Partial compact quantum groups, compact quantum homogeneous spaces and the dynamical quantum $SU(2)$ group

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Abstract

Compact quantum groups of face type, as introduced by T. Hayashi, form a class of quantum groupoids with a classical, finite set of objects. We generalize Hayashi's definition to allow for an infinite set of objects, and call the resulting objects partial compact quantum groups. We then show how any quantum homogeneous space of an ordinary compact quantum group leads to a partial compact quantum group. In particular, when this construction is applied to the non-standard Podleś spheres, we obtain partial compact quantum groups which are operator algebraic versions of the dynamical quantum $SU(2)$-group as studied by Etingof-Varchenko and Koelink-Rosengren.

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1 Partial compact quantum groups

We generalize Hayashi's definition of a compact quantum group of face type [6] to the case where the commutative base algebra is no longer finite-dimensional. We will present two approaches, based on *partial bialgebras* and *weak multiplier bialgebras* [2]. The first approach is piecewise and concrete, but requires some bookkeeping. The second approach is global but more abstract. As we will see from the general theory and the concrete examples, both approaches have their intrinsic value.

Let I be a set. We consider $I^2 = I \times I$ as the pair groupoid with $\cdot$ denoting composition. That is, an element $K = (k,l) \in I^2$ has source $K_l = k$ and target $K_r = l$, and if $K = (k,l)$ and $L = (l,m)$ we write $K \cdot L = (k,m)$.

Definition 1.1. A *partial algebra* $\mathscr{A} = (\mathscr{A}, M)$ (over $\mathbb{C}$) is a small $\mathbb{C}$-linear category, that is, a set I (the object set) together with

- for each $K = (k,l) \in I^2$ a vector space $A(K) = A\big(k,l\big) = {}_kA_l$ (possibly the zero vector space),
- for each K, L with $K_r = L_l$ a multiplication map

$$M(K,L) : A(K) \otimes A(L) \to A(K \cdot L), \qquad a \otimes b \mapsto ab$$

  and
- elements $\mathbf{1}(k) = \mathbf{1}_k \in A\big(k,k\big)$ (the units),

such that the obvious associativity and unit conditions are satisfied.

By *I-partial algebra* will be meant a partial algebra with object set I.

Remark 1.2. We allow the local units $\mathbf{1}_k$ to be zero.

Let $\mathscr{A}$ be an I-partial algebra. We define $A(K \cdot L)$ to be $\{0\}$ when $K \cdot L$ is ill-defined, i.e. $K_r \neq L_l$. We then let $M\big(K,L\big)$ be the zero map.

Definition 1.3. The *total algebra* A of an I-partial algebra $\mathscr{A}$ is the vector space

$$A = \oplus_{K \in I^2} A(K)$$

endowed with the unique multiplication whose restriction to $A(K) \otimes A(L)$ concides with $M(K,L)$.

Clearly A is an associative algebra. If I is infinite it will not possess a unit, but it is a *locally unital algebra* as there exist mutually orthogonal idempotents $\mathbf{1}_k$ with $A = \sum_{k,l}^{\oplus} \mathbf{1}_k A \mathbf{1}_l$. An element $a \in A$ can be interpreted as a function assigning to each element $(k,l) \in I^2$ an element $a_{kl} \in A(k,l)$, namely the (k,l)-th component of a. This identifies A with finite support I-indexed matrices whose (k,l)-th entry lies in $A(k,l)$, equipped with the natural matrix multiplication.

Remark 1.4. When $\mathscr{A}$ is an I-partial algebra with total algebra A, then $A \otimes A$ can be naturally identified with the total algebra of an $I \times I$-partial algebra $\mathscr{A} \otimes \mathscr{A}$, where

$$(A \otimes A)((k,k'),(l,l')) = A(k,l) \otimes A(k',l')$$

with the obvious tensor product multiplications and the $\mathbf{1}_{k,k'} = \mathbf{1}_k \otimes \mathbf{1}_{k'}$ as units.

The notion of partial algebra dualizes. For this we consider again I^2 as the pair groupoid, but now with elements considered as column vectors, and with $*$ denoting the (vertical) composition. So $K = \binom{k}{l}$ has source $K_u = k$ and target $K_d = l$, and if $K = \binom{k}{l}$ and $L = \binom{l}{m}$ then $K * L = \binom{k}{m}$.

There are two natural notions of morphism between partial algebras, functors and co-functors.

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Definition 1.5. A *partial coalgebra* $\mathscr{A} = (\mathscr{A}, \Delta)$ (over $\mathbb{C}$) consists of a set I (the object set) together with

- for each $K = \binom{k}{l} \in I^2$ a vector space $A(K) = A\binom{k}{l} = A^k_l$,
- for each K, L with $K_d = L_u$ a comultiplication map

$$\Delta\binom{K}{L} : A(K * L) \to A(K) \otimes A(L), \qquad a \mapsto a_{(1)K} \otimes a_{(2)L},$$

  and

- counit maps $\epsilon_k : A\binom{k}{k} \to \mathbb{C}$,

satisfying the obvious coassociativity and counitality conditions.

By *I-partial coalgebra* will be meant a partial coalgebra with object set I.

Notation 1.6. As the index of ϵ_k is determined by the element to which it is applied, there is no harm in dropping the index k and simply writing ϵ.

Similarly, if $K = \binom{k}{l}$ and $L = \binom{l}{m}$, we abbreviate $\Delta_l = \Delta\binom{K}{L}$, as the other indices are determined by the element to which Δ_l is applied.

We also make again the convention that $A(K * L) = \{0\}$ and $\Delta\binom{K}{L}$ the zero map when $K_d \neq L_u$. Similarly ϵ is seen as the zero functional on $A(K)$ when $K = \binom{k}{l}$ with $k \neq l$.

We can now superpose the notions of partial algebra and partial coalgebra. To formulate the condition that the coalgebra maps form a 'morphism of partial algebras', we will need to impose a finiteness condition which is automatically satisfied when the cardinality of I is finite.

Let I be a set, and let $M_2(I)$ be the set of 4-tuples of elements of I arranged as 2×2-matrices. We can endow $M_2(I)$ with two compositions, namely $\cdot$ (viewing $M_2(I)$ as a row vector of column vectors) and $*$ (viewing $M_2(I)$ as a column vector of row vectors). When $K \in M_2(I)$, we will write $K = (K_l, K_r) = \binom{K_u}{K_d} = \begin{pmatrix} K_{lu} & K_{ru} \\ K_{ld} & K_{rd}\end{pmatrix}$. One can view $M_2(I)$ as a double groupoid, and in fact as a *vacant* double groupoid in the sense of [1].

In the following, a vector (r, s) will sometimes be written simply as r, s (without parentheses) or rs in an index. We also follow Notation 1.6, but the reader should be aware that the index of Δ will now be a 1×2 vector in I^2 as we will work with partial coalgebras over I^2.

Definition 1.7. A *partial bialgebra* $\mathscr{A} = (\mathscr{A}, M, \Delta)$ consists of a set I and a collection of vector spaces $A(K)$ for $K \in M_2(I)$ such that

- the $A(K_l, K_r)$ form an I^2-partial algebra,
- the $A\binom{K_u}{K_d}$ form an I^2-partial coalgebra,

and for which the following compatibility relations are satisfied.

(a) (Comultiplication of Units) For all $k, l, m \in I$, one has

$$\Delta_{l,l}(\mathbf{1}\tbinom{k}{m}) = \mathbf{1}\tbinom{k}{l} \otimes \mathbf{1}\tbinom{l}{m}.$$

(b) (Counit of Multiplication) For all $K, L \in M_2(I)$ with $K_r = L_l$ and all $a \in A(K)$ and $b \in A(L)$,

$$\epsilon(ab) = \epsilon(a)\epsilon(b).$$

(c) (Non-degeneracy) For all $k \in I$, $\epsilon(\mathbf{1}\binom{k}{k}) = 1$.

(d) (Finiteness) For each $K \in M_2(I)$ and each $a \in A(K)$, the element $\Delta_{rs}(a)$ is zero except for a finite number of indices r (resp. s) when s (resp. r) is fixed.

(e) (Comultiplication is multiplicative) For all $a \in A(K)$ and $b \in A(L)$ with $K_r = L_l$,

$$\Delta_{rs}(ab) = \sum_t \Delta_{rt}(a)\Delta_{ts}(b).$$

Remark 1.8. By assumption (d), the sum on the right hand side in condition (e) is in fact finite and hence well-defined.

We want to relate the notion of partial bialgebra to the recently introduced notion of weak multiplier bialgebra [2]. We first recall some notions concerning non-unital algebras [3, 11].

Definition 1.9. Let A be an algebra over $\mathbb{C}$, not necessarily with unit. We call A *non-degenerate* if A is faithfully represented on itself by left and right multiplication. It is called *idempotent* if $A^2 = A$.

Definition 1.10. Let A be an algebra. A *multiplier* m for A consists of a couple of maps

$$\begin{aligned} L_m &: A \to A, \quad a \mapsto ma \\ R_m &: A \to A, \quad a \mapsto am \end{aligned}$$

such that $(am)b = a(mb)$ for all $a, b \in A$.

The set of all multipliers forms an algebra under composition for the L-maps and anti-composition for the R-maps. It is called the *multiplier algebra* of A, and is denoted $M(A)$.

One has a natural homomorphism $A \to M(A)$. When A is non-degenerate, this homomorphism is injective, and we can then identify A as a subalgebra of the (unital) algebra $M(A)$. We then also have inclusions

$$A \otimes A \subseteq M(A) \otimes M(A) \subseteq M(A \otimes A).$$

Example 1.11. 1. Let A be the total algebra of an I-partial algebra $\mathscr{A}$. As A has local units, it is non-degenerate and idempotent. Then one can identify $M(A)$ with

$$M(A) = \left(\prod_l \oplus_k A(k,l)\right) \bigcap \left(\prod_k \oplus_l A(k,l)\right) \subseteq \prod_{k,l} A(k,l),$$

i.e. with the space of functions

$$m : I^2 \to A, \quad m_{kl} \in A(k,l)$$

which have finite support in either one of the variables when the other variable has been fixed. The multiplication is given by the formula

$$(mn)_{kl} = \sum_p m_{kp} n_{pl}.$$

2. Let m_i be any collection of multipliers of A, and assume that for each $a \in A$, $m_i a = 0$ for almost all i, and similarly $a m_i = 0$ for almost all i. Then one can define a multiplier $\sum_i m_i$ in the obvious way by termwise multiplication. One says that the sum $\sum_i m_i$ converges in the *strict* topology.

Using the notion introduced in Example 1.11.2, we can introduce the following notation.

Notation 1.12. If $\mathscr{A}$ is an I-partial bialgebra, we write

$$\lambda_k = \sum_l \mathbf{1}\binom{k}{l}, \qquad \rho_l = \sum_k \mathbf{1}\binom{k}{l} \qquad \in M(A).$$

Remark 1.13. From Property (c) of Definition 1.7, it follows that $\lambda_k \neq 0 \neq \rho_k$ for any $k \in I$.

To show that the total algebra of a partial bialgebra becomes a weak multiplier bialgebra, we will need some easy lemmas.

Lemma 1.14. *Let $\mathscr{A}$ be an I-partial bialgebra. Then for each $a \in A$, there exists a unique multiplier $\Delta(a) \in M(A \otimes A)$ such that*

$$\begin{aligned} \Delta_{rs}(a) &= (1 \otimes \lambda_r)\Delta(a)(1 \otimes \lambda_s) \\ &= (\rho_r \otimes 1)\Delta(a)(\rho_s \otimes 1) \end{aligned} \tag{1.1}$$

for all $r, s \in I$, all $K \in M_2(I)$ and all $a \in A(K)$.

The resulting map

$$\Delta : A \to M(A \otimes A), \quad a \mapsto \Delta(a)$$

is a homomorphism.

Proof. For $a \in A$ homogeneous, we can define $\Delta(a) = \sum_{rs} \Delta_{rs}(a) \in M(A \otimes A)$, where the sum converges in the strict topology of $A \otimes A$ because of the property (d) of Definition 1.7. This expression clearly satisfies the identities stated in the lemma. In turn, these identities uniquely define $\Delta(a)$ as a multiplier, as they determine the value of $\Delta(a)$ when cut down to the left and right with the local units of $\mathscr{A} \otimes \mathscr{A}$.

We can then extend Δ to A by linearity. Since, for a, b homogeneous, $\Delta_{rt}(a)\Delta_{t's}(b) = 0$ unless $t = t'$, it follows from property (e) of Definition 1.7 that Δ is a homomorphism. $\square$

We will refer to $\Delta : A \to M(A \otimes A)$ as the *total comultiplication* of $\mathscr{A}$. We will then also use the suggestive Sweedler notation for this map,

$$\Delta(a) = a_{(1)} \otimes a_{(2)}.$$

Note for example that

$$\Delta(\mathbf{1}\tbinom{k}{m}) = \sum_l \mathbf{1}\tbinom{k}{l} \otimes \mathbf{1}\tbinom{l}{m} = \sum_l \lambda_k \rho_l \otimes \lambda_l \rho_m.$$

Lemma 1.15. *The element $E = \sum_{k,l,m} \mathbf{1}\binom{k}{l} \otimes \mathbf{1}\binom{l}{m} = \sum_l \rho_l \otimes \lambda_l$ is a well-defined idempotent in $A \otimes A$, and satisfies*

$$\Delta(A)(A \otimes A) = E(A \otimes A), \quad (A \otimes A)\Delta(A) = (A \otimes A)E.$$

Proof. Clearly the sum defining E is strictly convergent, and makes E into an idempotent. It is moreover immediate that $E\Delta(a) = \Delta(a) = \Delta(a)E$ for all $a \in A$. Since

$$E(\mathbf{1}\binom{k}{l} \otimes \mathbf{1}\binom{m}{n}) = \Delta(\mathbf{1}\binom{k}{n})(\mathbf{1}\binom{k}{l} \otimes \mathbf{1}\binom{m}{n})$$

by the property (a) of Definition 1.7, and analogously for multiplication with E on the right, the lemma is proven. $\square$

By [13, Proposition A.3], there is a unique homomorphism $\Delta : M(A) \to M(A \otimes A)$ extending Δ and such that $\Delta(1) = E$. Alternatively, if $m \in M(A)$, we can directly define $\Delta(m)$ as the strict limit of the series $\sum_{k,l,r,s} \Delta_{rs}(m_{kl})$. Similarly the maps $\mathrm{id} \otimes \Delta$ and $\Delta \otimes \mathrm{id}$ extend to maps from $M(A \otimes A)$ to $M(A \otimes A \otimes A)$.

For example, note that

$$\Delta(\lambda_k) = (\lambda_k \otimes 1)\Delta(1), \qquad \Delta(\rho_m) = (1 \otimes \rho_m)\Delta(1). \tag{1.2}$$

The following proposition gathers the properties of Δ, ϵ and $\Delta(1)$ which guarantee that (A, Δ) forms a weak multiplier bialgebra in the sense of [2, Definition 2.1].

Proposition 1.16. *Let $\mathscr{A}$ be a partial bialgebra with total algebra A, total comultiplication Δ and counit ϵ. Then the following properties are satisfied.*

1. *Coassociativity:* $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$ *(as maps $M(A) \to M(A^{\otimes 3})$).*
2. *Counitality:* $(\epsilon \otimes \mathrm{id})(\Delta(a)(1 \otimes b)) = ab = (\mathrm{id} \otimes \epsilon)((a \otimes 1)\Delta(b))$ *for all $a, b \in A$.*
3. *Weak Comultiplicativity of Unit:*
$$(\Delta(1) \otimes 1)(1 \otimes \Delta(1)) = (\Delta \otimes \mathrm{id})\Delta(1) = (\mathrm{id} \otimes \Delta)\Delta(1) = (1 \otimes \Delta(1))(\Delta(1) \otimes 1).$$
4. *Weak Multiplicativity of Counit: For all $a, b, c \in A$, one has*
$$(\epsilon \otimes \mathrm{id})(\Delta(a)(b \otimes c)) = (\epsilon \otimes \mathrm{id})((1 \otimes a)\Delta(1)(b \otimes c))$$
and
$$(\epsilon \otimes \mathrm{id})((a \otimes b)\Delta(c)) = (\epsilon \otimes \mathrm{id})((a \otimes b)\Delta(1)(1 \otimes c)).$$
5. *Strong multiplier property: For all $a, b \in A$, one has*
$$\Delta(A)(1 \otimes A) \cup (A \otimes 1)\Delta(A) \subseteq A \otimes A.$$

Proof. Most of these properties follow immediately from the definition of a partial bialgebra. For demonstrational purposes, let us check the first identity of property (4). Let us choose $a \in A(K)$, $b \in A(L)$ and $c \in A(M)$. Then

$$\Delta(a)(b \otimes c) = \delta_{K_{ru},L_{lu}}\delta_{M_{lu},L_{ld}} \sum_r \Delta_{r,L_{ld}}(a)(b \otimes c).$$

Applying $(\epsilon \otimes \mathrm{id})$ to both sides, we obtain by Proposition (b) of Definition 1.7 and counitality of ϵ that

$$(\epsilon \otimes \mathrm{id})(\Delta(a)(b \otimes c)) = \delta_{K_{ru},L_{lu},L_{ld},M_{lu}}\epsilon(b)ac.$$

On the other hand,

$$\begin{aligned}(1 \otimes a)\Delta(1)(b \otimes c) &= \sum_{r,s,t} \mathbf{1}\tbinom{r}{s} b \otimes a\mathbf{1}\tbinom{s}{t} c \\ &= \delta_{L_{ld},K_{ru},M_{lu}} b \otimes ac.\end{aligned}$$

Applying $(\epsilon \otimes \mathrm{id})$, we find

$$\begin{aligned}(\epsilon \otimes \mathrm{id})((1 \otimes a)\Delta(1)(b \otimes c)) &= \delta_{L_{ld},K_{ru},M_{lu}}\delta_{L_{lu},L_{ld}}\delta_{L_{ru},L_{rd}}\epsilon(b)ac \\ &= \delta_{L_{ld},L_{lu},K_{ru},M_{lu}}\epsilon(b)ac,\end{aligned}$$

which agrees with the expression above. $\square$

Remark 1.17. Since also the expressions $\Delta(a)(b \otimes 1)$ and $(1 \otimes a)\Delta(b)$ are in $A \otimes A$ for all $a, b \in A$, we see that (A, Δ) is in fact a *regular* weak multiplier bialgebra [2, Definition 2.3].

Recall from [2, Section 3] that a regular weak multiplier bialgebra admits four projections $A \to M(A)$, given by

$$\begin{aligned}\bar{\Pi}^L(a) &= (\epsilon \otimes \mathrm{id})((a \otimes 1)\Delta(1)), \quad \bar{\Pi}^R(a) = (\mathrm{id} \otimes \epsilon)(\Delta(1)(1 \otimes a)), \\ \Pi^L(a) &= (\epsilon \otimes \mathrm{id})(\Delta(1)(a \otimes 1)), \quad \Pi^R(a) = (\mathrm{id} \otimes \epsilon)((1 \otimes a)\Delta(1)),\end{aligned}$$

where the right hand side expressions are interpreted as multipliers in the obvious way. The relation $\Delta(1) = \sum_p \rho_p \otimes \lambda_p$ and condition (c) in Definition 1.7 imply

$$\bar{\Pi}^L(A) = \mathrm{span}\{\lambda_p : p \in I\} = \Pi^L(A), \qquad \bar{\Pi}^R(A) = \mathrm{span}\{\rho_p : p \in I\} = \Pi^R(A).$$

The *base algebra* of (A, Δ) is therefore just the algebra $\mathrm{Fun}_{\mathrm{f}}(I)$ of finite support functions on I, and the comultiplication of A is (left and right) *full* (meaning roughly that the legs of $\Delta(A)$ span A) by [2, Theorem 3.13].

The maps Π^L and Π^R can also be written in the form

$$\Pi^L(a) = \sum_p \epsilon(\lambda_p a)\lambda_p, \qquad \Pi^R(a) = \sum_p \epsilon(a\rho_p)\rho_p \tag{1.3}$$

because $\epsilon(\lambda_k\rho_m a\lambda_l\rho_n) = 0$ if $(k,l) \neq (m,n)$. These relations and (1.1), (1.2) imply

$$(\Pi^L \otimes \mathrm{id})(\Delta(a)) = \sum_p \lambda_p \otimes \lambda_p a, \qquad (\mathrm{id} \otimes \Pi^L)(\Delta(a)) = \sum_p \rho_p a \otimes \lambda_p, \tag{1.4}$$

$$(\Pi^R \otimes \mathrm{id})(\Delta(a)) = \sum_p \rho_p \otimes a\lambda_p, \qquad (\mathrm{id} \otimes \Pi^R)(\Delta(a)) = \sum_p a\rho_p \otimes \rho_p. \tag{1.5}$$

We now formulate the notion of partial Hopf algebra, whose total form will correspond to a weak multiplier Hopf algebra [2, 13, 14]. We will mainly refer to [2] for uniformity.

Let us denote $\circ$ for the inverse of $\cdot$, and $\bullet$ for the inverse of $*$, so

$$\begin{pmatrix} k & l \\ m & n \end{pmatrix}^{\circ} = \begin{pmatrix} l & k \\ n & m \end{pmatrix}, \quad \begin{pmatrix} k & l \\ m & n \end{pmatrix}^{\bullet} = \begin{pmatrix} m & n \\ k & l \end{pmatrix}, \quad \begin{pmatrix} k & l \\ m & n \end{pmatrix}^{\circ\bullet} = \begin{pmatrix} n & m \\ l & k \end{pmatrix}.$$

The notation $\circ$ (resp. $\bullet$) will also be used for row vectors (resp. column vectors).

Definition 1.18. An *antipode* for an I-partial bialgebra $\mathscr{A}$ consists of linear maps

$$S : A(K) \to A(K^{\circ\bullet})$$

such that the following property holds: for all $M, P \in M_2(I)$ and all $a \in A(M)$,

$$\sum_{\substack{K*L=M \\ K\cdot L^{\circ\bullet}=P}} a_{(1)K}S(a_{(2)L}) = \delta_{P_l,P_r}\epsilon(a)\mathbf{1}(P_l), \tag{1.6}$$

$$\sum_{\substack{K*L=M \\ K^{\circ\bullet}\cdot L=P}} S(a_{(1)K})a_{(2)L} = \delta_{P_l,P_r}\epsilon(a)\mathbf{1}(P_r). \tag{1.7}$$

A partial bialgebra $\mathscr{A}$ is called a *partial Hopf algebra* if it admits an antipode.

Remark 1.19. Note that condition (d) of Definition 1.7 again guarantees that the above sums are in fact finite.

If S is an antipode for a partial bialgebra, we can extend S to a linear map

$$S : A \to A$$

on the total algebra A. Conditions (1.6) and (1.7) then take the following simple form:

Lemma 1.20. *A family of maps $S\colon A(K) \to A(K^{\circ\bullet})$ satisfies (1.6) and (1.7) if and only if the total map $S\colon A \to A$ satisfies*

$$a_{(1)}S(a_{(2)}) = \Pi^L(a), \qquad S(a_{(1)})a_{(2)} = \Pi^R(a) \tag{1.8}$$

for all $a \in A$.

Note that these should be considered a priori as equalities of left (resp. right) multipliers on A.

Proof. For $M, P \in M_2(I)$ and $a \in A(M)$, the left and the right hand side of (1.6) are the P-homogeneous components of $a_{(1)}S(a_{(2)})$ and $\Pi^L(a) = \sum_p \epsilon(\lambda_p a)\lambda_p$, respectively. $\square$

Lemma 1.21. *Let $\mathscr{A}$ be a partial Hopf algebra with antipode S. For all $k, l \in I$, $S(\mathbf{1}\binom{k}{l}) = \mathbf{1}\binom{l}{k}$.*

Proof. For example the first identity in Equation (1.8) of Lemma 1.20 applied to $\mathbf{1}\binom{k}{k}$ gives

$$\sum_l S(\mathbf{1}\tbinom{l}{k}) = \sum_l \mathbf{1}\tbinom{k}{l} S(\mathbf{1}\tbinom{l}{k}) = \lambda_k,$$

as $S(\mathbf{1}\binom{l}{k}) \in {}^k_l A^k_l$ and $\Pi^L(\mathbf{1}\binom{k}{k}) = \lambda_k$. This implies the lemma. □

Remark 1.22. Let $\mathcal{A}$ be an I-partial Hopf algebra. Then the relation on I defined by

$$k \sim l \Leftrightarrow \mathbf{1}\tbinom{k}{l} \neq 0$$

is an equivalence relation. Indeed, it is reflexive and transitive by assumptions (c) and (a) in Definition 1.7, and symmetric by the preceding result.

The existence of an antipode is closely related to partial invertibility of the maps $T_1, T_2 \colon A \otimes A \to A \otimes A$ given by

$$T_1(a \otimes b) = \Delta(a)(1 \otimes b), \qquad T_2(a \otimes b) = (a \otimes 1)\Delta(b). \tag{1.9}$$

The precise formulation involves the linear maps $E_i, G_i \colon A \otimes A \to A \otimes A$ given by

$$G_1(a \otimes b) = \sum_p a\rho_p \otimes \rho_p b, \qquad E_1(a \otimes b) = E(a \otimes b) = \sum_p \rho_p a \otimes \lambda_p b, \tag{1.10}$$

$$G_2(a \otimes b) = \sum_p a\lambda_p \otimes \lambda_p b, \qquad E_2(a \otimes b) = (a \otimes b)E = \sum_p a\rho_p \otimes b\lambda_p. \tag{1.11}$$

Proposition 1.23. *Let $\mathscr{A}$ be a partial Hopf algebra with total algebra A, total comultiplication Δ and antipode S. Then the maps $R_1, R_2 \colon A \otimes A \to M(A \otimes A)$ given by*

$$R_1(a \otimes b) = a_{(1)} \otimes S(a_{(2)})b, \qquad R_2(a \otimes b) = aS(b_{(1)}) \otimes b_{(2)}$$

take values in $A \otimes A$ and satisfy for $i = 1, 2$ the relations

$$T_i R_i = E_i, \qquad R_i T_i = G_i, \qquad T_i R_i T_i = T_i, \qquad R_i T_i R_i = R_i. \tag{1.12}$$

Proof. The map R_1 takes values in $A \otimes A$ because

$$a_{(1)} \otimes S(a_{(2)})\lambda_k \rho_l = a_{(1)} \otimes S(\rho_l \lambda_k a_{(2)}) \in A \otimes A$$

for all $a \in A$, and Lemma 1.20, Equation (1.4) and Lemma 1.21 imply

$$T_1 R_1(a \otimes b) = a_{(1)} \otimes a_{(2)} S(a_{(3)})b = a_{(1)} \otimes \Pi^L(a_{(2)})b = \sum_p \rho_p a \otimes \lambda_p b,$$

$$R_1 T_1(a \otimes b) = a_{(1)} \otimes S(a_{(2)})a_{(3)}b = a_{(1)} \otimes \Pi^R(a_{(2)})b = \sum_p a\rho_p \otimes \rho_p b.$$

The relations $T_1 R_1 T_1 = T_1$ and $R_1 T_1 R_1 = R_1$ follow easily from (1.1) and (1.2). The assertions concerning R_2 and T_2 follow similarly. □

Theorem 1.24. *Let $\mathscr{A}$ be a partial bialgebra with total algebra A, total comultiplication Δ and counit ϵ. Then the following conditions are equivalent:*

(1) $\mathscr{A}$ is a partial Hopf algebra.

(2) There exist linear maps $R_1, R_2\colon A \otimes A \to A \otimes A$ satisfying (1.12).

(3) (A, Δ, ϵ) is a regular weak multiplier Hopf algebra in the sense of [14].

If these conditions hold, then the total antipode of $\mathscr{A}$ coincides with the antipode of (A, Δ, ϵ).

Proof. (1) implies (2) by Proposition 1.23. (2) is equivalent to (3) by Definition 1.14 in [14]. Indeed, the maps G_1, G_2 defined in (1.10) and (1.11) satisfy

$$G_1(a_{(1)} \otimes b) \otimes a_{(2)}c = \sum_p a_{(1)} \otimes \rho_p b \otimes a_{(2)}\lambda_p c,$$
$$ac_{(1)} \otimes G_2(b \otimes c_{(2)}) = \sum_p a\rho_p c_{(1)} \otimes b\lambda_p \otimes c_{(2)}$$

and therefore coincide with the maps introduced in Proposition 1.14 in [14]. Finally, assume (3). Then Lemma 6.14 and equation (6.14) in [2] imply that the antipode S of (A, Δ) satisfies $S(A(K)) \subseteq A(K^{\circ\bullet})$ and relation (1.8). Now, Lemma 1.20 implies (1). ☐

From [14, Proposition 3.5 and Proposition 3.7] or [2, Theorem 6.12 and Corollary 6.16], we can conclude that the antipode of a partial Hopf algebra reverses the multiplication and comultiplication. Denote by Δ^{op} the composition of Δ with the flip map.

Corollary 1.25. *Let $\mathscr{A}$ be a partial Hopf algebra. Then the total antipode $S : A \to A$ satisfies $S(ab) = S(b)S(a)$ and $\Delta(S(a)) = (S \otimes S)\Delta^{\mathrm{op}}(a)$ for all $a, b \in A$. In particular, $S(\mathbf{1}\binom{k}{m}) = \mathbf{1}\binom{m}{k}$ for all $k, m \in I$ if S is bijective.* `need for corepresentations somewhere that` $\epsilon \circ S = \epsilon$

We now turn towards the structures which will allow us to build operator algebraic quantum groupoids out of our partial Hopf algebras.

Definition 1.26. A *partial *-algebra $\mathscr{A}$* is a partial algebra whose total algebra A is equipped with an antilinear, antimultiplicative involution $*\colon A \to A$, $a \mapsto a^*$, such that the $\mathbf{1}_k$ are selfadjoint for all k in the object set.

One can of course give an alternative definition directly in terms of the partial algebra structure by requiring that we are given antilinear maps $A(k, l) \to A(l, k)$ satisfying the obvious antimultiplicativity and involution properties.

Definition 1.27. A *partial *-bialgebra $\mathscr{A}$* is a partial bialgebra whose underlying partial algebra has been endowed with a partial *-algebra structure such that $\Delta(a)^* = \Delta(a^*)$ for all $a \in A$. A *partial Hopf *-algebra* is a partial bialgebra which is at the same time a partial *-bialgebra and a partial Hopf algebra.

Thus, a partial bialgebra is a partial $*$-bialgebra if and only if the underlying weak multiplier bialgebra is a weak multiplier $*$-bialgebra.

From Theorem 1.24 and [2], [14], we can deduce:

Corollary 1.28. *An I-partial *-bialgebra $\mathscr{A}$ is an I-partial Hopf *-algebra if and only if the weak multiplier *-bialgebra (A, Δ) is a weak multiplier Hopf *-algebra. In that case, the counit and antipode satisfy $\epsilon(a^*) = \overline{\epsilon(a)}$ and $S(S(a)^*)^* = a$ for all $a \in A$. In particular, the total antipode is bijective.*

Proof. The if and only if part follows immediately from Theorem 1.24, the relation for the counit from uniqueness of the counit [2, Theorem 2.8], and the relation for the antipode from [14, Proposition 4.11]. □

Definition 1.29. Let $\mathscr{A}$ be an I-partial bialgebra. We call a family of functionals

$$\phi\binom{k}{m} \colon A\begin{pmatrix} k & k \\ m & m \end{pmatrix} \to \mathbb{C} \tag{1.13}$$

a *left invariant integral* if $\phi\binom{k}{k}(\mathbf{1}\binom{k}{k}) = 1$ for all $k \in I$ and

$$(\mathrm{id} \otimes \phi\tbinom{l}{m})(\Delta_{l,l}(a)) = \delta_{k,p}\phi\tbinom{k}{m}(a)\mathbf{1}\tbinom{k}{l} \tag{1.14}$$

for all $k, l, m, p \in I$, $a \in A\begin{pmatrix} k & p \\ m & m \end{pmatrix}$.

We call them a *right invariant integral* if instead one has

$$(\phi\tbinom{k}{l} \otimes \mathrm{id})(\Delta_{l,l}(a)) = \delta_{m,p}\phi\tbinom{k}{m}(a)\mathbf{1}\tbinom{l}{m} \tag{1.15}$$

for all $k, l, m, p \in I$, $a \in A\begin{pmatrix} k & k \\ m & p \end{pmatrix}$.

If ϕ is a left and a right invariant integral, we call ϕ an *invariant integral.*

Lemma 1.30. *(1) If ϕ is a left or right invariant integral, then $\phi\binom{k}{m}(\mathbf{1}\binom{k}{m}) = 1$ for all $k, m \in I$ with $\mathbf{1}\binom{k}{m} \neq 0$.*

(2) If a left integral and a right invariant integral exist, then the two are equal and unique.

(3) Assume that $\mathscr{A}$ is an I-partial Hopf algebra with bijective antipode S. If ϕ is a left or right invariant integral, then it is unique, an invariant integral, and equal to $\phi \circ S$.

Proof. To see (1), take $a = \mathbf{1}\binom{k}{k}$ in (1.14). For (2), assume that ϕ and ψ are a left and a right invariant integral. Then for all $k, l, m \in I$, $a \in A\begin{pmatrix} k & k \\ m & m \end{pmatrix}$,

$$\phi\tbinom{k}{m}(a) = (\psi\tbinom{k}{k} \otimes \phi\tbinom{k}{m})(\Delta_{k,k}(a)) = \psi\tbinom{k}{m}(a)\phi\tbinom{k}{m}(\mathbf{1}\tbinom{k}{m}) = \psi\tbinom{k}{m}(a).$$

To prove (3), assume that ϕ is a left or right invariant integral. Then Corollary 1.25 implies that $\phi \circ S$ is a right or left invariant integral, and (2) implies the claim. □

Given a family of functionals as in (1.13), we denote by $\phi\colon A \to \mathbb{C}$ the corresponding functional on the total algebra.

We are finally ready to formulate our main definition.

Definition 1.31. A *partial compact quantum group* $\mathscr{G}$ is a partial Hopf *-algebra $\mathscr{A} = P(\mathscr{G})$ with an invariant integral ϕ that is positive in the sense that $\phi(a^*a) \geqslant 0$ for all $a \in A$. We also say that $\mathscr{G}$ is the partial compact quantum group *defined by* $\mathscr{A}$.

Remark 1.32. Following [6], we could also have called our objects *compact quantum groups of face type*, but we feel this gives the wrong impression when the base algebra is infinite dimensional (i.e. the object set is not compact). When referring to partial compact quantum groups, we feel that it is better reflected that only the *parts* of this object are to be considered compact, not the total object.

The total form of the invariance conditions (1.14) reads as follows.

Lemma 1.33. *A family of functionals ϕ as in* (1.13) *satisfies the conditions in* (1.14) *if and only if for all $a \in A$,*

$$(\phi \otimes \mathrm{id})(\Delta(a)(1 \otimes b)) = \sum_n \phi(\rho_n a)\rho_n b \quad \big(\textit{or} \quad (\mathrm{id} \otimes \phi)((b \otimes 1)\Delta(a)) = \sum_k \phi(\lambda_k a) b \lambda_k .\big)$$

Proof. Straightforward. □

We will need the following lemma at some point, which is an almost verbatim transcription of the argument in [12, Proposition 3.4].

Lemma 1.34. *Let $\mathscr{A}$ be a partial Hopf algebra with bijective antipode and an integral ϕ. Then ϕ is faithful in the following sense: if $a \in A$ and $\phi(ab) = 0$ (resp. $\phi(ba) = 0$) for all $b \in A$, then $a = 0$.*

Proof. Suppose $\phi(ba) = 0$ for all b. Then for all $d \in A$ and all functionals ω on A, the element $p = (\omega \otimes \mathrm{id})((d \otimes 1)\Delta(a))$ satisfies

$$(\mathrm{id} \otimes \phi)((1 \otimes c)\Delta(p)) = 0.$$

Continuing as in the proof of [12, Proposition 3.4], we obtain from the antipode trick that

$$\sum_n \phi(cS(q)\rho_n)\epsilon(p\lambda_n) = 0.$$

Choosing now for c and q local units of the form $\lambda_k \rho_l$, the normalization condition on ϕ gives that $\epsilon(p\lambda_n) = 0$ for all n, hence $\epsilon(p) = 0$. This implies $\omega(da) = 0$. As ω and d were arbitrary, it follows that $a = 0$.

The other case follows similarly, or by considering the opposite comultiplication. □

Lemma 1.35. *Let $\mathscr{A}$ be a partial Hopf algebra with integral ϕ. Then for all $a \in A$,*

$$(\mathrm{id} \otimes \phi)((1 \otimes b)\Delta(a)) = (S \otimes \phi)(\Delta(b)(1 \otimes a)),$$
$$(\phi \otimes \mathrm{id})(\Delta(a)(b \otimes 1)) = (\phi \otimes S)(\Delta(a)(1 \otimes b)).$$

Proof. The counit property, the relations (1.1) and (1.8) and Lemma 1.33 imply

$$\begin{aligned} a_{(1)}\phi(ba_{(2)}) &= \sum_n a_{(1)}\phi(\epsilon(b_{(1)}\rho_n)b_{(2)}\lambda_n a_{(2)}) \\ &= \sum_n \epsilon(b_{(1)}\rho_n)\rho_n a_{(1)}\phi(b_{(2)}a_{(2)}) \\ &= S(b_{(1)})b_{(2)}a_{(1)}\phi(b_{(3)}a_{(2)}) = S(b_{(1)})\phi(b_{(2)}a) \end{aligned}$$

for all $a, b \in A$. The second equation follows similarly. □

2 Representation theory of partial compact quantum groups

In this section, the representation theory of partial compact quantum groups is investigated. As the situation is quite similar to the case already studied by Hayashi [6], we do not always provide fully written out proofs, but only draw attention to those parts of the theory which need modification.

In what follows, the homogeneous component $A(K) = A\begin{pmatrix} k & l \\ m & n \end{pmatrix}$ of a partial bialgebra will now be mainly written as $A(K) = {}^{k}_{m}A^{l}_{n}$.

2.1 Corepresentations of partial Hopf algebras

Let $\mathscr{A}$ be an I-partial bialgebra. We write $\mathrm{Hom}_{\mathbb{C}}(V, W)$ for the vector space of linear maps between two vector spaces V and W.

Denote by Vect_{I^2} the category whose objects are I^2-graded vector spaces $V = \bigoplus_{k,l \in I} {}^{k}V^{l}$ and whose morphisms are linear maps T that preserve the grading and therefore can be written $T = \bigoplus_{k,l \in I} {}^{k}T^{l}$. Clearly, this category is abelian and $\mathbb{C}$-linear. We call an I^2-graded vector space $V = \bigoplus_{k,l \in I} {}^{k}V^{l}$ *row- and column-finite* if $\oplus_l {}^{k}V^{l}$ (resp. $\oplus_l {}^{k}V^{l}$) is finite-dimensional for k (resp. l) fixed. We henceforth abbreviate "row- and column-finite" by rcf.

Definition 2.1. Let $\mathscr{A}$ be an I-partial bialgebra and let $V = \bigoplus_{k,l} {}^{k}V^{l}$ be an rcf I^2-graded vector space. A *corepresentation* $\mathscr{X} = ({}^{k}_{m}X^{l}_{n})_{k,l,m,n}$ of $\mathscr{A}$ on V is a family of elements

$${}^{k}_{m}X^{l}_{n} \in {}^{k}_{m}A^{l}_{n} \otimes \mathrm{Hom}_{\mathbb{C}}({}^{m}V^{n}, {}^{k}V^{l}) \tag{2.1}$$

satisfying

$$(\Delta_{pq} \otimes \mathrm{id})({}^{k}_{m}X^{l}_{n}) = \left({}^{k}_{p}X^{l}_{q}\right)_{13}\left({}^{p}_{m}X^{q}_{n}\right)_{23}, \tag{2.2}$$

$$(\epsilon \otimes \mathrm{id})({}^{k}_{m}X^{l}_{n}) = \delta_{k,m}\delta_{l,n}\,\mathrm{id}_{{}^{k}V^{l}} \tag{2.3}$$

for all possible indices. We also call $(V, \mathscr{X})$ an *rcf corepresentation.*

Here, we use here the standard leg numbering notation, e.g $a_{23} = 1 \otimes a$.

Example 2.2. Equip the vector space $\mathbb{C}^{(I)} = \bigoplus_{k \in I} \mathbb{C}$ with the diagonal I^2-grading. Then the family $\mathscr{U}$ given by

$${}^{k}_{m}U^{l}_{n} = \delta_{k,l}\delta_{m,n}\mathbf{1}\binom{k}{m} \in {}^{k}_{m}A^{l}_{n} \tag{2.4}$$

is a corepresentation of $\mathscr{A}$ on $\mathbb{C}^{(I)}$. We call it the *trivial corepresentation.*

Example 2.3. Assume given an rcf family of subspaces

$${}^{m}V^{n} \subseteq \bigoplus_{k,l} {}^{k}_{m}A^{l}_{n}$$

satisfying

$$\Delta_{pq}({}^{m}V^{n}) \subseteq {}^{p}V^{q} \otimes {}^{p}_{m}A^{q}_{n}. \tag{2.5}$$

Then the elements ${}^{k}_{m}X^{l}_{n} \in {}^{k}_{m}A^{l}_{n} \otimes \mathrm{Hom}_{\mathbb{C}}({}^{m}V^{n}, {}^{k}V^{l})$ defined by

$${}^{k}_{m}X^{l}_{n}(1 \otimes b) = \Delta^{\mathrm{op}}_{kl}(b) \in {}^{k}_{m}A^{l}_{n} \otimes {}^{k}V^{l} \quad \text{for all } b \in {}^{m}V^{n}$$

form a corepresentation $\mathscr{X}$ of $\mathscr{A}$ on V. Indeed,

$$\begin{aligned}(\Delta_{pq} \otimes \mathrm{id})({}^{k}_{m}X^{l}_{n})(1 \otimes 1 \otimes b) = (\Delta_{pq} \otimes \mathrm{id})(\Delta^{\mathrm{op}}_{kl}(b)) &= \left({}^{k}_{p}X^{l}_{q}\right)_{13}\left({}^{p}_{m}X^{q}_{n}\right)_{23}(1 \otimes 1 \otimes b), \\ (\epsilon \otimes \mathrm{id})({}^{k}_{m}X^{l}_{n})b = (\epsilon \otimes \mathrm{id})(\Delta^{\mathrm{op}}_{kl}(b)) &= b\end{aligned}$$

for all $b \in {}^{m}V^{n}$. We call $\mathscr{X}$ the *regular corepresentation on V.*

We next consider the total form of a corepresentation.

Let $\mathscr{A}$ be a partial bialgebra with total algebra A and let V be an rcf I^2-graded vector space. Denote by $\lambda^{V}_{k}, \rho^{V}_{l} \in \mathrm{Hom}_{\mathbb{C}}(V)$ the projections onto the summands ${}^{k}V = \bigoplus_{q} {}^{k}V^{q}$ and $V^{l} = \bigoplus_{p} {}^{p}V^{l}$, respectively, identify $\mathrm{Hom}_{\mathbb{C}}({}^{m}V^{n}, {}^{k}V^{l})$ with $\lambda^{V}_{k}\rho^{V}_{l}\,\mathrm{Hom}_{\mathbb{C}}(V)\lambda^{V}_{m}\rho^{V}_{n}$, denote by $\mathrm{Hom}^{0}_{\mathbb{C}}(V) \subseteq \mathrm{Hom}_{\mathbb{C}}(V)$ the sum of all these subspaces, and define a homomorphism

$$\Delta \otimes \mathrm{id} \colon M(A \otimes \mathrm{Hom}^{0}_{\mathbb{C}}(V)) \to M(A \otimes A \otimes \mathrm{Hom}^{0}_{\mathbb{C}}(V))$$

similarly as we defined $\Delta \colon A \to M(A \otimes A)$.

Lemma 2.4. *Let $\mathscr{A}$ be an I-partial bialgebra and V an rcf I^2-graded vector space. If $\mathscr{X}$ is a corepresentation of $\mathscr{A}$ on V, then the sum*

$$X := \sum_{k,l,m,n} {}^{k}_{m}X^{l}_{n} \in M(A \otimes \mathrm{Hom}^0_{\mathbb{C}}(V)) \tag{2.6}$$

converges strictly and satisfies the following conditions:

0. $(\lambda_k \rho_m \otimes \mathrm{id})X(\lambda_l \rho_n \otimes \mathrm{id}) = (1 \otimes \lambda^V_k \rho^V_l)X(1 \otimes \lambda^V_m \rho^V_n) = {}^{k}_{m}X^{l}_{n}$,
1. $(A \otimes 1)X$, $X(A \otimes 1)$ *and* $(1 \otimes \mathrm{Hom}^0_{\mathbb{C}}(V))X(1 \otimes \mathrm{Hom}^0_{\mathbb{C}}(V))$ *lie in* $A \otimes \mathrm{Hom}^0_{\mathbb{C}}(V)$,
2. $(\Delta \otimes \mathrm{id})(X) = X_{13}X_{23}$,
3. *the sum* $(\epsilon \otimes \mathrm{id})(X) := \sum(\epsilon \otimes \mathrm{id})({}^{k}_{m}X^{l}_{n})$ *converges in* $M(\mathrm{Hom}^0_{\mathbb{C}}(V))$ *strictly to* id_V.

Conversely, if $X \in M(A \otimes \mathrm{Hom}^0_{\mathbb{C}}(V))$ satisfies 0.–3. with ${}^{k}_{m}X^{l}_{n}$ defined by 0., then $\mathscr{X} = ({}^{k}_{m}X^{l}_{n})_{k,l,m,n}$ is a corepresentation of $\mathscr{A}$ on V.

Proof. Straightforward. □

If $\mathscr{A}$ is a partial Hopf algebra, then every corepresentation multiplier has a generalized inverse.

Lemma 2.5. *Let $(V, \mathscr{X})$ be an rcf corepresentation of an I-partial Hopf algebra $\mathscr{A}$. Then*

$${}^{k}_{m}X^{l}_{n}(S \otimes \mathrm{id})({}^{p}_{r}X^{q}_{s}) = 0 \text{ if } (l,m,n) \neq (s,p,q), \qquad \sum_n {}^{k}_{m}X^{l}_{n} \cdot (S \otimes \mathrm{id})({}^{m}_{k'}X^{n}_{l}) = \delta_{k,k'} \mathbf{1}\tbinom{k}{m} \otimes \mathrm{id}_{{}^{k}V^{l}},$$

$$(S \otimes \mathrm{id})({}^{k}_{m}X^{l}_{n}){}^{p}_{r}X^{q}_{s} = 0 \text{ if } (k,m,n) \neq (r,p,q), \qquad \sum_m (S \otimes \mathrm{id})({}^{k}_{m}X^{l}_{n}){}^{m}_{k}X^{n}_{l'} = \delta_{l,l'} \mathbf{1}\tbinom{n}{l} \otimes \mathrm{id}_{{}^{k}V^{l}}.$$

In particular, the multiplier $Z := (S \otimes \mathrm{id})(X) \in M(A \otimes \mathrm{Hom}^0_{\mathbb{C}}(V))$ satisfies

$$XZ = \sum_k \lambda_k \otimes \lambda^V_k, \qquad ZX = \sum_n \rho_n \otimes \rho^V_n, \tag{2.7}$$

and is a generalized inverse of X in the sense that $XZX = X$ and $ZXZ = Z$.

Proof. The first equation follows from (2.1) and the relation $S({}^{p}_{r}A^{q}_{s}) \subseteq {}^{s}_{q}A^{r}_{p}$. To verify the second one, we use relations 2. and 3. in Definition 2.1 and (1.6) and find

$$\begin{aligned}
\sum_n {}^{k}_{m}X^{l}_{n} \cdot (S \otimes \mathrm{id})({}^{m}_{k'}X^{n}_{l}) &= \sum_n (m_A \circ (\mathrm{id} \otimes S) \otimes \mathrm{id})(({}^{k}_{m}X^{l}_{n})_{13}({}^{m}_{k'}X^{n}_{l})_{23}) \\
&= \sum_n (m_A \circ (\mathrm{id} \otimes S) \circ \Delta_{m,n} \otimes \mathrm{id})({}^{k}_{k'}X^{l}_{l}) \\
&= \delta_{k,k'} \mathbf{1}\tbinom{k}{l} \otimes (\epsilon \otimes \mathrm{id})({}^{k}_{k'}X^{l}_{l}) \\
&= \delta_{k,k'} \mathbf{1}\tbinom{k}{m} \otimes \mathrm{id}_{{}^{k}V^{l}},
\end{aligned}$$

where m_A denotes the multiplication of A. The third and fourth equation follow similarly, and the assertions concerning Z are direct consequences. □

Definition 2.6. Let $\mathscr{X}$ be an rcf corepresentation of a partial Hopf algebra. We denote the generalized inverse $(S \otimes \mathrm{id})(X)$ of X by X^{-1} and let

$$ {}^{k}_{m}(X^{-1})^{l}_{n} = (S \otimes \mathrm{id})({}^{k}_{m}X^{l}_{n}) \in {}^{n}_{l}A^{m}_{k} \otimes \mathrm{Hom}_{\mathbb{C}}({}^{m}V^{n}, {}^{k}V^{l}) $$

For completeness, we mention the following following converse to Lemma 2.5

Lemma 2.7. *Let $\mathscr{A}$ be an I-partial bialgebra, V an rcf I^2-graded vector space and $X, Z \in M(A \otimes \mathrm{Hom}^0_{\mathbb{C}}(V))$. If conditions 0.–2. in Lemma 2.4 and* (2.7) *hold, then the corresponding family $\mathscr{X} = ({}^{k}_{m}X^{l}_{n})_{k,l,m,n}$ is a corepresentation of $\mathscr{A}$ on V.*

Proof. We have to verify condition 3. in Lemma 2.4. If $(k,l) \neq (p,q)$, then $\epsilon({}^{k}_{p}A^{l}_{q}) = 0$ and hence $(\epsilon \otimes \mathrm{id})({}^{k}_{p}X^{l}_{q}) = 0$. The counit property and condition 2. in Lemma 2.4 imply

$$ \begin{aligned} {}^{k}_{m}X^{l}_{n} &= ((\epsilon \otimes \mathrm{id}) \circ \Delta \otimes \mathrm{id})({}^{k}_{m}X^{l}_{n}) \\ &= \sum_{p,q} (\epsilon \otimes \mathrm{id} \otimes \mathrm{id}) \left(({}^{k}_{p}X^{l}_{q})_{13} ({}^{p}_{m}X^{q}_{n})_{23} \right) = (1 \otimes {}^{k}T^{l}) {}^{k}_{m}X^{l}_{n}, \end{aligned} $$

where ${}^{k}T^{l} = (\epsilon \otimes \mathrm{id})({}^{k}_{k}X^{l}_{l}) \in \mathrm{Hom}_{\mathbb{C}}({}^{k}V^{l})$. Therefore, $T = \sum_{k,l} T_{k,l}$ satisfies $(1 \otimes T)X = X$. Multiplying on the right by Z, we find $T\lambda^V_k = \lambda^V_k$ for all k. Thus, $T = \mathrm{id}_V$. $\square$

Morphisms of corepresentations are defined as follows.

Definition 2.8. Let $\mathscr{A}$ be an I-partial bialgebra. A *morphism* T between rcf corepresentations $(V, \mathscr{X})$ and $(W, \mathscr{Y})$ of $\mathscr{A}$ is a family of linear maps

$$ {}^{k}T^{l} \in \mathrm{Hom}_{\mathbb{C}}({}^{k}V^{l}, {}^{k}W^{l}) $$

satisfying

$$ (1 \otimes {}^{k}T^{l}) {}^{k}_{m}X^{l}_{n} = {}^{k}_{m}Y^{l}_{n} (1 \otimes {}^{m}T^{n}) $$

We denote the category of all corepresentations of $\mathscr{A}$ by $\mathrm{Corep}(\mathscr{A})$.

Remark 2.9. Equivalently, a morphism between $(V, \mathscr{X})$ and $(W, \mathscr{Y})$ is just a morphism of I^2-graded vector spaces $T\colon V \to W$ satisfying $(1 \otimes T)X = Y(1 \otimes T)$. If $\mathscr{A}$ is a partial Hopf algebra, this condition is equivalent to each of the relations

$$ Y^{-1}(1 \otimes T)X = \sum_{m,n} \rho_n \otimes {}^{m}T^{n}, \qquad Y(1 \otimes T)X^{-1} = \sum_{k,l} \lambda_k \otimes {}^{k}T^{l}. $$

The category $\mathrm{Corep}(\mathscr{A})$ is evidently $\mathbb{C}$-linear and the forgetful functor $\mathrm{Corep}(\mathscr{A}) \to \mathrm{Vect}_{I^2}$ is faithful.

Given an I^2-graded vector space $V = \bigoplus_{k,l} {}^{k}V^{l}$ and a family of subspaces ${}^{k}W^{l} \subseteq {}^{k}V^{l}$, we denote by $\iota_W\colon W \to V$ and $\pi_W\colon V \to V/W = \bigoplus_{k,l} {}^{k}V^{l}/{}^{k}W^{l}$ the embedding and the quotient map.

Definition 2.10. Let $(V, \mathscr{X})$ be an rcf corepresentation of a partial bialgebra $\mathscr{A}$. We call a family of subspaces ${}^kW^l \subseteq {}^kV^l$ *invariant (w.r.t. $\mathscr{X}$)* if

$$(1 \otimes {}^k\pi_W^l)\,{}^k_mX^l_n(1 \otimes {}^m\iota_W^n) = 0, \tag{2.8}$$

and $(V, \mathscr{X})$ *irreducible* if the only invariant families of subspaces are $(0)_{k,l}$ and $({}^kV^l)_{k,l}$.

If a corepresentation has an invariant subspace, it restricts to it and factorizes to the quotient:

Lemma 2.11. *Let $(V, \mathscr{X})$ be an rcf corepresentation of a partial bialgebra and let ${}^kW^l \subseteq {}^kV^l$ be an invariant family of subspaces. Then there exist unique rcf corepresentations $(W, \iota_W^*\mathscr{X})$ and $(V/W, (\pi_W)_*\mathscr{X})$ such that ι_W and π_W are morphisms $(W, \iota_W^*\mathscr{X}) \to (V, \mathscr{X}) \to (V/W, (\pi_W)_*\mathscr{X})$.*

Proof. Straightforward. □

The following analogue of Schur's Lemma holds.

Lemma 2.12. *Let T be a morphism of rcf corepresentations $(V, \mathscr{X})$ and $(W, \mathscr{Y})$ of a partial bialgebra. Then the families of subspaces $\ker {}^kT^l \subseteq {}^kV^l$ and $\mathrm{img}\, {}^kT^l \subseteq {}^kW^l$ are invariant. In particular, if $(V, \mathscr{X})$ and $(W, \mathscr{Y})$ are irreducible, then either all ${}^kT^l$ are zero or all ${}^kT^l$ are isomorphisms.*

Proof. Straightforward again. □

Given corepresentations $\mathscr{X}$ and $\mathscr{Y}$ of a partial bialgebra $\mathscr{A}$ on respective rcf I^2-graded vector spaces V and W, we obtain an I^2-graded vector space $V \oplus W$ by taking componentwise direct sums, and use the canonical embedding

$$\mathrm{Hom}({}^mV^n, {}^kV^l) \oplus \mathrm{Hom}({}^mW^n, {}^kW^l) \hookrightarrow \mathrm{Hom}({}^mV^n \oplus {}^mW^n, {}^kV^l \oplus {}^kW^l)$$

to define the *direct sum* $\mathscr{X} \oplus \mathscr{Y}$, which is a corepresentation of $\mathscr{A}$ on $V \oplus W$. Then the natural embeddings from V and W into $V \oplus W$ and the projections onto V and W are evidently morphisms of corepresentations. More generally, given a family of rcf corepresentations $((V_\alpha, \mathscr{X}_\alpha))_\alpha$ such that the sum $\bigoplus_\alpha V_\alpha$ is rcf again, one can form the direct sum $\bigoplus_\alpha \mathscr{X}_\alpha$, which is a corepresentation on $\bigoplus_\alpha V_\alpha$.

Proposition 2.13. *Let $\mathscr{A}$ be an I-partial bialgebra. Then $\mathrm{Corep}(\mathscr{A})$ is a $\mathbb{C}$-linear abelian category and the forgetful functor $\mathrm{Corep}(\mathscr{A}) \to \mathrm{Vect}_{I^2}$ lifts kernels, cokernels and biproducts.*

Proof. The preceding considerations show that the forgetful functor lifts kernels, cokernels and biproducts. Moreover, in $\mathrm{Corep}(\mathscr{A})$, every monic is a kernel and every epic is a cokernel because the same is true in Vect_{I^2} and because kernels and cokernels lift. □

2.2 The tensor product and duality

The category Vect_{I^2} is a tensor category, where the product of I^2-graded vector spaces V and W is the sum of the subspaces

$$^{k}(V \underset{I}{\otimes} W)^{l} = \bigoplus_{p} ({}^{k}V^{p} \otimes {}^{p}V^{l}) \subseteq V \otimes W,$$

which we denote by $V \underset{I}{\otimes} W$, and the product of morphisms is the restriction of the ordinary tensor product. We pretend this product to be strictly associative. The unit for this product is the vector space $\mathbb{C}^{(I)} = \bigoplus_{k\in I} \mathbb{C}$. Indeed, for every I^2-graded vector space V, there exist obvious natural isomorphisms $\mathbb{C}^{(I)} \underset{I}{\otimes} V \cong V \cong V \underset{I}{\otimes} \mathbb{C}^{(I)}$.

Note that $V \underset{I}{\otimes} W$ is rcf if V and W are.

Given V and W as above, we identify $\mathrm{Hom}_{\mathbb{C}}({}^{m}V^{n}, {}^{k}V^{l}) \otimes \mathrm{Hom}_{\mathbb{C}}({}^{n}W^{q}, {}^{l}W^{p})$ with a subspace of

$$\mathrm{Hom}_{\mathbb{C}}({}^{m}V^{n} \otimes {}^{n}W^{q}, {}^{k}V^{l} \otimes {}^{l}W^{p}) \subseteq \mathrm{Hom}_{\mathbb{C}}({}^{m}(V \underset{I}{\otimes} W)^{q}, {}^{k}(V \underset{I}{\otimes} W)^{p}).$$

We can now construct a product of corepresentations as follows.

Lemma 2.14. *Let $\mathscr{X}$ and $\mathscr{Y}$ be copresentations of $\mathscr{A}$ on respective rcf I^2-graded vector spaces V and W. Then the sum*

$$ {}^{k}_{m}(X \circledT Y)^{p}_{q} := \sum_{l,n} \left({}^{k}_{m}X^{l}_{n}\right)_{12} \left({}^{l}_{n}Y^{p}_{q}\right)_{13} \tag{2.9}$$

has only finitely many non-zero terms and the elements

$$ {}^{k}_{m}(X \circledT Y)^{p}_{q} \in {}^{k}_{m}A^{p}_{q} \otimes \mathrm{Hom}_{\mathbb{C}}({}^{m}(V \underset{I}{\otimes} W)^{q}, {}^{k}(V \underset{I}{\otimes} W)^{p})$$

define an rcf corepresentation $\mathscr{X} \circledT \mathscr{Y}$ of $\mathscr{A}$ on $V \underset{I}{\otimes} W$.

Proof. The sum (2.9) is finite because V and W are rcf. Using the identification above, we see that $\left({}^{k}_{m}X^{l}_{n}\right)_{12} \left({}^{l}_{n}Y^{p}_{q}\right)_{13}$ lies in the tensor product ${}^{k}_{m}A^{p}_{q} \otimes \mathrm{Hom}_{\mathbb{C}}({}^{m}(V \underset{I}{\otimes} W)^{q}, {}^{k}(V \underset{I}{\otimes} W)^{p})$. Now, the fact that ${}^{k}_{m}(X \circledT Y)^{p}_{q}$ is a corepresentation follows easily from the multiplicativity of Δ and the weak multiplicativity of ϵ. □

Remark 2.15. The "total" multiplier associated to $\mathscr{X} \circledT \mathscr{Y}$ is just $X_{12}Y_{13}$.

Proposition 2.16. *Let $\mathscr{A}$ be an I-partial bialgebra. Then* $\mathrm{Corep}(\mathscr{A})$ *carries the structure of strict tensor category such that the product of rcf corepresentations $(V, \mathscr{X})$ and $(W, \mathscr{Y})$ is the corepresentation $(V \underset{I}{\otimes} W, \mathscr{X} \circledT \mathscr{Y})$, the unit is the trivial corepresentation $(\mathbb{C}^{(I)}, \mathscr{U})$, and the forgetful functor* $\mathrm{Corep}(\mathscr{A}) \to \mathrm{Vect}_{I^2}$ *is a strict tensor functor.*

Proof. This is clear. □

Given a corepresentation of a partial Hopf algebra, one can use the antipode to define a contragredient corepresentation on a dual space. Denote the dual of vector spaces V and linear maps T by $V^\vee$ and $T^\vee$, respectively, and define the dual of an I^2-graded vector space $V = \bigoplus_{k,l} {}^kV^l$ to be the space

$$V^\vee = \bigoplus_{k,l} {}^k(V^\vee)^l, \quad \text{where } {}^k(V^\vee)^l = ({}^kV^l)^\vee.$$

Recall that an object X in a strict tensor category is called a *right dual* of an object Y and Y is called a *left dual* of X, if there are morphisms $X \otimes Y \to 1$ and $1 \to Y \otimes X$, where 1 denotes the tensor unit, such that the obvious compositions

$$X \otimes 1 \to X \otimes Y \otimes X \to 1 \otimes X \quad \text{and} \quad 1 \otimes Y \to Y \otimes X \otimes Y \to Y \otimes 1$$

are the identity of X and Y, respectively.

Proposition 2.17. *Let $\mathscr{A}$ be an I-partial Hopf algebra with antipode S and let $(V, \mathscr{X})$ be an rcf corepresentation of $\mathscr{A}$. Then $V^\vee$ and the family $\mathscr{X}^\vee$ given by*

$$ {}^k_m(X^\vee)^l_n := (S \otimes -^\vee)({}^n_lX^m_k) \tag{2.10}$$

form an rcf corepresentation of $\mathscr{A}$ which is a right dual of $(V, \mathscr{X})$. If the antipode S of $\mathscr{A}$ is bijective, then $V^\vee$ and the family ${}^\vee\mathscr{X}$ given by

$$ {}^k_m({}^\vee X)^l_n := (S^{-1} \otimes -^\vee)({}^n_lX^m_k) \tag{2.11}$$

form an rcf corepresentation of $\mathscr{A}$ which is a left dual of $(V, \mathscr{X})$.

Proof. We only prove the assertion concerning $(V^\vee, \mathscr{X}^\vee)$. To see that this is a corepresentation, note that the element (2.10) belongs to ${}^k_mA^l_n \otimes \mathrm{Hom}_{\mathbb{C}}({}^m(V^\vee)^n, {}^k(V^\vee)^l)$ and use the relations $\Delta \circ S = (S \otimes S)\Delta^{\mathrm{op}}$ and $\epsilon \circ S = \epsilon$ from Corollary 1.25. Let us show that $(V^\vee, \mathscr{X}^\vee)$ is a right dual of $(V, \mathscr{X})$.

Given a finite-dimensional vector space W, denote by $T_W : W^\vee \otimes W \to \mathbb{C}$ the evaluation map and by $R_W : \mathbb{C} \to W \otimes W^\vee$ the coevaluation map, given by $1 \mapsto \sum_i w_i \otimes w_i^\vee$ if $(w_i)_i$ and $(w_i^\vee)_i$ are dual bases of W and $W^\vee$. With respect to these maps, $W^\vee$ is a right dual of W. If $F : W_1 \to W_2$ is a linear map between finite-dimensional spaces, then

$$(\mathrm{id}_{W_2} \otimes F^\vee) \circ R_{W_2} = (F \otimes \mathrm{id}_{W_1^\vee}) \circ R_{W_1}, \quad T_{W_1}(F^\vee \otimes \mathrm{id}_{W_2}) = T_{W_2}(\mathrm{id}_{W_2^\vee} \otimes F). \tag{2.12}$$

Now, define morphisms $R : \mathbb{C}^{(I)} \to V \underset{I}{\otimes} V^\vee$ and $T : V^\vee \underset{I}{\otimes} V \to \mathbb{C}^{(I)}$ by

$$ {}^kR^l = \delta_{k,l} \sum_p R_{{}^kV^p} : \mathbb{C} \to {}^k(V \underset{I}{\otimes} V^\vee)^l, \qquad {}^kT^l = \delta_{k,l} \sum_{k,p} T_{({}^pV^k)} : {}^k(V \underset{I}{\otimes} V^\vee)^l \to \mathbb{C}.$$

One easily checks that with respect to these maps, $V^\vee$ is a right dual of V in Vect_{I^2}.

We therefore only need to show that R is a morphism from $\mathscr{X} ⓣ \mathscr{X}^\vee$ to $\mathscr{U}$ and that T is a morphism from $\mathscr{U}$ to $\mathscr{X}^\vee ⓣ \mathscr{X}$. But (2.12) and Lemma 2.5 imply

$$\begin{aligned}(1 \otimes {}^kT^k) \sum_{l,n} \left({}^k_m(X^\vee)^l_n\right)_{12} \left({}^l_n X^k_q\right)_{13} &= (1 \otimes {}^kT^k) \sum_{l,n} (S \otimes -^\vee)({}^n_l X^m_k)_{12}({}^l_n X^k_q)_{13} \\ &= (\mathrm{id} \otimes {}^mT^m) \sum_{l,n} (S \otimes \mathrm{id})({}^n_l X^m_k)_{13}({}^l_n X^k_q)_{13} \\ &= \delta_{m,q} \mathbf{1}\tbinom{k}{q} \otimes {}^mT^m \\ &= {}^k_m U^k_q (1 \otimes {}^mT^m),\end{aligned}$$

and a similar calculation shows that

$$\sum_{l,n} \left({}^k_m X^l_n\right)_{12} \left({}^l_n(X^\vee)^p_m\right)_{13} (1 \otimes {}^mR^m) = \delta_{k,p} \mathbf{1}\tbinom{k}{m} \otimes {}^kR^k = (1 \otimes {}^kR^k) {}^k_m U^p_m,$$

whence the claim follows. ☐

Corollary 2.18. *Let $\mathscr{A}$ be a partial Hopf algebra. Then* Corep($\mathscr{A}$) *is a tensor category with right duals and, if the antipode of $\mathscr{A}$ is invertible, with left duals.*

Let $\mathscr{A}$ be an I-partial Hopf algebra. Then the tensor unit in Corep($\mathscr{A}$), which is the trivial corepresentation $\mathscr{U}$ on $\mathbb{C}^{(I)}$, need not be irreducible and decomposes into irreducible corepresentations that correspond to equivalence classes for the relation $\sim$ on I given by $k \sim l \leftrightarrow \mathbf{1}\tbinom{k}{l} \neq 0$ (see Remark 1.22).

Lemma 2.19. *Let $\mathscr{A}$ be an I-partial Hopf algebra and let $(I_\alpha)_\alpha$ be a labelled partition of I into equivalence classes for the relation $\sim$. Then for each α, the subspace $\mathbb{C}^{(I_\alpha)} \subseteq \mathbb{C}^{(I)}$ is invariant and the restriction $\mathscr{U}_\alpha$ of $\mathscr{U}$ to $\mathbb{C}^{(I_\alpha)}$ is irreducible. In particular, $\mathscr{U} = \bigoplus_\alpha \mathscr{U}_\alpha$ is a decomposition into irreducible corepresentations.*

Proof. Immediate from the fact that ${}^k_m U^k_m = \mathbf{1}\tbinom{k}{m}$ is 1 if $k \sim m$ and 0 if $k \not\sim m$. ☐

The decomposition of the tensor unit leads to a decomposition of the whole tensor category into full subcategories, where the tensor product acts like the multiplication in a partial algebra. Let us briefly sketch the general categorical idea, which can conveniently be formulated using enriched categories, see, e.g. [], and apply it to Corep($\mathscr{A}$). Denote by $\mathbb{C}$-**Cat** the category of all small $\mathbb{C}$-linear abelian categories. This category is monoidal with respect to the cartesian product, where the unit is the category with just one object with endomorphism algebra $\mathbb{C}$.

Definition 2.20. A *strict partial tensor category* is a small $\mathbb{C}$-**Cat**-category $\mathscr{C}$, that is, a set J (the object set) together with

- for each $\alpha, \beta \in J$, a $\mathbb{C}$-linear abelian category ${}^\alpha C^\beta$,

- for each $\alpha, \beta, \gamma \in J$, a functor

$$- \underset{\beta}{\otimes} -: {}^{\alpha}C^{\beta} \times {}^{\beta}C^{\gamma} \to {}^{\alpha}C^{\gamma},$$

- for each $\alpha \in J$, a unit object $U_{\alpha} \in {}^{\alpha}C^{\alpha}$,

satisfying the obvious strict associativity and strict unitality conditions.

Let C be a strict tensor category, where the tensor unit U decomposes as a direct sum $U = \bigoplus_{\alpha \in J} U_{\alpha}$. Then there exists a unique strict partial tensor category $\mathscr{C}$, where each ${}^{\alpha}C^{\beta}$ is the full subcategory of C formed by all objects $U_{\alpha} \otimes X \otimes U_{\beta}$ for $X \in C$, and $- \underset{\beta}{\otimes} -$ is the restriction of the tensor product on C. Note that the natural maps $U_{\alpha} \leftrightarrows U$ induce corresponding natural transformations $C \leftrightarrows {}^{\alpha}C^{\beta}$ for all α, β.

Conversely, to every strict partial tensor category $\mathscr{C}$, one can associate a "total" category C, where the objects are all families of objects ${}^{\alpha}X^{\beta} \in {}^{\alpha}C^{\beta}$ with only finitely many non-zero components, and morphisms are all families of morphisms of the components. This total category C carries a product given by

$$({}^{\alpha}X^{\beta})_{\alpha,\beta} \otimes ({}^{\beta}Y^{\gamma})_{\beta,\gamma} = \big(\bigoplus_{\beta}({}^{\alpha}X^{\beta} \underset{\beta}{\otimes} {}^{\beta}X^{\gamma})\big)_{\alpha,\gamma},$$

which we can can pretend to be strict. Note that in general, this product has no unit but that the U_{α} can be used to from "local units".

2.3 Decomposition into irreducible corepresentations

An integral allows to average morphisms of vector spaces so that one obtains morphisms of corepresentations.

Lemma 2.21. *Let $(V, \mathscr{X})$ and $(W, \mathscr{Y})$ be rcf corepresentations of a partial Hopf algebra $\mathscr{A}$ with an integral ϕ and let ${}^{k}T^{l} \in \mathrm{Hom}_{\mathbb{C}}({}^{k}V^{l}, {}^{k}W^{l})$ for all $k, l \in I$. Then the families*

$$\begin{aligned} {}^{k}\check{T}^{l}_{n} &:= \sum_{m} (\phi \otimes \mathrm{id})({}^{k}_{m}(Y^{-1})^{l}_{n}(1 \otimes {}^{m}T^{n})^{m}_{k}X^{n}_{l}), \\ {}^{k}_{m}\hat{T}^{l} &:= \sum_{n} (\phi \otimes \mathrm{id})({}^{k}_{m}Y^{l}_{n}(1 \otimes {}^{m}T^{n})^{m}_{k}(X^{-1})^{n}_{l}) \end{aligned}$$

form morphisms $\check{T}_{n}$ and ${}_{m}\hat{T}$ from $(V, \mathscr{X})$ to $(W, \mathscr{Y})$.

Proof. In total form, $\check{T} = (\phi \otimes \mathrm{id})(Y^{-1}(1 \otimes T)X)$ and $\hat{T} = (\phi \otimes \mathrm{id})(Y(1 \otimes T)X^{-1})$. Now, Lemma 2.4 and 1.33 imply

$$\begin{aligned}
Y^{-1}(1\otimes\check{T})X &= (\phi\otimes\mathrm{id}\otimes\mathrm{id})((Y^{-1})_{23}(Y^{-1})_{13}(1\otimes 1\otimes T)X_{13}X_{23}) \\
&= ((\phi\otimes\mathrm{id})\circ\Delta\otimes\mathrm{id})(Y^{-1}(1\otimes T)X) \\
&= \sum_n \rho_n\otimes(\phi\otimes\mathrm{id})((\rho_n\otimes 1)Y^{-1}(1\otimes T)X) \\
&= \sum_{m,n}\rho_n\otimes {}^m\check{T}^n,
\end{aligned}$$

whence $\check{T}$ is a morphism from $\mathscr{X}$ to $\mathscr{Y}$ by Remark 2.9. The assertion on $\hat{T}$ follows similarly. □

Lemma 2.22. *Let $\mathscr{A}$ be a partial Hopf algebra with an integral ϕ. Let $(V,\mathscr{X})$ be an rcf corepresentation and ${}^kW^l\subseteq{}^kV^l$ and invariant family of subspaces. Then there exists an idempotent endomorphism T of $(V,\mathscr{X})$ such that ${}^kW^l=\mathrm{img}\,{}^kT^l$ for all k,l.*

Proof. Choose idempotent endomorphisms ${}^kT^l$ of ${}^kV^l$ with image ${}^kW^l$. We apply the preceding lemma, obtain an endomorphism $\check{T}_n$ of $(V,\mathscr{X})$, and show that $\mathrm{img}\,{}^k\check{T}^l_n=\mathrm{img}\,{}^kT^l={}^kW^l$. In total form, invariance of W implies $(1\otimes T)X(1\otimes T)=X(1\otimes T)$. Applying $(S\otimes\mathrm{id})$, we get $(1\otimes T)X^{-1}(1\otimes T)=X^{-1}(1\otimes T)$. Now, we combine Lemma 2.4, Lemma 2.5 and normalisation of ϕ, and find

$$\begin{aligned}
\check{T}_nT &= (\phi\otimes\mathrm{id})(X^{-1}(1\otimes T\rho^V_n)X(1\otimes T)) \\
&= (\phi\otimes\mathrm{id})(X^{-1}X(\lambda_n\otimes T)) = \sum_m\phi(\mathbf{1}\tbinom{n}{m})\rho^V_mT = T
\end{aligned}$$

and similarly $T\check{T}_n=T$. Therefore, $\mathrm{img}\,\check{T}_n=\mathrm{img}\,T$. `Need to insert a remark on the choice of` n`, taking into account the decomposition into components of corepresentations --- namely, there exists an` n `such that` $V^m=0$ `whenever` $\mathbf{1}\tbinom{n}{m}=0$ □

Proposition 2.23. *Let $\mathscr{A}$ be a partial Hopf algebra with an integral. Then every rcf corepresentation of $\mathscr{A}$ decomposes into a direct sum of irreducible rcf corepresentations.*

Proof. The preceding lemma shows that every corepresentation is either irreducible or the direct sum of two corepresentations.

`Put that into the first or second subsection` Let us now first show that the trivial representation decomposes into irreducibles. Let I be the object set of $\mathscr{A}$, and say $k\sim l$ if $\mathbf{1}\tbinom{k}{l}\neq 0$. Then $\sim$ is an equivalence relation: as

$$\Delta_{ll}(\mathbf{1}\tbinom{k}{m})=\mathbf{1}\tbinom{k}{l}\otimes\mathbf{1}\tbinom{l}{m},$$

the relation $\sim$ is transitive. As $S(\mathbf{1}\tbinom{k}{l})=\mathbf{1}\tbinom{l}{k}$, we have that $\sim$ is symmetric. And as $\varepsilon(\mathbf{1}\tbinom{k}{k})=1$, we also have that $\sim$ is reflexive.

Let then $I = \sqcup_{\alpha\in\mathscr{I}} I_\alpha$ be a labeled partition associated to $\sim$. Define $\mathbb{C}_{I_\alpha} \subseteq \mathbb{C}_I$ as the linear span of the homogeneous components with index in α. It is clear then that the $\mathbb{C}_{I_\alpha}$ are invariant and irreducible.

Consider now a general corepresentation $(X, \mathcal{H})$. Let ${}_\alpha\mathcal{H}_\beta$ be the closed linear span of the homogeneous components with index in $\alpha \times \beta$. As we can identify

$$ {}_\alpha\mathcal{H}_\beta \cong \mathbb{C}_{I_\alpha} \circledT \mathcal{H} \circledT \mathbb{C}_{I_\beta}, $$

we see that ${}_\alpha\mathcal{H}_\beta$ is an invariant subspace of $\mathcal{H}$. Hence we may as well suppose that $\mathcal{H} = {}_\alpha\mathcal{H}_\beta$.

But let then T be a bounded self-intertwiner of $\mathcal{H}$. Then from the two equations in Remark **??**, we see that $T \to {}^kT^l$ is injective for any choice of $k \in \alpha, l \in \beta$. It follows that the algebra of self-intertwiners of $\mathcal{H}$ is finite-dimensional. We then immediately conclude that $\mathcal{H}$ is a finite direct sum of irreducible invariant subspaces. □

Corollary 2.24. *Let $\mathscr{A}$ be a partial Hopf algebra with an integral. Then every irreducible rcf corepresentation $(V, \mathscr{X})$ of $\mathscr{A}$ is equivalent to its right bidual $(V, \mathscr{X}^{\vee\vee})$.*

Proof. The corepresentations above are a left and a right dual of $(V^\vee, \mathscr{X}^\vee)$. But in every semi-simple tensor category, left and right duals are isomorphic []. □

2.4 Matrix coefficients of irreducible corepresentations

Our next goal is to obtain the analogue of Schur's orthogonality relations for matrix coefficients of corepresentations.

Given finite-dimensional vector spaces V and W, the dual space of $\mathrm{Hom}_{\mathbb{C}}(V, W)$ is linearly spanned by functionals of the form

$$ \omega_{f,v} \colon \mathrm{Hom}_{\mathbb{C}}(V, W) \to \mathbb{C}, \quad T \mapsto (f|Tv), $$

where $v \in V$, $f \in W^\vee$, and $(-|-)$ denotes the natural pairing of $W^\vee$ with W.

Definition 2.25. Let $\mathscr{A}$ be a partial bialgebra. The space of *matrix coefficients* $\mathcal{C}(\mathscr{X})$ of an rcf corepresentation $(V, \mathscr{X})$ is the sum of the subspaces

$$ {}^k_m\mathcal{C}(\mathscr{X})^l_n = \left\{ (\mathrm{id} \otimes \omega_{f,v})({}^k_mX^l_n) \mid v \in {}^mV^n, f \in ({}^kV^l)^\vee \right\} \subseteq {}^k_mA^l_n. $$

Let $(V, \mathscr{X})$ be an rcf corepresentation of a partial bialgebra $\mathscr{A}$. Condition 2 in Definition 2.1 implies

$$ \Delta_{pq}({}^k_m\mathcal{C}(\mathscr{X})^l_n) \subseteq {}^k_p\mathcal{C}(\mathscr{X})^l_q \otimes {}^p_m\mathcal{C}(\mathscr{X})^q_n. \tag{2.13} $$

Thus, the ${}^{k}_{m}\mathcal{C}(\mathscr{X})^{l}_{n}$ form a partial coalgebra with respect to Δ and ϵ. Moreover, for each k, l, the I^2-graded vector space

$$ {}^{k}\mathcal{C}(\mathscr{X})^{l} := \bigoplus_{m,n} {}^{k}_{m}\mathcal{C}(\mathscr{X})^{l}_{n} $$

is rcf, and the inclusion above shows that one can form the regular corepresentation on this space.

Lemma 2.26. *Let $(V, \mathscr{X})$ be an rcf corepresentation of a partial bialgebra and let $f \in ({}^{k}V^{l})^{\vee}$. Then the family of maps*

$$ {}^{m}T^{n}_{(f)} \colon {}^{m}V^{n} \to {}^{k}_{m}\mathcal{C}(\mathscr{X})^{l}_{n}, \ w \mapsto (\mathrm{id} \otimes \omega_{f,w})({}^{k}_{m}X^{l}_{n}) = (\mathrm{id} \otimes f)({}^{k}_{m}X^{l}_{n}(1 \otimes w)), $$

is a morphism from $\mathscr{X}$ to the regular corepresentation on ${}^{k}\mathcal{C}(\mathscr{X})^{l}$.

Proof. Denote by $\mathscr{Y}$ the regular corepresentation on $\bigoplus_{m,n} {}^{k}_{m}\mathcal{C}(\mathscr{X})^{l}_{n}$. Then

$$ \begin{aligned} {}^{p}_{m}Y^{q}_{n}(1 \otimes {}^{m}T^{n}_{(f)}(v)) &= (\Delta^{\mathrm{op}}_{pq} \otimes \omega_{f,v})({}^{k}_{m}X^{l}_{n}) \\ &= (\mathrm{id} \otimes \mathrm{id} \otimes f)(({}^{k}_{p}X^{l}_{q})_{23}({}^{p}_{m}X^{q}_{n})_{13}(1 \otimes 1 \otimes v)) \\ &= (1 \otimes {}^{p}T^{q}_{(f)}) {}^{p}_{m}X^{q}_{n}(1 \otimes v) \end{aligned} $$

for all $v \in {}^{m}V^{n}$. □

Proposition 2.27. *Let $\mathscr{A}$ be a partial Hopf algebra with an integral. Then the total algebra A is the sum of the matrix coefficients of irreducible rcf corepresentations.*

Proof. Let $a \in {}^{k}_{m}A^{l}_{n}$. Write

$$ \Delta_{pq}(a) = \sum_{i} b^{i}_{pq} \otimes c^{i}_{pq} $$

with linearly independent $(c^{i}_{pq})_{i}$. Then the family of subspaces

$$ {}^{p}V^{q} = \mathrm{span}\{b^{i}_{pq} : i\} \subseteq \bigoplus_{k,l} {}^{k}_{m}A^{l}_{n} $$

is rcf, and the relation

$$ \sum_{i} \Delta_{rs}(b^{i}_{pq}) \otimes c^{i}_{pq} = (\Delta_{rs} \otimes \mathrm{id})\Delta_{pq}(a) = (\mathrm{id} \otimes \Delta_{pq})\Delta_{rs}(a) = \sum_{j} b^{j}_{rs} \otimes \Delta_{pq}(c^{j}_{rs}) $$

implies $\Delta_{rs}({}^{p}V^{q}) \subseteq {}^{r}V^{s} \otimes {}^{r}_{p}A^{s}_{q}$. We can therefore form the regular corepresentation $\mathscr{X}$ on V as in Example 2.3, and

$$ a = (\mathrm{id} \otimes \epsilon)(\Delta^{\mathrm{op}}_{mn}(a)) = (\mathrm{id} \otimes \epsilon)({}^{m}_{m}X^{n}_{n}(1 \otimes a)) \in {}^{m}_{m}\mathcal{C}(\mathscr{X})^{n}_{n}. $$

Decomposing $(V, \mathscr{X})$, we find that a is contained in the sum of matrix coefficients of irreducible rcf corepresentations. □

The first part of the orthogonality relations concerns matrix coefficients of inequivalent irreducible corepresentations.

Proposition 2.28. *Let $\mathcal{A}$ be a partial Hopf algebra with an integral ϕ and inequivalent irreducible rcf corepresentations $(V, \mathscr{X})$ and $(W, \mathscr{Y})$. Then for all $a \in \mathcal{C}(X), b \in \mathcal{C}(Y)$,*

$$\phi(S(b)a) = \phi(bS(a)) = 0.$$

Proof. Since ϕ vanishes on $S({}^{k}_{m}A^{l}_{n})^{p}_{r}A^{q}_{s}$ and on ${}^{p}_{r}A^{q}_{s}S({}^{k}_{m}A^{l}_{n})$ unless $(p,q,r,s) = (m,n,k,l)$, it suffices to prove the assertion for elements of the form

$$a = (\mathrm{id} \otimes \omega_{f,v})({}^{k}_{m}X^{l}_{n}) \qquad \text{and} \qquad b = (\mathrm{id} \otimes \omega_{g,w})({}^{k}_{m}Y^{l}_{n})$$

where $f \in ({}^{k}V^{l})^{\vee}, v \in {}^{m}V^{n}$ and $g \in ({}^{k}W^{l})^{\vee}, w \in {}^{m}W^{n}$. Lemma 2.21, applied to the family

$${}^{p}T^{q} \colon {}^{p}V^{q} \to {}^{p}W^{q}, \quad x \mapsto \delta_{p,k}\delta_{q,l}f(x)w,$$

yields morphisms $\check{T}_{l}, \hat{T}_{k}$ from $(V, \mathscr{X})$ to $(W, \mathscr{Y})$ which necessarily are 0. Inserting the definition of $\check{T}_{l}$, we find

$$\begin{aligned}\phi(S(b)a) &= \phi\big((S \otimes \omega_{g,w})({}^{m}_{k}Y^{n}_{l}) \cdot (\mathrm{id} \otimes \omega_{f,v})({}^{k}_{m}X^{l}_{n})\big) \\ &= (\phi \otimes \omega_{g,v})\left({}^{l}_{n}(Y^{-1})^{k}_{m}(1 \otimes {}^{k}T^{l})^{k}_{m}X^{l}_{n}\right) = \omega_{g,v}({}^{m}\check{T}_{l}^{n}) = 0.\end{aligned}$$

A similar calculation involving $\hat{T}$ shows that $\phi(bS(a)) = 0$. ∎

Theorem 2.29. *Let $\mathcal{A}$ be a partial Hopf algebra with an integral ϕ. Let $(V, \mathscr{X})$ be an irreducible rcf corepresentation of $\mathscr{A}$, let $F_{\mathscr{X}}$ be an isomorphism from $(V, \mathscr{X})$ to $(V, \mathscr{X}^{\vee\vee})$ and let $G_{\mathscr{X}} = F_{\mathscr{X}}^{-1}$.*

1. *The numbers $\alpha := \sum_{k} \mathrm{Tr}({}^{k}G^{l}_{\mathscr{X}})$ and $\beta := \sum_{n} \mathrm{Tr}({}^{m}F^{n}_{\mathscr{X}})$ do not depend on l or n.*
2. *For all k, l, m, n,*

$$\begin{aligned}(\phi \otimes \mathrm{id})({}^{m}_{k}(X^{-1})^{n}_{l}\,{}^{k}_{m}X^{l}_{n}) &= \alpha^{-1}\mathrm{Tr}({}^{k}G^{l}_{\mathscr{X}})\,\mathrm{id}_{{}^{m}V^{n}}, \\ (\phi \otimes \mathrm{id})({}^{k}_{m}X^{l}_{n}\,{}^{m}_{k}(X^{-1})^{n}_{l}) &= \beta^{-1}\mathrm{Tr}({}^{m}F^{n}_{\mathscr{X}})\,\mathrm{id}_{{}^{k}V^{l}}.\end{aligned}$$

3. *Denote by Σ_{klmn} the flip map ${}^{k}V^{l} \otimes {}^{m}V^{n} \to {}^{m}V^{n} \otimes {}^{k}V^{l}$. Then*

$$\begin{aligned}(\phi \otimes \mathrm{id} \otimes \mathrm{id})(({}^{m}_{k}(X^{-1})^{n}_{l})_{12}({}^{k}_{m}X^{l}_{n})_{13}) &= \alpha^{-1}(\mathrm{id}_{{}^{m}V^{n}} \otimes {}^{k}G^{l}_{\mathscr{X}}) \circ \Sigma_{klmn}, \\ (\phi \otimes \mathrm{id} \otimes \mathrm{id})(({}^{k}_{m}X^{l}_{n})_{13}({}^{m}_{k}(X^{-1})^{n}_{l})_{12}) &= \beta^{-1}({}^{m}F^{n}_{\mathscr{X}} \otimes \mathrm{id}_{{}^{k}V^{l}}) \circ \Sigma_{klmn}.\end{aligned}$$

Proof. We prove the assertions and equations involving α in (1), (2) and (3) simultaneously; the assertions involving β follow similarly.

Consider the following endomorphism $F_{m,n,k,l}$ of ${}^mV^n \otimes {}^kV^l$,

$$\begin{aligned} F_{m,n,k,l} &:= (\phi \otimes \mathrm{id} \otimes \mathrm{id})({}^m_k(X^{-1})^n_l)_{12}({}^k_mX^l_n)_{13}) \circ \Sigma_{mnkl} \\ &= (\phi \otimes \mathrm{id} \otimes \mathrm{id}) \left({}^m_k(X^{-1})^n_l)_{12}\Sigma_{klkl,23}({}^k_mX^l_n)_{12}\right). \end{aligned}$$

By applying Lemma 2.21 with respect to the flip map Σ_{klkl}, we see that the family $(F_{m,n,k,l})_{m,n}$ is an endomorphism of $(V \otimes {}^kV^l, X_{12})$ and hence

$$F_{m,n,k,l} = \mathrm{id}_{{}^mV^n} \otimes {}^kR^l \tag{2.14}$$

with some ${}^kR^l \in \mathrm{Hom}_{\mathbb{C}}({}^kV^l)$ not depending on m, n. On the other hand,

$$\begin{aligned} F_{m,n,k,l} &= (\phi \otimes \mathrm{id} \otimes \mathrm{id})((S \otimes \mathrm{id})({}^m_kX^n_l)_{12}({}^k_mX^l_n)_{13}) \circ \Sigma_{mnkl} \\ &= (\phi \circ S^{-1} \otimes \mathrm{id} \otimes \mathrm{id}) \left(((S \otimes \mathrm{id})({}^k_mX^l_n))_{13}((S^2 \otimes \mathrm{id})({}^m_kX^n_l))_{12}\right) \circ \Sigma_{mnkl} \\ &= (\phi \circ S^{-1} \otimes \mathrm{id} \otimes \mathrm{id}) \left(({}^k_m(X^{-1})^l_n)_{13}(\Sigma_{mnmn})_{23}({}^m_k(X^{\vee\vee})^n_l)_{13}\right). \end{aligned}$$

Since $\phi \circ S^{-1}$ is an invariant functional for $\mathscr{A}$, we can again apply Lemma 2.21 and find that the family $(F_{m,n,k,l})_{k,l}$ is a morphism

$$(F_{m,n,k,l})_{k,l} : ({}^mV^n \otimes V, (\mathscr{X}^{\vee\vee})_{13}) \to ({}^mV^n \otimes V, \mathscr{X}_{13}).$$

Therefore,

$$F_{m,n,k,l} = {}^mT^n \otimes {}^kG^l_{\mathscr{X}} \tag{2.15}$$

with some ${}^mT^n \in \mathrm{Hom}_{\mathbb{C}}({}^mV^n)$ not depending on k, l. Combining (2.14) and (2.15), we conclude that, for some $\lambda \in \mathbb{C}$,

$$F_{m,n,k,l} = \lambda(\mathrm{id}_{{}^mV^n} \otimes {}^kG^l_{\mathscr{X}})$$

Choose dual bases $(v_i)_i$ for ${}^kV^l$ and $(f_i)_i$ for $({}^kV^l)^\vee$. Then

$$\lambda \mathrm{Tr}({}^kG^l_{\mathscr{X}})\,\mathrm{id}_{{}^mV^n} = \sum_i (\mathrm{id} \otimes \omega_{f_i,v_i})(F_{m,n,k,l}) = (\phi \otimes \mathrm{id})(({}^k_mX^l_n)^* \, {}^k_mX^l_n).$$

Taking $n = l$ and summing over k, the relations $\sum_k ({}^k_mX^l_n)^* \, {}^k_mX^l_n = \mathbf{1}\binom{l}{n} \otimes \mathrm{id}_{{}^mV^n}$ and $\phi(\mathbf{1}\binom{l}{l}) = 1$ give

$$\lambda \cdot \sum_k \mathrm{Tr}({}^kG^l_{\mathscr{X}}) = 1.$$

Now all assertions in (1)–(3) concerning α follow. $\square$

Corollary 2.30. *Let $\mathscr{A}$ be a partial Hopf algebra with an integral ϕ, let $(V, \mathscr{X})$ be an irreducible rcf corepresentation of $\mathscr{A}$, let $F_{\mathscr{X}}$ be an isomorphism from $(V, \mathscr{X})$ to $(V, \mathscr{X}^{\vee\vee})$ and $G_{\mathscr{X}} = F_{\mathscr{X}}^{-1}$, and let $a = (\mathrm{id} \otimes \omega_{f,v})({}^{k}_{m}X^{l}_{n})$ and $b = (\mathrm{id} \otimes \omega_{g,w})({}^{m}_{k}X^{n}_{l})$, where $f \in ({}^{k}V^{l})^{\vee}$, $v \in {}^{m}V^{n}$, $g \in ({}^{m}V^{n})^{\vee}$, $w \in {}^{k}V^{l}$. Then*

$$\phi(S(b)a) = \frac{(g|v)(v|G_{\mathscr{X}}w)}{\sum_m \mathrm{Tr}({}^{m}G^{n}_{\mathscr{X}})}, \qquad \phi(aS(b)) = \frac{(g|F_{\mathscr{X}}v)(f|w)}{\sum_n \mathrm{Tr}({}^{m}F^{n}_{\mathscr{X}})}.$$

Proof. Apply $\omega_{g,w} \otimes \omega_{f,v}$ to the formulas in Theorem 2.29 3. □

Corollary 2.31. *Let $\mathscr{A}$ be a partial Hopf algebra with an integral and let $((V_\alpha, \mathscr{X}_\alpha))_\alpha$ be a representative family of all irreducible rcf corepresentations of $\mathscr{A}$. Then the map*

$$\bigoplus_{\alpha} \bigoplus_{k,l,m,n} (({}^{k}V^{l}_{\alpha})^{\vee} \otimes {}^{m}V^{n}_{\alpha}) \to A$$

that sends $v^{\vee} \otimes w \in ({}^{k}V^{l}_{\alpha})^{\vee} \otimes {}^{m}V^{n}_{\alpha}$ to $(\mathrm{id} \otimes \omega_{v^{\vee},w})({}^{k}_{m}(X_\alpha)^{l}_{n})$, is a linear isomorphism.

Proof. This follows from Proposition 2.27, Proposition 2.28 and Corollary 2.30. □

Corollary 2.32. *Let $\mathscr{A}$ be a partial Hopf algebra with an integral, let $((V_\alpha, \mathscr{X}_\alpha))_\alpha$ be a representative family of all irreducible rcf corepresentations of $\mathscr{A}$, fix α, k, l and denote by ${}^{k}\mathscr{Y}^{l}_{\alpha}$ the regular corepresentation on ${}^{k}\mathcal{C}(\mathscr{X}_\alpha)^{l}$. Then there exists a linear isomorphism*

$$({}^{k}V^{l})^{\vee} \to \mathrm{Mor}((V_\alpha, \mathscr{X}_\alpha), ({}^{k}\mathcal{C}(\mathscr{X}_\alpha)^{l}, {}^{k}\mathscr{Y}^{l}_{\alpha}))$$

assigning to each $v \in ({}^{k}V^{l})^{\vee}$ the morphism $T_{(v)}$ of Lemma 2.26.

2.5 Analogues of Woronowicz's characters

Suppose now $a \in {}^{k}_{m}A^{l}_{n}$ for some partial bialgebra $\mathscr{A}$. Then for $\omega \in \mathrm{Hom}_{\mathbb{C}}(A, \mathbb{C})$, we can define

$$\omega \underset{p,q}{*} a := (\mathrm{id} \otimes \omega)(\Delta_{pq}(a)), \qquad a \underset{r,s}{*} \omega := (\omega \otimes \mathrm{id})(\Delta_{rs}(a)).$$

Clearly we can define

$$\omega \underset{p,q}{*} a \underset{r,s}{*} \omega' := (\omega \underset{p,q}{*} a) \underset{r,s}{*} \omega' = \omega \underset{p,q}{*} (a \underset{r,s}{*} \omega').$$

When ω has support on the $A(K)$ with $K_u = K_d$, we can write, for $a \in {}^{k}_{m}A^{l}_{n}$,

$$\omega * a := \sum_{p,q} \omega \underset{p,q}{*} a = \omega \underset{m,n}{*} a, \quad a * \omega = \sum_{r,s} a \underset{r,s}{*} \omega = a \underset{k,l}{*} \omega.$$

We shall say that an entire function f has *exponential growth on the right half-plane* if there exist $C, d > 0$ such that $|f(x + iy)| \leqslant C\mathrm{e}^{dx}$ for all $x, y \in \mathbb{R}$ with $x > 0$.

Theorem 2.33. *Let $\mathscr{A}$ be a partial Hopf algebra with an integral ϕ. Then there exists a unique family of linear functionals $f_z \colon A \to \mathbb{C}$ such that*

(1) f_z vanishes on $A(K)$ when $K_u \neq K_d$.

(2) for each $a \in A$, the function $z \mapsto f_z(a)$ is entire and of exponential growth on the right half-plane.

(3) $f_0 = \epsilon$ and $(f_z \otimes f_{z'}) \circ \Delta = f_{z+z'}$ for all $z, z' \in \mathbb{C}$.

(4) $\phi(ab) = \phi(b(f_1 * a * f_1))$ for all $a, b \in A$.

This family furthermore satisfies

(5) $f_z(ab) = f_z(a)f_z(b)$ for $a \in A(K)$ and $b \in A(L)$ with $K_r = L_l$.

(6) $S^2(a) = f_{-1} * a * f_1$ for all $a \in A$.

(7) $f_z(\mathbf{1}\binom{l}{n}) = \delta_{l,n}$ and $f_z \circ S = f_{-z}$ for all $a \in A$.

Note that condition (3) is meaningful by condition (1).

Proof. We first prove uniqueness. Assume that $(f_z)_z$ is a family of functionals satisfying (1)–(4). Since ϕ is faithful, the map $\sigma \colon a \mapsto f_1 * a * f_1$ is uniquely determined by ϕ, and one easily sees that it is a homomorphism. Using (3), we find that $\epsilon \circ \sigma^n = f_{2n}$, which uniquely determines these functionals. Using (2) and the fact that every entire function of exponential growth on the right half-plane is uniquely determined by its values at $\mathbb{N} \subseteq \mathbb{C}$, we can conclude that the family f_z is uniquely determined. Moreover, since the property (5) holds for $z = 2n$, we also conclude by the same argument as above that it holds for all $z \in \mathbb{C}$.

Let us now prove existence. By Corollary 2.31, we can define for each $z \in \mathbb{C}$ a functional $f_z \colon A \to \mathbb{C}$ such that for every irreducible rcf corepresentation $(V, \mathscr{X})$,

$$f_z((\mathrm{id} \otimes \omega_{\xi,\eta})({}^k_m X^l_n)) = \delta_{k,m}\delta_{l,n} \cdot \omega_{\xi,\eta}(({}^k F^l_{\mathscr{X}})^z) \quad \text{for all } \xi \in {}^k V^l, \eta \in {}^m V^n,$$

or, equivalently,

$$(f_z \otimes \mathrm{id})({}^k_m X^l_n) = \delta_{k,m}\delta_{l,n} \cdot ({}^k F^l_{\mathscr{X}})^z,$$

where $F_{\mathscr{X}}$ is a non-zero positive operator implementing a morphism from $(V, \mathscr{X})$ to $(V, \mathscr{X}^{\vee\vee})$, scaled such that

$$\alpha_X := \sum_k \mathrm{Tr}({}^k(F_X^{-1})^l) = \sum_n \mathrm{Tr}({}^m F_X{}^n)$$

for all l, n (see Proposition **??** and Theorem 2.29). By construction, (1) and (2) hold. We show that the $(f_z)_z$ satisfy the assertions (3)–(7). Throughout the following arguments, let $(V, \mathscr{X})$ be a unitary irreducible corepresentation $(V, \mathscr{X})$ and let $F_{\mathscr{X}}$ be as above.

We first prove property (3). This follows from the relations

$$(f_0 \otimes \mathrm{id})({}^{k}_{m}X^{l}_{n}) = \delta_{k,m}\delta_{l,n}\,\mathrm{id}_{{}^{k}V^{l}} = (\epsilon \otimes \mathrm{id})({}^{k}_{m}X^{l}_{n})$$

and

$$\begin{aligned}(((f_z \otimes f_{z'}) \circ \Delta) \otimes \mathrm{id})({}^{k}_{m}X^{l}_{n}) &= \delta_{k,m}\delta_{l,n}(f_z \otimes f_{z'} \otimes \mathrm{id})\big(({}^{k}_{k}X^{l}_{l})_{13}({}^{k}_{k}X^{l}_{l})_{23}\big) \\ &= \delta_{k,m}\delta_{l,n}({}^{k}F_{\mathscr{X}}{}^{l})^{z} \cdot ({}^{k}F_{\mathscr{X}}{}^{l})^{z'} \\ &= (f_{z+z'} \otimes \mathrm{id})({}^{k}_{m}X^{l}_{n}).\end{aligned}$$

Applying slice maps of the form $\mathrm{id} \otimes \omega_{\xi,\xi'}$ and invoking Theorem 2.29, this proves (3).

To prove (4), write again $\Delta^{(2)} = (\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$, and put

$$\theta_{z,z'} := (f_{z'} \otimes \mathrm{id} \otimes f_z) \circ \Delta^{(2)}.$$

Then

$$\begin{aligned}(\theta_{z,z'} \otimes \mathrm{id})({}^{k}_{m}X^{l}_{n}) &= (f_{z'} \otimes \mathrm{id} \otimes f_z \otimes \mathrm{id})(({}^{k}_{k}X^{l}_{l})_{14}({}^{k}_{m}X^{l}_{n})_{24}({}^{m}_{m}X^{n}_{n})_{34}) \\ &= (1 \otimes ({}^{k}F_{\mathscr{X}}{}^{l})^{z'})\,{}^{k}_{m}X^{l}_{n}(1 \otimes ({}^{m}F_{\mathscr{X}}{}^{n})^{z}).\end{aligned}$$

We take $z = z' = 1$, use Theorem 2.29, where now $\alpha = \beta$ by our scaling of $F_{\mathscr{X}}$, and obtain

$$\begin{aligned}&(\phi \otimes \mathrm{id} \otimes \mathrm{id})(({}^{k}_{m}X^{l}_{n})^{*}_{12}((\theta_{1,1} \otimes \mathrm{id})({}^{k}_{m}X^{l}_{n}))_{13}) \\ &\qquad = \alpha^{-1}(\mathrm{id} \otimes {}^{k}F_{\mathscr{X}}{}^{l})(\mathrm{id} \otimes {}^{k}(F_{\mathscr{X}}^{-1})^{l})\Sigma_{k,l,m,n}(\mathrm{id} \otimes {}^{m}F_{\mathscr{X}}{}^{n}) \\ &\qquad = \beta^{-1}({}^{m}F_{\mathscr{X}}{}^{n} \otimes \mathrm{id})\Sigma_{k,l,m,n} \\ &\qquad = (\phi \otimes \mathrm{id} \otimes \mathrm{id})(({}^{k}_{m}X^{l}_{n})_{13}({}^{k}_{m}X^{l}_{n})^{*}_{12}).\end{aligned}$$

To conclude the proof of assertion (4), apply again slice maps of the form $\omega_{\xi,\xi'} \otimes \omega_{\eta,\eta'}$.

We have then already argued that the property (5) automatically holds. To show the property (6), note that by Proposition **??** and the calculation above,

$$(S^2 \otimes \mathrm{id})({}^{k}_{m}X^{l}_{n}) = (1 \otimes {}^{k}F_{\mathscr{X}}{}^{l})\,{}^{k}_{m}X^{l}_{n}(1 \otimes {}^{m}F_{\mathscr{X}}{}^{n})^{-1} = (\theta_{-1,1} \otimes \mathrm{id})({}^{k}_{m}X^{l}_{n}).$$

Assertion (6) follows again by applying slice maps.

Finally, (1), (2) and (4) immediately imply the relation $f_z(\mathbf{1}\binom{k}{m}) = \delta_{k,m}$. The concrete construction of f_z combined with property (3), the identity (**??**) and the partial character property (5) gives the equality

$$\begin{aligned}(f_{-z} \otimes \mathrm{id})({}^{k}_{k}X^{l}_{l}) = ({}^{k}(F_{\mathscr{X}})^{l})^{-z} &= \Big((f_z \otimes \mathrm{id})({}^{k}_{k}X^{l}_{l})\Big)^{-1} &\quad (2.16) \\ &= (f_z \otimes \mathrm{id})({}^{l}_{l}(X^{-1})^{k}_{k}) = ((f_z \circ S) \otimes \mathrm{id})({}^{k}_{k}X^{l}_{l}). &\quad (2.17)\end{aligned}$$

Therefore, $f_{-z} = f_z \circ S$. □

Corollary 2.34. *Let $\mathscr{A}$ be a partial Hopf algebra with integral ϕ and define $\theta_{z,z'}: A \to A$ by $a \mapsto f_z * a * f_{z'}$ for each $z, z' \in \mathbb{C}$, where the functionals f_z are as in Theorem 2.33. Then for all $z, z', w, w' \in \mathbb{C}$, the following conditions hold:*

1. *$\theta_{z,z'}$ is an algebra automorphism and preserves each subspace $A(K)$; in particular, $\theta_{z,z'}(\lambda_k \rho_m) = \lambda_k \rho_m$ for all $k, m \in I$;*
2. *$\theta_{z,z'} \circ \theta_{w,w'} = \theta_{z+w,z'+w'}$;*
3. *$(\theta_{w,z'} \otimes \theta_{z,-w}) \circ \Delta = \Delta \circ \theta_{z,z'}$, $\epsilon \circ \theta_{z,z'} = f_{z+z'}$, $\theta_{z,z'} \circ S = S \circ \theta_{-z',-z}$ and $\phi \circ \theta_{z,z'} = \phi$;*
4. *for every linear map $\omega: A \to \mathbb{C}$ and every $a \in A$, the map $(z, z') \mapsto \omega(\theta_{z,z'}(a))$ is entire.*

Proof. All of this follows easily from Theorem 2.33. □

Using the two-parameter group θ, we define the *modular automorphism group* σ, the *scaling group* τ and the *unitary antipode* of a partial compact quantum group A by

$$\sigma_z := \theta_{iz,iz}, \qquad \tau_z := \theta_{iz,-iz}, \qquad R := S \circ \tau_{i/2}. \tag{2.18}$$

Using Corollary 2.34, one verifies that σ, τ, R share all the main relations known for locally compact quantum groups and measured quantum groupoids, for example, σ and τ are complex one-parameter groups of algebra automorphisms of A, the map R is an anti-automorphism, τ_t and σ_t are automorphisms for all $t \in \mathbb{R}$, the family τ commutes with σ and with R in the obvious sense,

$$\phi \circ \sigma_z = \phi \circ \tau_z = \phi \circ R = \phi, \quad \phi(ab) = \phi(b\sigma_{-i}(a)), \tag{2.19}$$

$$\Delta \circ \tau_z = (\tau_z \otimes \tau_z) \circ \Delta, \quad (\tau_z \otimes \sigma_z) \circ \Delta = \Delta \circ \sigma_z = (\sigma_{-z} \otimes \tau_z) \circ \Delta, \tag{2.20}$$

$$R^2 = \mathrm{id}_A, \quad \Delta \circ R = (R \otimes R) \circ \Delta^{\mathrm{op}}. \tag{2.21}$$

2.6 Unitary corepresentations of partial compact quantum groups

Let us now enhance our partial Hopf algebras to partial compact quantum groups. One then considers corepresentations on sfd bigraded *Hilbert spaces* such that the inverse of the corepresentation coincides with its adjoint. More precisely, we have the following definition. We write $B(\mathcal{H}, \mathcal{G})$ for the linear space of bounded morphisms between Hilbert spaces $\mathcal{H}$ and $\mathcal{G}$.

Unitary corepresentations act on I^2-graded Hilbert spaces $\mathcal{H} = \bigoplus_{k,l} {}^k\mathcal{H}^l$ which are row- and column-finite, where the sum is a direct sum of Hilbert spaces. Associated to each such I^2-graded Hilbert space is the I^2-graded vector space which is obtained by taking the algebraic direct sum of the components.

Definition 2.35. Let $\mathscr{A}$ define a partial compact quantum group. We call an rcf corepresentation $\mathscr{X}$ on an rcf I^2-graded Hilbert space $\mathcal{H}$ *unitary* if

$$ {}^{n}_{l}(X^{-1})^{m}_{k} = ({}^{l}_{n}X^{k}_{m})^{*} \quad \text{in } {}^{k}_{m}A^{l}_{n} \otimes B({}^{l}\mathcal{H}^{k}, {}^{n}\mathcal{H}^{m}). $$

Remark 2.36. 1. In the Hilbert space setting, it is more natural to let $\mathcal{H}$ be the *closed* (instead of the purely algebraic) direct sum of all (finite-dimensional) ${}^{k}\mathcal{H}^{l}$. This does not change the notion of corepresentation, which had a local definition.

2. Concerning morphisms, we will say a collection of ${}^{k}T^{l}$ defines a *bounded* intertwiner or morphism if the total operator $T = \oplus {}^{k}T^{l}$ is bounded. We will denote by $\mathrm{Corep}_{u}(\mathscr{A})$ the category of unitary rcf corepresentations with arbitrary morphisms, and $\mathrm{Corep}^{\infty}_{u}(\mathscr{A})$ for the category with bounded morphisms.

Example 2.37. Regard $\mathbb{C}$ as a Hilbert space in the canonical way. Then the trivial corepresentation $\mathscr{U}$ on $l^2(I)$ is unitary.

Our aim now is to show that every irreducible rcf corepresentation is equivalent to a unitary one. We show this by embedding the corepresentation into a restriction of the regular corepresentation.

Lemma 2.38. *Let $\mathscr{A}$ define a partial compact quantum group with positive integral ϕ and let ${}^{m}V^{n} \subseteq \bigoplus_{k,l} {}^{k}_{m}A^{l}_{n}$ be subspaces such that $\Delta_{pq}({}^{m}V^{n}) \subseteq {}^{p}V^{q} \otimes {}^{p}_{m}A^{q}_{n}$ and $V = \bigoplus_{k,l} {}^{k}V^{l}$ is row- and column-finite. Then each ${}^{k}V^{l}$ is a Hilbert space with respect to the inner product given by $\langle a|b\rangle := \phi(a^{*}b)$, and the regular corepresentation $\mathscr{X}$ on V is unitary.*

Proof. By Lemma 2.5, it suffices to show that

$$ \sum_{k} ({}^{k}_{m}X^{l}_{n'})^{*}\, {}^{k}_{m}X^{l}_{n} = \delta_{n,n'} \mathbf{1}\tbinom{l}{n} \otimes \mathrm{id}_{{}^{m}\mathcal{H}^{n}}\,. \tag{2.22} $$

Let $a \in {}^{m}\mathcal{H}^{n}$, $b \in {}^{m}\mathcal{H}^{n'}$ and define $\omega_{b,a}\colon \mathrm{Hom}_{\mathbb{C}}({}^{m}\mathcal{H}^{n}, {}^{m}\mathcal{H}^{n}) \to \mathbb{C}$ by $T \mapsto \langle b|Ta\rangle$. Then

$$ \begin{aligned} \sum_{k} (\mathrm{id} \otimes \omega_{b,a})(({}^{k}_{m}X^{l}_{n'})^{*}\, {}^{k}_{m}X^{l}_{n}) &= \sum_{k} (\mathrm{id} \otimes \phi)(\Delta^{\mathrm{op}}_{kl}(b)^{*} \Delta^{\mathrm{op}}_{kl}(a)) \\ &= \sum_{k} (\phi \otimes \mathrm{id})(\Delta_{lk}(b^{*}) \Delta_{kl}(a)) \\ &= (\phi \otimes \mathrm{id})(\Delta_{ll}(b^{*}a)) \\ &= \phi(b^{*}a) \mathbf{1}\tbinom{l}{n} \\ &= \delta_{n',n} \mathbf{1}\tbinom{l}{n} \otimes \langle b|a\rangle. \end{aligned} $$

This proves (2.22). ☐

Proposition 2.39. *Every rcf corepresentation of a partial compact quantum group $\mathscr{A}$ is isomorphic to a unitary rcf corepresentation.*

Proof. By Proposition 2.23, it suffices to prove the assertion for every sfd corepresentation $(V, \mathscr{X})$ that is irreducible. For some k, l and $v^\vee \in ({}^kV^l)^\vee$, the operator $T_{(v^\vee)}$ defined in Lemma 2.26 has to be non-zero and hence, by Schur's Lemma, injective. Thus, it forms an equivalence between $(V, \mathscr{X})$ and a restriction of the regular corepresentation on ${}^k\mathcal{C}(\mathscr{X})^l$, which is unitary by Lemma 2.38. $\square$

This result and Proposition 2.23 imply that the category $\mathrm{Corep}_u(\mathscr{A})$ is semisimple:

Corollary 2.40. *Every unitary rcf corepresentation of a partial compact quantum group decomposes into a direct sum of irreducible unitary rcf corepresentations.*

The tensor product of rcf corepresentations lifts to a tensor product of unitary corepresentations as follows. We define the tensor product of rcf I^2-graded Hilbert spaces similarly as for rcf I^2-graded vector spaces and pretend it to be strict again.

Lemma 2.41. *Let $(\mathcal{H}, \mathscr{X})$ and $(\mathcal{G}, \mathscr{Y})$ be unitary rcf corepresentations of a partial compact quantum group. Then the tensor product $(\mathcal{H} \underset{I}{\otimes} \mathcal{G}, \mathscr{X} ⓣ \mathscr{Y})$ is unitary again.*

Proof. In total form, $(X ⓣ Y)^{-1} = Y_{13}^{-1}X_{12}^{-1} = Y_{13}^*X_{12}^* = (X ⓣ Y)^*$ by Remark 2.15. $\square$

With the evident definition on morphisms, we obtain a tensor product on $\mathrm{Corep}_u(\mathscr{A})$. This tensor category is *rigid* in the sense that every object has a left and a right dual:

Corollary 2.42. *Let $\mathscr{A}$ define a partial compact quantum group. Then the category* $\mathrm{Corep}_u(\mathscr{A})$ *is a rigid tensor category.*

Proof. Since the antipode of $\mathscr{A}$ is invertible (Corollary 1.28), every unitary rcf corepresentation $(\mathcal{H}, \mathscr{X})$ has a left and a right dual in $\mathrm{Corep}(\mathscr{A})$ (Corollary 2.18), and these are equivalent isomorphic to unitary rcf corepresentations (Proposition 2.39). $\square$

Note that in $\mathrm{Corep}(\mathscr{A})$, the right dual of a unitary rcf corepresentation $(\mathcal{H}, \mathscr{X})$ is given by the I^2-graded vector space $\mathcal{H}^\vee$ and the corepresentation multiplier

$$(S \otimes -^\vee)(X) = (\mathrm{id} \otimes -^\vee)(X^{-1})) = (-^* \otimes -^{*\vee})(X). \tag{2.23}$$

By Corollary 2.24, $(\mathcal{H}, \mathscr{X})$ is isomorphic to the right bidual, which is given by the $\mathcal{H}$ and $(S^2 \otimes \mathrm{id})(X)$. This isomorphism can be chosen to be positive:

Proposition 2.43. *Let $\mathscr{A}$ define a partial compact quantum group and let $(\mathcal{H}, \mathscr{X})$ be an irreducible unitary rcf corepresentation of $\mathscr{A}$. Then there exists an isomorphism $F_{\mathscr{X}}$ from $(\mathcal{H}, \mathscr{X})$ to $(\mathcal{H}, (S^2 \otimes \mathrm{id})(\mathscr{X}))$ in* $\mathrm{Corep}(\mathscr{A})$ *such that each ${}^kF^l_{\mathscr{X}}$ is positive.*

Proof. By Proposition 2.39, there exists an isomorphism $T \colon \mathscr{X}^\vee \to \mathscr{Y}$ for some unitary rcf corepresentation $\mathscr{Y}$, so that in total form, $(1 \otimes T)X^\vee = Y(1 \otimes T)$. We apply $S \otimes -^\vee$ and $-^* \otimes -^{*\vee}$, respectively, use (2.23), and find

$$X^{\vee\vee}(1 \otimes T^\vee) = (1 \otimes T^\vee)Y^\vee, \qquad (1 \otimes T^{*\vee})X = Y^\vee(1 \otimes T^{*\vee}).$$

Here, we identify the the dual of a Hilbert space with its conjugate Hilbert space to make sense of $T^{*\vee}$. Combining both equations, we find $X^{\vee\vee}(1 \otimes T^{\vee}T^{*\vee})) = (1 \otimes T^{\vee}T^{*\vee})X$. Thus, we can take $F_{\mathscr{X}} := T^{\vee}T^{*\vee}$. □

The Schur orthogonality relations in Corollary 2.30 can be rewritten using the involution instead of the antipode as follows. Let $(\mathcal{H}, \mathscr{X})$ be a unitary rcf corepresentation of $\mathscr{A}$. Since $(S \otimes \mathrm{id})(X) = X^{-1} = X^*$, the space of matrix coefficients $\mathcal{C}(\mathscr{X})$ satisfies

$$S({}^{k}_{m}\mathcal{C}(\mathscr{X})^{l}_{n}) = ({}^{m}_{k}\mathcal{C}(\mathscr{X})^{n}_{l})^* \subseteq {}^{n}_{l}A^{m}_{k}. \tag{2.24}$$

More precisely, let $v \in {}^{k}\mathcal{H}^{l}$, $v' \in {}^{m}\mathcal{H}^{n}$ and denote by $\omega_{v,v'}$ and $\omega_{v',v}$ the functionals given by $T \mapsto \langle v|Tv'\rangle$ and $T \mapsto \langle v'|Tv\rangle$, respectively. Then

$$\begin{aligned} S((\mathrm{id} \otimes \omega_{v,v'})({}^{k}_{m}X^{l}_{n})) &= (\mathrm{id} \otimes \omega_{v,v'})({}^{k}_{m}(X^{-1})^{l}_{n})) \\ &= (\mathrm{id} \otimes \omega_{v,v'})(({}^{m}_{k}X^{n}_{l})^*) = (\mathrm{id} \otimes \omega_{v',v})({}^{m}_{k}X^{n}_{l})^*. \end{aligned}$$

This equation and Corollary 2.30 imply:

Corollary 2.44. *Let $\mathscr{A}$ define a partial compact quantum group with invertible integral ϕ, let $(\mathcal{H}, \mathscr{X})$ be an irreducible rcf corepresentation of $\mathscr{A}$, let $F_{\mathscr{X}}$ be an isomorphism from $(\mathcal{H}, \mathscr{X})$ to $(\mathcal{H}, \mathscr{X}^{\vee\vee})$ and $G_{\mathscr{X}} = F_{\mathscr{X}}^{-1}$, and let $a = (\mathrm{id} \otimes \omega_{v,v'})({}^{k}_{m}X^{l}_{n})$ and $b = (\mathrm{id} \otimes \omega_{w,w'})({}^{k}_{m}X^{l}_{n})$, where $v, w \in {}^{k}\mathcal{H}^{l}$ and $v', w' \in {}^{m}\mathcal{H}^{n}$. Then*

$$\phi(b^*a) = \frac{\langle w|v'\rangle\langle v|G_{\mathscr{X}}w'\rangle}{\sum_m \mathrm{Tr}({}^{m}G^{n}_{\mathscr{X}})}, \qquad \phi(ab^*) = \frac{\langle w|F_{\mathscr{X}}v'\rangle\langle v|w'\rangle}{\sum_n \mathrm{Tr}({}^{m}F^{n}_{\mathscr{X}})}.$$

As a consequence of Proposition 2.27 and Proposition 2.39 or Lemma 2.38, the matrix coefficients of irreducible unitary rcf corepresentations span $\mathscr{A}$, and in the Corollary 2.31, we may assume the irreducible rcf corepresentations $(V_\alpha, \mathscr{X}_\alpha)$ to be unitary if $\mathscr{A}$ defines a partial compact quantum group.

Finally, we consider the functionals f_z, the automorphisms $\theta_{z,z'}$, τ_z, σ_z and the anti-isomorphism R introduced in Theorem 2.33, Corollary 2.34 and (2.18).

Proposition 2.45. *Let $\mathscr{A}$ define a partial compact quantum group. Then $f_z \circ * = * \circ f_{-\bar{z}}$ and $\theta_{z,z'} \circ * = * \circ \theta_{-\bar{z},-\bar{z}'}$ for all $z \in \mathbb{C}$. In particular, R is a $*$-anti-automorphism and $\theta_{it,is}$, τ_t and σ_t are $*$-automorphisms for all $s, t \in \mathbb{R}$,*

Proof. We only have to prove the first equation. Write $\bar{f}_z(a) = \overline{f_z(a^*)}$. Using the relations $({}^{k}_{k}X^{l}_{l})^* = (S \otimes \mathrm{id})({}^{k}_{k}X^{l}_{l})$, $f_z \circ S = f_{-z}$ (Theorem 2.33) and positivity of ${}^{k}F^{l}_{\mathscr{X}}$ (Proposition 2.43), we conclude

$$\begin{aligned} (\bar{f}_z \otimes \mathrm{id})({}^{k}_{k}X^{l}_{l}) &= \Big((f_z \otimes \mathrm{id})(({}^{k}_{k}X^{l}_{l})^*)\Big)^* \\ &= \Big((f_{-z} \otimes \mathrm{id})({}^{k}_{k}X^{l}_{l})\Big)^* = (({}^{k}F^{l}_{\mathscr{X}})^{-z})^* = ({}^{k}F^{l}_{\mathscr{X}})^{-\bar{z}} = (f_{-\bar{z}} \otimes \mathrm{id})({}^{k}_{k}X^{l}_{l}), \end{aligned}$$

whence $\bar{f}_z(a) = f_{-\bar{z}}(a)$ for all $a \in {}^k_k\mathcal{C}(\mathscr{X})^l_l$. Since f_z and $f_{-\bar{z}}$ vanish on ${}^k_mA^l_n$ if $(k,l) \neq (m,n)$ and the matrix coefficients of unitary corepresentations span A, we can conclude $\bar{f}_z = f_{-\bar{z}}$. ☐

3 Tannaka-Krein for partial compact quantum groups

The notion of partial algebra has a nice categorification. Recall first that the appropriate (vertical) categorification of a unital $\mathbb{C}$-algebra is a $\mathbb{C}$-linear additive tensor category. From now on, by 'category' we will by default mean a $\mathbb{C}$-linear additive category.

Definition 3.1. A *partial tensor category* $\mathscr{C}$ over a set I_0 consists of

(a) a collection of categories $\mathcal{C}_{ij}$ with $i, j \in I_0$,

(b) $\mathbb{C}$-linear bi-additive functors

$$\otimes : \mathcal{C}_{ij} \times \mathcal{C}_{jk} \to \mathcal{C}_{ik},$$

(c) natural isomorphisms

$$\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z), \qquad X \in \mathcal{C}_{ij}, Y \in \mathcal{C}_{jk}, Z \in \mathcal{C}_{kl},$$

(d) objects $\mathbf{1}_i \in \mathcal{C}_{ii}$,

(e) natural isomorphisms

$$U^{(l)}_X : \mathbf{1}_i \otimes X \to X, \qquad U^{(r)}_X : X \otimes \mathbf{1}_j \to X, \qquad X \in \mathcal{C}_{ij},$$

satisfying the obvious associativity and unit constraints.

The corresponding total notion is as follows.

Definition 3.2. A *tensor category with local units (indexed by I_0)* consists of

(a) a category $\mathcal{C}$,

(b) a $\mathbb{C}$-bilinear bi-additive functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ with compatible associativity constraint α,

(c) a collection $\{\mathbf{1}_i\}_{i\in I_0}$ of objects such that

- $\mathbf{1}_i \otimes \mathbf{1}_j = 0$ for each $i \neq j$, and
- for each X, $\mathbf{1}_i \otimes X = 0 = X \otimes \mathbf{1}_i$ for all but a finite set of i,

(d) natural isomorphisms $U^{(l)}_X : \oplus_i(\mathbf{1}_i \otimes X) \to X$ and $U^{(r)}_X : \oplus_i(X \otimes \mathbf{1}_i) \to X$ satisfying the obvious unit conditions.

Note that the ((d)) makes sense because of the local support condition in ((c)).

Remark 3.3. There is no problem in modifying Maclane's coherence theorem, and we will henceforth assume that our partial tensor categories and tensor categories with local units are strict, just to lighten notation.

Notation 3.4. If $(\mathcal{C}, \otimes, \{\mathbf{1}_i\})$ is a tensor category with local units, and $X \in \mathcal{C}$, we define

$$X_{ij} = \mathbf{1}_i \otimes X \otimes \mathbf{1}_j,$$

and we denote by

$$\eta_{ij} : X_{ij} \to \oplus_{k,l} (\mathbf{1}_k \otimes X \otimes \mathbf{1}_l) \cong X$$

the natural inclusion maps.

Lemma 3.5. *Up to the appropriate notion of equivalence, there is a canonical one-to-one correspondence between partial tensor categories and tensor categories with local units.*

We will not expand upon the appropriate notion of equivalence, as it can easily be furnished by the reader.

Proof. Let $(\mathcal{C}, \otimes, \{\mathbf{1}_i\}_{i \in I_0})$ be a tensor category with local units indexed by I_0. Then the $\mathcal{C}_{ij} = \{X \in \mathcal{C} \mid X_{ij} \underset{\eta_{ij}}{\cong} X\}$, seen as full subcategories of $\mathcal{C}$, form a partial tensor category upon restriction of $\otimes$.

Conversely, let $\mathscr{C}$ be a partial tensor category. Then we let $\mathcal{C}$ be the category formed by formal finite direct sums $\oplus X_{ij}$ with $X_{ij} \in \mathcal{C}_{ij}$, and with $\mathrm{Mor}(\oplus X_{ij}, \oplus Y_{ij}) = \oplus_{ij} \mathrm{Mor}(X_{ij}, Y_{ij})$. The tensor product can be extended to $\mathcal{C}$ by putting $X_{ij} \otimes X_{kl} = 0$ when $k \neq j$. The associativity constraints can then be summed to find an associativity constraint for $\mathcal{C}$. It is evident that the $\mathbf{1}_i$ provide local units for $\mathcal{C}$. $\square$

Continuing the analogy with the algebra case, we define the enveloping *multiplier tensor category* of a tensor category with local units. This notion is important to formulate the appropriate notion of morphism between tensor categories with local units.

Definition 3.6. Let $\mathscr{C}$ be a partial tensor category with total tensor category $\mathcal{C}$. The *multiplier tensor category* $M(\mathcal{C})$ of $\mathcal{C}$ is defined to be the category consisting of formal sums $\oplus X_{ij}$ which are column- and row-finite, and with

$$\mathrm{Mor}(\oplus X_{ij}, \oplus Y_{ij}) = \left(\oplus_i \prod_j \mathrm{Mor}(X_{ij}, Y_{ij})\right) \cap \left(\oplus_j \prod_i \mathrm{Mor}(X_{ij}, Y_{ij})\right).$$

The tensor product of $\mathcal{C}$ can be extended to $M(\mathcal{C})$ by putting

$$(\oplus X_{ij}) \otimes (\oplus Y_{ij}) = \oplus_{i,j,k} (X_{ij} \otimes Y_{jk}).$$

This makes sense because of the column- and row finitedness of the objects of $M(\mathcal{C})$. The resulting tensor category $M(\mathcal{C})$ is then a tensor category with unit. With some effort, a more intrinsic construction of the multiplier tensor category can be given in terms of couples of endofunctors, in the same vein as the construction of the multiplier algebra of a non-unital algebra.

We can now formulate the appropriate notion of functor between partial tensor categories.

Definition 3.7. Let $\mathscr{C}$ and $\mathscr{D}$ be partial tensor categories. A *morphism* from $\mathscr{C}$ to $\mathscr{D}$ is a strong monoidal functor $F : \mathscr{C} \to M(\mathscr{D})$ with accompanying coherence map $\alpha_{X,Y} : F(X \otimes Y) \to F(X) \otimes F(Y)$

We will be interested in partial tensor categories with some analytic structure.

Definition 3.8. A *partial tensor C*-category* is a partial tensor category $(\mathcal{C}_{ij}, \otimes)$ such that all $\mathcal{C}_{ij}$ are semi-simple C*-categories with finite-dimensional morphism spaces, with all functors $\otimes$ being *-functors, and with the associativity and unit constraints given by unitary maps.

Remark 3.9.

1. As in the case of partial algebras, to a partial tensor C*-category corresponds a total category $\mathcal{C}$, whose objects are formal finite-support direct sums $\oplus_{ij} X_{ij}$, and with direct sums of morphisms spaces. However, if I_0 is infinite, the morphisms spaces are only pre-C*-algebras, and there are only local units. Although these are not serious issues, we will continue to work with partial tensor C*-categories, as they are more in the spirit of the paper.
2. Another global viewpoint is to see the collection of $\mathcal{C}_{ij}$ as a C*-2-category with 0-cells indexed by the set I_0, the objects of the C_{ij} as 1-cells, and the morphisms of the C_{ij} as 2-cells. Again we will not emphasize this way of looking, as it is not compatible with the notion of monoidal functor between partial tensor C*-categories.

The following is the natural definition of monoidal functor between partial tensor C*-categories.

Definition 3.10. Let $\mathcal{C}$ and $\mathcal{D}$ be partial tensor C*-categories over respective sets I_0 and J_0. A *morphism* from $\mathcal{C}$ to $\mathcal{D}$ consists of a partition $\{j_i\}_{i \in I_0}$ of J_0 (with corresponding projection function $j \mapsto i(j)$), functors

$$F_{jl} : \mathcal{C}_{i(j),k(l)} \to \mathcal{D}_{jl}$$

and natural isometries

$$\phi^{(j,l,n)}_{X,Y} : F_{jl}(X) \otimes F_{ln}(Y) \to F_{jn}(X \otimes Y), \qquad X \in \mathcal{C}_{i(j),k(l)}, Y \in \mathcal{C}_{k(l),m(n)}$$

commuting with associativity and unit constraints and such that $\oplus_{l,k_l=k} \phi^{(j,l,n)}_{X,Y}$ is unitary.

In this section, we prove a Tannaka-Krein result for partial compact quantum groupoids. In the previous section, we saw that a partial Hopf algebra $(\mathscr{A}, \Delta)$ gives rise to a tensor category with duals $(\mathcal{C}, \otimes)$, its category of row and column-finite corepresentations. Its unit is in general not an irreducible object. When $\mathscr{A}$ defnes a compact quantum groupoid, we moreover have that endomorphism spaces are *-algebras, and upon imposing a uniform boundedness condition (as in ...), they are finite von Neumann algebras of type I with discrete center, i.e. a von Neumann algebraic direct product of matrix algebras. General morphism spaces are then Hilbert W*-bimodules between the endomorphism algebras of their source and target. Let us formalize this notion.

Definition 3.11. A tensor W*-category $(\mathcal{C}, \otimes)$ with duals is said to have a *decomposable unit* if $\mathrm{End}(\mathbf{1}_\mathcal{C}) = l^\infty(\mathcal{I})$ for some set $\mathcal{I}$. In this case, we call $(\mathcal{C}, \otimes)$ a *multi*-fusion W*-category.

As we have remarked before, tensor W*-categories with decomposable unit can also be described as a collection of tensor W*-categories with duality and irreducible units, and a collection of bimodule W*-categories between them.

Example 3.12. Let I be a set. The category $\mathrm{Hilb}_f^{I\times I}$ of row and column-finite $I \times I$-graded Hilbert spaces with tensor product $\underset{I}{\otimes}$ forms a tensor W*-category with duals and decomposable unit.

Example 3.13. Weak Morita equivalence.

Now if $\mathscr{A}$ defines a partial compact quantum group over the base set I, then its $\mathcal{C} = \mathrm{Corep}_{u,f}(\mathscr{A})$ comes equipped with a faithful forgetful *-functor $F : \mathcal{C} \to \mathrm{Hilb}_f^{I\times I}$. This F is equipped with a natural unitary strong monoidal structure. Such pairs $((\mathcal{C}, \otimes), F)$ also arise in other situations.

Example 3.14. Any ergodic action of a compact quantum group $\mathbb{G}$ on a unital C*-algebra provides a tensor C*-functor of $\mathcal{C}$ into Hilb_{I^2} for some set I.

Our aim is to show that this establishes a one-to-one correspondence between partial compact quantum groups over the base space I and multifusion W*-categories endowed with a unitary strong monoidal *-functor into $\mathrm{Hilb}_f^{I\times I}$. We will need some preparation. As the situation is quite analogous to the one in ..., we will not verify all steps.

In the next part, we will fix once and for all a multifusion W*-category $(\mathcal{C}, \otimes)$ equipped with a faithful forgetful unitary monoidal *-functor F into $\mathrm{Hilb}_f^{I\times I}$. We will in general view the tensor product of $\mathcal{C}$ as being strict. We choose a maximal family of mutually inequivalent irreducible objects $\{u_a\}_{a\in J}$ in $\mathcal{C}$. We let $J_0 \subseteq J$ be the set of indices a such that u_a is a subrepresentation of $\mathbf{1}_\mathcal{C}$. Whenever convenient, we will replace u_a by its associated index symbol a. We will also fix once and for all orthonormal bases $f_{c,j}^{a,b}$ for $\mathrm{Mor}(u_c, u_a \otimes u_b)$, where j runs over an index set $J_c^{a,b}$.

Definition 3.15. For $k, l, m, n \in I$, define vector spaces

$$_{m}^{k}A_{n}^{l}(a) = B(F(u_a)_{kl}, F(u_a)_{mn}).$$

Write

$$ {}_m^kA_n^l = \bigoplus_{a\in J} {}_m^kA_n^l(a), \qquad A = \bigoplus_{k,l,m,n} {}_m^kA_n^l. $$

The a-spectral subspace $A(a)$ of A is defined as

$$ A(a) = \sum_{k,l,m,n}{}^{\oplus} {}_m^kA_n^l(a). $$

For any element $x \in A$, its component in the a-spectral subspace is written x_a.

Our goal is to turn A into to the total weak multiplier Hopf *-algebra of a generalized compact Hopf face algebra with base set I.

Let us denote the unitary compatibility morphisms of F by $\phi_{X,Y} : F(X) \underset{I}{\otimes} F(Y) \to F(X \otimes Y)$, where we recall that they are assumed to satisfy the coherence conditions

$$ \phi_{X,Y\otimes Z}(\mathrm{id}_X \otimes \phi_{Y,Z}) = \phi_{X\otimes Y,Z}(\phi_{X,Y} \otimes \mathrm{id}_Z), \qquad \phi_{o,a} = \phi_{a,o} = \mathrm{id}_a\,. $$

It will be convenient to extend $\phi_{X,Y}$ to a coisometry $F(X)\otimes F(Y) \to F(X\otimes Y)$, defining it to be zero on the orthogonal complement of $F(X) \underset{I}{\otimes} F(Y)$. Note however that then $\phi_{X,\mathbf{1}_\mathcal{C}}$ becomes the coisometry $F(X) \otimes l^2(I) \to F(X)$ sending $F(X)_{rs} \otimes \mathbb{C}\delta_t$ canonically onto $\delta_{s,t}F(X)_{rs}$, and similarly for $\phi_{\mathbf{1}_\mathcal{C},X}$. Whenever X, Y are clear, we will abbreviate $\phi_{X,Y}$ as ϕ. We will use the notation

$$ F_{c,j}^{a,b} = \phi^* F(f_{c,j}^{a,b}) \in B(F(u_c), F(u_a) \otimes F(u_b)). $$

We first turn A into the total algebra of a partial algebra.

Definition 3.16. Let $x \in {}_m^kA_n^l(a)$ and $y \in {}_n^lA_s^q(b)$. We define $x \cdot y$ as the element in ${}_m^kA_s^q$ with

$$ (x\cdot y)_c = \sum_{j\in J_c^{a,b}} \left(F_{c,j}^{a,b}\right)^* (x\otimes y) \left(F_{c,j}^{a,b}\right). $$

Note that the product is independent of the specific choice of orthogonal bases $f_{c,j}^{a,b}$.

Lemma 3.17. *With the above product, A becomes a faithful strong I^2-algebra.*

Proof. Let $x \in {}_m^kA_n^l(a)$, $y \in {}_r^pA_s^q(b)$ and $z \in {}_s^qA_v^t$. From the fact that ϕ is a natural transformation, we find that

$$ ((x\cdot y)\cdot z)_d = \sum_{e\in J}\sum_{k\in J_d^{e,c}}\sum_{j\in J_e^{a,b}} \left(\phi^*(\phi^*\otimes \mathrm{id})F(f_{d,e,j,k}^{1,a,b,c})\right)^* (x\otimes y\otimes z) \left((\phi^*\otimes \mathrm{id})\phi^*F(f_{d,e,j,k}^{1,a,b,c})\right) $$

where $f_{d,e,j,k}^{1,a,b,c} = (f_{e,j}^{a,b}\otimes \mathrm{id})f_{d,k}^{e,c}$. On the other hand,

$$ (x\cdot(y\cdot z))_d = \sum_{e\in J}\sum_{k\in J_d^{a,e}}\sum_{j\in J_e^{b,c}} \left(\phi(\mathrm{id}\otimes\phi)F(f_{d,e,j,k}^{2,a,b,c})\right)^* (x\otimes y\otimes z) \left((\mathrm{id}\otimes\phi)\phi F(f_{d,e,j,k}^{2,a,b,c})\right) $$

where $f^{2,a,b,c}_{d,e,j,k} = (\mathrm{id} \otimes f^{b,c}_{e,j}) f^{a,e}_{d,k}$. As $\phi(\phi \otimes \mathrm{id})$ by $\phi(\mathrm{id} \otimes \phi)$ by assumption, and as the orthonormal bases $\{f^{1,a,b,c}_{d,e,j,k} \mid e,j,k\}$ or $\{f^{2,a,b,c}_{d,e,j,k} \mid e,j,k\}$ can clearly be replaced by any other orthonormal basis of $\mathrm{Mor}(u_d, u_a \otimes u_b \otimes u_c)$, it follows that $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

Define $1_{rs} \in B(F(u_o)_{rr}, F(u_o)_{ss}) = {}^{r}_{s}A^{r}_{s}(o)$ as the map sending δ_r to δ_s. By the compatibility assumption for $\phi_{a,o}$ and $\phi_{o,a}$, the map $\mathrm{Fun}_{\mathrm{f}}(I^2) \to A$ mapping $\delta_{(r,s)}$ to 1_{rs} is an algebra homomorphism. Thus A becomes a faithful strong I^2-algebra. □

We will continue to use the $\cdot$-notation to distinguish this product from the ordinary multiplication of operators.

In the following, we will again write $\lambda_r = \sum_s 1_{rs}$ and $\rho_s = \sum_r 1_{rs}$ inside $M(A)$, using the notation as at the end of the proof of the previous lemma.

We turn to the coproduct. Let $\{e_{a,i} \mid i \in B_a\}$ denote an orthonormal basis of $F(u_a)$ over an index set B_a which is adapted to the bigrading (in the sense that each $e_{a,i}$ is inside exactly one component). Write $B_{a,rs} \subseteq B_a$ for the set of indices for which $e_{a,i} \in F(u_a)_{rs}$. Define elements

$$P^{kl}_{mn}(a) \in {}^{k}_{m}A^{l}_{n}(a) \otimes {}^{m}_{k}A^{n}_{l}(a)$$

by

$$P^{kl}_{mn}(a) = \sum_{i \in B_{a,kl}, j \in B_{a,mn}} e_{a,j} e^*_{a,i} \otimes e_{a,i} e^*_{a,j}.$$

As each $F(u_a)_{kl}$ is finite-dimensional, the above sums are finite.

Define now maps

$$\Delta_{rs} : {}^{k}_{m}A^{l}_{n}(a) \to {}^{k}_{r}A^{l}_{s}(a) \otimes {}^{r}_{m}A^{s}_{n}(a)$$

by the application

$$x \mapsto P^{mn}_{rs}(a)(x \otimes 1) = (1 \otimes x) P^{kl}_{rs}(a).$$

They obviously extend to linear maps Δ_{rs} from A to $A \underset{I}{\otimes} A$.

Lemma 3.18. *For each $x \in A$, the element $\Delta(x) = \sum_{rs} \Delta_{rs}(x)$ gives a well-defined multiplier of $A \underset{I}{\otimes} A$. The resulting map $\Delta : A \to M(A \underset{I}{\otimes} A)$ is an I^2-coproduct.*

Proof. As the grading on each $F(u_a)$ is column-finite, it follows at once that for each fixed p, q and $x \in A$, the element $\Delta_{rs}(x)(1 \underset{I}{\otimes} \lambda_p \rho_q)$ is zero except for finitely many r and s. Similarly, $(1 \underset{I}{\otimes} \lambda_p \rho_q)\Delta_{rs}(x)$ is zero except for finitely many r, s because of row-finiteness of $F(u_a)$. Hence $\Delta(x)$ is well-defined as a multiplier for each $x \in A$. Once we show that Δ is multiplicative, it will be immediate that Δ is coassociative, since each Δ_{rs} is coassociative. Moreover, also the fact that Δ then is an I^2-morphism is clear from the definition.

To obtain the multiplicativity of Δ, or rather of the coextension $\widetilde{\Delta}$, choose $x \in {}_mA_n(a)$ and $y \in {}_nA_q(b)$. Then

$$\begin{aligned}\widetilde{\Delta}(x) \cdot \widetilde{\Delta}(y) &= \sum_{rstv} (P^{mn}_{rs}(a)(x \otimes 1)) \cdot (P^{nq}_{tv}(b)(y \otimes 1)) \\ &= \sum_{cd} \sum_{ij} \sum_{klpt} \left(F^{a,b}_{c,i} \otimes F^{a,b}_{d,j}\right)^* \left(e_{a,k}e^*_{a,l}x \otimes e_{b,p}e^*_{b,t}y \otimes e_{a,l}e^*_{a,k} \otimes e_{b,t}e^*_{b,p}\right) \left(F^{a,b}_{c,i} \otimes F^{a,b}_{d,j}\right),\end{aligned}$$

where we may take the sum over all $k, l \in I_a$, $p, t \in I_b$ (and where the composition of operators with mismatching target and source is considered to be zero). Note that the infinite sums are convergent inside $M(A \otimes A)$ by the argument in the first paragraph.

Plugging in the identity operator $\sum_{cd} \sum_{rs} \left(e_{c,r}e^*_{c,r} \otimes e_{d,s}e^*_{d,s}\right)$ at the front, we obtain that the expression becomes

$$\sum_{cd} \sum_{rs} \sum_{ij} \sum_{klpt} X^{c,r,i}_{k,p} Y^{d,s,j}_{l,t} (e_{c,r} \otimes e_{d,s}) \left(e^*_{a,l}x \otimes e^*_{b,t}y \otimes e^*_{a,k} \otimes e^*_{b,p}\right) \left(F^{a,b}_{c,i} \otimes F^{a,b}_{d,j}\right)^*$$

where $X^{c,r,i}_{k,p} = e^*_{c,r}(F^{a,b}_{c,i})^*(e_{a,k} \otimes e_{b,p})$ and $Y^{d,s,j}_{l,t} = e^*_{d,t}(F^{a,b}_{d,j})^*(e_{a,l} \otimes e_{b,t})$. Resumming over the k, l, p, t, we obtain

$$\sum_{cd} \sum_{rs} \sum_{ij} \left(e_{c,r}e^*_{d,s}(F^{a,b}_{d,j})^*(x \otimes y)F^{a,b}_{c,i}\right) \otimes \left(e_{d,s}e^*_{c,r}(F^{a,b}_{c,i})^*F^{a,b}_{d,j}\right).$$

As ϕ is a coisometry and the $f^{a,b}_{c,i}$ are orthonormal, this expression simplifies to

$$\sum_{c} \sum_{rs} \sum_{i} e_{c,r}e^*_{c,s} \left((F^{a,b}_{c,i})^*(x \otimes y)F^{a,b}_{c,i}\right) \otimes e_{c,s}e^*_{c,r},$$

which is precisely $\widetilde{\Delta}(x \cdot y)$. ☐

Proposition 3.19. *The couple (A, Δ) is a generalized face algebra over I.*

Proof. Let ε assign to any $x \in {}^k_mA^l_n(a)$ the number $\mathrm{Tr}(x) = \sum_{i \in B_a}(e^*_{a,i}xe_{a,i})$ (where we keep the convention that mismatching operators compose to zero). We claim that ε is a counit, satisfying the conditions in the definition of a generalized face algebra. The fact that ε is a counit is immediate from the definition of Δ. It is also computed directly that for $x \in {}^k_mA^l_n$ and $y \in {}^l_nA^r_s$, we have $\varepsilon(x \cdot y) = \varepsilon(x)\varepsilon(y)$, since the $\{\phi^*F(f^{a,b}_{c,i})e_{c,j} \mid c, i, j\}$ form an orthonormal basis of $F(u_a) \underset{I}{\otimes} F(u_b)$. From this formula, the second identity for the counit will hold true once we show that

$$\varepsilon(\lambda_k\rho_m x \lambda_l\rho_n) = \varepsilon(\lambda_k\rho_m x_{(1)})\varepsilon(x_{(2)}\lambda_l\rho_n).$$

But both left and right hand side are zero unless $k = m$, $n = l$ and $x \in {}^k_mA^l_n$, in which case both sides equal $\varepsilon(x)$. ☐

Our next job is to define a suitable antipode for (A,Δ). Here the rigidity of $\mathcal{C}$ will come into play, so we first fix our conventions. Let $a \mapsto \bar{a}$ be the involution induced by the rigidity on the index set J. We assume that $\overline{u_a} = u_{\bar{a}}$. For each u_a, we will fix duality morphisms $R_a : u_0 \to u_{\bar{a}} \otimes u_a$ and $\bar{R}_a : u_0 \to u_a \otimes u_{\bar{a}}$. By means of F and ϕ, they induce I^2-grading preserving maps $\mathscr{R}_a : l^2(I) \to F(u_{\bar{a}}) \underset{I}{\otimes} F(u_a)$ and $\bar{\mathscr{R}}_a : l^2(I) \to F(u_a) \underset{I}{\otimes} F(u_{\bar{a}})$. These in turn provide an invertible anti-linear map $I_a : F(u_a)_{kl} \to F(u_{\bar{a}})_{lk}$ and $J_a : F(u_{\bar{a}})_{lk} \to F(u_a)_{kl}$ such that $\langle I_a\xi_a, \eta_{\bar{a}}\rangle = \sum_r \delta_r^* \bar{\mathscr{R}}_a^*(\xi_a \otimes \eta_{\bar{a}})$ and $\langle J_a\eta_{\bar{a}}, \xi_a\rangle = \sum_s \delta_s^* \mathscr{R}_a^*(\eta_{\bar{a}} \otimes \xi_a)$. The snake identities for R_a and $\bar{R}_a$ guarantee that J_a is the inverse of I_a.

We define

$$S : {}_m^kA_n^l(a) \to {}_l^nA_k^m(\bar{a})$$

by

$$x \mapsto I_a x^* J_a.$$

Lemma 3.20. *By means of the map S, the couple (A,Δ) becomes a generalized Hopf face algebra.*

Proof. It is clear that S is invertible. We also have $S(\lambda_k\rho_l) = \lambda_l\rho_k$ as $I_o\delta_k = \delta_k$.

Let us check that S satisfies the condition $S(x_{(1)}) \cdot x_{(2)} = \sum_p \varepsilon(x\cdot\lambda_p)\rho_p$ in the multiplier algebra for $x \in {}_m^kA_n^l(a)$. By definition, we have

$$S(x_{(1)}) \cdot x_{(2)} = \sum_c \sum_i \sum_{p,q\in B_a} \left(F_{c,i}^{\bar{a},a}\right)^* \left(I_a e_{a,q} e_{a,p}^* J_a \otimes x e_{a,q} e_{a,p}^*\right) \left(F_{c,i}^{\bar{a},a}\right).$$

Let $C : \mathbb{C} \to \mathbb{C}$ be complex conjugation. Then we can write $I_a e_{a,q} e_{a,p}^* J_a = (I_a e_{a,q} C)(C e_{a,p}^* J_a)$. We now calculate, by definition of J_a and $F_{c,i}^{\bar{a},a}$, that

$$\sum_{p\in B_a} (C e_{a,p}^* J_a \otimes e_{a,p}^*) \left(F_{c,i}^{\bar{a},a}\right) = (R_a^* f_{c,i}^{\bar{a},a}) \sum_{s\in I} \delta_s^*,$$

since ϕ is a coisometry. Plugging this into our expression for $S(x_{(1)}) \cdot x_{(2)}$, we obtain

$$\sum_s \sum_{q\in B_a} \left(\sum_c \sum_i \left((f_{c,i}^{\bar{a},a})^* R_a \right) F_{c,i}^{\bar{a},a} \right)^* (I_a e_{a,q} C \otimes x e_{a,q} \delta_s^*).$$

As ϕ is a coisometry and $\phi^*\phi\mathscr{R}_a = \mathscr{R}_a$, we can write $\left((f_{c,i}^{\bar{a},a})^* R_a \right) F_{c,i}^{\bar{a},a} = F_{c,i}^{\bar{a},a} (F_{c,i}^{\bar{a},a})^* \mathscr{R}_a$. As the $f_{c,i}^{\bar{a},a}$ form an orthonormal basis, we thus get

$$S(x_{(1)}) \cdot x_{(2)} = \sum_s \sum_{q\in B_a} \mathscr{R}_a^* (I_a e_{a,q} C \otimes x e_{a,q} \delta_s^*).$$

Now the composition $I_a e_{a,q} C$ is the creation operator for the vector $I_a e_{a,q}$. Hence using again the definition of J_a, and using that $x \in {}_m^k A_n^l$, we get

$$\begin{aligned} S(x_{(1)}) \cdot x_{(2)} &= \sum_s \sum_q \delta_n \delta_s^* e_{a,q}^* x e_{a,q} \\ &= \sum_s \mathrm{Tr}(x) \delta_n \delta_s^* \\ &= \sum_p \varepsilon(x\lambda_p)\rho_p, \end{aligned}$$

since $\mathrm{Tr}(x) = \delta_{k,m}\delta_{n,l}\varepsilon(x)$.

The identity $x_{(1)} \cdot S(x_{(2)}) = \sum_p \varepsilon(\rho_p x)\lambda_p$ is proven in a similar way. □

In the next step, we determine an invariant functional for (A, Δ).

Definition 3.21. We define $\varphi : {}_m^k A_n^l \to \mathbb{C}$ as the projection onto the component ${}_m^k A_n^l(o) \cong \delta_{kl}\delta_{mn}\mathbb{C}$.

Lemma 3.22. *The functional φ is an invariant normalized functional.*

Proof. The fact that φ is normalized is immediate, so let us check that it is invariant. Let $x \in {}_m^k A_n^l(a)$. Then

$$\begin{aligned} (\mathrm{id} \otimes \varphi)\widetilde{\Delta}(x) &= \sum_{i,j} \varphi(e_{a,j} e_{a,i}^*) e_{a,i} e_{a,j}^* x \\ &= \delta_{a,o} \sum_{r,s} \delta_r \delta_s^* x \\ &= \varphi(x) \sum_r \delta_r \delta_k^* \\ &= \sum_p \varphi(\lambda_p x)\lambda_p. \end{aligned}$$

The proof of right invariance follows similarly. □

Finally, we introduce the *-structure and show that (A, Δ) is a generalized compact Hopf face algebra. To distinguish the new *-operation from the ordinary operator algebraic one, we will denote it by $\dagger$.

Definition 3.23. We define the anti-linear map $\dagger : {}_m^k A_n^l \to {}_k^m A_l^n$ by the formula

$$x^\dagger = S(x^*)$$

Lemma 3.24. *The map $x \mapsto x^\dagger$ is an anti-multiplicative anti-linear involution on A.*

Proof. It is clear that $x \mapsto x^\dagger$ is anti-linear. It is also immediate from the definition of the product that $(x \cdot y)^* = x^* \cdot y^*$. Together with the anti-multiplicativity of S, this proves the anti-multiplicativity of $\dagger$.

Let us proof that $\dagger$ is an involution. It is sufficient to prove that $I_{\bar{a}} I_a = \lambda \,\mathrm{id}$ and $J_a J_{\bar{a}} = \lambda^{-1}\,\mathrm{id}$ for some scalar λ. But this follows from the fact that $(\bar{R}_a, R_a)$ and $(R_{\bar{a}}, \bar{R}_{\bar{a}})$ are both solutions to the conjugate equations for $\bar{a}$. ☐

The last property which needs to be proven is the positivity of φ. For this, recall that $R_a^* R_a$ and $\bar{R}_a^* \bar{R}_a$ are scalars as u_a is irreducible. One can then rescale R_a and $\bar{R}_a$ such that the scalar in both expressions is the same. This scalar is then a uniquely determined number $\dim_q(a)$, called the *quantum dimension* of a. It follows that $\frac{1}{\dim_q(a)} F(R_a R_a^*)$ is the projection of $F(u_{\bar{a}} \otimes u_a)$ onto the copy of $F(u_o)$ inside, and a similar statement holds for $\bar{R}_a$.

Proposition 3.25. *For any $x \in A$, the scalar $\varphi(x^\dagger \cdot x)$ is positive.*

Proof. It is straightforward to see that the blocks ${}^{k}_{m}A^{l}_{n}$ are mutually orthogonal, and that moreover the spectral subspaces inside are mutually orthogonal. Let then $\xi, \zeta \in F(u_a)_{kl}$ and $\eta, \mu \in F(u_a)_{mn}$. We have, using the remark above,

$$
\begin{aligned}
\varphi(y^\dagger \cdot x) &= \varphi(\sum_c \sum_i \left(F^{\bar{a},a}_{c,i}\right)^* (I_a y J_a \otimes x) \left(F^{\bar{a},a}_{c,i}\right)) \\
&= \delta_n^* \sum_i \left(F^{\bar{a},a}_{o,i}\right)^* (I_a y J_a \otimes x) \left(F^{\bar{a},a}_{o,i}\right) \delta_l \\
&= \frac{1}{\dim_q(u_a)} \delta_n^* \mathscr{R}_a^* (I_a y J_a \otimes x) \mathscr{R}_a \delta_l \\
&= \frac{1}{\dim_q(u_a)} \sum_{p,q} \delta_n^* \mathscr{R}_a^* (I_a y J_a e_{\bar{a},p} e_{\bar{a},p}^* \otimes x e_{a,q} e_{a,q}^*) \mathscr{R}_a \delta_l .
\end{aligned}
$$

By the defining properties of I_a and J_a, this expression becomes $\dim_q(u_a)^{-1} \sum_p \langle e_{\bar{a},p}, J_a^* x^* y J_a e_{\bar{a},p} \rangle$, thus clearly φ will be positive on A. ☐

Corepresentations of generalized compact Hopf face algebras

Let (A, Δ) be a generalized compact Hopf face algebra over an index set I. A *locally finite-dimensional unitary corepresentation* of (A, Δ) consists of a row and column-finite I^2-graded Hilbert space $\mathcal{H} = \sum_{k,l \in I}^{\oplus} \mathcal{H}_{kl}$ together with elements ${}^{k}_{m}U^{l}_{n} \in {}^{k}_{m}A^{l}_{n} \otimes B({}^{m}\mathcal{H}^{n}, {}^{k}\mathcal{H}^{l})$ such that

$$\sum_k \left({}^{k}_{m}U^{l}_{n}\right)^* {}^{k}_{m}U^{l}_{n} = \lambda_l \rho_n \otimes \mathrm{id}_{{}^{m}\mathcal{H}^{n}}$$

and

$$\sum_n {}^k_mU^l_n \left({}^k_mU^l_n\right)^* = \lambda_k\rho_m \otimes \mathrm{id}_{{}^k\mathcal{H}^l},$$

and

$$(\tilde{\Delta}\otimes\mathrm{id})({}^k_mU^l_n) = \sum_{p,q}\left({}^k_pU^l_q\right)_{13}({}^p_mU^q_n)_{23}.$$

Note that in the first two identities, the sums are finite, while in the finite identity the possibly infinite sum is meaningful inside the multiplier algebra sense.

By a morphism between two locally finite-dimensional unitary corepresentations $(\mathscr{H}, U)$ and $(\mathscr{G}, V)$ is meant a grading-preserving bounded map $T = \sum_{k,l}^{\oplus} {}^kT^l : \mathscr{H} \to \mathscr{G}$ such that $(1 \otimes {}^kT^l){}^k_mU^l_n = {}^k_mV^l_n(1\otimes {}^mT^n)$. The collection of all locally finite-dimensional unitary corepresentations clearly forms a semi-simple C*-category Corep(A). We will say that (A,Δ) is *of finite type* if the morphisms in Corep(A) are finite-dimensional.

One can define a tensor product $\textcircled{T}$ between locally finite-dimensional corepresentations by means of the $\underset{I}{\otimes}$-product of bigraded Hilbert spaces and the operation

$${}^k_m(U\textcircled{T}V)^l_n = \left({}^k_mU^r_s\right)_{12}\left({}^r_sV^l_n\right)_{13}.$$

In this way, the category Corep(A) becomes a monoidal category. The unit object consists of the I^2-graded Hilbert space $l^2(I)$ together with the elements ${}^k_mU^l_n = \delta_{kl}\delta_{mn}\lambda_k\rho_m\otimes 1$.

Assume now that $\mathcal{C}$ is a semi-simple tensor C*-category with irreducible unit, and $F : \mathcal{C} \to \mathrm{Hilb}$ a strong tensor C*-functor. Let (A,Δ) be the associated generalized compact Hopf face algebra. Let us show that $\mathcal{C} \cong \mathrm{Corep}(A)$ by means of an equivalence functor G.

For X an object of $\mathcal{C}$, we build a locally finite-dimensional unitary corepresentation U on $F(X)$. Consider the canonical isomorphism $F(X) \cong \oplus_{a\in J}X_a \otimes \mathrm{Mor}(X_a, X)$. Let

$${}^k_mU^l_n(a) \in {}^k_mA^l_n(a)\otimes B(F(u_a)_{mn}, F(u_a)_{kl}) = B(F(u_a)_{kl}, F(u_a)_{mn})\otimes B(F(u_a)_{mn}, F(u_a)_{kl})$$

be determined as the element implementing the non-degenerate pairing $B(F(u_a)_{kl}, F(u_a)_{mn})\otimes B(F(u_a)_{nm}, F(u_a)_{lk}) \to \mathbb{C}$ sending $S\otimes T$ to $\mathrm{Tr}(ST)$. Using notation as before, this means that

$${}^k_mU^l_n(a) = \sum_{p\in B_{a,mn}, q\in B_{a,kl}} e_pe_q^* \otimes e_qe_p^*.$$

Monoidal equivalence of generalized compact Hopf face algebras

Let (A,Δ) be a generalized Hopf face algebra over a set I. Assume that $I = I_1 \sqcup I_2$, and let $\Lambda_j = \sum_{i\in I_j}\lambda_i$, resp. $\mathrm{P}_j = \sum_{i\in I_j}\rho_j$. If the Λ_j and P_j are central in $M(A)$, then we can write $A = \sum_{i,j}^{\oplus} A(ij)$ where $A(ij) = \Lambda_i \mathrm{P}_j A$ are subalgebras. Moreover, the comultiplication $\widetilde{\Delta}$ splits into comultiplications

$$\widetilde{\Delta}_{ij}^{k} : A(ij) \to M(A(ik) \otimes A(kj)) \text{ s.t. } \widetilde{\Delta} = \widetilde{\Delta}_{ij}^{1} + \widetilde{\Delta}_{ij}^{2} \text{ on } A(ij).$$

A similar decomposition holds for Δ.

It is immediate to see that the $(A(ii), \Delta_{ii}^{i})$ are two generalized Hopf face algebras over the respective I_i.

Definition 3.26. We say (A, Δ) is a *co-linking generalized (compact) Hopf face algebra* between $(A(11), \Delta_{11}^{1})$ and $(A(22), \Delta_{22}^{2})$ if $\lambda_i \mathrm{P}_2 \neq 0$ for any $i \in I_1$.

Upon applying the antipode, we see that then $\rho_j \Lambda_1 \neq 0$ for any $j \in I_2$ as well.

Definition 3.27. Two generalized (compact) Hopf face algebras are called *comonoidally Morita equivalent* if they are isomorphic to the components $(A_{ii}, \Delta_{ii}^{i})$ of some co-linking generalized (compact) Hopf face algebra.

As an example, consider two sets I_i, and two tensor functors (F_i, ϕ_i) of a semi-simple rigid C*-category $\mathcal{C}$ with irreducible unit into $\mathrm{Hilb}_{I_i^2}$. Then with $I = I_1 \sqcup I_2$, we can form a new C*-functor $F = F_1 \oplus F_2$ of $\mathcal{C}$ into Hilb_{I^2} by putting $F(X) = F_1(X) \oplus F_2(X)$ with the obvious I^2-grading (and the obvious direct sum operation on morphisms). It becomes monoidal by means of the unitaries

$$F(X \otimes Y) = F_1(X \otimes Y) \oplus F_2(X \otimes Y) \underset{\phi_1 \oplus \phi_2}{\cong} (F_1(X) \underset{I_1}{\otimes} F_1(Y)) \oplus (F_2(X) \otimes F_2(Y)) \cong F(X) \underset{I}{\otimes} F(Y)$$

(where the last map is unitary since $(F(X) \underset{I}{\otimes} F(Y))_{ij} = 0$ for example for $i \in I_1$ and $j \in I_2$).

If we then consider the generalized compact Hopf face algebra (A, Δ) associated to F, we have immediately from the construction that the Λ_i and P_i associated to the decomposition $I = I_1 \sqcup I_2$ are indeed central elements in $M(A)$. Moreover, the parts $(A_{ii}^{i}, \Delta_{ii}^{i})$ are seen to arise from applying the Tannaka-Krein construction to the respective functors F_1 and F_2. The fact that (A, Δ) is co-linking is immediate from the fact that *none* of the $\lambda_i \rho_j$ are zero in this particular case (since ${}_{m}^{k}A_{m}^{k}(o) = B(F(u_o)_{kk}, F(u_o)_{mm}) \cong \mathbb{C}$).

We will exploit the above extra structure in the following section to say something about the algebra A appearing in ... This is the component $\tilde{A}(1,1)$ of the above algebra. The following lemma will be needed.

Lemma 3.28. *Assume (A, Δ) is a co-linking generalized Hopf face algebra. Then any of the maps $\widetilde{\Delta}_{ij}^{k}$ is injective.*

Proof. Take a non-zero $x \in A_n(ij)$ where $n \in I_j$. Then for any $l \in I$ with $\rho_n \lambda_l \neq 0$, we know that $\widetilde{\Delta}(x)(1 \otimes \rho_n \lambda_l) \neq 0$. Hence $\widetilde{\Delta}_{ij}^{k}(x)(1 \otimes \rho_n \lambda_l) \neq 0$ for $l \in I_k$, and hence $\widetilde{\Delta}_{ij}^{k}(x) \neq 0$. Now if $j = k$, the condition $\rho_n \lambda_l \neq 0$ is satisfied by taking $l = n$ (since $\varepsilon(\lambda_n \rho_n) = 1$). If $j \neq k$, it is satisfied for at least one l by the co-linking assumption. $\square$

4 Compact Hopf face algebras on the level of operator algebras

Let $\mathscr{G}$ be a partial compact quantum group. We now construct completions of the underlying $*$-algebra $P(\mathscr{G})$ in the form of a universal C^*-algebra $C_0^u(\mathscr{G})$, a reduced C^*-algebra $C_0^r(\mathscr{G})$ and a von Neumann algebra $L^\infty(\mathscr{G})$. The existence of the first one follows from the Peter-Weyl theorem **??**, and the second and third one arise from a GNS-representation of $P(\mathscr{G})$ on the Hilbert space $L^2(\mathscr{G})$ associated to the invariant functional of $\mathscr{G}$. We then lift the comultiplication, the invariant functional, the unitary antipode and the scaling group to level of operator algebras and show that $L^\infty(\mathscr{G})$ becomes a measured quantum groupoid in the sense of Lesieur [] and Enock [].

Let us start with the construction of $C_0^u(\mathscr{G})$. Denote by A the underlying total $*$-algebra of the partial $*$-algebra $P(\mathscr{G})$ and define a map $|\cdot|_u\colon A \to [0,\infty]$ by

$$|a|_u := \sup\{\|\pi(a)\| : \pi \text{ is a } *\text{-homomorphism from } A \text{ into some } C^*\text{-algebra } B\}.$$

Lemma 4.1. *$|a|_u < \infty$ for each $a \in A$.*

Proof. By Corollary 2.31, we can write each $a \in A$ in the form $a = (\mathrm{id} \otimes \omega_{\xi,\eta})(X(K))$, where X is a unitary sfd corepresentation of $P(\mathscr{G})$ on some sfd I^2-graded Hilbert space $\mathcal{H}$ and $K = \left(\begin{smallmatrix} k & l \\ m & n \end{smallmatrix}\right) \in M_2(I)$, $\xi \in {}^m\mathcal{H}^n$, $\eta \in {}^k\mathcal{H}^l$. Since X is unitary,

$$\sum_p X\left(\begin{smallmatrix} p & l \\ m & n \end{smallmatrix}\right)^* X\left(\begin{smallmatrix} p & l \\ m & n \end{smallmatrix}\right) = \lambda_l \rho_n \otimes \mathrm{id}_{{}^m\mathcal{H}^n}$$

by **??**, where the sum is finite because X is sfd. As $\pi(\lambda_k\rho_m) \in \mathbb{C}$ is a projection, we can conclude

$$\begin{aligned}\|\pi(a)\| &\leqslant \|\xi\|\|\eta\|\|(\pi\otimes\mathrm{id})(X(K))\| \\ &= \|\xi\|\|\eta\|\|(\pi\otimes\mathrm{id})(X(K)^*X(K))\| \leqslant \|\xi\|\|\eta\|\|\pi(\lambda_l\rho_n)\otimes\mathrm{id}_{{}^m\mathcal{H}^n}\| = \|\xi\|\|\eta\|.\end{aligned}$$

□

Clearly, $|\cdot|_u$ defines a C^*-semi-norm on $P(\mathscr{G})$, and the separated completion of $P(\mathscr{G})$ with respect to this norm is a C^*-algebra. We denote this C^*-algebra by $C_0^u(\mathscr{G})$. By construction, every $*$-homorphism π of A into some C^*-algebra C factorises through $C_0^u(\mathscr{G})$. We shall see that the canonical $*$-homomorphism $\pi_u\colon A \to C_0^u(\mathscr{G})$ is injective.

Remark 4.2. The universal property of $C_0^u(\mathscr{G})$ implies that the modular automorphism group σ and the scaling group τ for real parameters and the unitary antipode R of $\mathscr{G}$ introduced after Corollary 2.34 lift to one-parameter groups τ^u, σ^u and a $*$-anti-automorphism R_u of $C_0^u(\mathscr{G})$, that is, $\tau_t^u \circ \pi_u = \pi_u \circ \tau_t$, $\sigma_t^u \circ \pi_u = \pi_u \circ \sigma_t$, $R_u \circ \pi_u = \pi_u \circ R$. Corollary 2.34 5. and A.1 in [] [Takesaki:2] imply that elements in $\pi_u(A)$ are analytic for τ^u and σ^u; in particular, τ^u and σ^u are strongly continuous.

We now turn to the construction of the reduced C^*-algebra $C_0^r(\mathscr{G})$ and the von Neumann algebra $L^\infty(\mathscr{G})$. Denote by $L^2(\mathscr{G})$ the completion of A with respect to the norm associated to the inner product given by

$$\langle a|b\rangle := \phi(a^*b) \quad \text{for all } a, b \in A,$$

and by $\Lambda\colon A \to L^2(\mathscr{G})$ the natural embedding. This product is definite because ϕ is faithful by 1.34, and it extends to the space $L^2(\mathscr{G})$ which thus becomes a Hilbert space. As such, $L^2(\mathscr{G})$ is the orthogonal direct sum of the subspaces $\Lambda(A(K)) \subseteq L^2(\mathscr{G})$, where $K \in M_2(I)$, because $\phi(A(K)^*A(L)) = 0$ if $K \neq L$ by **??**. In particular, there exist operators $\lambda_k, \lambda_k^{\mathrm{op}}, \rho_k, \rho_k^{\mathrm{op}} \in \mathcal{B}(L^2(\mathscr{G}))$ for each $k \in I$ such that

$$\lambda_k\Lambda(a) = \Lambda(\lambda_k a), \quad \lambda_k^{\mathrm{op}}\Lambda(a) = \Lambda(a\lambda_k), \quad \rho_k\Lambda(a) = \Lambda(\rho_k a), \quad \rho_k^{\mathrm{op}}\Lambda(a) = \Lambda(a\rho_k)$$

for all $a \in A$, and faithful, normal $*$-homomorphisms

$$\lambda, \rho\colon l^\infty(I) \to \mathcal{B}(L^2(\mathscr{G})) \tag{4.1}$$

that send the delta function at $k \in I$ to the operators λ_k or ρ_k, respectively.

Define $\overline{E}, \overline{G} \in \mathcal{B}(L^2(\mathscr{G}) \otimes L^2(\mathscr{G}))$ by

$$\overline{E} := \sum_k \rho_k \otimes \lambda_k, \qquad \overline{G} := \sum_k \rho_k^{\mathrm{op}} \otimes \rho_k,$$

where the sums converge with respect to the strong operator topology.

Lemma 4.3. *There exists a unique partial isometry V on $L^2(\mathscr{G}) \otimes L^2(\mathscr{G})$ such that*

$$V(\Lambda(a) \otimes \Lambda(b)) = \Lambda(a_{(1)}) \otimes \Lambda(a_{(2)}b)$$

for all $a, b \in A$. Its range and domain projections are given by $VV^* = \overline{E}$ and $V^*V = \overline{G}$.

Proof. Let $a, b \in A$. Since Δ is a $*$-homomorphism and ϕ is invariant,

$$\begin{aligned}\langle\Lambda(a_{(1)}) \otimes \Lambda(a_{(2)}b)|\Lambda(a'_{(1)}) \otimes \Lambda(a'_{(2)}b')\rangle &= \phi(a^*_{(1)}a'_{(1)})\phi(b^*a^*_{(2)}a'_{(2)}b') \\ &= \sum_p \phi(b^*\rho_p\phi(\rho_p a^*a'\rho_p)\rho_p b') \\ &= \sum_p \langle\Lambda(a\rho_p) \otimes \Lambda(\rho_p b)|\Lambda(a'\rho_p) \otimes \Lambda(b'\rho_p)\rangle.\end{aligned}$$

Now, the assertion follows from Proposition 1.23. $\square$

Proposition 4.4. *Let $\mathscr{G}$ be a partial compact quantum group with underlying total $*$-algebra A and associated Hilbert space $L^2(\mathscr{G})$. Then there exists a unique $*$-homomorphism $\pi_r\colon A \to \mathcal{B}(L^2(\mathscr{G}))$ such that $\pi_r(a)\Lambda(b) = \Lambda(ab)$ for all $a, b \in A$, and this π_r is faithful.*

Proof. Let $a, c \in A$. Then the formula $x \mapsto \langle \Lambda(c) | x\Lambda(a) \rangle$ defines a bounded linear functional $\omega_{\Lambda(c),\Lambda(a)}$ on $\mathcal{B}(L^2(\mathscr{G}))$ and a straightforward computation shows that

$$(\omega_{\Lambda(c),\Lambda(a)} \otimes \mathrm{id})(V)\Lambda(b) = \Lambda(\varphi(c^* a_{(1)})a_{(2)}b) \tag{4.2}$$

for all $b \in A$. Therefore, left multiplication by $\varphi(c^* a_{(1)})a_{(2)}$ extends to a bounded linear operator on $L^2(\mathscr{G})$. Since $(A \otimes 1)\Delta(A) = (A \otimes A)\Delta(1)$ by Proposition 1.23 and ϕ is normalized, elements of the form $\phi(c^* a_{(1)})a_{(2)}$ span A. □

Corollary 4.5. *Let $\mathscr{G}$ be a partial compact quantum group with underlying total algebra A. Then the canonical $*$-homomorphism $\pi_u \colon A \to C_0^u(\mathscr{G})$ is injective.*

Proof. The injective $*$-homomorphism π_r factorises through π_u. □

Given a partial compact quantum group $\mathscr{G}$, we call $(L^2(\mathscr{G}), \Lambda, \pi)$ the *associated GNS-construction* and denote by

$$C_0^r(\mathscr{G}) \subseteq \mathcal{B}(L^2(\mathscr{G})) \qquad \text{and} \qquad L^\infty(\mathscr{G}) \subseteq \mathcal{B}(L^2(\mathscr{G})) \tag{4.3}$$

the C^*-algebra and the von Neumann algebra generated by $\pi_r(A) \subseteq L^2(\mathscr{G})$, respectively, and identify $M(C_0^r(\mathscr{G}))$ with a C^*-subalgebra of $L^2(\mathscr{G})$. Since π_r extends to a $*$-homomorphism on $C_0^u(\mathscr{G})$, we get a sequence of $*$-homomorphisms

$$A \hookrightarrow C_0^u(\mathscr{G}) \to C_0^r(\mathscr{G}) \hookrightarrow M(C_0^r(\mathscr{G})) \hookrightarrow L^\infty(\mathscr{G}) \hookrightarrow \mathcal{B}(L^2(\mathscr{G})).$$

Note that the $*$-homomorphisms λ, ρ in (4.1) send $l^\infty(I)$ to $M(C_0^r(\mathscr{G}))$, and that

$$\overline{E} \in M(C_0^r(\mathscr{G}) \otimes C_0^r(\mathscr{G})) \subseteq L^\infty(\mathscr{G}) \otimes L^\infty(\mathscr{G}) \subseteq \mathcal{B}(L^2(\mathscr{G}) \otimes L^2(\mathscr{G})),$$

where $\otimes$ denotes the minimal tensor product of C^*-algebras, the tensor product of von Neumann algebras, and the tensor product of Hilbert spaces, respectively.

Consider the map

$$\overline{\Delta} \colon L^\infty(\mathscr{G}) \to \mathcal{B}(L^2(\mathscr{G}) \otimes L^2(\mathscr{G})), \ x \mapsto V(x \otimes 1)V^*.$$

Lemma 4.6. *1. $\overline{\Delta}(\pi_r(a))(\Lambda(b) \otimes \Lambda(c)) = \Lambda(a_{(1)}b) \otimes \Lambda(a_{(2)}b)$ for all $a, b, c \in A$;*

2. $\overline{\Delta}$ is a normal, faithful $*$-homomorphism;

3. $\overline{\Delta}(C_0^r(\mathscr{G})) \subseteq \overline{E}M(C_0^r(\mathscr{G}) \otimes C_0^r(\mathscr{G}))\overline{E}$ and $\overline{\Delta}(L^\infty(\mathscr{G})) \subseteq \overline{E}(L^\infty(\mathscr{G}) \otimes L^\infty(\mathscr{G}))\overline{E}$.

Proof. The equation in 1. is easily verified. The map $\overline{\Delta}$ is normal by construction, a $*$-homomorphism by 1., and faithful because $\overline{\Delta}(x) = 0$ implies $x \otimes 1 = 0$ on $V^*V(L^2(\mathscr{G}) \otimes L^2(\mathscr{G}))$ and hence $x = 0$ on $\bigoplus_k \rho_k^{\mathrm{op}} L^2(\mathscr{G}) = L^2(\mathscr{G})$. Finally, 3. follows from the relation $\Delta(a) = E\Delta(a)E$, which holds for all $a \in A$. □

Next, we lift the invariant functional ϕ of $\mathscr{G}$ to $L^\infty(\mathscr{G})$ and define associated operator-valued weight T_λ, T_ρ from $L^\infty(\mathscr{G})$ to $l^\infty(I)$. Since ϕ is normalized, each $\Lambda(\lambda_k, \rho_m)$ is a unit vector and the associated vector functional

$$\overline{\phi}\tbinom{k}{m}\colon L^\infty(\mathscr{G}) \to \mathbb{C}, \quad x \mapsto \langle \Lambda(\lambda_k\rho_m)|x\Lambda(\lambda_k\rho_m)\rangle$$

is a state. Then the formulas

$$\overline{\phi}(x) := \sum_{k,m} \overline{\phi}\tbinom{k}{m}(x), \qquad T_\lambda(x) := \sum_{k,m} \overline{\phi}\tbinom{k}{m}(x)\lambda_k, \qquad T_\rho(x) := \sum_{k,m} \overline{\phi}\tbinom{k}{m}(x)\rho_m, \tag{4.4}$$

where $x \in L^\infty(\mathscr{G})_+$, define a a normal semi-finite weight $\overline{\phi}$ on $L^\infty(\mathscr{G})$ and normal semi-finite conditional expectations T_λ and T_ρ from $L^\infty(\mathscr{G})$ to $\lambda(l^\infty(I))$ and $\rho(l^\infty(I))$, respectively. These maps are determined by their restrictions to $\pi_r(A)$:

Lemma 4.7. *The normal weight $\overline{\phi}$ and the normal conditional expectations T_λ, T_ρ satisfy $\pi_r(A) \subseteq \mathfrak{M}_{\overline{\phi}} \cap \mathfrak{M}_T \cap \mathfrak{M}_{T'}$ and*

$$\overline{\phi}(\pi_r(a)) = \phi(a), \quad T_\lambda(\pi_r(a))\Lambda(b) = \sum_k \Lambda(\phi(\lambda_k a)\lambda_k b), \quad T_\rho(\pi_r(b))\Lambda(b) = \sum_m \Lambda(\phi(\rho_m a)\rho_m b)$$

for all $a, b \in A$, and are uniquely determined by these equations.

Proof. The equations follow immediately from the definition and the relation $\phi(a) = \sum_{k,m} \phi(\lambda_k\rho_m a\lambda_k\rho_m)$, see **??**. To prove uniqueness, observe that the $p_{k,m} := \pi_r(\lambda_k\rho_m)$ are pairwise orthogonal projections in $\mathfrak{M}_{\overline{\phi}} \cap \mathfrak{M}_{T_\lambda} \cap \mathfrak{M}_{T_\rho}$ summing up to 1, whence $\overline{\phi}$, T_λ and T_ρ are the sums of the bounded linear maps that send an $x \in L^\infty(\mathscr{G})_+$ to $\overline{\phi}(p_{k,m}xp_{l,n})$, $T_\lambda(p_{k,m}xp_{l,n})$, or $T_\rho(p_{k,m}xp_{l,n})$, respectively, which are determined by their restriction to $\pi_r(A)$. $\square$

Invariance of ϕ implies invariance of $\overline{\phi}$ as follows.

Proposition 4.8. *Let $\mathscr{G}$ be a partial compact quantum group. Then for all $x \in L^\infty(\mathscr{G})_+$, the normal, semi-finite weight $\overline{\phi}$ on $L^\infty(\mathscr{G})$ satisfies*

$$(\mathrm{id} \otimes \overline{\phi})(\overline{\Delta}(x)) = T_\lambda(x), \qquad\qquad (\overline{\phi} \otimes \mathrm{id})(\overline{\Delta}(x)) = T_\rho(x).$$

Proof. Let $a \in A$. Then (1.14) and the relation $\overline{\phi}\tbinom{k}{m} \circ \pi = \phi\tbinom{k}{m}$ imply

$$(\mathrm{id} \otimes \overline{\phi}\tbinom{l}{m})(\overline{\Delta}(\pi_r(a))) = \sum_k \overline{\phi}\tbinom{k}{m}(\pi_r(a))\lambda_k\rho_l.$$

Since each $\overline{\phi}\tbinom{k}{m}$ is a vector state and $\pi_r(A)$ is weakly dense in $L^\infty(\mathscr{G})$, this equations remains true if we replace $\pi_r(a)$ by arbitrary $x \in L^\infty(\mathscr{G})$. Summing over l and m, we obtain the first equation which we have to prove. The second one follows similarly. $\square$

The next result gives a fairly complete description of the objects of Tomita-Takesaki theory associated to $\overline{\phi}$.

Lemma 4.9. *The subspace $\Lambda(A) \subseteq L^2(\mathscr{G})$ is a Hilbert algebra with respect to the operations $\Lambda(a)\Lambda(b) = \Lambda(ab)$ and $\Lambda(a)^* = \Lambda(a^*)$ for all $a, b \in A$, and a Tomita algebra with respect to the family of operators ∇_z given by $\nabla_z\Lambda(a) = \Lambda(\sigma_z(a))$ for all $a \in A$, $z \in \mathbb{C}$. The associated left von Neumann algebra is $L^\infty(\mathscr{G})$, the associated normal, semifinite, faithful weight is $\overline{\phi}$, the modular operator $\Delta_{\overline{\phi}}$ is the closure of ∇_{-i}, the modular conjugation $J_{\overline{\phi}}$ is given by $J_{\overline{\phi}}\Lambda(a) = \Lambda(\sigma_{i/2}(a)^*)$ for all $a \in A$, and the modular automorphism group $\sigma^{\overline{\phi}}$ satisfies $\sigma^{\overline{\phi}}_t \circ \pi_r = \pi_r \circ \sigma_t$ for all $t \in \mathbb{R}$.*

Proof. We first show that $\Lambda(A)$ is a Hilbert algebra. Indeed, the map $\pi_r(a)\colon \Lambda(b) \to \Lambda(ab)$ is bounded for each $a \in A$ by Proposition 4.4, and the involution is pre-closed because for all $a, b \in A$,

$$\langle \Lambda(a)|\Lambda(b^*)\rangle = \phi(a^*b^*) = \phi(b^*\sigma(a^*)) = \langle \Lambda(b)|\Lambda(\sigma(a^*))\rangle$$

To see that $\Lambda(A)$ and $(\nabla_z)_z$ form a Tomita algebra, we have to verify that the map $z \mapsto \langle \Lambda(a)|\nabla_z\Lambda(b)\rangle = \phi(a^*\sigma_z(b))$ is entire for all $a, b \in A$ and that

$$\nabla_z\Lambda(a)^* = \nabla_{\overline{z}}\Lambda(a)^*, \quad \langle \Lambda(a)|\Lambda(b)\rangle = \langle \nabla_{-\overline{z}}\Lambda(a)|\Lambda(b)\rangle, \quad \langle \Lambda(a)^*|\Lambda(b)^*\rangle = \langle \Lambda(b)|\nabla_{-i}\Lambda(a)\rangle$$

for all $a, b \in A$, $z \in \mathbb{C}$. But all of this follows immediately from Corollary 2.34.

The left von Neumann algebra of $\Lambda(A)$ is $\pi_r(A)'' = L^\infty(\mathscr{G})$ and the associated weight $\tilde{\phi}$ satisfies $\tilde{\phi}(\pi_r(a^*a)) = \langle \Lambda(a)|\Lambda(a)\rangle = \phi(a^*a)$ for all $a \in A$. By Lemma 4.7, it coincides with $\overline{\phi}$. By [] [Takesaki:2, Thm. VI.2.2 and its proof], the modular operator $\Delta_{\overline{\phi}}$ is the closure of ∇_{-i} and the modular automorphism group is implemented by $(\nabla_t)_t$. □

The general theory of Hilbert algebras [] implies now:

Proposition 4.10. *Let $\mathscr{G}$ be a partial compact quantum group. Then the extension $\overline{\phi}$ of the invariant functional to $L^\infty(\mathscr{G})$ is faithful.* □

Remark 4.11. Without using the theory of Hilbert algebras, one could also check directly that the formula for $J_{\overline{\phi}}$ defines an anti-linear isometry, that $J_{\overline{\phi}}\pi_r(A)J_{\overline{\phi}}$ commutes with $\pi_r(A)$ and hence with $L^\infty(\mathscr{G})$, and that the family $(\Lambda(\lambda_k\rho_m))_{k,m}$ is cyclic for $J\pi_r(A)J$. Then this family is separating for $L^\infty(\mathscr{G})$ and $\overline{\phi}$ is faithful.

The scaling group τ and the unitary antipode R of $\mathscr{G}$ can easily be lifted to $C^r_0(\mathscr{G})$ and $L^\infty(\mathscr{G})$ using the following result. Let us call a conjugate-linear map on a Hilbert space an *anti-symmetry* if it is isometric and its square is the identity.

Lemma 4.12. *There exist a unique anti-symmetry I and a strongly continuous one-parameter group $P = (P_t)_t$ on on $L^2(\mathscr{G})$ such that for all $a \in A$, $t \in \mathbb{R}$,*

$$I\Lambda(a) = \Lambda(R(a)^*), \qquad P_t\Lambda(a) = \Lambda(\tau_t(a)).$$

Proof. Corollary 2.34 implies that the formulas above define an anti-symmetry I and unitaries P_t; for example, $\|I\Lambda(a)\|^2) = \phi(R(a)R(a)^*) = \phi(a^*a) = \|\Lambda(a)\|^2$, and $I^2 = \mathrm{id}$ because $* \circ R \circ * \circ R = R^2 = \mathrm{id}$. By A.1 in [] [Takesaki:2] and Corollary 2.34 5., elements of $\Lambda(A)$ are analytic with respect to P; in particular, P is strongly continuous. □

Proposition 4.13. *Let $\mathscr{G}$ be an I-partial compact quantum group.*

1. *There exists a unique $*$-anti-automorphism $\overline{R}$ of $L^\infty(\mathscr{G})$ such that $\overline{R} \circ \pi_r = \pi_r \circ R$. This $\overline{R}$ restricts to a $*$-anti-automorphism of $C_0^r(\mathscr{G})$.*
2. *There exists a unique strongly continuous one-parameter group $\overline{\tau}$ on $L^\infty(\mathscr{G})$ such that $\overline{\tau}_t \circ \pi_r = \pi_r \circ \theta_{-it,it}$ for all $t \in \mathbb{R}$, and this $\overline{\tau}$ restricts to a strongly continuous one-parameter group on $C_0^r(\mathscr{G})$.*

Proof. Short calculations show that the maps $\overline{R}\colon x \mapsto Ix^*I$ and $\overline{\tau}_t\colon x \mapsto P_t x P_t^*$ have the desired properties. □

Note that the relations (2.20) and (2.21) can be lifted to $C_0^r(\mathscr{G})$ and $L^\infty(\mathscr{G})$ by continuity. The next result will allow us to relate $\overline{R}$ to the unitary antipode of the measured quantum groupoid that we are going to construct.

Lemma 4.14. *For all $a, b \in A$,*

$$\overline{R}(\mathrm{id} \otimes \omega_{J\Lambda(b),J\Lambda(b)})(\overline{\Delta}(\pi(a^*a))) = (\mathrm{id} \otimes \omega_{J\Lambda(a),J\Lambda(a)})(\overline{\Delta}(\pi(b^*b))).$$

Proof. Let $c = a^*a$ and $d = b^*b$. A short calculation using (2.19) shows that the right hand side is equal to

$$d_{(1)}\phi(\sigma_{i/2}(a)d_{(2)}\sigma_{i/2}(a)^*) = d_{(1)}\phi(\sigma_{i/2}(c)d_{(2)}).$$

By Lemma 1.35 and (2.20), (2.19), this equals $S(\tau_{i/2}(c_{(1)}))\phi(\sigma_{i/2}(c_{(2)})d)$ which is the left hand side. □

The operator-algebraic structures constructed so far fit into the theory of measured quantum groupoids as follows.

Denote by ν the normal, faithful, semifinite weight on $l^\infty(I)$ given by

$$\nu(f) = \sum_k f(k) \quad \text{for all } f \in l^\infty(I)_+. \tag{4.5}$$

Then the relative tensor product of $L^2(\mathscr{G})$ with itself, relative to the representations ρ, λ of $l^\infty(I)$ and the weight ν, takes the simple form

$$L^\infty(\mathscr{G}) \mathbin{{}_\rho\underset{\nu}{\otimes}_\lambda} L^\infty(\mathscr{G}) \cong \bigoplus_k (\rho_k L^2(\mathscr{G}) \otimes \lambda_k L^2(\mathscr{G})) = \overline{E}(L^2(\mathscr{G}) \otimes L^2(\mathscr{G})),$$

see [], the relative tensor product of operators $S \in \rho(l^\infty(I))'$ and $T \in \lambda(l^\infty(I))'$ gets identified with the compression

$$S \underset{\nu}{{}_\rho\!\otimes_\lambda} T \equiv \overline{E}(S \otimes T) = (S \otimes T)\overline{E} \subseteq \mathcal{B}(\overline{E}(L^2(\mathscr{G}) \otimes L^2(\mathscr{G}))),$$

and the fiber product of $L^\infty(\mathscr{G})$ with itself, relative to ρ and λ, gets identified with

$$\begin{aligned} L^\infty(\mathscr{G}) \underset{l^\infty(I)}{{}_\rho\!*_\lambda} L^\infty(\mathscr{G}) &= (L^\infty(\mathscr{G})' \underset{\nu}{{}_\rho\!\otimes_\lambda} L^\infty(\mathscr{G})')' \\ &\equiv (\overline{E}(L^\infty(\mathscr{G})' \otimes L^\infty(\mathscr{G})'))' = \overline{E}(L^\infty(\mathscr{G}) \otimes L^\infty(\mathscr{G}))\overline{E}. \end{aligned} \tag{4.6}$$

By Lemma 4.6 3., we can co-restrict $\overline{\Delta}$ to a normal, faithful $*$-homomorphism

$$\tilde{\Delta} \colon L^\infty(\mathscr{G}) \to L^\infty(\mathscr{G}) \underset{l^\infty(I)}{{}_\rho\!*_\lambda} L^\infty(\mathscr{G}).$$

We now obtain a Hopf-von Neumann bimodule in the sense of [].

Proposition 4.15. *Let $\mathscr{G}$ be an I-partial compact quantum group. Then*

1. $\tilde{\Delta}(\lambda(x)) = \lambda(x) \underset{\nu}{{}_\rho\!\otimes_\lambda} 1$ *and* $\tilde{\Delta}(\rho(x)) = 1 \underset{\nu}{{}_\rho\!\otimes_\lambda} \rho(x)$ *for all* $x \in l^\infty(I)$, *and*
2. $(\tilde{\Delta} * \mathrm{id}) \circ \tilde{\Delta} = (\mathrm{id} * \tilde{\Delta}) \circ \tilde{\Delta}$.

In particular, $(l^\infty(I), L^\infty(\mathscr{G}), \lambda, \rho, \tilde{\Delta})$ *is a Hopf-von Neumann bimodule.*

Proof. Assertion 1. follows from (1.2) and Lemma 4.6 1. and ensures that the $*$-homomorphisms

$$\tilde{\Delta} * \mathrm{id}, \mathrm{id} * \tilde{\Delta} \colon L^\infty(\mathscr{G}) \underset{l^\infty(I)}{{}_\rho\!*_\lambda} L^\infty(\mathscr{G}) \to L^\infty(\mathscr{G}) \underset{l^\infty(I)}{{}_\rho\!*_\lambda} L^\infty(\mathscr{G}) \underset{l^\infty(I)}{{}_\rho\!*_\lambda} L^\infty(\mathscr{G})$$

are well-defined. As in (4.6), we can identify

$$L^\infty(\mathscr{G}) \underset{l^\infty(I)}{{}_\rho\!*_\lambda} L^\infty(\mathscr{G}) \underset{l^\infty(I)}{{}_\rho\!*_\lambda} L^\infty(\mathscr{G}) \cong \overline{E}^{(2)}(L^\infty(\mathscr{G}) \otimes L^\infty(\mathscr{G}) \otimes L^\infty(\mathscr{G}))\overline{E}^{(2)},$$

where $\overline{E}^{(2)} = (\overline{E} \otimes 1)(1 \otimes \overline{E})$, and then the $*$-homomorphisms become restrictions of the maps $\tilde{\Delta} \otimes \mathrm{id}$ and $\mathrm{id} \otimes \tilde{\Delta}$, respectively. Now, 2. follows from Lemma 4.6 1. and co-associativity of Δ. □

This Hopf-von Neumann bimodule is a measured quantum groupoid in the sense of [].

Theorem 4.16. *Let $\mathscr{G}$ be an I-partial compact quantum group. Then the Hopf-von Neumann bimodule* $(l^\infty(I), L^\infty(\mathscr{G}), \lambda, \rho, \tilde{\Delta})$ *and the weights* (T_λ, T_ρ *and* ν *defined in* (4.4) *and* (4.5) *form a measured quantum groupoid. It is unimodular and its unitary antipode and scaling group coincide with* $\overline{R}$ *and* $\overline{\tau}$, *respectively.*

Proof. First, observe that the the modular automorphism groups of the weights $\nu \circ \lambda^{-1} \circ T_\lambda$ and $\nu \circ \rho^{-1} \circ T_\rho$ commute because the two compositions coincide with $\overline{\phi}$. Next, we need to show that T_λ is left-invariant in the sense that

$$(\mathrm{id} \underset{\nu}{{}_\rho*_\lambda} \overline{\phi})(\tilde{\Delta}(x)) = T_\lambda(x)$$

for all $x \in L^\infty(\mathscr{G})_+$. But it is easy to see that the left hand side coincides with $(\mathrm{id} * \overline{\phi})(\overline{\Delta}(x))$ so that the equation above follows from Proposition 4.8. Likewise T_ρ is right-invariant in the appropriate sense. We thus obtain a measured quantum groupoid as claimed. Denote by $\tilde{R}$ its unitary antipode and by $\tilde{\tau}$ its scaling group.

Let us prove that $\tilde{\tau}_t = \overline{\tau}$ for all $t \in \mathbb{R}$. By [] and (2.20),

$$(\tilde{\tau}_t \underset{l^\infty(I)}{{}_\rho*_\lambda} \sigma_t^{\overline{\tau}}) \circ \tilde{\Delta} = \tilde{\Delta} \circ \sigma_t^{\overline{\tau}}, \qquad\qquad (\overline{\tau}_t \otimes \sigma_t^{\overline{\tau}}) \circ \overline{\Delta} = \overline{\Delta} \circ \overline{\tau}_t.$$

The second equation implies that the first one remains true if we replace $\tilde{\tau}_t$ by $\overline{\tau}_t$. Using Theorem A.7 in [] [enock:action], we can conclude that $\tilde{\tau}_t = \overline{\tau}_t$.

To prove that $\tilde{R} = \overline{R}$, we use the relations

$$\tilde{R}(\mathrm{id} \underset{\nu}{{}_\rho*_\lambda} \omega_{J\Lambda(b),J\Lambda(b)})(\overline{\Delta}(\pi(a^*a))) = (\mathrm{id} \underset{\nu}{{}_\rho*_\lambda} \omega_{J\Lambda(a),J\Lambda(a)})(\overline{\Delta}(\pi(b^*b)))$$

from [] and Lemma 4.14. □

5 Partial compact quantum groups from reciprocal random walks

We recall some notions introduced in [4]. We slightly change the terminology for the sake of convenience.

Definition 5.1. Let $t \in \mathbb{R}_0$. A *t-reciprocal random walk* consists of a quadruple $(\Gamma, w, \mathrm{sgn}, i)$ with

- $\Gamma = (\Gamma^{(0)}, \Gamma^{(1)}, s, t)$ a graph with *source* and *target* maps
$$s, t : \Gamma^{(1)} \to \Gamma^{(0)},$$
- w a function (the *weight* function) $w : \Gamma^{(1)} \to \mathbb{R}_0^+$,
- sgn a function (the *sign* function) $\mathrm{sgn} : \Gamma^{(1)} \to \{\pm 1\}$,
- i an involution
$$i : \Gamma^{(1)} \to \Gamma^{(1)}, \quad e \mapsto \overline{e}$$
with $s(\overline{e}) = t(e)$ for all edges e,

such that the following conditions are satisfied:

- $w(e)w(\bar{e}) = 1$ for all edges e,
- $\operatorname{sgn}(e)\operatorname{sgn}(\bar{e}) = \operatorname{sgn}(t)$ for all edges e,
- $p(e) = \frac{1}{|t|}w(e)$ defines a *random walk*: $\sum_{s(e)=v} p(e) = 1$ for all $v \in \Gamma^{(0)}$.

Note that, by [4, Proposition 3.1], there is a uniform bound on the number of edges leaving from any given vertex v.

For examples of t-reciprocal random walks, we refer to [4]. One particular example which will be needed for our construction of dynamical quantum $SU(2)$ is the following.

Example 5.2. Take $0 < |q| < 1$ and $x \in \mathbb{R}$. Write $2_q = q + q^{-1}$. Then we have the reciprocal -2_q-random walk

$$\Gamma_x = (\Gamma_x, w, \operatorname{sgn}, i)$$

with

$$\Gamma^{(0)} = \mathbb{Z}, \quad \Gamma^{(1)} = \{(k,l) \mid |k-l| = 1\} \subseteq \mathbb{Z} \times \mathbb{Z}$$

with projection on the first (resp. second) leg as source (resp. target) map, with weight function

$$w(k, k \pm 1) = \frac{|q|^{x+k\pm 1} + |q|^{-(x+k\pm 1)}}{|q|^{x+k} + |q|^{-(x+k)}},$$

sign function

$$\operatorname{sgn}(k, k+1) = 1, \quad \operatorname{sgn}(k, k-1) = -\operatorname{sgn}(q),$$

and involution $\overline{(k, k+1)} = (k+1, k)$.

By translation we shift the value of x by an integer, and by inversion we change x into $-x$ and multiply the sign function with a fixed sign. It follows that by some graph isomorphism, we can always arrange to have $x \in [0, \frac{1}{2}]$ at the cost of having a different sign function.

Let now $0 < |q| \leqslant 1$, and let $SU_q(2)$ be Woronowicz's twisted $SU(2)$ group [15]. Then $SU_q(2)$ is a compact quantum group, and its category of finite-dimensional unitary representations $\operatorname{Rep}(SU_q(2))$ is generated by the spin 1/2-representation $\pi_{1/2}$ on $\mathbb{C}^2$.

Let $\Gamma = (\Gamma, w, \operatorname{sgn}, i)$ be a -2_q-reciprocal random walk. Define $\mathcal{H}_\Gamma$ as the $\Gamma^{(0)}$-bigraded Hilbert space $l^2(\Gamma^{(1)})$, where the $\Gamma^{(0)}$-bigrading is given by

$$\delta_e \in {}^{s(e)}\mathcal{H}_\Gamma{}^{t(e)}$$

for the obvious Dirac functions. Further define R_Γ as the (bounded) map

$$R_\Gamma : l^2(\Gamma^{(0)}) \to \mathcal{H}_\Gamma \underset{\Gamma^{(0)}}{\otimes} \mathcal{H}_\Gamma$$

given by

$$R_\Gamma \delta_v \quad = \sum_{e, s(e)=v} \operatorname{sgn}(e)\sqrt{w(e)}\delta_e \otimes \delta_{\bar{e}}.$$

Then $R_\Gamma^* R_\Gamma = |q| + |q|^{-1}$ and

$$(R_\Gamma^* \otimes \mathrm{id})(\mathrm{id} \otimes R_\Gamma) = -\operatorname{sgn}(q)\,\mathrm{id}\,.$$

Hence, by the universal property of $\mathrm{Rep}(SU_q(2))$ ([4, Theorem 1.4], based on [10, 5, 16, 8, 9]), we have a strongly monoidal *-functor

$$F_\Gamma : \mathrm{Rep}(SU_q(2)) \to {}^{\Gamma^{(0)}}\mathrm{Hilb}_f^{\Gamma^{(0)}}$$

such that $F_\Gamma(\pi_{1/2}) = \mathcal{H}_\Gamma$ and $F_\Gamma(\mathscr{R}) = R_\Gamma$, with

$$(\pi_{1/2}, \mathscr{R}, -\operatorname{sgn}(q)\mathscr{R})$$

a solution for the conjugate equations for $\pi_{1/2}$. Up to equivalence, F_Γ only depends upon the isomorphism class of (Γ, w), and is independent of the chosen involution or sign structure.

It follows from our main theorem that for each reciprocal random walk on a graph Γ, one obtains a $\Gamma^{(0)}$-partial compact quantum group. Our aim is to give a direct representation of it by generators and relations. We will write $\Gamma_{vw} \subseteq \Gamma^{(1)}$ for the set of edges with source v and target w.

Theorem 5.3. *Let $0 < |q| \leqslant 1$, and let Γ be a -2_q-reciprocal random walk. Let $A(\Gamma)$ be the total *-algebra associated to the $\Gamma^{(0)}$-partial compact quantum group constructed from the fiber functor F_Γ. Then $A(\Gamma)$ is the universal *-algebra generated by a copy of the *-algebra of finite support functions on $\Gamma^{(0)} \times \Gamma^{(0)}$ (with the Dirac functions written as $\mathbf{1}\binom{v}{w}$) and elements $(u_{e,f})_{e,f\in\Gamma^{(1)}}$ where $u_{e,f} \in {}^{s(e)}_{s(f)}A(\Gamma)^{t(e)}_{t(f)}$ and*

$$\sum_{v\in\Gamma^{(0)}} \sum_{g\in\Gamma_{vw}} u_{g,e}^* u_{g,f} = \delta_{e,f}\mathbf{1}\tbinom{w}{t(e)}, \qquad \forall w \in \Gamma^{(0)}, e, f \in \Gamma^{(1)}, \tag{5.1}$$

$$\sum_{w\in\Gamma^{(0)}} \sum_{g\in\Gamma_{vw}} u_{e,g} u_{f,g}^* = \delta_{e,f}\mathbf{1}\tbinom{s(e)}{v} \qquad \forall v \in \Gamma(0), e, f \in \Gamma^{(1)}, \tag{5.2}$$

$$u_{e,f}^* \;=\; \operatorname{sgn}(e)\operatorname{sgn}(f)\sqrt{\frac{w(f)}{w(e)}}u_{\bar{e},\bar{f}}, \qquad \forall e, f \in \Gamma^{(1)}. \tag{5.3}$$

If moreover $v, w \in \Gamma^{(0)}$ and $e, f \in \Gamma^{(1)}$, we have

$$\Delta_{vw}(u_{e,f}) = \sum_{\substack{s(g)=v\\ t(g)=w}} u_{e,g} \otimes u_{g,f},$$

$$\varepsilon(u_{e,f}) = \delta_{e,f}$$

and

$$S(u_{e,f}) = u_{f,e}^*.$$

Proof. Let the $(v_{e,f})_{e,f\in\Gamma^{(1)}}$ be the matrix coefficients of the generating corepresentation of $A(\Gamma)$ on $F_\Gamma(\pi_{1/2}) = l^2(\Gamma^{(1)})$. By construction $V = (v_{e,f})_{e,f\in\Gamma^{(1)}}$ defines a unitary corepresentation of $A(\Gamma)$, hence the relations (5.1) and (5.2) are satisfied for the $v_{e,f}$. Now as R_Γ is an intertwiner between $\mathbb{C}_{\Gamma^{(0)}}$ and $V \underset{\Gamma^{(0)}}{\textcircled{\scriptsize T}} V$, we have for all $v \in \Gamma^{(0)}$ that

$$\sum_{\substack{e,f,g,h\in\Gamma(1)\\ t(f)=s(g),t(e)=s(f)}} v_{e,f}v_{g,h} \otimes ((e_{e,f}\otimes e_{g,h})\circ R_\Gamma\delta_v) = \sum_w \mathbf{1}\tbinom{w}{v} \otimes R_\Gamma\delta_v, \tag{5.4}$$

hence

$$\sum_{\substack{e,g,k\\ t(e)=s(g),s(k)=v}} \operatorname{sgn}(k)\sqrt{w(k)}\left(v_{e,k}v_{g,\bar{k}}\otimes\delta_e\otimes\delta_g\right) = \sum_{k,s(k)=v,w} \operatorname{sgn}(k)\sqrt{w(k)}\left(\mathbf{1}\tbinom{w}{v}\otimes\delta_k\otimes\delta_{\bar{k}}\right).$$

Hence if $t(e) = s(g) = z$, we have

$$\sum_{k,s(k)=v} \operatorname{sgn}(k)\sqrt{w(k)}v_{e,k}v_{g,\bar{k}} = \sum_w \delta_{s(e),v}\delta_{e,\bar{g}}\operatorname{sgn}(e)\sqrt{w(e)}\mathbf{1}\tbinom{w}{v}.$$

Multiplying to the left with $v_{e,l}^*$ and summing over all e with $t(e) = z$, we see from (5.1) that also relation (5.3) is satisfied. Hence the $v_{e,f}$ satisfy the universal relations in the statement of the theorem. The formulas for comultiplication, counit and antipode then follow immediately from the fact that V is a unitary corepresentation.

Let us now a priori denote by $B(\Gamma)$ the *-algebra determined by the relations (5.1),(5.2) and (5.3) above, and write $\mathscr{B}(\Gamma)$ for the associated $\Gamma^{(0)}\times\Gamma^{(0)}$-partial *-algebra. Write

$$\Delta(u_{e,f}) = \sum_{g\in\Gamma^{(1)}} u_{e,g}\otimes u_{g,f},$$

which makes sense in $M(B(\Gamma)\otimes B(\Gamma))$ as the number of edges leaving or arriving at a fixed vertex is uniformly bounded. Then we compute

$$\begin{aligned}
\sum_{v\in\Gamma^{(0)}}\sum_{g\in\Gamma_{vw}} \Delta(u_{g,e})^*\Delta(u_{g,f}) &= \sum_{v\in\Gamma^{(0)}}\sum_{g\in\Gamma_{vw}}\sum_{h,k\in\Gamma^{(1)}} u_{g,h}^*u_{g,k}\otimes u_{h,e}^*u_{k,f}\\
&= \sum_{h,k\in\Gamma^{(1)}} \delta_{h,k}\mathbf{1}\tbinom{w}{t(h)}\otimes u_{h,e}^*u_{k,f}\\
&= \sum_{z\in\Gamma^{(0)}}\sum_{\substack{h\in\Gamma^{(1)}\\ t(h)=z}} \mathbf{1}\tbinom{w}{z}\otimes u_{h,e}^*u_{h,f}\\
&= \delta_{e,f}\sum_{z\in\Gamma^{(0)}} \mathbf{1}\tbinom{w}{z}\otimes\mathbf{1}\tbinom{z}{t(e)}\\
&= \delta_{e,f}\Delta(1).
\end{aligned}$$

Similarly, the analogue of (5.2) holds for $\Delta(u_{e,f})$. As also (5.3) holds for $\Delta(u_{e,f})$ by its very form, it follows that we can define a *-algebra homomorphism

$$\Delta : B(\Gamma) \to M(B(\Gamma) \otimes B(\Gamma))$$

sending $u_{e,f}$ to $\Delta(u_{e,f})$ and $\mathbf{1}\binom{v}{w}$ to $\sum_{z\in\Gamma^{(0)}} \mathbf{1}\binom{v}{z} \otimes \mathbf{1}\binom{z}{w}$. Cutting down, we obtain maps

$$\Delta_{vw} : {}^{r}_{t}B(\Gamma)^{s}_{z} \to {}^{r}_{v}B(\Gamma)^{s}_{w} \otimes {}^{v}_{t}B(\Gamma)^{w}_{z}$$

which then satisfy the properties (a),(d) and (e) of Definition 1.7. Moreover, the Δ_{vw} are coassociative as they are coassociative on generators.

Let now $e_{\binom{v}{w},\binom{v'}{w'}}$ be the matrix units for $l^2(\Gamma^{(0)} \otimes \Gamma^{(0)})$. Then one verifies again directly from the defining relations of $B(\Gamma)$ that one can define a *-homorphism

$$\tilde{\varepsilon} : B(\Gamma) \to B(l^2(\Gamma^{(0)})), \qquad \begin{cases} \mathbf{1}\binom{v}{w} & \mapsto & \delta_{v,w}\, e_{\binom{v}{v},\binom{v}{v}} \\ u_{e,f} & \mapsto & \delta_{e,f}\, e_{\binom{s(e)}{s(f)},\binom{t(e)}{t(f)}} \end{cases}$$

We can hence define a map $\varepsilon : B(\Gamma) \to \mathbb{C}$ such that

$$\tilde{\varepsilon}(x) = \varepsilon(x) e_{\binom{k}{m},\binom{l}{n}}, \qquad \forall x \in {}^{k}_{m}B(\Gamma)^{l}_{n}.$$

Clearly it satisfies the conditions (b) and (c) of a partial *-bialgebra. As ε satisfies the counit condition on generators, it follows by multiplicativity that it satisfies the counit condition on the whole of $B(\Gamma)$, i.e. $B(\Gamma)$ is a partial *-bialgebra.

It is clear now that the $u_{e,f}$ define a corepresentation U of $B(\Gamma)$. Moreover, from (5.1) and (5.3) we can deduce that $R_\Gamma : \mathbb{C}_{\Gamma^{(0)}} \to \mathcal{H}_\Gamma \underset{\Gamma^{(0)}}{\otimes} \mathcal{H}_\Gamma$ is a morphism from $\mathbb{C}_{\Gamma^{(0)}}$ to $U \underset{\Gamma(0)}{\textcircled{\scriptsize T}} U$ in $\mathrm{Corep}_{\mathrm{sfd},u}(\mathscr{B}(\Gamma))$, cf. (5.4). From the universal property of $\mathrm{Rep}(SU_q(2))$, it then follows that we have a (unique) monoidal functor

$$G_\Gamma : \mathrm{Rep}(SU_q(2)) \to \mathrm{Corep}_{\mathrm{sfd},u}(\mathscr{B}(\Gamma))$$

such that $G_\Gamma(\pi_{1/2}) = U$. This functor is faithful as $\mathrm{Rep}(SU_q(2))$ has no non-trivial ideals. On the other hand, as we have a Δ-preserving *-homomorphism $\mathscr{B}(\Gamma) \to \mathscr{A}(\Gamma)$ by the universal property of $\mathscr{B}(\Gamma)$, we have a monoidal functor $H_\Gamma : \mathrm{Corep}_{\mathrm{sfd},u}(\mathscr{B}(\Gamma)) \to \mathrm{Rep}(SU_q(2))$ which is inverse to G_Γ. Then since the commutation relations of $\mathscr{A}(\Gamma)$ are completely determined by the morphism spaces of $\mathrm{Rep}(SU_q(2))$, it follows that we have a *-homomorphism $\mathscr{A}(\Gamma) \to \mathscr{B}(\Gamma)$ sending $v_{e,f}$ to $u_{e,f}$. This proves the theorem. $\square$

6 Dynamical quantum $SU(2)$

6.1 Dynamical quantum $SU(2)$ from the Podleś graph

Let us now consider the particular case of the Podleś graph of Example 5.2. We assume in the following that $x \in [0, \frac{1}{2}]$.

Let us denote

$$w_+(k) = w(k, k+1),$$
$$w_-(k) = w(k, k-1) = w_+(k-1)^{-1}.$$

Let $A_x = A(\Gamma_x)$ be the total *-algebra of the associated partial compact quantum group. Using Theorem 5.3, we have the following presentation of A_x. Let B be the *-algebra of finite support functions on $\mathbb{Z} \times \mathbb{Z}$, whose Dirac functions we write as $\mathbf{1}\binom{k}{l}$. Let $s_q = \frac{1}{2}(1 + \mathrm{sgn}(q))$. Then A_x is generated by a copy of B and elements

$$(u_{\epsilon,\nu})_{k,l} = u_{(k,k+\epsilon),(l,l+\nu)}$$

for $\epsilon, \nu \in \{-1, 1\} = \{-, +\}$ and $k, l \in \mathbb{Z}$ with defining relations

$$\begin{aligned}
\sum_{\mu \in \{\pm\}} (u_{\mu,\epsilon})^*_{m-\mu,k} (u_{\mu,\nu})_{m-\mu,l} &= \delta_{k,l}\delta_{\epsilon,\nu}\mathbf{1}\binom{m}{k+\epsilon}, \\
\sum_{\mu \in \{\pm\}} (u_{\epsilon,\mu})_{k,m} (u_{\nu,\mu})^*_{l,m} &= \delta_{\epsilon,\nu}\delta_{k,l}\mathbf{1}\binom{k}{m} \\
(u_{\epsilon,\nu})^*_{k,l} &= (\epsilon\nu)^{s_q} \left(\frac{w_\nu(l)}{w_\epsilon(k)}\right)^{1/2} (u_{-\epsilon,-\nu})_{k+\epsilon,l+\nu}.
\end{aligned}$$

The element $(u_{\epsilon,\nu})_{k,l}$ lives inside the component ${}^{k}_{l}(A_x)^{k+\epsilon}_{l+\nu}$.

Consider now $M(A_x)$, the multiplier algebra of A_x. We can form in $M(A_x)$ the elements $u_{\epsilon,\nu} = \sum_{k,l} (u_{\epsilon,\nu})_{k,l}$. Then $u = (u_{\epsilon,\nu})$ is a unitary 2×2 matrix. Moreover,

$$u^*_{\epsilon,\nu} = (\epsilon\nu)^{s_q} u_{-\epsilon,-\nu} \frac{w_\nu^{1/2}(\rho)}{w_\epsilon^{1/2}(\lambda)}, \tag{6.1}$$

where $w_\pm^{1/2}(k) = w_\pm(k)^{1/2}$ and where for a function f on $\mathbb{Z}$ we write $f(\lambda)(k,l) = f(k)$, $f(\rho)(k,l) = f(l)$. In the following, we then also use the notation $f(\lambda, \rho)$ for a function f on $\mathbb{Z} \times \mathbb{Z}$ interpreted as an element of $M(A)$, and for example $f(\lambda + 1, \rho)$ corresponds to the function $(k, l) \mapsto f(k+1, l)$. We then have the following commutation relations between functions on $\mathbb{Z} \times \mathbb{Z}$ and the entries of u:

$$f(\lambda, \rho) u_{\epsilon,\nu} = u_{\epsilon,\nu} f(\lambda - \epsilon, \rho - \nu). \tag{6.2}$$

Let us write

$$F(k) = |q|^{-1} w_+(k) = |q|^{-1} \frac{|q|^{x+k+1} + |q|^{-x-k-1}}{|q|^{x+k} + |q|^{-x-k}},$$

and further put

$$\alpha = \frac{F^{1/2}(\rho - 1)}{F^{1/2}(\lambda - 1)} u_{--}, \qquad \beta = \frac{1}{F^{1/2}(\lambda - 1)} u_{-+}.$$

Then the unitarity of $(u_{\epsilon,\nu})_{\epsilon,\nu}$ together with (6.1) and (6.2) are equivalent to the commutation relations

$$\alpha\beta = qF(\rho-1)\beta\alpha \qquad \alpha\beta^* = qF(\lambda)\beta^*\alpha \tag{6.3}$$

$$\begin{aligned} \alpha\alpha^* + F(\lambda)\beta^*\beta = 1, &\qquad \alpha^*\alpha + q^{-2}F(\rho-1)^{-1}\beta^*\beta = 1, \\ F(\rho-1)^{-1}\alpha\alpha^* + \beta\beta^* = F(\lambda-1)^{-1}, &\qquad F(\lambda)\alpha^*\alpha + q^{-2}\beta\beta^* = F(\rho), \end{aligned} \tag{6.4}$$

$$f(\lambda)g(\rho)\alpha = \alpha f(\lambda+1)g(\rho+1), \qquad f(\lambda)g(\rho)\beta = \beta f(\lambda+1)g(\rho-1). \tag{6.5}$$

These are precisely the commutation relations for the dynamical quantum $SU(2)$-group as in [7, Definition 2.6], except that the precise value of F has been changed by a shift in the parameter domain by a complex constant. Clearly, by Theorem 5.3 the (total) coproduct on A_x also agrees with the one on the dynamical quantum $SU(2)$-group, namely

$$\begin{aligned} \Delta(\alpha) &= \Delta(1)(\alpha\otimes\alpha - q^{-1}\beta\otimes\beta^*), \\ \Delta(\beta) &= \Delta(1)(\beta\otimes\alpha^* + \alpha\otimes\beta) \end{aligned}$$

where $\Delta(1) = \sum_{k\in\mathbb{Z}} \rho_k \otimes \lambda_k$.

6.2 Representation theory of the function algebra of dynamical quantum $SU(2)$

In this section we classify the irreducible *-representations of A_x. The parametrisation will hinge on the classification of what we call irreducible (x,c)-adapted sets, which we will now discuss. In the following, we fix $0 < |q| < 1$.

Definition 6.1. Let $x \in [0, \frac{1}{2}]$, and let $c \in \mathbb{R}$. For $\epsilon \in \{\pm\}$, an integer $m \in \mathbb{Z}$ will be called $(x,c)_\epsilon$*-adapted* if

$$c \leqslant |q|^{2x+m-\epsilon} + |q|^{-2x-m+\epsilon}, \tag{6.6}$$

and *strictly* $(x,c)_\epsilon$-adapted if this holds strictly. An integer is called (x,c)*-adapted* if it is both $(x,c)_+$ and $(x,c)_-$-adapted.

A set of integers Z is called an (x,c)*-set* if the following conditions hold:

- Z is not empty.
- Z consists of (x,c)-adapted points.
- If $m \in Z$ is strictly $(x,c)_\epsilon$-adapted, then $m - 2\epsilon$ is in Z.

An (x,c)-set Z is called *even* (resp. *odd*) if $Z \subseteq 2\mathbb{Z}$ (resp. $Z \subseteq 2\mathbb{Z}+1$).

An (x,c)-set is called *irreducible* if it can not be written as the union of two disjoint (x,c)-sets.

We aim to classify irreducible (x, c)-sets. We start of with the following lemma.

Lemma 6.2. *Let $c \in \mathbb{R}$ and $x \in [0, \frac{1}{2}]$.*

1. *Any irreducible (x, c)-set is either even or odd.*
2. *Z is an irreducible (x, c)-set if and only if $-Z - 1$ is an irreducible $(\frac{1}{2} - x, c)$-set.*

Proof. Immediate. □

Hence it suffices to classify even irreducible (x, c)-sets. This is achieved in the following proposition. (We use the convention $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{N}_0 = \{1, 2, \ldots\}$).

Proposition 6.3. *Let $c \in \mathbb{R}$ and $x \in [0, \frac{1}{2}]$.*

- *If $c < |q|^{2x-1} + |q|^{-2x+1}$, then $2\mathbb{Z}$ is an irreducible (x, c)-set.*
- *If $c \geqslant |q|^{2x-1} + |q|^{-2x+1}$, write $c = |q|^y + |q|^{-y}$ for some unique $y \geqslant 1 - 2x$.*
    - * *Assume $x = 0$.*
        - ○ *If $y = 1$, then $-2\mathbb{N}_0$, $\{0\}$ and $2\mathbb{N}_0$ are irreducible (x, c)-sets.*
        - ○ *If $y \in 2\mathbb{N}_0 + 1$, then $y + 1 + 2\mathbb{N}$ and $-y - 1 - 2\mathbb{N}$ are irreducible (x, c)-sets.*
    - * *Assume $x \in (0, \frac{1}{2})$. Assume y is of the form $y = |2x + M - 1|$, where M is a (uniquely determined) even integer.*
        - ○ *If $M > 0$, then $M + 2\mathbb{N}$ is an irreducible (x, c)-set.*
        - ○ *If $M < 0$, then $M - 2\mathbb{N}_0$ is an irreducible (x, c)-set.*
        - ○ *If $M = 0$, then $2\mathbb{N}$ and $-2\mathbb{N}_0$ are irreducible (x, c)-sets.*
    - * *Assume $x = \frac{1}{2}$.*
        - ○ *If $y \in 2\mathbb{N}$, then $y + 2\mathbb{N}$ and $-y - 2\mathbb{N}_0$ are irreducible (x, c)-sets.*

The above (x, c)-sets exhaust all possible even irreducible (x, c)-sets.

Proof. Clearly

$$\min\{|q|^{2x+m-\epsilon} + |q|^{-2x-m+\epsilon} \mid m \in 2\mathbb{Z}, \epsilon \in \{-1, 1\}\} = |q|^{2x-1} + |q|^{-2x+1}.$$

Hence if $c < |q|^{2x-1} + |q|^{-2x+1}$, all $m \in 2\mathbb{Z}$ are strictly $(x, c)_\epsilon$-adapted for any $\epsilon \in \{-1, 1\}$, and we deduce that $2\mathbb{Z}$ is the only irreducible (x, c)-set.

If $c \geqslant |q|^{2x-1} + |q|^{-2x+1}$, we can write $c = |q|^y + |q|^{-y}$ for a uniquely determined $y \geqslant 1 - 2x$, and then we see m is $(x, c)_\epsilon$-adapted if and only if

$$y \leqslant |2x + m - \epsilon|.$$

If $y = 1 - 2x$, we can still conclude that $2\mathbb{Z}$ is an (x, c)-set, but it will no longer be irreducible. If $m \in 2\mathbb{N}_0$ and $x \neq 0$, then it is strictly $(x, c)_\epsilon$-adapted for $\epsilon \in \{-, +\}$, hence if an irreducible (x, c)-set contains a strictly positive even integer, it contains $2\mathbb{N}$. Since $m = 0$ is (x, c)-adapted but not strictly $(x, c)_+$-adapted, we conclude that $2\mathbb{N}$ is an irreducible (x, c)-set. In case $x = 0$ however, $m = 2$ is not strictly $(x, c)_+$-adapted, and we can conclude that in fact $2\mathbb{N}_0$ is an irreducible (x, c)-set. The set $\{0\}$ then is an irreducible (x, c)-set by itself. Considering again a general x, we similarly see that if an irreducible (x, c)-set contains an even integer < 2, then it contains $-2\mathbb{N}_0$. Hence $-2\mathbb{N}_0$ is an irreducible (x, c)-set. Hence for $x \neq 0$, $2\mathbb{N}$ and $-2\mathbb{N}_0$ exhaust the irreducible (x, c)-sets, while for $x = 0$ we have the three sets $2\mathbb{N}_0, \{0\}$ and $-2\mathbb{N}_0$.

If $y > 1-2x$, it is clear that an (x, c)-set exists if and only if the equation $y = |2x+m-1|$ admits a solution in $2\mathbb{Z}$.

Let us first assume that $x \in (0, \frac{1}{2})$. Then this solution is unique, say $y = |2x + M - 1|$ with $M \in 2\mathbb{Z}$.

Assume first that $M > 0$. Then any even $m < 0$ which is $(x, c)_-$-adapted is strictly $(x, c)_-$-adapted. It follows that if an (x, c)-set contains a strictly negative integer, it contains $m = 0$. But then $m = 0$ is $(x, c)_+$-adapted, which leads to the contradictory statement $x \leqslant 0$. Hence any (x, c)-set necessarily lies in $2\mathbb{N}_0$. It is then easily seen that the only (x, c)-set is $M + 2\mathbb{N}$, which is then necessarily irreducible.

Assume now that $M \leqslant 0$. From the condition $y > 1-2x$, we may in fact assume $M < 0$. Then as above, we conclude that if an (x, c)-set contains a strictly positive even integer, it contains $m = 0$, which is not $(x, c)_-$-adapted. It then follows that $M - 2\mathbb{N}_0$ is the only irreducible (x, c)-set.

Let us now assume that $x = 0$. Then we may assume $y = M + 1$ with $M \in 2\mathbb{N}_0$. Then $m > 0$ is $(x, c)_+$-adapted if and only if $m \geqslant M + 2$, and it is strictly so if and only if $m > M + 2$. As $m = M + 2$ is (x, c)-adapted, it follows that $M + 2\mathbb{N}_0$ is an irreducible (x, c)-set, and the only one which intersects the strictly positive integers non-trivially. On the other hand, an $m < 0$ is $(x, c)_-$-adapted if and only if $m \leqslant -M - 2$, and strictly so if and only if $m < -M - 2$. As $m = -M - 2$ is (x, c)-adapted, it follows that $-M - 2\mathbb{N}_0$ is an irreducible (x, c)-set, and the only one which intersects the strictly negative integers non-trivially. Finally, since we have assumed $M > 0$, the integer $m = 0$ is never (x, c)-adapted.

Finally, let us assume that $x = \frac{1}{2}$. Then we may assume $y = M$ with $M \in 2\mathbb{N}_0$. By a similar reasoning as in the case $x = 0$, we conclude that the only irreducible (x, c)-sets are $M + 2\mathbb{N}$ and $-M - 2\mathbb{N}_0$. □

Let us now return to the representation theory of the *-algebra A_x.

Let π be any (necessarily bounded) non-degenerate *-representation of A_x on a Hilbert space $\mathcal{H}_\pi$. Then

$$\mathcal{H}_\pi = \oplus \mathcal{H}_n^m, \qquad \mathcal{H}_n^m = \pi(\mathbf{1}\tbinom{m}{n})\mathcal{H}.$$

Write

$$V_\pi = \text{the (non-closed) linear span of all } \mathcal{H}_n^m.$$

Then $\pi(A_x)V_\pi = V_\pi$. It follows that one can extend π to a map

$$\pi : M(A_x) \to \mathrm{End}_{\mathrm{adj}}(V_\pi),$$

where $\mathrm{End}_{\mathrm{adj}}(V_\pi)$ denotes the *-algebra of adjointable operators on the pre-Hilbert space V_π. As the $u_{\epsilon,\nu}$ form a unitary matrix, we can then in fact make sense of the $\pi(u_{\epsilon,\nu})$ as contractions on $\mathcal{H}_\pi$. Hence the generators α, β and their adjoints give rise to endomorphisms $V_\pi \to V_\pi$ which are bounded when restricted to any $\mathcal{H}_n^m$.

We then have the following easy lemma.

Lemma 6.4. *There is a one-to-one correspondence between*

- *non-degenerate *-representations $(\mathcal{H}_\pi, \pi)$ of A_x on Hilbert spaces, and*
- *$\mathbb{Z}^2$-bigraded pre-Hilbert spaces V_π with norm-complete components and equipped with adjointable maps $\alpha, \beta : V_\pi \to V_\pi$ satisfying the commutation relations as in* (6.3), (6.4) *and* (6.5) *and with $f(\lambda,\rho)\xi = f(k,l)\xi$ for f a function on $\mathbb{Z}\times\mathbb{Z}$ and $\xi \in V_l^k$.*

Definition 6.5. The *Casimir* of A_x is defined to be the following element $\Omega \in M(A_x)$,

$$\Omega = q^{\lambda-\rho+1} + q^{\rho-\lambda-1} - \mathrm{sgn}(q)^{\lambda-\rho}q^{-1}(|q|^{x+\lambda+1} + |q|^{-x-\lambda-1})(|q|^{x+\rho-1} + |q|^{-x-\rho+1})\beta^*\beta.$$

Lemma 6.6. *The element Ω is self-adjoint central in $M(A_x)$.*

Proof. Straightforward, cf. [7, Lemma 3.3]. □

Corollary 6.7. *If π is an irreducible *-representation of A_x, there exists $c \in \mathbb{R}$ such that $\pi(\Omega)\xi = c\xi$ for all $\xi \in V_\pi$.*

Proof. As $\pi(\Omega)$ is bounded when restricted to any V_l^k, this follows immediately from a spectral argument. □

The following lemma follows from a straightforward computation, using the relations (6.4).

Lemma 6.8. *Inside $M(A_x)$, we have the following identities:*

$$\begin{aligned}
\alpha^*\alpha &= \frac{|q|^{2x+\lambda+\rho+1} + |q|^{-2x-\lambda-\rho-1} + \mathrm{sgn}(q)^{\lambda-\rho+1}\Omega}{(|q|^{x+\lambda+1} + |q|^{-x-\lambda-1})(|q|^{x+\rho} + |q|^{-x-\rho})} \\
\alpha\alpha^* &= \frac{|q|^{2x+\lambda+\rho-1} + |q|^{-2x-\lambda-\rho+1} + \mathrm{sgn}(q)^{\lambda-\rho-1}\Omega}{(|q|^{x+\lambda} + |q|^{-x-\lambda})(|q|^{x+\rho-1} + |q|^{-x-\rho+1})} \\
\beta^*\beta &= |q|\frac{|q|^{\lambda-\rho+1} + |q|^{-\lambda+\rho-1} - \mathrm{sgn}(q)^{\lambda-\rho+1}\Omega}{(|q|^{x+\lambda+1} + |q|^{-x-\lambda-1})(|q|^{x+\rho-1} + |q|^{-x-\rho+1})} \\
\beta\beta^* &= |q|\frac{|q|^{\lambda-\rho-1} + |q|^{-\lambda+\rho+1} - \mathrm{sgn}(q)^{\lambda-\rho-1}\Omega}{(|q|^{x+\lambda} + |q|^{-x-\lambda})(|q|^{x+\rho} + |q|^{-x-\rho})}.
\end{aligned}$$

Note that the right hand sides are well-defined because of centrality of Ω.

Corollary 6.9. *If π is an irreducible *-representation of A_x on a Hilbert space $\mathcal{H}_\pi$, then $\mathcal{H}_n^m$ is at most one-dimensional for each $m, n \in \mathbb{Z}$. Moreover, either all $\mathcal{H}_l^k$ with $k-l$ odd are zero, or all $\mathcal{H}_l^k$ with $k-l$ even are zero.*

Proof. Using Corollary 6.7, the first assertion follows immediately from (6.3), the grading relations (6.5) and Lemma 6.8. The second assertion follows immediately from the grading relations (6.5). $\square$

Definition 6.10. Let $(\mathcal{H}_\pi, \pi)$ be an irreducible *-representation of A_x. We call π even (resp. odd) if all $\mathcal{H}_l^k$ with $k-l$ odd (resp. even) are zero.

With the above preliminaries, we can now classify the irreducible *-representations of A_x. We first extend the terminology of Definition 6.1.

Definition 6.11. Fix $c \in \mathbb{R}$ and $x \in [0, \frac{1}{2}]$. For $\epsilon, \nu \in \{-, +\}$, a couple $(k, l) \in \mathbb{Z}^2$ is called $(x, c)_{\epsilon,\nu}$*-adapted* if the following inequality holds:

$$(|q|^{(x+k)+\epsilon\nu(x+l)-\epsilon} + |q|^{-(x+k)-\epsilon\nu(x+l)+\epsilon}) + \operatorname{sgn}(q)^{k-l+1}\epsilon\nu c \geqslant 0. \tag{6.7}$$

A couple (k, l) is called *strictly* $(x, c)_{\epsilon,\nu}$-adapted if this is a strict equality. We call (k, l) (x, c)*-adapted* if it is $(x, c)_{\epsilon,\nu}$-adapted for all $\epsilon, \nu \in \{+, -\}$.

Definition 6.12. Fix $c \in \mathbb{R}$ and $x \in [0, \frac{1}{2}]$. We call a subset $T \subseteq \mathbb{Z}^2$ an (x, c)*-set* if the following conditions are satisfied:

- T is not empty.
- T consists of (x, c)-adapted points.
- If $(k, l) \in T$ is strictly $(x, c)_{\epsilon,\nu}$-adapted, then $(k - \epsilon, l - \nu)$ is in T.

We say that T *irreducible* if it is not the disjoint union of two (x, c)-sets.

Writing $\mathbb{Z}^2_{\text{even}} = \{(k, l) \mid k - l \text{ even}\}$ and $\mathbb{Z}^2_{\text{odd}} = \mathbb{Z}^2 \backslash \mathbb{Z}^2_{\text{even}}$, we call a (x, c)-set even or odd according to whether it lies in $\mathbb{Z}^2_{\text{even}}$ or $\mathbb{Z}^2_{\text{odd}}$.

Definition 6.13. Fix $x \in [0, \frac{1}{2}]$. For π an irreducible representation of A_x, a couple $(k, l) \in \mathbb{Z}^2$ is called π-compatible if $\mathcal{H}_l^k \neq 0$.

For $c \in \mathbb{R}$, a subset $T \subseteq \mathbb{Z}^2$ is called (x, c)*-compatible* if there exists an irreducible representation π of A_x with $\pi(\Omega) = c$ and $T = \{(k, l) \in \mathbb{Z}^2 \mid \mathcal{H}_l^k \neq \{0\}\}$. In this case, we say that π is T*-adapted*.

Proposition 6.14. *A set $T \subseteq \mathbb{Z}^2$ is an irreducible (x, c)-set if and only if it is a (x, c)-compatible set. Moreover, for any (x, c)-compatible set T there is only one irreducible *-representation π of A_x, up to unitary equivalence, which is T-compatible.*

Proof. Assume first that T is (x,c)-compatible, and let π be a T-compatible irreducible *-representation of A_x. If $(k,l) \in T$, then it follows from Lemma 6.8 that (k,l) is (x,c)-adapted. Moreover, if $(k,l) \in T$ is strictly $(x,c)_{\epsilon,\nu}$-adapted, then we have that $\|u_{\epsilon,\nu}\xi\| \neq 0$ for a non-zero $\xi \in \mathcal{H}_l^k$, hence also $\mathcal{H}_{l-\nu}^{k-\epsilon} \neq \{0\}$. It follows that T is a (x,c)-set. Now if $T = T_1 \cup T_2$ a disjoint union of (x,c)-sets, it would follow that π restricts to the direct sum of all $\mathcal{H}_l^k$ with $(k,l) \in T_1$, contradicting irreducibility. It follows that T is an irreducible (x,c)-set.

Conversely, let T be an irreducible (x,c)-set. Put $\mathcal{H}_\pi = l^2(T)$ with

$$\begin{aligned}
\pi(\alpha)e_{k,l} &= \left(\frac{|q|^{2x+k+l+1} + |q|^{-2x-k-l-1} + \operatorname{sgn}(q)^{k-l+1}c}{(|q|^{x+k+1} + |q|^{-x-k-1})(|q|^{x+l} + |q|^{-x-l})}\right)^{1/2} e_{k+1,l+1}, \\
\pi(\beta)e_{k,l} &= \operatorname{sgn}(q)^k \left(|q|\frac{|q|^{k-l+1} + |q|^{-k+l-1} - \operatorname{sgn}(q)^{k-l+1}c}{(|q|^{x+k+1} + |q|^{-x-k-1})(|q|^{x+l-1} + |q|^{-x-l+1})}\right)^{1/2} e_{k+1,l-1},
\end{aligned}$$

where the right hand side is considered as the zero vector when the scalar factor on the right is zero. Note that the roots on the right hand side are well-defined precisely because T is a (x,c)-set.

By direct computation, using the defining commutation relations (6.3) and (6.4), we see that π defines a *-representation of A_x with $\pi(\Omega) = c$. Moreover, π is irreducible since otherwise, by Corollary 6.9, T would split as a disjoint union of (x,c)-compatible sets. Hence T is an (x,c)-compatible set.

Now the formulas for $\pi(\alpha)$ and $\pi(\beta)$ are uniquely determined up to a unimodular gauge factor. As any non-zero $\mathcal{H}_l^k$ is cyclic for π, it follows that these gauge factors are determined by their value at one component. We then easily conclude that π is in fact the unique T-compatible *-representation, up to unitary equivalence. $\square$

What remains is to classify irreducible (x,c)-sets for each $c \in \mathbb{R}$.

Lemma 6.15. *A set $T \subseteq \mathbb{Z}^2_{\text{even}}$ is an irreducible (x,c)-set if and only if there exists an even irreducible $(x,-\operatorname{sgn}(q)c)$-set $Z_+ \subseteq \mathbb{Z}$ and an even irreducible $(0,\operatorname{sgn}(q)c)$-set $Z_- \subseteq \mathbb{Z}$ such that $(k,l) \in T$ if and only if $k+l \in Z_+$ and $k-l \in Z_-$.*

A set $T \subseteq \mathbb{Z}^2_{\text{odd}}$ is an irreducible (x,c)-set if and only if there exists an odd irreducible $(x,-c)$-set $Z_+ \subseteq \mathbb{Z}$ and an odd irreducible $(0,c)$-set $Z_- \subseteq \mathbb{Z}$ such that $(k,l) \in T$ if and only if $k+l \in Z_+$ and $k-l \in Z_-$.

Proof. It is immediate that $(k,l) \in \mathbb{Z}^2_{\text{even}}$ is (strictly) $(x,c)_{\epsilon,\nu}$-adapted if and only if $k+\epsilon\nu l$ is $((1+\epsilon\nu)x, -\operatorname{sgn}(q)\epsilon\nu c)_\epsilon$-adapted. Similarly, $(k,l) \in \mathbb{Z}^2_{\text{even}}$ is (strictly) $(x,c)_{\epsilon,\nu}$-adapted if and only if $k+\epsilon\nu l$ is $((1+\epsilon\nu)x, -\epsilon\nu c)_\epsilon$-adapted. The conclusion of the lemma then follows immediately. $\square$

Combining Proposition 6.14 with Proposition 6.3 and Lemma 6.15, we thus obtain a concrete description of the spectrum of A_x. The following pictures illustrate the form of the spectrum of A_x for the case $q > 0$.

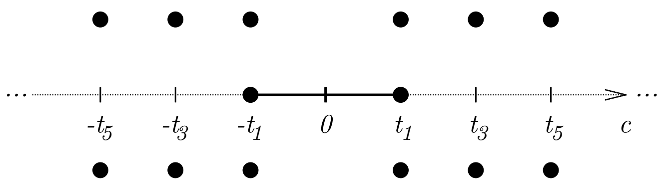

Figure 1: Case $x = 0$, even

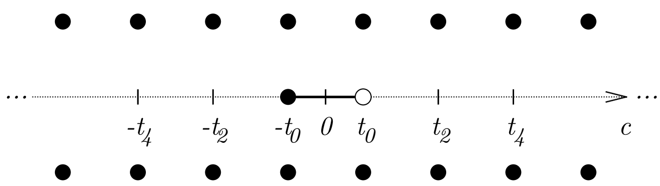

Figure 2: Case $x = 0$, odd

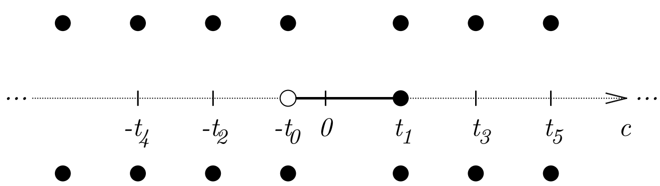

Figure 3: Case $x = \frac{1}{2}$, even

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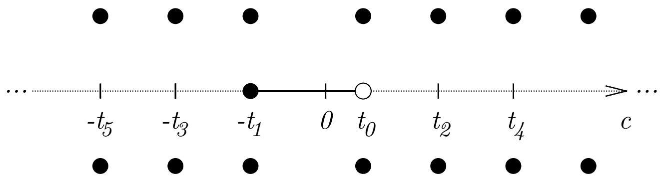

Figure 4: Case $x = \frac{1}{2}$, odd

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