

Partial compact quantum groups, compact quantum homogeneous spaces and the dynamical quantum $SU(2)$ group

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Abstract

Compact quantum groups of face type, as introduced by T. Hayashi, form a class of quantum groupoids with a classical, finite set of objects. We generalize Hayashi's definition to allow for an infinite set of objects, and call the resulting objects partial compact quantum groups. We then show how any quantum homogeneous space of an ordinary compact quantum group leads to a partial compact quantum group. In particular, when this construction is applied to the non-standard Podleś spheres, we obtain partial compact quantum groups which are operator algebraic versions of the dynamical quantum $SU(2)$ -group as studied by Etingof-Varchenko and Koelink-Rosengren.

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1 Partial compact quantum groups

We generalize Hayashi's definition of a compact quantum group of face type [6] to the case where the commutative base algebra is no longer finite-dimensional. We will present two approaches, based on *partial bialgebras* and *weak multiplier bialgebras* [2]. The first approach is piecewise and concrete, but requires some bookkeeping. The second approach is global but more abstract. As we will see from the general theory and the concrete examples, both approaches have their intrinsic value.

Let I be a set. We consider $I^2 = I \times I$ as the pair groupoid with \cdot denoting composition. That is, an element $K = (k, l) \in I^2$ has source $K_l = k$ and target $K_r = l$, and if $K = (k, l)$ and $L = (l, m)$ we write $K \cdot L = (k, m)$.

Definition 1.1. A *partial algebra* $\mathcal{A} = (\mathcal{A}, M)$ (over \mathbb{C}) is a small \mathbb{C} -linear category, that is, a set I (the object set) together with

- for each $K = (k, l) \in I^2$ a vector space $A(K) = A(k, l) = {}_k A_l$ (possibly the zero vector space),
- for each K, L with $K_r = L_l$ a multiplication map

$$M(K, L) : A(K) \otimes A(L) \rightarrow A(K \cdot L), \quad a \otimes b \mapsto ab$$

and

- elements $\mathbf{1}(k) = \mathbf{1}_k \in A(k, k)$ (the units),

such that the obvious associativity and unit conditions are satisfied.

By *I-partial algebra* will be meant a partial algebra with object set I .

Remark 1.2. We allow the local units $\mathbf{1}_k$ to be zero.

Let \mathcal{A} be an I -partial algebra. We define $A(K \cdot L)$ to be $\{0\}$ when $K \cdot L$ is ill-defined, i.e. $K_r \neq L_l$. We then let $M(K, L)$ be the zero map.

Definition 1.3. The *total algebra* A of an I -partial algebra \mathcal{A} is the vector space

$$A = \bigoplus_{K \in I^2} A(K)$$

endowed with the unique multiplication whose restriction to $A(K) \otimes A(L)$ concides with $M(K, L)$.

Clearly A is an associative algebra. If I is infinite it will not possess a unit, but it is a *locally unital algebra* as there exist mutually orthogonal idempotents $\mathbf{1}_k$ with $A = \sum_{k,l}^{\oplus} \mathbf{1}_k A \mathbf{1}_l$. An element $a \in A$ can be interpreted as a function assigning to each element

$(k, l) \in I^2$ an element $a_{kl} \in A(k, l)$, namely the (k, l) -th component of a . This identifies A with finite support I -indexed matrices whose (k, l) -th entry lies in $A(k, l)$, equipped with the natural matrix multiplication.

Remark 1.4. When \mathcal{A} is an I -partial algebra with total algebra A , then $A \otimes A$ can be naturally identified with the total algebra of an $I \times I$ -partial algebra $\mathcal{A} \otimes \mathcal{A}$, where

$$(A \otimes A)((k, k'), (l, l')) = A(k, l) \otimes A(k', l')$$

with the obvious tensor product multiplications and the $\mathbf{1}_{k, k'} = \mathbf{1}_k \otimes \mathbf{1}_{k'}$ as units.

The notion of partial algebra dualizes. For this we consider again I^2 as the pair groupoid, but now with elements considered as column vectors, and with $*$ denoting the (vertical) composition. So $K = \begin{pmatrix} k \\ l \end{pmatrix}$ has source $K_u = k$ and target $K_d = l$, and if $K = \begin{pmatrix} k \\ l \end{pmatrix}$ and $L = \begin{pmatrix} l \\ m \end{pmatrix}$ then $K * L = \begin{pmatrix} k \\ m \end{pmatrix}$.

Definition 1.5. A *partial coalgebra* $\mathcal{A} = (\mathcal{A}, \Delta)$ (over \mathbb{C}) consists of a set I (the object set) together with

- for each $K = \begin{pmatrix} k \\ l \end{pmatrix} \in I^2$ a vector space $A(K) = A\left(\begin{pmatrix} k \\ l \end{pmatrix}\right) = A_l^k$,
- for each K, L with $K_d = L_u$ a comultiplication map

$$\Delta\left(\begin{pmatrix} K \\ L \end{pmatrix}\right) : A(K * L) \rightarrow A(K) \otimes A(L), \quad a \mapsto a_{(1)K} \otimes a_{(2)L},$$

and

- counit maps $\epsilon_k : A\left(\begin{pmatrix} k \\ k \end{pmatrix}\right) \rightarrow \mathbb{C}$,

satisfying the obvious coassociativity and counitality conditions.

By *I -partial coalgebra* will be meant a partial coalgebra with object set I .

Notation 1.6. As the index of ϵ_k is determined by the element to which it is applied, there is no harm in dropping the index k and simply writing ϵ .

Similarly, if $K = \begin{pmatrix} k \\ l \end{pmatrix}$ and $L = \begin{pmatrix} l \\ m \end{pmatrix}$, we abbreviate $\Delta_l = \Delta\left(\begin{pmatrix} K \\ L \end{pmatrix}\right)$, as the other indices are determined by the element to which Δ_l is applied.

We also make again the convention that $A(K * L) = \{0\}$ and $\Delta\left(\begin{pmatrix} K \\ L \end{pmatrix}\right)$ the zero map when $K_d \neq L_u$. Similarly ϵ is seen as the zero functional on $A(K)$ when $K = \begin{pmatrix} k \\ l \end{pmatrix}$ with $k \neq l$.

We can now superpose the notions of partial algebra and partial coalgebra. To formulate the condition that the coalgebra maps form a ‘morphism of partial algebras’, we will need to impose a finiteness condition which is automatically satisfied when the cardinality of I is finite.

Let I be a set, and let $M_2(I)$ be the set of 4-tuples of elements of I arranged as 2×2 -matrices. We can endow $M_2(I)$ with two compositions, namely \cdot (viewing $M_2(I)$ as a row vector of column vectors) and $*$ (viewing $M_2(I)$ as a column vector of row vectors). When $K \in M_2(I)$, we will write $K = (K_l, K_r) = \begin{pmatrix} K_u \\ K_d \end{pmatrix} = \begin{pmatrix} K_{lu} & K_{ru} \\ K_{ld} & K_{rd} \end{pmatrix}$. One can view $M_2(I)$ as a double groupoid, and in fact as a *vacant* double groupoid in the sense of [1].

In the following, a vector (r, s) will sometimes be written simply as r, s (without parentheses) or rs in an index. We also follow Notation 1.6, but the reader should be aware

that the index of Δ will now be a 1×2 vector in I^2 as we will work with partial coalgebras over I^2 .

Definition 1.7. A *partial bialgebra* $\mathcal{A} = (\mathcal{A}, M, \Delta)$ consists of a set I and a collection of vector spaces $A(K)$ for $K \in M_2(I)$ such that

- the $A(K_l, K_r)$ form an I^2 -partial algebra,
- the $A\left(\begin{smallmatrix} K_u \\ K_d \end{smallmatrix}\right)$ form an I^2 -partial coalgebra,

and for which the following compatibility relations are satisfied.

- (a) (Comultiplication of Units) For all $k, l, m \in I$, one has

$$\Delta_{l,l}(\mathbf{1}\binom{k}{m}) = \mathbf{1}\binom{k}{l} \otimes \mathbf{1}\binom{l}{m}.$$

- (b) (Counit of Multiplication) For all $K, L \in M_2(I)$ with $K_r = L_l$ and all $a \in A(K)$ and $b \in A(L)$,

$$\epsilon(ab) = \epsilon(a)\epsilon(b).$$

- (c) (Non-degeneracy) For all $k \in I$, $\epsilon(\mathbf{1}\binom{k}{k}) = 1$.

- (d) (Finiteness) For each $K \in M_2(I)$ and each $a \in A(K)$, the element $\Delta_{rs}(a)$ is zero except for a finite number of indices r (resp. s) when s (resp. r) is fixed.

- (e) (Comultiplication is multiplicative) For all $a \in A(K)$ and $b \in A(L)$ with $K_r = L_l$,

$$\Delta_{rs}(ab) = \sum_t \Delta_{rt}(a)\Delta_{ts}(b).$$

Remark 1.8. By assumption (d), the sum on the right hand side in condition (e) is in fact finite and hence well-defined.

We want to relate the notion of partial bialgebra to the recently introduced notion of weak multiplier bialgebra [2]. We first recall some notions concerning non-unital algebras [3, 11].

Definition 1.9. Let A be an algebra over \mathbb{C} , not necessarily with unit. We call A *non-degenerate* if A is faithfully represented on itself by left and right multiplication. It is called *idempotent* if $A^2 = A$.

Definition 1.10. Let A be an algebra. A *multiplier* m for A consists of a couple of maps

$$\begin{aligned} L_m : A &\rightarrow A, & a &\mapsto ma \\ R_m : A &\rightarrow A, & a &\mapsto am \end{aligned}$$

such that $(am)b = a(mb)$ for all $a, b \in A$.

The set of all multipliers forms an algebra under composition for the L -maps and anti-composition for the R -maps. It is called the *multiplier algebra* of A , and is denoted $M(A)$.

One has a natural homomorphism $A \rightarrow M(A)$. When A is non-degenerate, this homomorphism is injective, and we can then identify A as a subalgebra of the (unital) algebra $M(A)$. We then also have inclusions

$$A \otimes A \subseteq M(A) \otimes M(A) \subseteq M(A \otimes A).$$

Example 1.11. 1. Let A be the total algebra of an I -partial algebra \mathcal{A} . As A has local units, it is non-degenerate and idempotent. Then one can identify $M(A)$ with

$$M(A) = \left(\prod_l \oplus_k A(k, l) \right) \cap \left(\prod_k \oplus_l A(k, l) \right) \subseteq \prod_{k, l} A(k, l),$$

i.e. with the space of functions

$$m : I^2 \rightarrow A, \quad m_{kl} \in A(k, l)$$

which have finite support in either one of the variables when the other variable has been fixed. The multiplication is given by the formula

$$(mn)_{kl} = \sum_p m_{kp} n_{pl}.$$

2. Let m_i be any collection of multipliers of A , and assume that for each $a \in A$, $m_i a = 0$ for almost all i , and similarly $a m_i = 0$ for almost all i . Then one can define a multiplier $\sum_i m_i$ in the obvious way by termwise multiplication. One says that the sum $\sum_i m_i$ converges in the *strict* topology.

Using the notion introduced in Example 1.11.2, we can introduce the following notation.

Notation 1.12. If \mathcal{A} is an I -partial bialgebra, we write

$$\lambda_k = \sum_l \mathbf{1} \binom{k}{l}, \quad \rho_l = \sum_k \mathbf{1} \binom{k}{l} \quad \in M(A).$$

Remark 1.13. From Property (c) of Definition 1.7, it follows that $\lambda_k \neq 0 \neq \rho_k$ for any $k \in I$.

To show that the total algebra of a partial bialgebra becomes a weak multiplier bialgebra, we will need some easy lemmas.

Lemma 1.14. *Let \mathcal{A} be an I -partial bialgebra. Then for each $a \in A$, there exists a unique multiplier $\Delta(a) \in M(A \otimes A)$ such that*

$$\begin{aligned} \Delta_{rs}(a) &= (1 \otimes \lambda_r) \Delta(a) (1 \otimes \lambda_s) \\ &= (\rho_r \otimes 1) \Delta(a) (\rho_s \otimes 1) \end{aligned} \tag{1.1}$$

for all $r, s \in I$, all $K \in M_2(I)$ and all $a \in A(K)$.

The resulting map

$$\Delta : A \rightarrow M(A \otimes A), \quad a \mapsto \Delta(a)$$

is a homomorphism.

Proof. For $a \in A$ homogeneous, we can define $\Delta(a) = \sum_{rs} \Delta_{rs}(a) \in M(A \otimes A)$, where the sum converges in the strict topology of $A \otimes A$ because of the property (d) of Definition 1.7. This expression clearly satisfies the identities stated in the lemma. In turn, these identities uniquely define $\Delta(a)$ as a multiplier, as they determine the value of $\Delta(a)$ when cut down to the left and right with the local units of $\mathcal{A} \otimes \mathcal{A}$.

We can then extend Δ to A by linearity. Since, for a, b homogeneous, $\Delta_{rt}(a)\Delta_{t's}(b) = 0$ unless $t = t'$, it follows from property (e) of Definition 1.7 that Δ is a homomorphism. \square

We will refer to $\Delta : A \rightarrow M(A \otimes A)$ as the *total comultiplication* of \mathcal{A} . We will then also use the suggestive Sweedler notation for this map,

$$\Delta(a) = a_{(1)} \otimes a_{(2)}.$$

Note for example that

$$\Delta(\mathbf{1}_{\binom{k}{m}}) = \sum_l \mathbf{1}_{\binom{k}{l}} \otimes \mathbf{1}_{\binom{l}{m}} = \sum_l \lambda_k \rho_l \otimes \lambda_l \rho_m.$$

Lemma 1.15. *The element $E = \sum_{k,l,m} \mathbf{1}_{\binom{k}{l}} \otimes \mathbf{1}_{\binom{l}{m}} = \sum_l \rho_l \otimes \lambda_l$ is a well-defined idempotent in $A \otimes A$, and satisfies*

$$\Delta(A)(A \otimes A) = E(A \otimes A), \quad (A \otimes A)\Delta(A) = (A \otimes A)E.$$

Proof. Clearly the sum defining E is strictly convergent, and makes E into an idempotent. It is moreover immediate that $E\Delta(a) = \Delta(a) = \Delta(a)E$ for all $a \in A$. Since

$$E(\mathbf{1}_{\binom{k}{l}} \otimes \mathbf{1}_{\binom{m}{n}}) = \Delta(\mathbf{1}_{\binom{k}{n}})(\mathbf{1}_{\binom{k}{l}} \otimes \mathbf{1}_{\binom{m}{n}})$$

by the property (a) of Definition 1.7, and analogously for multiplication with E on the right, the lemma is proven. \square

By [13, Proposition A.3], there is a unique homomorphism $\Delta : M(A) \rightarrow M(A \otimes A)$ extending Δ and such that $\Delta(1) = E$. Alternatively, if $m \in M(A)$, we can directly define $\Delta(m)$ as the strict limit of the series $\sum_{k,l,r,s} \Delta_{rs}(m_{kl})$. Similarly the maps $\text{id} \otimes \Delta$ and $\Delta \otimes \text{id}$ extend to maps from $M(A \otimes A)$ to $M(A \otimes A \otimes A)$.

For example, note that

$$\Delta(\lambda_k) = (\lambda_k \otimes 1)\Delta(1), \quad \Delta(\rho_m) = (1 \otimes \rho_m)\Delta(1). \quad (1.2)$$

The following proposition gathers the properties of Δ , ϵ and $\Delta(1)$ which guarantee that (A, Δ) forms a weak multiplier bialgebra in the sense of [2, Definition 2.1].

Proposition 1.16. *Let \mathcal{A} be a partial bialgebra with total algebra A , total comultiplication Δ and counit ϵ . Then the following properties are satisfied.*

- (1) *Coassociativity:* $(\Delta \otimes \text{id})\Delta = (\text{id} \otimes \Delta)\Delta$ (as maps $M(A) \rightarrow M(A^{\otimes 3})$).

(2) *Counitality*: $(\epsilon \otimes \text{id})(\Delta(a)(1 \otimes b)) = ab = (\text{id} \otimes \epsilon)((a \otimes 1)\Delta(b))$ for all $a, b \in A$.

(3) *Weak Comultiplicativity of Unit*:

$$(\Delta(1) \otimes 1)(1 \otimes \Delta(1)) = (\Delta \otimes \text{id})\Delta(1) = (\text{id} \otimes \Delta)\Delta(1) = (1 \otimes \Delta(1))(\Delta(1) \otimes 1).$$

(4) *Weak Multiplicativity of Counit*: For all $a, b, c \in A$, one has

$$(\epsilon \otimes \text{id})(\Delta(a)(b \otimes c)) = (\epsilon \otimes \text{id})((1 \otimes a)\Delta(1)(b \otimes c))$$

and

$$(\epsilon \otimes \text{id})((a \otimes b)\Delta(c)) = (\epsilon \otimes \text{id})((a \otimes b)\Delta(1)(1 \otimes c)).$$

(5) *Strong multiplier property*: For all $a, b \in A$, one has

$$\Delta(A)(1 \otimes A) \cup (A \otimes 1)\Delta(A) \subseteq A \otimes A.$$

Proof. Most of these properties follow immediately from the definition of a partial bialgebra. For demonstrational purposes, let us check the first identity of property (4). Let us choose $a \in A(K)$, $b \in A(L)$ and $c \in A(M)$. Then

$$\Delta(a)(b \otimes c) = \delta_{K_{ru}, L_{lu}} \delta_{M_{lu}, L_{ld}} \sum_r \Delta_{r, L_{ld}}(a)(b \otimes c).$$

Applying $(\epsilon \otimes \text{id})$ to both sides, we obtain by Proposition (b) of Definition 1.7 and counitality of ϵ that

$$(\epsilon \otimes \text{id})(\Delta(a)(b \otimes c)) = \delta_{K_{ru}, L_{lu}, L_{ld}, M_{lu}} \epsilon(b)ac.$$

On the other hand,

$$\begin{aligned} (1 \otimes a)\Delta(1)(b \otimes c) &= \sum_{r,s,t} \mathbf{1} \binom{r}{s} b \otimes a \mathbf{1} \binom{s}{t} c \\ &= \delta_{L_{ld}, K_{ru}, M_{lu}} b \otimes ac. \end{aligned}$$

Applying $(\epsilon \otimes \text{id})$, we find

$$\begin{aligned} (\epsilon \otimes \text{id})((1 \otimes a)\Delta(1)(b \otimes c)) &= \delta_{L_{ld}, K_{ru}, M_{lu}} \delta_{L_{lu}, L_{ld}} \delta_{L_{ru}, L_{rd}} \epsilon(b)ac \\ &= \delta_{L_{ld}, L_{lu}, K_{ru}, M_{lu}} \epsilon(b)ac, \end{aligned}$$

which agrees with the expression above. \square

Remark 1.17. Since also the expressions $\Delta(a)(b \otimes 1)$ and $(1 \otimes a)\Delta(b)$ are in $A \otimes A$ for all $a, b \in A$, we see that (A, Δ) is in fact a *regular* weak multiplier bialgebra [2, Definition 2.3].

Recall from [2, Section 3] that a regular weak multiplier bialgebra admits four projections $A \rightarrow M(A)$, given by

$$\begin{aligned}\bar{\Pi}^L(a) &= (\epsilon \otimes \text{id})((a \otimes 1)\Delta(1)), & \bar{\Pi}^R(a) &= (\text{id} \otimes \epsilon)(\Delta(1)(1 \otimes a)), \\ \Pi^L(a) &= (\epsilon \otimes \text{id})(\Delta(1)(a \otimes 1)), & \Pi^R(a) &= (\text{id} \otimes \epsilon)((1 \otimes a)\Delta(1)),\end{aligned}$$

where the right hand side expressions are interpreted as multipliers in the obvious way. The relation $\Delta(1) = \sum_p \rho_p \otimes \lambda_p$ and condition (c) in Definition 1.7 imply

$$\bar{\Pi}^L(A) = \text{span}\{\lambda_p : p \in I\} = \Pi^L(A), \quad \bar{\Pi}^R(A) = \text{span}\{\rho_p : p \in I\} = \Pi^R(A).$$

The *base algebra* of (A, Δ) is therefore just the algebra $\text{Fun}_f(I)$ of finite support functions on I , and the comultiplication of A is (left and right) *full* (meaning roughly that the legs of $\Delta(A)$ span A) by [2, Theorem 3.13].

The maps Π^L and Π^R can also be written in the form

$$\Pi^L(a) = \sum_p \epsilon(\lambda_p a) \lambda_p, \quad \Pi^R(a) = \sum_p \epsilon(a \rho_p) \rho_p \quad (1.3)$$

because $\epsilon(\lambda_k \rho_m a \lambda_l \rho_n) = 0$ if $(k, l) \neq (m, n)$. These relations and (1.1), (1.2) imply

$$(\Pi^L \otimes \text{id})(\Delta(a)) = \sum_p \lambda_p \otimes \lambda_p a, \quad (\text{id} \otimes \Pi^L)(\Delta(a)) = \sum_p \rho_p a \otimes \lambda_p, \quad (1.4)$$

$$(\Pi^R \otimes \text{id})(\Delta(a)) = \sum_p \rho_p \otimes a \lambda_p, \quad (\text{id} \otimes \Pi^R)(\Delta(a)) = \sum_p a \rho_p \otimes \rho_p. \quad (1.5)$$

We now formulate the notion of partial Hopf algebra, whose total form will correspond to a weak multiplier Hopf algebra [2, 13, 14]. We will mainly refer to [2] for uniformity.

Let us denote \circ for the inverse of \cdot , and \bullet for the inverse of $*$, so

$$\begin{pmatrix} k & l \\ m & n \end{pmatrix}^\circ = \begin{pmatrix} l & k \\ n & m \end{pmatrix}, \quad \begin{pmatrix} k & l \\ m & n \end{pmatrix}^\bullet = \begin{pmatrix} m & n \\ k & l \end{pmatrix}, \quad \begin{pmatrix} k & l \\ m & n \end{pmatrix}^{\circ\bullet} = \begin{pmatrix} n & m \\ l & k \end{pmatrix}.$$

The notation \circ (resp. \bullet) will also be used for row vectors (resp. column vectors).

Definition 1.18. An *antipode* for an I -partial bialgebra \mathcal{A} consists of linear maps

$$S : A(K) \rightarrow A(K^{\circ\bullet})$$

such that the following property holds: for all $M, P \in M_2(I)$ and all $a \in A(M)$,

$$\sum_{\substack{K * L = M \\ K \cdot L^{\circ\bullet} = P}} a_{(1)K} S(a_{(2)L}) = \delta_{P, P_r} \epsilon(a) \mathbf{1}(P_l), \quad (1.6)$$

$$\sum_{\substack{K * L = M \\ K^{\circ\bullet} \cdot L = P}} S(a_{(1)K}) a_{(2)L} = \delta_{P, P_r} \epsilon(a) \mathbf{1}(P_r). \quad (1.7)$$

A partial bialgebra \mathcal{A} is called a *partial Hopf algebra* if it admits an antipode.

Remark 1.19. Note that condition (d) of Definition 1.7 again guarantees that the above sums are in fact finite.

If S is an antipode for a partial bialgebra, we can extend S to a linear map

$$S : A \rightarrow A$$

on the total algebra A . Conditions (1.6) and (1.7) then take the following simple form:

Lemma 1.20. *A family of maps $S : A(K) \rightarrow A(K^\circ \bullet)$ satisfies (1.6) and (1.7) if and only if the total map $S : A \rightarrow A$ satisfies*

$$a_{(1)}S(a_{(2)}) = \Pi^L(a), \quad S(a_{(1)})a_{(2)} = \Pi^R(a) \quad (1.8)$$

for all $a \in A$.

Note that these should be considered a priori as equalities of left (resp. right) multipliers on A .

Proof. For $M, P \in M_2(I)$ and $a \in A(M)$, the left and the right hand side of (1.6) are the P -homogeneous components of $a_{(1)}S(a_{(2)})$ and $\Pi^L(a) = \sum_p \epsilon(\lambda_p a) \lambda_p$, respectively. \square

Lemma 1.21. *Let \mathcal{A} be a partial Hopf algebra with antipode S . For all $k, l \in I$, $S(\mathbf{1}_{(l)}^{(k)}) = \mathbf{1}_{(k)}^{(l)}$.*

Proof. For example the first identity in Equation (1.8) of Lemma 1.20 applied to $\mathbf{1}_{(k)}^{(l)}$ gives

$$\sum_l S(\mathbf{1}_{(k)}^{(l)}) = \sum_l \mathbf{1}_{(l)}^{(k)} S(\mathbf{1}_{(k)}^{(l)}) = \lambda_k,$$

as $S(\mathbf{1}_{(k)}^{(l)}) \in {}^k A_l^k$ and $\Pi^L(\mathbf{1}_{(k)}^{(k)}) = \lambda_k$. This implies the lemma. \square

The existence of an antipode is closely related to partial invertibility of the maps $T_1, T_2 : A \otimes A \rightarrow A \otimes A$ given by

$$T_1(a \otimes b) = \Delta(a)(1 \otimes b), \quad T_2(a \otimes b) = (a \otimes 1)\Delta(b). \quad (1.9)$$

The precise formulation involves the linear maps $E_i, G_i : A \otimes A \rightarrow A \otimes A$ given by

$$G_1(a \otimes b) = \sum_p a \rho_p \otimes \rho_p b, \quad E_1(a \otimes b) = E(a \otimes b) = \sum_p \rho_p a \otimes \lambda_p b, \quad (1.10)$$

$$G_2(a \otimes b) = \sum_p a \lambda_p \otimes \lambda_p b, \quad E_2(a \otimes b) = (a \otimes b)E = \sum_p a \rho_p \otimes b \lambda_p. \quad (1.11)$$

Proposition 1.22. *Let \mathcal{A} be a partial Hopf algebra with total algebra A , total comultiplication Δ and antipode S . Then the maps $R_1, R_2: A \otimes A \rightarrow M(A \otimes A)$ given by*

$$R_1(a \otimes b) = a_{(1)} \otimes S(a_{(2)})b, \quad R_2(a \otimes b) = aS(b_{(1)}) \otimes b_{(2)}$$

take values in $A \otimes A$ and satisfy for $i = 1, 2$ the relations

$$T_i R_i = E_i, \quad R_i T_i = G_i, \quad T_i R_i T_i = T_i, \quad R_i T_i R_i = R_i. \quad (1.12)$$

Proof. The map R_1 takes values in $A \otimes A$ because

$$a_{(1)} \otimes S(a_{(2)})\lambda_k \rho_l = a_{(1)} \otimes S(\rho_l \lambda_k a_{(2)}) \in A \otimes A$$

for all $a \in A$, and Lemma 1.20, Equation (1.4) and Lemma 1.21 imply

$$\begin{aligned} T_1 R_1(a \otimes b) &= a_{(1)} \otimes a_{(2)} S(a_{(3)})b = a_{(1)} \otimes \Pi^L(a_{(2)})b = \sum_p \rho_p a \otimes \lambda_p b, \\ R_1 T_1(a \otimes b) &= a_{(1)} \otimes S(a_{(2)})a_{(3)}b = a_{(1)} \otimes \Pi^R(a_{(2)})b = \sum_p a \rho_p \otimes \rho_p b. \end{aligned}$$

The relations $T_1 R_1 T_1 = T_1$ and $R_1 T_1 R_1 = R_1$ follow easily from (1.1) and (1.2). The assertions concerning R_2 and T_2 follow similarly. \square

Theorem 1.23. *Let \mathcal{A} be a partial bialgebra with total algebra A , total comultiplication Δ and counit ϵ . Then the following conditions are equivalent:*

- (1) \mathcal{A} is a partial Hopf algebra.
- (2) There exist linear maps $R_1, R_2: A \otimes A \rightarrow A \otimes A$ satisfying (1.12).
- (3) (A, Δ, ϵ) is a regular weak multiplier Hopf algebra in the sense of [14].

If these conditions hold, then the total antipode of \mathcal{A} coincides with the antipode of (A, Δ, ϵ) .

Proof. (1) implies (2) by Proposition 1.22. (2) is equivalent to (3) by Definition 1.14 in [14]. Indeed, the maps G_1, G_2 defined in (1.10) and (1.11) satisfy

$$\begin{aligned} G_1(a_{(1)} \otimes b) \otimes a_{(2)} c &= \sum_p a_{(1)} \otimes \rho_p b \otimes a_{(2)} \lambda_p c, \\ a c_{(1)} \otimes G_2(b \otimes c_{(2)}) &= \sum_p a \rho_p c_{(1)} \otimes b \lambda_p \otimes c_{(2)} \end{aligned}$$

and therefore coincide with the maps introduced in Proposition 1.14 in [14]. Finally, assume (3). Then Lemma 6.14 and equation (6.14) in [2] imply that the antipode S of (A, Δ) satisfies $S(A(K)) \subseteq A(K^{\circ\bullet})$ and relation (1.8). Now, Lemma 1.20 implies (1). \square

From [14, Proposition 3.5 and Proposition 3.7] or [2, Theorem 6.12 and Corollary 6.16], we can conclude that the antipode of a partial Hopf algebra reverses the multiplication and comultiplication. Denote by Δ^{op} the composition of Δ with the flip map.

Corollary 1.24. *Let \mathcal{A} be a partial Hopf algebra. Then the total antipode $S : A \rightarrow A$ satisfies $S(ab) = S(b)S(a)$ and $\Delta(S(a)) = (S \otimes S)\Delta^{\text{op}}(a)$ for all $a, b \in A$.*

We now turn towards the structures which will allow us to build operator algebraic quantum groupoids out of our partial Hopf algebras.

Definition 1.25. A *partial $*$ -algebra* \mathcal{A} is a partial algebra whose total algebra A is equipped with an antilinear, antimultiplicative involution $*$: $A \rightarrow A$, $a \mapsto a^*$, such that the $\mathbf{1}_k$ are selfadjoint for all k in the object set.

One can of course give an alternative definition directly in terms of the partial algebra structure by requiring that we are given antilinear maps $A(k, l) \rightarrow A(l, k)$ satisfying the obvious antimultiplicativity and involution properties.

Definition 1.26. A *partial $*$ -bialgebra* \mathcal{A} is a partial bialgebra whose underlying partial algebra has been endowed with a partial $*$ -algebra structure such that $\Delta(a)^* = \Delta(a^*)$ for all $a \in A$. A *partial Hopf $*$ -algebra* is a partial bialgebra which is at the same time a partial $*$ -bialgebra and a partial Hopf algebra.

Thus, a partial bialgebra is a partial $*$ -bialgebra if and only if the underlying weak multiplier bialgebra is a weak multiplier $*$ -bialgebra.

From Theorem 1.23 and [2], [14], we can deduce:

Corollary 1.27. *An I -partial $*$ -bialgebra \mathcal{A} is an I -partial Hopf $*$ -algebra if and only if the weak multiplier $*$ -bialgebra (A, Δ) is a weak multiplier Hopf $*$ -algebra. In that case, the counit and antipode satisfy $\epsilon(a^*) = \overline{\epsilon(a)}$ and $S(S(a)^*)^* = a$ for all $a \in A$. In particular, the total antipode is bijective.*

Proof. The if and only if part follows immediately from Theorem 1.23, the relation for the counit from uniqueness of the counit [2, Theorem 2.8], and the relation for the antipode from [14, Proposition 4.11]. \square

Definition 1.28. Let \mathcal{A} be an I -partial bialgebra. We call a family of functionals

$$\phi \binom{k}{m} : A \binom{k}{m} \binom{k}{m} \rightarrow \mathbb{C} \quad (1.13)$$

a *left invariant integral* if $\phi \binom{k}{k} (\mathbf{1} \binom{k}{k}) = 1$ for all $k \in I$ and

$$(\text{id} \otimes \phi \binom{l}{m})(\Delta_{l,l}(a)) = \delta_{k,p} \phi \binom{k}{m}(a) \mathbf{1} \binom{k}{l} \quad (1.14)$$

for all $k, l, m, p \in I$, $a \in A \binom{k}{m} \binom{p}{m}$.

We call them a *right invariant integral* if instead one has

$$(\phi \binom{k}{l} \otimes \text{id})(\Delta_{l,l}(a)) = \delta_{m,p} \phi \binom{k}{m}(a) \mathbf{1} \binom{l}{m} \quad (1.15)$$

for all $k, l, m, p \in I$, $a \in A\left(\begin{smallmatrix} k & k \\ m & p \end{smallmatrix}\right)$.

If ϕ is a left and a right invariant integral, we call ϕ an *invariant integral*.

Lemma 1.29. (1) If ϕ is a left or right invariant integral, then $\phi\left(\begin{smallmatrix} k \\ m \end{smallmatrix}\right)(\mathbf{1}\left(\begin{smallmatrix} k \\ m \end{smallmatrix}\right)) = 1$ for all $k, m \in I$ with $\mathbf{1}\left(\begin{smallmatrix} k \\ m \end{smallmatrix}\right) \neq 0$.

(2) If a left integral and a right invariant integral exist, then the two are equal and unique.

(3) Assume that \mathcal{A} is an I -partial Hopf algebra with bijective antipode S . If ϕ is a left or right invariant integral, then it is unique, an invariant integral, and equal to $\phi \circ S$.

Proof. To see (1), take $a = \mathbf{1}\left(\begin{smallmatrix} k \\ k \end{smallmatrix}\right)$ in (1.14). For (2), assume that ϕ and ψ are a left and a right invariant integral. Then for all $k, l, m \in I$, $a \in A\left(\begin{smallmatrix} k & k \\ m & m \end{smallmatrix}\right)$,

$$\phi\left(\begin{smallmatrix} k \\ m \end{smallmatrix}\right)(a) = (\psi\left(\begin{smallmatrix} k \\ k \end{smallmatrix}\right) \otimes \phi\left(\begin{smallmatrix} k \\ m \end{smallmatrix}\right))(\Delta_{k,k}(a)) = \psi\left(\begin{smallmatrix} k \\ m \end{smallmatrix}\right)(a)\phi\left(\begin{smallmatrix} k \\ m \end{smallmatrix}\right)(\mathbf{1}\left(\begin{smallmatrix} k \\ m \end{smallmatrix}\right)) = \psi\left(\begin{smallmatrix} k \\ m \end{smallmatrix}\right)(a).$$

To prove (3), assume that ϕ is a left or right invariant integral. Then Corollary 1.24 implies that $\phi \circ S$ is a right or left invariant integral, and (2) implies the claim. \square

Given a family of functionals as in (1.13), we denote by $\phi: A \rightarrow \mathbb{C}$ the corresponding functional on the total algebra.

We are finally ready to formulate our main definition.

Definition 1.30. A *partial compact quantum group* \mathcal{G} is a partial Hopf $*$ -algebra $\mathcal{A} = P(\mathcal{G})$ with an invariant integral ϕ that is positive in the sense that $\phi(a^*a) \geq 0$ for all $a \in A$. We also say that \mathcal{G} is the partial compact quantum group defined by \mathcal{A} .

Remark 1.31. Following [6], we could also have called our objects *compact quantum groups of face type*, but we feel this gives the wrong impression when the base algebra is infinite dimensional (i.e. the object set is not compact). When referring to partial compact quantum groups, we feel that it is better reflected that only the *parts* of this object are to be considered compact, not the total object.

The total form of the invariance conditions (1.14) reads as follows.

Lemma 1.32. A family of functionals ϕ as in (1.13) satisfies the conditions in (1.14) if and only if for all $a \in A$,

$$(\phi \otimes \text{id})(\Delta(a)(1 \otimes b)) = \sum_n \phi(\rho_n a) \rho_n b \quad (\text{or} \quad (\text{id} \otimes \phi)((b \otimes 1)\Delta(a)) = \sum_k \phi(\lambda_k a) b \lambda_k.)$$

Proof. Straightforward. \square

We will need the following lemma at some point, which is an almost verbatim transcription of the argument in [12, Proposition 3.4].

Lemma 1.33. *Let \mathcal{A} be a partial Hopf algebra with bijective antipode and an integral ϕ . Then ϕ is faithful in the following sense: if $a \in A$ and $\phi(ab) = 0$ (resp. $\phi(ba) = 0$) for all $b \in A$, then $a = 0$.*

Proof. Suppose $\phi(ba) = 0$ for all b . Then for all $d \in A$ and all functionals ω on A , the element $p = (\omega \otimes \text{id})((d \otimes 1)\Delta(a))$ satisfies

$$(\text{id} \otimes \phi)((1 \otimes c)\Delta(p)) = 0.$$

Continuing as in the proof of [12, Proposition 3.4], we obtain from the antipode trick that

$$\sum_n \phi(cS(q)\rho_n)\epsilon(p\lambda_n) = 0.$$

Choosing now for c and q local units of the form $\lambda_k\rho_l$, the normalization condition on ϕ gives that $\epsilon(p\lambda_n) = 0$ for all n , hence $\epsilon(p) = 0$. This implies $\omega(da) = 0$. As ω and d were arbitrary, it follows that $a = 0$.

The other case follows similarly, or by considering the opposite comultiplication. \square

Lemma 1.34. *Let \mathcal{A} be a partial Hopf algebra with integral ϕ . Then for all $a \in A$,*

$$\begin{aligned} (\text{id} \otimes \phi)((1 \otimes b)\Delta(a)) &= (S \otimes \phi)(\Delta(b)(1 \otimes a)), \\ (\phi \otimes \text{id})(\Delta(a)(b \otimes 1)) &= (\phi \otimes S)(\Delta(a)(1 \otimes b)). \end{aligned}$$

Proof. The counit property, the relations (1.1) and (1.8) and Lemma 1.32 imply

$$\begin{aligned} a_{(1)}\phi(ba_{(2)}) &= \sum_n a_{(1)}\phi(\epsilon(b_{(1)}\rho_n)b_{(2)}\lambda_n a_{(2)}) \\ &= \sum_n \epsilon(b_{(1)}\rho_n)\rho_n a_{(1)}\phi(b_{(2)}a_{(2)}) \\ &= S(b_{(1)})b_{(2)}a_{(1)}\phi(b_{(3)}a_{(2)}) = S(b_{(1)})\phi(b_{(2)}a) \end{aligned}$$

for all $a, b \in A$. The second equation follows similarly. \square

2 Representation theory

In this section, the representation theory of partial compact quantum groups is investigated. As the situation is quite similar to the case already studied by Hayashi [6], we do not always provide fully written out proofs, but only draw attention to those parts of the theory which need modification.

In what follows, the homogeneous component $A(K) = A\left(\begin{smallmatrix} k & l \\ m & n \end{smallmatrix}\right)$ of a partial bialgebra will now be mainly written as $A(K) = {}^k_m A^l_n$.

2.1 Corepresentations of partial Hopf algebras

Let \mathcal{A} be an I -partial bialgebra. We denote $\text{Hom}_{\mathbb{C}}(V, W)$ for the vector space of linear maps between two vector spaces.

We denote by Vect_{I^2} the category whose objects are I^2 -graded vector spaces $V = \bigoplus_{k,l \in I} {}^k V^l$ and whose morphisms are linear maps T that preserve the grading and therefore can be written $T = \bigoplus_{k,l \in I} {}^k T^l$.

Definition 2.1. Let \mathcal{A} be an I -partial bialgebra and let $V = \bigoplus_{k,l} {}^k V^l$ be an I^2 -graded vector space. A *corepresentation* $\mathcal{X} = ({}_m X_n^l)_{k,l,m,n}$ of \mathcal{A} on V is a family of elements

$${}_m X_n^l \in {}_m A_n^l \otimes \text{Hom}_{\mathbb{C}}({}^m V^n, {}^k V^l) \quad (2.1)$$

satisfying the following conditions:

1. ${}_m X_n^l = 0$ for almost all l, n (resp. almost all k, m) when k, m (resp. l, n) are fixed,
2. $(\Delta_{pq} \otimes \text{id})({}_m X_n^l) = \left({}_p X_q^l\right)_{13} \left({}_m X_n^q\right)_{23},$
3. $(\epsilon \otimes \text{id})({}_m X_n^l) = \delta_{k,m} \delta_{l,n} \text{id}_k V^l$

for all possible indices.

Here, we use here the standard leg numbering notation, e.g $a_{23} = 1 \otimes a$.

Remark 2.2. 1. If 3. holds, then 1. is satisfied if and only if V is *separately finitely supported (sfs)* in the sense that ${}^k V^l = \{0\}$ for almost all k (resp. almost all l) when l (resp. k) is fixed.

2.

Example 2.3. Equip the vector space $\mathbb{C}^{(I)} = \bigoplus_{k \in I} \mathbb{C}$ with the diagonal I^2 -grading. Then the family \mathcal{U} given by

$${}_m U_n^l = \delta_{k,l} \delta_{m,n} \mathbf{1}_{\binom{k}{m}} \in {}_m A_n^l \quad (2.2)$$

is a corepresentation of \mathcal{A} on $\mathbb{C}^{(I)}$. We call it the *trivial corepresentation*.

Let V be an I^2 -graded vector space. Then corepresentations of \mathcal{A} on V can be summed up to “total” corepresentation multipliers as follows. We denote by A the total algebra of \mathcal{A} and by $\lambda_k^V, \rho_l^V \in \text{Hom}_{\mathbb{C}}(V)$ the projections onto the summands ${}^k V = \bigoplus_q {}^k V^q$ and $V^l = \bigoplus_p {}^p V^l$, respectively, identify $\text{Hom}_{\mathbb{C}}({}^m V^n, {}^k V^l)$ with $\lambda_k^V \rho_l^V \text{Hom}_{\mathbb{C}}(V) \lambda_m^V \rho_n^V$, denote by $\text{Hom}_{\mathbb{C}}^0(V) \subseteq \text{Hom}_{\mathbb{C}}(V)$ the sum of all these subspaces, and define a homomorphism

$$\Delta \otimes \text{id}: M(A \otimes \text{Hom}_{\mathbb{C}}^0(V)) \rightarrow M(A \otimes A \otimes \text{Hom}_{\mathbb{C}}^0(V))$$

similarly as we defined $\Delta: A \rightarrow M(A \otimes A)$.

Lemma 2.4. *Let V be an I^2 -graded vector space. If (V, X) is a corepresentation of \mathcal{A} , then the sum*

$$X := \sum_{k,l,m,n} {}^k X_n^l \in M(A \otimes \text{Hom}_{\mathbb{C}}^0(V)) \quad (2.3)$$

converges strictly and satisfies the following conditions:

0. $(\lambda_k \rho_m \otimes \text{id})X(\lambda_l \rho_n \otimes \text{id}) = (1 \otimes \lambda_k^V \rho_l^V)X(1 \otimes \lambda_m^V \rho_n^V) = {}^k X_n^l$,
1. $(A \otimes 1)X$ and $X(A \otimes 1)$ are contained in $A \otimes \text{Hom}_{\mathbb{C}}^0(V)$,
2. $(\Delta \otimes \text{id})(X) = X_{13}X_{23}$,
3. the sum $(\epsilon \otimes \text{id})(X) := \sum (\epsilon \otimes \text{id})({}^k X_n^l)$ converges in $M(\text{Hom}_0(V))$ strictly to id_V .

Conversely, if $X \in M(A \otimes \text{Hom}_{\mathbb{C}}^0(V))$ satisfies 0.–3. with ${}^k X_n^l$ defined by 0., then $\mathcal{X} = ({}^k X_n^l)_{k,l,m,n}$ is a corepresentation of \mathcal{A} on V .

Proof. Straightforward. □

Morphisms of corepresentations are defined as follows.

Definition 2.5. Let \mathcal{A} be an I -partial bialgebra. A *morphism* T between two corepresentations (V, \mathcal{X}) and (W, \mathcal{Y}) of \mathcal{A} is a family of linear maps

$${}^k T^l \in \text{Hom}_{\mathbb{C}}({}^k V^l, {}^k W^l)$$

satisfying

$$(1 \otimes {}^k T^l) {}^k X_n^l = {}^k Y_n^l (1 \otimes {}^m T^m)$$

We denote the category of all corepresentations of \mathcal{A} by $\text{Corep}(\mathcal{A})$.

Remark 2.6. Equivalently, a morphism between corepresentations (V, \mathcal{X}) and (W, \mathcal{Y}) is just a morphism of I^2 -graded vector spaces $T: V \rightarrow W$ satisfying $(1 \otimes T)X = Y(1 \otimes T)$.

Note that the categories Vect_{I^2} and $\text{Corep}(\mathcal{A})$ are \mathbb{C} -linear and that the forgetful functor $\text{Corep}(\mathcal{A}) \rightarrow \text{Vect}_{I^2}$ is faithful.

The category Vect_{I^2} is a tensor category, where the product of I^2 -graded vector spaces V and W is the sum of the subspaces

$${}^k (V \otimes_I W)^l = \bigoplus_p ({}^k V^p \otimes {}^p W^l) \subseteq V \otimes W,$$

which we denote by $V \otimes_I W$, and the product of morphisms is the restriction of the ordinary tensor product. We pretend this product to be strictly associative. The unit for this product is the vector space $\mathbb{C}^{(I)} = \bigoplus_{k \in I} \mathbb{C}$. Indeed, for every I^2 -graded vector space V , there exist obvious natural isomorphisms $\mathbb{C}^{(I)} \otimes_I V \cong V \cong V \otimes_I \mathbb{C}^{(I)}$.

Given V and W as above, we identify $\text{Hom}_{\mathbb{C}}({}^m V^n, {}^k V^l) \otimes \text{Hom}_{\mathbb{C}}({}^n W^q, {}^l W^p)$ with a subspace of

$$\text{Hom}_{\mathbb{C}}({}^m V^n \otimes {}^n W^q, {}^k V^l \otimes {}^l W^p) \subseteq \text{Hom}_{\mathbb{C}}({}^m (V \otimes_I W)^q, {}^k (V \otimes_I W)^p).$$

We can now construct a product of corepresentations as follows.

Lemma 2.7. *Let \mathcal{X} and \mathcal{Y} be copresentations of \mathcal{A} on respective I^2 -graded vector spaces V and W . Then the sum*

$${}_m^k (X \oplus Y)_q^p := \sum_{l,n} \binom{k}{m} X_n^l \binom{l}{n} Y_q^p \quad (2.4)$$

has only finitely many non-zero terms and the elements

$${}_m^k (X \oplus Y)_q^p \in {}_m^k A_q^p \otimes \text{Hom}_{\mathbb{C}}({}^m (V \otimes_I W)^q, {}^k (V \otimes_I W)^p)$$

define a corepresentation $\mathcal{X} \oplus \mathcal{Y}$ of \mathcal{A} on $V \otimes_I W$.

Proof. The sum (2.4) is finite by condition 1. in 2.1. Using the identification above, we see that $\binom{k}{m} X_n^l \binom{l}{n} Y_q^p$ lies in ${}_m^k A_q^p \otimes \text{Hom}_{\mathbb{C}}({}^m (V \otimes_I W)^q, {}^k (V \otimes_I W)^p)$. Now, the fact that ${}_m^k (X \oplus Y)_q^p$ is a corepresentation follows easily from the multiplicativity of Δ and the weak multiplicativity of ϵ . \square

Remark 2.8. The “total” multiplier associated to $\mathcal{X} \oplus \mathcal{Y}$ is just $X_{12} Y_{13}$.

Proposition 2.9. *Let \mathcal{A} be an I -partial bialgebra. The $\text{Corep}(\mathcal{A})$ carries the structure of strict tensor category such that the product of corepresentations (V, \mathcal{X}) and (W, \mathcal{Y}) is the corepresentation $(V \otimes_I W, \mathcal{X} \oplus \mathcal{Y})$, the unit is the trivial corepresentation $(\mathbb{C}^{(I)}, \mathcal{U})$, and the forgetful functor $\text{Corep}(\mathcal{A}) \rightarrow \text{Vect}_{I^2}$ is a strict tensor functor.*

Proof. This is clear. \square

If \mathcal{A} is a partial Hopf algebra, every corepresentation multiplier has a generalized inverse.

Lemma 2.10. *Let \mathcal{X} be a corepresentation of an I -partial Hopf algebra \mathcal{A} on an I^2 -graded vector space V . Then*

$$\begin{aligned} {}_m^k X_n^l (S \otimes \text{id})({}_r^p X_s^q) &= 0 \text{ if } (l, m, n) \neq (s, p, q), \quad \sum_n {}_m^k X_n^l \cdot (S \otimes \text{id})({}_{k'}^m X_l^n) = \delta_{k,k'} \mathbf{1}_m^k \otimes \text{id}_k V^l, \\ (S \otimes \text{id})({}_m^k X_n^l) {}_r^p X_s^q &= 0 \text{ if } (k, m, n) \neq (r, p, q), \quad \sum_m (S \otimes \text{id})({}_m^k X_n^l) {}_k^m X_{l'}^n = \delta_{l,l'} \mathbf{1}_l^k \otimes \text{id}_k V^{l'}. \end{aligned}$$

In particular, the multiplier $Z := (S \otimes \text{id})(X) \in M(A \otimes \text{Hom}_{\mathbb{C}}^0(V))$ satisfies

$$XZ = \sum_k \lambda_k \otimes \lambda_k^V, \quad ZX = \sum_n \rho_n \otimes \rho_n^V, \quad (2.5)$$

and is a generalized inverse of X in the sense that $XZX = X$ and $ZXZ = Z$.

Proof. The first equation follows from (2.1) and the relation $S({}_r^p A_s^q) \subseteq {}_q^s A_p^r$. To verify the second one, we use relations 2. and 3. in Definition 2.1 and (1.6) and find

$$\begin{aligned} \sum_n {}^k X_n^l \cdot (S \otimes \text{id})({}_k^m X_l^n) &= \sum_n (m_A \circ (\text{id} \otimes S) \otimes \text{id})({}_k^m X_n^l)_{13}({}_k^m X_l^n)_{23} \\ &= \sum_n (m_A \circ (\text{id} \otimes S) \circ \Delta_{m,n} \otimes \text{id})({}_k^m X_l^n) \\ &= \delta_{k,k'} \mathbf{1}_l^{(k)} \otimes (\epsilon \otimes \text{id})({}_k^m X_l^n) \\ &= \delta_{k,k'} \mathbf{1}_m^{(k)} \otimes \text{id}_k V^l, \end{aligned}$$

where m_A denotes the multiplication of A . The third and fourth equation follow similarly, and the assertions concerning Z are direct consequences. \square

For completeness, we mention the following following converse.

Lemma 2.11. *Let \mathcal{A} be an I -partial bialgebra, let V be an I^2 -graded vector space and let $X \in M(A \otimes \text{Hom}_{\mathbb{C}}^0(V))$. If X satisfies conditions 0.–2. in Lemma 2.4 and (2.5) for some $Z \in M(A \otimes \text{Hom}_{\mathbb{C}}^0(V))$, then the corresponding family $\mathcal{X} = ({}_k^m X_n^l)_{k,l,m,n}$ is a corepresentation of \mathcal{A} on V .*

Proof. We have to check condition 3. in Lemma 2.4. Let ${}_m^k T_n^l := (\epsilon \otimes \text{id})({}_m^k X_n^l) \in \text{Hom}({}_m^k V^n, {}_k^m V^l)$. If $(k, l) \neq (m, n)$, then $\epsilon({}_m^k A_n^l) = 0$ and hence ${}_m^k T_n^l = 0$. Now, the counit property and condition 2. in Lemma 2.4 imply

$${}_m^k X_n^l = ((\epsilon \otimes \text{id}) \circ \Delta \otimes \text{id})({}_m^k X_n^l) = \sum_{p,q} (\epsilon \otimes \text{id} \otimes \text{id}) \left(({}_p^k X_q^l)_{13} ({}_m^p X_n^q)_{23} \right) = (1 \otimes {}_k^m T_l^n) {}_m^k X_n^l.$$

The morphism of V given by the family $T = ({}_k^m T_l^n)_{k,l}$ therefore satisfies $(1 \otimes T)X = X$. Multiplying on the right by Z , we can conclude $T\lambda_k^V = \lambda_k^V$ for all k . Thus, $T = \text{id}_V$. \square

If (A, Δ) is a generalized Hopf face algebra over an index set I , one can use the antipode to construct a left and a right dual for each corepresentation that satisfies a certain finiteness assumption. We first recall the general notion of duality in monoidal categories, see also [1], and then discuss the cases Vect_{I^2} and $\text{Corep}(A, \Delta)$.

Let X and Y be objects in a strict tensor category \mathcal{C} . We say that X is a *right dual* of Y and Y a *left dual* of X if there are morphisms $X \otimes Y \rightarrow 1$ and $1 \rightarrow Y \otimes X$, where 1 denotes the tensor unit of \mathcal{C} , such that the obvious compositions

$$X \otimes 1 \rightarrow X \otimes Y \otimes X \rightarrow 1 \otimes X \quad \text{and} \quad 1 \otimes Y \rightarrow Y \otimes X \otimes Y \rightarrow Y \otimes 1$$

are the identity of X and Y , respectively.

The next example fixes some notation.

Example 2.12. A vector space V has a left dual or a right dual if and only if it has finite dimension. In that case, a left dual of V is given by the dual space V^\vee with the morphisms

$$: V^\vee \otimes V \rightarrow \mathbb{C}, \quad f \otimes v \mapsto f(v), \quad : \mathbb{C} \rightarrow V \otimes V^\vee, \quad \lambda \mapsto \sum_k \lambda \xi_k \otimes \xi_k^\vee,$$

where $(\xi_k)_k$ is an arbitrary basis of V and $(\xi_k^\vee)_k$ is the dual basis of V^\vee . Note that $\text{coev}_V(1)$ corresponds to id_V under the natural identification $V \otimes V^\vee \cong \text{Hom}(V)$. Identifying V with $(V^\vee)^\vee$, we find that $(V^\vee, \text{ev}_{V^\vee}, \text{coev}_{V^\vee})$ is a right dual of V .

If T is a linear map between finite-dimensional vector spaces V and W and $T^\vee : W^\vee \rightarrow V^\vee$ denotes the dual map, then

$$(T \otimes \text{id}_{V^\vee}) \circ \text{coev}_V = (\text{id}_W \otimes T^\vee) \circ \text{coev}_W, \quad \text{ev}_W(\text{id}_{W^\vee} \otimes T) = \text{ev}_V(T^\vee \otimes \text{id}_V). \quad (2.6)$$

The next step is to consider the category Vect_{I^2} . We need some notation.

Given an I^2 -graded vector space V , denote by ${}^k\overline{V}^l$ the dual space of ${}^lV^k$ and identify it with the space of all functionals on V that vanish on ${}^mV^n$ for $(m, n) \neq (k, l)$. Then $\overline{V} := \bigoplus_{k,l} {}^k\overline{V}^l \subseteq V^\vee$ is an I^2 -graded vector space. Note that V^\vee is row-/column-finite if and only if V is column-/row-finite, and that the assignments $V \mapsto \overline{V}$ and $T \mapsto T^\vee$ define an isomorphism of Vect_{I^2} with its opposite category.

An I^2 -graded vector space V has a right dual if and only if V is row-finite, and in that case, a right dual is given by \overline{V} and the morphisms

$$\begin{aligned} \text{ev}_V &\in \text{Hom}_{I^2}(\overline{V} \otimes_I V, \mathbb{C}^{(I)}), \quad \text{where } {}^k(\text{ev}_V)^l = \delta_{k,l} \text{ev}_V({}^kV), \\ \text{coev}_V &\in \text{Hom}_{I^2}(\mathbb{C}^{(I)}, V \otimes_I \overline{V}), \quad \text{where } {}^k(\text{coev}_V)^l = \delta_{k,l} \text{coev}_V({}^kV). \end{aligned}$$

This follows easily from Example 2.12. Likewise, an I^2 -graded vector space has a left dual if and only if it is column-finite. In that case, \overline{V} is row-finite, we can identify V with $\overline{\overline{V}}$, and $(\overline{V}, \text{ev}_{\overline{V}}, \text{coev}_{\overline{V}})$ is a left dual of V .

The monoidal category $\text{Vect}_{I^2}^f$ is therefore rigid.

We next show that if (A, Δ) is a generalized Hopf face algebra over an index set I , then $\text{Corep}(A, \Delta)$ is rigid as well. The left and right dual of a locally finite corepresentation can be constructed with the help of the antipode as follows.

Proposition 2.13. *Let (A, Δ) be a generalized Hopf face algebra over an index set I , where A is a strong I^2 -algebra.*

1. *Let (V, X) be a locally finite corepresentation of (A, Δ) . Then $X^{S \otimes \vee} := (S \otimes (-)^\vee)(X) \in M(A \otimes \text{Hom}_0(\overline{V}))$ is locally finite corepresentation of (A, Δ) on \overline{V} .*
2. *$((\overline{V}, X^{S \otimes \vee}), \text{ev}_V, \text{coev}_V)$ is a right dual of dual of (V, X) .*

3. The assignments $(V, X) \mapsto (\bar{V}, X^{S \otimes \top})$ and $T \mapsto T^\vee$ form an isomorphism between $\text{Corep}(A, \Delta)$ and its opposite category.
4. The monoidal category $(\text{Corep}^f(A, \Delta), \otimes, (\mathbb{C}^{(I)}, \tau))$ is rigid.

Proof. 1. Using (??), the verification is straightforward:

$$\begin{aligned}
{}_m^k(X^{S \otimes \vee})_n^l &= (S \otimes (-)^\vee)({}_n^l X_k^m) \\
&\in S({}_l^m A_k^m) \otimes \text{Hom}({}_n^m V^m, {}_l^l V^k)^\vee = {}_m^k A_n^l \otimes \text{Hom}({}_m^k \bar{V}^l, {}_n^m \bar{V}^n), \\
(\epsilon \otimes \text{id})({}_m^k(X^{S \otimes \vee})_n^l) &= (\epsilon \circ S \otimes \vee)({}_n^l X_k^m) \\
&= (\epsilon \otimes \text{id})({}_n^l X_k^m) = \delta_{k,l} \delta_{m,n} \text{id}_{{}_l^l V^k} = \delta_{k,l} \delta_{m,n} \text{id}_{{}_m^k \bar{V}^l}, \\
(\tilde{\Delta} \otimes \text{id})(X^{S \otimes \vee}) &= (\sigma \circ (S \otimes S) \otimes (-)^\vee)(\tilde{\Delta} \otimes \text{id})(X) \\
&= (\sigma \circ (S \otimes S) \otimes (-)^\vee)(X_{13} X_{23}) = (X^{S \otimes \vee})_{13} (X^{S \otimes \vee})_{23}.
\end{aligned}$$

2. We first show that coev_V is a morphism from $(\mathbb{C}^{(I)}, \tau)$ to $(\bar{V}, X^{S \otimes \vee}) \oplus (V, X)$. Indeed, the relations (??), (2.6) and Lemma ?? imply

$$\begin{aligned}
{}_m^k(X \oplus X^{S \otimes \vee})_n^l (1 \otimes {}^m(\text{coev}_V)^n) &= \sum_{p,q} ({}_m^k X_q^p)_{12} (S \otimes (-)^\vee)({}_l^q X_p^m)_{13} (1 \otimes \delta_{m,n} \text{coev}_V^m) \\
&= \delta_{m,n} \sum_{p,q} ({}_m^k X_q^p)_{12} (S \otimes \text{id})({}_l^q X_p^m)_{13} (1 \otimes \text{coev}_V^m) \\
&= \delta_{m,n} \delta_{k,l} (\lambda_k \rho_m \otimes \lambda_k^V)_{12} (1 \otimes \text{coev}_V^m) \\
&= \delta_{m,n} \delta_{k,l} \lambda_k \rho_m \otimes \text{coev}_V^m \\
&= (1 \otimes {}^k(\text{coev}_V)^l) {}_m^k \tau_n^l.
\end{aligned}$$

A similar calculation shows that ev_V is a morphism from $(V, X) \oplus (\bar{V}, X^{S \otimes \vee})$ to $(\mathbb{C}^{(I)}, \tau)$. Now, the assertion follows because $(\bar{V}, \text{ev}_V, \text{coev}_V)$ is a right dual of V in Vect_2^f .

3. Clearly, the assignments for a functor. Similar calculations as above show that one can define for every corepresentation (V, X) a corepresentation $(\bar{V}, X^{S^{-1} \otimes \vee})$, and the assignments $(V, X) \mapsto (\bar{V}, X^{S^{-1} \otimes \vee})$ and $T \mapsto T^\vee$ form the inverse of the first functor.

4. The discussion above shows that every locally finite corepresentation (V, X) has a right dual and is the right dual of the locally finite corepresentation $(\bar{V}, X^{S^{-1} \otimes \vee})$, which then is a left dual of (V, X) . \square

The tensor category

Definition 2.14. An I^2 -graded vector space V is called *seperately finite dimensional (sfd)* if $\oplus_l {}^k V^l$ (resp. $\oplus_l {}^k V^l$) is finite dimensional for k (resp. l) fixed. Correspondingly, we talk of an sfd corepresentation of a partial bialgebra \mathcal{A} , and we then denote by $\text{Corep}_{\text{sfd}}(\mathcal{A})$ the full subcategory of $\text{Corep}_{\text{sfs}}(\mathcal{A})$ consisting of sfd representations.

One easily sees that $\text{Corep}_{\text{sfd}}(\mathcal{A})$ is closed under \oplus .

Lemma 2.15. *Let \mathcal{A} be a partial Hopf algebra. Then $\text{Corep}_{\text{sfd}}(\mathcal{A})$ is a tensor category with left duality.*

Proof. Let X be an sfd corepresentation on a bigraded vector space V . Put

$${}^k(V^*)^l = ({}^kV^l)^*,$$

and let V^* denote their direct sum bigraded vector space. Using the natural contravariant identification

$$\text{Hom}_{\mathbb{C}}({}^lV^k, {}^nV^m) \cong \text{Hom}_{\mathbb{C}}({}^m(V^*)^n, {}^k(V^*)^l),$$

we see (by means of Lemma ??) that X^{-1} gets transformed into a corepresentation X^d on V^* .

We claim that X^d is the left dual of X . To see this, consider the evaluation maps

$${}^kT^m : {}^k(V^* \otimes_I V)^m \supseteq ({}^lV^k)^* \otimes {}^lV^m \rightarrow \delta_{k,m}\mathbb{C} = {}^k\mathbb{C}_I^m.$$

Then from Lemma ??, we obtain

$$\begin{aligned} (1 \otimes {}^kT^p) \left(\sum_{l,n} \binom{k}{m} (X^d)_n^l \binom{l}{n} X_q^p \right)_{13} &= \delta_{p,k} (\text{id} \otimes {}^mT^m) \left(\sum_{l,n} \binom{k}{m} (X^{-1})_{nn}^l {}^lX_q^k \right)_{13} \\ &= \delta_{p,k} \mathbf{1}_{\binom{k}{q}} \otimes {}^mT^q. \end{aligned}$$

Hence the ${}^kT^l$ define an intertwiner between $V^* \otimes_I V$ and \mathbb{C}_I . Similarly, the maps

$${}^kR^k : {}^k\mathbb{C}_I^k = \mathbb{C} \rightarrow {}^k(V \otimes_I V^*)^k, \quad 1 \mapsto \sum_{l,i} {}^k v_i^l \otimes {}^l \omega_i^k,$$

where the $\{{}^k v_i^l \mid i\}$ and $\{{}^l \omega_i^k \mid i\}$ form a dual basis of ${}^kV^l$, can be shown to form an intertwiner. It is then easy to check that T and R make X^d into the left dual of X . \square

Let us now enhance our partial Hopf algebras to partial compact quantum groups. One then considers corepresentations on sfd bigraded *Hilbert spaces* such that the inverse of the corepresentation coincides with its adjoint. More precisely, we have the following definition. We denote $B(\mathcal{H}, \mathcal{G})$ for the linear space of bounded morphisms between Hilbert spaces.

Definition 2.16. Let \mathcal{A} define a partial compact quantum group. We call an sfd corepresentation (\mathcal{H}, X) on an sfd I^2 -graded Hilbert space \mathcal{H} *unitary* if

$${}^k_m(X^{-1})_n^l = ({}^lX_m^k)^* \quad \text{in } {}^kA_n^l \otimes B({}^l\mathcal{H}^k, {}^n\mathcal{H}^m).$$

Remark 2.17. 1. In the Hilbert space setting, it is more natural to let \mathcal{H} be the *closed* (instead of the purely algebraic) direct sum of all (finite-dimensional) ${}^k\mathcal{H}^l$. This does not change the notion of corepresentation, which had a local definition.

2. Concerning morphisms, we will say a collection of ${}^kT^l$ defines a *bounded* intertwiner or morphism if the total operator $T = \oplus {}^kT^l$ is bounded. We will denote by $\text{Corep}_{\text{sfd},u}(\mathcal{A})$ the category of unitary sfd corepresentations with arbitrary morphisms, and $\text{Corep}_{\text{sfd},u}^\infty(\mathcal{A})$ for the category with bounded morphisms.

Our aim now is to show that every irreducible sfd corepresentation is equivalent to a unitary one. We show this by embedding the corepresentation into a restriction of the regular corepresentation.

Example 2.18. Let \mathcal{A} define a partial compact quantum group with normalized positive invariant functional ϕ .

Let ${}^m\mathcal{H}^n \subseteq \bigoplus_{k,l} {}^kA_n^l$ be finite dimensional subspaces satisfying

$$\Delta_{pq}^{\text{op}}({}^m\mathcal{H}^n) \subseteq {}^pA_n^q \otimes {}^p\mathcal{H}^q.$$

for all indices. Equip each ${}^k\mathcal{H}^l$ with the scalar product $\langle a|b \rangle := \phi(a^*b)$. By Lemma 1.33, these are finite-dimensional Hilbert spaces. Take the Hilbert space direct sum $\mathcal{H} := \bigoplus_{k,l} {}^k\mathcal{H}^l$. Define

$${}^kV_n^l \in \text{Hom}_{\mathbb{C}}({}^m\mathcal{H}^n, {}^kA_n^l \otimes {}^k\mathcal{H}^l) \cong {}^kA_n^l \otimes \mathcal{B}({}^m\mathcal{H}^n, {}^k\mathcal{H}^l)$$

by the equation

$${}^kV_n^l a = \Delta_{kl}^{\text{co}}(a).$$

Lemma 2.19. *The couple (\mathcal{H}, V) defines a unitary corepresentation.*

Proof. It is clear that V defines a corepresentation. It then suffices to prove that

$$\sum_k ({}^kV_{n'}^l)^* {}^kV_n^l = \delta_{n,n'} \mathbf{1}_n^{(l)} \otimes \text{id}^m \mathcal{H}^n. \quad (2.7)$$

Take $a \in {}^m\mathcal{H}^n$ and $b \in {}^m\mathcal{H}^{n'}$. Then writing

$$\Lambda(a) : \mathbb{C} \rightarrow {}^m\mathcal{H}^n, \quad 1 \mapsto a,$$

and similarly for b , we compute

$$\begin{aligned}
(1 \otimes \Lambda(b)^*) \left(\sum_k ({}^k V_{n'}^l)^* {}^k V_n^l \right) (1 \otimes \Lambda(a)) &= \sum_k (\text{id} \otimes \phi)(\Delta_{kl}^{\text{op}}(b)^* \Delta_{kl}^{\text{op}}(a)) \\
&= \sum_k (\text{id} \otimes \phi)(\Delta_{lk}^{\text{op}}(b^*) \Delta_{kl}^{\text{op}}(a)) \\
&= (\text{id} \otimes \phi)(\Delta_{ll}^{\text{op}}(b^* a)) \\
&= (\phi \otimes \text{id})(\Delta_{ll}(b^* a)) \\
&= \phi(b^* a) \mathbf{1}_{\binom{l}{n}} \\
&= \delta_{n',n} \mathbf{1}_{\binom{l}{n}} \otimes \Lambda(b)^* \Lambda(a).
\end{aligned}$$

This proves (2.7). \square

We will call (\mathcal{H}, V) the *sfd restriction of the regular corepresentation* determined by the family $({}^k \mathcal{H}^l)_{k,l}$.

In the following, we will use the notation

$$\omega_{\xi,\eta} : B(\mathcal{H}, \mathcal{G}) \rightarrow \mathbb{C}, \quad x \mapsto \langle \xi, x\eta \rangle, \quad \xi \in \mathcal{G}, \eta \in \mathcal{H}.$$

Lemma 2.20. *Let \mathcal{A} define a partial compact quantum group. Let (\mathcal{H}, X) be an sfd corepresentation on a Hilbert space, and let $\xi \in {}^k \mathcal{H}^l$. Then the family of finite-dimensional subspaces*

$${}^m \mathcal{K}^n = \{(\text{id} \otimes \omega_{\xi,\eta})({}^k X_n^l) : \eta \in {}^m \mathcal{H}^n\} \subseteq {}^k A_n^l$$

defines an sfd restriction (\mathcal{K}, V) of the regular corepresentation, and the family of maps

$${}^m T_{(\xi)}^n : {}^m \mathcal{H}^n \rightarrow {}^m \mathcal{K}^n, \quad \eta \mapsto (\text{id} \otimes \omega_{\xi,\eta})({}^k X_n^l),$$

is a morphism from (\mathcal{H}, X) to (\mathcal{K}, V) in $\text{Corep}_{\text{sfd},u}(\mathcal{A})$.

Note that the family $({}^m \mathcal{K}^n)_{m,n}$ is sfd because $({}^m \mathcal{H}^n)_{m,n}$ is.

Proof. Both assertions follow from the fact that for all $\eta \in {}^p \mathcal{H}^q$,

$$\begin{aligned}
\Delta_{pq}^{\text{op}}({}^m T_{(\xi)}^n(\eta)) &= \Delta_{pq}^{\text{op}}((\text{id} \otimes \omega_{\xi,\eta})({}^k X_n^l)) \\
&= (\text{id} \otimes \text{id} \otimes \omega_{\xi,\eta})(({}^k X_q^l)_{23} ({}^p X_n^q)_{13}) \\
&= (1 \otimes {}^p T_{(\xi)}^q) {}^p X_n^q \eta.
\end{aligned}$$

\square

Proposition 2.21. *Let \mathcal{A} define a partial compact quantum group. Then every irreducible sfd corepresentation on a Hilbert space is equivalent to a unitary sfd corepresentation.*

Proof. Let (\mathcal{H}, X) be an irreducible sfd corepresentation. Then for some k, l and $\xi \in {}^k\mathcal{H}^l$, the operator $T_{(\xi)}$ defined in Lemma 2.20 has to be non-zero and hence, by Schur's Lemma, injective. Thus, it forms an equivalence between (\mathcal{H}, X) and a sub-corepresentation of an sfd restriction of the regular corepresentation, which is unitary by Example 2.18. \square

Our next goal is to obtain the analogue of Schur's orthogonality relations for matrix coefficients of corepresentations.

Definition 2.22. Let \mathcal{A} define a partial compact quantum group. The space of *matrix coefficients* $\mathcal{C}(X)$ of an sfd corepresentation (\mathcal{H}, X) is the sum of the subspaces

$${}_m^k\mathcal{C}(X)_n^l = \left\{ (\text{id} \otimes \omega_{\xi, \eta})({}_m^k X_n^l) \mid \xi \in {}^k\mathcal{H}^l, \eta \in {}^m\mathcal{H}^n \right\} \subseteq {}_m^k A_n^l.$$

Lemma 2.23. *Every sfd unitary corepresentation (X, \mathcal{H}) of \mathcal{A} decomposes into a direct sum of irreducible sfd unitary corepresentations.*

Proof. From the unarity assumption, it follows immediately that an invariant subspace of \mathcal{H} also has an invariant orthogonal complement. Hence irreducibility and indecomposability of unitary corepresentations coincide. More generally, one deduces that the bounded self-intertwiners of \mathcal{H} form a (von Neumann) $*$ -algebra.

Let us now first show that the trivial representation decomposes into irreducibles. Let I be the object set of \mathcal{A} , and say $k \sim l$ if $\mathbf{1} \binom{k}{l} \neq 0$. Then \sim is an equivalence relation: as

$$\Delta_U(\mathbf{1} \binom{k}{m}) = \mathbf{1} \binom{k}{l} \otimes \mathbf{1} \binom{l}{m},$$

the relation \sim is transitive. As $S(\mathbf{1} \binom{k}{l}) = \mathbf{1} \binom{l}{k}$, we have that \sim is symmetric. And as $\varepsilon(\mathbf{1} \binom{k}{k}) = 1$, we also have that \sim is reflexive.

Let then $I = \sqcup_{\alpha \in \mathcal{A}} I_\alpha$ be a labeled partition associated to \sim . Define $\mathbb{C}_{I_\alpha} \subseteq \mathbb{C}_I$ as the linear span of the homogeneous components with index in α . It is clear then that the \mathbb{C}_{I_α} are invariant and irreducible.

Consider now a general corepresentation (X, \mathcal{H}) . Let ${}_ \alpha \mathcal{H}_\beta$ be the closed linear span of the homogeneous components with index in $\alpha \times \beta$. As we can identify

$${}_ \alpha \mathcal{H}_\beta \cong \mathbb{C}_{I_\alpha} \oplus \mathcal{H} \oplus \mathbb{C}_{I_\beta},$$

we see that ${}_ \alpha \mathcal{H}_\beta$ is an invariant subspace of \mathcal{H} . Hence we may as well suppose that $\mathcal{H} = {}_ \alpha \mathcal{H}_\beta$.

But let then T be a bounded self-intertwiner of \mathcal{H} . Then from the two equations in Remark ??, we see that $T \rightarrow {}^k T^l$ is injective for any choice of $k \in \alpha, l \in \beta$. It follows that the algebra of self-intertwiners of \mathcal{H} is finite-dimensional. We then immediately conclude that \mathcal{H} is a finite direct sum of irreducible invariant subspaces. \square

Proposition 2.24. *Assume that \mathcal{A} defines a partial compact quantum group. Then the total algebra A is the sum of the matrix coefficients of irreducible unitary sfd corepresentations.*

Proof. Let $a \in {}^k A_n^l$. Then $\Delta_{pq}^{\text{co}}(a) \in {}^p A_n^q \otimes {}^k A_q^l$, and the subspace

$${}^p \mathcal{H}^q := \{(\omega \otimes \text{id})(\Delta_{pq}^{\text{co}}(a)) : \omega \in \text{Hom}_{\mathbb{C}}({}^p A_n^q, \mathbb{C})\} \subseteq {}^k A_q^l$$

has finite dimension. Since a is fixed, the $({}^p \mathcal{H}^q)_{p,q}$ are in fact sfd. Using co-associativity, one checks that this family defines an sfd restriction (\mathcal{H}, V) of the regular corepresentation. Evidently, $a \in \mathcal{C}(V)$. Decomposing (\mathcal{H}, V) , we find that a is contained in the sum of matrix coefficients of unitary irreducible corepresentations. \square

The key to the orthogonality relations is the following averaging procedure.

Lemma 2.25. *Let \mathcal{A} define a partial compact quantum group, and let ϕ be any invariant functional for \mathcal{A} . Let (\mathcal{H}, X) and (\mathcal{K}, Y) be sfd corepresentations of \mathcal{A} and let T be a family of operators ${}^k T^l \in \mathcal{B}({}^k \mathcal{H}^l, {}^k \mathcal{K}^l)$.*

Then for any fixed n , the family of linear maps

$${}^k \check{T}_n^l := \sum_m (\phi \otimes \text{id})({}^n_l(Y^{-1})^m (1 \otimes {}^m T^n)_k X_l^n)$$

define a morphism \check{T}_n from (\mathcal{H}, X) to (\mathcal{K}, Y) in $\text{Corep}_{\text{sfd}, u}(\mathcal{A})$.

Similarly, for fixed m , the

$${}^k \hat{T}_m^l := \sum_n (\phi \otimes \text{id})({}^k_m Y_n^l (1 \otimes {}^m T^n)_n (X^{-1})_m^k)$$

define a morphism from (\mathcal{H}, X) to (\mathcal{K}, Y) .

Proof. Using Remark ??, the assertion concerning the \check{T} follows from the calculation

$$\begin{aligned} & \sum_m {}^n_{l'}(Y^{-1})_k^m (1 \otimes {}^m \check{T}_q^n)_k X_l^n = \\ &= \sum_{m,p} (\phi \otimes \text{id} \otimes \text{id}) \left(({}^n_{l'}(Y^{-1})_k^m)_{23} ({}^q_n(Y^{-1})_m^p)_{13} (1 \otimes 1 \otimes {}^p T^q) ({}^p X_n^q)_{13} ({}^m_k X_l^n)_{23} \right) \\ &= \sum_{m,p} (((\phi \otimes \text{id}) \circ \Delta_{mn}) \otimes \text{id}) \left(({}^q_{l'}(Y^{-1})_k^p) (1 \otimes {}^p T^q)_k X_l^q \right) \\ &= \delta_{l',l} \mathbf{1} \binom{n}{l} \otimes \left(\sum_{p,q} (\phi \otimes \text{id}) \left(({}^q_l(Y^{-1})_k^p) (1 \otimes {}^p T^q)_k X_l^q \right) \right) \\ &= \delta_{l,l'} \mathbf{1} \binom{n}{l} \otimes {}^k \check{T}_q^l, \end{aligned}$$

where we used the relation $\phi({}^r_l A_l^s) = 0$ for $l' \neq l$ for the last equality.

A similar calculation proves the assertion concerning the \hat{T} . \square

The first part of the orthogonality relations concerns matrix coefficients of inequivalent irreducible corepresentations.

Proposition 2.26. *Let (\mathcal{H}, X) and (\mathcal{K}, Y) be inequivalent unitary irreducible sfd corepresentations, and let ϕ be an invariant functional for (A, Δ) . Then*

$$\phi(S(b)a) = \phi(b^*a) = \phi(bS(a)) = \phi(ba^*) = 0$$

for all $a \in \mathcal{C}(X), b \in \mathcal{C}(Y)$.

Proof. Let $a = (\text{id} \otimes \omega_{\xi, \xi'})({}^k_m X_n^l)$ and $b = (\text{id} \otimes \omega_{\eta, \eta'})({}^p_r Y_s^q)$, where $\xi \in {}^k \mathcal{H}^l, \xi' \in {}^m \mathcal{H}^n$ and $\eta \in {}^p \mathcal{K}^q, \eta' \in {}^r \mathcal{K}^s$.

If $(p, q, r, s) \neq (m, n, k, l)$, then clearly $\phi(S(b)a) = 0 = \phi(bS(a))$.

If $(p, q, r, s) = (m, n, k, l)$, then Lemma 2.25, applied to the family ${}^p T^q = \delta_{p,k} \delta_{q,l} |\eta'\rangle\langle\xi|$, yields morphisms \tilde{T}_l, \hat{T}_k from (\mathcal{H}, X) to (\mathcal{K}, Y) which necessarily are 0. Inserting the definition of \tilde{T}_l , we find

$$\begin{aligned} \phi(S(b)a) &= \phi((S \otimes \omega_{\eta, \eta'})({}^m_k Y_l^n) \cdot (\text{id} \otimes \omega_{\xi, \xi'})({}^k_m X_n^l)) \\ &= (\phi \otimes \text{id}) \left((1 \otimes \langle \eta |)_n (Y^{-1})_m^k (1 \otimes |\eta'\rangle\langle\xi|)_m^k X_n^l (1 \otimes |\xi'\rangle) \right) \\ &= (1 \otimes \langle \eta |)^m \tilde{T}_l^n (1 \otimes |\xi'\rangle) = 0. \end{aligned}$$

A similar calculation involving \hat{T} shows that $\phi(bS(a)) = 0$.

Using the relation $X^* = X^{-1} = (S \otimes \text{id})(X)$ and $Y^* = (S \otimes \text{id})(Y)$, we conclude $\phi(b^*a) = \phi(ba^*) = 0$. \square

The second part of the orthogonality relations concerns inner products as above but with $a, b \in \mathcal{C}(X)$ for some irreducible corepresentation X . It involves the conjugate corepresentation, which is defined as follows.

Given Hilbert spaces \mathcal{H}, \mathcal{K} , we denote by $\overline{\mathcal{H}}, \overline{\mathcal{K}}$ the conjugate Hilbert spaces, by $T \mapsto \overline{T}$ the canonical conjugate-linear isomorphism $\mathcal{B}(\mathcal{H}, \mathcal{K}) \rightarrow \mathcal{B}(\overline{\mathcal{H}}, \overline{\mathcal{K}})$, and by $T \mapsto T^\top := \overline{T}^*$ the linear anti-isomorphism $\mathcal{B}(\mathcal{H}, \mathcal{K}) \rightarrow \mathcal{B}(\overline{\mathcal{K}}, \overline{\mathcal{H}})$.

Lemma 2.27. *Let $\text{Corep}_{\text{sfd}, \text{Hilb}}(\mathcal{A})$ denote the category of (not necessarily unitary) corepresentations of \mathcal{A} on sfd bigraded Hilbert spaces. Then on $\text{Corep}_{\text{sfd}, \text{Hilb}}(\mathcal{A})$ there exist*

1. a covariant antilinear functor $(\mathcal{H}, X) \mapsto (\overline{\mathcal{H}}, \overline{X})$ and $T \mapsto \overline{T}$, where

$${}^k \overline{\mathcal{H}}^l = \overline{{}^l \mathcal{H}^k}, \quad {}^k_m \overline{X}_n^l = ({}^l X_m^k)^{(* \otimes (\cdot)^\top)} = (({}^l X_m^k)^*)^{\text{id} \otimes \top}, \quad {}^k \overline{T}^l = \overline{{}^l T^k};$$

2. a contravariant linear functor $(\mathcal{H}, X) \mapsto (\overline{\mathcal{H}}, X^{S \otimes \top})$ and $T \mapsto T^\top$, where

$${}^k \overline{\mathcal{H}}^l = \overline{{}^l \mathcal{H}^k}, \quad {}^k_m (X^{S \otimes \top})^l_n = (S \otimes (\cdot)^\top)({}^n_l X^m_k), \quad {}^k (T^\top)^l = ({}^l T^k)^\top;$$

3. a covariant linear functor $(\mathcal{H}, X) \mapsto (\mathcal{H}, X^{S^2 \otimes \text{id}})$ and $T \mapsto T$, where the grading is unchanged and

$${}^k_m (X^{S^2 \otimes \text{id}})^l_n = (S^2 \otimes \text{id})({}^k_m X^l_n).$$

If (\mathcal{H}, X) is unitary, then $\overline{X} = X^{S \otimes \top}$.

Proof. The first assertion follows immediately from the fact that $\Delta_{rs}(a^*) = \Delta_{sr}(a)^*$ and the $*$ -compatibility of ϵ . The second assertion follows from the fact that $\Delta_{pq} \circ S = (S \otimes S) \circ \Delta_{qp}^{\text{op}}$ and $\epsilon \circ S = \epsilon$. The final functor is just the square of the second functor.

The fact that $\overline{X} = X^{S \otimes \top}$ for X unitary is just by definition. □

We call $(\overline{\mathcal{H}}, \overline{X})$ the *conjugate* of (\mathcal{H}, X) .

Proposition 2.28. *Let \mathcal{A} define a partial compact quantum group, and let ϕ be a positive normalized invariant functional. Let (\mathcal{H}, X) be a unitary irreducible sfd corepresentation.*

1. *The conjugate $\overline{\mathcal{H}}$ with the family ${}^k_m (\overline{X}^{-*})^l_n := ({}^l_n (\overline{X}^{-1})^k_m)^*$ form an sfd corepresentation, and there exists an invertible, positive morphism \overline{F}_X from $(\overline{\mathcal{H}}, \overline{X})$ to $(\overline{\mathcal{H}}, \overline{X}^{-*})$.*
2. *The family ${}^k F_X^l := \overline{{}^l F_X^k}$ is an invertible, positive operator implementing a morphism from (\mathcal{H}, X) to $(\mathcal{H}, X^{S^2 \otimes \text{id}})$.*

Proof. (1) By Proposition 2.21, $(\overline{\mathcal{H}}, \overline{X})$ is equivalent to a unitary corepresentation, that is, there exists a family of invertible operators ${}^k T^l \in \mathcal{B}({}^k \overline{\mathcal{H}}^l)$ such that the family

$${}^k_m Z_n^l := (1 \otimes {}^k T^l) {}^k_m \overline{X}_n^l (1 \otimes {}^m T^n)^{-1}$$

is a unitary corepresentation. The relation ${}^n_l (Z^{-1})_k^m = ({}^m_k Z_l^n)^*$ then implies

$${}^k_m Z_n^l = (1 \otimes ({}^k T^l)^{-1})^* ({}^l_n (\overline{X}^{-1})^k_m)^* (1 \otimes {}^m T^n)^*$$

and hence the family ${}^k_m (\overline{X}^{-*})^l_n := ({}^l_n (\overline{X}^{-1})^k_m)^*$ is an irreducible sfd corepresentation. The maps ${}^k \overline{F}_X^l := ({}^k T^l)^* {}^k T^l \in \mathcal{B}({}^k \overline{\mathcal{H}}^l)$ then form an isomorphism from $(\overline{\mathcal{H}}, \overline{X})$ to $(\overline{\mathcal{H}}, \overline{X}^{-*})$.

(2) The morphism T from $(\overline{\mathcal{H}}, \overline{X})$ to $(\overline{\mathcal{H}}, Z)$ yields morphisms \overline{T} from (\mathcal{H}, X) to $(\mathcal{H}, \overline{Z})$ and T^\top from $(\mathcal{H}, Z^{S \otimes \top})$ to $(\mathcal{H}, \overline{X}^{S \otimes \top})$. Since X and Z are unitary, $\overline{Z} = Z^{S \otimes \top}$ and $\overline{X}^{S \otimes \top} = X^{S^2 \otimes \text{id}}$. Thus $T^\top \overline{T} = \overline{T}^* T$ is a morphism from (\mathcal{H}, X) to $(\mathcal{H}, X^{S^2 \otimes \text{id}})$. □

Theorem 2.29. *Let \mathcal{A} define a partial compact quantum group. Let ϕ be a positive normalized invariant functional. Let (\mathcal{H}, X) be a unitary irreducible sfd corepresentation of \mathcal{A} , and let F_X be a non-zero morphism from (\mathcal{H}, X) to $(\mathcal{H}, X^{S^2 \otimes \text{id}})$.*

1. *The numbers $\alpha := \sum_k \text{Tr}({}^k(F_X^{-1})^l)$ and $\beta := \sum_n \text{Tr}({}^m F_X^n)$ do not depend on l or n .*
2. *For all k, l, m, n ,*

$$\begin{aligned} (\phi \otimes \text{id})(({}^k X_n^l)^* {}^k X_n^l) &= \alpha^{-1} \text{Tr}({}^k(F_X^{-1})^l) \cdot \text{id}_m \mathcal{H}^n, \\ (\phi \otimes \text{id})({}^k X_n^l ({}^k X_n^l)^*) &= \beta^{-1} \text{Tr}({}^m(F_X)^n) \cdot \text{id}_k \mathcal{H}^l. \end{aligned}$$

3. *Denote by Σ_{klmn} the flip map ${}^k \mathcal{H}^l \otimes {}^m \mathcal{H}^n \rightarrow {}^m \mathcal{H}^n \otimes {}^k \mathcal{H}^l$. Then*

$$\begin{aligned} (\phi \otimes \text{id} \otimes \text{id})(({}^k X_n^l)^* ({}^k X_n^l)_{13}) &= \alpha^{-1} (\text{id}_m \mathcal{H}^n \otimes {}^k(F_X^{-1})^l) \circ \Sigma_{klmn}, \\ (\phi \otimes \text{id} \otimes \text{id})(({}^k X_n^l)_{13} ({}^k X_n^l)^*) &= \beta^{-1} ({}^m F_X^n \otimes \text{id}_k \mathcal{H}^l) \circ \Sigma_{klmn}. \end{aligned}$$

Proof. We prove the assertions and equations involving α in (1), (2) and (3) simultaneously; the assertions involving β follow similarly.

Consider the following endomorphism $F_{m,n,k,l}$ of ${}^m \mathcal{H}^n \otimes {}^k \mathcal{H}^l$,

$$\begin{aligned} F_{m,n,k,l} &:= (\phi \otimes \text{id} \otimes \text{id})(({}^k X_n^l)^* ({}^k X_n^l)_{13}) \circ \Sigma_{mnkl} \\ &= (\phi \otimes \text{id} \otimes \text{id}) \left(({}^l(X^{-1})_m^k)_{12} \Sigma_{klkl,23} ({}^k X_n^l)_{12} \right). \end{aligned}$$

By applying Lemma 2.25 with respect to the flip map Σ_{klkl} , we see that the family $(F_{m,n,k,l})_{m,n}$ is an endomorphism of $(\mathcal{H} \otimes {}^k \mathcal{H}^l, X_{12})$ and hence

$$F_{m,n,k,l} = \text{id}_m \mathcal{H}^n \otimes {}^k R^l \quad (2.8)$$

with some ${}^k R^l \in \mathcal{B}({}^k \mathcal{H}^l)$ not depending on m, n . On the other hand,

$$\begin{aligned} F_{m,n,k,l} &= (\phi \otimes \text{id} \otimes \text{id})((S \otimes \text{id})({}^m X_l^n)_{12} ({}^k X_n^l)_{13}) \circ \Sigma_{mnkl} \\ &= (\phi \circ S^{-1} \otimes \text{id} \otimes \text{id}) \left(((S \otimes \text{id})({}^k X_n^l)_{13}) ((S^2 \otimes \text{id})({}^m X_l^n)_{12}) \right) \circ \Sigma_{mnkl} \\ &= (\phi \circ S^{-1} \otimes \text{id} \otimes \text{id}) \left(((S \otimes \text{id})({}^k X_n^l)_{13}) (\Sigma_{mnmn})_{23} ((S^2 \otimes \text{id})({}^m X_l^n)_{13}) \right). \end{aligned}$$

Since $\phi \circ S^{-1}$ is an invariant functional for \mathcal{A} , we can again apply Lemma 2.25 and find that the family $(F_{m,n,k,l})_{k,l}$ is a morphism

$$(F_{m,n,k,l})_{k,l} : ({}^m \mathcal{H}^n \otimes \mathcal{H}, (X^{S^2 \otimes \text{id}})_{13}) \rightarrow ({}^m \mathcal{H}^n \otimes \mathcal{H}, X_{13}).$$

Therefore,

$$F_{m,n,k,l} = {}^m T^n \otimes ({}^k F_X^l)^{-1} \quad (2.9)$$

with some ${}^mT^n \in \mathcal{B}({}^m\mathcal{H}^n)$ not depending on k, l . Combining (2.8) and (2.9), we conclude that, for some $\lambda \in \mathbb{C}$,

$$F_{m,n,k,l} = \lambda(\text{id}^m \mathcal{H}^n \otimes ({}^kF_X^l)^{-1})$$

Choose a basis $(\zeta_i)_i$ for ${}^k\mathcal{H}^l$. Then

$$\lambda \cdot \text{id}^m \mathcal{H}^n \cdot \text{Tr}({}^kF_X^l)^{-1} = \sum_i (\text{id} \otimes \omega_{\zeta_i, \zeta_i})(F_{m,n,k,l}) = (\phi \otimes \text{id})({}^kX_n^l \otimes {}^kX_n^l).$$

Taking $n = l$ and summing over k , the relations $\sum_k ({}^kX_n^l)^* {}^kX_n^l = \mathbf{1} \binom{l}{n} \otimes \text{id}^m \mathcal{H}^n$ and $\phi(\mathbf{1} \binom{l}{i}) = 1$ give

$$\lambda \cdot \sum_k \text{Tr}({}^kF_X^l)^{-1} = 1.$$

Now all assertions in (1)–(3) concerning α follow. \square

Corollary 2.30. *Assume that \mathcal{A} defines a partial compact quantum group, and let ϕ be a normalized positive invariant functional. Let (\mathcal{H}, X) be a unitary irreducible sfd corepresentation of \mathcal{A} , let F_X be a non-zero morphism from (\mathcal{H}, X) to $(\mathcal{H}, (S^2 \otimes \text{id})(X))$, and let $a = (\text{id} \otimes \omega_{\xi, \xi'})({}^mX_n^l)$ and $b = (\text{id} \otimes \omega_{\eta, \eta'})({}^mX_n^l)$, where $\xi, \eta \in {}^k\mathcal{H}^l$ and $\xi', \eta' \in {}^m\mathcal{H}^n$. Then*

$$\phi(b^*a) = \frac{\langle \eta' | \xi' \rangle \langle \xi | F_X^{-1} \eta \rangle}{\sum_m \text{Tr}({}^m(F_X^{-1})^n)}, \quad \phi(ab^*) = \frac{\langle \eta' | F_X \xi' \rangle \langle \xi | \eta \rangle}{\sum_n \text{Tr}({}^mF_X^n)}.$$

Proof. By Theorem 2.29,

$$\begin{aligned} \phi(b^*a) &= (\phi \otimes \omega_{\eta', \eta} \otimes \omega_{\xi, \xi'})({}^kX_n^l \otimes {}^mX_n^l)_{12}({}^mX_n^l)_{13} \\ &= \frac{1}{\sum_k \text{Tr}({}^k(F_X^{-1})^l)} (\omega_{\eta', \eta} \otimes \omega_{\xi, \xi'})((\text{id}^m \mathcal{H}^n \otimes {}^k(F_X^{-1})^l) \circ \Sigma_{k,l,m,n}). \end{aligned}$$

The formula for $\phi(ab^*)$ follows similarly or by considering the co-opposite of \mathcal{A} . \square

Corollary 2.31. *Let \mathcal{A} define a partial compact quantum group. Let $(\mathcal{H}_\alpha, X_\alpha)_\alpha$ be a representative family of all irreducible sfd corepresentations of \mathcal{A} . Then the map*

$$\bigoplus_\alpha \bigoplus_{k,l,m,n} (\overline{{}^k\mathcal{H}_\alpha^l} \otimes {}^m\mathcal{H}_\alpha^n) \rightarrow A$$

that sends $\bar{\xi} \otimes \eta \in \overline{{}^k\mathcal{H}_\alpha^l} \otimes {}^m\mathcal{H}_\alpha^n$ to $(\text{id} \otimes \omega_{\xi, \eta})({}^k(X_\alpha)_n^l)$, is a linear isomorphism.

Proof. This follows from Proposition 2.24, Proposition 2.26 and Corollary 2.30. \square

Suppose now $a \in {}^k_m A_n^l$ for some partial bialgebra \mathcal{A} . Then for $\omega \in \text{Hom}_{\mathbb{C}}(A, \mathbb{C})$, we can define

$$\omega *_{p,q} a := (\text{id} \otimes \omega)(\Delta_{pq}(a)), \quad a *_{r,s} \omega := (\omega \otimes \text{id})(\Delta_{rs}(a)).$$

Clearly we can define

$$\omega *_{p,q} a *_{r,s} \omega' := (\omega *_{p,q} a) *_{r,s} \omega' = \omega *_{p,q} (a *_{r,s} \omega').$$

When ω has support on the $A(K)$ with $K_u = K_d$, we can write, for $a \in {}^k_m A_n^l$,

$$\omega * a := \sum_{p,q} \omega *_{p,q} a = \omega *_{m,n} a, \quad a * \omega = \sum_{r,s} a *_{r,s} \omega = a *_{k,l} \omega.$$

We shall say that an entire function f has *exponential growth on the right half-plane* if there exist $C, d > 0$ such that $|f(x + iy)| \leq Ce^{dx}$ for all $x, y \in \mathbb{R}$ with $x > 0$.

Theorem 2.32. *Assume that \mathcal{A} defines a partial compact quantum group with positive normalized invariant functional ϕ . There exists a unique family of linear functionals $f_z: A \rightarrow \mathbb{C}$ such that*

- (1) f_z vanishes on $A(K)$ when $K_u \neq K_d$.
- (2) for each $a \in A$, the function $z \mapsto f_z(a)$ is entire and of exponential growth on the right half-plane.
- (3) $f_0 = \epsilon$ and $(f_z \otimes f_{z'}) \circ \Delta = f_{z+z'}$ for all $z, z' \in \mathbb{C}$.
- (4) $\phi(ab) = \phi(b(f_1 * a * f_1))$ for all $a, b \in A$.

This family furthermore satisfies

- (5) $f_z(ab) = f_z(a)f_z(b)$ for $a \in A(K)$ and $b \in A(L)$ with $K_r = L_l$.
- (6) $S^2(a) = f_{-1} * a * f_1$ for all $a \in A$.
- (7) $f_z(\mathbf{1}_n^l) = \delta_{l,n}$, $f_z \circ S = f_{-z}$, and $f_z(a^*) = \overline{f_{-\bar{z}}(a)}$ for all $a \in A$.

Note that condition (3) is meaningful by condition (1).

Proof. We first prove uniqueness. Assume that $(f_z)_z$ is a family of functionals satisfying (1)–(4). Since ϕ is faithful, the map $\sigma: a \mapsto f_1 * a * f_1$ is uniquely determined by ϕ , and one easily sees that it is a homomorphism. Using (3), we find that $\epsilon \circ \sigma^n = f_{2n}$, which uniquely determines these functionals. Using (2) and the fact that every entire function of exponential growth on the right half-plane is uniquely determined by its values at $\mathbb{N} \subseteq \mathbb{C}$, we can conclude that the family f_z is uniquely determined. Moreover, since the property (5) holds for $z = 2n$, we also conclude by the same argument as above that it holds for all $z \in \mathbb{C}$.

Let us now prove existence. By Corollary 2.31, we can define for each $z \in \mathbb{C}$ a functional $f_z: A \rightarrow \mathbb{C}$ such that for every unitary irreducible sfd corepresentation (\mathcal{H}, X) ,

$$f_z((\text{id} \otimes \omega_{\xi, \eta})({}^k X_n^l)) = \delta_{k,m} \delta_{l,n} \cdot \omega_{\xi, \eta}({}^k F_X^l)^z \quad \text{for all } \xi \in {}^k \mathcal{H}^l, \eta \in {}^m \mathcal{H}^n,$$

or, equivalently,

$$(f_z \otimes \text{id})({}^k X_n^l) = \delta_{k,m} \delta_{l,n} \cdot ({}^k F_X^l)^z,$$

where F_X is a non-zero positive operator implementing a morphism from (\mathcal{H}, X) to $(\mathcal{H}, (S^2 \otimes \text{id})(X))$, scaled such that

$$\alpha_X := \sum_k \text{Tr}({}^k (F_X^{-1})^l) = \sum_n \text{Tr}({}^m F_X^n)$$

for all l, n (see Proposition 2.28 and Theorem 2.29). By construction, (1) and (2) hold. We show that the $(f_z)_z$ satisfy the assertions (3)–(7). Throughout the following arguments, let (\mathcal{H}, X) be a unitary irreducible corepresentation (\mathcal{H}, X) and let F_X be as above.

We first prove property (3). This follows from the relations

$$(f_0 \otimes \text{id})({}^k X_n^l) = \delta_{k,m} \delta_{l,n} \text{id}_k \mathcal{H}^l = (\epsilon \otimes \text{id})({}^k X_n^l)$$

and

$$\begin{aligned} (((f_z \otimes f_{z'}) \circ \Delta) \otimes \text{id})({}^k X_n^l) &= \delta_{k,m} \delta_{l,n} (f_z \otimes f_{z'} \otimes \text{id})({}^k X_l^l)_{13} ({}^k X_l^l)_{23} \\ &= \delta_{k,m} \delta_{l,n} ({}^k F_X^l)^z \cdot ({}^k F_X^l)^{z'} \\ &= (f_{z+z'} \otimes \text{id})({}^k X_n^l). \end{aligned}$$

Applying slice maps of the form $\text{id} \otimes \omega_{\xi, \xi'}$ and invoking Theorem 2.29, this proves (3).

To prove (4), write again $\Delta^{(2)} = (\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta$, and put

$$\theta_{z, z'} := (f_{z'} \otimes \text{id} \otimes f_z) \circ \Delta^{(2)}.$$

Then

$$\begin{aligned} (\theta_{z, z'} \otimes \text{id})({}^k X_n^l) &= (f_{z'} \otimes \text{id} \otimes f_z \otimes \text{id})({}^k X_l^l)_{14} ({}^k X_n^l)_{24} ({}^m X_n^n)_{34} \\ &= (1 \otimes ({}^k F_X^l)^{z'}) {}^k X_n^l (1 \otimes ({}^m F_X^n)^z). \end{aligned}$$

We take $z = z' = 1$, use Theorem 2.29, where now $\alpha = \beta$ by our scaling of F_X , and obtain

$$\begin{aligned} (\phi \otimes \text{id} \otimes \text{id})({}^k X_n^l)_{12}^* ((\theta_{1,1} \otimes \text{id})({}^k X_n^l))_{13} &= \alpha^{-1} (\text{id} \otimes {}^k F_X^l) (\text{id} \otimes {}^k (F_X^{-1})^l)_{\Sigma_{k,l,m,n}} (\text{id} \otimes {}^m F_X^n) \\ &= \beta^{-1} ({}^m F_X^n \otimes \text{id})_{\Sigma_{k,l,m,n}} \\ &= (\phi \otimes \text{id} \otimes \text{id})({}^k X_n^l)_{13} ({}^k X_n^l)_{12}^*. \end{aligned}$$

To conclude the proof of assertion (4), apply again slice maps of the form $\omega_{\xi,\xi'} \otimes \omega_{\eta,\eta'}$.

We have then already argued that the property (5) automatically holds. To show the property (6), note that by Proposition 2.28 and the calculation above,

$$(S^2 \otimes \text{id})({}^k_m X_n^l) = (1 \otimes {}^k F_X^l) {}^k_m X_n^l (1 \otimes {}^m F_X^n)^{-1} = (\theta_{-1,1} \otimes \text{id})({}^k_m X_n^l).$$

Assertion (6) follows again by applying slice maps.

Finally, (1), (2) and (4) immediately imply the relation $f_z(\mathbf{1}({}^k_m)) = \delta_{k,m}$. The concrete construction of f_z combined with property (3), the identity (??) and the partial character property (5) gives the equality

$$\begin{aligned} (f_{-z} \otimes \text{id})({}^k_l X_l^l) &= ({}^k(F_X)^l)^{-z} = \left((f_z \otimes \text{id})({}^k_l X_l^l) \right)^{-1} \\ &= (f_z \otimes \text{id})({}_l^l(X^{-1})_k^k) = ((f_z \circ S) \otimes \text{id})({}^k_l X_l^l). \end{aligned}$$

Therefore, $f_{-z} = f_z \circ S$. Let us now write $\bar{f}_z(a) = \overline{f_z(a^*)}$. Using the preceding calculation, the relation $(S \otimes \text{id})({}^k_l X_l^l) = ({}^k_l X_l^l)^*$ and positivity of ${}^k F_X^l$, we conclude

$$\begin{aligned} (\bar{f}_z \otimes \text{id})({}^k_l X_l^l) &= \left((f_z \otimes \text{id})(({}^k_l X_l^l)^*) \right)^* \\ &= \left((f_{-z} \otimes \text{id})({}^k_l X_l^l) \right)^* = (({}^k F_X^l)^{-z})^* = ({}^k F_X^l)^{-\bar{z}} = (f_{-\bar{z}} \otimes \text{id})({}^k_l X_l^l), \end{aligned}$$

whence $\bar{f}_z = f_{-\bar{z}}$. □

Corollary 2.33. *Let \mathcal{G} be an partial compact quantum group with underlying total algebra A and define $\theta_{z,z'}: A \rightarrow A$ by $a \mapsto f_z * a * f_{z'}$ for each $z, z' \in \mathbb{C}$, where the functionals f_z are as in Theorem 2.32. Then for all $z, z', w, w' \in \mathbb{C}$, the following conditions hold:*

1. $\theta_{z,z'}$ is an algebra automorphism and preserves each subspace $A(K)$; in particular, $\theta_{z,z'}(\lambda_k \rho_m) = \lambda_k \rho_m$ for all $k, m \in I$;
2. $\theta_{z,z'} \circ * = * \circ \theta_{-\bar{z}, -\bar{z}'}$; in particular, $\theta_{it, is}$ is a $*$ -automorphism for each $t, s \in \mathbb{R}$;
3. $\theta_{z,z'} \circ \theta_{w,w'} = \theta_{z+w, z'+w'}$;
4. $(\theta_{w,z'} \otimes \theta_{z,-w}) \circ \Delta = \Delta \circ \theta_{z,z'}$, $\epsilon \circ \theta_{z,z'} = f_{z+z'}$, $\theta_{z,z'} \circ S = S \circ \theta_{-z', -z}$ and $\phi \circ \theta_{z,z'} = \phi$;
5. for every linear map $\omega: A \rightarrow \mathbb{C}$ and every $a \in A$, the map $(z, z') \mapsto \omega(\theta_{z,z'}(a))$ is entire.

Proof. All of this follows easily from Theorem 2.32. □

Using the two-parameter group θ , we define the *modular automorphism group* σ , the *scaling group* τ and the *unitary antipode* of a partial compact quantum group A by

$$\sigma_z := \theta_{iz, iz}, \quad \tau_z := \theta_{iz, -iz}, \quad R := S \circ \tau_{i/2}.$$

Using Corollary 2.33, one verifies that σ, τ, R share all the main relations known for locally compact quantum groups and measured quantum groupoids, for example, σ and τ are complex one-parameter groups of algebra automorphisms of A , the map R is a $*$ -anti-automorphism, τ_t and σ_t are $*$ -automorphisms for all $t \in \mathbb{R}$, the family τ commutes with σ and with R in the obvious sense,

$$\phi \circ \sigma_z = \phi \circ \tau_z = \phi \circ R = \phi, \quad \phi(ab) = \phi(b\sigma_{-i}(a)), \quad (2.10)$$

$$\Delta \circ \tau_z = (\tau_z \otimes \tau_z) \circ \Delta, \quad (\tau_z \otimes \sigma_z) \circ \Delta = \Delta \circ \sigma_z = (\sigma_{-z} \otimes \tau_z) \circ \Delta, \quad (2.11)$$

$$R^2 = \text{id}_A, \quad \Delta \circ R = (R \otimes R) \circ \Delta^{\text{op}}. \quad (2.12)$$

3 Tannaka-Krein duality for compact Hopf face algebras

Let \mathcal{C} be a rigid tensor C^* -category with irreducible unit. For example, one can take $\mathcal{C} = \text{Rep}(\mathbb{G})$, the category of finite-dimensional unitary representations of a compact quantum group \mathbb{G} . We will in general view the tensor product of \mathcal{C} as being strict. Let J be an index set for a maximal set of mutually non-isomorphic irreducible objects u_a in \mathcal{C} . The unit object of \mathcal{C} will be written u_o . Whenever convenient, we will replace u_a by its associated index symbol a . We will also fix once and for all orthonormal bases $f_{c,j}^{a,b}$ for $\text{Mor}(u_c, u_a \otimes u_b)$, where j runs over an index set $J_c^{a,b}$.

Let I be a (countable) set. We will write Hilb_{I^2} for the monoidal tensor C^* -category of I -bigraded Hilbert spaces $\mathcal{H} = \sum_{r,s}^{\oplus} \mathcal{H}_{rs}$, where the direct sum on the right is understood as the completion of the ordinary algebraic one. The tensor product \otimes_I in Hilb_{I^2} is defined by $(\mathcal{H} \otimes_I \mathcal{G})_{rs} = \bar{\oplus}_t (\mathcal{H}_{rt} \otimes \mathcal{G}_{ts})$. The unit of Hilb_{I^2} is $l^2(I)$ with the obvious I^2 -grading. We will view this monoidal category as being strict.

We will be interested in strong tensor C^* -functors F from \mathcal{C} to Hilb_{I^2} . As shown in [DCY], any ergodic action of a compact quantum group \mathbb{G} on a unital C^* -algebra provides a tensor C^* -functor of \mathcal{C} into Hilb_{I^2} for some set I .

For $F : \mathcal{C} \rightarrow \text{Hilb}_{I^2}$ a strong tensor C^* -functor, we denote the unitary compatibility morphisms by $\phi_{X,Y} : F(X) \otimes_I F(Y) \rightarrow F(X \otimes Y)$, where we recall that they are assumed to satisfy the coherence conditions

$$\phi_{X,Y \otimes Z}(\text{id}_X \otimes \phi_{Y,Z}) = \phi_{X \otimes Y, Z}(\phi_{X,Y} \otimes \text{id}_Z), \quad \phi_{o,a} = \phi_{a,o} = \text{id}_a.$$

It will be convenient to extend $\phi_{X,Y}$ to a coisometry $F(X) \otimes F(Y) \rightarrow F(X \otimes Y)$, defining it to be zero on the orthogonal complement of $F(X) \otimes_I F(Y)$. Note however that then $\phi_{X,o}$ becomes the coisometry $F(X) \otimes l^2(I) \rightarrow F(X)$ sending $F(X)_{rs} \otimes \mathbb{C}\delta_t$ canonically onto $\delta_{s,t}F(X)_{rs}$, and similarly for $\phi_{o,X}$. Whenever X, Y are clear, we will abbreviate $\phi_{X,Y}$ as ϕ . We will use the notation

$$F_{c,j}^{a,b} = \phi^* F(f_{c,j}^{a,b}) \in B(F(u_c), F(u_a) \otimes F(u_b)).$$

As \mathcal{C} is rigid, each $F(X)$ will be column-finite in the sense that for each X in \mathcal{C} and each fixed s in I , the direct sum $\sum_r^{\oplus} F(X)_{rs}$ will be finite-dimensional. Similarly, each $F(X)$ will be row-finite. See ...

Define vector spaces

$${}_m^k A_n^l(a) = B(F(u_a)_{kl}, F(u_a)_{mn}).$$

Write ${}_m^k A_n^l = \oplus_{a \in J} {}_m^k A_n^l(a)$ and $A = \oplus_{k,l,m,n} {}_m^k A_n^l$. The a -spectral subspace $A(a)$ of A is defined as

$$A(a) = \sum_{k,l,m,n}^{\oplus} {}_m^k A_n^l(a).$$

For any element $x \in A$, its component in the a -spectral subspace is written x_a .

Our goal is to turn A into a generalized compact Hopf face algebra.

We first turn A into an algebra. The multiplication $x \cdot y$ of $x \in {}_m^k A_n^l(a)$ and $y \in {}_r^p A_s^q(b)$ is the element in ${}_m^k A_s^q$ defined by the formula

$$(x \cdot y)_c = \sum_{j \in J_c^{a,b}} \left(F_{c,j}^{a,b} \right)^* (x \otimes y) \left(F_{c,j}^{a,b} \right).$$

Note that the product is independent of the specific choice of orthogonal bases $f_{c,j}^{a,b}$. We will continue to use the \cdot -notation to distinguish this product from the ordinary multiplication of operators.

Lemma 3.1. *With the above product, A becomes a faithful strong I^2 -algebra.*

Proof. Let $x \in {}_m^k A_n^l(a)$, $y \in {}_r^p A_s^q(b)$ and $z \in {}_s^q A_v^t$. From the fact that ϕ is a natural transformation, we find that

$$((x \cdot y) \cdot z)_d = \sum_{e \in J} \sum_{k \in J_d^{e,c}} \sum_{j \in J_e^{a,b}} \left(\phi^*(\phi^* \otimes \text{id}) F(f_{d,e,j,k}^{1,a,b,c}) \right)^* (x \otimes y \otimes z) \left((\phi^* \otimes \text{id}) \phi^* F(f_{d,e,j,k}^{1,a,b,c}) \right)$$

where $f_{d,e,j,k}^{1,a,b,c} = (f_{e,j}^{a,b} \otimes \text{id}) f_{d,k}^{e,c}$. On the other hand,

$$(x \cdot (y \cdot z))_d = \sum_{e \in J} \sum_{k \in J_d^{a,e}} \sum_{j \in J_e^{b,c}} \left(\phi(\text{id} \otimes \phi) F(f_{d,e,j,k}^{2,a,b,c}) \right)^* (x \otimes y \otimes z) \left((\text{id} \otimes \phi) \phi F(f_{d,e,j,k}^{2,a,b,c}) \right)$$

where $f_{d,e,j,k}^{2,a,b,c} = (\text{id} \otimes f_{e,j}^{b,c}) f_{d,k}^{a,e}$. As $\phi(\phi \otimes \text{id})$ by $\phi(\text{id} \otimes \phi)$ by assumption, and as the orthonormal bases $\{f_{d,e,j,k}^{1,a,b,c} \mid e, j, k\}$ or $\{f_{d,e,j,k}^{2,a,b,c} \mid e, j, k\}$ can clearly be replaced by any other orthonormal basis of $\text{Mor}(u_d, u_a \otimes u_b \otimes u_c)$, it follows that $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

Define $1_{rs} \in B(F(u_o)_{rr}, F(u_o)_{ss}) = {}_s^r A_s^r(o)$ as the map sending δ_r to δ_s . By the compatibility assumption for $\phi_{a,o}$ and $\phi_{o,a}$, the map $\text{Fun}_f(I^2) \rightarrow A$ mapping $\delta_{(r,s)}$ to 1_{rs} is an algebra homomorphism. Thus A becomes a faithful strong I^2 -algebra. \square

In the following, we will again write $\lambda_r = \sum_s 1_{rs}$ and $\rho_s = \sum_r 1_{rs}$ inside $M(A)$, using the notation as at the end of the proof of the previous lemma.

We turn to the coproduct. Let $\{e_{a,i} \mid i \in B_a\}$ denote an orthonormal basis of $F(u_a)$ over an index set B_a which is adapted to the bigrading (in the sense that each $e_{a,i}$ is inside exactly one component). Write $B_{a,rs} \subseteq B_a$ for the set of indices for which $e_{a,i} \in F(u_a)_{rs}$. Define elements

$$P_{mn}^{kl}(a) \in {}^k_m A_n^l(a) \otimes {}^m_k A_l^n(a)$$

by

$$P_{mn}^{kl}(a) = \sum_{i \in B_{a,kl}, j \in B_{a,mn}} e_{a,j} e_{a,i}^* \otimes e_{a,i} e_{a,j}^*.$$

As each $F(u_a)_{kl}$ is finite-dimensional, the above sums are finite.

Define now maps

$$\Delta_{rs} : {}^k_m A_n^l(a) \rightarrow {}^k_r A_s^l(a) \otimes {}^r_m A_n^s(a)$$

by the application

$$x \mapsto P_{rs}^{mn}(a)(x \otimes 1) = (1 \otimes x)P_{rs}^{kl}(a).$$

They obviously extend to linear maps Δ_{rs} from A to $A \otimes_I A$.

Lemma 3.2. *For each $x \in A$, the element $\Delta(x) = \sum_{rs} \Delta_{rs}(x)$ gives a well-defined multiplier of $A \otimes_I A$. The resulting map $\Delta : A \rightarrow M(A \otimes_I A)$ is an I^2 -coproduct.*

Proof. As the grading on each $F(u_a)$ is column-finite, it follows at once that for each fixed p, q and $x \in A$, the element $\Delta_{rs}(x)(1 \otimes_I \lambda_p \rho_q)$ is zero except for finitely many r and s . Similarly, $(1 \otimes_I \lambda_p \rho_q) \Delta_{rs}(x)$ is zero except for finitely many r, s because of row-finiteness of $F(u_a)$. Hence $\Delta(x)$ is well-defined as a multiplier for each $x \in A$. Once we show that Δ is multiplicative, it will be immediate that Δ is coassociative, since each Δ_{rs} is coassociative. Moreover, also the fact that Δ then is an I^2 -morphism is clear from the definition.

To obtain the multiplicativity of Δ , or rather of the coextension $\tilde{\Delta}$, choose $x \in {}_m A_n(a)$ and $y \in {}_n A_q(b)$. Then

$$\begin{aligned} \tilde{\Delta}(x) \cdot \tilde{\Delta}(y) &= \sum_{rstv} (P_{rs}^{mn}(a)(x \otimes 1)) \cdot (P_{tv}^{nq}(b)(y \otimes 1)) \\ &= \sum_{cd} \sum_{ij} \sum_{klpt} \left(F_{c,i}^{a,b} \otimes F_{d,j}^{a,b} \right)^* (e_{a,k} e_{a,l}^* x \otimes e_{b,p} e_{b,t}^* y \otimes e_{a,l} e_{a,k}^* \otimes e_{b,t} e_{b,p}^*) \left(F_{c,i}^{a,b} \otimes F_{d,j}^{a,b} \right), \end{aligned}$$

where we may take the sum over all $k, l \in I_a$, $p, t \in I_b$ (and where the composition of operators with mismatching target and source is considered to be zero). Note that the infinite sums are convergent inside $M(A \otimes A)$ by the argument in the first paragraph.

Plugging in the identity operator $\sum_{cd} \sum_{rs} (e_{c,r} e_{c,r}^* \otimes e_{d,s} e_{d,s}^*)$ at the front, we obtain that the expression becomes

$$\sum_{cd} \sum_{rs} \sum_{ij} \sum_{klpt} X_{k,p}^{c,r,i} Y_{l,t}^{d,s,j} (e_{c,r} \otimes e_{d,s}) (e_{a,l}^* x \otimes e_{b,t}^* y \otimes e_{a,k}^* \otimes e_{b,p}^*) (F_{c,i}^{a,b} \otimes F_{d,j}^{a,b})^*$$

where $X_{k,p}^{c,r,i} = e_{c,r}^* (F_{c,i}^{a,b})^* (e_{a,k} \otimes e_{b,p})$ and $Y_{l,t}^{d,s,j} = e_{d,t}^* (F_{d,j}^{a,b})^* (e_{a,l} \otimes e_{b,t})$. Resumming over the k, l, p, t , we obtain

$$\sum_{cd} \sum_{rs} \sum_{ij} (e_{c,r} e_{d,s}^* (F_{d,j}^{a,b})^* (x \otimes y) F_{c,i}^{a,b}) \otimes (e_{d,s} e_{c,r}^* (F_{c,i}^{a,b})^* F_{d,j}^{a,b}).$$

As ϕ is a coisometry and the $f_{c,i}^{a,b}$ are orthonormal, this expression simplifies to

$$\sum_c \sum_{rs} \sum_i e_{c,r} e_{c,s}^* \left((F_{c,i}^{a,b})^* (x \otimes y) F_{c,i}^{a,b} \right) \otimes e_{c,s} e_{c,r}^*,$$

which is precisely $\tilde{\Delta}(x \cdot y)$. □

Proposition 3.3. *The couple (A, Δ) is a generalized face algebra over I .*

Proof. Let ε assign to any $x \in {}^k_m A_n^l(a)$ the number $\text{Tr}(x) = \sum_{i \in B_a} (e_{a,i}^* x e_{a,i})$ (where we keep the convention that mismatching operators compose to zero). We claim that ε is a counit, satisfying the conditions in the definition of a generalized face algebra. The fact that ε is a counit is immediate from the definition of Δ . It is also computed directly that for $x \in {}^k_m A_n^l$ and $y \in {}^l_n A_s^r$, we have $\varepsilon(x \cdot y) = \varepsilon(x) \varepsilon(y)$, since the $\{\phi^* F(f_{c,i}^{a,b}) e_{c,j} \mid c, i, j\}$ form an orthonormal basis of $F(u_a) \otimes_I F(u_b)$. From this formula, the second identity for the counit will hold true once we show that

$$\varepsilon(\lambda_k \rho_m x \lambda_l \rho_n) = \varepsilon(\lambda_k \rho_m x_{(1)}) \varepsilon(x_{(2)} \lambda_l \rho_n).$$

But both left and right hand side are zero unless $k = m$, $n = l$ and $x \in {}^k_m A_n^l$, in which case both sides equal $\varepsilon(x)$. □

Our next job is to define a suitable antipode for (A, Δ) . Here the rigidity of \mathcal{C} will come into play, so we first fix our conventions. Let $a \mapsto \bar{a}$ be the involution induced by the rigidity on the index set J . We assume that $\overline{u_a} = u_{\bar{a}}$. For each u_a , we will fix duality morphisms $R_a : u_0 \rightarrow u_{\bar{a}} \otimes u_a$ and $\bar{R}_a : u_0 \rightarrow u_a \otimes u_{\bar{a}}$. By means of F and ϕ , they induce I^2 -grading preserving maps $\mathcal{R}_a : l^2(I) \rightarrow F(u_{\bar{a}}) \otimes_I F(u_a)$ and $\bar{\mathcal{R}}_a : l^2(I) \rightarrow F(u_a) \otimes_I F(u_{\bar{a}})$. These in turn provide an invertible anti-linear map $I_a : F(u_a)_{kl} \rightarrow F(u_{\bar{a}})_{lk}$ and $J_a : F(u_{\bar{a}})_{lk} \rightarrow F(u_a)_{kl}$ such that $\langle I_a \xi_a, \eta_{\bar{a}} \rangle = \sum_r \delta_r^* \bar{\mathcal{R}}_a^* (\xi_a \otimes \eta_{\bar{a}})$ and $\langle J_a \eta_{\bar{a}}, \xi_a \rangle = \sum_s \delta_s^* \mathcal{R}_a^* (\eta_{\bar{a}} \otimes \xi_a)$. The snake identities for R_a and \bar{R}_a guarantee that J_a is the inverse of I_a .

We define

$$S : {}^k_m A_n^l(a) \rightarrow {}^n_l A_k^m(\bar{a})$$

by

$$x \mapsto I_a x^* J_a.$$

Lemma 3.4. *By means of the map S , the couple (A, Δ) becomes a generalized Hopf face algebra.*

Proof. It is clear that S is invertible. We also have $S(\lambda_k \rho_l) = \lambda_l \rho_k$ as $I_o \delta_k = \delta_k$.

Let us check that S satisfies the condition $S(x_{(1)}) \cdot x_{(2)} = \sum_p \varepsilon(x \cdot \lambda_p) \rho_p$ in the multiplier algebra for $x \in {}^k_m A_n^l(a)$. By definition, we have

$$S(x_{(1)}) \cdot x_{(2)} = \sum_c \sum_i \sum_{p,q \in B_a} \left(F_{c,i}^{\bar{a},a} \right)^* (I_a e_{a,q} e_{a,p}^* J_a \otimes x e_{a,q} e_{a,p}^*) \left(F_{c,i}^{\bar{a},a} \right).$$

Let $C : \mathbb{C} \rightarrow \mathbb{C}$ be complex conjugation. Then we can write $I_a e_{a,q} e_{a,p}^* J_a = (I_a e_{a,q} C)(C e_{a,p}^* J_a)$. We now calculate, by definition of J_a and $F_{c,i}^{\bar{a},a}$, that

$$\sum_{p \in B_a} (C e_{a,p}^* J_a \otimes e_{a,p}^*) \left(F_{c,i}^{\bar{a},a} \right) = (R_a^* f_{c,i}^{\bar{a},a}) \sum_{s \in I} \delta_s^*,$$

since ϕ is a coisometry. Plugging this into our expression for $S(x_{(1)}) \cdot x_{(2)}$, we obtain

$$\sum_s \sum_{q \in B_a} \left(\sum_c \sum_i \left(f_{c,i}^{\bar{a},a} \right)^* R_a \right) F_{c,i}^{\bar{a},a} (I_a e_{a,q} C \otimes x e_{a,q} \delta_s^*).$$

As ϕ is a coisometry and $\phi^* \phi \mathcal{R}_a = \mathcal{R}_a$, we can write $\left(f_{c,i}^{\bar{a},a} \right)^* R_a F_{c,i}^{\bar{a},a} = F_{c,i}^{\bar{a},a} (F_{c,i}^{\bar{a},a})^* \mathcal{R}_a$. As the $f_{c,i}^{\bar{a},a}$ form an orthonormal basis, we thus get

$$S(x_{(1)}) \cdot x_{(2)} = \sum_s \sum_{q \in B_a} \mathcal{R}_a^* (I_a e_{a,q} C \otimes x e_{a,q} \delta_s^*).$$

Now the composition $I_a e_{a,q} C$ is the creation operator for the vector $I_a e_{a,q}$. Hence using again the definition of J_a , and using that $x \in {}^k_m A_n^l$, we get

$$\begin{aligned} S(x_{(1)}) \cdot x_{(2)} &= \sum_s \sum_q \delta_n \delta_s^* e_{a,q}^* x e_{a,q} \\ &= \sum_s \text{Tr}(x) \delta_n \delta_s^* \\ &= \sum_p \varepsilon(x \lambda_p) \rho_p, \end{aligned}$$

since $\text{Tr}(x) = \delta_{k,m} \delta_{n,l} \varepsilon(x)$.

The identity $x_{(1)} \cdot S(x_{(2)}) = \sum_p \varepsilon(\rho_p x) \lambda_p$ is proven in a similar way. \square

In the next step, we determine an invariant functional for (A, Δ) .

Definition 3.5. We define $\varphi : {}^k A_n^l \rightarrow \mathbb{C}$ as the projection onto the component ${}^k A_n^l(o) \cong \delta_{kl} \delta_{mn} \mathbb{C}$.

Lemma 3.6. *The functional φ is an invariant normalized functional.*

Proof. The fact that φ is normalized is immediate, so let us check that it is invariant. Let $x \in {}^k A_n^l(a)$. Then

$$\begin{aligned} (\text{id} \otimes \varphi) \tilde{\Delta}(x) &= \sum_{i,j} \varphi(e_{a,j} e_{a,i}^*) e_{a,i} e_{a,j}^* x \\ &= \delta_{a,o} \sum_{r,s} \delta_r \delta_s^* x \\ &= \varphi(x) \sum_r \delta_r \delta_r^* \\ &= \sum_p \varphi(\lambda_p x) \lambda_p. \end{aligned}$$

The proof of right invariance follows similarly. \square

Finally, we introduce the $*$ -structure and show that (A, Δ) is a generalized compact Hopf face algebra. To distinguish the new $*$ -operation from the ordinary operator algebraic one, we will denote it by \dagger .

Definition 3.7. We define the anti-linear map $\dagger : {}^k A_n^l \rightarrow {}^m A_l^n$ by the formula

$$x^\dagger = S(x^*)$$

Lemma 3.8. *The map $x \mapsto x^\dagger$ is an anti-multiplicative anti-linear involution on A .*

Proof. It is clear that $x \mapsto x^\dagger$ is anti-linear. It is also immediate from the definition of the product that $(x \cdot y)^* = x^* \cdot y^*$. Together with the anti-multiplicativity of S , this proves the anti-multiplicativity of \dagger .

Let us proof that \dagger is an involution. It is sufficient to prove that $I_{\bar{a}} I_a = \lambda \text{id}$ and $J_a J_{\bar{a}} = \lambda^{-1} \text{id}$ for some scalar λ . But this follows from the fact that (\bar{R}_a, R_a) and $(R_{\bar{a}}, \bar{R}_{\bar{a}})$ are both solutions to the conjugate equations for \bar{a} . \square

The last property which needs to be proven is the positivity of φ . For this, recall that $R_a^* R_a$ and $\bar{R}_a^* \bar{R}_a$ are scalars as u_a is irreducible. One can then rescale R_a and \bar{R}_a such that the scalar in both expressions is the same. This scalar is then a uniquely determined number $\dim_q(a)$, called the *quantum dimension* of a . It follows that $\frac{1}{\dim_q(a)} F(R_a R_a^*)$ is the projection of $F(u_{\bar{a}} \otimes u_a)$ onto the copy of $F(u_o)$ inside, and a similar statement holds for \bar{R}_a .

Proposition 3.9. *For any $x \in A$, the scalar $\varphi(x^\dagger \cdot x)$ is positive.*

Proof. It is straightforward to see that the blocks ${}_m^k A_n^l$ are mutually orthogonal, and that moreover the spectral subspaces inside are mutually orthogonal. Let then $\xi, \zeta \in F(u_a)_{kl}$ and $\eta, \mu \in F(u_a)_{mn}$. We have, using the remark above,

$$\begin{aligned}
\varphi(y^\dagger \cdot x) &= \varphi\left(\sum_c \sum_i \left(F_{c,i}^{\bar{a},a}\right)^* (I_a y J_a \otimes x) \left(F_{c,i}^{\bar{a},a}\right)\right) \\
&= \delta_n^* \sum_i \left(F_{o,i}^{\bar{a},a}\right)^* (I_a y J_a \otimes x) \left(F_{o,i}^{\bar{a},a}\right) \delta_l \\
&= \frac{1}{\dim_q(u_a)} \delta_n^* \mathcal{R}_a^* (I_a y J_a \otimes x) \mathcal{R}_a \delta_l \\
&= \frac{1}{\dim_q(u_a)} \sum_{p,q} \delta_n^* \mathcal{R}_a^* (I_a y J_a e_{\bar{a},p} e_{\bar{a},p}^* \otimes x e_{a,q} e_{a,q}^*) \mathcal{R}_a \delta_l.
\end{aligned}$$

By the defining properties of I_a and J_a , this expression becomes $\dim_q(u_a)^{-1} \sum_p \langle e_{\bar{a},p}, J_a^* x^* y J_a e_{\bar{a},p} \rangle$, thus clearly φ will be positive on A . \square

Corepresentations of generalized compact Hopf face algebras

Let (A, Δ) be a generalized compact Hopf face algebra over an index set I . A *locally finite-dimensional unitary corepresentation* of (A, Δ) consists of a row and column-finite I^2 -graded Hilbert space $\mathcal{H} = \sum_{k,l \in I} {}^\oplus \mathcal{H}_{kl}$ together with elements ${}_m^k U_n^l \in {}_m^k A_n^l \otimes B({}^m \mathcal{H}^n, {}^k \mathcal{H}^l)$ such that

$$\sum_k \left({}_m^k U_n^l\right)^* {}_m^k U_n^l = \lambda_l \rho_n \otimes \text{id}_m \mathcal{H}^n$$

and

$$\sum_n {}_m^k U_n^l \left({}_m^k U_n^l\right)^* = \lambda_k \rho_m \otimes \text{id}_k \mathcal{H}^l,$$

and

$$(\tilde{\Delta} \otimes \text{id})({}_m^k U_n^l) = \sum_{p,q} \left({}_p^k U_q^l\right)_{13} ({}_m^p U_n^q)_{23}.$$

Note that in the first two identities, the sums are finite, while in the finite identity the possibly infinite sum is meaningful inside the multiplier algebra sense.

By a morphism between two locally finite-dimensional unitary corepresentations (\mathcal{H}, U) and (\mathcal{G}, V) is meant a grading-preserving bounded map $T = \sum_{k,l} {}^\oplus {}^k T^l : \mathcal{H} \rightarrow \mathcal{G}$ such that $(1 \otimes {}^k T^l) {}_m^k U_n^l = {}_m^k V_n^l (1 \otimes {}^m T^n)$. The collection of all locally finite-dimensional unitary

corepresentations clearly forms a semi-simple C*-category $\text{Corep}(A)$. We will say that (A, Δ) is *of finite type* if the morphisms in $\text{Corep}(A)$ are finite-dimensional.

One can define a tensor product \oplus between locally finite-dimensional corepresentations by means of the \otimes -product of bigraded Hilbert spaces and the operation

$${}_m^k(U \oplus V)_n^l = \left({}_m^k U_s^r \right)_{12} \left({}_s^r V_n^l \right)_{13}.$$

In this way, the category $\text{Corep}(A)$ becomes a monoidal category. The unit object consists of the I^2 -graded Hilbert space $l^2(I)$ together with the elements ${}_m^k U_n^l = \delta_{kl} \delta_{mn} \lambda_k \rho_m \otimes 1$.

Assume now that \mathcal{C} is a semi-simple tensor C*-category with irreducible unit, and $F : \mathcal{C} \rightarrow \text{Hilb}$ a strong tensor C*-functor. Let (A, Δ) be the associated generalized compact Hopf face algebra. Let us show that $\mathcal{C} \cong \text{Corep}(A)$ by means of an equivalence functor G .

For X an object of \mathcal{C} , we build a locally finite-dimensional unitary corepresentation U on $F(X)$. Consider the canonical isomorphism $F(X) \cong \oplus_{a \in J} X_a \otimes \text{Mor}(X_a, X)$. Let

$${}_m^k U_n^l(a) \in {}_m^k A_n^l(a) \otimes B(F(u_a)_{mn}, F(u_a)_{kl}) = B(F(u_a)_{kl}, F(u_a)_{mn}) \otimes B(F(u_a)_{mn}, F(u_a)_{kl})$$

be determined as the element implementing the non-degenerate pairing $B(F(u_a)_{kl}, F(u_a)_{mn}) \otimes B(F(u_a)_{nm}, F(u_a)_{lk}) \rightarrow \mathbb{C}$ sending $S \otimes T$ to $\text{Tr}(ST)$. Using notation as before, this means that

$${}_m^k U_n^l(a) = \sum_{p \in B_{a,mn}, q \in B_{a,kl}} e_p e_q^* \otimes e_q e_p^*.$$

Monoidal equivalence of generalized compact Hopf face algebras

Let (A, Δ) be a generalized Hopf face algebra over a set I . Assume that $I = I_1 \sqcup I_2$, and let $\Lambda_j = \sum_{i \in I_j} \lambda_i$, resp. $P_j = \sum_{i \in I_j} \rho_j$. If the Λ_j and P_j are central in $M(A)$, then we can write $A = \sum_{i,j}^{\oplus} A(ij)$ where $A(ij) = \Lambda_i P_j A$ are subalgebras. Moreover, the comultiplication $\tilde{\Delta}$ splits into comultiplications

$$\tilde{\Delta}_{ij}^k : A(ij) \rightarrow M(A(ik) \otimes A(kj)) \text{ s.t. } \tilde{\Delta} = \tilde{\Delta}_{ij}^1 + \tilde{\Delta}_{ij}^2 \text{ on } A(ij).$$

A similar decomposition holds for Δ .

It is immediate to see that the $(A(ii), \Delta_{ii}^i)$ are two generalized Hopf face algebras over the respective I_i .

Definition 3.10. We say (A, Δ) is a *co-linking generalized (compact) Hopf face algebra* between $(A(11), \Delta_{11}^1)$ and $(A(22), \Delta_{22}^2)$ if $\lambda_i P_2 \neq 0$ for any $i \in I_1$.

Upon applying the antipode, we see that then $\rho_j \Lambda_1 \neq 0$ for any $j \in I_2$ as well.

Definition 3.11. Two generalized (compact) Hopf face algebras are called *comonoidally Morita equivalent* if they are isomorphic to the components (A_{ii}, Δ_{ii}^i) of some co-linking generalized (compact) Hopf face algebra.

As an example, consider two sets I_i , and two tensor functors (F_i, ϕ_i) of a semi-simple rigid C^* -category \mathcal{C} with irreducible unit into $\text{Hilb}_{I_i^2}$. Then with $I = I_1 \sqcup I_2$, we can form a new C^* -functor $F = F_1 \oplus F_2$ of \mathcal{C} into Hilb_{I^2} by putting $F(X) = F_1(X) \oplus F_2(X)$ with the obvious I^2 -grading (and the obvious direct sum operation on morphisms). It becomes monoidal by means of the unitaries

$$F(X \otimes Y) = F_1(X \otimes Y) \oplus F_2(X \otimes Y) \underset{\phi_1 \oplus \phi_2}{\cong} (F_1(X) \otimes_{I_1} F_1(Y)) \oplus (F_2(X) \otimes_{I_2} F_2(Y)) \cong F(X) \otimes_I F(Y)$$

(where the last map is unitary since $(F(X) \otimes_I F(Y))_{ij} = 0$ for example for $i \in I_1$ and $j \in I_2$).

If we then consider the generalized compact Hopf face algebra (A, Δ) associated to F , we have immediately from the construction that the Λ_i and P_i associated to the decomposition $I = I_1 \sqcup I_2$ are indeed central elements in $M(A)$. Moreover, the parts $(A_{ii}^i, \Delta_{ii}^i)$ are seen to arise from applying the Tannaka-Krein construction to the respective functors F_1 and F_2 . The fact that (A, Δ) is co-linking is immediate from the fact that *none* of the $\lambda_i \rho_j$ are zero in this particular case (since ${}_m^k A_m^k(o) = B(F(u_o)_{kk}, F(u_o)_{mm}) \cong \mathbb{C}$).

We will exploit the above extra structure in the following section to say something about the algebra A appearing in ... This is the component $\tilde{A}(1, 1)$ of the above algebra. The following lemma will be needed.

Lemma 3.12. *Assume (A, Δ) is a co-linking generalized Hopf face algebra. Then any of the maps $\tilde{\Delta}_{ij}^k$ is injective.*

Proof. Take a non-zero $x \in A_n(ij)$ where $n \in I_j$. Then for any $l \in I$ with $\rho_n \lambda_l \neq 0$, we know that $\tilde{\Delta}(x)(1 \otimes \rho_n \lambda_l) \neq 0$. Hence $\tilde{\Delta}_{ij}^k(x)(1 \otimes \rho_n \lambda_l) \neq 0$ for $l \in I_k$, and hence $\tilde{\Delta}_{ij}^k(x) \neq 0$. Now if $j = k$, the condition $\rho_n \lambda_l \neq 0$ is satisfied by taking $l = n$ (since $\varepsilon(\lambda_n \rho_n) = 1$). If $j \neq k$, it is satisfied for at least one l by the co-linking assumption. \square

4 Compact Hopf face algebras on the level of operator algebras

Let \mathcal{G} be a partial compact quantum group. We now construct completions of the underlying $*$ -algebra $P(\mathcal{G})$ in the form of a universal C^* -algebra $C_0^u(\mathcal{G})$, a reduced C^* -algebra $C_0^r(\mathcal{G})$ and a von Neumann algebra $L^\infty(\mathcal{G})$. The existence of the first one follows from the Peter-Weyl theorem ??, and the second and third one arise from a GNS-representation of $P(\mathcal{G})$ on the Hilbert space $L^2(\mathcal{G})$ associated to the invariant functional

of \mathcal{G} . We then lift the comultiplication, the invariant functional, the unitary antipode and the scaling group to level of operator algebras and show that $L^\infty(\mathcal{G})$ becomes a measured quantum groupoid in the sense of Lesieur [] and Enock [].

Let us start with the construction of $C_0^u(\mathcal{G})$. Denote by A the underlying total $*$ -algebra of the partial $*$ -algebra $P(\mathcal{G})$ and define a map $|\cdot|_u : A \rightarrow [0, \infty]$ by

$$|a|_u := \sup\{\|\pi(a)\| : \pi \text{ is a } *\text{-homomorphism from } A \text{ into some } C^*\text{-algebra } B\}.$$

Lemma 4.1. $|a|_u < \infty$ for each $a \in A$.

Proof. By Corollary 2.31, we can write each $a \in A$ in the form $a = (\text{id} \otimes \omega_{\xi, \eta})(X(K))$, where X is a unitary sfd corepresentation of $P(\mathcal{G})$ on some sfd I^2 -graded Hilbert space \mathcal{H} and $K = \begin{pmatrix} k & l \\ m & n \end{pmatrix} \in M_2(I)$, $\xi \in {}^m\mathcal{H}^n$, $\eta \in {}^k\mathcal{H}^l$. Since X is unitary,

$$\sum_p X \begin{pmatrix} p & l \\ m & n \end{pmatrix}^* X \begin{pmatrix} p & l \\ m & n \end{pmatrix} = \lambda_l \rho_n \otimes \text{id}_m \mathcal{H}^n$$

by ??, where the sum is finite because X is sfd. As $\pi(\lambda_k \rho_m) \in \mathbb{C}$ is a projection, we can conclude

$$\begin{aligned} \|\pi(a)\| &\leq \|\xi\| \|\eta\| \|(\pi \otimes \text{id})(X(K))\| \\ &= \|\xi\| \|\eta\| \|(\pi \otimes \text{id})(X(K)^* X(K))\| \leq \|\xi\| \|\eta\| \|\pi(\lambda_l \rho_n) \otimes \text{id}_m \mathcal{H}^n\| = \|\xi\| \|\eta\|. \end{aligned}$$

□

Clearly, $|\cdot|_u$ defines a C^* -semi-norm on $P(\mathcal{G})$, and the separated completion of $P(\mathcal{G})$ with respect to this norm is a C^* -algebra. We denote this C^* -algebra by $C_0^u(\mathcal{G})$. By construction, every $*$ -homomorphism π of A into some C^* -algebra C factorises through $C_0^u(\mathcal{G})$. We shall see that the canonical $*$ -homomorphism $\pi_u : A \rightarrow C_0^u(\mathcal{G})$ is injective.

Remark 4.2. The universal property of $C_0^u(\mathcal{G})$ implies that the modular automorphism group σ and the scaling group τ for real parameters and the unitary antipode R of \mathcal{G} introduced after Corollary 2.33 lift to one-parameter groups τ^u, σ^u and a $*$ -anti-automorphism R_u of $C_0^u(\mathcal{G})$, that is, $\tau_t^u \circ \pi_u = \pi_u \circ \tau_t$, $\sigma_t^u \circ \pi_u = \pi_u \circ \sigma_t$, $R_u \circ \pi_u = \pi_u \circ R$. Corollary 2.33 5. and A.1 in [] [Takesaki:2] imply that elements in $\pi_u(A)$ are analytic for τ^u and σ^u ; in particular, τ^u and σ^u are strongly continuous.

We now turn to the construction of the reduced C^* -algebra $C_0^r(\mathcal{G})$ and the von Neumann algebra $L^\infty(\mathcal{G})$. Denote by $L^2(\mathcal{G})$ the completion of A with respect to the norm associated to the inner product given by

$$\langle a|b \rangle := \phi(a^*b) \quad \text{for all } a, b \in A,$$

and by $\Lambda : A \rightarrow L^2(\mathcal{G})$ the natural embedding. This product is definite because ϕ is faithful by 1.33, and it extends to the space $L^2(\mathcal{G})$ which thus becomes a Hilbert space. As such, $L^2(\mathcal{G})$ is the orthogonal direct sum of the subspaces $\Lambda(A(K)) \subseteq L^2(\mathcal{G})$, where

$K \in M_2(I)$, because $\phi(A(K)^*A(L)) = 0$ if $K \neq L$ by ???. In particular, there exist operators $\lambda_k, \lambda_k^{\text{op}}, \rho_k, \rho_k^{\text{op}} \in \mathcal{B}(L^2(\mathcal{G}))$ for each $k \in I$ such that

$$\lambda_k \Lambda(a) = \Lambda(\lambda_k a), \quad \lambda_k^{\text{op}} \Lambda(a) = \Lambda(a \lambda_k), \quad \rho_k \Lambda(a) = \Lambda(\rho_k a), \quad \rho_k^{\text{op}} \Lambda(a) = \Lambda(a \rho_k)$$

for all $a \in A$, and faithful, normal $*$ -homomorphisms

$$\lambda, \rho: l^\infty(I) \rightarrow \mathcal{B}(L^2(\mathcal{G})) \quad (4.1)$$

that send the delta function at $k \in I$ to the operators λ_k or ρ_k , respectively.

Define $\bar{E}, \bar{G} \in \mathcal{B}(L^2(\mathcal{G}) \otimes L^2(\mathcal{G}))$ by

$$\bar{E} := \sum_k \rho_k \otimes \lambda_k, \quad \bar{G} := \sum_k \rho_k^{\text{op}} \otimes \rho_k,$$

where the sums converge with respect to the strong operator topology.

Lemma 4.3. *There exists a unique partial isometry V on $L^2(\mathcal{G}) \otimes L^2(\mathcal{G})$ such that*

$$V(\Lambda(a) \otimes \Lambda(b)) = \Lambda(a_{(1)}) \otimes \Lambda(a_{(2)}b)$$

for all $a, b \in A$. Its range and domain projections are given by $VV^* = \bar{E}$ and $V^*V = \bar{G}$.

Proof. Let $a, b \in A$. Since Δ is a $*$ -homomorphism and ϕ is invariant,

$$\begin{aligned} \langle \Lambda(a_{(1)}) \otimes \Lambda(a_{(2)}b) | \Lambda(a'_{(1)}) \otimes \Lambda(a'_{(2)}b') \rangle &= \phi(a_{(1)}^* a'_{(1)}) \phi(b^* a_{(2)}^* a'_{(2)} b') \\ &= \sum_p \phi(b^* \rho_p \phi(\rho_p a^* a' \rho_p) \rho_p b') \\ &= \sum_p \langle \Lambda(a \rho_p) \otimes \Lambda(\rho_p b) | \Lambda(a' \rho_p) \otimes \Lambda(b' \rho_p) \rangle. \end{aligned}$$

Now, the assertion follows from Proposition 1.22. \square

Proposition 4.4. *Let \mathcal{G} be a partial compact quantum group with underlying total $*$ -algebra A and associated Hilbert space $L^2(\mathcal{G})$. Then there exists a unique $*$ -homomorphism $\pi_r: A \rightarrow \mathcal{B}(L^2(\mathcal{G}))$ such that $\pi_r(a)\Lambda(b) = \Lambda(ab)$ for all $a, b \in A$, and this π_r is faithful.*

Proof. Let $a, c \in A$. Then the formula $x \mapsto \langle \Lambda(c) | x \Lambda(a) \rangle$ defines a bounded linear functional $\omega_{\Lambda(c), \Lambda(a)}$ on $\mathcal{B}(L^2(\mathcal{G}))$ and a straightforward computation shows that

$$(\omega_{\Lambda(c), \Lambda(a)} \otimes \text{id})(V)\Lambda(b) = \Lambda(\varphi(c^* a_{(1)}) a_{(2)} b) \quad (4.2)$$

for all $b \in A$. Therefore, left multiplication by $\varphi(c^* a_{(1)}) a_{(2)}$ extends to a bounded linear operator on $L^2(\mathcal{G})$. Since $(A \otimes 1)\Delta(A) = (A \otimes A)\Delta(1)$ by Proposition 1.22 and ϕ is normalized, elements of the form $\phi(c^* a_{(1)}) a_{(2)}$ span A . \square

Corollary 4.5. *Let \mathcal{G} be a partial compact quantum group with underlying total algebra A . Then the canonical $*$ -homomorphism $\pi_u: A \rightarrow C_0^u(\mathcal{G})$ is injective.*

Proof. The injective $*$ -homomorphism π_r factorises through π_u . \square

Given a partial compact quantum group \mathcal{G} , we call $(L^2(\mathcal{G}), \Lambda, \pi)$ the *associated GNS-construction* and denote by

$$C_0^r(\mathcal{G}) \subseteq \mathcal{B}(L^2(\mathcal{G})) \quad \text{and} \quad L^\infty(\mathcal{G}) \subseteq \mathcal{B}(L^2(\mathcal{G})) \quad (4.3)$$

the C^* -algebra and the von Neumann algebra generated by $\pi_r(A) \subseteq L^2(\mathcal{G})$, respectively, and identify $M(C_0^r(\mathcal{G}))$ with a C^* -subalgebra of $L^2(\mathcal{G})$. Since π_r extends to a $*$ -homomorphism on $C_0^u(\mathcal{G})$, we get a sequence of $*$ -homomorphisms

$$A \hookrightarrow C_0^u(\mathcal{G}) \rightarrow C_0^r(\mathcal{G}) \hookrightarrow M(C_0^r(\mathcal{G})) \hookrightarrow L^\infty(\mathcal{G}) \hookrightarrow \mathcal{B}(L^2(\mathcal{G})).$$

Note that the $*$ -homomorphisms λ, ρ in (4.1) send $l^\infty(I)$ to $M(C_0^r(\mathcal{G}))$, and that

$$\overline{E} \in M(C_0^r(\mathcal{G}) \otimes C_0^r(\mathcal{G})) \subseteq L^\infty(\mathcal{G}) \otimes L^\infty(\mathcal{G}) \subseteq \mathcal{B}(L^2(\mathcal{G}) \otimes L^2(\mathcal{G})),$$

where \otimes denotes the minimal tensor product of C^* -algebras, the tensor product of von Neumann algebras, and the tensor product of Hilbert spaces, respectively.

Consider the map

$$\overline{\Delta}: L^\infty(\mathcal{G}) \rightarrow \mathcal{B}(L^2(\mathcal{G}) \otimes L^2(\mathcal{G})), \quad x \mapsto V(x \otimes 1)V^*.$$

Lemma 4.6. 1. $\overline{\Delta}(\pi_r(a))(\Lambda(b) \otimes \Lambda(c)) = \Lambda(a_{(1)}b) \otimes \Lambda(a_{(2)}b)$ for all $a, b, c \in A$;

2. $\overline{\Delta}$ is a normal, faithful $*$ -homomorphism;

3. $\overline{\Delta}(C_0^r(\mathcal{G})) \subseteq \overline{E}M(C_0^r(\mathcal{G}) \otimes C_0^r(\mathcal{G}))\overline{E}$ and $\overline{\Delta}(L^\infty(\mathcal{G})) \subseteq \overline{E}(L^\infty(\mathcal{G}) \otimes L^\infty(\mathcal{G}))\overline{E}$.

Proof. The equation in 1. is easily verified. The map $\overline{\Delta}$ is normal by construction, a $*$ -homomorphism by 1., and faithful because $\overline{\Delta}(x) = 0$ implies $x \otimes 1 = 0$ on $V^*V(L^2(\mathcal{G}) \otimes L^2(\mathcal{G}))$ and hence $x = 0$ on $\bigoplus_k \rho_k^{\text{op}} L^2(\mathcal{G}) = L^2(\mathcal{G})$. Finally, 3. follows from the relation $\Delta(a) = E\Delta(a)E$, which holds for all $a \in A$. \square

Next, we lift the invariant functional ϕ of \mathcal{G} to $L^\infty(\mathcal{G})$ and define associated operator-valued weight T_λ, T_ρ from $L^\infty(\mathcal{G})$ to $l^\infty(I)$. Since ϕ is normalized, each $\Lambda(\lambda_k, \rho_m)$ is a unit vector and the associated vector functional

$$\overline{\phi}^{(k)}_m: L^\infty(\mathcal{G}) \rightarrow \mathbb{C}, \quad x \mapsto \langle \Lambda(\lambda_k \rho_m) | x \Lambda(\lambda_k \rho_m) \rangle$$

is a state. Then the formulas

$$\overline{\phi}(x) := \sum_{k,m} \overline{\phi}^{(k)}_m(x), \quad T_\lambda(x) := \sum_{k,m} \overline{\phi}^{(k)}_m(x) \lambda_k, \quad T_\rho(x) := \sum_{k,m} \overline{\phi}^{(k)}_m(x) \rho_m, \quad (4.4)$$

where $x \in L^\infty(\mathcal{G})_+$, define a normal semi-finite weight $\overline{\phi}$ on $L^\infty(\mathcal{G})$ and normal semi-finite conditional expectations T_λ and T_ρ from $L^\infty(\mathcal{G})$ to $\lambda(l^\infty(I))$ and $\rho(l^\infty(I))$, respectively. These maps are determined by their restrictions to $\pi_r(A)$:

Lemma 4.7. *The normal weight $\bar{\phi}$ and the normal conditional expectations T_λ, T_ρ satisfy $\pi_r(A) \subseteq \mathfrak{M}_{\bar{\phi}} \cap \mathfrak{M}_T \cap \mathfrak{M}_{T'}$ and*

$$\bar{\phi}(\pi_r(a)) = \phi(a), \quad T_\lambda(\pi_r(a))\Lambda(b) = \sum_k \Lambda(\phi(\lambda_k a) \lambda_k b), \quad T_\rho(\pi_r(b))\Lambda(b) = \sum_m \Lambda(\phi(\rho_m a) \rho_m b)$$

for all $a, b \in A$, and are uniquely determined by these equations.

Proof. The equations follow immediately from the definition and the relation $\phi(a) = \sum_{k,m} \phi(\lambda_k \rho_m a \lambda_k \rho_m)$, see ???. To prove uniqueness, observe that the $p_{k,m} := \pi_r(\lambda_k \rho_m)$ are pairwise orthogonal projections in $\mathfrak{M}_{\bar{\phi}} \cap \mathfrak{M}_{T_\lambda} \cap \mathfrak{M}_{T_\rho}$ summing up to 1, whence $\bar{\phi}, T_\lambda$ and T_ρ are the sums of the bounded linear maps that send an $x \in L^\infty(\mathcal{G})_+$ to $\bar{\phi}(p_{k,m} x p_{l,n})$, $T_\lambda(p_{k,m} x p_{l,n})$, or $T_\rho(p_{k,m} x p_{l,n})$, respectively, which are determined by their restriction to $\pi_r(A)$. \square

Invariance of ϕ implies invariance of $\bar{\phi}$ as follows.

Proposition 4.8. *Let \mathcal{G} be a partial compact quantum group. Then for all $x \in L^\infty(\mathcal{G})_+$, the normal, semi-finite weight $\bar{\phi}$ on $L^\infty(\mathcal{G})$ satisfies*

$$(\text{id} \otimes \bar{\phi})(\bar{\Delta}(x)) = T_\lambda(x), \quad (\bar{\phi} \otimes \text{id})(\bar{\Delta}(x)) = T_\rho(x).$$

Proof. Let $a \in A$. Then (1.14) and the relation $\bar{\phi}\left(\begin{smallmatrix} k \\ m \end{smallmatrix}\right) \circ \pi = \phi\left(\begin{smallmatrix} k \\ m \end{smallmatrix}\right)$ imply

$$(\text{id} \otimes \bar{\phi}\left(\begin{smallmatrix} l \\ m \end{smallmatrix}\right))(\bar{\Delta}(\pi_r(a))) = \sum_k \bar{\phi}\left(\begin{smallmatrix} k \\ m \end{smallmatrix}\right)(\pi_r(a)) \lambda_k \rho_l.$$

Since each $\bar{\phi}\left(\begin{smallmatrix} k \\ m \end{smallmatrix}\right)$ is a vector state and $\pi_r(A)$ is weakly dense in $L^\infty(\mathcal{G})$, this equations remains true if we replace $\pi_r(a)$ by arbitrary $x \in L^\infty(\mathcal{G})$. Summing over l and m , we obtain the first equation which we have to prove. The second one follows similarly. \square

The next result gives a fairly complete description of the objects of Tomita-Takesaki theory associated to $\bar{\phi}$.

Lemma 4.9. *The subspace $\Lambda(A) \subseteq L^2(\mathcal{G})$ is a Hilbert algebra with respect to the operations $\Lambda(a)\Lambda(b) = \Lambda(ab)$ and $\Lambda(a)^* = \Lambda(a^*)$ for all $a, b \in A$, and a Tomita algebra with respect to the family of operators ∇_z given by $\nabla_z \Lambda(a) = \Lambda(\sigma_z(a))$ for all $a \in A, z \in \mathbb{C}$. The associated left von Neumann algebra is $L^\infty(\mathcal{G})$, the associated normal, semifinite, faithful weight is $\bar{\phi}$, the modular operator $\Delta_{\bar{\phi}}$ is the closure of ∇_{-i} , the modular conjugation $J_{\bar{\phi}}$ is given by $J_{\bar{\phi}} \Lambda(a) = \Lambda(\sigma_{i/2}(a)^*)$ for all $a \in A$, and the modular automorphism group $\sigma^{\bar{\phi}}$ satisfies $\sigma_t^{\bar{\phi}} \circ \pi_r = \pi_r \circ \sigma_t$ for all $t \in \mathbb{R}$.*

Proof. We first show that $\Lambda(A)$ is a Hilbert algebra. Indeed, the map $\pi_r(a): \Lambda(b) \rightarrow \Lambda(ab)$ is bounded for each $a \in A$ by Proposition 4.4, and the involution is pre-closed because for all $a, b \in A$,

$$\langle \Lambda(a) | \Lambda(b^*) \rangle = \phi(a^* b^*) = \phi(b^* \sigma(a^*)) = \langle \Lambda(b) | \Lambda(\sigma(a^*)) \rangle$$

To see that $\Lambda(A)$ and $(\nabla_z)_z$ form a Tomita algebra, we have to verify that the map $z \mapsto \langle \Lambda(a) | \nabla_z \Lambda(b) \rangle = \phi(a^* \sigma_z(b))$ is entire for all $a, b \in A$ and that

$$\nabla_z \Lambda(a)^* = \nabla_{\bar{z}} \Lambda(a)^*, \quad \langle \Lambda(a) | \Lambda(b) \rangle = \langle \nabla_{-\bar{z}} \Lambda(a) | \Lambda(b) \rangle, \quad \langle \Lambda(a)^* | \Lambda(b)^* \rangle = \langle \Lambda(b) | \nabla_{-i} \Lambda(a) \rangle$$

for all $a, b \in A$, $z \in \mathbb{C}$. But all of this follows immediately from Corollary 2.33.

The left von Neumann algebra of $\Lambda(A)$ is $\pi_r(A)'' = L^\infty(\mathcal{G})$ and the associated weight $\tilde{\phi}$ satisfies $\tilde{\phi}(\pi_r(a^*a)) = \langle \Lambda(a) | \Lambda(a) \rangle = \phi(a^*a)$ for all $a \in A$. By Lemma 4.7, it coincides with $\bar{\phi}$. By [Takesaki:2, Thm. VI.2.2 and its proof], the modular operator $\Delta_{\bar{\phi}}$ is the closure of ∇_{-i} and the modular automorphism group is implemented by $(\nabla_t)_t$. \square

The general theory of Hilbert algebras [] implies now:

Proposition 4.10. *Let \mathcal{G} be a partial compact quantum group. Then the extension $\bar{\phi}$ of the invariant functional to $L^\infty(\mathcal{G})$ is faithful.* \square

Remark 4.11. Without using the theory of Hilbert algebras, one could also check directly that the formula for $J_{\bar{\phi}}$ defines an anti-linear isometry, that $J_{\bar{\phi}}\pi_r(A)J_{\bar{\phi}}$ commutes with $\pi_r(A)$ and hence with $L^\infty(\mathcal{G})$, and that the family $(\Lambda(\lambda_k \rho_m))_{k,m}$ is cyclic for $J\pi_r(A)J$. Then this family is separating for $L^\infty(\mathcal{G})$ and $\bar{\phi}$ is faithful.

The scaling group τ and the unitary antipode R of \mathcal{G} can easily be lifted to $C_0^r(\mathcal{G})$ and $L^\infty(\mathcal{G})$ using the following result. Let us call a conjugate-linear map on a Hilbert space an *anti-symmetry* if it is isometric and its square is the identity.

Lemma 4.12. *There exist a unique anti-symmetry I and a strongly continuous one-parameter group $P = (P_t)_t$ on $L^2(\mathcal{G})$ such that for all $a \in A$, $t \in \mathbb{R}$,*

$$I\Lambda(a) = \Lambda(R(a)^*), \quad P_t\Lambda(a) = \Lambda(\tau_t(a)).$$

Proof. Corollary 2.33 implies that the formulas above define an anti-symmetry I and unitaries P_t ; for example, $\|I\Lambda(a)\|^2 = \phi(R(a)R(a)^*) = \phi(a^*a) = \|\Lambda(a)\|^2$, and $I^2 = \text{id}$ because $* \circ R \circ * \circ R = R^2 = \text{id}$. By A.1 in [Takesaki:2] and Corollary 2.33 5., elements of $\Lambda(A)$ are analytic with respect to P ; in particular, P is strongly continuous. \square

Proposition 4.13. *Let \mathcal{G} be an I -partial compact quantum group.*

1. *There exists a unique $*$ -anti-automorphism \bar{R} of $L^\infty(\mathcal{G})$ such that $\bar{R} \circ \pi_r = \pi_r \circ R$. This \bar{R} restricts to a $*$ -anti-automorphism of $C_0^r(\mathcal{G})$.*
2. *There exists a unique strongly continuous one-parameter group $\bar{\tau}$ on $L^\infty(\mathcal{G})$ such that $\bar{\tau}_t \circ \pi_r = \pi_r \circ \theta_{-it, it}$ for all $t \in \mathbb{R}$, and this $\bar{\tau}$ restricts to a strongly continuous one-parameter group on $C_0^r(\mathcal{G})$.*

Proof. Short calculations show that the maps $\bar{R}: x \mapsto Ix^*I$ and $\bar{\tau}_t: x \mapsto P_t x P_t^*$ have the desired properties. \square

Note that the relations (2.11) and (2.12) can be lifted to $C_0^r(\mathcal{G})$ and $L^\infty(\mathcal{G})$ by continuity. The next result will allow us to relate \overline{R} to the unitary antipode of the measured quantum groupoid that we are going to construct.

Lemma 4.14. *For all $a, b \in A$,*

$$\overline{R}(\text{id} \otimes \omega_{J\Lambda(b), J\Lambda(b)})(\overline{\Delta}(\pi(a^*a))) = (\text{id} \otimes \omega_{J\Lambda(a), J\Lambda(a)})(\overline{\Delta}(\pi(b^*b))).$$

Proof. Let $c = a^*a$ and $d = b^*b$. A short calculation using (2.10) shows that the right hand side is equal to

$$d_{(1)}\phi(\sigma_{i/2}(a)d_{(2)}\sigma_{i/2}(a)^*) = d_{(1)}\phi(\sigma_{i/2}(c)d_{(2)}).$$

By Lemma 1.34 and (2.11), (2.10), this equals $S(\tau_{i/2}(c_{(1)}))\phi(\sigma_{i/2}(c_{(2)})d)$ which is the left hand side. \square

The operator-algebraic structures constructed so far fit into the theory of measured quantum groupoids as follows.

Denote by ν the normal, faithful, semifinite weight on $l^\infty(I)$ given by

$$\nu(f) = \sum_k f(k) \quad \text{for all } f \in l^\infty(I)_+. \quad (4.5)$$

Then the relative tensor product of $L^2(\mathcal{G})$ with itself, relative to the representations ρ, λ of $l^\infty(I)$ and the weight ν , takes the simple form

$$L^\infty(\mathcal{G})_{\rho \otimes_\lambda \nu} L^\infty(\mathcal{G}) \cong \bigoplus_k (\rho_k L^2(\mathcal{G}) \otimes \lambda_k L^2(\mathcal{G})) = \overline{E}(L^2(\mathcal{G}) \otimes L^2(\mathcal{G})),$$

see [], the relative tensor product of operators $S \in \rho(l^\infty(I))'$ and $T \in \lambda(l^\infty(I))'$ gets identified with the compression

$$S_{\rho \otimes_\lambda \nu} T \equiv \overline{E}(S \otimes T) = (S \otimes T)\overline{E} \subseteq \mathcal{B}(\overline{E}(L^2(\mathcal{G}) \otimes L^2(\mathcal{G}))),$$

and the fiber product of $L^\infty(\mathcal{G})$ with itself, relative to ρ and λ , gets identified with

$$\begin{aligned} L^\infty(\mathcal{G})_{\rho *_{\lambda} l^\infty(I)} L^\infty(\mathcal{G}) &= (L^\infty(\mathcal{G})'_{\rho \otimes_\lambda \nu} L^\infty(\mathcal{G})')' \\ &\equiv (\overline{E}(L^\infty(\mathcal{G})' \otimes L^\infty(\mathcal{G})'))' = \overline{E}(L^\infty(\mathcal{G}) \otimes L^\infty(\mathcal{G}))\overline{E}. \end{aligned} \quad (4.6)$$

By Lemma 4.6 3., we can co-restrict $\overline{\Delta}$ to a normal, faithful $*$ -homomorphism

$$\tilde{\Delta}: L^\infty(\mathcal{G}) \rightarrow L^\infty(\mathcal{G})_{\rho *_{\lambda} l^\infty(I)} L^\infty(\mathcal{G}).$$

We now obtain a Hopf-von Neumann bimodule in the sense of [].

Proposition 4.15. *Let \mathcal{G} be an I -partial compact quantum group. Then*

1. $\tilde{\Delta}(\lambda(x)) = \lambda(x) \underset{\nu}{\rho} \otimes_{\lambda} 1$ and $\tilde{\Delta}(\rho(x)) = 1 \underset{\nu}{\rho} \otimes_{\lambda} \rho(x)$ for all $x \in l^{\infty}(I)$, and
2. $(\tilde{\Delta} * \text{id}) \circ \tilde{\Delta} = (\text{id} * \tilde{\Delta}) \circ \tilde{\Delta}$.

In particular, $(l^{\infty}(I), L^{\infty}(\mathcal{G}), \lambda, \rho, \tilde{\Delta})$ is a Hopf-von Neumann bimodule.

Proof. Assertion 1. follows from (1.2) and Lemma 4.6 1. and ensures that the $*$ -homomorphisms

$$\tilde{\Delta} * \text{id}, \text{id} * \tilde{\Delta}: L^{\infty}(\mathcal{G}) \underset{l^{\infty}(I)}{\underset{\rho}{\rho} * \lambda} L^{\infty}(\mathcal{G}) \rightarrow L^{\infty}(\mathcal{G}) \underset{l^{\infty}(I)}{\underset{\rho}{\rho} * \lambda} L^{\infty}(\mathcal{G}) \underset{l^{\infty}(I)}{\underset{\rho}{\rho} * \lambda} L^{\infty}(\mathcal{G})$$

are well-defined. As in (4.6), we can identify

$$L^{\infty}(\mathcal{G}) \underset{l^{\infty}(I)}{\underset{\rho}{\rho} * \lambda} L^{\infty}(\mathcal{G}) \underset{l^{\infty}(I)}{\underset{\rho}{\rho} * \lambda} L^{\infty}(\mathcal{G}) \cong \overline{E}^{(2)}(L^{\infty}(\mathcal{G}) \otimes L^{\infty}(\mathcal{G}) \otimes L^{\infty}(\mathcal{G})) \overline{E}^{(2)},$$

where $\overline{E}^{(2)} = (\overline{E} \otimes 1)(1 \otimes \overline{E})$, and then the $*$ -homomorphisms become restrictions of the maps $\tilde{\Delta} \otimes \text{id}$ and $\text{id} \otimes \tilde{\Delta}$, respectively. Now, 2. follows from Lemma 4.6 1. and co-associativity of Δ . \square

This Hopf-von Neumann bimodule is a measured quantum groupoid in the sense of [1].

Theorem 4.16. *Let \mathcal{G} be an I -partial compact quantum group. Then the Hopf-von Neumann bimodule $(l^{\infty}(I), L^{\infty}(\mathcal{G}), \lambda, \rho, \tilde{\Delta})$ and the weights $(T_{\lambda}, T_{\rho}$ and ν defined in (4.4) and (4.5) form a measured quantum groupoid. It is unimodular and its unitary antipode and scaling group coincide with \bar{R} and $\bar{\tau}$, respectively.*

Proof. First, observe that the the modular automorphism groups of the weights $\nu \circ \lambda^{-1} \circ T_{\lambda}$ and $\nu \circ \rho^{-1} \circ T_{\rho}$ commute because the two compositions coincide with $\bar{\phi}$. Next, we need to show that T_{λ} is left-invariant in the sense that

$$(\text{id} \underset{\nu}{\rho} * \lambda \bar{\phi})(\tilde{\Delta}(x)) = T_{\lambda}(x)$$

for all $x \in L^{\infty}(\mathcal{G})_{+}$. But it is easy to see that the left hand side coincides with $(\text{id} * \bar{\phi})(\tilde{\Delta}(x))$ so that the equation above follows from Proposition 4.8. Likewise T_{ρ} is right-invariant in the appropriate sense. We thus obtain a measured quantum groupoid as claimed. Denote by \tilde{R} its unitary antipode and by $\tilde{\tau}$ its scaling group.

Let us prove that $\tilde{\tau}_t = \bar{\tau}$ for all $t \in \mathbb{R}$. By [1] and (2.11),

$$(\tilde{\tau}_t \underset{l^{\infty}(I)}{\underset{\rho}{\rho} * \lambda} \sigma_t^{\bar{\tau}}) \circ \tilde{\Delta} = \tilde{\Delta} \circ \sigma_t^{\bar{\tau}}, \quad (\bar{\tau}_t \otimes \sigma_t^{\bar{\tau}}) \circ \bar{\Delta} = \bar{\Delta} \circ \bar{\tau}_t.$$

The second equation implies that the first one remains true if we replace $\tilde{\tau}_t$ by $\bar{\tau}_t$. Using Theorem A.7 in [1] [enock:action], we can conclude that $\tilde{\tau}_t = \bar{\tau}_t$.

To prove that $\tilde{R} = \overline{R}$, we use the relations

$$\tilde{R}(\text{id}_{\rho^* \lambda \omega_{J\Lambda(b), J\Lambda(b)}})(\overline{\Delta}(\pi(a^* a))) = (\text{id}_{\rho^* \lambda \omega_{J\Lambda(a), J\Lambda(a)}})(\overline{\Delta}(\pi(b^* b)))$$

from [] and Lemma 4.14. □

5 Partial compact quantum groups from reciprocal random walks

We recall some notions introduced in [4]. We slightly change the terminology for the sake of convenience.

Definition 5.1. Let $t \in \mathbb{R}_0$. A t -reciprocal random walk consists of a quadruple $(\Gamma, w, \text{sgn}, i)$ with

- $\Gamma = (\Gamma^{(0)}, \Gamma^{(1)}, s, t)$ a graph with *source* and *target* maps

$$s, t : \Gamma^{(1)} \rightarrow \Gamma^{(0)},$$

- w a function (the *weight* function) $w : \Gamma^{(1)} \rightarrow \mathbb{R}_0^+$,
- sgn a function (the *sign* function) $\text{sgn} : \Gamma^{(1)} \rightarrow \{\pm 1\}$,
- i an involution

$$i : \Gamma^{(1)} \rightarrow \Gamma^{(1)}, \quad e \mapsto \bar{e}$$

with $s(\bar{e}) = t(e)$ for all edges e ,

such that the following conditions are satisfied:

- $w(e)w(\bar{e}) = 1$ for all edges e ,
- $\text{sgn}(e)\text{sgn}(\bar{e}) = \text{sgn}(t)$ for all edges e ,
- $p(e) = \frac{1}{|t|}w(e)$ defines a *random walk*: $\sum_{s(e)=v} p(e) = 1$ for all $v \in \Gamma^{(0)}$.

Note that, by [4, Proposition 3.1], there is a uniform bound on the number of edges leaving from any given vertex v .

For examples of t -reciprocal random walks, we refer to [4]. One particular example which will be needed for our construction of dynamical quantum $SU(2)$ is the following.

Example 5.2. Take $0 < |q| < 1$ and $x \in \mathbb{R}$. Write $2_q = q + q^{-1}$. Then we have the reciprocal -2_q -random walk

$$\Gamma_x = (\Gamma_x, w, \text{sgn}, i)$$

with

$$\Gamma^{(0)} = \mathbb{Z}, \quad \Gamma^{(1)} = \{(k, l) \mid |k - l| = 1\} \subseteq \mathbb{Z} \times \mathbb{Z}$$

with projection on the first (resp. second) leg as source (resp. target) map, with weight function

$$w(k, k \pm 1) = \frac{|q|^{x+k\pm 1} + |q|^{-(x+k\pm 1)}}{|q|^{x+k} + |q|^{-(x+k)}},$$

sign function

$$\text{sgn}(k, k+1) = 1, \quad \text{sgn}(k, k-1) = -\text{sgn}(q),$$

and involution $\overline{(k, k+1)} = (k+1, k)$.

By translation we shift the value of x by an integer, and by inversion we change x into $-x$ and multiply the sign function with a fixed sign. It follows that by some graph isomorphism, we can always arrange to have $x \in [0, \frac{1}{2}]$ at the cost of having a different sign function.

Let now $0 < |q| \leq 1$, and let $SU_q(2)$ be Woronowicz's twisted $SU(2)$ group [15]. Then $SU_q(2)$ is a compact quantum group, and its category of finite-dimensional unitary representations $\text{Rep}(SU_q(2))$ is generated by the spin 1/2-representation $\pi_{1/2}$ on \mathbb{C}^2 .

Let $\Gamma = (\Gamma, w, \text{sgn}, i)$ be a -2_q -reciprocal random walk. Define \mathcal{H}_Γ as the $\Gamma^{(0)}$ -bigraded Hilbert space $l^2(\Gamma^{(1)})$, where the $\Gamma^{(0)}$ -bigrading is given by

$$\delta_e \in {}^{s(e)}\mathcal{H}_\Gamma {}^{t(e)}$$

for the obvious Dirac functions. Further define R_Γ as the (bounded) map

$$R_\Gamma : l^2(\Gamma^{(0)}) \rightarrow \mathcal{H}_\Gamma \underset{\Gamma^{(0)}}{\otimes} \mathcal{H}_\Gamma$$

given by

$$R_\Gamma \delta_v = \sum_{e, s(e)=v} \text{sgn}(e) \sqrt{w(e)} \delta_e \otimes \delta_{\bar{e}}.$$

Then $R_\Gamma^* R_\Gamma = |q| + |q|^{-1}$ and

$$(R_\Gamma^* \otimes \text{id})(\text{id} \otimes R_\Gamma) = -\text{sgn}(q) \text{id}.$$

Hence, by the universal property of $\text{Rep}(SU_q(2))$ ([4, Theorem 1.4], based on [10, 5, 16, 8, 9]), we have a strongly monoidal $*$ -functor

$$F_\Gamma : \text{Rep}(SU_q(2)) \rightarrow {}^{\Gamma^{(0)}}\text{Hilb}_f^{\Gamma^{(0)}}$$

such that $F_\Gamma(\pi_{1/2}) = \mathcal{H}_\Gamma$ and $F_\Gamma(\mathcal{R}) = R_\Gamma$, with

$$(\pi_{1/2}, \mathcal{R}, -\text{sgn}(q)\mathcal{R})$$

a solution for the conjugate equations for $\pi_{1/2}$. Up to equivalence, F_Γ only depends upon the isomorphism class of (Γ, w) , and is independent of the chosen involution or sign structure.

It follows from our main theorem that for each reciprocal random walk on a graph Γ , one obtains a $\Gamma^{(0)}$ -partial compact quantum group. Our aim is to give a direct representation of it by generators and relations. We will write $\Gamma_{vw} \subseteq \Gamma^{(1)}$ for the set of edges with source v and target w .

Theorem 5.3. *Let $0 < |q| \leq 1$, and let Γ be a -2_q -reciprocal random walk. Let $A(\Gamma)$ be the total $*$ -algebra associated to the $\Gamma^{(0)}$ -partial compact quantum group constructed from the fiber functor F_Γ . Then $A(\Gamma)$ is the universal $*$ -algebra generated by a copy of the $*$ -algebra of finite support functions on $\Gamma^{(0)} \times \Gamma^{(0)}$ (with the Dirac functions written as $\mathbf{1}_{\binom{v}{w}}$) and elements $(u_{e,f})_{e,f \in \Gamma^{(1)}}$ where $u_{e,f} \in {}^{s(e)}_{s(f)}A(\Gamma)_{t(f)}^{t(e)}$ and*

$$\sum_{v \in \Gamma^{(0)}} \sum_{g \in \Gamma_{vw}} u_{g,e}^* u_{g,f} = \delta_{e,f} \mathbf{1}_{\binom{w}{t(e)}}, \quad \forall w \in \Gamma^{(0)}, e, f \in \Gamma^{(1)}, \quad (5.1)$$

$$\sum_{w \in \Gamma^{(0)}} \sum_{g \in \Gamma_{vw}} u_{e,g} u_{f,g}^* = \delta_{e,f} \mathbf{1}_{\binom{s(e)}{v}} \quad \forall v \in \Gamma^{(0)}, e, f \in \Gamma^{(1)}, \quad (5.2)$$

$$u_{e,f}^* = \text{sgn}(e) \text{sgn}(f) \sqrt{\frac{w(f)}{w(e)}} u_{\bar{e}, \bar{f}}, \quad \forall e, f \in \Gamma^{(1)}. \quad (5.3)$$

If moreover $v, w \in \Gamma^{(0)}$ and $e, f \in \Gamma^{(1)}$, we have

$$\Delta_{vw}(u_{e,f}) = \sum_{\substack{s(g)=v \\ t(g)=w}} u_{e,g} \otimes u_{g,f},$$

$$\varepsilon(u_{e,f}) = \delta_{e,f}$$

and

$$S(u_{e,f}) = u_{f,e}^*.$$

Proof. Let the $(v_{e,f})_{e,f \in \Gamma^{(1)}}$ be the matrix coefficients of the generating corepresentation of $A(\Gamma)$ on $F_\Gamma(\pi_{1/2}) = l^2(\Gamma^{(1)})$. By construction $V = (v_{e,f})_{e,f \in \Gamma^{(1)}}$ defines a unitary corepresentation of $A(\Gamma)$, hence the relations (5.1) and (5.2) are satisfied for the $v_{e,f}$. Now as R_Γ is an intertwiner between $\mathbb{C}_{\Gamma^{(0)}}$ and $V \oplus_{\Gamma^{(0)}} V$, we have for all $v \in \Gamma^{(0)}$ that

$$\sum_{\substack{e,f,g,h \in \Gamma^{(1)} \\ t(f)=s(g), t(e)=s(f)}} v_{e,f} v_{g,h} \otimes ((e_{e,f} \otimes e_{g,h}) \circ R_\Gamma \delta_v) = \sum_w \mathbf{1}_{\binom{w}{v}} \otimes R_\Gamma \delta_v, \quad (5.4)$$

hence

$$\sum_{\substack{e,g,k \\ t(e)=s(g), s(k)=v}} \text{sgn}(k) \sqrt{w(k)} (v_{e,k} v_{g,\bar{k}} \otimes \delta_e \otimes \delta_g) = \sum_{k, s(k)=v, w} \text{sgn}(k) \sqrt{w(k)} (\mathbf{1}_{\binom{w}{v}} \otimes \delta_k \otimes \delta_{\bar{k}}).$$

Hence if $t(e) = s(g) = z$, we have

$$\sum_{k, s(k)=v} \text{sgn}(k) \sqrt{w(k)} v_{e,k} v_{g,\bar{k}} = \sum_w \delta_{s(e),v} \delta_{e,\bar{g}} \text{sgn}(e) \sqrt{w(e)} \mathbf{1}_v^{(w)}.$$

Multiplying to the left with $v_{e,l}^*$ and summing over all e with $t(e) = z$, we see from (5.1) that also relation (5.3) is satisfied. Hence the $v_{e,f}$ satisfy the universal relations in the statement of the theorem. The formulas for comultiplication, counit and antipode then follow immediately from the fact that V is a unitary corepresentation.

Let us now a priori denote by $B(\Gamma)$ the $*$ -algebra determined by the relations (5.1), (5.2) and (5.3) above, and write $\mathcal{B}(\Gamma)$ for the associated $\Gamma^{(0)} \times \Gamma^{(0)}$ -partial $*$ -algebra. Write

$$\Delta(u_{e,f}) = \sum_{g \in \Gamma^{(1)}} u_{e,g} \otimes u_{g,f},$$

which makes sense in $M(B(\Gamma) \otimes B(\Gamma))$ as the number of edges leaving or arriving at a fixed vertex is uniformly bounded. Then we compute

$$\begin{aligned} \sum_{v \in \Gamma^{(0)}} \sum_{g \in \Gamma_{vw}^{(1)}} \Delta(u_{g,e})^* \Delta(u_{g,f}) &= \sum_{v \in \Gamma^{(0)}} \sum_{g \in \Gamma_{vw}^{(1)}} \sum_{h,k \in \Gamma^{(1)}} u_{g,h}^* u_{g,k} \otimes u_{h,e}^* u_{k,f} \\ &= \sum_{h,k \in \Gamma^{(1)}} \delta_{h,k} \mathbf{1}_{t(h)}^{(w)} \otimes u_{h,e}^* u_{k,f} \\ &= \sum_{z \in \Gamma^{(0)}} \sum_{\substack{h \in \Gamma^{(1)} \\ t(h)=z}} \mathbf{1}_z^{(w)} \otimes u_{h,e}^* u_{h,f} \\ &= \delta_{e,f} \sum_{z \in \Gamma^{(0)}} \mathbf{1}_z^{(w)} \otimes \mathbf{1}_{t(e)}^{(z)} \\ &= \delta_{e,f} \Delta(1). \end{aligned}$$

Similarly, the analogue of (5.2) holds for $\Delta(u_{e,f})$. As also (5.3) holds for $\Delta(u_{e,f})$ by its very form, it follows that we can define a $*$ -algebra homomorphism

$$\Delta : B(\Gamma) \rightarrow M(B(\Gamma) \otimes B(\Gamma))$$

sending $u_{e,f}$ to $\Delta(u_{e,f})$ and $\mathbf{1}_w^{(v)}$ to $\sum_{z \in \Gamma^{(0)}} \mathbf{1}_z^{(v)} \otimes \mathbf{1}_w^{(z)}$. Cutting down, we obtain maps

$$\Delta_{vw} : {}^r B(\Gamma)_z^s \rightarrow {}^r B(\Gamma)_w^s \otimes {}^v B(\Gamma)_z^w$$

which then satisfy the properties (a),(d) and (e) of Definition 1.7. Moreover, the Δ_{vw} are coassociative as they are coassociative on generators.

Let now $e_{\binom{v}{w}, \binom{v'}{w'}}$ be the matrix units for $l^2(\Gamma^{(0)} \otimes \Gamma^{(0)})$. Then one verifies again directly from the defining relations of $B(\Gamma)$ that one can define a $*$ -homomorphism

$$\tilde{\varepsilon} : B(\Gamma) \rightarrow B(l^2(\Gamma^{(0)})), \quad \begin{cases} \mathbf{1}_w^{(v)} & \mapsto \delta_{v,w} e_{\binom{v}{w}, \binom{v}{w}} \\ u_{e,f} & \mapsto \delta_{e,f} e_{\binom{s(e)}{s(f)}, \binom{t(e)}{t(f)}} \end{cases}$$

We can hence define a map $\varepsilon : B(\Gamma) \rightarrow \mathbb{C}$ such that

$$\tilde{\varepsilon}(x) = \varepsilon(x) e_{\binom{k}{m}, \binom{l}{n}}, \quad \forall x \in {}^k_m B(\Gamma)_n^l.$$

Clearly it satisfies the conditions (b) and (c) of a partial $*$ -bialgebra. As ε satisfies the counit condition on generators, it follows by multiplicativity that it satisfies the counit condition on the whole of $B(\Gamma)$, i.e. $B(\Gamma)$ is a partial $*$ -bialgebra.

It is clear now that the $u_{e,f}$ define a corepresentation U of $B(\Gamma)$. Moreover, from (5.1) and (5.3) we can deduce that $R_\Gamma : \mathbb{C}_{\Gamma(0)} \rightarrow \mathcal{H}_\Gamma \otimes_{\Gamma(0)} \mathcal{H}_\Gamma$ is a morphism from $\mathbb{C}_{\Gamma(0)}$ to $U \oplus_{\Gamma(0)} U$ in $\text{Corep}_{\text{sfd},u}(\mathcal{B}(\Gamma))$, cf. (5.4). From the universal property of $\text{Rep}(SU_q(2))$, it then follows that we have a (unique) monoidal functor

$$G_\Gamma : \text{Rep}(SU_q(2)) \rightarrow \text{Corep}_{\text{sfd},u}(\mathcal{B}(\Gamma))$$

such that $G_\Gamma(\pi_{1/2}) = U$. This functor is faithful as $\text{Rep}(SU_q(2))$ has no non-trivial ideals. On the other hand, as we have a Δ -preserving $*$ -homomorphism $\mathcal{B}(\Gamma) \rightarrow \mathcal{A}(\Gamma)$ by the universal property of $\mathcal{B}(\Gamma)$, we have a monoidal functor $H_\Gamma : \text{Corep}_{\text{sfd},u}(\mathcal{B}(\Gamma)) \rightarrow \text{Rep}(SU_q(2))$ which is inverse to G_Γ . Then since the commutation relations of $\mathcal{A}(\Gamma)$ are completely determined by the morphism spaces of $\text{Rep}(SU_q(2))$, it follows that we have a $*$ -homomorphism $\mathcal{A}(\Gamma) \rightarrow \mathcal{B}(\Gamma)$ sending $v_{e,f}$ to $u_{e,f}$. This proves the theorem. \square

6 Dynamical quantum $SU(2)$

6.1 Dynamical quantum $SU(2)$ from the Podleś graph

Let us now consider the particular case of the Podleś graph of Example 5.2. We assume in the following that $x \in [0, \frac{1}{2}]$.

Let us denote

$$\begin{aligned} w_+(k) &= w(k, k+1), \\ w_-(k) &= w(k, k-1) = w_+(k-1)^{-1}. \end{aligned}$$

Let $A_x = A(\Gamma_x)$ be the total $*$ -algebra of the associated partial compact quantum group. Using Theorem 5.3, we have the following presentation of A_x . Let B be the $*$ -algebra of finite support functions on $\mathbb{Z} \times \mathbb{Z}$, whose Dirac functions we write as $\mathbf{1} \binom{k}{l}$. Let $s_q = \frac{1}{2}(1 + \text{sgn}(q))$. Then A_x is generated by a copy of B and elements

$$(u_{\epsilon,\nu})_{k,l} = u_{(k,k+\epsilon),(l,l+\nu)}$$

for $\epsilon, \nu \in \{-1, 1\} = \{-, +\}$ and $k, l \in \mathbb{Z}$ with defining relations

$$\begin{aligned} \sum_{\mu \in \{\pm\}} (u_{\mu, \epsilon})_{m-\mu, k}^* (u_{\mu, \nu})_{m-\mu, l} &= \delta_{k, l} \delta_{\epsilon, \nu} \mathbf{1}_{\binom{m}{k+\epsilon}}, \\ \sum_{\mu \in \{\pm\}} (u_{\epsilon, \mu})_{k, m} (u_{\nu, \mu})_{l, m}^* &= \delta_{\epsilon, \nu} \delta_{k, l} \mathbf{1}_{\binom{k}{m}} \\ (u_{\epsilon, \nu})_{k, l}^* &= (\epsilon \nu)^{s_q} \left(\frac{w_\nu(l)}{w_\epsilon(k)} \right)^{1/2} (u_{-\epsilon, -\nu})_{k+\epsilon, l+\nu}. \end{aligned}$$

The element $(u_{\epsilon, \nu})_{k, l}$ lives inside the component ${}^k_l(A_x)_{l+\nu}^{k+\epsilon}$.

Consider now $M(A_x)$, the multiplier algebra of A_x . We can form in $M(A_x)$ the elements $u_{\epsilon, \nu} = \sum_{k, l} (u_{\epsilon, \nu})_{k, l}$. Then $u = (u_{\epsilon, \nu})$ is a unitary 2×2 matrix. Moreover,

$$u_{\epsilon, \nu}^* = (\epsilon \nu)^{s_q} u_{-\epsilon, -\nu} \frac{w_\nu^{1/2}(\rho)}{w_\epsilon^{1/2}(\lambda)}, \quad (6.1)$$

where $w_\pm^{1/2}(k) = w_\pm(k)^{1/2}$ and where for a function f on \mathbb{Z} we write $f(\lambda)(k, l) = f(k)$, $f(\rho)(k, l) = f(l)$. In the following, we then also use the notation $f(\lambda, \rho)$ for a function f on $\mathbb{Z} \times \mathbb{Z}$ interpreted as an element of $M(A)$, and for example $f(\lambda + 1, \rho)$ corresponds to the function $(k, l) \mapsto f(k + 1, l)$. We then have the following commutation relations between functions on $\mathbb{Z} \times \mathbb{Z}$ and the entries of u :

$$f(\lambda, \rho) u_{\epsilon, \nu} = u_{\epsilon, \nu} f(\lambda - \epsilon, \rho - \nu). \quad (6.2)$$

Let us write

$$F(k) = |q|^{-1} w_+(k) = |q|^{-1} \frac{|q|^{x+k+1} + |q|^{-x-k-1}}{|q|^{x+k} + |q|^{-x-k}},$$

and further put

$$\alpha = \frac{F^{1/2}(\rho - 1)}{F^{1/2}(\lambda - 1)} u_{--}, \quad \beta = \frac{1}{F^{1/2}(\lambda - 1)} u_{-+}.$$

Then the unitarity of $(u_{\epsilon, \nu})_{\epsilon, \nu}$ together with (6.1) and (6.2) are equivalent to the commutation relations

$$\alpha \beta = q F(\rho - 1) \beta \alpha \quad \alpha \beta^* = q F(\lambda) \beta^* \alpha \quad (6.3)$$

$$\begin{aligned} \alpha \alpha^* + F(\lambda) \beta^* \beta &= 1, & \alpha^* \alpha + q^{-2} F(\rho - 1)^{-1} \beta^* \beta &= 1, \\ F(\rho - 1)^{-1} \alpha \alpha^* + \beta \beta^* &= F(\lambda - 1)^{-1}, & F(\lambda) \alpha^* \alpha + q^{-2} \beta \beta^* &= F(\rho), \end{aligned} \quad (6.4)$$

$$f(\lambda) g(\rho) \alpha = \alpha f(\lambda + 1) g(\rho + 1), \quad f(\lambda) g(\rho) \beta = \beta f(\lambda + 1) g(\rho - 1). \quad (6.5)$$

These are precisely the commutation relations for the dynamical quantum $SU(2)$ -group as in [7, Definition 2.6], except that the precise value of F has been changed by a shift in the parameter domain by a complex constant. Clearly, by Theorem 5.3 the (total) coproduct on A_x also agrees with the one on the dynamical quantum $SU(2)$ -group, namely

$$\begin{aligned}\Delta(\alpha) &= \Delta(1)(\alpha \otimes \alpha - q^{-1}\beta \otimes \beta^*), \\ \Delta(\beta) &= \Delta(1)(\beta \otimes \alpha^* + \alpha \otimes \beta)\end{aligned}$$

where $\Delta(1) = \sum_{k \in \mathbb{Z}} \rho_k \otimes \lambda_k$.

6.2 Representation theory of the function algebra of dynamical quantum $SU(2)$

In this section we classify the irreducible $*$ -representations of A_x . The parametrisation will hinge on the classification of what we call irreducible (x, c) -adapted sets, which we will now discuss. In the following, we fix $0 < |q| < 1$.

Definition 6.1. Let $x \in [0, \frac{1}{2}]$, and let $c \in \mathbb{R}$. For $\epsilon \in \{\pm\}$, an integer $m \in \mathbb{Z}$ will be called $(x, c)_\epsilon$ -adapted if

$$c \leq |q|^{2x+m-\epsilon} + |q|^{-2x-m+\epsilon}, \quad (6.6)$$

and *strictly* $(x, c)_\epsilon$ -adapted if this holds strictly. An integer is called (x, c) -adapted if it is both $(x, c)_+$ and $(x, c)_-$ -adapted.

A set of integers Z is called an (x, c) -set if the following conditions hold:

- Z is not empty.
- Z consists of (x, c) -adapted points.
- If $m \in Z$ is strictly $(x, c)_\epsilon$ -adapted, then $m - 2\epsilon$ is in Z .

An (x, c) -set Z is called *even* (resp. *odd*) if $Z \subseteq 2\mathbb{Z}$ (resp. $Z \subseteq 2\mathbb{Z} + 1$).

An (x, c) -set is called *irreducible* if it can not be written as the union of two disjoint (x, c) -sets.

We aim to classify irreducible (x, c) -sets. We start of with the following lemma.

Lemma 6.2. Let $c \in \mathbb{R}$ and $x \in [0, \frac{1}{2}]$.

1. Any irreducible (x, c) -set is either even or odd.
2. Z is an irreducible (x, c) -set if and only if $-Z - 1$ is an irreducible $(\frac{1}{2} - x, c)$ -set.

Proof. Immediate. □

Hence it suffices to classify even irreducible (x, c) -sets. This is achieved in the following proposition. (We use the convention $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\mathbb{N}_0 = \{1, 2, \dots\}$).

Proposition 6.3. *Let $c \in \mathbb{R}$ and $x \in [0, \frac{1}{2}]$.*

- *If $c < |q|^{2x-1} + |q|^{-2x+1}$, then $2\mathbb{Z}$ is an irreducible (x, c) -set.*
- *If $c \geq |q|^{2x-1} + |q|^{-2x+1}$, write $c = |q|^y + |q|^{-y}$ for some unique $y \geq 1 - 2x$.*
 - * *Assume $x = 0$.*
 - *If $y = 1$, then $-2\mathbb{N}_0$, $\{0\}$ and $2\mathbb{N}_0$ are irreducible (x, c) -sets.*
 - *If $y \in 2\mathbb{N}_0 + 1$, then $y + 1 + 2\mathbb{N}$ and $-y - 1 - 2\mathbb{N}$ are irreducible (x, c) -sets.*
 - * *Assume $x \in (0, \frac{1}{2})$. Assume y is of the form $y = |2x + M - 1|$, where M is a (uniquely determined) even integer.*
 - *If $M > 0$, then $M + 2\mathbb{N}$ is an irreducible (x, c) -set.*
 - *If $M < 0$, then $M - 2\mathbb{N}_0$ is an irreducible (x, c) -set.*
 - *If $M = 0$, then $2\mathbb{N}$ and $-2\mathbb{N}_0$ are irreducible (x, c) -sets.*
 - * *Assume $x = \frac{1}{2}$.*
 - *If $y \in 2\mathbb{N}$, then $y + 2\mathbb{N}$ and $-y - 2\mathbb{N}_0$ are irreducible (x, c) -sets.*

The above (x, c) -sets exhaust all possible even irreducible (x, c) -sets.

Proof. Clearly

$$\min\{|q|^{2x+m-\epsilon} + |q|^{-2x-m+\epsilon} \mid m \in 2\mathbb{Z}, \epsilon \in \{-1, 1\}\} = |q|^{2x-1} + |q|^{-2x+1}.$$

Hence if $c < |q|^{2x-1} + |q|^{-2x+1}$, all $m \in 2\mathbb{Z}$ are strictly $(x, c)_\epsilon$ -adapted for any $\epsilon \in \{-1, 1\}$, and we deduce that $2\mathbb{Z}$ is the only irreducible (x, c) -set.

If $c \geq |q|^{2x-1} + |q|^{-2x+1}$, we can write $c = |q|^y + |q|^{-y}$ for a uniquely determined $y \geq 1 - 2x$, and then we see m is $(x, c)_\epsilon$ -adapted if and only if

$$y \leq |2x + m - \epsilon|.$$

If $y = 1 - 2x$, we can still conclude that $2\mathbb{Z}$ is an (x, c) -set, but it will no longer be irreducible. If $m \in 2\mathbb{N}_0$ and $x \neq 0$, then it is strictly $(x, c)_\epsilon$ -adapted for $\epsilon \in \{-, +\}$, hence if an irreducible (x, c) -set contains a strictly positive even integer, it contains $2\mathbb{N}$. Since $m = 0$ is (x, c) -adapted but not strictly $(x, c)_+$ -adapted, we conclude that $2\mathbb{N}$ is an irreducible (x, c) -set. In case $x = 0$ however, $m = 2$ is not strictly $(x, c)_+$ -adapted, and we can conclude that in fact $2\mathbb{N}_0$ is an irreducible (x, c) -set. The set $\{0\}$ then is an irreducible (x, c) -set by itself. Considering again a general x , we similarly see that if an irreducible (x, c) -set contains an even integer < 2 , then it contains $-2\mathbb{N}_0$. Hence

$-2\mathbb{N}_0$ is an irreducible (x, c) -set. Hence for $x \neq 0$, $2\mathbb{N}$ and $-2\mathbb{N}_0$ exhaust the irreducible (x, c) -sets, while for $x = 0$ we have the three sets $2\mathbb{N}_0$, $\{0\}$ and $-2\mathbb{N}_0$.

If $y > 1 - 2x$, it is clear that an (x, c) -set exists if and only if the equation $y = |2x + m - 1|$ admits a solution in $2\mathbb{Z}$.

Let us first assume that $x \in (0, \frac{1}{2})$. Then this solution is unique, say $y = |2x + M - 1|$ with $M \in 2\mathbb{Z}$.

Assume first that $M > 0$. Then any even $m < 0$ which is $(x, c)_-$ -adapted is strictly $(x, c)_-$ -adapted. It follows that if an (x, c) -set contains a strictly negative integer, it contains $m = 0$. But then $m = 0$ is $(x, c)_+$ -adapted, which leads to the contradictory statement $x \leq 0$. Hence any (x, c) -set necessarily lies in $2\mathbb{N}_0$. It is then easily seen that the only (x, c) -set is $M + 2\mathbb{N}$, which is then necessarily irreducible.

Assume now that $M \leq 0$. From the condition $y > 1 - 2x$, we may in fact assume $M < 0$. Then as above, we conclude that if an (x, c) -set contains a strictly positive even integer, it contains $m = 0$, which is not $(x, c)_-$ -adapted. It then follows that $M - 2\mathbb{N}_0$ is the only irreducible (x, c) -set.

Let us now assume that $x = 0$. Then we may assume $y = M + 1$ with $M \in 2\mathbb{N}_0$. Then $m > 0$ is $(x, c)_+$ -adapted if and only if $m \geq M + 2$, and it is strictly so if and only if $m > M + 2$. As $m = M + 2$ is (x, c) -adapted, it follows that $M + 2\mathbb{N}_0$ is an irreducible (x, c) -set, and the only one which intersects the strictly positive integers non-trivially. On the other hand, an $m < 0$ is $(x, c)_-$ -adapted if and only if $m \leq -M - 2$, and strictly so if and only if $m < -M - 2$. As $m = -M - 2$ is (x, c) -adapted, it follows that $-M - 2\mathbb{N}_0$ is an irreducible (x, c) -set, and the only one which intersects the strictly negative integers non-trivially. Finally, since we have assumed $M > 0$, the integer $m = 0$ is never (x, c) -adapted.

Finally, let us assume that $x = \frac{1}{2}$. Then we may assume $y = M$ with $M \in 2\mathbb{N}_0$. By a similar reasoning as in the case $x = 0$, we conclude that the only irreducible (x, c) -sets are $M + 2\mathbb{N}$ and $-M - 2\mathbb{N}_0$. \square

Let us now return to the representation theory of the $*$ -algebra A_x .

Let π be any (necessarily bounded) non-degenerate $*$ -representation of A_x on a Hilbert space \mathcal{H}_π . Then

$$\mathcal{H}_\pi = \oplus \mathcal{H}_n^m, \quad \mathcal{H}_n^m = \pi(\mathbf{1}(\binom{m}{n}))\mathcal{H}.$$

Write

$$V_\pi = \text{the (non-closed) linear span of all } \mathcal{H}_n^m.$$

Then $\pi(A_x)V_\pi = V_\pi$. It follows that one can extend π to a map

$$\pi : M(A_x) \rightarrow \text{End}_{\text{adj}}(V_\pi),$$

where $\text{End}_{\text{adj}}(V_\pi)$ denotes the $*$ -algebra of adjointable operators on the pre-Hilbert space V_π . As the $u_{\epsilon, \nu}$ form a unitary matrix, we can then in fact make sense of the $\pi(u_{\epsilon, \nu})$ as

contractions on \mathcal{H}_π . Hence the generators α, β and their adjoints give rise to endomorphisms $V_\pi \rightarrow V_\pi$ which are bounded when restricted to any \mathcal{H}_n^m .

We then have the following easy lemma.

Lemma 6.4. *There is a one-to-one correspondence between*

- *non-degenerate $*$ -representations (\mathcal{H}_π, π) of A_x on Hilbert spaces, and*
- *\mathbb{Z}^2 -bigraded pre-Hilbert spaces V_π with norm-complete components and equipped with adjointable maps $\alpha, \beta : V_\pi \rightarrow V_\pi$ satisfying the commutation relations as in (6.3), (6.4) and (6.5) and with $f(\lambda, \rho)\xi = f(k, l)\xi$ for f a function on $\mathbb{Z} \times \mathbb{Z}$ and $\xi \in V_l^k$.*

Definition 6.5. The *Casimir* of A_x is defined to be the following element $\Omega \in M(A_x)$,

$$\Omega = q^{\lambda-\rho+1} + q^{\rho-\lambda-1} - \operatorname{sgn}(q)^{\lambda-\rho} q^{-1} (|q|^{x+\lambda+1} + |q|^{-x-\lambda-1}) (|q|^{x+\rho-1} + |q|^{-x-\rho+1}) \beta^* \beta.$$

Lemma 6.6. *The element Ω is self-adjoint central in $M(A_x)$.*

Proof. Straightforward, cf. [7, Lemma 3.3]. □

Corollary 6.7. *If π is an irreducible $*$ -representation of A_x , there exists $c \in \mathbb{R}$ such that $\pi(\Omega)\xi = c\xi$ for all $\xi \in V_\pi$.*

Proof. As $\pi(\Omega)$ is bounded when restricted to any V_l^k , this follows immediately from a spectral argument. □

The following lemma follows from a straightforward computation, using the relations (6.4).

Lemma 6.8. *Inside $M(A_x)$, we have the following identities:*

$$\begin{aligned} \alpha^* \alpha &= \frac{|q|^{2x+\lambda+\rho+1} + |q|^{-2x-\lambda-\rho-1} + \operatorname{sgn}(q)^{\lambda-\rho+1} \Omega}{(|q|^{x+\lambda+1} + |q|^{-x-\lambda-1})(|q|^{x+\rho} + |q|^{-x-\rho})} \\ \alpha \alpha^* &= \frac{|q|^{2x+\lambda+\rho-1} + |q|^{-2x-\lambda-\rho+1} + \operatorname{sgn}(q)^{\lambda-\rho-1} \Omega}{(|q|^{x+\lambda} + |q|^{-x-\lambda})(|q|^{x+\rho-1} + |q|^{-x-\rho+1})} \\ \beta^* \beta &= |q| \frac{|q|^{\lambda-\rho+1} + |q|^{-\lambda+\rho-1} - \operatorname{sgn}(q)^{\lambda-\rho+1} \Omega}{(|q|^{x+\lambda+1} + |q|^{-x-\lambda-1})(|q|^{x+\rho-1} + |q|^{-x-\rho+1})} \\ \beta \beta^* &= |q| \frac{|q|^{\lambda-\rho-1} + |q|^{-\lambda+\rho+1} - \operatorname{sgn}(q)^{\lambda-\rho-1} \Omega}{(|q|^{x+\lambda} + |q|^{-x-\lambda})(|q|^{x+\rho} + |q|^{-x-\rho})}. \end{aligned}$$

Note that the right hand sides are well-defined because of centrality of Ω .

Corollary 6.9. *If π is an irreducible $*$ -representation of A_x on a Hilbert space \mathcal{H}_π , then \mathcal{H}_n^m is at most one-dimensional for each $m, n \in \mathbb{Z}$. Moreover, either all \mathcal{H}_l^k with $k - l$ odd are zero, or all \mathcal{H}_l^k with $k - l$ even are zero.*

Proof. Using Corollary 6.7, the first assertion follows immediately from (6.3), the grading relations (6.5) and Lemma 6.8. The second assertion follows immediately from the grading relations (6.5). \square

Definition 6.10. Let (\mathcal{H}_π, π) be an irreducible $*$ -representation of A_x . We call π even (resp. odd) if all \mathcal{H}_l^k with $k - l$ odd (resp. even) are zero.

With the above preliminaries, we can now classify the irreducible $*$ -representations of A_x . We first extend the terminology of Definition 6.1.

Definition 6.11. Fix $c \in \mathbb{R}$ and $x \in [0, \frac{1}{2}]$. For $\epsilon, \nu \in \{-, +\}$, a couple $(k, l) \in \mathbb{Z}^2$ is called $(x, c)_{\epsilon, \nu}$ -adapted if the following inequality holds:

$$(|q|^{(x+k)+\epsilon\nu(x+l)-\epsilon} + |q|^{-(x+k)-\epsilon\nu(x+l)+\epsilon}) + \operatorname{sgn}(q)^{k-l+1}\epsilon\nu c \geq 0. \quad (6.7)$$

A couple (k, l) is called *strictly* $(x, c)_{\epsilon, \nu}$ -adapted if this is a strict equality. We call (k, l) (x, c) -adapted if it is $(x, c)_{\epsilon, \nu}$ -adapted for all $\epsilon, \nu \in \{+, -\}$.

Definition 6.12. Fix $c \in \mathbb{R}$ and $x \in [0, \frac{1}{2}]$. We call a subset $T \subseteq \mathbb{Z}^2$ an (x, c) -set if the following conditions are satisfied:

- T is not empty.
- T consists of (x, c) -adapted points.
- If $(k, l) \in T$ is strictly $(x, c)_{\epsilon, \nu}$ -adapted, then $(k - \epsilon, l - \nu)$ is in T .

We say that T *irreducible* if it is not the disjoint union of two (x, c) -sets.

Writing $\mathbb{Z}_{\text{even}}^2 = \{(k, l) \mid k - l \text{ even}\}$ and $\mathbb{Z}_{\text{odd}}^2 = \mathbb{Z}^2 \setminus \mathbb{Z}_{\text{even}}^2$, we call a (x, c) -set even or odd according to whether it lies in $\mathbb{Z}_{\text{even}}^2$ or $\mathbb{Z}_{\text{odd}}^2$.

Definition 6.13. Fix $x \in [0, \frac{1}{2}]$. For π an irreducible representation of A_x , a couple $(k, l) \in \mathbb{Z}^2$ is called π -compatible if $\mathcal{H}_l^k \neq 0$.

For $c \in \mathbb{R}$, a subset $T \subseteq \mathbb{Z}^2$ is called (x, c) -compatible if there exists an irreducible representation π of A_x with $\pi(\Omega) = c$ and $T = \{(k, l) \in \mathbb{Z}^2 \mid \mathcal{H}_l^k \neq \{0\}\}$. In this case, we say that π is T -adapted.

Proposition 6.14. A set $T \subseteq \mathbb{Z}^2$ is an irreducible (x, c) -set if and only if it is a (x, c) -compatible set. Moreover, for any (x, c) -compatible set T there is only one irreducible $*$ -representation π of A_x , up to unitary equivalence, which is T -compatible.

Proof. Assume first that T is (x, c) -compatible, and let π be a T -compatible irreducible $*$ -representation of A_x . If $(k, l) \in T$, then it follows from Lemma 6.8 that (k, l) is (x, c) -adapted. Moreover, if $(k, l) \in T$ is strictly $(x, c)_{\epsilon, \nu}$ -adapted, then we have that $\|u_{\epsilon, \nu}\xi\| \neq 0$ for a non-zero $\xi \in \mathcal{H}_l^k$, hence also $\mathcal{H}_{l-\nu}^{k-\epsilon} \neq \{0\}$. It follows that T is a (x, c) -set. Now if $T = T_1 \cup T_2$ a disjoint union of (x, c) -sets, it would follow that π restricts to the direct sum of all \mathcal{H}_l^k with $(k, l) \in T_1$, contradicting irreducibility. It follows that T is an irreducible (x, c) -set.

Conversely, let T be an irreducible (x, c) -set. Put $\mathcal{H}_\pi = l^2(T)$ with

$$\begin{aligned}\pi(\alpha)e_{k,l} &= \left(\frac{|q|^{2x+k+l+1} + |q|^{-2x-k-l-1} + \operatorname{sgn}(q)^{k-l+1}c}{(|q|^{x+k+1} + |q|^{-x-k-1})(|q|^{x+l} + |q|^{-x-l})} \right)^{1/2} e_{k+1,l+1}, \\ \pi(\beta)e_{k,l} &= \operatorname{sgn}(q)^k \left(|q| \frac{|q|^{k-l+1} + |q|^{-k+l-1} - \operatorname{sgn}(q)^{k-l+1}c}{(|q|^{x+k+1} + |q|^{-x-k-1})(|q|^{x+l-1} + |q|^{-x-l+1})} \right)^{1/2} e_{k+1,l-1},\end{aligned}$$

where the right hand side is considered as the zero vector when the scalar factor on the right is zero. Note that the roots on the right hand side are well-defined precisely because T is a (x, c) -set.

By direct computation, using the defining commutation relations (6.3) and (6.4), we see that π defines a $*$ -representation of A_x with $\pi(\Omega) = c$. Moreover, π is irreducible since otherwise, by Corollary 6.9, T would split as a disjoint union of (x, c) -compatible sets. Hence T is an (x, c) -compatible set.

Now the formulas for $\pi(\alpha)$ and $\pi(\beta)$ are uniquely determined up to a unimodular gauge factor. As any non-zero \mathcal{H}_l^k is cyclic for π , it follows that these gauge factors are determined by their value at one component. We then easily conclude that π is in fact the unique T -compatible $*$ -representation, up to unitary equivalence. \square

What remains is to classify irreducible (x, c) -sets for each $c \in \mathbb{R}$.

Lemma 6.15. *A set $T \subseteq \mathbb{Z}_{\text{even}}^2$ is an irreducible (x, c) -set if and only if there exists an even irreducible $(x, -\operatorname{sgn}(q)c)$ -set $Z_+ \subseteq \mathbb{Z}$ and an even irreducible $(0, \operatorname{sgn}(q)c)$ -set $Z_- \subseteq \mathbb{Z}$ such that $(k, l) \in T$ if and only if $k + l \in Z_+$ and $k - l \in Z_-$.*

A set $T \subseteq \mathbb{Z}_{\text{odd}}^2$ is an irreducible (x, c) -set if and only if there exists an odd irreducible $(x, -c)$ -set $Z_+ \subseteq \mathbb{Z}$ and an odd irreducible $(0, c)$ -set $Z_- \subseteq \mathbb{Z}$ such that $(k, l) \in T$ if and only if $k + l \in Z_+$ and $k - l \in Z_-$.

Proof. It is immediate that $(k, l) \in \mathbb{Z}_{\text{even}}^2$ is (strictly) $(x, c)_{\epsilon, \nu}$ -adapted if and only if $k + \epsilon \nu l$ is $((1 + \epsilon \nu)x, -\operatorname{sgn}(q)\epsilon \nu c)_{\epsilon}$ -adapted. Similarly, $(k, l) \in \mathbb{Z}_{\text{even}}^2$ is (strictly) $(x, c)_{\epsilon, \nu}$ -adapted if and only if $k + \epsilon \nu l$ is $((1 + \epsilon \nu)x, -\epsilon \nu c)_{\epsilon}$ -adapted. The conclusion of the lemma then follows immediately. \square

Combining Proposition 6.14 with Proposition 6.3 and Lemma 6.15, we thus obtain a concrete description of the spectrum of A_x . The following pictures illustrate the form of the spectrum of A_x for the case $q > 0$.

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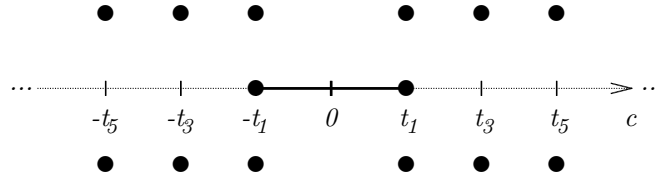


Figure 1: Case $x = 0$, even

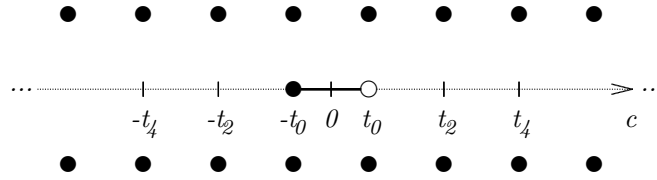


Figure 2: Case $x = 0$, odd

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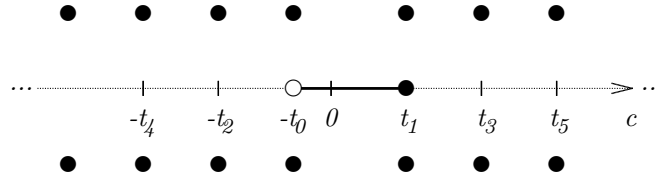


Figure 3: Case $x = \frac{1}{2}$, even

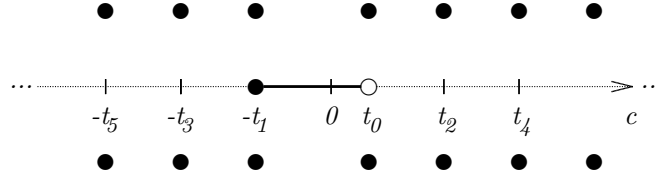


Figure 4: Case $x = \frac{1}{2}$, odd

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