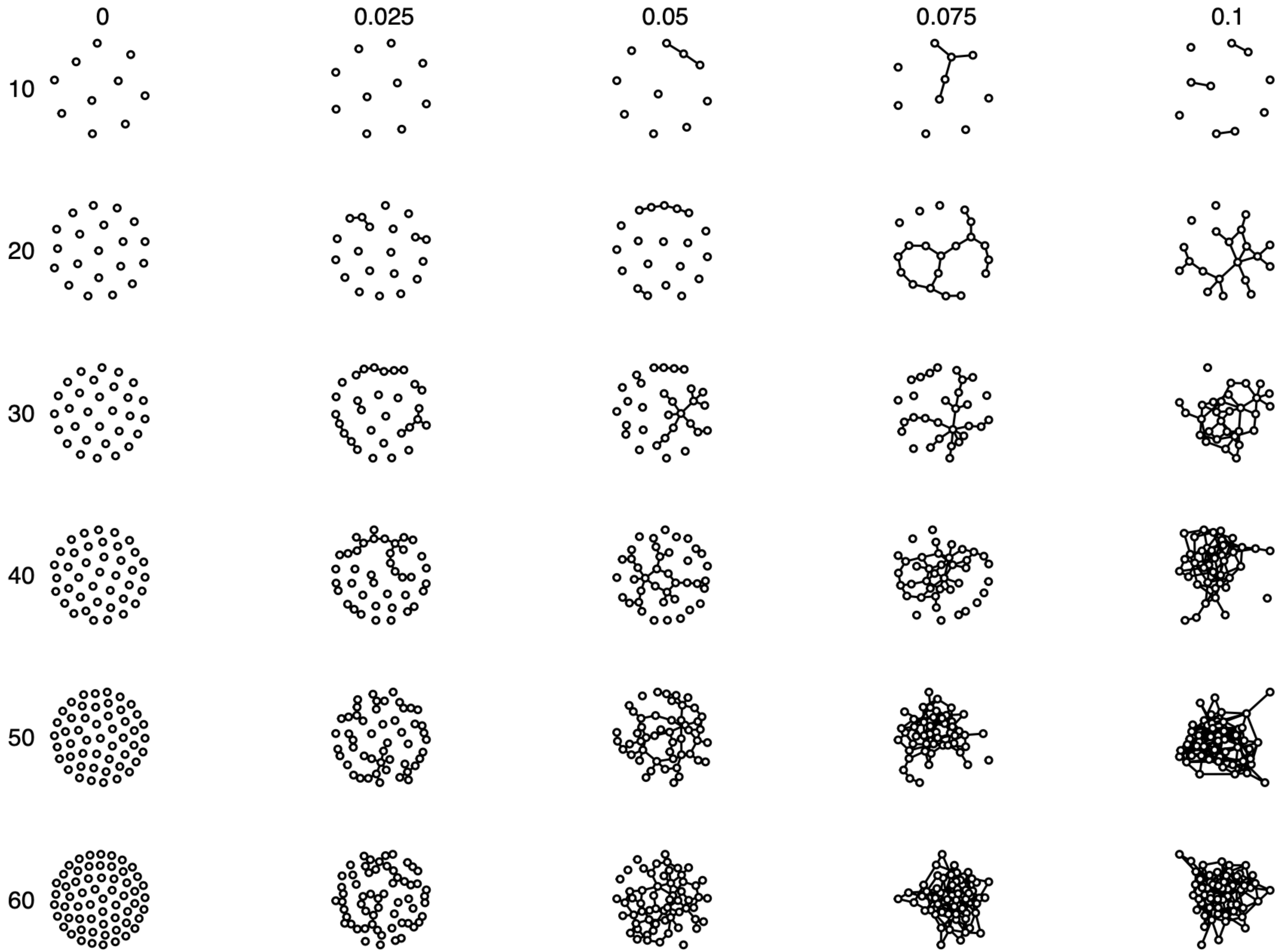
G(N,E) — Define N nodes and probability of edge (or density)

This is *the* standard model for a random graph.



```
igraph::sample_gnp(N, p)
```

ER Random Graphs —what happens as p and N increase?

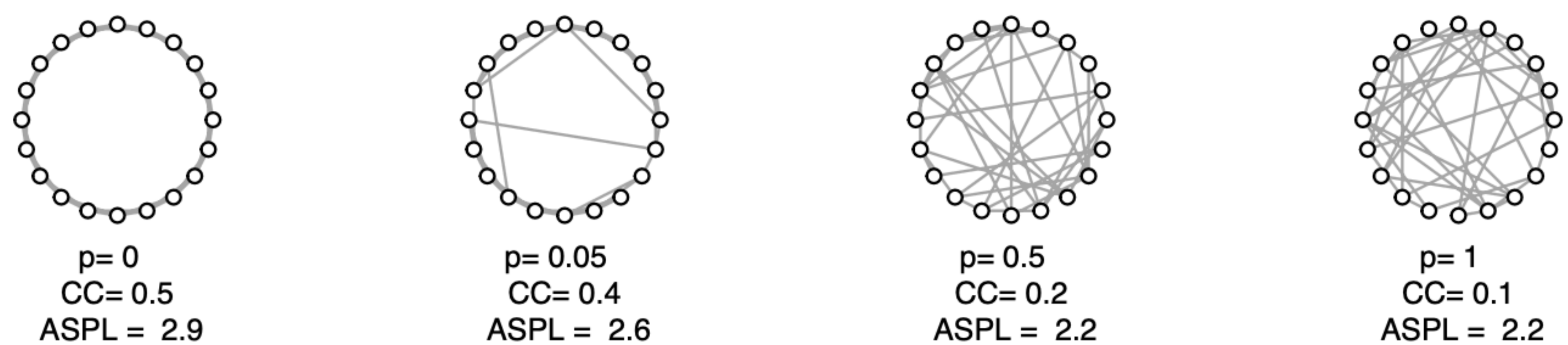


recall: I mentioned comparing graphs of different sizes is evil.

Small world networks

(Watts Strogatz)

Small world networks: nodes are closer together than they would be in a random network.



Start with structured lattice (ring lattice) and rewire edges with probability p

Small World Networks
(Watts Strogatz)

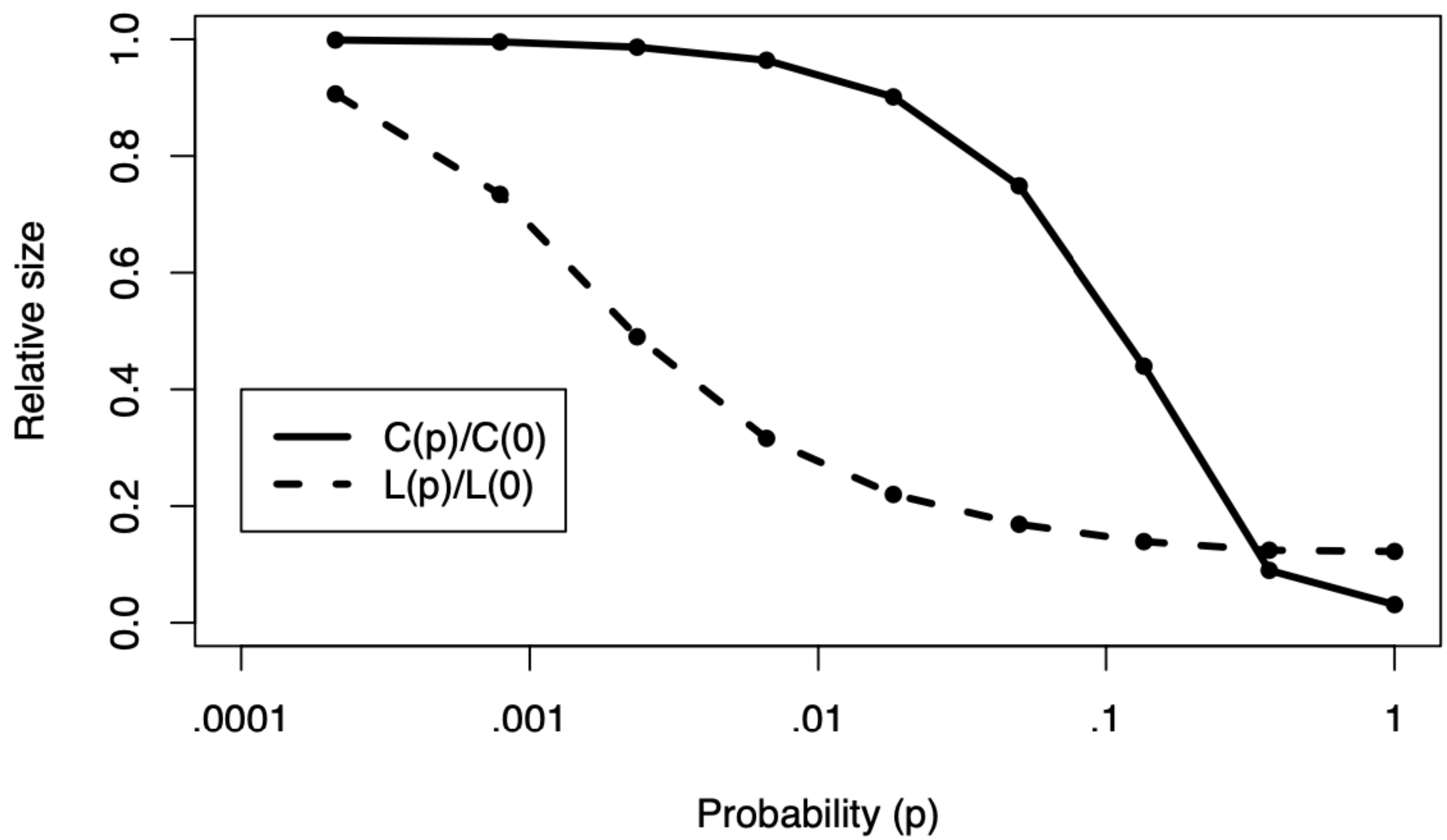
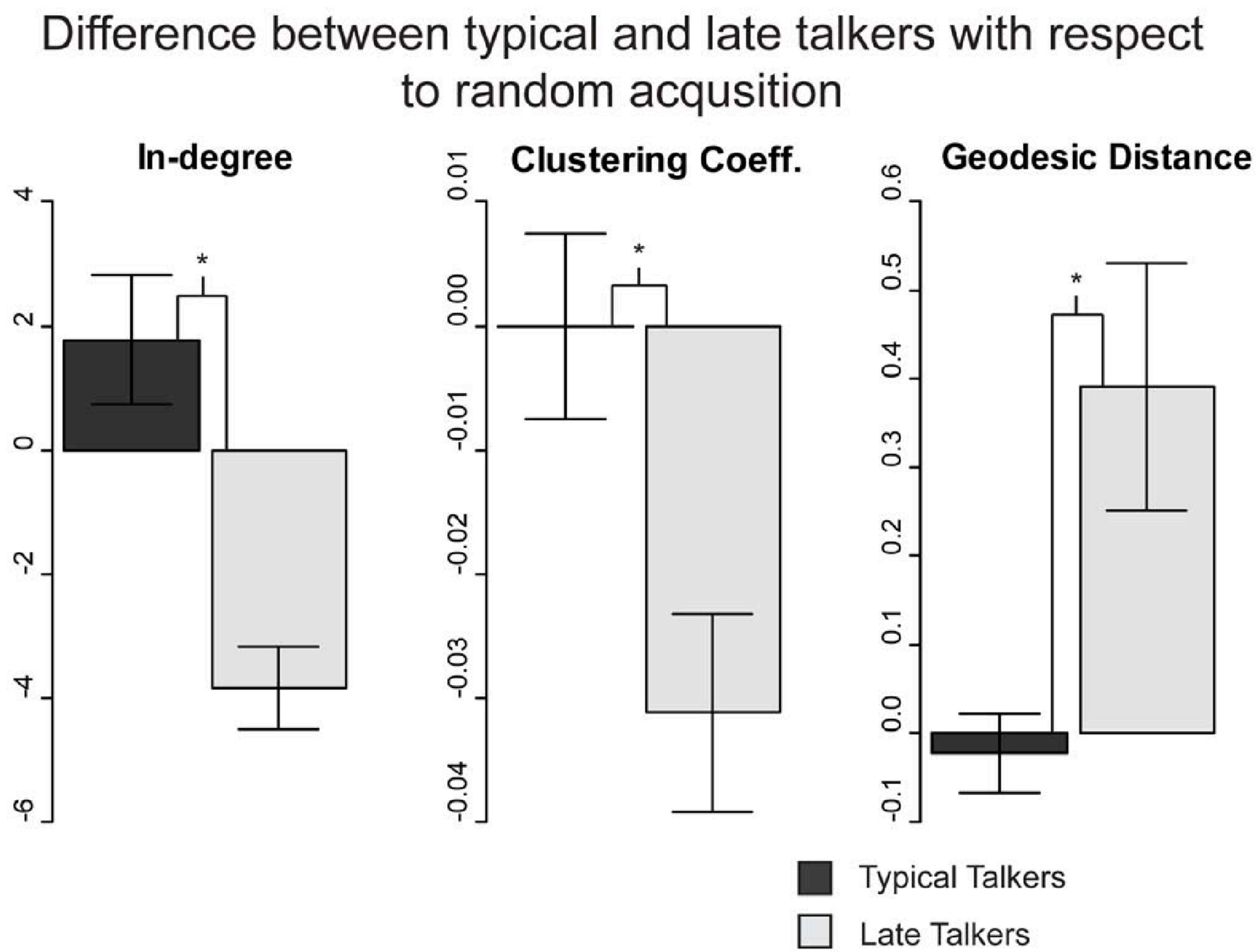


Figure 2: Relative clustering coefficient and average shortest path length for the Watts-Strogatz small world model across a range of rewiring probabilities, p.

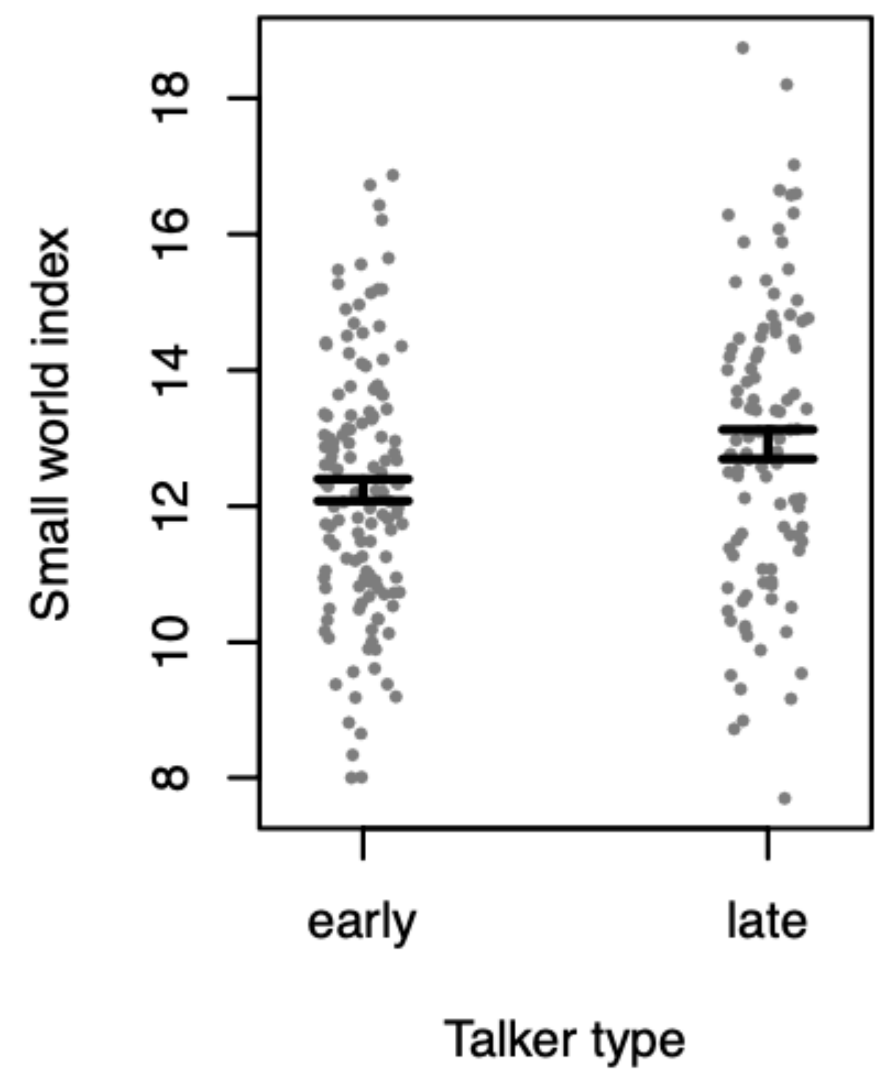
Clustering coefficient and the words children learn



Late talkers have lower degree and lower clustering coefficient and have average shortest path length (ASPL = geodesic distance)

Small world Index

$$SWI = \frac{\frac{C_{observed}}{C_{random}}}{\frac{L_{observed}}{L_{random}}} = \frac{\gamma}{\lambda}$$

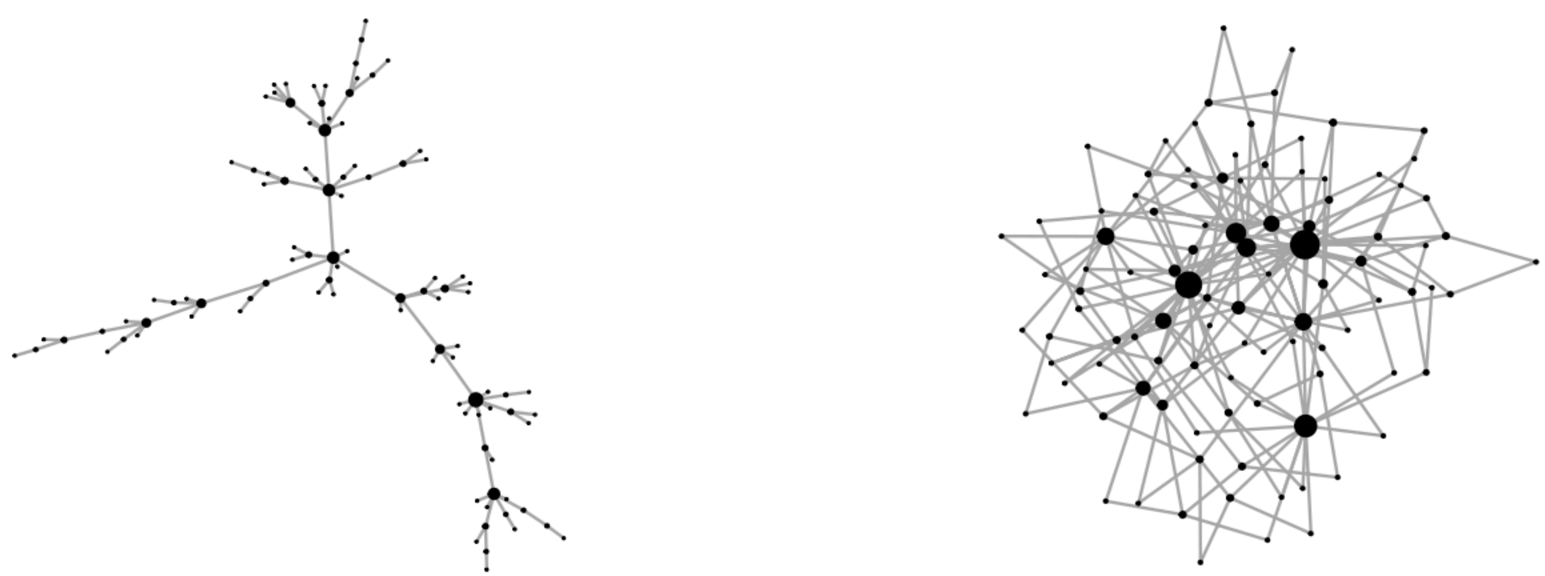


Chapter 7 discusses the 'small world spectrum' and multiple different measures.

Preferential Attachment

(Barabasi)

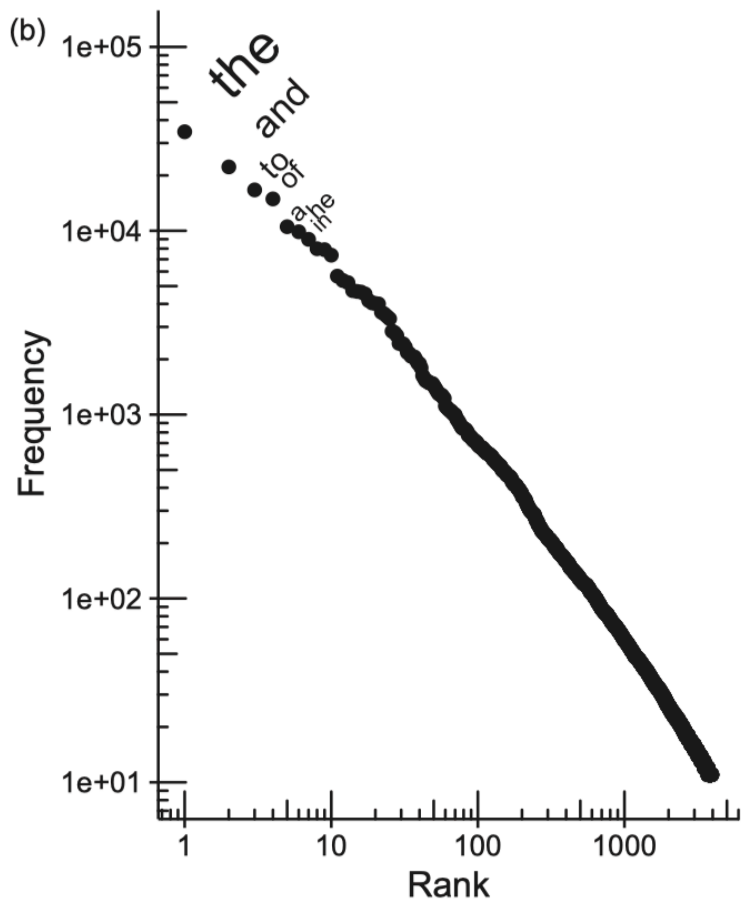
Nodes are added one at a time and preferentially attach to existing nodes ~ f(degree)



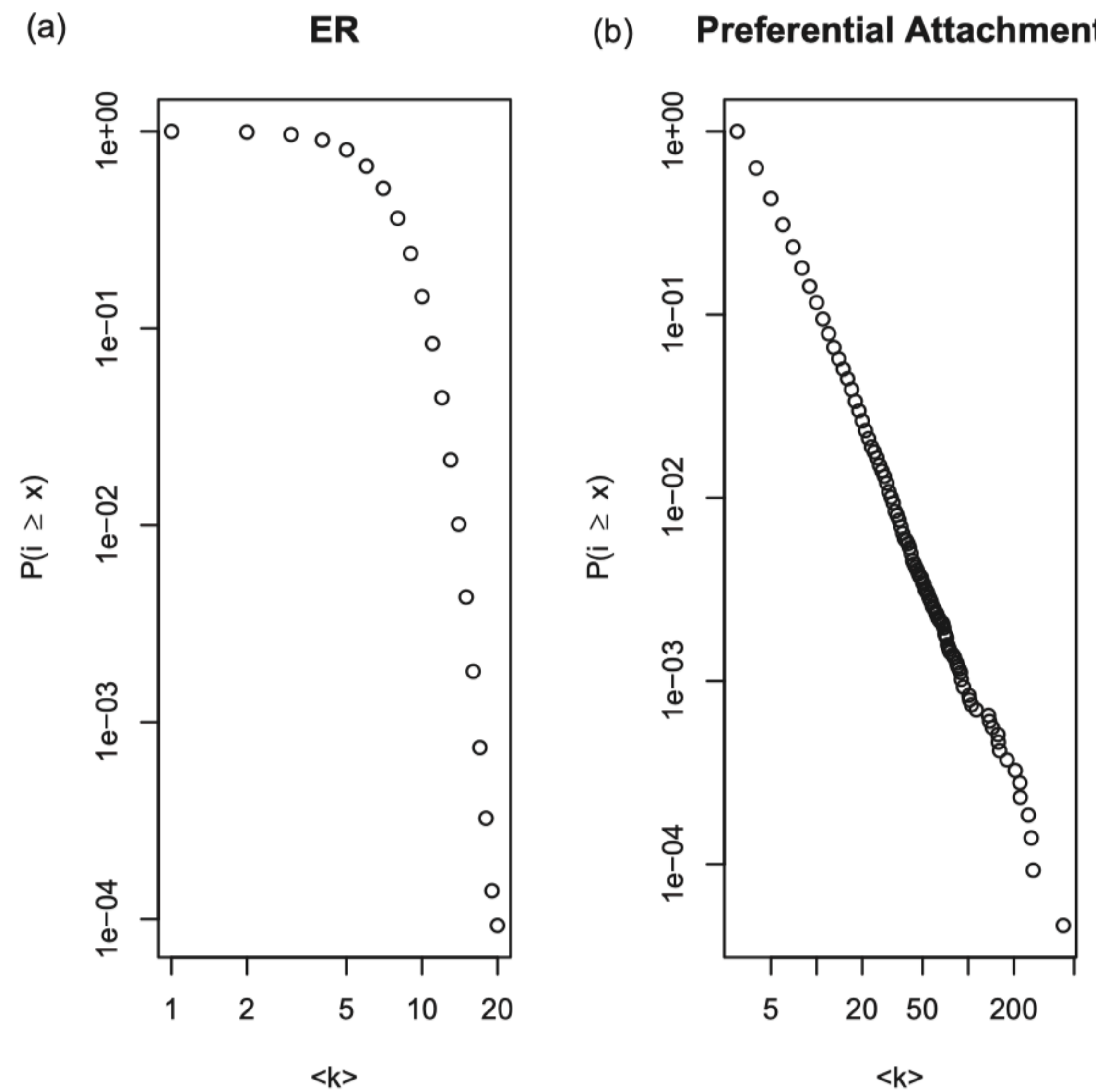
$$P(i) = \frac{k_i^{\alpha} + a}{\sum_{j \in N}^{N} k_j^{\alpha} + a}$$

Degree distributions for Scale-free Networks

Scale-free networks are linear on a log-log plot



Zipf's Law for words



Degree distributions for various network models

Scale-free Networks

Fractals are scale-free

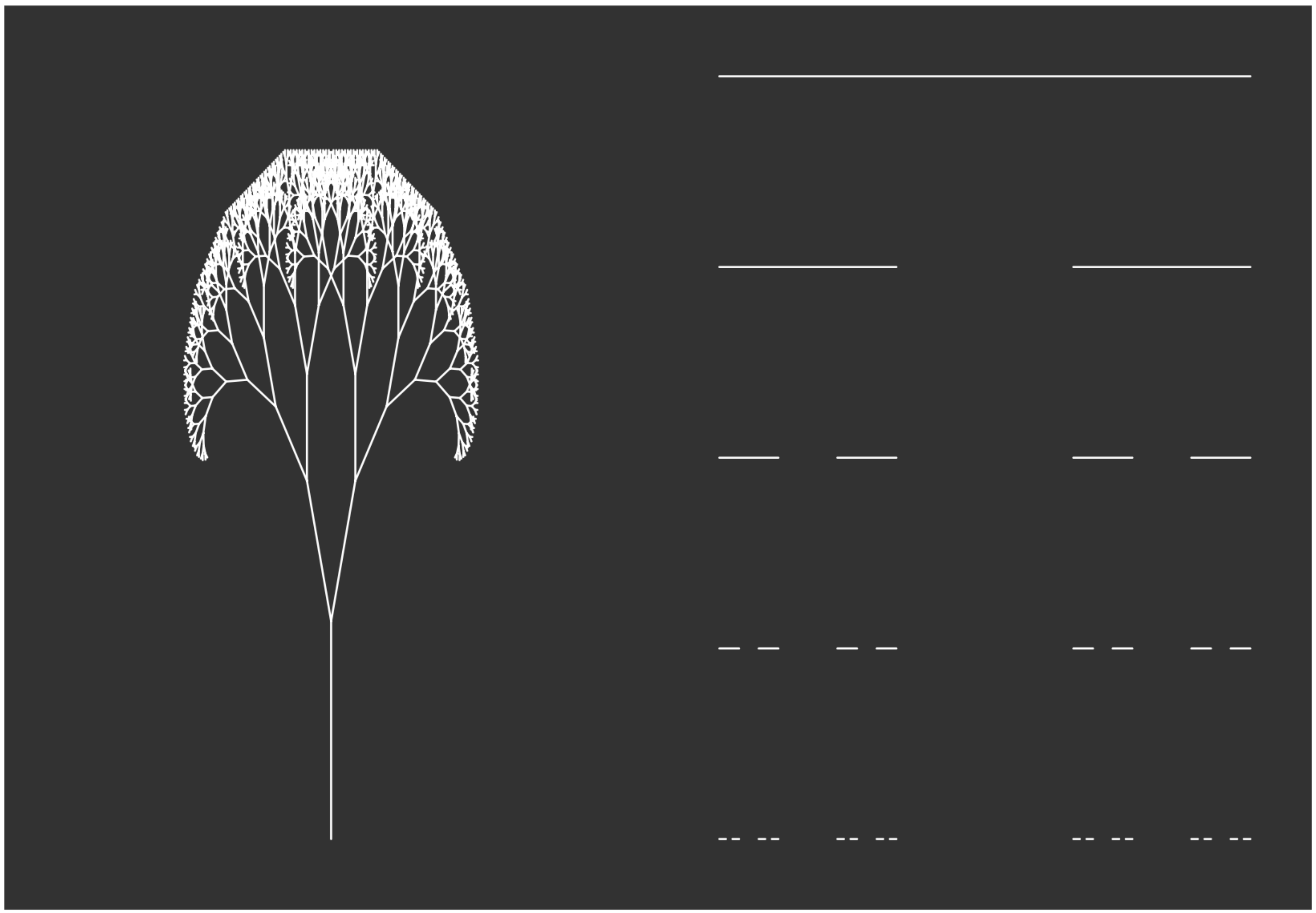
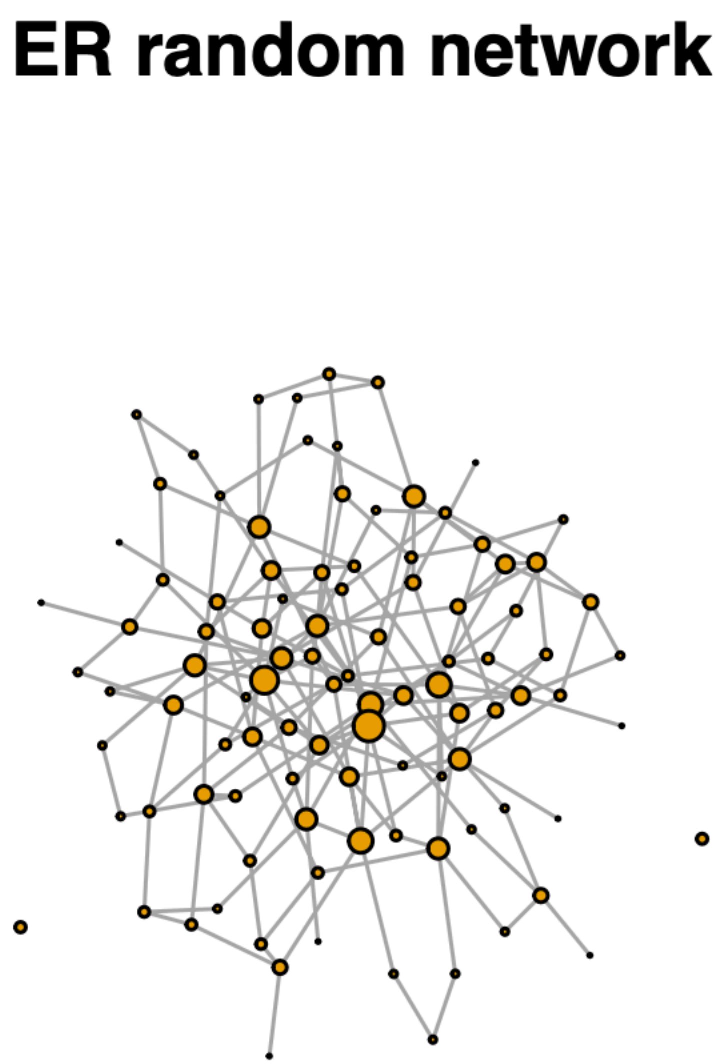


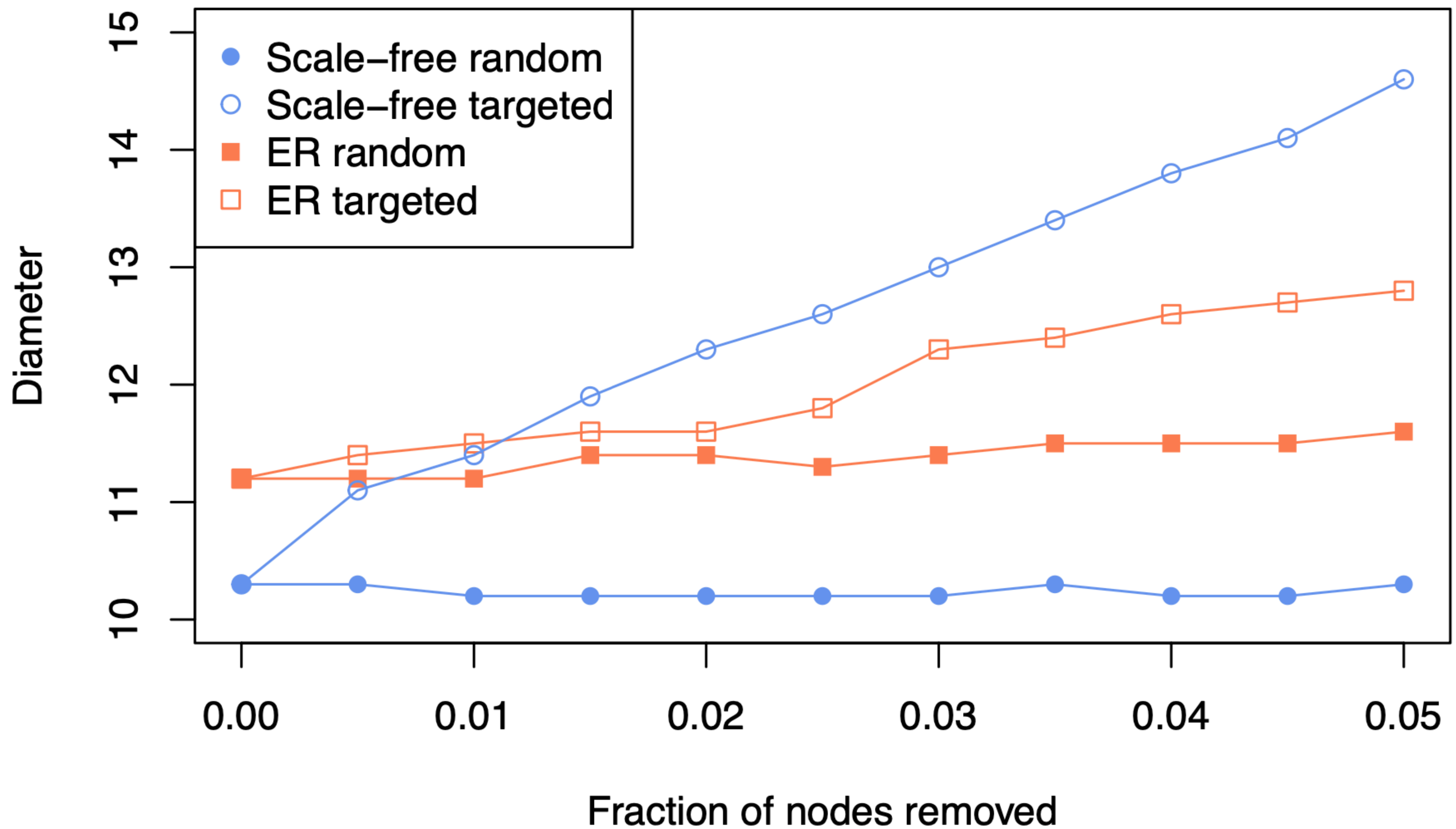
Figure 2: Fractals exhibiting scale-free and power-law distributions of edge lengths. On the left is a recursive tree: each branch leads to two new lines of reduced size. On the right is a Cantor Set: each line produces two new lines of 1/3 the size as we move downwards. Note that as we move up in size, the number of larger lines is always half the number of lines one-size smaller.

Scale-free Networks



Attack Tolerance

Scale-free networks are resilient to random attacks, but not to targeted attacks



This is because HUBS have a large influence on scale free networks.

Figure 5: Targeted and random attacks on random and scale-free networks. All networks have N=5000 nodes and E=10000 edges. Random error attacks remove a fraction, f, of nodes with a uniform probability across all nodes. Targeted attacks remove the fraction, f, of nodes with the highest degree. Scale-free networks are more resilient to random attacks, but more susceptible to targeted attacks.

Albert & Barabasi, 2000

Additional null models

Comparing a network with variations of itself

- We can compare a network’s statistics with variations of itself
- We want to compare networks of a similar size and perhaps other statistical properties (e.g., exponential random graphs).

- Configuration networks (rewire edges, keeping degree distribution)
- Random acquisition networks (acquire nodes randomly from a larger subgraph) (Haebig et al., 2025)
- bootstrapping (subgraphs) (Wulff et al., 2022)

- go to part 4