Partial compact quantum groups, compact quantum homogeneous spaces and the dynamical quantum $SU(2)$ group

Kenny De Commer* Thomas Timmermann†

Abstract

Compact quantum groups of face type, as introduced by T. Hayashi, form a class of quantum groupoids with a classical, finite set of objects. We generalize Hayashi's definition to allow for an infinite set of objects, and call the resulting objects partial compact quantum groups. We then show how any quantum homogeneous space of an ordinary compact quantum group leads to a partial compact quantum group. In particular, when this construction is applied to the non-standard Podleś spheres, we obtain partial compact quantum groups which are operator algebraic versions of the dynamical quantum $SU(2)$-group as studied by Etingof-Varchenko and Koelink-Rosengren.

1 Partial compact quantum groups

We generalize Hayashi's definition of a compact quantum group of face type [7] to the case where the commutative base algebra is no longer finite-dimensional. We will present two approaches, based on *partial bialgebras* and *weak multiplier bialgebras* [3]. The first approach is piecewise and concrete, but requires some bookkeeping. The second approach is global but more abstract. As we will see from the general theory and the concrete examples, both approaches have their intrinsic value.

Let I be a set. We consider $I^2 = I \times I$ as the pair groupoid with $\cdot$ denoting composition. That is, an element $K = (k,l) \in I^2$ has source $K_l = k$ and target $K_r = l$, and if $K = (k,l)$ and $L = (l,m)$ we write $K \cdot L = (k,m)$.

Definition 1.1. A *partial algebra* $\mathscr{A} = (\mathscr{A}, M)$ (over $\mathbb{C}$) is a small $\mathbb{C}$-linear category, that is, a set I (the object set) together with

*Department of Mathematics, Vrije Universiteit Brussel, VUB, B-1050 Brussels, Belgium, email: kenny.de.commer@vub.ac.be

†University of Münster

- for each $K = (k,l) \in I^2$ a vector space $A(K) = A\big(k,l\big) = {}_kA_l$ (possibly the zero vector space),
- for each K, L with $K_r = L_l$ a multiplication map

$$M(K,L) : A(K) \otimes A(L) \to A(K \cdot L), \qquad a \otimes b \mapsto ab$$

and
- elements $\mathbf{1}(k) = \mathbf{1}_k \in A\big(k,k\big)$ (the units),

such that the obvious associativity and unit conditions are satisfied.

By *I-partial algebra* will be meant a partial algebra with object set I.

Remark 1.2. We allow the local units $\mathbf{1}_k$ to be zero.

Let $\mathscr{A}$ be an I-partial algebra. We define $A(K \cdot L)$ to be $\{0\}$ when $K \cdot L$ is ill-defined, i.e. $K_r \neq L_l$. We then let $M\big(K,L\big)$ be the zero map.

Definition 1.3. The *total algebra* A of an I-partial algebra $\mathscr{A}$ is the vector space

$$A = \oplus_{K \in I^2} A(K)$$

endowed with the unique multiplication whose restriction to $A(K) \otimes A(L)$ concides with $M(K,L)$.

Clearly A is an associative algebra. If I is infinite it will not possess a unit, but it is a *locally unital algebra* as there exist mutually orthogonal idempotents $\mathbf{1}_k$ with $A = \sum_{k,l}^{\oplus} \mathbf{1}_k A \mathbf{1}_l$. An element $a \in A$ can be interpreted as a function assigning to each element $(k,l) \in I^2$ an element $a_{kl} \in A(k,l)$, namely the (k,l)-th component of a. This identifies A with finite support I-indexed matrices whose (k,l)-th entry lies in $A(k,l)$, equipped with the natural matrix multiplication.

Remark 1.4. When $\mathscr{A}$ is an I-partial algebra with total algebra A, then $A \otimes A$ can be naturally identified with the total algebra of an $I \times I$-partial algebra $\mathscr{A} \otimes \mathscr{A}$, where

$$(A \otimes A)((k,k'),(l,l')) = A(k,l) \otimes A(k',l')$$

with the obvious tensor product multiplications and the $\mathbf{1}_{k,k'} = \mathbf{1}_k \otimes \mathbf{1}_{k'}$ as units.

The notion of partial algebra dualizes. For this we consider again I^2 as the pair groupoid, but now with elements considered as column vectors, and with $*$ denoting the (vertical) composition. So $K = \binom{k}{l}$ has source $K_u = k$ and target $K_d = l$, and if $K = \binom{k}{l}$ and $L = \binom{l}{m}$ then $K * L = \binom{k}{n}$.

Definition 1.5. A *partial coalgebra* $\mathscr{A} = (\mathscr{A}, \Delta)$ (over $\mathbb{C}$) consists of a set I (the object set) together with

- for each $K = \begin{pmatrix} k \\ l \end{pmatrix} \in I^2$ a vector space $A(K) = A\binom{k}{l} = A_l^k$,

- for each K, L with $K_d = L_u$ a comultiplication map

$$\Delta\binom{K}{L} : A(K * L) \to A(K) \otimes A(L), \qquad a \mapsto a_{(1)K} \otimes a_{(2)L},$$

  and

- counit maps $\epsilon_k : A\binom{k}{k} \to \mathbb{C}$,

satisfying the obvious coassociativity and counitality conditions.

By *I-partial coalgebra* will be meant a partial coalgebra with object set I.

Notation 1.6. As the index of ϵ_k is determined by the element to which it is applied, there is no harm in dropping the index k and simply writing ϵ.

Similarly, if $K = \binom{k}{l}$ and $L = \binom{l}{m}$, we abbreviate $\Delta_l = \Delta\binom{K}{L}$, as the other indices are determined by the element to which Δ_l is applied.

We also make again the convention that $A(K * L) = \{0\}$ and $\Delta\binom{K}{L}$ the zero map when $K_d \neq L_u$. Similarly ϵ is seen as the zero functional on $A(K)$ when $K = \binom{k}{l}$ with $k \neq l$.

We can now superpose the notions of partial algebra and partial coalgebra. To formulate the condition that the coalgebra maps form a 'morphism of partial algebras', we will need to impose a finiteness condition which is automatically satisfied when the cardinality of I is finite.

Let I be a set, and let $M_2(I)$ be the set of 4-tuples of elements of I arranged as 2×2-matrices. We can endow $M_2(I)$ with two compositions, namely $\cdot$ (viewing $M_2(I)$ as a row vector of column vectors) and $*$ (viewing $M_2(I)$ as a column vector of row vectors). When $K \in M_2(I)$, we will write $K = (K_l, K_r) = \binom{K_u}{K_d} = \begin{pmatrix} K_{lu} & K_{ru} \\ K_{ld} & K_{rd}\end{pmatrix}$. One can view $M_2(I)$ as a double groupoid, and in fact as a *vacant* double groupoid in the sense of [1].

In the following, a vector (r, s) will sometimes be written simply as r, s (without parentheses) or rs in an index. We also follow Notation 1.6, but the reader should be aware that the index of Δ will now be a 1×2 vector in I^2 as we will work with partial coalgebras over I^2.

Definition 1.7. A *partial bialgebra* $\mathscr{A} = (\mathscr{A}, M, \Delta)$ consists of a set I and a collection of vector spaces $A(K)$ for $K \in M_2(I)$ such that

- the $A(K_l, K_r)$ form an I^2-partial algebra,
- the $A\binom{K_u}{K_d}$ form an I^2-partial coalgebra,

and for which the following compatibility relations are satisfied.

(a) (Comultiplication of Units) For all $k, l, m \in I$, one has

$$\Delta_{l,l}(\mathbf{1}\begin{pmatrix} k \\ m \end{pmatrix}) = \mathbf{1}\begin{pmatrix} k \\ l \end{pmatrix} \otimes \mathbf{1}\begin{pmatrix} l \\ m \end{pmatrix}.$$

(b) (Counit of Multiplication) For all $K, L \in M_2(I)$ with $K_r = L_l$ and all $a \in A(K)$ and $b \in A(L)$,

$$\epsilon(ab) = \epsilon(a)\epsilon(b).$$

(c) (Non-degeneracy) For all $k \in I$, $\epsilon(\mathbf{1}\begin{pmatrix} k \\ k \end{pmatrix}) = 1$.

(d) (Finiteness) For each $K \in M_2(I)$ and each $a \in A(K)$, the element $\Delta_{rs}(a)$ is zero except for a finite number of indices r (resp. s) when s (resp. r) is fixed.

(e) (Comultiplication is multiplicative) For all $a \in A(K)$ and $b \in A(L)$ with $K_r = L_l$,

$$\Delta_{rs}(ab) = \sum_t \Delta_{rt}(a)\Delta_{ts}(b).$$

Remark 1.8. By assumption (d), the sum on the right hand side in condition (e) is well-defined.

We want to relate the notion of partial bialgebra to the recently introduced notion of weak multiplier bialgebra [3]. We first recall some notions concerning non-unital algebras [4, 11].

Definition 1.9. Let A be an algebra over $\mathbb{C}$, not necessarily with unit. We call A *non-degenerate* if A is faithfully represented on itself by left and right multiplication. It is called *idempotent* if $A^2 = A$.

Definition 1.10. Let A be an algebra. A *multiplier* m for A consists of a couple of maps

$$\begin{aligned} L_m &: A \to A, \quad a \mapsto ma \\ R_m &: A \to A, \quad a \mapsto am \end{aligned}$$

such that $(am)b = a(mb)$ for all $a, b \in A$.

The set of all multipliers forms an algebra under composition for the L-maps and anti-composition for the R-maps. It is called the *multiplier algebra* of A, and is denoted $M(A)$.

One has a natural homomorphism $A \to M(A)$. When A is non-degenerate, this homomorphism is injective, and we can then identify A as a subalgebra of the (unital) algebra $M(A)$. We then also have inclusions

$$A \otimes A \subseteq M(A) \otimes M(A) \subseteq M(A \otimes A).$$

Example 1.11. 1. Let A be the total algebra of an I-partial algebra $\mathscr{A}$. As A has local units, it is non-degenerate and idempotent. Then one can identify $M(A)$ with

$$M(A) = \left(\prod_l \oplus_k A(k,l)\right) \bigcap \left(\prod_k \oplus_l A(k,l)\right) \subseteq \prod_{k,l} A(k,l),$$

i.e. with the space of functions

$$m : I^2 \to A, \quad m_{kl} \in A(k,l)$$

which have finite support in either one of the variables when the other variable has been fixed. The multiplication is given by the formula

$$(mn)_{kl} = \sum_p m_{kp} n_{pl}.$$

2. Let m_i be any collection of multipliers of A, and assume that for each $a \in A$, $m_i a = 0$ for almost all i, and similarly $a m_i = 0$ for almost all i. Then one can define a multiplier $\sum_i m_i$ in the obvious way by termwise multiplication. One says that the sum $\sum_i m_i$ converges in the *strict* topology.

Using the notion introduced in Example 1.11.2, we can introduce the following notation.

Notation 1.12. If $\mathscr{A}$ is an I-partial bialgebra, we write

$$\lambda_k = \sum_l \mathbf{1}\binom{k}{l}, \qquad \rho_l = \sum_k \mathbf{1}\binom{k}{l} \qquad \in M(A).$$

Remark 1.13. From Property (c) of Definition 1.7, it follows that $\lambda_k \neq 0 \neq \rho_k$ for any $k \in I$.

To show that the total algebra of a partial bialgebra becomes a weak multiplier bialgebra, we will need some easy lemmas.

Lemma 1.14. *Let $\mathscr{A}$ be an I-partial bialgebra. Then for each $a \in A$, there exists a unique multiplier $\Delta(a) \in M(A \otimes A)$ such that*

$$\begin{aligned} \Delta_{rs}(a) &= (1 \otimes \lambda_r)\Delta(a)(1 \otimes \lambda_s) &(1.1)\\ &= (\rho_r \otimes 1)\Delta(a)(\rho_s \otimes 1) &(1.2) \end{aligned}$$

for all $r, s \in I$, all $K \in M_2(I)$ and all $a \in A(K)$.

The resulting map

$$\Delta : A \to M(A \otimes A), \quad a \mapsto \Delta(a)$$

is a homomorphism.

Proof. For $a \in A$ homogeneous, we can define $\Delta(a) = \sum_{rs} \Delta_{rs}(a) \in M(A \otimes A)$, where the sum converges in the strict topology of $A \otimes A$ because of the property (d) of Definition 1.7. This expression clearly satisfies the identities stated in the lemma. In turn, these identities uniquely define $\Delta(a)$ as a multiplier, as they determine the value of $\Delta(a)$ when cut down to the left and right with the local units of $\mathscr{A} \otimes \mathscr{A}$.

We can then extend Δ to A by linearity. Since, for a, b homogeneous, $\Delta_{rt}(a)\Delta_{t's}(b) = 0$ unless $t = t'$, it follows from property (e) of Definition 1.7 that Δ is a homomorphism. □

We will refer to $\Delta : A \to M(A \otimes A)$ as the *total comultiplication* of $\mathscr{A}$. We will then also use the suggestive Sweedler notation for this map,

$$\Delta(a) = a_{(1)} \otimes a_{(2)}.$$

Lemma 1.15. *The element $E = \sum_{k,l,m} \mathbf{1}\begin{pmatrix} k \\ l \end{pmatrix} \otimes \mathbf{1}\begin{pmatrix} l \\ m \end{pmatrix}$ is a well-defined idempotent in $A \otimes A$, and satisfies*

$$\Delta(A)(A \otimes A) = E(A \otimes A), \quad (A \otimes A)\Delta(A) = (A \otimes A)E.$$

Proof. Clearly the sum defining E is strictly convergent, and makes E into an idempotent. It is moreover immediate that $E\Delta(a) = \Delta(a) = \Delta(a)E$ for all $a \in A$. Since

$$E(\mathbf{1}\begin{pmatrix} k \\ l \end{pmatrix} \otimes \mathbf{1}\begin{pmatrix} m \\ n \end{pmatrix}) = \Delta(\mathbf{1}\begin{pmatrix} k \\ n \end{pmatrix})(\mathbf{1}\begin{pmatrix} k \\ l \end{pmatrix} \otimes \mathbf{1}\begin{pmatrix} m \\ n \end{pmatrix})$$

by the property (a) of Definition 1.7, and analogously for multiplication with E on the right, the lemma is proven. $\square$

By [13, Proposition A.3], there is a unique homomorphism $\Delta : M(A) \to M(A \otimes A)$ extending Δ and such that $\Delta(1) = E$. Alternatively, if $m \in M(A)$, we can directly define $\Delta(m)$ as the strict limit of the series $\sum_{k,l,r,s} \Delta_{rs}(m_{kl})$. Similarly the maps $\mathrm{id} \otimes \Delta$ and $\Delta \otimes \mathrm{id}$ extend to maps from $M(A \otimes A)$ to $M(A \otimes A \otimes A)$. The following proposition gathers the properties of Δ, ϵ and $\Delta(1)$ which guarantee that (A, Δ) forms a weak multiplier bialgebra in the sense of [3, Definition 2.1].

Proposition 1.16. *Let $\mathscr{A}$ be a partial bialgebra with total algebra A, total comultiplication Δ and counit ϵ. Then the following properties are satisfied.*

(1) Coassociativity: $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$ (as maps $M(A) \to M(A^{\otimes 3})$).

(2) Counitality: $(\epsilon \otimes \mathrm{id})(\Delta(a)(1 \otimes b)) = ab = (\mathrm{id} \otimes \epsilon)((a \otimes 1)\Delta(b))$ for all $a, b \in A$.

(3) Weak Comultiplicativity of Unit:

$$(\Delta(1) \otimes 1)(1 \otimes \Delta(1)) = (\Delta \otimes \mathrm{id})\Delta(1) = (\mathrm{id} \otimes \Delta)\Delta(1) = (1 \otimes \Delta(1))(\Delta(1) \otimes 1).$$

(4) Weak Multiplicativity of Counit: For all $a, b, c \in A$, one has

$$(\epsilon \otimes \mathrm{id})(\Delta(a)(b \otimes c)) = (\epsilon \otimes \mathrm{id})((1 \otimes a)\Delta(1)(b \otimes c))$$

and

$$(\epsilon \otimes \mathrm{id})((a \otimes b)\Delta(c)) = (\epsilon \otimes \mathrm{id})((a \otimes b)\Delta(1)(1 \otimes c)).$$

(5) Strong multiplier property: For all $a, b \in A$, one has

$$\Delta(A)(1 \otimes A) \cup (A \otimes 1)\Delta(A) \subseteq A \otimes A.$$

Proof. Most of these properties follow immediately from the definition of a partial bialgebra. For demonstrational purposes, let us check the first identity of property (4). Let us choose $a \in A(K)$, $b \in A(L)$ and $c \in A(M)$. Then

$$\Delta(a)(b \otimes c) = \delta_{K_{ru},L_{lu}}\delta_{M_{lu},L_{ld}}\sum_{r}\Delta_{r,L_{ld}}(a)(b \otimes c).$$

Applying $(\epsilon \otimes \mathrm{id})$ to both sides, we obtain by Proposition (b) of Definition 1.7 and counitality of ϵ that

$$(\epsilon \otimes \mathrm{id})(\Delta(a)(b \otimes c)) = \delta_{K_{ru},L_{lu},L_{ld},M_{lu}}\epsilon(b)ac.$$

On the other hand,

$$\begin{aligned}(1 \otimes a)\Delta(1)(b \otimes c) &= \sum_{r,s,t}\mathbf{1}\begin{pmatrix} r \\ s \end{pmatrix} b \otimes a\mathbf{1}\begin{pmatrix} s \\ t \end{pmatrix} c \\ &= \delta_{L_{ld},K_{ru},M_{lu}} b \otimes ac.\end{aligned}$$

Applying $(\epsilon \otimes \mathrm{id})$, we find

$$\begin{aligned}(\epsilon \otimes \mathrm{id})((1 \otimes a)\Delta(1)(b \otimes c)) &= \delta_{L_{ld},K_{ru},M_{lu}}\delta_{L_{lu},L_{ld}}\delta_{L_{ru},L_{rd}}\epsilon(b)ac \\ &= \delta_{L_{ld},L_{lu},K_{ru},M_{lu}}\epsilon(b)ac,\end{aligned}$$

which agrees with the expression above. □

Remark 1.17. Since also the expressions $\Delta(a)(b \otimes 1)$ and $(1 \otimes a)\Delta(b)$ are in $A \otimes A$ for all $a, b \in A$, we see that (A, Δ) is in fact a *regular* weak multiplier bialgebra [3, Definition 2.3].

Remark 1.18. Recall from [3, Section 3] that a regular weak multiplier bialgebra admits four projections $A \to M(A)$, given by

$$\begin{aligned}&\bar{\Pi}^L(a) = (\epsilon \otimes \mathrm{id})((a \otimes 1)\Delta(1)), \quad \bar{\Pi}^R(a) = (\mathrm{id} \otimes \epsilon)(\Delta(1)(1 \otimes a)), \\ &\Pi^L(a) = (\epsilon \otimes \mathrm{id})(\Delta(1)(a \otimes 1)), \quad \Pi^R(a) = (\mathrm{id} \otimes \epsilon)((1 \otimes a)\Delta(1)),\end{aligned}$$

where the right hand side expressions are interpreted as multipliers in the obvious way. A trivial computation shows that if $a \in A\begin{pmatrix} k & l \\ m & n \end{pmatrix}$, then

$$\overline{\Pi}^L(a) = \epsilon(a)\lambda_n, \quad \overline{\Pi}^R(a) = \epsilon(a)\rho_k, \quad \Pi^L(a) = \epsilon(a)\lambda_m, \quad \Pi^R(a) = \epsilon(a)\rho_l.$$

In particular, we see that the *base algebra* of (A, Δ) is just the algebra $\mathrm{Fun}_{\mathrm{f}}(I)$ of finite support functions on I. From [3, Theorem 3.13], we conclude that the comultiplication of A is (left and right) *full* (meaning roughly that the legs of $\Delta(A)$ span A).

We now formulate the notion of partial Hopf algebra, whose total form will correspond to a weak multiplier Hopf algebra [3, 13, 14]. We will mainly refer to [3] for uniformity.

Let us denote $\circ$ for the inverse of $\cdot$, and $\bullet$ for the inverse of $*$, so

$$\begin{pmatrix} k & l \\ m & n \end{pmatrix}^{\circ} = \begin{pmatrix} l & k \\ n & m \end{pmatrix}, \quad \begin{pmatrix} k & l \\ m & n \end{pmatrix}^{\bullet} = \begin{pmatrix} m & n \\ k & l \end{pmatrix}, \quad \begin{pmatrix} k & l \\ m & n \end{pmatrix}^{\circ\bullet} = \begin{pmatrix} n & m \\ l & k \end{pmatrix}.$$

The notation $\circ$ (resp. $\bullet$) will also be used for row vectors (resp. column vectors).

Definition 1.19. An *antipode* for an I-partial bialgebra $\mathscr{A}$ consists of maps

$$S : A(K) \to A(K^{\circ\bullet})$$

such that the following property holds: for all $M, P \in M_2(I)$ and all $a \in A(M)$,

$$\begin{aligned}
\sum_{\substack{K*L=M \\ K\cdot L^{\circ\bullet}=P}} a_{(1)K}S(a_{(2)L}) &= \delta_{P_l,P_r}\epsilon(a)\mathbf{1}(P_l), \\
\sum_{\substack{K*L=M \\ K^{\circ\bullet}\cdot L=P}} S(a_{(1)K})a_{(2)L} &= \delta_{P_l,P_r}\epsilon(a)\mathbf{1}(P_r).
\end{aligned}$$

A partial multiplier bialgebra $\mathscr{A}$ is called a *partial Hopf algebra* if it admits an antipode.

Remark 1.20. Note that condition (d) of Definition 1.7 again guarantees that the above sums are in fact finite.

If S is an antipode for a partial bialgebra, we can extends S to a linear map

$$S : A \to A$$

on the total algebra A.

Lemma 1.21. *Let S be an antipode for a partial Hopf algebra $\mathscr{A}$. Then for all $a, b, c \in A$, one has*

$$\begin{aligned}
(a_{(1)}c \otimes ba_{(2)}S(a_{(3)})) &= (1 \otimes b)\Delta(1)(ac \otimes 1) \\
(S(a_{(1)})a_{(2)}c \otimes ba_{(3)}) &= (1 \otimes ba)\Delta(1)(c \otimes 1).
\end{aligned}$$

Proof. Take $a \in A\left(\begin{smallmatrix} k & l \\ m & n \end{smallmatrix}\right)$. Then in the strict topology, we obtain from the first identity in Definition 1.19 that

$$\begin{aligned}
a_{(1)} \otimes a_{(2)}S(a_{(3)}) &= \sum_{r,s,t,u} a_{(1)\left(\begin{smallmatrix} k & l \\ r & s \end{smallmatrix}\right)} \otimes a_{(2)\left(\begin{smallmatrix} r & s \\ t & u \end{smallmatrix}\right)} S(a_{(3)\left(\begin{smallmatrix} t & u \\ m & n \end{smallmatrix}\right)}) \\
&= \sum_{r,t} a_{(1)\left(\begin{smallmatrix} k & l \\ r & n \end{smallmatrix}\right)} \otimes \left(\sum_u a_{(2)\left(\begin{smallmatrix} r & n \\ t & u \end{smallmatrix}\right)} S(a_{(3)\left(\begin{smallmatrix} t & u \\ m & n \end{smallmatrix}\right)})\right) \\
&= \sum_{r,t} a_{(1)\left(\begin{smallmatrix} k & l \\ r & n \end{smallmatrix}\right)} \otimes \left(\delta_{r,m}\epsilon(a_{(2)\left(\begin{smallmatrix} m & n \\ m & n \end{smallmatrix}\right)})\mathbf{1}\begin{pmatrix} r \\ t \end{pmatrix}\right) \\
&= a \otimes \lambda_m \\
&= \Delta(1)(a \otimes 1).
\end{aligned}$$

The second identity is proven similarly. $\square$

Proposition 1.22. *A collection of maps $S : A(K) \to A(K^{\circ\bullet})$ defines an antipode for a partial bialgebra $\mathscr{A}$ if and only if the total map $S : A \to A$ defines an antipode for the total weak multiplier bialgebra (A, Δ) in the sense of [3, Section 6].*

Proof. We verify that the total antipode $S : A \to A$ satisfies the conditions of an antipode as stated in [3, Theorem 6.8.(2)]. In fact, the two first identities of their Theorem 6.8.(2) are precisely our identities in Lemma 1.21. The third and last identity of their Theorem 6.8.(2) says that formally one should have

$$\sum_k S(\rho_k a)\lambda_k = S(a), \quad \forall a \in A,$$

but this follows immediately from the way S acts on homogeneous components. □

From the remarks following [3, Theorem 6.8.(2)], we conclude the following.

Corollary 1.23. *If $\mathscr{A}$ is a partial Hopf algebra, its total algebra A with total comultiplication Δ forms a weak multiplier Hopf algebra in the sense of [14, Definition 1.14].*

From [3, Theorem 6.12 and Corollary 6.16], we obtain the following corollary. The notation $\Delta_{rs}^{\mathrm{op}}$ signifies the composition of Δ_{rs} with the switch map.

Corollary 1.24. *The map $S : A \to A$ is an antihomomorphism s.t.*

$$\Delta_{rs}(S(a)) = (S \otimes S)\Delta_{sr}^{\mathrm{op}}(a).$$

We now turn towards the structures which will allow us to build operator algebraic quantum groupoids out of our partial Hopf algebras.

Definition 1.25. A partial *-algebra $\mathscr{A}$ is a partial algebra whose total algebra A is equipped with an antilinear, antimultiplicative involution

$$* : A \to A, \quad a \mapsto a^*$$

such that the $\mathbf{1}_k$ are selfadjoint for all k in the object set.

One can of course give an alternative definition directly in terms of the partial algebra structure by requiring that we are given antilinear maps $A(k, l) \to A(l, k)$ satisfying the obvious antimultiplicativity and involution properties.

Definition 1.26. An I-partial *-bialgebra $\mathscr{A}$ is an I-partial bialgebra whose underlying partial algebra has been endowed with a partial *-algebra structure for which

$$\Delta_{rs}(a)^* = \Delta_{sr}(a^*), \qquad \forall K \in M_2(I), \forall a \in A(K), \forall r, s \in I.$$

A partial Hopf *-algebra is a partial bialgebra which is at the same time a partial *-bialgebra and a partial Hopf algebra.

Remark 1.27. 1. By uniqueness of the counit [3, Theorem 2.8], it follows automatically that ϵ satisfies $\epsilon(a^*) = \overline{\epsilon(a)}$ for all a.

2. From [14, Proposition 4.11], it follows automatically that $S(S(a)^*)^* = a$ for all a. In particular, S is invertible.

Definition 1.28. Let $\mathscr{A}$ be a partial bi-algebra. An *invariant functional* φ for $\mathscr{A}$ consists of a functional $\varphi : A \to \mathbb{C}$ such that for all $a \in A\binom{k\ \ l}{m\ \ n}$ and all b, one has

$$(\mathrm{id} \otimes \varphi)((b \otimes 1)\Delta(a)) = \varphi(a)b\lambda_k,$$

and

$$(\varphi \otimes \mathrm{id})(\Delta(a)(1 \otimes b)) = \varphi(a)\rho_n b.$$

We call φ *normalized* if $\varphi(\mathbf{1}\begin{pmatrix} k \\ k \end{pmatrix}) = 1$ for all k.

Remark 1.29. 1. It follows from the defining property of φ (and the fact that $\lambda_k \neq 0 \neq \rho_k$ for any $k \in I$) that φ is zero on the $A(K)$ with $K_l \neq K_r$.

2. If φ is normalized, it also follows immediately from the definining property that $\varphi(\mathbf{1}\begin{pmatrix} k \\ l \end{pmatrix}) = 1$ whenever $\mathbf{1}\begin{pmatrix} k \\ l \end{pmatrix} \neq 0$.

We are finally ready to give our main definition.

Definition 1.30. A *partial compact quantum group* $\mathscr{G}$ is a partial Hopf *-algebra $\mathscr{A} = P(\mathscr{G})$ with a *positive* normalized invariant functional φ, that is $\varphi(a^*a) \geqslant 0$ for all $a \in A$. We also say that $\mathscr{G}$ is the partial compact quantum group *defined by* $\mathscr{A}$.

Remark 1.31. Following [7], we could also have called our objects *compact quantum groups of face type*, but we feel this gives the wrong impression when the base algebra is infinite dimensional (i.e. the object set is not compact). When referring to partial compact quantum groups, we feel that it is better reflected that only the *parts* of this object are to be considered compact, not the total object.

We will need the following lemma at some point, which is an almost verbatim transcription of the argument in [12, Proposition 3.4].

Lemma 1.32. *Let $\mathscr{A}$ be a partial Hopf algebra, and let φ be a normalized invariant functional for $\mathscr{A}$. Then φ is faithful: if $a \in A$ and $\varphi(ab) = 0$ (resp. $\varphi(ba) = 0$) for all $b \in A$, then $a = 0$.*

Proof. Suppose $\varphi(ba) = 0$ for all b, we arrive at the conclusion that for all $d \in A$ and all functionals ω on A, the element $p = (\omega \otimes \mathrm{id})((d \otimes 1)\Delta(b))$ satisfies

$$(\mathrm{id} \otimes \varphi)((1 \otimes c)\Delta(p)) = 0.$$

Continuing as in the proof of [12, Proposition 3.4], we obtain from the antipode trick that

$$\sum_n \varphi(cS(q)\rho_n)\epsilon(p\lambda_n) = 0.$$

Choosing now for c and q local units of the form $\lambda_k\rho_l$, the normalization condition on φ gives that $\epsilon(p\lambda_n) = 0$ for all n, hence $\epsilon(p) = 0$. This implies $\omega(da) = 0$. As ω and d were arbitrary, it follows that $a = 0$.

The other case follows similarly, or by considering the opposite comultiplication. $\square$

2 Representation theory

In this section, the representation theory of partial compact quantum groups is investigated. As the situation is quite similar to the case already studied by Hayashi [7], we do not always provide fully written out proofs, but only draw attention to those parts of the theory which need modification.

In what follows, the homogeneous component $A(K) = A\begin{pmatrix} k & l \\ m & n \end{pmatrix}$ of a partial bialgebra will now be mainly written as $A(K) = {}^{k}_{m}A^{l}_{n}$.

2.1 Corepresentations of partial Hopf algebras

Let $\mathscr{A}$ be an I-partial bialgebra. We denote $\mathrm{Hom}_{\mathbb{C}}(V, W)$ for the vector space of linear maps between two vector spaces.

Definition 2.1. Let $\mathscr{A}$ be an I-partial bialgebra, and let $V = \bigoplus_{k,l} {}^{k}V^{l}$ be an I^2-graded vector space. A *corepresentation* $X = ({}^{k}_{m}X^{l}_{n})_{k,l,m,n}$ of $\mathscr{A}$ on V consists of a family of elements ${}^{k}_{m}X^{l}_{n} \in {}^{k}_{m}A^{l}_{n} \otimes \mathrm{Hom}_{\mathbb{C}}({}^{m}V^{n}, {}^{k}V^{l})$ satisfying

$$(\Delta_{pq} \otimes \mathrm{id})({}^{k}_{m}X^{l}_{n}) = \left({}^{k}_{p}X^{l}_{q}\right)_{13}\left({}^{p}_{m}X^{q}_{n}\right)_{23} \tag{2.1}$$

and

$$(\epsilon \otimes \mathrm{id})({}^{k}_{m}X^{l}_{n}) = \delta_{k,m}\delta_{l,n}\,\mathrm{id}_{{}^{k}V^{l}}$$

for all possible indices.

We use here the standard leg numbering notation, e.g. $a_{12} = a \otimes 1$.

We will sometimes for convenience consider ${}^{k}_{m}X^{l}_{n}$ as a map in $\mathrm{Hom}_{\mathbb{C}}({}^{m}V^{n}, {}^{k}_{m}A^{l}_{n} \otimes {}^{k}V^{l})$.

Definition 2.2. A *morphism* T between two corepresentations (V, X) and (W, Y) of a partial bialgebra $\mathscr{A}$ is a family of linear maps

$${}^{k}T^{l} \in \mathrm{Hom}_{\mathbb{C}}({}^{k}V^{l}, {}^{k}W^{l})$$

satisfying

$$(1 \otimes {}^kT^l)\, {}^k_mX^l_n = {}^k_mY^l_n(1 \otimes {}^mT^n).$$

Definition 2.3. Let (V, X) be a corepresentation of a partial bialgebra $\mathscr{A}$. A family of subspaces

$${}^kW^l \subseteq {}^kV^l$$

is called *invariant* if

$$(1 \otimes {}^kQ^l)\, {}^k_mX^l_n(1 \otimes {}^mP^n) = 0,$$

where ${}^mP^n : {}^mW^n \to {}^mV^n$ is the inclusion map and ${}^kQ^l : {}^kV^l \to {}^kV^l/{}^kW^l$ denotes the quotient map.

The following analogue of Schur's Lemma holds.

Lemma 2.4. *Let T be a morphism of corepresentations (V, X) and (W, Y). Then $\bigoplus_{k,l} \ker {}^kT^l$ and $\bigoplus_{k,l} \operatorname{img} {}^kT^l$ are invariant subspaces of V and W, respectively.*

In particular, if (V, X) and (W, Y) are irreducible, then a morphism T from V to W either has all ${}^kT^l = 0$ or all ${}^kT^l$ isomorphisms.

Definition 2.5. Let $\mathscr{A}$ be a partial bialgebra. We denote by $\mathrm{Corep}(\mathscr{A})$ the category of corepresentations of $\mathscr{A}$ with morphisms as in Definition 2.2.

Clearly $\mathscr{A}$ is a $\mathbb{C}$-linear category. We will be interested in a full subcategory of it.

Definition 2.6. An I^2-graded vector space $V = \bigoplus_{k,l} {}^kV^l$ is called *separately finitely supported (sfs)* if ${}^kV^l = \{0\}$ for almost all k (resp. almost all l) when l (resp. k) is fixed.

If $\mathscr{A}$ is an I-partial bialgebra, a corepresentation X will be called sfs if its carrier I^2-graded vector space is sfs.

We will denote by $\mathrm{Corep}_{\mathrm{sfs}}(\mathscr{A})$ the full category of $\mathrm{Corep}(\mathscr{A})$ consisting of sfs corepresentations.

The next lemma provides one with a tensor product structure on the category $\mathrm{Corep}_{\mathrm{sfs}}(\mathscr{A})$.

Lemma 2.7. *Let X and Y be two sfs copresentations on respective I^2-graded vector spaces V and W. Put*

$${}^k(V \underset{I}{\otimes} W)^m = \oplus_l \left({}^kV^l \otimes {}^lW^m \right),$$

and consider their direct sum $V \underset{I}{\otimes} W$ with its natural I^2-grading. Then $V \underset{I}{\otimes} W$ is sfs, the sum

$${}^k_m(X \underset{I}{\textcircled{\scriptsize T}} Y)^p_q := \sum_{l,n} \left({}^k_mX^l_n \right)_{12} \left({}^l_nY^p_q \right)_{13}$$

has only finitely many non-zero terms, and

$${}^k_m(X \underset{I}{\textcircled{\scriptsize T}} Y)^p_q \in {}^k_mA^p_q \otimes \mathrm{Hom}_{\mathbb{C}}({}^m(V \underset{I}{\otimes} W)^q,\ {}^k(V \underset{I}{\otimes} W)^p)$$

defines a corepresentation.

Proof. It is trivially checked that $V \underset{I}{\otimes} W$ is sfs. Then for k, m fixed, there are only finitely many l, n for which ${}^{k}_{m}X^{l}_{n}$ is non-zero. This shows that the sum defining ${}^{k}_{m}(X \underset{I}{\textcircled{\scriptsize T}} Y)^{p}_{q}$ is in fact finite. Using the natural inclusions

$$\begin{aligned}
&{}^{k}_{m}A^{p}_{q} \otimes \mathrm{Hom}_{\mathbb{C}}({}^{m}V^{n}, {}^{k}V^{l}) \otimes \mathrm{Hom}_{\mathbb{C}}({}^{n}W^{q}, {}^{l}W^{p}) \\
&\qquad\qquad \subseteq {}^{k}_{m}A^{p}_{q} \otimes \mathrm{Hom}_{\mathbb{C}}({}^{m}V^{n} \otimes {}^{n}W^{q}, {}^{k}V^{l} \otimes {}^{l}W^{p}) \\
&\qquad\qquad \subseteq {}^{k}_{m}A^{p}_{q} \otimes \mathrm{Hom}_{\mathbb{C}}({}^{m}(V \underset{I}{\otimes} W)^{q}, {}^{k}(V \underset{I}{\otimes} W)^{p}),
\end{aligned}$$

we see that ${}^{k}_{m}(X \underset{I}{\textcircled{\scriptsize T}} Y)^{p}_{q}$ is indeed an element of ${}^{k}_{m}A^{p}_{q} \otimes \mathrm{Hom}_{\mathbb{C}}({}^{m}(V \underset{I}{\otimes} W)^{q}, {}^{k}(V \underset{I}{\otimes} W)^{p})$. The fact that ${}^{k}_{m}(X \underset{I}{\textcircled{\scriptsize T}} Y)^{p}_{q}$ is a corepresentation then follows straightforwardly from the multiplicativity of Δ and the weak multiplicativity of ϵ. $\square$

Corollary 2.8. *The category* $\mathrm{Corep}_{\mathrm{sfs}}(\mathscr{A})$ *with the above tensor product* $\underset{I}{\textcircled{\scriptsize T}}$ *forms a tensor category.*

Proof. Clear. We only remark that the unit object of the category is given by the graded vector space $\mathbb{C}_I$ with components $\delta_{kl}\mathbb{C}$ and associated corepresentation

$${}^{k}_{m}X^{l}_{n} = \delta_{m,n}\delta_{k,l}\mathbf{1}\begin{pmatrix} k \\ m \end{pmatrix}.$$

$\square$

The following lemma shows that when $\mathscr{A}$ is a partial *Hopf* algebra, an sfs corepresentation is invertible.

Lemma 2.9. *Let* $\mathscr{A}$ *be a partial Hopf algebra, and let* X *be an sfs corepresentation of* $\mathscr{A}$ *on an* I^2*-graded vector space* V*. Then the elements*

$${}^{k}_{m}Z^{l}_{n} = (S \otimes \mathrm{id})({}^{n}_{l}X^{m}_{k}) \in {}^{k}_{m}A^{l}_{n} \otimes \mathrm{Hom}_{\mathbb{C}}({}^{l}V^{k}, {}^{n}V^{m})$$

are the unique elements such that

$$\sum_{k} {}^{l}_{n'}Z^{k}_{m}\, {}^{k}_{m}X^{l}_{n} = \delta_{n,n'}\mathbf{1}\begin{pmatrix} l \\ n \end{pmatrix} \otimes \mathrm{id}_{{}^{m}V^{n}}, \quad \sum_{n} {}^{k}_{m}X^{l}_{n}\, {}^{l}_{n}Z^{k'}_{m} = \delta_{k,k'}\mathbf{1}\begin{pmatrix} k \\ m \end{pmatrix} \otimes \mathrm{id}_{{}^{k}V^{l}}. \tag{2.2}$$

In particular, they satisfy

$$(\Delta_{pq} \otimes \mathrm{id})({}^{k}_{m}Z^{l}_{n}) = ({}^{p}_{m}Z^{q}_{n})_{23}\,({}^{p}_{k}Z^{q}_{l})_{13}.$$

Note that the sums in (2.2) are finite because $\mathcal{H}$ is sfs.

Proof. Let ${}^{k}_{m}Z^{l}_{n} = (S \otimes \mathrm{id})({}^{k}_{m}X^{l}_{n})$, and write the multiplication of A as M_A. Then the antipode axiom implies

$$\begin{aligned}
\sum_k {}^{l}_{n'}Z^{k}_{m}\, {}^{k}_{m}X^{l}_{n} &= \sum_k (S \otimes \mathrm{id})({}^{m}_{k}X^{n'}_{l})\, {}^{k}_{m}X^{l}_{n} \\
&= \sum_k (M_A \otimes \mathrm{id})(S \otimes \mathrm{id} \otimes \mathrm{id})(({}^{m}_{k}X^{n'}_{l})_{13}({}^{k}_{m}X^{l}_{n})_{23}) \\
&= (M_A \otimes \mathrm{id})(S \otimes \mathrm{id} \otimes \mathrm{id}) \sum_k ((\Delta_{kl} \otimes \mathrm{id})({}^{m}_{m}X^{n'}_{n}) \\
&= \delta_{n,n'} \mathbf{1}\binom{l}{n} \otimes (\epsilon \otimes \mathrm{id})({}^{m}_{m}X^{n}_{n}) \\
&= \delta_{n,n'} \mathbf{1}\binom{l}{n} \otimes \mathrm{id}_{{}^{m}V^{n}}\,.
\end{aligned}$$

The second identity follows similarly.

Uniqueness follows immediately from the identities (2.2). □

Definition 2.10. Let $\mathscr{A}$ and X be as in Lemma 2.9. The element Z, when it exists, will be denoted X^{-1}.

Remark 2.11. Using (2.2), one finds that a collection of linear maps ${}^{k}T^{l} \in \mathcal{L}({}^{k}V^{l}, {}^{k}W^{l})$ forms a morphism between two sfs corepresentations (V, X) and (W, Y) if and only if

$$\sum_m {}^{n}_{l'}(Y^{-1})^{m}_{k}(1 \otimes {}^{m}T^{n})\, {}^{m}_{k}X^{n}_{l} = \delta_{l,l'} \mathbf{1}\binom{n}{l} \otimes {}^{k}T^{l}, \tag{2.3}$$

$$\sum_n {}^{k'}_{m}Y^{l}_{n}(1 \otimes {}^{m}T^{n})\, {}^{l}_{n}(X^{-1})^{k}_{m} = \delta_{k,k'} \mathbf{1}\binom{m}{k} \otimes {}^{k}T^{l}. \tag{2.4}$$

For example, if the first equation holds, then

$${}^{m}_{k}Y^{n}_{l}(1 \otimes {}^{k}T^{l}) = \sum_m {}^{m}_{k}Y^{n}_{l}\, {}^{n}_{l}(Y^{-1})^{m}_{k}(1 \otimes {}^{m}T^{n})\, {}^{m}_{k}X^{n}_{l} = (\mathbf{1}\binom{k}{m} \otimes {}^{m}T^{n})\, {}^{m}_{k}X^{n}_{l}.$$

Conversely, if T is a morphism, then

$$\sum_m {}^{n}_{l'}(Y^{-1})^{m}_{k}(1 \otimes {}^{m}T^{n})\, {}^{m}_{k}X^{n}_{l} = \sum_m {}^{n}_{l'}(Y^{-1})^{m}_{k}\, {}^{m}_{k}Y^{n}_{l}(1 \otimes {}^{k}T^{l}) = \delta_{l',l} \mathbf{1}\binom{n}{l} \otimes {}^{k}T^{l}.$$

To actually have a tensor category with duality, we need something stronger than the sfs condition.

Definition 2.12. An I^2-graded vector space V is called *seperately finite dimensional (sfd)* if $\oplus_l {}^{k}V^{l}$ (resp. $\oplus_l {}^{k}V^{l}$) is finite dimensional for k (resp. l) fixed. Correspondingly, we talk of an sfd corepresentation of a partial bialgebra $\mathscr{A}$, and we then denote by $\mathrm{Corep}_{\mathrm{sfd}}(\mathscr{A})$ the full subcategory of $\mathrm{Corep}_{\mathrm{sfs}}(\mathscr{A})$ consisting of sfd representations.

One easily sees that $\mathrm{Corep}_{\mathrm{sfd}}(\mathscr{A})$ is closed under $\underset{I}{\textcircled{\scriptsize T}}$.

Lemma 2.13. *Let $\mathscr{A}$ be a partial Hopf algebra. Then* $\mathrm{Corep}_{\mathrm{sfd}}(\mathscr{A})$ *is a tensor category with left duality.*

Proof. Let X be an sfd corepresentation on a bigraded vector space V. Put

$$^{k}(V^{*})^{l} = (\,^{k}V^{l})^{*},$$

and let V^{*} denote their direct sum bigraded vector space. Using the natural contravariant identification

$$\mathrm{Hom}_{\mathbb{C}}(\,^{l}V^{k},\, ^{n}V^{m}) \cong \mathrm{Hom}_{\mathbb{C}}(\,^{m}(V^{*})^{n},\, ^{k}(V^{*})^{l}),$$

we see (by means of Lemma 2.9) that X^{-1} gets transformed into a corepresentation X^{d} on V^{*}.

We claim that X^{d} is the left dual of X. To see this, consider the evaluation maps

$$^{k}T^{m} : \,^{k}(V^{*} \underset{I}{\otimes} V)^{m} \supseteq (\,^{l}V^{k})^{*} \otimes \,^{l}V^{m} \to \delta_{k,m}\mathbb{C} = \,^{k}\mathbb{C}_{I}^{m}.$$

Then from Lemma 2.9, we obtain

$$\begin{aligned}(1 \otimes \,^{k}T^{p})\Big(\sum_{l,n} \big(\,^{k}_{m}(X^{d})^{l}_{n}\big)_{12}\big(\,^{l}_{n}X^{p}_{q}\big)_{13}\Big) &= \delta_{p,k}(\mathrm{id} \otimes \,^{m}T^{m})\left(\sum_{l,n} \,^{k}_{m}(X^{-1})^{l}_{n}\,\,^{l}_{n}X^{k}_{q}\right)_{13} \\ &= \delta_{p,k}\mathbf{1}\begin{pmatrix} k \\ q \end{pmatrix} \otimes \,^{m}T^{q}.\end{aligned}$$

Hence the $^{k}T^{l}$ define an intertwiner between $V^{*} \underset{I}{\otimes} V$ and $\mathbb{C}_{I}$. Similarly, the maps

$$^{k}R^{k} : \,^{k}\mathbb{C}_{I}^{k} = \mathbb{C} \to \,^{k}(V \underset{I}{\otimes} V^{*})^{k}, \quad 1 \mapsto \sum_{l,i} \,^{k}v_{i}^{l} \otimes \,^{l}\omega_{i}^{k},$$

where the $\{\,^{k}v_{i}^{l} \mid i\}$ and $\{\,^{l}\omega_{i}^{k} \mid i\}$ form a dual basis of $^{k}V^{l}$, can be shown to form an intertwiner. It is then easy to check that T and R make X^{d} into the left dual of X. $\square$

Let us now enhance our partial Hopf algebras to partial compact quantum groups. One then considers corepresentations on sfd bigraded *Hilbert spaces* such that the inverse of the corepresentation coincides with its adjoint. More precisely, we have the following definition. We denote $B(\mathcal{H}, \mathcal{G})$ for the linear space of bounded morphisms between Hilbert spaces.

Definition 2.14. Let $\mathscr{A}$ define a partial compact quantum group. We call an sfd corepresentation $(\mathcal{H}, X)$ on an sfd I^{2}-graded Hilbert space $\mathcal{H}$ *unitary* if

$$^{k}_{m}(X^{-1})^{l}_{n} = (\,^{l}_{n}X^{k}_{m})^{*} \quad \text{in } \,^{k}_{m}A^{l}_{n} \otimes B(\,^{l}\mathcal{H}^{k},\, ^{n}\mathcal{H}^{m}).$$

Remark 2.15. 1. In the Hilbert space setting, it is more natural to let $\mathcal{H}$ be the *closed* (instead of the purely algebraic) direct sum of all (finite-dimensional) ${}^{k}\mathcal{H}^{l}$. This does not change the notion of corepresentation, which had a local definition.

2. Concerning morphisms, we will say a collection of ${}^{k}T^{l}$ defines a *bounded* intertwiner or morphism if the total operator $T = \oplus\, {}^{k}T^{l}$ is bounded. We will denote by $\mathrm{Corep}_{\mathrm{sfd},u}(\mathscr{A})$ the category of unitary sfd corepresentations with arbitrary morphisms, and $\mathrm{Corep}^{\infty}_{\mathrm{sfd},u}(\mathscr{A})$ for the category with bounded morphisms.

Our aim now is to show that every irreducible sfd corepresentation is equivalent to a unitary one. We show this by embedding the corepresentation into a restriction of the regular corepresentation.

Example 2.16. Let $\mathscr{A}$ define a partial compact quantum group with normalized positive invariant functional ϕ.

Let ${}^{m}\mathcal{H}^{n} \subseteq \bigoplus_{k,l}\, {}^{k}_{m}A^{l}_{n}$ be finite dimensional subspaces satisfying

$$\Delta^{\mathrm{op}}_{pq}({}^{m}\mathcal{H}^{n}) \subseteq {}^{p}_{m}A^{q}_{n} \otimes {}^{p}\mathcal{H}^{q}.$$

for all indices. Equip each ${}^{k}\mathcal{H}^{l}$ with the scalar product $\langle a|b\rangle := \phi(a^{*}b)$. By Lemma 1.32, these are finite-dimensional Hilbert spaces. Take the Hilbert space direct sum $\mathcal{H} := \bigoplus_{k,l}\, {}^{k}\mathcal{H}^{l}$. Define

$${}^{k}_{m}V^{l}_{n} \in \mathrm{Hom}_{\mathbb{C}}({}^{m}\mathcal{H}^{n},\, {}^{k}_{m}A^{l}_{n} \otimes {}^{k}\mathcal{H}^{l}) \cong {}^{k}_{m}A^{l}_{n} \otimes \mathcal{B}({}^{m}\mathcal{H}^{n},\, {}^{k}\mathcal{H}^{l})$$

by the equation

$${}^{k}_{m}V^{l}_{n}a = \Delta^{\mathrm{co}}_{kl}(a).$$

Lemma 2.17. *The couple $(\mathcal{H}, V)$ defines a unitary corepresentation.*

Proof. It is clear that V defines a corepresentation. It then suffices to prove that

$$\sum_{k} ({}^{k}_{m}V^{l}_{n'})^{*}\, {}^{k}_{m}V^{l}_{n} = \delta_{n,n'} \mathbf{1}\binom{l}{n} \otimes \mathrm{id}_{\,{}^{m}\mathcal{H}^{n}}\,. \tag{2.5}$$

Take $a \in {}^{m}\mathcal{H}^{n}$ and $b \in {}^{m}\mathcal{H}^{n'}$. Then writing

$$\Lambda(a) : \mathbb{C} \to {}^{m}\mathcal{H}^{n}, \quad 1 \mapsto a,$$

and similarly for b, we compute

$$\begin{aligned}
(1\otimes\Lambda(b)^*)\left(\sum_k ({}^{k}_{m}V^{l}_{n'})^* \, {}^{k}_{m}V^{l}_{n}\right)(1\otimes\Lambda(a)) &= \sum_k (\mathrm{id}\otimes\phi)(\Delta^{\mathrm{op}}_{kl}(b)^*\Delta^{\mathrm{op}}_{kl}(a)) \\
&= \sum_k (\mathrm{id}\otimes\phi)(\Delta^{\mathrm{op}}_{lk}(b^*)\Delta^{\mathrm{op}}_{kl}(a)) \\
&= (\mathrm{id}\otimes\phi)(\Delta^{\mathrm{op}}_{ll}(b^*a)) \\
&= (\phi\otimes\mathrm{id})(\Delta_{ll}(b^*a)) \\
&= \phi(b^*a)\mathbf{1}\binom{l}{n} \\
&= \delta_{n',n}\mathbf{1}\binom{l}{n}\otimes\Lambda(b)^*\Lambda(a).
\end{aligned}$$

This proves (2.5). □

We will call $(\mathcal{H},V)$ the *sfd restriction of the regular corepresentation* determined by the family $({}^{k}\mathcal{H}^{l})_{k,l}$.

In the following, we will use the notation

$$\omega_{\xi,\eta}: B(\mathcal{H},\mathcal{G})\to\mathbb{C},\quad x\mapsto\langle\xi,x\eta\rangle,\quad \xi\in\mathcal{G},\eta\in\mathcal{H}.$$

Lemma 2.18. *Let $\mathscr{A}$ define a partial compact quantum group. Let $(\mathcal{H},X)$ be an sfd corepresentation on a Hilbert space, and let $\xi\in{}^{k}\mathcal{H}^{l}$. Then the family of finite-dimensional subspaces*

$${}^{m}\mathcal{K}^{n} = \{(\mathrm{id}\otimes\omega_{\xi,\eta})({}^{k}_{m}X^{l}_{n}) : \eta\in{}^{m}\mathcal{H}^{n}\}\subseteq {}^{k}_{m}A^{l}_{n}$$

defines an sfd restriction $(\mathcal{K},V)$ of the regular corepresentation, and the family of maps

$${}^{m}T^{n}_{(\xi)}: {}^{m}\mathcal{H}^{n}\to{}^{m}\mathcal{K}^{n},\ \eta\mapsto(\mathrm{id}\otimes\omega_{\xi,\eta})({}^{k}_{m}X^{l}_{n}),$$

is a morphism from $(\mathcal{H},X)$ to $(\mathcal{K},V)$ in $\mathrm{Corep}_{\mathrm{sfd},u}(\mathscr{A})$.

Note that the family $({}^{m}\mathcal{K}^{n})_{m,n}$ is sfd because $({}^{m}\mathcal{H}^{n})_{m,n}$ is.

Proof. Both assertions follow from the fact that for all $\eta\in{}^{p}\mathcal{H}^{q}$,

$$\begin{aligned}
\Delta^{\mathrm{op}}_{pq}({}^{m}T^{n}_{(\xi)}(\eta)) &= \Delta^{\mathrm{op}}_{pq}\big((\mathrm{id}\otimes\omega_{\xi,\eta})({}^{k}_{m}X^{l}_{n})\big) \\
&= (\mathrm{id}\otimes\mathrm{id}\otimes\omega_{\xi,\eta})\big(({}^{k}_{p}X^{l}_{q})_{23}({}^{p}_{m}X^{q}_{n})_{13}\big) \\
&= (1\otimes{}^{p}T^{q}_{(\xi)})\,{}^{p}_{m}X^{q}_{n}\eta.
\end{aligned}$$

□

Proposition 2.19. *Let $\mathscr{A}$ define a partial compact quantum group. Then every irreducible sfd corepresentation on a Hilbert space is equivalent to a unitary sfd corepresentation.*

Proof. Let $(\mathcal{H}, X)$ be an irreducible sfd corepresentation. Then for some k, l and $\xi \in {}^k\mathcal{H}^l$, the operator $T_{(\xi)}$ defined in Lemma 2.18 has to be non-zero and hence, by Schur's Lemma, injective. Thus, it forms an equivalence between $(\mathcal{H}, X)$ and a sub-corepresentation of an sfd restriction of the regular corepresentation, which is unitary by Example 2.16. $\square$

Our next goal is to obtain the analogue of Schur's orthogonality relations for matrix coefficients of corepresentations.

Definition 2.20. Let $\mathscr{A}$ define a partial compact quantum group. The space of *matrix coefficients* $\mathcal{C}(X)$ of an sfd corepresentation $(\mathcal{H}, X)$ is the sum of the subspaces

$$ {}^k_m\mathcal{C}(X)^l_n = \left\{ (\mathrm{id} \otimes \omega_{\xi,\eta})({}^k_m X^l_n) \mid \xi \in {}^k\mathcal{H}^l, \eta \in {}^m\mathcal{H}^n \right\} \subseteq {}^k_m A^l_n. $$

Lemma 2.21. *Every sfd unitary corepresentation $(X, \mathcal{H})$ of $\mathscr{A}$ decomposes into a direct sum of irreducible sfd unitary corepresentations.*

Proof. From the unarity assumption, it follows immediately that an invariant subspace of $\mathcal{H}$ also has an invariant orthogonal complement. Hence irreducibility and indecomposability of unitary corepresentations coincide. More generally, one deduces that the bounded self-interwiners of $\mathcal{H}$ form a (von Neumann) *-algebra.

Let us now first show that the trivial representation decomposes into irreducibles. Let I be the object set of $\mathscr{A}$, and say $k \sim l$ if $\mathbf{1}\begin{pmatrix} k \\ l \end{pmatrix} \neq 0$. Then $\sim$ is an equivalence relation: as

$$ \Delta_{ll}(\mathbf{1}\begin{pmatrix} k \\ m \end{pmatrix}) = \mathbf{1}\begin{pmatrix} k \\ l \end{pmatrix} \otimes \mathbf{1}\begin{pmatrix} l \\ m \end{pmatrix}, $$

the relation $\sim$ is transitive. As $S(\mathbf{1}\begin{pmatrix} k \\ l \end{pmatrix}) = \mathbf{1}\begin{pmatrix} l \\ k \end{pmatrix}$, we have that $\sim$ is symmetric. And as $\varepsilon(\mathbf{1}\begin{pmatrix} k \\ k \end{pmatrix}) = 1$, we also have that $\sim$ is reflexive.

Let then $I = \sqcup_{\alpha \in \mathscr{I}} I_\alpha$ be a labeled partition associated to $\sim$. Define $\mathbb{C}_{I_\alpha} \subseteq \mathbb{C}_I$ as the linear span of the homogeneous components with index in α. It is clear then that the $\mathbb{C}_{I_\alpha}$ are invariant and irreducible.

Consider now a general corepresentation $(X, \mathcal{H})$. Let ${}_\alpha\mathcal{H}_\beta$ be the closed linear span of the homogeneous components with index in $\alpha \times \beta$. As we can identify

$$ {}_\alpha\mathcal{H}_\beta \cong \mathbb{C}_{I_\alpha} \underset{I}{\oplus} \mathcal{H} \underset{I}{\oplus} \mathbb{C}_{I_\beta}, $$

we see that ${}_\alpha\mathcal{H}_\beta$ is an invariant subspace of $\mathcal{H}$. Hence we may as well suppose that $\mathcal{H} = {}_\alpha\mathcal{H}_\beta$.

But let then T be a bounded self-intertwiner of $\mathcal{H}$. Then from the two equations in Remark 2.11, we see that $T \to {}^kT^l$ is injective for any choice of $k \in \alpha, l \in \beta$. It follows that the algebra of self-intertwiners of $\mathcal{H}$ is finite-dimensional. We then immediately conclude that $\mathcal{H}$ is a finite direct sum of irreducible invariant subspaces. $\square$

Proposition 2.22. *Assume that $\mathscr{A}$ defines a partial compact quantum group. Then the total algebra A is the sum of the matrix coefficients of irreducible unitary sfd corepresentations.*

Proof. Let $a \in {}_{m}^{k}A_{n}^{l}$. Then $\Delta_{pq}^{\mathrm{co}}(a) \in {}_{m}^{p}A_{n}^{q} \otimes {}_{p}^{k}A_{q}^{l}$, and the subspace

$$ {}^{p}\mathcal{H}^{q} := \{(\omega \otimes \mathrm{id})(\Delta_{pq}^{\mathrm{co}}(a)) : \omega \in \mathrm{Hom}_{\mathbb{C}}({}_{m}^{p}A_{n}^{q}, \mathbb{C})\} \subseteq {}_{p}^{k}A_{q}^{l} $$

has finite dimension. Since a is fixed, the $({}^{p}\mathcal{H}^{q})_{p,q}$ are in fact sfd. Using co-associativity, one checks that this family defines an sfd restriction $(\mathcal{H}, V)$ of the regular corepresentation. Evidently, $a \in \mathcal{C}(V)$. Decomposing $(\mathcal{H}, V)$, we find that a is contained in the sum of matrix coefficients of unitary irreducible corepresentations. □

The key to the orthogonality relations is the following averaging procedure.

Lemma 2.23. *Let $\mathscr{A}$ define a partial compact quantum group, and let ϕ be any invariant functional for $\mathscr{A}$. Let $(\mathcal{H}, X)$ and $(\mathcal{K}, Y)$ be sfd corepresentations of $\mathscr{A}$ and let T be a family of operators ${}^{k}T^{l} \in \mathcal{B}({}^{k}\mathcal{H}^{l}, {}^{k}\mathcal{K}^{l})$.*

Then for any fixed n, the family of linear maps

$$ {}^{k}\check{T}_{n}^{l} := \sum_{m} (\phi \otimes \mathrm{id})({}_{l}^{n}(Y^{-1})_{k}^{m}(1 \otimes {}^{m}T^{n})\, {}_{k}^{m}X_{l}^{n}) $$

define a morphism $\check{T}_{n}$ from $(\mathcal{H}, X)$ to $(\mathcal{K}, Y)$ in $\mathrm{Corep}_{\mathrm{sfd},u}(\mathscr{A})$.

Similarly, for fixed m, the

$$ {}^{k}\hat{T}_{m}^{l} := \sum_{n} (\phi \otimes \mathrm{id})({}_{m}^{k}Y_{n}^{l}(1 \otimes {}^{m}T^{n})\, {}_{n}^{l}(X^{-1})_{m}^{k}) $$

define a morphism from $(\mathcal{H}, X)$ to $(\mathcal{K}, Y)$.

Proof. Using Remark 2.11, the assertion concerning the $\check{T}$ follows from the calculation

$$
\begin{aligned}
&\sum_{m} {}_{l'}^{n}(Y^{-1})_{k}^{m}(1 \otimes {}^{m}\check{T}_{q}^{n})\, {}_{k}^{m}X_{l}^{n} = \\
&= \sum_{m,p} (\phi \otimes \mathrm{id} \otimes \mathrm{id}) \left(\left({}_{l'}^{n}(Y^{-1})_{k}^{m} \right)_{23} \left({}_{n}^{q}(Y^{-1})_{m}^{p} \right)_{13} (1 \otimes 1 \otimes {}^{p}T^{q}) \left({}_{m}^{p}X_{n}^{q} \right)_{13} \left({}_{k}^{m}X_{l}^{n} \right)_{23} \right) \\
&= \sum_{m,p} (((\phi \otimes \mathrm{id}) \circ \Delta_{mn}) \otimes \mathrm{id}) \left({}_{l'}^{q}(Y^{-1})_{k}^{p}(1 \otimes {}^{p}T^{q})\, {}_{k}^{p}X_{l}^{q} \right) \\
&= \delta_{l',l} \mathbf{1}\binom{n}{l} \otimes \left(\sum_{p,q} (\phi \otimes \mathrm{id}) \left({}_{l}^{q}(Y^{-1})_{k}^{p}(1 \otimes {}^{p}T^{q})\, {}_{k}^{p}X_{l}^{q} \right) \right) \\
&= \delta_{l,l'} \mathbf{1}\binom{n}{l} \otimes {}^{k}\check{T}_{q}^{l},
\end{aligned}
$$

where we used the relation $\phi({}_{l'}^{\ r}A_l^s) = 0$ for $l' \neq l$ for the last equality.

A similar calculation proves the assertion concerning the $\hat{T}$. $\square$

The first part of the orthogonality relations concerns matrix coefficients of inequivalent irreducible corepresentations.

Proposition 2.24. *Let $(\mathcal{H}, X)$ and $(\mathcal{K}, Y)$ be inequivalent unitary irreducible sfd corepresentations, and let ϕ be an invariant functional for (A, Δ). Then*

$$\phi(S(b)a) = \phi(b^*a) = \phi(bS(a)) = \phi(ba^*) = 0$$

for all $a \in \mathcal{C}(X), b \in \mathcal{C}(Y)$.

Proof. Let $a = (\mathrm{id} \otimes \omega_{\xi,\xi'})({}_{m}^{k}X_n^l)$ and $b = (\mathrm{id} \otimes \omega_{\eta,\eta'})({}_{r}^{p}Y_s^q)$, where $\xi \in {}^k\mathcal{H}^l, \xi' \in {}^m\mathcal{H}^n$ and $\eta \in {}^p\mathcal{K}^q, \eta' \in {}^r\mathcal{K}^s$.

If $(p, q, r, s) \neq (m, n, k, l)$, then clearly $\phi(S(b)a) = 0 = \phi(bS(a))$.

If $(p, q, r, s) = (m, n, k, l)$, then Lemma 2.23, applied to the family ${}^pT^q = \delta_{p,k}\delta_{q,l}|\eta'\rangle\langle\xi|$, yields morphisms $\check{T}_l, \hat{T}_k$ from $(\mathcal{H}, X)$ to $(\mathcal{K}, Y)$ which necessarily are 0. Inserting the definition of $\check{T}_l$, we find

$$\begin{aligned}
\phi(S(b)a) &= \phi\big((S \otimes \omega_{\eta,\eta'})({}_{k}^{m}Y_l^n) \cdot (\mathrm{id} \otimes \omega_{\xi,\xi'})({}_{m}^{k}X_n^l)\big) \\
&= (\phi \otimes \mathrm{id})\Big((1 \otimes \langle\eta|)\ {}_{n}^{l}(Y^{-1})_m^k(1 \otimes |\eta'\rangle\langle\xi|)\ {}_{m}^{k}X_n^l(1 \otimes |\xi'\rangle)\Big) \\
&= (1 \otimes \langle\eta|)\ {}^m\check{T}_l^n(1 \otimes |\xi'\rangle) = 0.
\end{aligned}$$

A similar calculation involving $\hat{T}$ shows that $\phi(bS(a)) = 0$.

Using the relation $X^* = X^{-1} = (S \otimes \mathrm{id})(X)$ and $Y^* = (S \otimes \mathrm{id})(Y)$, we conclude $\phi(b^*a) = \phi(ba^*) = 0$. $\square$

The second part of the orthogonality relations concerns inner products as above but with $a, b \in \mathcal{C}(X)$ for some irreducible corepresentation X. It involves the conjugate corepresentation, which is defined as follows.

Given Hilbert spaces $\mathcal{H}, \mathcal{K}$, we denote by $\overline{\mathcal{H}}, \overline{\mathcal{K}}$ the conjugate Hilbert spaces, by $T \mapsto \overline{T}$ the canonical conjugate-linear isomorphism $\mathcal{B}(\mathcal{H}, \mathcal{K}) \to \mathcal{B}(\overline{\mathcal{H}}, \overline{\mathcal{K}})$, and by $T \mapsto T^\top := \overline{T}^*$ the linear anti-isomorphism $\mathcal{B}(\mathcal{H}, \mathcal{K}) \to \mathcal{B}(\overline{\mathcal{K}}, \overline{\mathcal{H}})$.

Lemma 2.25. *Let $\mathrm{Corep}_{\mathrm{sfd,Hilb}}(\mathscr{A})$ denote the category of (not necessarily unitary) corepresentations of $\mathscr{A}$ on sfd bigraded Hilbert spaces. Then on $\mathrm{Corep}_{\mathrm{sfd,Hilb}}(\mathscr{A})$ there exist*

1. *a covariant antilinear functor $(\mathcal{H}, X) \mapsto (\overline{\mathcal{H}}, \overline{X})$ and $T \mapsto \overline{T}$, where*

$${}^k\overline{\mathcal{H}}^l = \overline{{}^l\mathcal{H}^k}, \qquad {}_{m}^{k}\overline{X}_n^l = ({}_{n}^{l}X_m^k)^{(*\otimes\overline{(\,\cdot\,)})} = (({}_{n}^{l}X_m^k)^*)^{\mathrm{id}\otimes\top}, \qquad {}^k\overline{T}^l = \overline{{}^lT^k};$$

2. *a contravariant linear functor* $(\mathcal{H}, X) \mapsto (\overline{\mathcal{H}}, X^{S\otimes\top})$ *and* $T \mapsto T^\top$*, where*

$$^k\overline{\mathcal{H}}^l = \overline{^l\mathcal{H}^k}, \qquad {}^k_m(X^{S\otimes\top})^l_n = (S \otimes (\,\cdot\,)^\top)({}^n_l X^m_k), \qquad {}^k(T^\top)^l = ({}^l T^k)^\top;$$

3. *a covariant linear functor* $(\mathcal{H}, X) \mapsto (\mathcal{H}, X^{S^2\otimes\mathrm{id}})$ *and* $T \mapsto T$*, where the grading is unchanged and*

$${}^k_m(X^{S^2\otimes\mathrm{id}})^l_n = (S^2 \otimes \mathrm{id})({}^k_m X^l_n).$$

If $(\mathcal{H}, X)$ *is unitary, then* $\overline{X} = X^{S\otimes\top}$.

Proof. The first assertion follows immediately from the fact that $\Delta_{rs}(a^*) = \Delta_{sr}(a)^*$ and the *-compatibility of ϵ. The second assertion follows from the fact that $\Delta_{pq} \circ S = (S \otimes S) \circ \Delta^{\mathrm{op}}_{qp}$ and $\epsilon \circ S = \epsilon$. The final functor is just the square of the second functor.

The fact that $\overline{X} = X^{S\otimes\top}$ for X unitary is just by definition.

□

We call $(\overline{\mathcal{H}}, \overline{X})$ the *conjugate* of $(\mathcal{H}, X)$.

Proposition 2.26. *Let* $\mathscr{A}$ *define a partial compact quantum group, and let* ϕ *be a positive normalized invariant functional. Let* $(\mathcal{H}, X)$ *be a unitary irreducible sfd corepresentation.*

1. *The conjugate* $\overline{\mathcal{H}}$ *with the family* ${}^k_m(\overline{X}^{-*})^l_n := \big({}^l_n(\overline{X}^{-1})^k_m\big)^*$ *form an sfd corepresentation, and there exists an invertible, positive morphism* $\overline{F_X}$ *from* $(\overline{\mathcal{H}}, \overline{X})$ *to* $(\overline{\mathcal{H}}, \overline{X}^{-*})$.

2. *The family* ${}^kF^l_X := \overline{{}^l\overline{F_X}^k}$ *is an invertible, positive operator implementing a morphism from* $(\mathcal{H}, X)$ *to* $(\mathcal{H}, X^{S^2\otimes\mathrm{id}})$.

Proof. (1) By Proposition 2.19, $(\overline{\mathcal{H}}, \overline{X})$ is equivalent to a unitary corepresentation, that is, there exists a family of invertible operators ${}^kT^l \in \mathcal{B}({}^k\overline{\mathcal{H}}^l)$ such that the family

$${}^k_m Z^l_n := (1 \otimes {}^kT^l)\, {}^k_m\overline{X}^l_n (1 \otimes {}^mT^n)^{-1}$$

is a unitary corepresentation. The relation ${}^n_l(Z^{-1})^m_k = \big({}^m_k Z^n_l\big)^*$ then implies

$${}^k_m Z^l_n = (1 \otimes ({}^kT^l)^{-1})^* \big({}^l_n(\overline{X}^{-1})^k_m\big)^* (1 \otimes {}^mT^n)^*$$

and hence the family ${}^k_m(\overline{X}^{-*})^l_n := \big({}^l_n(\overline{X}^{-1})^k_m\big)^*$ is an irreducible sfd corepresentation. The maps ${}^k\overline{F}^l_X := ({}^kT^l)^*\, {}^kT^l \in \mathcal{B}({}^k\overline{\mathcal{H}}^l)$ then form an isomorphism from $(\overline{\mathcal{H}}, \overline{X})$ to $(\overline{\mathcal{H}}, \overline{X}^{-*})$.

(2) The morphism T from $(\overline{\mathcal{H}}, \overline{X})$ to $(\overline{\mathcal{H}}, Z)$ yields morphisms $\overline{T}$ from $(\mathcal{H}, X)$ to $(\mathcal{H}, \overline{Z})$ and $T^\top$ from $(\mathcal{H}, Z^{S\otimes\top})$ to $(\mathcal{H}, \overline{X}^{S\otimes\top})$. Since X and Z are unitary, $\overline{Z} = Z^{S\otimes\top}$ and $\overline{X}^{S\otimes\top} = X^{S^2\otimes\mathrm{id}}$. Thus $T^\top\overline{T} = \overline{T^*T}$ is a morphism from $(\mathcal{H}, X)$ to $(\mathcal{H}, X^{S^2\otimes\mathrm{id}})$. □

Theorem 2.27. *Let $\mathscr{A}$ define a partial compact quantum group. Let ϕ be a positive normalized invariant functional. Let $(\mathcal{H}, X)$ be a unitary irreducible sfd corepresentation of $\mathscr{A}$, and let F_X be a non-zero morphism from $(\mathcal{H}, X)$ to $(\mathcal{H}, X^{S^2\otimes\mathrm{id}})$.*

1. *The numbers $\alpha := \sum_k \mathrm{Tr}(\,^k(F_X^{-1})^l)$ and $\beta := \sum_n \mathrm{Tr}(\,^m F_X^n)$ do not depend on l or n.*

2. *For all k, l, m, n,*

$$(\phi\otimes\mathrm{id})((\,^k_m X^l_n)^*\,^k_m X^l_n) = \alpha^{-1}\mathrm{Tr}(\,^k(F_X^{-1})^l)\cdot\mathrm{id}_{\,^m\mathcal{H}^n},$$
$$(\phi\otimes\mathrm{id})(\,^k_m X^l_n(\,^k_m X^l_n)^*) = \beta^{-1}\mathrm{Tr}(\,^m(F_X)^n)\cdot\mathrm{id}_{\,^k\mathcal{H}^l}.$$

3. *Denote by Σ_{klmn} the flip map $\,^k\mathcal{H}^l\otimes\,^m\mathcal{H}^n\to\,^m\mathcal{H}^n\otimes\,^k\mathcal{H}^l$. Then*

$$(\phi\otimes\mathrm{id}\otimes\mathrm{id})((\,^k_m X^l_n)^*_{12}(\,^k_m X^l_n)_{13}) = \alpha^{-1}(\mathrm{id}_{\,^m\mathcal{H}^n}\otimes\,^k(F_X^{-1})^l)\circ\Sigma_{klmn},$$
$$(\phi\otimes\mathrm{id}\otimes\mathrm{id})((\,^k_m X^l_n)_{13}(\,^k_m X^l_n)^*_{12}) = \beta^{-1}(\,^m F_X^n\otimes\mathrm{id}_{\,^k\mathcal{H}^l})\circ\Sigma_{klmn}.$$

Proof. We prove the assertions and equations involving α in (1), (2) and (3) simultaneously; the assertions involving β follow similarly.

Consider the following endomorphism $F_{m,n,k,l}$ of $\,^m\mathcal{H}^n\otimes\,^k\mathcal{H}^l$,

$$\begin{aligned} F_{m,n,k,l} &:= (\phi\otimes\mathrm{id}\otimes\mathrm{id})((\,^k_m X^l_n)^*_{12}(\,^k_m X^l_n)_{13})\circ\Sigma_{mnkl} \\ &= (\phi\otimes\mathrm{id}\otimes\mathrm{id})\left((\,^l_n(X^{-1})^k_m)_{12}\Sigma_{klkl,23}(\,^k_m X^l_n)_{12}\right). \end{aligned}$$

By applying Lemma 2.23 with respect to the flip map Σ_{klkl}, we see that the family $(F_{m,n,k,l})_{m,n}$ is an endomorphism of $(\mathcal{H}\otimes\,^k\mathcal{H}^l, X_{12})$ and hence

$$F_{m,n,k,l} = \mathrm{id}_{\,^m\mathcal{H}^n}\otimes\,^k R^l \tag{2.6}$$

with some $\,^k R^l\in\mathcal{B}(\,^k\mathcal{H}^l)$ not depending on m, n. On the other hand,

$$\begin{aligned} F_{m,n,k,l} &= (\phi\otimes\mathrm{id}\otimes\mathrm{id})((S\otimes\mathrm{id})(\,^m_k X^n_l)_{12}(\,^k_m X^l_n)_{13})\circ\Sigma_{mnkl} \\ &= (\phi\circ S^{-1}\otimes\mathrm{id}\otimes\mathrm{id})\left(((S\otimes\mathrm{id})(\,^k_m X^l_n))_{13}((S^2\otimes\mathrm{id})(\,^m_k X^n_l))_{12}\right)\circ\Sigma_{mnkl} \\ &= (\phi\circ S^{-1}\otimes\mathrm{id}\otimes\mathrm{id})\left(((S\otimes\mathrm{id})(\,^k_m X^l_n))_{13}(\Sigma_{mnmn})_{23}((S^2\otimes\mathrm{id})(\,^m_k X^n_l))_{13}\right). \end{aligned}$$

Since $\phi\circ S^{-1}$ is an invariant functional for $\mathscr{A}$, we can again apply Lemma 2.23 and find that the family $(F_{m,n,k,l})_{k,l}$ is a morphism

$$(F_{m,n,k,l})_{k,l} : (\,^m\mathcal{H}^n\otimes\mathcal{H}, (X^{S^2\otimes\mathrm{id}})_{13})\to(\,^m\mathcal{H}^n\otimes\mathcal{H}, X_{13}).$$

Therefore,

$$F_{m,n,k,l} = \,^m T^n\otimes(\,^k F_X^l)^{-1} \tag{2.7}$$

with some ${}^{m}T^{n} \in \mathcal{B}({}^{m}\mathcal{H}^{n})$ not depending on k, l. Combining (2.6) and (2.7), we conclude that, for some $\lambda \in \mathbb{C}$,

$$F_{m,n,k,l} = \lambda(\mathrm{id}_{{}^{m}\mathcal{H}^{n}} \otimes ({}^{k}F_{X}^{l})^{-1})$$

Choose a basis $(\zeta_i)_i$ for ${}^{k}\mathcal{H}^{l}$. Then

$$\lambda \cdot \mathrm{id}_{{}^{m}\mathcal{H}^{n}} \cdot \mathrm{Tr}(({}^{k}F_{X}^{l})^{-1}) = \sum_{i} (\mathrm{id} \otimes \omega_{\zeta_i,\zeta_i})(F_{m,n,k,l}) = (\phi \otimes \mathrm{id})(({}^{k}_{m}X^{l}_{n})^{*}\, {}^{k}_{m}X^{l}_{n}).$$

Taking $n = l$ and summing over k, the relations $\sum_k ({}^{k}_{m}X^{l}_{n})^{*}\, {}^{k}_{m}X^{l}_{n} = \mathbf{1}\binom{l}{n} \otimes \mathrm{id}_{{}^{m}\mathcal{H}^{n}}$ and $\phi(\mathbf{1}\binom{l}{l}) = 1$ give

$$\lambda \cdot \sum_{k} \mathrm{Tr}(({}^{k}F_{X}^{l})^{-1}) = 1.$$

Now all assertions in (1)–(3) concerning α follow. □

Corollary 2.28. *Assume that $\mathscr{A}$ defines a partial compact quantum group, and let ϕ be a normalized positive invariant functional. Let $(\mathcal{H}, X)$ be a unitary irreducible sfd corepresentation of $\mathscr{A}$, let F_X be a non-zero morphism from $(\mathcal{H}, X)$ to $(\mathcal{H}, (S^2 \otimes \mathrm{id})(X))$, and let $a = (\mathrm{id} \otimes \omega_{\xi,\xi'})({}^{k}_{m}X^{l}_{n})$ and $b = (\mathrm{id} \otimes \omega_{\eta,\eta'})({}^{k}_{m}X^{l}_{n})$, where $\xi, \eta \in {}^{k}\mathcal{H}^{l}$ and $\xi', \eta' \in {}^{m}\mathcal{H}^{n}$. Then*

$$\phi(b^{*}a) = \frac{\langle \eta' | \xi' \rangle \langle \xi | F_X^{-1} \eta \rangle}{\sum_m \mathrm{Tr}({}^{m}(F_X^{-1})^{n})}, \qquad \phi(ab^{*}) = \frac{\langle \eta' | F_X \xi' \rangle \langle \xi | \eta \rangle}{\sum_n \mathrm{Tr}({}^{m}F_X^{n})}.$$

Proof. By Theorem 2.27,

$$\begin{aligned} \phi(b^{*}a) &= (\phi \otimes \omega_{\eta',\eta} \otimes \omega_{\xi,\xi'})(({}^{k}_{m}X^{l}_{n})^{*}_{12}({}^{k}_{m}X^{l}_{n})_{13}) \\ &= \frac{1}{\sum_k \mathrm{Tr}({}^{k}(F_X^{-1})^{l})} (\omega_{\eta',\eta} \otimes \omega_{\xi,\xi'})((\mathrm{id}_{{}^{m}\mathcal{H}^{n}} \otimes {}^{k}(F_X^{-1})^{l}) \circ \Sigma_{k,l,m,n}). \end{aligned}$$

The formula for $\phi(ab^{*})$ follows similarly or by considering the co-opposite of $\mathscr{A}$. □

Corollary 2.29. *Let $\mathscr{A}$ define a partial compact quantum group. Let $(\mathcal{H}_\alpha, X_\alpha)_\alpha$ be a representative family of all irreducible sfd corepresentations of $\mathscr{A}$. Then the map*

$$\bigoplus_{\alpha} \bigoplus_{k,l,m,n} (\overline{{}^{k}\mathcal{H}^{l}_{\alpha}} \otimes {}^{m}\mathcal{H}^{n}_{\alpha}) \to A$$

that sends $\overline{\xi} \otimes \eta \in \overline{{}^{k}\mathcal{H}^{l}_{\alpha}} \otimes {}^{m}\mathcal{H}^{n}_{\alpha}$ to $(\mathrm{id} \otimes \omega_{\xi,\eta})({}^{k}_{m}(X_\alpha)^{l}_{n})$, is a linear isomorphism.

Proof. This follows from Proposition 2.22, Proposition 2.24 and Corollary 2.28. □

Suppose now $a \in {}_m^kA_n^l$ for some partial bialgebra $\mathscr{A}$. Then for $\omega \in \mathrm{Hom}_{\mathbb{C}}(A, \mathbb{C})$, we can define

$$\omega \underset{p,q}{*} a := (\mathrm{id} \otimes \omega)(\Delta_{pq}(a)), \qquad a \underset{r,s}{*} \omega := (\omega \otimes \mathrm{id})(\Delta_{rs}(a)).$$

Clearly we can define

$$\omega \underset{p,q}{*} a \underset{r,s}{*} \omega' := (\omega \underset{p,q}{*} a) \underset{r,s}{*} \omega' = \omega \underset{p,q}{*} (a \underset{r,s}{*} \omega').$$

When ω has support on the $A(K)$ with $K_u = K_d$, we can write, for $a \in {}_m^kA_n^l$,

$$\omega * a := \sum_{p,q} \omega \underset{p,q}{*} a = \omega \underset{m,n}{*} a, \quad a * \omega = \sum_{r,s} a \underset{r,s}{*} \omega = a \underset{k,l}{*} \omega.$$

We shall say that an entire function f has *exponential growth on the right half-plane* if there exist $C, d > 0$ such that $|f(x + iy)| \leqslant C\mathrm{e}^{dx}$ for all $x, y \in \mathbb{R}$ with $x > 0$.

Theorem 2.30. *Assume that $\mathscr{A}$ defines a partial compact quantum group with positive normalized invariant functional ϕ. There exists a unique family of linear functionals $f_z \colon A \to \mathbb{C}$ such that*

(1) f_z vanishes on $A(K)$ when $K_u \neq K_d$.

(2) for each $a \in A$, the function $z \mapsto f_z(a)$ is entire and of exponential growth on the right half-plane.

(3) $f_0 = \epsilon$ and $(f_z \otimes f_{z'}) \circ \Delta = f_{z+z'}$ for all $z, z' \in \mathbb{C}$.

(4) $\phi(ab) = \phi(b(f_1 * a * f_1))$ for all $a, b \in A$.

This family furthermore satisfies

(5) $f_z(ab) = f_z(a)f_z(b)$ for $a \in A(K)$ and $b \in A(L)$ with $K_r = L_l$.

(6) $S^2(a) = f_{-1} * a * f_1$ for all $a \in A$.

(7) $f_z(\mathbf{1}\begin{pmatrix} l \\ n \end{pmatrix}) = \delta_{l,n}$, $f_z \circ S = f_{-z}$, and $f_z(a^*) = \overline{f_{-\overline{z}}(a)}$ for all $a \in A$.

Note that condition (3) is meaningful by condition (1).

Proof. We first prove uniqueness. Assume that $(f_z)_z$ is a family of functionals satisfying (1)–(4). Since ϕ is faithful, the map $\sigma \colon a \mapsto f_1 * a * f_1$ is uniquely determined by ϕ, and one easily sees that it is a homomorphism. Using (3), we find that $\epsilon \circ \sigma^n = f_{2n}$, which uniquely determines these functionals. Using (2) and the fact that every entire function of exponential growth on the right half-plane is uniquely determined by its values at $\mathbb{N} \subseteq \mathbb{C}$, we can conclude that the family f_z is uniquely determined. Moreover, since the property (5) holds for $z = 2n$, we also conclude by the same argument as above that it holds for all $z \in \mathbb{C}$.

Let us now prove existence. By Corollary 2.29, we can define for each $z \in \mathbb{C}$ a functional $f_z \colon A \to \mathbb{C}$ such that for every unitary irreducible sfd corepresentation $(\mathcal{H}, X)$,

$$f_z((\mathrm{id} \otimes \omega_{\xi,\eta})({}^{k}_{m}X^{l}_{n})) = \delta_{k,m}\delta_{l,n} \cdot \omega_{\xi,\eta}(({}^{k}F^{l}_{X})^{z}) \quad \text{for all } \xi \in {}^{k}\mathcal{H}^{l}, \eta \in {}^{m}\mathcal{H}^{n},$$

or, equivalently,

$$(f_z \otimes \mathrm{id})({}^{k}_{m}X^{l}_{n}) = \delta_{k,m}\delta_{l,n} \cdot ({}^{k}F^{l}_{X})^{z},$$

where F_X is a non-zero positive operator implementing a morphism from $(\mathcal{H}, X)$ to $(\mathcal{H}, (S^2 \otimes \mathrm{id})(X))$, scaled such that

$$\alpha_X := \sum_{k} \mathrm{Tr}({}^{k}(F_X^{-1})^{l}) = \sum_{n} \mathrm{Tr}({}^{m}F^{n}_{X})$$

for all l, n (see Proposition 2.26 and Theorem 2.27). By construction, (1) and (2) hold. We show that the $(f_z)_z$ satisfy the assertions (3)–(7). Throughout the following arguments, let $(\mathcal{H}, X)$ be a unitary irreducible corepresentation $(\mathcal{H}, X)$ and let F_X be as above.

We first prove property (3). This follows from the relations

$$(f_0 \otimes \mathrm{id})({}^{k}_{m}X^{l}_{n}) = \delta_{k,m}\delta_{l,n} \, \mathrm{id}_{\,{}^{k}\mathcal{H}^{l}} = (\epsilon \otimes \mathrm{id})({}^{k}_{m}X^{l}_{n})$$

and

$$\begin{aligned}(((f_z \otimes f_{z'}) \circ \Delta) \otimes \mathrm{id})({}^{k}_{m}X^{l}_{n}) &= \delta_{k,m}\delta_{l,n}(f_z \otimes f_{z'} \otimes \mathrm{id})\big(({}^{k}_{k}X^{l}_{l})_{13}({}^{k}_{k}X^{l}_{l})_{23}\big) \\ &= \delta_{k,m}\delta_{l,n}({}^{k}F^{l}_{X})^{z} \cdot ({}^{k}F^{l}_{X})^{z'} \\ &= (f_{z+z'} \otimes \mathrm{id})({}^{k}_{m}X^{l}_{n}).\end{aligned}$$

Applying slice maps of the form $\mathrm{id} \otimes \omega_{\xi,\xi'}$ and invoking Theorem 2.27, this proves (3).

To prove (4), write again $\Delta^{(2)} = (\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$, and put

$$\rho_{z,z'} := (f_{z'} \otimes \mathrm{id} \otimes f_z) \circ \Delta^{(2)}.$$

Then

$$\begin{aligned}(\rho_{z,z'} \otimes \mathrm{id})({}^{k}_{m}X^{l}_{n}) &= (f_{z'} \otimes \mathrm{id} \otimes f_z \otimes \mathrm{id})(({}^{k}_{k}X^{l}_{l})_{14}({}^{k}_{m}X^{l}_{n})_{24}({}^{m}_{m}X^{n}_{n})_{34}) \\ &= (1 \otimes ({}^{k}F^{l}_{X})^{z'}) \, {}^{k}_{m}X^{l}_{n}(1 \otimes ({}^{m}F^{n}_{X})^{z}).\end{aligned}$$

We take $z = z' = 1$, use Theorem 2.27, where now $\alpha = \beta$ by our scaling of F_X, and obtain

$$\begin{aligned}&(\phi \otimes \mathrm{id} \otimes \mathrm{id})(({}^{k}_{m}X^{l}_{n})^{*}_{12}((\rho_{1,1} \otimes \mathrm{id})({}^{k}_{m}X^{l}_{n}))_{13}) \\ &\quad = \alpha^{-1}(\mathrm{id} \otimes {}^{k}F^{l}_{X})(\mathrm{id} \otimes {}^{k}(F_X^{-1})^{l})\Sigma_{k,l,m,n}(\mathrm{id} \otimes {}^{m}F^{n}_{X}) \\ &\quad = \beta^{-1}({}^{m}F^{n}_{X} \otimes \mathrm{id})\Sigma_{k,l,m,n} \\ &\quad = (\phi \otimes \mathrm{id} \otimes \mathrm{id})(({}^{k}_{m}X^{l}_{n})_{13}({}^{k}_{m}X^{l}_{n})^{*}_{12}).\end{aligned}$$

To conclude the proof of assertion (4), apply again slice maps of the form $\omega_{\xi,\xi'} \otimes \omega_{\eta,\eta'}$.

We have then already argued that the property (5) automatically holds. To show the property (6), note that by Proposition 2.26 and the calculation above,

$$(S^2 \otimes \mathrm{id})({}^{k}_{m}X^{l}_{n}) = (1 \otimes {}^{k}F^{l}_{X})\, {}^{k}_{m}X^{l}_{n}(1 \otimes {}^{m}F^{n}_{X})^{-1} = (\rho_{-1,1} \otimes \mathrm{id})({}^{k}_{m}X^{l}_{n}).$$

Assertion (6) follows again by applying slice maps.

Finally, (1), (2) and (4) immediately imply the relation $f_z(\mathbf{1}\begin{pmatrix} k \\ m \end{pmatrix}) = \delta_{k,m}$. The concrete construction of f_z combined with property (3), the identity (2.2) and the partial character property (5) gives the equality

$$\begin{aligned}(f_{-z} \otimes \mathrm{id})({}^{k}_{k}X^{l}_{l}) = ({}^{k}(F_X)^{l})^{-z} &= \Big((f_z \otimes \mathrm{id})({}^{k}_{k}X^{l}_{l})\Big)^{-1} \\ &= (f_z \otimes \mathrm{id})({}^{l}_{l}(X^{-1})^{k}_{k}) = ((f_z \circ S) \otimes \mathrm{id})({}^{k}_{k}X^{l}_{l}).\end{aligned}$$

Therefore, $f_{-z} = f_z \circ S$. Let us now write $\bar{f}_z(a) = \overline{f_z(a^*)}$. Using the preceding calculation, the relation $(S \otimes \mathrm{id})({}^{k}_{k}X^{l}_{l}) = ({}^{k}_{k}X^{l}_{l})^*$ and positivity of ${}^{k}F^{l}_{X}$, we conclude

$$\begin{aligned}(\bar{f}_z \otimes \mathrm{id})({}^{k}_{k}X^{l}_{l}) &= \Big((f_z \otimes \mathrm{id})(({}^{k}_{k}X^{l}_{l})^*)\Big)^* \\ &= \Big((f_{-z} \otimes \mathrm{id})({}^{k}_{k}X^{l}_{l})\Big)^* = (({}^{k}F^{l}_{X})^{-z})^* = ({}^{k}F^{l}_{X})^{-\bar{z}} = (f_{-\bar{z}} \otimes \mathrm{id})({}^{k}_{k}X^{l}_{l}),\end{aligned}$$

whence $\bar{f}_z = f_{-\bar{z}}$. □

3 Tannaka-Krein duality for compact Hopf face algebras

Let $\mathcal{C}$ be a rigid tensor C*-category with irreducible unit. For example, one can take $\mathcal{C} = \mathrm{Rep}(\mathbb{G})$, the category of finite-dimensional unitary representations of a compact quantum group $\mathbb{G}$. We will in general view the tensor product of $\mathcal{C}$ as being strict. Let J be an index set for a maximal set of mutually non-isomorphic irreducible objects u_a in $\mathcal{C}$. The unit object of $\mathcal{C}$ will be written u_o. Whenever convenient, we will replace u_a by its associated index symbol a. We will also fix once and for all orthonormal bases $f^{a,b}_{c,j}$ for $\mathrm{Mor}(u_c, u_a \otimes u_b)$, where j runs over an index set $J^{a,b}_{c}$.

Let I be a (countable) set. We will write Hilb_{I^2} for the monoidal tensor C*-category of I-bigraded Hilbert spaces $\mathcal{H} = \sum_{r,s}^{\oplus}\mathcal{H}_{rs}$, where the direct sum on the right is understood as the completion of the ordinary algebraic one. The tensor product $\underset{I}{\otimes}$ in Hilb_{I^2} is defined by $(\mathcal{H} \underset{I}{\otimes} \mathscr{G})_{rs} = \bar{\oplus}_t (\mathcal{H}_{rt} \otimes \mathscr{G}_{ts})$. The unit of Hilb_{I^2} is $l^2(I)$ with the obvious I^2-grading. We will view this monoidal category as being strict.

We will be interested in strong tensor C*-functors F from $\mathcal{C}$ to Hilb_{I^2}. As shown in [DCY], any ergodic action of a compact quantum group $\mathbb{G}$ on a unital C*-algebra provides a tensor C*-functor of $\mathcal{C}$ into Hilb_{I^2} for some set I.

For $F : \mathcal{C} \to \mathrm{Hilb}_{I^2}$ a strong tensor C*-functor, we denote the unitary compatibility morphisms by $\phi_{X,Y} : F(X) \underset{I}{\otimes} F(Y) \to F(X \otimes Y)$, where we recall that they are assumed to satisfy the coherence conditions

$$\phi_{X,Y\otimes Z}(\mathrm{id}_X \otimes \phi_{Y,Z}) = \phi_{X\otimes Y,Z}(\phi_{X,Y} \otimes \mathrm{id}_Z), \qquad \phi_{o,a} = \phi_{a,o} = \mathrm{id}_a\,.$$

It will be convenient to extend $\phi_{X,Y}$ to a coisometry $F(X)\otimes F(Y) \to F(X\otimes Y)$, defining it to be zero on the orthogonal complement of $F(X) \underset{I}{\otimes} F(Y)$. Note however that then $\phi_{X,o}$ becomes the coisometry $F(X) \otimes l^2(I) \to F(X)$ sending $F(X)_{rs} \otimes \mathbb{C}\delta_t$ canonically onto $\delta_{s,t}F(X)_{rs}$, and similarly for $\phi_{o,X}$. Whenever X, Y are clear, we will abbreviate $\phi_{X,Y}$ as ϕ. We will use the notation

$$F^{a,b}_{c,j} = \phi^* F(f^{a,b}_{c,j}) \in B(F(u_c), F(u_a) \otimes F(u_b)).$$

As $\mathcal{C}$ is rigid, each $F(X)$ will be column-finite in the sense that for each X in $\mathcal{C}$ and each fixed s in I, the direct sum $\sum_r^{\oplus} F(X)_{rs}$ will be finite-dimensional. Similarly, each $F(X)$ will be row-finite. See ...

Define vector spaces

$${}^{k}_{m}A^{l}_{n}(a) = B(F(u_a)_{kl}, F(u_a)_{mn}).$$

Write ${}^{k}_{m}A^{l}_{n} = \oplus_{a\in J}\, {}^{k}_{m}A^{l}_{n}(a)$ and $A = \oplus_{k,l,m,n}\, {}^{k}_{m}A^{l}_{n}$. The a-spectral subspace $A(a)$ of A is defined as

$$A(a) = \sum_{k,l,m,n}^{\oplus} {}^{k}_{m}A^{l}_{n}(a).$$

For any element $x \in A$, its component in the a-spectral subspace is written x_a.

Our goal is to turn A into a generalized compact Hopf face algebra.

We first turn A into an algebra. The multiplication $x \cdot y$ of $x \in {}^{k}_{m}A^{l}_{n}(a)$ and $y \in {}^{p}_{r}A^{q}_{s}(b)$ is the element in ${}^{k}_{m}A^{q}_{s}$ defined by the formula

$$(x\cdot y)_c = \sum_{j\in J^{a,b}_c} \left(F^{a,b}_{c,j}\right)^* (x\otimes y)\left(F^{a,b}_{c,j}\right).$$

Note that the product is independent of the specific choice of orthogonal bases $f^{a,b}_{c,j}$. We will continue to use the $\cdot$-notation to distinguish this product from the ordinary multiplication of operators.

Lemma 3.1. *With the above product, A becomes a faithful strong I^2-algebra.*

Proof. Let $x \in {}^k_m A^l_n(a)$, $y \in {}^p_r A^q_s(b)$ and $z \in {}^q_s A^t_v$. From the fact that ϕ is a natural transformation, we find that

$$((x\cdot y)\cdot z)_d = \sum_{e\in J}\sum_{k\in J^{e,c}_d}\sum_{j\in J^{a,b}_e}\left(\phi^*(\phi^*\otimes \mathrm{id})F(f^{1,a,b,c}_{d,e,j,k})\right)^*(x\otimes y\otimes z)\left((\phi^*\otimes\mathrm{id})\phi^*F(f^{1,a,b,c}_{d,e,j,k})\right)$$

where $f^{1,a,b,c}_{d,e,j,k} = (f^{a,b}_{e,j}\otimes \mathrm{id})f^{e,c}_{d,k}$. On the other hand,

$$(x\cdot(y\cdot z))_d = \sum_{e\in J}\sum_{k\in J^{a,e}_d}\sum_{j\in J^{b,c}_e}\left(\phi(\mathrm{id}\otimes\phi)F(f^{2,a,b,c}_{d,e,j,k})\right)^*(x\otimes y\otimes z)\left((\mathrm{id}\otimes\phi)\phi F(f^{2,a,b,c}_{d,e,j,k})\right)$$

where $f^{2,a,b,c}_{d,e,j,k} = (\mathrm{id}\otimes f^{b,c}_{e,j})f^{a,e}_{d,k}$. As $\phi(\phi\otimes\mathrm{id})$ by $\phi(\mathrm{id}\otimes\phi)$ by assumption, and as the orthonormal bases $\{f^{1,a,b,c}_{d,e,j,k}\mid e,j,k\}$ or $\{f^{2,a,b,c}_{d,e,j,k}\mid e,j,k\}$ can clearly be replaced by any other orthonormal basis of $\mathrm{Mor}(u_d, u_a\otimes u_b\otimes u_c)$, it follows that $(x\cdot y)\cdot z = x\cdot(y\cdot z)$.

Define $1_{rs} \in B(F(u_o)_{rr}, F(u_o)_{ss}) = {}^r_s A^r_s(o)$ as the map sending δ_r to δ_s. By the compatibility assumption for $\phi_{a,o}$ and $\phi_{o,a}$, the map $\mathrm{Fun}_{\mathrm{f}}(I^2)\to A$ mapping $\delta_{(r,s)}$ to 1_{rs} is an algebra homomorphism. Thus A becomes a faithful strong I^2-algebra. □

In the following, we will again write $\lambda_r = \sum_s 1_{rs}$ and $\rho_s = \sum_r 1_{rs}$ inside $M(A)$, using the notation as at the end of the proof of the previous lemma.

We turn to the coproduct. Let $\{e_{a,i}\mid i\in B_a\}$ denote an orthonormal basis of $F(u_a)$ over an index set B_a which is adapted to the bigrading (in the sense that each $e_{a,i}$ is inside exactly one component). Write $B_{a,rs}\subseteq B_a$ for the set of indices for which $e_{a,i}\in F(u_a)_{rs}$. Define elements

$$P^{kl}_{mn}(a) \in {}^k_m A^l_n(a)\otimes {}^m_k A^n_l(a)$$

by

$$P^{kl}_{mn}(a) = \sum_{i\in B_{a,kl}, j\in B_{a,mn}} e_{a,j}e^*_{a,i}\otimes e_{a,i}e^*_{a,j}.$$

As each $F(u_a)_{kl}$ is finite-dimensional, the above sums are finite.

Define now maps

$$\Delta_{rs}: {}^k_m A^l_n(a) \to {}^k_r A^l_s(a)\otimes {}^r_m A^s_n(a)$$

by the application

$$x\mapsto P^{mn}_{rs}(a)(x\otimes 1) = (1\otimes x)P^{kl}_{rs}(a).$$

They obviously extend to linear maps Δ_{rs} from A to $A\underset{I^2}{\otimes}A$.

Lemma 3.2. *For each $x\in A$, the element $\Delta(x) = \sum_{rs}\Delta_{rs}(x)$ gives a well-defined multiplier of $A\underset{I^2}{\otimes}A$. The resulting map $\Delta: A\to M(A\underset{I^2}{\otimes}A)$ is an I^2-coproduct.*

Proof. As the grading on each $F(u_a)$ is column-finite, it follows at once that for each fixed p, q and $x \in A$, the element $\Delta_{rs}(x)(1 \underset{I^2}{\otimes} \lambda_p\rho_q)$ is zero except for finitely many r and s. Similarly, $(1 \underset{I^2}{\otimes} \lambda_p\rho_q)\Delta_{rs}(x)$ is zero except for finitely many r, s because of row-finiteness of $F(u_a)$. Hence $\Delta(x)$ is well-defined as a multiplier for each $x \in A$. Once we show that Δ is multiplicative, it will be immediate that Δ is coassociative, since each Δ_{rs} is coassociative. Moreover, also the fact that Δ then is an I^2-morphism is clear from the definition.

To obtain the multiplicativity of Δ, or rather of the coextension $\widetilde{\Delta}$, choose $x \in {}_mA_n(a)$ and $y \in {}_nA_q(b)$. Then

$$\begin{aligned}\widetilde{\Delta}(x) \cdot \widetilde{\Delta}(y) &= \sum_{rstv} (P^{mn}_{rs}(a)(x \otimes 1)) \cdot (P^{nq}_{tv}(b)(y \otimes 1)) \\ &= \sum_{cd}\sum_{ij}\sum_{klpt} \left(F^{a,b}_{c,i} \otimes F^{a,b}_{d,j}\right)^* \left(e_{a,k}e^*_{a,l}x \otimes e_{b,p}e^*_{b,t}y \otimes e_{a,l}e^*_{a,k} \otimes e_{b,t}e^*_{b,p}\right) \left(F^{a,b}_{c,i} \otimes F^{a,b}_{d,j}\right),\end{aligned}$$

where we may take the sum over all $k, l \in I_a$, $p, t \in I_b$ (and where the composition of operators with mismatching target and source is considered to be zero). Note that the infinite sums are convergent inside $M(A \otimes A)$ by the argument in the first paragraph.

Plugging in the identity operator $\sum_{cd}\sum_{rs}\left(e_{c,r}e^*_{c,r} \otimes e_{d,s}e^*_{d,s}\right)$ at the front, we obtain that the expression becomes

$$\sum_{cd}\sum_{rs}\sum_{ij}\sum_{klpt} X^{c,r,i}_{k,p} Y^{d,s,j}_{l,t} (e_{c,r} \otimes e_{d,s}) \left(e^*_{a,l}x \otimes e^*_{b,t}y \otimes e^*_{a,k} \otimes e^*_{b,p}\right) \left(F^{a,b}_{c,i} \otimes F^{a,b}_{d,j}\right)^*$$

where $X^{c,r,i}_{k,p} = e^*_{c,r}(F^{a,b}_{c,i})^*(e_{a,k} \otimes e_{b,p})$ and $Y^{d,s,j}_{l,t} = e^*_{d,t}(F^{a,b}_{d,j})^*(e_{a,l} \otimes e_{b,t})$. Resumming over the k, l, p, t, we obtain

$$\sum_{cd}\sum_{rs}\sum_{ij} \left(e_{c,r}e^*_{d,s}(F^{a,b}_{d,j})^*(x \otimes y)F^{a,b}_{c,i}\right) \otimes \left(e_{d,s}e^*_{c,r}(F^{a,b}_{c,i})^*F^{a,b}_{d,j}\right).$$

As ϕ is a coisometry and the $f^{a,b}_{c,i}$ are orthonormal, this expression simplifies to

$$\sum_{c}\sum_{rs}\sum_{i} e_{c,r}e^*_{c,s}\left((F^{a,b}_{c,i})^*(x \otimes y)F^{a,b}_{c,i}\right) \otimes e_{c,s}e^*_{c,r},$$

which is precisely $\widetilde{\Delta}(x \cdot y)$. □

Proposition 3.3. *The couple (A, Δ) is a generalized face algebra over I.*

Proof. Let ε assign to any $x \in {}^k_mA^l_n(a)$ the number $\mathrm{Tr}(x) = \sum_{i \in B_a}(e^*_{a,i}xe_{a,i})$ (where we keep the convention that mismatching operators compose to zero). We claim that ε is a counit, satisfying the conditions in the definition of a generalized face algebra. The fact that ε is a counit is immediate from the definition of Δ. It is also computed directly that for $x \in {}_m^kA_n^l$ and $y \in {}_n^lA_s^r$, we have $\varepsilon(x \cdot y) = \varepsilon(x)\varepsilon(y)$, since the $\{\phi^* F(f_{c,i}^{a,b})e_{c,j} \mid c, i, j\}$ form an orthonormal basis of $F(u_a) \underset{I}{\otimes} F(u_b)$. From this formula, the second identity for the counit will hold true once we show that

$$\varepsilon(\lambda_k\rho_m x \lambda_l\rho_n) = \varepsilon(\lambda_k\rho_m x_{(1)})\varepsilon(x_{(2)}\lambda_l\rho_n).$$

But both left and right hand side are zero unless $k = m$, $n = l$ and $x \in {}_m^kA_n^l$, in which case both sides equal $\varepsilon(x)$. $\square$

Our next job is to define a suitable antipode for (A, Δ). Here the rigidity of $\mathcal{C}$ will come into play, so we first fix our conventions. Let $a \mapsto \bar{a}$ be the involution induced by the rigidity on the index set J. We assume that $\overline{u_a} = u_{\bar{a}}$. For each u_a, we will fix duality morphisms $R_a : u_0 \to u_{\bar{a}} \otimes u_a$ and $\bar{R}_a : u_0 \to u_a \otimes u_{\bar{a}}$. By means of F and ϕ, they induce I^2-grading preserving maps $\mathscr{R}_a : l^2(I) \to F(u_{\bar{a}}) \underset{I}{\otimes} F(u_a)$ and $\bar{\mathscr{R}}_a : l^2(I) \to F(u_a) \underset{I}{\otimes} F(u_{\bar{a}})$. These in turn provide an invertible anti-linear map $I_a : F(u_a)_{kl} \to F(u_{\bar{a}})_{lk}$ and $J_a : F(u_{\bar{a}})_{lk} \to F(u_a)_{kl}$ such that $\langle I_a\xi_a, \eta_{\bar{a}}\rangle = \sum_r \delta_r^* \bar{\mathscr{R}}_a^*(\xi_a \otimes \eta_{\bar{a}})$ and $\langle J_a\eta_{\bar{a}}, \xi_a\rangle = \sum_s \delta_s^* \mathscr{R}_a^*(\eta_{\bar{a}} \otimes \xi_a)$. The snake identities for R_a and $\bar{R}_a$ guarantee that J_a is the inverse of I_a.

We define

$$S : {}_m^kA_n^l(a) \to {}_l^nA_k^m(\bar{a})$$

by

$$x \mapsto I_a x^* J_a.$$

Lemma 3.4. *By means of the map S, the couple (A, Δ) becomes a generalized Hopf face algebra.*

Proof. It is clear that S is invertible. We also have $S(\lambda_k\rho_l) = \lambda_l\rho_k$ as $I_o\delta_k = \delta_k$.

Let us check that S satisfies the condition $S(x_{(1)}) \cdot x_{(2)} = \sum_p \varepsilon(x \cdot \lambda_p)\rho_p$ in the multiplier algebra for $x \in {}_m^kA_n^l(a)$. By definition, we have

$$S(x_{(1)}) \cdot x_{(2)} = \sum_c \sum_i \sum_{p,q \in B_a} \left(F_{c,i}^{\bar{a},a}\right)^* \left(I_a e_{a,q} e_{a,p}^* J_a \otimes x e_{a,q} e_{a,p}^*\right) \left(F_{c,i}^{\bar{a},a}\right).$$

Let $C : \mathbb{C} \to \mathbb{C}$ be complex conjugation. Then we can write $I_a e_{a,q} e_{a,p}^* J_a = (I_a e_{a,q} C)(C e_{a,p}^* J_a)$. We now calculate, by definition of J_a and $F_{c,i}^{\bar{a},a}$, that

$$\sum_{p \in B_a} (C e_{a,p}^* J_a \otimes e_{a,p}^*) \left(F_{c,i}^{\bar{a},a}\right) = (R_a^* f_{c,i}^{\bar{a},a}) \sum_{s \in I} \delta_s^*,$$

since ϕ is a coisometry. Plugging this into our expression for $S(x_{(1)}) \cdot x_{(2)}$, we obtain

$$\sum_s \sum_{q \in B_a} \left(\sum_c \sum_i \left(f_{c,i}^{\bar{a},a})^* R_a\right) F_{c,i}^{\bar{a},a}\right)^* (I_a e_{a,q} C \otimes x e_{a,q} \delta_s^*).$$

As ϕ is a coisometry and $\phi^*\phi\mathscr{R}_a = \mathscr{R}_a$, we can write $\left(f_{c,i}^{\bar{a},a})^* R_a\right) F_{c,i}^{\bar{a},a} = F_{c,i}^{\bar{a},a}(F_{c,i}^{\bar{a},a})^*\mathscr{R}_a$. As the $f_{c,i}^{\bar{a},a}$ form an orthonormal basis, we thus get

$$S(x_{(1)}) \cdot x_{(2)} = \sum_s \sum_{q \in B_a} \mathscr{R}_a^*(I_a e_{a,q} C \otimes x e_{a,q} \delta_s^*).$$

Now the composition $I_a e_{a,q} C$ is the creation operator for the vector $I_a e_{a,q}$. Hence using again the definition of J_a, and using that $x \in {}_m^k A_n^l$, we get

$$\begin{aligned} S(x_{(1)}) \cdot x_{(2)} &= \sum_s \sum_q \delta_n \delta_s^* e_{a,q}^* x e_{a,q} \\ &= \sum_s \mathrm{Tr}(x) \delta_n \delta_s^* \\ &= \sum_p \varepsilon(x\lambda_p)\rho_p, \end{aligned}$$

since $\mathrm{Tr}(x) = \delta_{k,m}\delta_{n,l}\varepsilon(x)$.

The identity $x_{(1)} \cdot S(x_{(2)}) = \sum_p \varepsilon(\rho_p x)\lambda_p$ is proven in a similar way. □

In the next step, we determine an invariant functional for (A, Δ).

Definition 3.5. We define $\varphi : {}_m^k A_n^l \to \mathbb{C}$ as the projection onto the component ${}_m^k A_n^l(o) \cong \delta_{kl}\delta_{mn}\mathbb{C}$.

Lemma 3.6. *The functional φ is an invariant normalized functional.*

Proof. The fact that φ is normalized is immediate, so let us check that it is invariant. Let $x \in {}_m^k A_n^l(a)$. Then

$$\begin{aligned} (\mathrm{id} \otimes \varphi)\widetilde{\Delta}(x) &= \sum_{i,j} \varphi(e_{a,j} e_{a,i}^*) e_{a,i} e_{a,j}^* x \\ &= \delta_{a,o} \sum_{r,s} \delta_r \delta_s^* x \\ &= \varphi(x) \sum_r \delta_r \delta_k^* \\ &= \sum_p \varphi(\lambda_p x)\lambda_p. \end{aligned}$$

The proof of right invariance follows similarly. □

Finally, we introduce the *-structure and show that (A, Δ) is a generalized compact Hopf face algebra. To distinguish the new *-operation from the ordinary operator algebraic one, we will denote it by †.

Definition 3.7. We define the anti-linear map $\dagger : {}_{m}^{k}A_{n}^{l} \to {}_{k}^{m}A_{l}^{n}$ by the formula

$$x^\dagger = S(x^*)$$

Lemma 3.8. *The map $x \mapsto x^\dagger$ is an anti-multiplicative anti-linear involution on A.*

Proof. It is clear that $x \mapsto x^\dagger$ is anti-linear. It is also immediate from the definition of the product that $(x \cdot y)^* = x^* \cdot y^*$. Together with the anti-multiplicativity of S, this proves the anti-multiplicativity of $\dagger$.

Let us proof that $\dagger$ is an involution. It is sufficient to prove that $I_{\bar{a}}I_a = \lambda\,\mathrm{id}$ and $J_aJ_{\bar{a}} = \lambda^{-1}\,\mathrm{id}$ for some scalar λ. But this follows from the fact that $(\bar{R}_a, R_a)$ and $(R_{\bar{a}}, \bar{R}_{\bar{a}})$ are both solutions to the conjugate equations for $\bar{a}$. □

The last property which needs to be proven is the positivity of φ. For this, recall that $R_a^*R_a$ and $\bar{R}_a^*\bar{R}_a$ are scalars as u_a is irreducible. One can then rescale R_a and $\bar{R}_a$ such that the scalar in both expressions is the same. This scalar is then a uniquely determined number $\dim_q(a)$, called the *quantum dimension* of a. It follows that $\frac{1}{\dim_q(a)}F(R_aR_a^*)$ is the projection of $F(u_{\bar{a}} \otimes u_a)$ onto the copy of $F(u_o)$ inside, and a similar statement holds for $\bar{R}_a$.

Proposition 3.9. *For any $x \in A$, the scalar $\varphi(x^\dagger \cdot x)$ is positive.*

Proof. It is straightforward to see that the blocks ${}_{m}^{k}A_{n}^{l}$ are mutually orthogonal, and that moreover the spectral subspaces inside are mutually orthogonal. Let then $\xi, \zeta \in F(u_a)_{kl}$ and $\eta, \mu \in F(u_a)_{mn}$. We have, using the remark above,

$$\begin{aligned}
\varphi(y^\dagger \cdot x) &= \varphi(\sum_c \sum_i \left(F_{c,i}^{\bar{a},a}\right)^* (I_ayJ_a \otimes x) \left(F_{c,i}^{\bar{a},a}\right)) \\
&= \delta_n^* \sum_i \left(F_{o,i}^{\bar{a},a}\right)^* (I_ayJ_a \otimes x) \left(F_{o,i}^{\bar{a},a}\right) \delta_l \\
&= \frac{1}{\dim_q(u_a)} \delta_n^* \mathscr{R}_a^* (I_ayJ_a \otimes x) \mathscr{R}_a \delta_l \\
&= \frac{1}{\dim_q(u_a)} \sum_{p,q} \delta_n^* \mathscr{R}_a^* (I_ayJ_ae_{\bar{a},p}e_{\bar{a},p}^* \otimes xe_{a,q}e_{a,q}^*) \mathscr{R}_a \delta_l.
\end{aligned}$$

By the defining properties of I_a and J_a, this expression becomes $\dim_q(u_a)^{-1}\sum_p \langle e_{\bar{a},p}, J_a^*x^*yJ_ae_{\bar{a},p}\rangle$, thus clearly φ will be positive on A. □

Corepresentations of generalized compact Hopf face algebras

Let (A, Δ) be a generalized compact Hopf face algebra over an index set I. A *locally finite-dimensional unitary corepresentation* of (A, Δ) consists of a row and column-finite I^2-graded Hilbert space $\mathcal{H} = \sum\limits_{k,l\in I}^{\oplus} \mathcal{H}_{kl}$ together with elements ${}_m^kU_n^l \in {}_m^kA_n^l \otimes B({}^m\mathcal{H}^n, {}^k\mathcal{H}^l)$ such that

$$\sum_k \left({}_m^kU_n^l\right)^* {}_m^kU_n^l = \lambda_l\rho_n \otimes \mathrm{id}_{{}^m\mathcal{H}^n}$$

and

$$\sum_n {}_m^kU_n^l \left({}_m^kU_n^l\right)^* = \lambda_k\rho_m \otimes \mathrm{id}_{{}^k\mathcal{H}^l},$$

and

$$(\widetilde{\Delta} \otimes \mathrm{id})({}_m^kU_n^l) = \sum_{p,q} \left({}_p^kU_q^l\right)_{13} ({}_m^pU_n^q)_{23}\,.$$

Note that in the first two identities, the sums are finite, while in the finite identity the possibly infinite sum is meaningful inside the multiplier algebra sense.

By a morphism between two locally finite-dimensional unitary corepresentations $(\mathscr{H}, U)$ and $(\mathscr{G}, V)$ is meant a grading-preserving bounded map $T = \sum\limits_{k,l}^{\oplus} {}^kT^l : \mathscr{H} \to \mathscr{G}$ such that $(1 \otimes {}^kT^l)\, {}_m^kU_n^l = {}_m^kV_n^l(1 \otimes {}^mT^n)$. The collection of all locally finite-dimensional unitary corepresentations clearly forms a semi-simple C*-category Corep(A). We will say that (A, Δ) is *of finite type* if the morphisms in Corep(A) are finite-dimensional.

One can define a tensor product $\underset{I}{\textcircled{T}}$ between locally finite-dimensional corepresentations by means of the $\underset{I}{\otimes}$-product of bigraded Hilbert spaces and the operation

$${}_m^k(U \underset{I}{\textcircled{T}} V)_n^l = \left({}_m^kU_s^r\right)_{12} \left({}_s^rV_n^l\right)_{13}.$$

In this way, the category Corep(A) becomes a monoidal category. The unit object consists of the I^2-graded Hilbert space $l^2(I)$ together with the elements ${}_m^kU_n^l = \delta_{kl}\delta_{mn}\lambda_k\rho_m \otimes 1$.

Assume now that $\mathcal{C}$ is a semi-simple tensor C*-category with irreducible unit, and $F : \mathcal{C} \to \mathrm{Hilb}$ a strong tensor C*-functor. Let (A, Δ) be the associated generalized compact Hopf face algebra. Let us show that $\mathcal{C} \cong \mathrm{Corep}(A)$ by means of an equivalence functor G.

For X an object of $\mathcal{C}$, we build a locally finite-dimensional unitary corepresentation U on $F(X)$. Consider the canonical isomorphism $F(X) \cong \oplus_{a\in J} X_a \otimes \mathrm{Mor}(X_a, X)$. Let

$${}_m^kU_n^l(a) \in {}_m^kA_n^l(a) \otimes B(F(u_a)_{mn}, F(u_a)_{kl}) = B(F(u_a)_{kl}, F(u_a)_{mn}) \otimes B(F(u_a)_{mn}, F(u_a)_{kl})$$

be determined as the element implementing the non-degenerate pairing $B(F(u_a)_{kl}, F(u_a)_{mn}) \otimes B(F(u_a)_{nm}, F(u_a)_{lk}) \to \mathbb{C}$ sending $S \otimes T$ to $\mathrm{Tr}(ST)$. Using notation as before, this means that

$${}_m^kU_n^l(a) = \sum_{p\in B_{a,mn}, q\in B_{a,kl}} e_pe_q^* \otimes e_qe_p^*.$$

Monoidal equivalence of generalized compact Hopf face algebras

Let (A, Δ) be a generalized Hopf face algebra over a set I. Assume that $I = I_1 \sqcup I_2$, and let $\Lambda_j = \sum_{i \in I_j} \lambda_i$, resp. $\mathrm{P}_j = \sum_{i \in I_j} \rho_j$. If the Λ_j and P_j are central in $M(A)$, then we can write $A = \sum_{i,j}^{\oplus} A(ij)$ where $A(ij) = \Lambda_i \mathrm{P}_j A$ are subalgebras. Moreover, the comultiplication $\widetilde{\Delta}$ splits into comultiplications

$$\widetilde{\Delta}_{ij}^k : A(ij) \to M(A(ik) \otimes A(kj)) \text{ s.t. } \widetilde{\Delta} = \widetilde{\Delta}_{ij}^1 + \widetilde{\Delta}_{ij}^2 \text{ on } A(ij).$$

A similar decomposition holds for Δ.

It is immediate to see that the $(A(ii), \Delta_{ii}^i)$ are two generalized Hopf face algebras over the respective I_i.

Definition 3.10. We say (A, Δ) is a *co-linking generalized (compact) Hopf face algebra* between $(A(11), \Delta_{11}^1)$ and $(A(22), \Delta_{22}^2)$ if $\lambda_i \mathrm{P}_2 \neq 0$ for any $i \in I_1$.

Upon applying the antipode, we see that then $\rho_j \Lambda_1 \neq 0$ for any $j \in I_2$ as well.

Definition 3.11. Two generalized (compact) Hopf face algebras are called *comonoidally Morita equivalent* if they are isomorphic to the components (A_{ii}, Δ_{ii}^i) of some co-linking generalized (compact) Hopf face algebra.

As an example, consider two sets I_i, and two tensor functors (F_i, ϕ_i) of a semi-simple rigid C*-category $\mathcal{C}$ with irreducible unit into $\mathrm{Hilb}_{I_i^2}$. Then with $I = I_1 \sqcup I_2$, we can form a new C*-functor $F = F_1 \oplus F_2$ of $\mathcal{C}$ into Hilb_{I^2} by putting $F(X) = F_1(X) \oplus F_2(X)$ with the obvious I^2-grading (and the obvious direct sum operation on morphisms). It becomes monoidal by means of the unitaries

$$F(X \otimes Y) = F_1(X \otimes Y) \oplus F_2(X \otimes Y) \underset{\phi_1 \oplus \phi_2}{\cong} (F_1(X) \underset{I_1}{\otimes} F_1(Y)) \oplus (F_2(X) \otimes F_2(Y)) \cong F(X) \underset{I}{\otimes} F(Y)$$

(where the last map is unitary since $(F(X) \underset{I}{\otimes} F(Y))_{ij} = 0$ for example for $i \in I_1$ and $j \in I_2$).

If we then consider the generalized compact Hopf face algebra (A, Δ) associated to F, we have immediately from the construction that the Λ_i and P_i associated to the decomposition $I = I_1 \sqcup I_2$ are indeed central elements in $M(A)$. Moreover, the parts $(A_{ii}^i, \Delta_{ii}^i)$ are seen to arise from applying the Tannaka-Krein construction to the respective functors F_1 and F_2. The fact that (A, Δ) is co-linking is immediate from the fact that *none* of the $\lambda_i \rho_j$ are zero in this particular case (since ${}_m^k A_m^k(o) = B(F(u_o)_{kk}, F(u_o)_{mm}) \cong \mathbb{C}$).

We will exploit the above extra structure in the following section to say something about the algebra A appearing in ... This is the component $\tilde{A}(1,1)$ of the above algebra. The following lemma will be needed.

Lemma 3.12. *Assume* (A, Δ) *is a co-linking generalized Hopf face algebra. Then any of the maps* $\widetilde{\Delta}_{ij}^{k}$ *is injective.*

Proof. Take a non-zero $x \in A_n(ij)$ where $n \in I_j$. Then for any $l \in I$ with $\rho_n\lambda_l \neq 0$, we know that $\widetilde{\Delta}(x)(1 \otimes \rho_n\lambda_l) \neq 0$. Hence $\widetilde{\Delta}_{ij}^{k}(x)(1 \otimes \rho_n\lambda_l) \neq 0$ for $l \in I_k$, and hence $\widetilde{\Delta}_{ij}^{k}(x) \neq 0$. Now if $j = k$, the condition $\rho_n\lambda_l \neq 0$ is satisfied by taking $l = n$ (since $\varepsilon(\lambda_n\rho_n) = 1$). If $j \neq k$, it is satisfied for at least one l by the co-linking assumption. □

4 Compact Hopf face algebras on the level of operator algebras

It then follows by symmetry that also the maps

$$(W_{m,n,u,v}^{k,t})^* : \oplus_l \; {}_{m}^{k}A_n^l \otimes \; {}_{u}^{l}A_v^t \to \oplus_r \; {}_{m}^{k}A_r^t \otimes \; {}_{u}^{n}A_v^r$$

defined by the formula

$$a \otimes b \to \Delta(b)(a \otimes 1)$$

are unitaries, with inverse map $a \otimes b \mapsto S^{-1}(b_{(1)})a \otimes b_{(2)}$.

Lemma 4.1. *Let* (A, Δ) *be a generalized compact face algebra. Then each* $V_{m,v}^{k,l,s,t}$ *is a unitary, and similarly for the* $W_{m,n,u,v}^{k,t}$.

Proof. It is immediately checked that $V_{m,v}^{k,l,s,t}$ is isometric. □

Let us write $\mathscr{L}^2(A, \varphi)$ for the completion of A with respect to the inner product $\langle a, b\rangle = \varphi(a^*b)$. The canonical inclusion of A into $\mathscr{L}^2(A)$ will be denoted Λ.

Lemma 4.2. *Assume* (A, Δ) *is a generalized compact face algebra. The representation of* A *by left multiplication on itself extends to a representation by bounded operators on the completion* $\mathscr{L}^2(A, \varphi)$.

Proof. Denote $\omega_{\xi,\eta}(x) = \langle \xi, x\eta\rangle$ for ξ, η vectors and x a bounded operator. Then a straightforward computation shows that

$$(\omega_{\Lambda(a),\Lambda(b)} \otimes \mathrm{id})(V) = \varphi(a^*b_{(1)})b_{(2)}$$

as a left multiplication operator. As $(A \otimes 1)\Delta(A) = (A \otimes A)\Delta(1)$ by Lemma 4.1 (applied to the opposite algebra), it follows by normalization of φ that each element of A can be represented in the form $(\omega_{\Lambda(a),\Lambda(b)} \otimes \mathrm{id})(V)$, and hence extends to a bounded operator on $\mathscr{L}^2(A, \varphi)$. □

In the following, we will abbreviate $\mathscr{L}^2(A)$ by L^2A.

Let (A, Δ) be a generalized compact face algebra. Denote the von Neumann algebraic completion of $A \subseteq B(L^2A)$ by M. Denote $L^2A \underset{I}{\otimes} L^2A = E(L^2A \otimes L^2A)$, where $E = \sum_p \rho_p \otimes \lambda_p$ is extended to a bounded operator (in fact, a self-adjoint projection). Finally, denote $M \underset{I^2}{\otimes} M = E(M \otimes M)E$. Then $M \underset{I^2}{\otimes} M$ is the von Neumann algebraic completion of $A \underset{I^2}{\otimes} A$.

Extend now the $V_{m,v}^{k,l,s,t}$ to unitaries

$$V : \oplus_p L^2(A_p) \otimes L^2({}_pA) \to \oplus_p L^2({}_pA) \otimes L^2({}^pA) = E(L^2A \otimes L^2A).$$

Then we can construct a map

$$\Delta : M \to M \underset{I^2}{\otimes} M, \quad x \to V(x \otimes 1)V^*.$$

By direct computation, we see that Δ extends the comultiplication map on A. It is then immediate to check that Δ is in fact coassociative (where one may as well consider Δ as a non-unital map from M to $M \otimes M$).

We aim to show that (M, Δ) can be fitted into the theory of measured quantum groupoids.

5 Partial compact quantum groups from reciprocal random walks

We recall some notions introduced in [5]. We slightly change the terminology for the sake of convenience.

Definition 5.1. Let $t \in \mathbb{R}_0$. A *t-reciprocal random walk* consists of a quadruple $(\Gamma, w, \mathrm{sgn}, i)$ with

- $\Gamma = (\Gamma^{(0)}, \Gamma^{(1)}, s, t)$ a graph with *source* and *target* maps
$$s, t : \Gamma^{(1)} \to \Gamma^{(0)},$$
- w a function (the *weight* function) $w : \Gamma^{(1)} \to \mathbb{R}_0^+$,
- sgn a function (the *sign* function) $\mathrm{sgn} : \Gamma^{(1)} \to \{\pm 1\}$,
- i an involution
$$i : \Gamma^{(1)} \to \Gamma^{(1)}, \quad e \mapsto \overline{e}$$
with $s(\bar{e}) = t(e)$ for all edges e,

such that the following conditions are satisfied:

- $w(e)w(\bar{e}) = 1$ for all edges e,
- $\operatorname{sgn}(e)\operatorname{sgn}(\bar{e}) = \operatorname{sgn}(t)$ for all edges e,
- $p(e) = \frac{1}{|t|}w(e)$ defines a *random walk*: $\sum_{s(e)=v} p(e) = 1$ for all $v \in \Gamma^{(0)}$.

For some examples of t-reciprocal random walks, we refer to [5]. One particular example which will be needed for our construction of dynamical quantum $SU(2)$ is the following.

Example 5.2. Take $0 < |q| < 1$ and $x \in \mathbb{R}$. Write $2_q = q + q^{-1}$. Then we have the reciprocal -2_q-random walk

$$\Gamma_x = (\Gamma_x, w, \operatorname{sgn}, i)$$

with

$$\Gamma^{(0)} = \mathbb{Z}, \quad \Gamma^{(1)} = \{(k,l) \mid |k-l| = 1\} \subseteq \mathbb{Z} \times \mathbb{Z}$$

with projection on the first (resp. second) leg as source (resp. target) map, with weight function

$$w(k, k \pm 1) = \frac{|q|^{x+k\pm 1} + |q|^{-(x+k\pm 1)}}{|q|^{x+k} + |q|^{-(x+k)}},$$

sign function

$$\operatorname{sgn}(k, k+1) = 1, \quad \operatorname{sgn}(k, k-1) = -\operatorname{sgn}(q),$$

and involution $\overline{(k, k+1)} = (k+1, k)$.

By translation we shift the value of x by an integer, and by inversion we change x into $-x$ and multiply the sign function with a fixed sign. It follows that by some graph isomorphism, we can always arrange to have $x \in [0, \frac{1}{2}]$ at the cost of having a different sign function.

Let now $0 < |q| \leqslant 1$, and let $SU_q(2)$ be Woronowicz's twisted $SU(2)$ group [16]. Then $SU_q(2)$ is a compact quantum group, and its category of finite-dimensional unitary representations $\operatorname{Rep}(SU_q(2))$ is generated by the spin $1/2$-representation $\pi_{1/2}$ on $\mathbb{C}^2$.

Let $\Gamma = (\Gamma, w, \operatorname{sgn}, i)$ be a -2_q-reciprocal random walk. Define $\mathcal{H}_\Gamma$ as the $\Gamma^{(0)}$-bigraded Hilbert space $l^2(\Gamma^{(1)})$, where the $\Gamma^{(0)}$-bigrading is given by

$$\delta_e \in {}^{s(e)}\mathcal{H}_\Gamma^{t(e)}$$

for the obvious Dirac functions. Further define R_Γ as the (bounded) map

$$R_\Gamma : l^2(\Gamma^{(0)}) \to \mathcal{H}_\Gamma \underset{\Gamma^{(0)}}{\otimes} \mathcal{H}_\Gamma$$

given by

$$R_\Gamma \delta_v \quad = \sum_{e, s(e)=v} \operatorname{sgn}(e)\sqrt{w(e)}\delta_e \otimes \delta_{\bar{e}}.$$

Then $R_\Gamma^* R_\Gamma = |q| + |q|^{-1}$ and

$$(R_\Gamma^* \otimes \mathrm{id})(\mathrm{id} \otimes R_\Gamma) = -\operatorname{sgn}(q)\,\mathrm{id}\,.$$

Hence, by the universal property of $\mathrm{Rep}(SU_q(2))$ ([5, Theorem 1.4], based on [10, 6, 15, 8, 9]), we have a strongly monoidal *-functor

$$F_\Gamma : \mathrm{Rep}(SU_q(2)) \to {}^{\Gamma^{(0)}}\mathrm{Hilb}_f^{\Gamma^{(0)}}$$

such that $F_\Gamma(\pi_{1/2}) = \mathcal{H}_\Gamma$ and $F_\Gamma(\mathscr{R}) = R_\Gamma$, with

$$(\pi_{1/2}, \mathscr{R}, -\operatorname{sgn}(q)\mathscr{R})$$

a solution for the conjugate equations for $\pi_{1/2}$. Up to equivalence, F_Γ only depends upon the isomorphism class of (Γ, w), and is independent of the chosen involution or sign structure.

It follows from our main theorem that for each reciprocal random walk on a graph Γ, one obtains a $\Gamma^{(0)}$-partial compact quantum group. Our aim is to give a direct representation of it by generators and relations.

Theorem 5.3. *Let $0 < |q| \leqslant 1$, and let Γ be a -2_q-reciprocal random walk. Let $A(\Gamma)$ be the total *-algebra associated to the $\Gamma^{(0)}$-partial compact quantum group constructed from the fiber functor F_Γ. Then $A(\Gamma)$ is the universal *-algebra generated by a copy of the *-algebra of finite support functions on $\Gamma^{(0)} \times \Gamma^{(0)}$ (with the Dirac functions written as $\mathbf{1}\begin{pmatrix} v \\ w \end{pmatrix}$) and elements $(u_{e,f})_{e,f\in\Gamma^{(1)}}$ where $u_{e,f} \in {}^{s(e)}_{s(f)}A(\Gamma)^{t(e)}_{t(f)}$ and*

$$\sum_{v\in\Gamma^{(0)}} \sum_{g\in\Gamma_{vw}} u_{g,e}^* u_{g,f} = \delta_{e,f}\mathbf{1}\begin{pmatrix} w \\ t(e) \end{pmatrix}, \qquad \forall w \in \Gamma^{(0)}, e, f \in \Gamma^{(1)}, \tag{5.1}$$

$$\sum_{w\in\Gamma^{(0)}} \sum_{g\in\Gamma_{vw}} u_{e,g} u_{f,g}^* = \delta_{e,f}\mathbf{1}\begin{pmatrix} s(e) \\ v \end{pmatrix} \qquad \forall v \in \Gamma(0), e, f \in \Gamma^{(1)}, \tag{5.2}$$

$$u_{e,f}^* \;=\; \operatorname{sgn}(e)\operatorname{sgn}(f)\sqrt{\frac{w(f)}{w(e)}}\,u_{\bar{e},\bar{f}}, \qquad \forall e, f \in \Gamma^{(1)}. \tag{5.3}$$

If moreover $v, w \in \Gamma^{(0)}$ and $e, f \in \Gamma^{(1)}$, we have

$$\Delta_{vw}(u_{e,f}) = \sum_{\substack{s(g)=v\\ t(g)=w}} u_{e,g} \otimes u_{g,f},$$

$$\varepsilon(u_{e,f}) = \delta_{e,f}$$

and

$$S(u_{e,f}) = u_{f,e}^*.$$

Proof. We follow the proof of [2, Theorem 5.5].

First of all, it is straightforward to verify that $\mathscr{A}(\Gamma)$ indeed has a unique coproduct, counit and antipode for which it becomes a partial Hopf *-algebra, and such that on the generators the formulas above for them are satisfied.

On the other hand, denote by $\widetilde{\mathscr{A}}(\Gamma)$ the partial Hopf *-algebra constructed from F_Γ. Let $\widetilde{U}$ be the corepresentation corresponding to $F_\Gamma(\pi_{1/2})$. Then the matrix entries $\widetilde{u}_{e,f}$ of $\widetilde{U}$ with respect to the canonical basis of $\mathcal{H}_\Gamma = l^2(\Gamma^{(1)})$ satisfy the relations (5.1) and (5.2) because of unitarity of $\widetilde{U}$. On the other hand, the relation (5.3) follows immediately from the fact that R_Γ is an intertwiner. As $\pi_{1/2}$ is a generating representation for $\mathrm{Rep}(SU_q(2))$, it follows that we have a surjective *-representation $\mathscr{A}(\Gamma) \to \widetilde{\mathscr{A}}(\Gamma)$.

Let now $P_{n,m}$ for $n \in \mathbb{N}, m \in \frac{1}{2}\mathbb{N}$ with $2m = n \mod 2$ and $2m \leqslant n$ be the (Jones-Wenzl) projection in $\mathrm{Mor}(\pi_{1/2}^n, \pi_{1/2}^n)$ which projects onto the spin m-representation π_m. As U is a (real or anti-real) selfconjugate corepresentation of $\mathscr{A}$, it follows that $P_{n,m}$ is an endomorphism of $U^{(n)} := U^{\textcircled{\top}_I n}$. Hence we obtain a well-defined linear map

$$\theta : \widetilde{\mathscr{A}}(\Gamma) \to \mathscr{A}(\Gamma)$$

such that, with $\widetilde{U}_m$ the corepresentation of $\widetilde{\mathscr{A}}(\Gamma)$ associated to $F(\pi_m)$,

$$(\theta \otimes \mathrm{id})\widetilde{U}_m = (\mathrm{id} \otimes P_{n,m})U^{(n)}.$$

As in [2], we then infer that

$$(\theta \otimes \mathrm{id})(\widetilde{U}_m)_{12}(\theta \otimes \mathrm{id})(\widetilde{U}_{m'})_{13} = (\theta \otimes \mathrm{id})((\widetilde{U}_m)_{12}(\widetilde{U}_{m'})_{13})$$

inside $\mathrm{End}(\mathcal{H}_\Gamma^{\otimes_I n}) \otimes \mathscr{A}(\Gamma)$, so that θ is multiplicative and hence an inverse to the quotient map $\mathscr{A}(\Gamma) \to \widetilde{\mathscr{A}}(\Gamma)$. $\square$

6 Dynamical quantum $SU(2)$

6.1 Dynamical quantum $SU(2)$ from the Podleś graph

Let us now consider the particular case of the Podleś graph of Example 5.2. We assume in the following that $x \in [0, \frac{1}{2}]$.

Let us denote

$$w_+(k) = w(k, k+1),$$
$$w_-(k) = w(k, k-1) = w_+(k-1)^{-1}.$$

Let $A_x = A(\Gamma_x)$ be the total *-algebra of the associated partial compact quantum group. Using Theorem 5.3, we have the following presentation of A_x. Let B be the *-algebra of finite support functions on $\mathbb{Z} \times \mathbb{Z}$, whose Dirac functions we write as $\mathbf{1}\begin{pmatrix} k \\ l \end{pmatrix}$. Let $s_q = \frac{1}{2}(1 + \mathrm{sgn}(q))$. Then A_x is generated by a copy of B and elements

$$(u_{\epsilon,\nu})_{k,l} = u_{(k,k+\epsilon),(l,l+\nu)}$$

for $\epsilon, \nu \in \{-1, 1\} = \{-, +\}$ and $k, l \in \mathbb{Z}$ with defining relations

$$\begin{aligned}
\sum_{\mu \in \{\pm\}} (u_{\mu,\epsilon})^*_{m-\mu,k}(u_{\mu,\nu})_{m-\mu,l} &= \delta_{k,l}\delta_{\epsilon,\nu}\mathbf{1}\begin{pmatrix} m \\ k+\epsilon \end{pmatrix}, \\
\sum_{\mu \in \{\pm\}} (u_{\epsilon,\mu})_{k,m}(u_{\nu,\mu})^*_{l,m} &= \delta_{\epsilon,\nu}\delta_{k,l}\mathbf{1}\begin{pmatrix} k \\ m \end{pmatrix} \\
(u_{\epsilon,\nu})^*_{k,l} &= (\epsilon\nu)^{s_q}\left(\frac{w_\nu(l)}{w_\epsilon(k)}\right)^{1/2}(u_{-\epsilon,-\nu})_{k+\epsilon,l+\nu}.
\end{aligned}$$

The element $(u_{\epsilon,\nu})_{k,l}$ lives inside the component $A_x\begin{pmatrix} k & k+\epsilon & \\ & l & & l+\nu \end{pmatrix}$.

Consider now $M(A_x)$, the multiplier algebra of A_x. We can form in $M(A_x)$ the elements $u_{\epsilon,\nu} = \sum_{k,l}(u_{\epsilon,\nu})_{k,l}$. Then $u = (u_{\epsilon,\nu})$ is a unitary $2{\times}2$ matrix. Moreover,

$$u^*_{\epsilon,\nu} = (\epsilon\nu)^{s_q}u_{-\epsilon,-\nu}\frac{w_\nu^{1/2}(\rho)}{w_\epsilon^{1/2}(\lambda)},$$

where $w_\pm^{1/2}(k) = w_\pm(k)^{1/2}$ and where for a function f on $\mathbb{Z}$ we write $f(\lambda)(k,l) = f(k)$, $f(\rho)(k,l) = f(l)$. In the following, we then also use the notation $f(\lambda, \rho)$ for a function f on $\mathbb{Z} \times \mathbb{Z}$ interpreted as an element of $M(A)$, and for example $f(\lambda + 1, \rho)$ corresponds to the function $(k,l) \mapsto f(k+1,l)$. We then have the following commutation relations between functions on $\mathbb{Z} \times \mathbb{Z}$ and the entries of u:

$$f(\lambda, \rho)u_{\epsilon,\nu} = u_{\epsilon,\nu}f(\lambda - \epsilon, \rho - \nu).$$

$$\begin{aligned}
u_{++} &= \frac{w_+(\rho)^{1/2}}{w_+(\lambda)^{1/2}}u^*_{--}, \\
u_{+-} &= (-1)^{s_q}\frac{w_-^{1/2}(\rho)}{w_+^{1/2}(\lambda)}u^*_{-+}.
\end{aligned}$$

Let us write

$$F(k) = |q|^{-1}w_+(k) = |q|^{-1}\frac{|q|^{x+k+1} + |q|^{-x-k-1}}{|q|^{x+k} + |q|^{-x-k}},$$

and further put

$$\alpha = \frac{F^{1/2}(\rho-1)}{F^{1/2}(\lambda-1)}u_{--}, \qquad \beta = \frac{1}{F^{1/2}(\lambda-1)}u_{-+}.$$

Then the above commutation relations are equivalent to

$$\alpha\beta = qF(\rho-1)\beta\alpha \qquad\qquad \alpha\beta^* = qF(\lambda)\beta^*\alpha$$

$$\begin{aligned} \alpha\alpha^* + F(\lambda)\beta^*\beta = 1, &\qquad \alpha^*\alpha + q^{-2}F(\rho-1)^{-1}\beta^*\beta = 1,\\ F(\rho-1)^{-1}\alpha\alpha^* + \beta\beta^* = F(\lambda-1)^{-1}, &\qquad F(\lambda)\alpha^*\alpha + q^{-2}\beta\beta^* = F(\rho), \end{aligned}$$

$$f(\lambda)g(\rho)\alpha = \alpha f(\lambda+1)g(\rho+1), \qquad f(\lambda)g(\rho)\beta = \beta f(\lambda+1)g(\rho-1).$$

These are precisely the commutation relations for the dynamical quantum $SU(2)$-group as in for example [KR], except that the precise value of F has been changed by a shift in the parameter domain by a complex constant. Clearly, by ... the (total) coproduct on A_x also agrees with the one on the dynamical quantum $SU(2)$-group, namely

$$\begin{aligned} \Delta(\alpha) &= \alpha \underset{I^2}{\otimes} \alpha - q^{-1}\beta \underset{I^2}{\otimes} \beta^*,\\ \Delta(\beta) &= \beta \underset{I^2}{\otimes} \alpha^* + \alpha \underset{I^2}{\otimes} \beta. \end{aligned}$$

6.2 Representation theory of the function algebra of dynamical quantum $SU(2)$

Let us classify the irreducible representations of A_x. The parametrisation will hinge on the classification of what we call irreducible (x,c)-admissible sets, which we will now discuss.

Let $x \in [0, \frac{1}{2}]$, and let $c \geqslant 0$. For $\epsilon \in \{\pm\}$, an integer $m \in \mathbb{Z}$ will be called $(x,c)_\epsilon$-adapted if

$$c \leqslant |q|^{2x+m-\epsilon} + |q|^{-2x-m+\epsilon}, \tag{6.1}$$

and *strongly* $(x,c)_\epsilon$-adapted if this holds strictly. An integer is called (x,c)-adapted if it is both $(x,c)_+$ and $(x,c)_-$-adapted.

A set of integers Z is called an (x,c)-set if the following conditions hold:

- Z is not empty.
- Z consists of (x,c)-adapted points.
- If $m \in Z$ is strongly $(x,c)_\epsilon$-adapted, then $m - 2\epsilon$ is in Z.

An (x,c)-set is called *irreducible* if it can not be written as the union of two (x,c)-sets.

Note that if Z is an irreducible $(\frac{1}{2},c)$-set, then $Z+1$ is an irreducible $(0,c)$-set. We can hence assume that $x \in [0,\frac{1}{2})$. Also remark that clearly any irreducible (x,c)-set consists completely of either even or odd integers.

First note now that if $c < q^{2x-1} + q^{-2x+1}$, then clearly $2\mathbb{Z}$ and $2\mathbb{Z}+1$ are irreducible (x,c)-sets. If $c \geqslant |q|^{2x-1} + |q|^{-2x+1}$ we can uniquely write $c = |q|^{y} + |q|^{-y}$ for some $y \geqslant 1-2x$. We will treat the cases $x=0$ and $x \neq 0$ separately.

First assume that $x=0$, so that we may assume $y \geqslant 1$. Then it is easy to see that there are only (x,c)-sets if y is an integer. If $y=1$, there are three irreducible (x,c)-sets $-2\mathbb{N}_0$, $\{0\}$ and $2\mathbb{N}_0$. If $y \geqslant 2$, there are the two irreducible (x,c)-sets $y+1+2\mathbb{N}$ and $-y-1-2\mathbb{N}$.

For $x \in (0,\frac{1}{2})$, it is again easily seen that we only have (x,c)-sets when y is of the form $y = |2x+M|$, where M is a uniquely determined integer. If M is positive, we only have one irreducible (x,c)-set $2\mathbb{N}_0 + M$. If M is negative and $M \neq -1$, we have only one irreducible (x,c)-set $M - 2\mathbb{N}$. If $M=-1$, we have two irreducible (x,c)-sets, namely $2\mathbb{N}$ and $-2\mathbb{N}_0$.

Let us now return to the representation theory of our quantum groupoid.

Let $(\mathcal{H}_\pi,\pi)$ be any (bounded) non-degenerate *-representation of A_x on a Hilbert space. Then $\mathcal{H}_\pi = \oplus \mathcal{H}_{m,n}$ with $\mathcal{H}_{m,n} = \pi(e\binom{m}{n})\mathcal{H}$. Let V_π be the non-closed linear span of all $\mathcal{H}_{m,n}$. Then $\pi(A)V_\pi = V_\pi$. It follows that one can extends π to a map $M(A) \rightarrow \mathrm{End}(V_\pi)$. As the $u_{\epsilon,\eta}$ form a unitary matrix, we can then in fact make sense of the $\pi(u_{\epsilon,\eta})$ as contractions on $\mathcal{H}_\pi$. On the other hand, the generators α,β and their adjoints give rise to endomorphisms $V_\pi \rightarrow V_\pi$ which are bounded when restricted to any $\mathcal{H}_{m,n}$. It is easy to see that non-degenerate *-representations of A_x are in one-to-one correspondence with $\mathbb{Z}^2$-direct sums V of finite-dimensional Hilbert spaces equipped with maps $\alpha,\beta : V_\pi \rightarrow V_\pi$ satisfying the commutation relations as in ...

Consider

$$\Omega = q^{\lambda-\rho+1} + q^{\rho-\lambda-1} - \mathrm{sgn}(q)^{\lambda-\rho} q^{-1}(|q|^{x+\lambda+1} + |q|^{-x-\lambda-1})(|q|^{x+\rho-1} + |q|^{-x-\rho+1})\beta^*\beta.$$

As in [KR], one shows that Ω is a central element. It then follows immediately that if π is an irreducible representation of A_x, there exists $c \in \mathbb{R}$ such that $\Omega\xi = c\xi$ for all $\xi \in V_\pi$. Moreover, since for any ϵ,ν one has that $u_{\epsilon,\nu}^* u_{\epsilon,\nu}$ can be expressed as an element of the form $g_{\epsilon,\nu}(\lambda,\rho) + h_{\epsilon,\nu}\Omega$, we easily deduce from the commutation relations and irreducibility that $\mathcal{H}_{m,n}$ is at most one-dimensional. Because of the commutation relations of the generators with the $f(\lambda,\mu)$ it is also clear that either all $\mathcal{H}_{m,n}$ with $m-n$ odd are zero, or all $\mathcal{H}_{m,n}$ with $m-n$ even are zero. In the first case we call π even, in the second case we call π odd.

Note now that we have the following identities (where the right hand sides are unambiguously defined because of centrality of Ω):

$$\begin{aligned}
\alpha^*\alpha &= \frac{|q|^{2x+\lambda+\rho+1}+|q|^{-2x-\lambda-\rho-1}+\mathrm{sgn}(q)^{\lambda-\rho+1}\Omega}{(|q|^{x+\lambda+1}+|q|^{-x-\lambda-1})(|q|^{x+\rho}+|q|^{-x-\rho})}\\
\alpha\alpha^* &= \frac{|q|^{2x+\lambda+\rho-1}+|q|^{-2x-\lambda-\rho+1}+\mathrm{sgn}(q)^{\lambda-\rho-1}\Omega}{(|q|^{x+\lambda}+|q|^{-x-\lambda})(|q|^{x+\rho-1}+|q|^{-x-\rho+1})}\\
\beta^*\beta &= |q|\frac{|q|^{\lambda-\rho+1}+|q|^{-\lambda+\rho-1}-\mathrm{sgn}(q)^{\lambda-\rho+1}\Omega}{(|q|^{x+\lambda+1}+|q|^{-x-\lambda-1})(|q|^{x+\rho-1}+|q|^{-x-\rho+1})}\\
\beta\beta^* &= |q|\frac{|q|^{\lambda-\rho-1}+|q|^{-\lambda+\rho+1}-\mathrm{sgn}(q)^{\lambda-\rho-1}\Omega}{(|q|^{x+\lambda}+|q|^{-x-\lambda})(|q|^{x+\rho}+|q|^{-x-\rho})}.
\end{aligned}$$

For $c \in \mathbb{R}$, let us call a couple $(k,l) \in \mathbb{Z}^2$ $(\epsilon,\nu)_c$ adapted if the following inequality holds:

$$(|q|^{(x+k)+\epsilon\nu(x+l)-\epsilon}+|q|^{-(x+k)-\epsilon\nu(x+l)+\epsilon})+\mathrm{sgn}(q)^{k-l+1}\epsilon\nu c \geqslant 0. \tag{6.2}$$

Let us call (k,l) *strongly* $(\epsilon,\nu)_c$-adapted if this is a strict equality. Let us call (k,l) c-adapted if it is $(\epsilon,\nu)_c$-adapted for all ϵ,ν. Finally, let us call (k,l) c-compatible if there exists an irreducible representation π of A_x with $\pi(\Omega)=c$ and $\mathcal{H}_{k,l}\neq\{0\}$.

Let us call a subset $T\subseteq\mathbb{Z}^2$ a c-set if the following conditions are satisfied:

- T is not empty.
- T consists of c-adapted points.
- If $(k,l)\in T$ is strongly $(\epsilon,\nu)_c$-adapted, then $(k-\epsilon,l-\nu)$ is in T.

Call T irreducible if it is not the disjoint union of two c-sets. Let us write $\mathbb{Z}^2_{\text{even}}=\{(k,l)\mid k-l \text{ even}\}$ and $\mathbb{Z}^2_{\text{odd}}=\mathbb{Z}^2\backslash\mathbb{Z}^2_{\text{even}}$, and call a c-set even or odd according to whether it lies in $\mathbb{Z}^2_{\text{even}}$ or $\mathbb{Z}^2_{\text{odd}}$. Then it is easily seen that for any even (resp. odd) irreducible representation π of A, the set of $c=\pi(\Omega)$-compatible (k,l) forms an irreducible even (resp. odd) c-set. Conversely, if T is an irreducible c-set, then necessarily T is either an even or odd c-set, and we can construct an irreducible even/odd representation of A on $l^2(T)$ by putting

$$\begin{aligned}
\alpha e_{k,l} &= \left(\frac{|q|^{2x+k+l+1}+|q|^{-2x-k-l-1}+\mathrm{sgn}(q)^{k-l+1}c}{(|q|^{x+k+1}+|q|^{-x-k-1})(|q|^{x+l}+|q|^{-x-l})}\right)^{1/2} e_{k+1,l+1},\\
\beta e_{k,l} &= \mathrm{sgn}(q)^k\left(|q|\frac{|q|^{k-l+1}+|q|^{-k+l-1}-\mathrm{sgn}(q)^{k-l+1}c}{(|q|^{x+k+1}+|q|^{-x-k-1})(|q|^{x+l-1}+|q|^{-x-l+1})}\right)^{1/2} e_{k+1,l-1},
\end{aligned}$$

where the right hand side is considered as the zero vector when the scalar factor on the right is zero. Moreover, this then establishes a one-to-one correspondence between irreducible c-sets and irreducible representations of A_x with $\pi(\Omega)=c$.

Hence what remains is to classify irreducible c-sets for each $c\in\mathbb{R}$. But clearly T is an even irreducible c-set if and only if there exists an even irreducible $(x,-\mathrm{sgn}(q)c)$-set

$Z_+ \subseteq \mathbb{Z}$ and an even irreducible $(0, \mathrm{sgn}(q)c)$-set $Z_- \subseteq \mathbb{Z}$ such that $(k,l) \in T$ if and only if $k-l \in Z_-$ and $k+l \in Z_+$. Similary, T is an odd irreducible c-set if and only if there exists an odd irreducible $(x,-c)$-set $Z_+ \subseteq \mathbb{Z}$ and an irreducible $(0,c)$-set $Z_- \subseteq \mathbb{Z}$ such that $(k,l) \in T$ if and only if $k-l \in Z_-$ and $k+l \in Z_+$.

6.3 Representation theory of the intertwiner function algebra on the dynamical quantum $SU(2)$ group (to be modified)

Lemma 6.1. *There are faithful* **-representations* $\pi_\pm$ *of* $\mathrm{P_e}(\mathbb{X})$ *as operators* $\mathscr{D}^\pm \to \mathscr{D}^\pm$*, given by the following formulas (where we suppress the explicit notations* $\pi_\pm$*):*

$$\alpha \cdot e^+_{n,y} = \left(\frac{1+q^{2n-2y}}{1+q^{-2y-2}}\right)^{1/2} e^+_{n,y+1}, \qquad \beta \cdot e^+_{n,y} = \left(\frac{q^{-2y}-q^{2n-2y+2}}{1+q^{-2y-2}}\right)^{1/2} e^+_{n+1,y+1},$$

$$\alpha \cdot e^-_{n,y} = \left(\frac{1-q^{2n}}{1+q^{-2y-2}}\right)^{1/2} e^-_{n-1,y+1}, \qquad \beta \cdot e^-_{n,y} = \left(\frac{q^{2n+2}+q^{-2y}}{1+q^{-2y-2}}\right)^{1/2} e^-_{n,y+1},$$

the functions in $C_c(\mathbb{R})$ *simply acting by* $fe^\pm_{n,y} = f(y)e^\pm_{n,y}$.

Both representations are bounded when restricted to $\mathrm{P}(\mathbb{X})$.

6.4 Representation theory of the function algebra on the dynamical quantum $SU(2)$ group

Lemma 6.2. *There are faithful* **-representations* $\pi_\pm$ *of* $\mathrm{P_e}(\mathbb{X})$ *as operators* $\mathscr{D}^\pm \to \mathscr{D}^\pm$*, given by the following formulas (where we suppress the explicit notations* $\pi_\pm$*):*

$$\alpha \cdot e^+_{n,y} = \left(\frac{1+q^{2n-2y}}{1+q^{-2y-2}}\right)^{1/2} e^+_{n,y+1}, \qquad \beta \cdot e^+_{n,y} = \left(\frac{q^{-2y}-q^{2n-2y+2}}{1+q^{-2y-2}}\right)^{1/2} e^+_{n+1,y+1},$$

$$\alpha \cdot e^-_{n,y} = \left(\frac{1-q^{2n}}{1+q^{-2y-2}}\right)^{1/2} e^-_{n-1,y+1}, \qquad \beta \cdot e^-_{n,y} = \left(\frac{q^{2n+2}+q^{-2y}}{1+q^{-2y-2}}\right)^{1/2} e^-_{n,y+1},$$

the functions in $C_c(\mathbb{R})$ *simply acting by* $fe^\pm_{n,y} = f(y)e^\pm_{n,y}$.

Both representations are bounded when restricted to $\mathrm{P}(\mathbb{X})$.

References

[1] N. Andruskiewitsch and S. Natale, Double categories and quantum groupoids, *Publ. Mat. Urug.* **10**, 11–51 (2005).

[2] J. Bichon, A. De Rijdt and S. Vaes, Ergodic coactions with large multiplicity and monoidal equivalence of quantum groups, *Comm. Math. Phys.* **262** (2006), 703–728.

[3] G. Böhm, J. Gmez-Torrecillas and E. Lpez-Centella, Weak multiplier bialgebras, *Trans. Amer. Math. Soc.*, in press., arXiv:1306.1466.

[4] J. Dauns, Multiplier rings and primitive ideals, *Trans. Amer. Math. Soc* **145** (1969), 124–158.

[5] K. De Commer and M. Yamashita, Tannaka-Kreĭn duality for compact quantum homogeneous spaces II. Classification of quantum homogeneous spaces for quantum $SU(2)$, J. Reine Angew. Math., DOI: 10.1515/crelle-2013-0074 (2013).

[6] P. Etingof and V. Ostrik, Module categories over representations of $\mathrm{SL}_q(2)$ and graphs, *Math. Res. Lett.* **11** (1) (2004), 103–114.

[7] T. Hayashi, Compact Quantum Groups of Face Type, *PRIMS* **32** (1996), 351–369.

[8] C. Pinzari, The representation category of the Woronowicz quantum group $S_\mu U(d)$ as a braided tensor C*-category, *Int. J. Math.* **18** (2) (2007), 113–136.

[9] C. Pinzari and J.E. Roberts, Ergodic actions of compact quantum groups from solutions of the conjugate equations, preprint (2008) `arXiv:0808.3326 [math.OA]`.

[10] V. Turaev, Quantum invariants of knots and 3-manifolds, *de Gruyter Studies in Mathematics* **18**, Walter de Gruyter & Co., Berlin (1994).

[11] A. Van Daele, Multiplier Hopf algebras, *Trans. Amer. Math. Soc.* **342** (1994), 917–932.

[12] A. Van Daele, An algebraic framework for group duality, *Adv. in Math.* **140** (1998), 323–366.

[13] A. Van Daele and S. Wang, Weak multiplier Hopf algebras. Preliminaries, motivation and basic examples, *Banach Center Publ.* **98** (2012), 367–415.

[14] A. Van Daele and S. Wang: Weak multiplier Hopf algebras I. The main theory, *Preprint University of Leuven and Southeast University of Nanjing* (2012), to appear in *Crelles Journal*, `arXiv:math/1210.4395 [math.RA]`.

[15] S. Yamagami, A categorical and diagrammatical approach to Temperley–Lieb algebras, preprint (2004) `arXiv:math/0405267 [math.QA]`.

[16] S.L. Woronowicz, Twisted SU(2) group. An example of a non-commutative differential calculus, *Publ. Res. Inst. Math. Sci.* **23** (1) (1987), 117–181.