Compact quantum groups of face type, compact quantum homogeneous spaces and the dynamical quantum $SU(2)$ group

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Abstract

Compact quantum groups of face type, as introduced by Hayashi, form a class of quantum groupoids with a classical, finite set of objects. We generalize Hayashi's definition to allow for an infinite set of objects. We then show how any quantum homogeneous space of an ordinary compact quantum group leads to a compact quantum group of face type. In particular, when this construction is applied to the non-standard Podleś spheres, we obtain compact quantum groups of face type which are operator algebraic versions of the dynamical quantum $SU(2)$-group as studied by Etingof-Varchenko and Koelink-Rosengren.

1   Compact quantum groups of face type

We generalize Hayashi's definition of a compact quantum group of face type [Hayashi] to the case where the commutative base algebra is no longer finite-dimensional. We will present two approaches, based on *partial bialgebras* and *weak multiplier bialgebras* [Böhm]. The first approach is piecewise and concrete, but requires some bookkeeping. The second approach is global but more abstract. As we will see from the general theory and the concrete examples, both approaches have their intrinsic value.

Let I be a set. We consider $I^2 = I \times I$ as the pair groupoid with $\cdot$ denoting composition. That is, an element $K = (k,l) \in I^2$ has source k and target l, and if $K = (k,l)$ and $L = (l,m)$ we write $K \cdot L = (k,m)$. For general $K, L \in I^2$, we write the property 'K and L are composable' as $K \to L$.

Definition 1.1. A *partial algebra* $\mathscr{A} = (\mathscr{A}, M)$ (over $\mathbb{C}$) is a small $\mathbb{C}$-linear category, that is, a set I (the object set) together with

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- for each $K = (k, l) \in I^2$ a vector space $A(K) = A(k, l) = {}_kA_l$ (possibly the zero vector space),

- for each K, L with $K \to L$ a multiplication map

$$M(K, L) : A(K) \otimes A(L) \to A(K \cdot L), \qquad a \otimes b \mapsto ab$$

  and

- elements $\mathbf{1}(k) = \mathbf{1}_k \in A(k, k)$ (the units),

such that the obvious associativity and unit conditions are satisfied.

By I-*partial algebra* will be meant a partial algebra with object set I.

Remark 1.2. For the moment we will allow the local units $\mathbf{1}_k$ to be zero.

Let $\mathscr{A}$ be an I-partial algebra. We define $A(K \cdot L)$ to be $\{0\}$ when $\neg (K \to L)$, and we then let $M(K, L)$ be the zero map.

Definition 1.3. The *total algebra* A of an I-partial algebra $\mathscr{A}$ is the vector space

$$A = \oplus_{K \in I^2} A(K)$$

endowed with the unique multiplication whose restriction to $A(K) \otimes A(L)$ concides with $M(K, L)$.

Clearly A is an associative algebra. If I is infinite it will not possess a unit, but it is a *locally unital algebra* [Quil] as there exist mutually orthogonal idempotents $\mathbf{1}_k$ with $A = \sum_{k,l}^{\oplus} \mathbf{1}_k A \mathbf{1}_l$. An element $a \in A$ can be interpreted as a function assigning to each element $(k, l) \in I^2$ an element $a_{kl} \in A(k, l)$, namely the (k, l)-th component of a. This identifies A with finite support I-indexed matrices whose (k, l)-th entry lies in $A(k, l)$, equipped with the natural matrix multiplication.

Remark 1.4. When $\mathscr{A}$ is an I-partial algebra with total algebra A, then $A \otimes A$ can be naturally identified with the total algebra of an $I \times I$-partial algebra $\mathscr{A} \otimes \mathscr{A}$, where

$$(A \otimes A)((k, k'), (l, l')) = A(k, l) \otimes A(k', l')$$

with the obvious tensor product multiplications and the $\mathbf{1}_{k,k'} = \mathbf{1}_k \otimes \mathbf{1}_{k'}$ as units.

The notion of partial algebra dualizes. For this we consider again I^2 as the pair groupoid, but now with elements considered as column vectors, and with $*$ denoting the (vertical) composition. So $K = \binom{k}{l}$ has source k and target l, and if $K = \binom{k}{l}$ and $L = \binom{l}{m}$ then $K * L = \binom{k}{n}$. We write $K \downarrow L$ if K and L are composable.

Definition 1.5. A *partial coalgebra* $\mathscr{A} = (\mathscr{A}, \Delta)$ (over $\mathbb{C}$) consists of a set I (the object set) together with

- for each $K = \binom{k}{l} \in I^2$ a vector space $A(K) = A\binom{k}{l} = A_l^k$,

- for each K, L with $K \downarrow L$ a comultiplication map

$$\Delta\left(\begin{smallmatrix} K \\ L \end{smallmatrix}\right) : A(K * L) \to A(K) \otimes A(L), \qquad a \mapsto a_{(1)K} \otimes a_{(2)L},$$

  and

- counit maps $\varepsilon_k : A\left(\begin{smallmatrix} k \\ k \end{smallmatrix}\right) \to \mathbb{C}$,

satisfying the obvious coassociativity and counitality conditions.

By I-*partial coalgebra* will be meant a partial coalgebra with object set I.

Notation 1.6. As the index of ε_k is determined by the element to which it is applied, there is no harm in dropping the index k and simply writing ε.

Similarly, if $K = \left(\begin{smallmatrix} k \\ l \end{smallmatrix}\right)$ and $L = \left(\begin{smallmatrix} l \\ m \end{smallmatrix}\right)$, we abbreviate $\Delta_l = \Delta\left(\begin{smallmatrix} K \\ L \end{smallmatrix}\right)$, as the other indices are determined by the element to which Δ_l is applied.

We also make again the convention that $A(K * L) = \{0\}$ and $\Delta\left(\begin{smallmatrix} K \\ L \end{smallmatrix}\right)$ the zero map when $\neg (K \downarrow L)$. Similarly ε is seen as the zero functional on $A(K)$ when $K = \left(\begin{smallmatrix} k \\ l \end{smallmatrix}\right)$ with $k \neq l$.

We can now superpose the notions of partial algebra and partial coalgebra. To formulate the condition that the coalgebra maps form a 'morphism of partial algebras', we will need to impose a finiteness condition which is automatically satisfied when the cardinality of I is finite.

Let I be a set, and let $M_2(I)$ be the set of 4-tuples of elements of I arranged as 2×2-matrices. We can endow $M_2(I)$ with two compositions, namely $\cdot$ (viewing $M_2(I)$ as a row vector of column vectors) and $*$ (viewing $M_2(I)$ as a column vector of row vectors). When $K \in M_2(I)$, we will write $K = \left(K_l, K_r\right) = \left(\begin{smallmatrix} K_u \\ K_d \end{smallmatrix}\right) = \left(\begin{smallmatrix} K_{lu} & K_{ru} \\ K_{ld} & K_{rd} \end{smallmatrix}\right)$. One can view $M_2(I)$ as a double groupoid, and in fact a *vacant* double groupoid in the sense of [Andruskiewitsch-Natale].

In the following, a vector (r, s) will sometimes be written simply as r, s or rs in an index. We also follow Notation 1.6, but the reader should be aware that the index of Δ will now be a 1×2 vector in I^2 as we will work with partial coalgebras over I^2.

Definition 1.7. A *partial bialgebra* $\mathscr{A} = (\mathscr{A}, M, \Delta)$ consists of a set I and a collection of vector spaces $A(K)$ for $K \in M_2(I)$ such that

- the $A\left(K_l, K_r\right)$ form an I^2-partial algebra,

- the $A\left(\begin{smallmatrix} K_u \\ K_d \end{smallmatrix}\right)$ form an I^2-partial coalgebra,

and for which the following compatibility relations are satisfied.

(a) (Comultiplication of Units) For all $k, l, l', m \in I$, one has

$$\Delta_{l,l'}(\mathbf{1}\left(\begin{smallmatrix} k \\ m \end{smallmatrix}\right)) = \delta_{l,l'} \mathbf{1}\left(\begin{smallmatrix} k \\ l \end{smallmatrix}\right) \otimes \mathbf{1}\left(\begin{smallmatrix} l \\ m \end{smallmatrix}\right).$$

3

(b) (Counit of Multiplication) For all $K, L \in M_2(I)$ with $K \to L$ and all $a \in A(K)$ and $b \in A(L)$,

$$\varepsilon(ab) = \varepsilon(a)\varepsilon(b).$$

(c) (Non-degeneracy) For all $k \in I$, $\varepsilon(\mathbf{1}\begin{pmatrix} k \\ k \end{pmatrix}) = 1$.

(d) (Finiteness) For each $K \in M_2(I)$ and each $a \in A(K)$, the element $\Delta_{rs}(a)$ is zero except for a finite number of indices r (resp. s) when s (resp. r) is fixed.

(e) (Comultiplication is multiplicative) For all $a \in A(K)$ and $b \in A(L)$ with $K \to L$,

$$\Delta_{rs}(ab) = \sum_t \Delta_{rt}(a)\Delta_{ts}(b).$$

Remark 1.8. By assumption (d), the sum on the right hand side in condition (e) is well-defined.

We want to relate the notion of partial bialgebra to the recently introduced notion of weak multiplier bialgebra [Böhm]. We first recall some notions concerning non-unital algebras [VDae].

Definition 1.9. Let A be an algebra over $\mathbb{C}$, not necessarily with unit. We call A *non-degenerate* if A is faithfully represented on itself by left and right multiplication. It is called *idempotent* if $A^2 = A$.

Definition 1.10. Let A be an algebra. A *multiplier m* for A consists of a couple of maps

$$L_m : A \to A, \quad a \mapsto am$$
$$R_m : A \to A, \quad a \mapsto ma$$

such that $(am)b = a(mb)$ for all $a, b \in A$.

The set of all multipliers forms an algebra under composition of L_m and anti-composition of R_m. It is called the *multiplier algebra* of A, and is denoted $M(A)$.

One has a natural homomorphism $A \to M(A)$. When A is non-degenerate, this homomorphism is injective, and we can then identify A as a subalgebra of the (unital) algebra $M(A)$. We then also have inclusions $A \otimes A \subseteq M(A) \otimes M(A) \subseteq M(A \otimes A)$.

Example 1.11. 1. Let A be the total algebra of a partial algebra $\mathscr{A}$. As A has local units, it is non-degenerate and idempotent. Then one can identify $M(A)$ with

$$M(A) = \left(\prod_l \oplus_k A(k,l) \right) \bigcap \left(\prod_k \oplus_l A(k,l) \right) \subseteq \prod_{k,l} A(k,l),$$

i.e. with the space of functions

$$m : I^2 \to A, \quad m_{kl} \in A(k,l)$$

4

which have finite support when either one of the variables has been fixed. The multiplication is given by the formula

$$(mn)_{kl} = \sum_p m_{kp} n_{pl}.$$

2. Let m_i be any collection of multipliers of A, and assume that for each $a \in A$, $m_i a = 0$ for almost all i, and similarly $a m_i = 0$ for almost all i. Then one can define a multiplier $\sum_i m_i$ in the obvious way by termwise multiplication. One says that the sum $\sum_i m_i$ converges in the *strict* topology.

Using the notion introduced in Example 1.11.2, we can introduce the following notation.

Notation 1.12. If $\mathscr{A}$ is an I-partial bialgebra, we write

$$\lambda_k = \sum_l \mathbf{1}\binom{k}{l}, \qquad \rho_l = \sum_k \mathbf{1}\binom{k}{l} \qquad \in M(A).$$

To show that the total algebra of a partial bialgebra becomes a weak multiplier bialgebra, we will need some easy lemmas.

Lemma 1.13. *Let $\mathscr{A}$ be an I-partial bialgebra. Then for each $a \in A$, there exists a unique multiplier $\Delta(a) \in M(A \otimes A)$ such that*

$$
\begin{aligned}
\Delta_{rs}(a) &= (1 \otimes \lambda_r)\Delta(a)(1 \otimes \lambda_s) & (1.1) \\
&= (\rho_r \otimes 1)\Delta(a)(\rho_s \otimes 1) & (1.2)
\end{aligned}
$$

for all $r, s \in I$, all $K \in M_2(I)$ and all $a \in A(K)$.

The resulting map

$$\Delta : A \to M(A \otimes A), \quad a \mapsto \Delta(a)$$

is a homomorphism.

Proof. For $a \in A$ homogeneous, we can define $\Delta(a) = \sum_{rs} \Delta_{rs}(a) \in M(A \otimes A)$, where the sum converges in the strict topology of $A \otimes A$ because of the property (d) of Definition 1.7. This expression clearly satisfies the identities stated in the lemma, and these in turn uniquely define it (as they determine the value of $\Delta(a)$ multiplied to the left and right with the local units of $\mathscr{A} \otimes \mathscr{A}$). We can then extend Δ by linearity to A. Since, for a, b homogeneous, $\Delta_{rt}(a)\Delta_{t's}(b) = 0$ unless $t = t'$, it follows from (e) of that definition that Δ is a homomorphism. $\square$

We will refer to $\Delta : A \to M(A \otimes A)$ as the *total comultiplication* of $\mathscr{A}$. We will then also use the suggestive Sweedler notation for this map,

$$\Delta(a) = a_{(1)} \otimes a_{(2)}.$$

Lemma 1.14. *The element $E = \sum_{k,l,m} \mathbf{1}\begin{pmatrix} k \\ l \end{pmatrix} \otimes \mathbf{1}\begin{pmatrix} l \\ m \end{pmatrix}$ is a well-defined idempotent in $A \otimes A$, and satisfies*

$$\Delta(A)(A \otimes A) = E(A \otimes A), \quad (A \otimes A)\Delta(A) = (A \otimes A)E.$$

Proof. Clearly the sum defining E is strictly convergent, and makes E into an idempotent. It is moreover immediate that $E\Delta(a) = \Delta(a) = \Delta(a)E$ for all $a \in A$. Since

$$E(\mathbf{1}\begin{pmatrix} k \\ l \end{pmatrix} \otimes \mathbf{1}\begin{pmatrix} m \\ n \end{pmatrix}) = \Delta(\mathbf{1}\begin{pmatrix} k \\ n \end{pmatrix})(\mathbf{1}\begin{pmatrix} k \\ l \end{pmatrix} \otimes \mathbf{1}\begin{pmatrix} m \\ n \end{pmatrix})$$

by the property (a) of Definition 1.7, and analogously for multiplication with E on the right, the lemma is proven. $\qquad\square$

By [Van Daele-Wang], there is a unique homomorphism $\Delta : M(A) \to M(A \otimes A)$ extending Δ and such that $\Delta(1) = E$. Alternatively, if $m \in M(A)$, we can directly define $\Delta(m)$ as the strict limit of the series $\sum_{k,l,r,s} \Delta_{rs}(m_{kl})$. Similarly the maps $\mathrm{id} \otimes \Delta$ and $\Delta \otimes \mathrm{id}$ extend to maps from $M(A \otimes A)$ to $M(A \otimes A \otimes A)$. The following proposition then gathers the properties of Δ, ε and $\Delta(1)$ which guarantee that (A, Δ) forms a weak multiplier bialgebra in the sense of [Bohm].

Proposition 1.15. *Let $\mathscr{A}$ be a partial bialgebra with total algebra A, total comultiplication Δ and counit ε. Then the following properties are satisfied.*

- *Coassociativity:* $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$.

- *Counitality:* $(\varepsilon \otimes \mathrm{id})(\Delta(a)(1 \otimes b)) = ab = (\mathrm{id} \otimes \varepsilon)((a \otimes 1)\Delta(b))$ *for all $a, b \in A$.*

- *Weak unitality:*

  $$(\Delta(1) \otimes 1)(1 \otimes \Delta(1)) = (\Delta \otimes \mathrm{id})\Delta(1) = (\mathrm{id} \otimes \Delta)\Delta(1) = (1 \otimes \Delta(1))(\Delta(1) \otimes 1).$$

- *Weak counitality: For all $a, b, c \in A$, one has*

  $$(\varepsilon \otimes \mathrm{id})(\Delta(a)(b \otimes c)) = (\varepsilon \otimes \mathrm{id})((1 \otimes a)\Delta(1)(b \otimes c))$$

  and

  $$(\varepsilon \otimes \mathrm{id})((a \otimes b)\Delta(c)) = (\varepsilon \otimes \mathrm{id})((a \otimes b)\Delta(1)(1 \otimes c)).$$

- *Strong multiplier property: For all $a, b \in A$, one has*

  $$\Delta(A)(1 \otimes A) \cup (A \otimes 1)\Delta(A) \subseteq A \otimes A.$$

Proof. Most of these properties follow immediately from the definition of a partial bialgebra. For demonstrational purposes, let us check the first weak counitality condition. Let us choose $a \in A(K)$, $b \in A(L)$ and $c \in A(M)$. Then

$$\Delta(a)(b \otimes c) = \delta_{K_{ru}, L_{lu}} \delta_{M_{lu}, L_{ld}} \sum_r \Delta_{r, L_{ld}}(a)(b \otimes c).$$

Applying $(\varepsilon \otimes \mathrm{id})$ to both sides, we obtain by Proposition (b) of Definition 1.7 and counitality of Δ that

$$(\varepsilon \otimes \mathrm{id})(\Delta(a)(b \otimes c)) = \delta_{K_{ru},L_{lu},L_{ld},M_{lu}}\varepsilon(b)ac.$$

On the other hand,

$$
\begin{aligned}
(1 \otimes a)\Delta(1)(b \otimes c) &= \sum_{r,s,t} \mathbf{1}\begin{pmatrix} r \\ s \end{pmatrix} b \otimes a\mathbf{1}\begin{pmatrix} s \\ t \end{pmatrix} c \\
&= \delta_{L_{ld},K_{ru},M_{lu}} b \otimes ac.
\end{aligned}
$$

Applying $(\varepsilon \otimes \mathrm{id})$, we find

$$
\begin{aligned}
(\varepsilon \otimes \mathrm{id})((1 \otimes a)\Delta(1)(b \otimes c)) &= \delta_{L_{ld},K_{ru},M_{lu}}\delta_{L_{lu},L_{ld}}\delta_{L_{ru},L_{rd}}\varepsilon(b)ac \\
&= \delta_{L_{ld},L_{lu},K_{ru},M_{lu}}\varepsilon(b)ac,
\end{aligned}
$$

which agrees with the expression above. $\qquad\square$

Remark 1.16. Since also the expressions $\Delta(a)(b \otimes 1)$ and $(1 \otimes a)\Delta(b)$ are in $A \otimes A$ for all $a, b \in A$, we see that (A, Δ) is in fact a *regular* weak multiplier bialgebra [Bohm].

Remark 1.17. We have the following formulas for the maps $\overline{\Pi}^L, \overline{\Pi}^R, \Pi^L$ and Π^R onto the base algebra (see [Bohm]): if $a \in A\begin{pmatrix} k & l \\ m & n \end{pmatrix}$, then

$$\overline{\Pi}^L(a) = \varepsilon(a)\lambda_n, \quad \overline{\Pi}^R(a) = \varepsilon(a)\rho_k, \quad \Pi^L(a) = \varepsilon(a)\lambda_m, \quad \Pi^R(a) = \varepsilon(a)\rho_l.$$

In particular, we see that the base algebra consists of the algebra $\mathrm{Fun}_{\mathrm{f}}(I)$ of finite support functions on I.

We now formulate the notion of partial Hopf algebra, whose total form will correspond to a weak (multiplier) Hopf algebra. Let us denote $\circ$ for the inverse of $\cdot$, and $\bullet$ for the inverse of $*$, so

$$\begin{pmatrix} k & l \\ m & n \end{pmatrix}^{\circ} = \begin{pmatrix} l & k \\ n & m \end{pmatrix}, \quad \begin{pmatrix} k & l \\ m & n \end{pmatrix}^{\bullet} = \begin{pmatrix} m & n \\ k & l \end{pmatrix}, \quad \begin{pmatrix} k & l \\ m & n \end{pmatrix}^{\circ\bullet} = \begin{pmatrix} n & m \\ l & k \end{pmatrix}.$$

The notation $\circ$ (resp. $\bullet$) will also be used for row vectors (resp. column vectors).

Definition 1.18. An *antipode* for an I-partial multiplier bialgebra $\mathscr{A}$ consists of maps $S : A(K) \to A(K^{\circ\bullet})$ such that the following property holds: for all $M, P \in M_2(I)$ and all $a \in A(M)$,

$$\sum_{\substack{K*L=M \\ K\cdot L^{\circ\bullet}=P}} a_{(1)K}S(a_{(2)L}) = \delta_{P_l,P_r}\varepsilon(a)\mathbf{1}(P_l),$$

$$\sum_{\substack{K*L=M \\ K^{\circ\bullet}\cdot L=P}} S(a_{(1)K})a_{(2)L} = \delta_{P_l,P_r}\varepsilon(a)\mathbf{1}(P_r).$$

A partial multiplier bialgebra $\mathscr{A}$ is called a *partial Hopf algebra* if it admits an antipode.

Remark 1.19. Note that condition (d) of Definition 1.7 again guarantees that the above sums are in fact finite.

If S is an antipode for a partial bialgebra, we can extends S to a linear map

$$S : A \to A$$

on the total algebra A.

Lemma 1.20. *Let S be an antipode for a partial multiplier bialgebra. Then for all $a, b, c \in A$, one has*

$$
\begin{aligned}
(a_{(1)}c \otimes ba_{(2)}S(a_{(3)})) &= (1 \otimes b)\Delta(1)(ac \otimes 1) \\
(S(a_{(1)})a_{(2)}c \otimes ba_{(3)}) &= (1 \otimes ba)\Delta(1)(c \otimes 1).
\end{aligned}
$$

Proof. Take $a \in A\left(\begin{smallmatrix} k & l \\ m & n \end{smallmatrix}\right)$. Then in the strict topology, we obtain from the first identity in Definition 1.18 that

$$
\begin{aligned}
a_{(1)} \otimes a_{(2)}S(a_{(3)}) &= \sum_{r,s,t,u} a_{(1)\left(\begin{smallmatrix} k & l \\ r & s \end{smallmatrix}\right)} \otimes a_{(2)\left(\begin{smallmatrix} r & s \\ t & u \end{smallmatrix}\right)}S(a_{(3)\left(\begin{smallmatrix} t & u \\ m & n \end{smallmatrix}\right)}) \\
&= \sum_{r,t} a_{(1)\left(\begin{smallmatrix} k & l \\ r & n \end{smallmatrix}\right)} \otimes \left(\sum_u a_{(2)\left(\begin{smallmatrix} r & n \\ t & u \end{smallmatrix}\right)}S(a_{(3)\left(\begin{smallmatrix} t & u \\ m & n \end{smallmatrix}\right)}) \right) \\
&= \sum_{r,t} a_{(1)\left(\begin{smallmatrix} k & l \\ r & n \end{smallmatrix}\right)} \otimes \left(\delta_{r,m}\varepsilon(a_{(2)\left(\begin{smallmatrix} m & n \\ m & n \end{smallmatrix}\right)})\mathbf{1}\binom{r}{t} \right) \\
&= \sum_t a_{\left(\begin{smallmatrix} k & l \\ m & n \end{smallmatrix}\right)} \otimes \mathbf{1}\binom{m}{t} \\
&= \Delta(1)(a \otimes 1).
\end{aligned}
$$

The second identity is proven similarly. $\qquad\square$

Proposition 1.21. *A partial bialgebra $\mathscr{A}$ is a partial Hopf algebra if and only if the total weak multiplier bialgebra (A, Δ) is a weak multiplier Hopf algebra (in the sense of [Böhm]).*

Proof. We verify that the total antipode $S : A \to A$ satsisfies the conditions as in [Böhm]. In fact, the two first identities are precisely our identities in Lemma 1.20. The last identity says that formally one should have

$$\sum_k S(\rho_k a)\lambda_k = S(a), \quad \forall a \in A,$$

but this follows immediately from the fact the way S sends homogeneous components to homogeneous components. $\qquad\square$

From ..., we obtain the following corollary.

Corollary 1.22. *The map $S : A \to A$ is an anti-homomorphism.*

We now turn towards the structures which will allow us to build operator algebraic quantum groupoids out of our partial Hopf algebras.

Definition 1.23. A partial *-algebra $\mathscr{A}$ is a partial algebra whose total algebra A is equipped with an antilinear, anti-multiplicative involution

$$* : A \to A, \quad a \mapsto a^*$$

such that the $\mathbf{1}_k$ are self-adjoint for all k in the object set.

One can of course give an alternative definition directly in terms of the partial algebra structure by requiring that we are given anti-linear maps $A(k, l) \to A(l, k)$ satisfying the obvious anti-multiplicativity and involution properties.

Definition 1.24. A partial *-bialgebra $\mathscr{A}$ is a partial bialgebra whose underlying partial algebra has been endowed with a *-involution for which

$$\Delta_{rs}(a)^* = \Delta_{sr}(a^*), \qquad \forall K \in M_2(I), a \in A(K).$$

A partial Hopf *-algebra is a partial bialgebra which is at the same time a partial *-bialgebra and a partial Hopf algebra.

Remark 1.25. 1. By uniqueness of the counit, cf. Bohm, it follows automatically that ε satisfies $\varepsilon(a^*) = \overline{\varepsilon(a)}$ for all a.

2. By uniqueness of the antipode, cf. Bohm, it follows automatically that $S(S(a)^*)^* = a$ for all a. In particular, S is invertible.

Definition 1.26. Let $\mathscr{A}$ be a partial bi-algebra. An *invariant functional* φ for $\mathscr{A}$ consists of a functional $\varphi : A \to \mathbb{C}$ with support on the $A\begin{pmatrix} k & k \\ m & m \end{pmatrix}$ such that for all $a \in A\begin{pmatrix} k & l \\ m & n \end{pmatrix}$ and homogeneous b, one has

$$(\mathrm{id} \otimes \varphi)(b \otimes 1)\Delta(a)) = \varphi(a)b\lambda_k,$$

and

$$(\varphi \otimes \mathrm{id})(\Delta(a)(1 \otimes b) = \varphi(a)\rho_n b.$$

We call φ *normalized* if $\varphi(\mathbf{1}\begin{pmatrix} k \\ l \end{pmatrix}) = 1$ whenever $\mathbf{1}\begin{pmatrix} k \\ l \end{pmatrix} \neq 0$.

We are finally ready to give our main definition.

Definition 1.27. A *compact quantum group of face type* is a partial Hopf *-algebra with a *positive* invariant functional φ, that is $\varphi(a^*a) \geqslant 0$ for all $a \in A$.

2 Structure theory for compact quantum groups of face type

Definition 2.1. A *(right) comodule* for a generalized face bialgebra (A, Δ) over I is an I^2-graded vector space $V = \sum_{K \in I^2}^{\oplus} V(K)$ together with linear maps

$$\delta \begin{pmatrix} K \\ L \end{pmatrix} : V(L) \to V(K) \otimes A \begin{pmatrix} K \\ L \end{pmatrix}$$

satisfying

$$(\mathrm{id} \otimes \Delta \begin{pmatrix} K \\ L \end{pmatrix}) \delta(K * L) = (\delta(K) \otimes \mathrm{id}) \delta(L)$$

and

$$(\mathrm{id} \otimes \varepsilon_K) \delta \begin{pmatrix} K \\ K \end{pmatrix} = \mathrm{id}_{V(K)} .$$

We say (V, δ) is *of finite type* (or simply *finite*) if the support of $M \mapsto V(M)$ is finite in one variable if the other variable is held fixed.

For example, each $\oplus_L A \begin{pmatrix} K \\ L \end{pmatrix}$ is a right comodule under Δ. We will write

$$\delta(M)(v) = v_{(0)M_u} \otimes v_{(1)M},$$

to be interpreted as zero if $v \notin V(M_u)$.

We have a tensor product $\boxtimes$ on finite comodules by putting

$$(V \boxtimes W)(M) = \bigoplus_{K \cdot L = M} (V(K) \otimes V(L))$$

with comodule structure

$$\delta(M)(v \otimes w) = \sum_{K \cdot L = M} v_{(0)K_u} \otimes w_{(0)L_u} \otimes v_{(1)K} w_{(1)L}.$$

We then obtain a tenor category with unit the vector space $\mathbf{1} = \mathrm{Fun}_{\mathrm{f}}(I)$ of finite support functions on I with grading $\mathbf{1}(k, l) = \delta_{k,l} \mathbb{C} \delta_k$ (with δ_k the Dirac function at k) and comodule structure

$$\delta(K, K)(\delta_{K_d}) = \delta_{K_u} \otimes e(K).$$

When (A, Δ) is a Hopf face algebra, this tensor category admits left duals. Indeed, define $(V^*)(M) = V(M^\circ)^*$ with coaction

$$(\delta(M)(\omega))(v) = \omega(v_{(0)M_d^\circ}) S(v_{(1)M^\circ \bullet}), \qquad \omega \in V(M_d^\circ)^*, v \in V(M_u^\circ).$$

Then the natural dualities between the $V(K)$ and $V^*(K^\circ)$ lead to comodule maps

$$\oplus_K V^*(K^\circ) \otimes V(K) \to \mathbf{1}_K, \qquad \mathbf{1}_K \to \oplus_K V(K) \otimes V^*(K^\circ).$$

Let us now turn to the notion of invariant functional.

Definition 2.2. Let I be a set. An *invariant functional* for a Hopf face algebra (A, Δ) over I is a functional $\varphi : A \to \mathbb{C}$ such that for all K, L and $a \in A(K * L)$ we have

$$(\mathrm{id} \otimes \varphi)\Delta \begin{pmatrix} K \\ L \end{pmatrix}(a) = \delta_{K_l, K_r} \varphi(a) e(K_l), \qquad (\varphi \otimes \mathrm{id})\Delta \begin{pmatrix} K \\ L \end{pmatrix}(a) = \delta_{L_l, L_r} \varphi(a) e(L_r).$$

We say that φ is normalized if $\varphi(\lambda_k \rho_l) = 1$ for all $k, l \in I$ with $\lambda_k \rho_l \neq 0$.

Lemma 2.3. *An invariant normalized functional φ is faithful, i.e. $\varphi(ab) = 0$ for all b implies $b = 0$, and $\varphi(ab) = 0$ for all a implies $b = 0$.*

Proof. We follow ad verbatim the proof of Proposition 3.4 in [VDae, Algebraic framework]: if $\varphi(ba) = 0$ for all a, we arrive at the conclusion that for all $d \in A$ and all functionals ω on A, the element $p = (\omega \otimes \mathrm{id})((d \otimes 1)\widetilde{\Delta}(a))$ satisfies $(\mathrm{id} \otimes \varphi)((1 \otimes c)\widetilde{\Delta}(p)) = 0$. Continuing as in that proof, we obtain from the antipode trick that $\sum_n \varphi(cS(q)\rho_n)\varepsilon(p\lambda_n) = 0$. Choosing now for c and q local units of the form $\lambda_k \rho_l$, the normalization condition on φ gives that $\varepsilon(p\lambda_n) = 0$ for all n, hence $\varepsilon(p) = 0$. This implies $\omega(da) = 0$. As ω and d were arbitrary, it follows that $a = 0$.

The other case follows similarly, considering the opposite algebra. $\qquad \square$

Our next aim is to prove that a normalized invariant functional is modular, that is, there exists an automorphism $\sigma : A \to A$ such that for all $a, b \in A$, we have

$$\varphi(ba) = \varphi(a\sigma(b)).$$

Lemma 2.4. *Let φ be a normalized invariant functional. For all $a \in A$ and $k, m \in I$, we have*

$$\varphi(a\lambda_k) = \varphi(\lambda_k a), \qquad \varphi(a\rho_m) = \varphi(\rho_m a).$$

Proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

Definition 2.5. Define

$$V : \sum_n^{\oplus} A_n \otimes {}_n A \to \sum_r^{\oplus} {}_r A \otimes {}^r A$$

by the formula

$$a \otimes b \to \widetilde{\Delta}(a)(1 \otimes b).$$

Lemma 2.6. *The map V is an isomorphism.*

Proof. As $\widetilde{\Delta}(A)(A \otimes A) \subseteq E(A \otimes A)$ with $E = \sum_p \rho_p \otimes \lambda_p$, it is clear that V has the proper range. Define

$$\widetilde{V} : \sum_r^{\oplus} {}_r A \otimes {}^r A \to \sum_n^{\oplus} A_n \otimes {}_n A$$

11

by means of the formula

$$a \otimes b \mapsto a_{(1)} \otimes S(a_{(2)})b = a_{(1)} \otimes S(S^{-1}(b)a_{(2)}).$$

By the defining property of S, we find that for all $a, b, c \in A$, we have

$$(c \otimes 1) \cdot (\widetilde{V}V)(a \otimes b) = \sum_p ca_{(1)}\varepsilon(a_{(2)}\lambda_p) \otimes \rho_p b.$$

By the previous lemma, this equals $\sum_p ca_{(1)}\varepsilon(a_{(2)}\rho_p) \otimes \rho_p b$. But as $a \otimes b = \sum_p a\rho_p \otimes \rho_p b$ by assumption, we obtain that

$$(c \otimes 1) \cdot (\widetilde{V}V)(a \otimes b) = ca_{(1)}\varepsilon(a_{(2)}) \otimes b = ca \otimes b,$$

proving that $\widetilde{V}V(a \otimes b) = a \otimes b$.

The identity $V\widetilde{V} = \mathrm{id}$ is proven similarly. $\square$

Corollary 2.7. *Define*

$$W : \sum_n^{\oplus} {}_nA \otimes {}^nA \to \sum_r^{\oplus} {}^rA \otimes A^r$$

by the formula

$$a \otimes b \to S^{-1}(b_{(1)})a \otimes d_{(2)} = S^{-1}(S(a)b_{(1)}) \otimes b_{(2)}.$$

Then W is invertible, its inverse being given as

$$W^{-1}(a \otimes b) = \widetilde{\Delta}(b)(a \otimes 1).$$

Proof. Apply the previous Lemma to $(A, \Delta^{\mathrm{op}})$. $\square$

2.1 Generalized compact Hopf face algebras

A non-degenerate algebra A is called a *-algebra if it comes equipped with an anti-linear involutive anti-homomorphism $A \to A, a \mapsto a^*$. In this case, $M(A)$ becomes a *-algebra in a natural way. For example, we always consider $\mathrm{Fun}_{\mathrm{f}}(I)$ as a *-algebra by the ordinary complex conjugation of functions, $f^*(k) = \overline{f(k)}$.

Definition 2.8. A couple (A, Δ) consisting of a generalized Hopf face *-algebra with an invertible antipode invariant normalized functional φ is called a *generalized compact face algebra*.

One proves that a generalized Hopf face *-algebra has $S(S(x)^*)^* = x$ for all x, so S is automatically invertible. It then follows by symmetry that also the maps

$$(W^{k,t}_{m,n,u,v})^* : \oplus_l {}^k_m A^l_n \otimes {}^l_u A^t_v \to \oplus_r {}^k_m A^t_r \otimes {}^n_u A^r_v$$

defined by the formula

$$a \otimes b \to \Delta(b)(a \otimes 1)$$

are unitaries, with inverse map $a \otimes b \mapsto S^{-1}(b_{(1)})a \otimes b_{(2)}$.

Lemma 2.9. *Let (A, Δ) be a generalized compact face algebra. Then each $V_{m,v}^{k,l,s,t}$ is a unitary, and similarly for the $W_{m,n,u,v}^{k,t}$.*

Proof. It is immediately checked that $V_{m,v}^{k,l,s,t}$ is isometric. $\qquad\qquad\square$

Let us write $\mathscr{L}^2(A, \varphi)$ for the completion of A with respect to the inner product $\langle a, b \rangle = \varphi(a^*b)$. The canonical inclusion of A into $\mathscr{L}^2(A)$ will be denoted Λ.

Lemma 2.10. *Assume (A, Δ) is a generalized compact face algebra. The representation of A by left multiplication on itself extends to a representation by bounded operators on the completion $\mathscr{L}^2(A, \varphi)$.*

Proof. Denote $\omega_{\xi,\eta}(x) = \langle \xi, x\eta \rangle$ for ξ, η vectors and x a bounded operator. Then a straightforward computation shows that

$$(\omega_{\Lambda(a),\Lambda(b)} \otimes \mathrm{id})(V) = \varphi(a^*b_{(1)})b_{(2)}$$

as a left multiplication operator. As $(A \otimes 1)\Delta(A) = (A \otimes A)\Delta(1)$ by Lemma 5.1 (applied to the opposite algebra), it follows by normalization of φ that each element of A can be represented in the form $(\omega_{\Lambda(a),\Lambda(b)} \otimes \mathrm{id})(V)$, and hence extends to a bounded operator on $\mathscr{L}^2(A, \varphi)$. $\qquad\qquad\square$

In the following, we will abbreviate $\mathscr{L}^2(A)$ by L^2A.

Let (A, Δ) be a generalized compact face algebra. Denote the von Neumann algebraic completion of $A \subseteq B(L^2A)$ by M. Denote $L^2A \underset{I}{\otimes} L^2A = E(L^2A \otimes L^2A)$, where $E = \sum_p \rho_p \otimes \lambda_p$ is extended to a bounded operator (in fact, a self-adjoint projection). Finally, denote $M \underset{I^2}{\otimes} M = E(M \otimes M)E$. Then $M \underset{I^2}{\otimes} M$ is the von Neumann algebraic completion of $A \underset{I^2}{\otimes} A$.

Extend now the $V_{m,v}^{k,l,s,t}$ to unitaries

$$V : \oplus_p L^2(A_p) \otimes L^2({}_pA) \to \oplus_p L^2({}_pA) \otimes L^2({}^pA) = E(L^2A \otimes L^2A).$$

Then we can construct a map

$$\Delta : M \to M \underset{I^2}{\otimes} M, \quad x \to V(x \otimes 1)V^*.$$

By direct computation, we see that Δ extends the comultiplication map on A. It is then immediate to check that Δ is in fact coassociative (where one may as well consider Δ as a non-unital map from M to $M \otimes M$).

We aim to show that (M, Δ) can be fitted into the theory of measured quantum groupoids.

3 Representation theory

3.1 Corepresentations of generalized compact Hopf face algebras

Let (A, Δ) be a generalized compact Hopf face algebra over an index set I.

Lemma 3.1. *Let $\mathcal{H} = \bigoplus_{k,l} {}^k\mathcal{H}^l$ be a row- and column-finite I^2-graded Hilbert space and let $X = ({}^k_m X^l_n)_{k,l,m,n}$ be a family of elements ${}^k_m X^l_n \in {}^k_m A^l_n \otimes \mathcal{B}({}^m\mathcal{H}^n, {}^k\mathcal{H}^l)$ satisfying*

$$(\tilde{\Delta} \otimes \mathrm{id})({}^k_m X^l_n) = \sum_{p,q} \left({}^k_p X^l_q \right)_{13} ({}^p_m X^q_n)_{23} . \tag{3.1}$$

Then the following conditions are equivalent:

1. *$(\epsilon \otimes \mathrm{id})({}^k_m X^l_n) = \delta_{k,m}\delta_{l,n} \, \mathrm{id} \, {}_k\mathcal{H}^l$;*

2. *there exist elements ${}^k_m Z^l_n \in {}^n_l A^m_k \otimes \mathcal{B}({}^m\mathcal{H}^n, {}^k\mathcal{H}^l)$ such that*

$$\sum_k {}^m_k Z^{n'}_l \, {}^k_m X^l_n = \delta_{n,n'}\lambda_l \rho_n \otimes \mathrm{id} \, {}_m\mathcal{H}^n, \qquad \sum_n {}^k_m X^l_n \, {}^m_{k'} Z^n_l = \delta_{k,k'}\lambda_k \rho_m \otimes \mathrm{id} \, {}_k\mathcal{H}^l .$$

If these conditions hold, then the family $X^{-1} := Z$ is unique and given by ${}^k_m(X^{-1})^l_n = (S \otimes \mathrm{id})({}^k_m X^l_n)$. In particular, it satisfies

$$(\tilde{\Delta} \otimes \mathrm{id})({}^k_m(X^{-1})^l_n) = \sum_{p,q} \left({}^k_m(X^{-1})^l_n \right)_{23} ({}^m_p(X^{-1})^n_q)_{13} .$$

Note that the sum in (3.1) makes sense in the multiplier algebra, and that the sums in (2) are finite because $\mathcal{H}$ is row- and column-finite. If it exists, we denote the family Z in condition (2) by X^{-1}.

Proof. Assume that (1) holds and let ${}^k_m Z^l_n = (S \otimes \mathrm{id})({}^k_m X^l_n)$. Then the antipode axiom implies

$$
\begin{aligned}
\sum_k {}^m_k Z^{n'}_l \, {}^k_m X^l_n &= \sum_k (S \otimes \mathrm{id})({}^m_k X^{n'}_l) \, {}^k_m X^l_n \\
&= \sum_k (m_A \otimes \mathrm{id})(S \otimes \mathrm{id} \otimes \mathrm{id})(({}^m_k X^{n'}_l)_{13}({}^k_m X^l_n)_{23}) \\
&= (m_A \otimes \mathrm{id})(S \otimes \mathrm{id} \otimes \mathrm{id})((\tilde{\Delta} \otimes \mathrm{id})({}^m_m X^{n'}_n)(1 \otimes \lambda_l \otimes 1)) \\
&= \sum_p \rho_p \lambda_l \otimes (\epsilon(-\lambda_p) \otimes \mathrm{id})({}^m_m X^{n'}_n) \\
&= \rho_n \lambda_l \otimes \delta_{n',n} \, \mathrm{id} \, {}_m\mathcal{H}^n,
\end{aligned}
$$

which is the first equation in (2). The second one follows similarly.

14

Conversely, assume that (2) holds. Then uniqueness of the family Z is easily verified. Let $\begin{smallmatrix}k\\m\end{smallmatrix}c_n^l := (\epsilon \otimes \mathrm{id})(\begin{smallmatrix}k\\m\end{smallmatrix}X_n^l)$. If $(k,l) \neq (m,n)$, then $\begin{smallmatrix}k\\m\end{smallmatrix}c_n^l = 0$ because $\epsilon(\begin{smallmatrix}k\\m\end{smallmatrix}A_n^l) = 0$. Relation (3.1) and the counit property imply

$$
\begin{aligned}
\begin{smallmatrix}k\\m\end{smallmatrix}X_n^l &= (\lambda_k \otimes 1)\,\begin{smallmatrix}k\\m\end{smallmatrix}X_n^l(\lambda_l \otimes 1)\\
&= (\epsilon \otimes \mathrm{id} \otimes \mathrm{id})(((1 \otimes \lambda_k \otimes 1)(\hat{\Delta} \otimes \mathrm{id})(\begin{smallmatrix}k\\m\end{smallmatrix}X_n^l)(1 \otimes \lambda_l \otimes 1))\\
&= (\epsilon \otimes \mathrm{id} \otimes \mathrm{id})\left(\left(\begin{smallmatrix}k\\k\end{smallmatrix}X_l^l\right)_{13}\left(\begin{smallmatrix}k\\m\end{smallmatrix}X_n^l\right)_{23}\right)\\
&= (1 \otimes \begin{smallmatrix}k\\k\end{smallmatrix}c_l^l)\,\begin{smallmatrix}k\\m\end{smallmatrix}X_n^l.
\end{aligned}
$$

We multiply on the right by $\begin{smallmatrix}m\\k\end{smallmatrix}Z_l^n$, sum over n, use condition (2) and find $\begin{smallmatrix}k\\k\end{smallmatrix}c_l^l = \mathrm{id}\,{}_k\mathcal{H}^l$. $\qquad\square$

A *locally finite corepresentation of* (A, Δ) is a pair $(\mathcal{H}, X)$ as above satisfying conditions (1) and (2). A *morphism* T between two such corepresentations $(\mathcal{H}, X)$ and $(\mathcal{K}, Y)$ is a family of morphisms $\begin{smallmatrix}k\end{smallmatrix}T^l \in \mathcal{B}(\,{}^k\mathcal{H}^l,\ {}^k\mathcal{K}^l)$ satisfying $(1 \otimes {}^kT^l)\,\begin{smallmatrix}k\\m\end{smallmatrix}X_n^l = \begin{smallmatrix}k\\m\end{smallmatrix}Y_n^l(1 \otimes {}^mT^n)$. Using condition (2) in Lemma 3.1, one finds that the last equation holds if and only if

$$
\sum_m \begin{smallmatrix}k\\m\end{smallmatrix}(Y^{-1})_n^{l'}(1 \otimes {}^mT^n)\,\begin{smallmatrix}k\\k\end{smallmatrix}X_l^n = \delta_{l,l'}\lambda_n\rho_l \otimes {}^kT^l, \tag{3.2}
$$

$$
\sum_n \begin{smallmatrix}k'\\m\end{smallmatrix}Y_n^l(1 \otimes {}^mT^n)\,\begin{smallmatrix}m\\k\end{smallmatrix}(X^{-1})_l^n = \delta_{k,k'}\lambda_m\rho_k \otimes {}^kT^l. \tag{3.3}
$$

For example, if the first equation holds, then

$$
\begin{smallmatrix}m\\k\end{smallmatrix}Y_l^n(1 \otimes {}^kT^l) = \sum_m \begin{smallmatrix}m\\k\end{smallmatrix}Y_l^n\,\begin{smallmatrix}k\\m\end{smallmatrix}(Y^{-1})_n^l(1 \otimes {}^mT^n)\,\begin{smallmatrix}m\\k\end{smallmatrix}X_l^n = (\lambda_k\rho_m \otimes {}^mT^n)\,\begin{smallmatrix}m\\k\end{smallmatrix}X_l^n.
$$

Conversely, if T is a morphism, then

$$
\sum_m \begin{smallmatrix}k\\m\end{smallmatrix}(Y^{-1})_n^{l'}(1 \otimes {}^mT^n)\,\begin{smallmatrix}k\\k\end{smallmatrix}X_l^n = \sum_m \begin{smallmatrix}k\\m\end{smallmatrix}(Y^{-1})_n^{l'}\,\begin{smallmatrix}m\\k\end{smallmatrix}Y_l^n(1 \otimes {}^kT^l) = \delta_{l',l}\lambda_n\rho_l \otimes {}^kT^l.
$$

We denote by $\mathrm{Corep_f}(A, \Delta)$ the category of locally finite corepresentations of (A, Δ) with morphisms as above.

Let $(\mathcal{H}, X)$ be a locally finite corepresentation. We call a family of subspaces ${}^k\mathcal{K}^l \subseteq {}^k\mathcal{H}^l$ *invariant* if $\begin{smallmatrix}k\\m\end{smallmatrix}X_n^l(1 \otimes {}^mP^n) = (1 \otimes {}^kP^l)\,\begin{smallmatrix}k\\m\end{smallmatrix}X_n^l$, where ${}^kP^l$ denotes the projection ${}^k\mathcal{H}^l \to {}^k\mathcal{K}^l$.

The following analogue of Schur's Lemma holds. Let T be a morphism of locally finite corepresentations $(\mathcal{H}, X)$ and $(\mathcal{K}, Y)$. Then $\bigoplus_{k,l} \ker {}^kT^l$ and $\bigoplus_{k,l} \mathrm{img}\ {}^kT^l$ are invariant subspaces of $\mathcal{H}$ and $\mathcal{K}$, respectively. In particular, if $(\mathcal{H}, X)$ and $(\mathcal{K}, Y)$ are irreducible, then T is either 0 or an isomorphism.

Unitarity of corepresentations We call a locally finite corepresentation $(\mathcal{H}, X)$ *unitary* if ${}_m^k(X^{-1})_n^l = ({}_k^m X_l^n)^*$. In this paragraph, we assume that (A, Δ) has a faithful positive functional, and show that then every irreducible locally finite-dimensional corepresentation is equivalent to a unitary one, by embedding it into a restriction of the regular corepresentation.

Example 3.2. Assume that (A, Δ) has a positive invariant functional ϕ that is faithful. Let ${}^k\mathcal{H}^l \subseteq \bigoplus_{m,n} {}_k^m A_l^n$ be a row- and column-finite family of finite-dimensional subspaces satisfying

$$\tilde{\Delta}^{\mathrm{co}}({}^m\mathcal{H}^n) \subseteq \sum_{p,q} {}_m^p A_n^q \otimes {}^p\mathcal{H}^q.$$

Equip each ${}^k\mathcal{H}^l$ with the scalar product $\langle a | b \rangle := \phi(a^* b)$ and take the Hilbert space direct sum $\mathcal{H} := \bigoplus_{k,l} {}^k\mathcal{H}^l$. Define ${}_m^k V_n^l \in {}_m^k A_n^l \otimes \mathcal{B}({}^m\mathcal{H}^n, {}^k\mathcal{H}^l)$ by the equation

$${}_m^k V_n^l |a\rangle_2 = \tilde{\Delta}^{\mathrm{co}}(a),$$

where ${}_m^k V_n^l |a\rangle_2$ denotes the application of the second leg of ${}_m^k V_n^l$ to $a \in {}^m\mathcal{H}^n$. Then $(\mathcal{H}, V)$ is a unitary corepresentation. We call it the *locally finite restriction of the regular corepresentation* determined by the family $({}^k\mathcal{H}^l)_{k,l}$.

Lemma 3.3. *Assume that (A, Δ) has a faithful and positive invariant functional. Let $(\mathcal{H}, X)$ be a locally finite-dimensional corepresentation and let $\xi \in {}^k\mathcal{H}^l$. Then the family of finite-dimensional subspaces*

$${}^m\mathcal{K}^n = \{(\mathrm{id} \otimes \omega_{\xi,\eta})({}_m^k X_n^l) : \eta \in {}^m\mathcal{H}^n\} \subseteq {}_m^k A_n^l$$

defines a restriction of the regular corepresentation $(\mathcal{K}, V)$, and the family of maps

$${}^m T_{(\xi)}^n : \; {}^m\mathcal{H}^n \to {}^m\mathcal{K}^n, \; \eta \mapsto (\mathrm{id} \otimes \omega_{\xi,\eta})({}_m^k X_n^l),$$

is a morphism from $(\mathcal{H}, X)$ to $(\mathcal{K}, V)$.

Note that the family $({}^m\mathcal{K}^n)_{m,n}$ is row- and column-finite because $({}^m\mathcal{H}^n)_{m,n}$ is.

Proof. Both assertions follow from the fact that for all $\eta \in {}^p\mathcal{H}^q$,

$$(1 \otimes \rho_m) \cdot \tilde{\Delta}^{\mathrm{co}}\big((\mathrm{id} \otimes \omega_{\xi,\eta})({}_p^k X_q^l)\big) \cdot (1 \otimes \rho_n) = (\mathrm{id} \otimes \mathrm{id} \otimes \omega_{\xi,\eta})\big(({}_m^k X_n^l)_{23}({}_p^m X_q^n)_{13}\big)$$
$$= (1 \otimes {}^m T_{(\xi)}^n) {}_p^m X_q^n |\eta\rangle_2.$$

$\square$

Proposition 3.4. *Assume that (A, Δ) has a faithful and positive invariant functional. Then every irreducible locally finite corepresentation is equivalent to a unitary one.*

Proof. Let $(\mathcal{H}, X)$ be an irreducible locally finite corepresentation. Then for some k, l and $\xi \in {}^k\mathcal{H}^l$, the operator $T_{(\xi)}$ defined in Lemma 3.3 has to be non-zero and hence, by Schur's Lemma, injective. Thus, it forms an equivalence between $(\mathcal{H}, X)$ and a sub-corepresentation of a locally finite restriction of the regular corepresentation, which is unitary by Example 3.2. $\square$

Schur orthogonality relations   In this paragraph, we obtain the analogue of Schur's orthogonality relations for matrix coefficients of corepresentations.

The space of *matrix coefficients* $\mathcal{C}(X)$ of a locally finite corepresentation $(\mathcal{H}, X)$ is the sum of the subspaces

$$_m^k\mathcal{C}(X)_n^l = \left\{ (\mathrm{id} \otimes \omega_{\xi,\eta})(\,_m^k X_n^l) \mid \xi \in \,^k\mathcal{H}^l, \eta \in \,^m\mathcal{H}^n \right\} \subseteq \,_m^k A_n^l.$$

Proposition 3.5. *Assume that (A, Δ) has a faithful and positive invariant functional. Then A is the sum of the matrix coefficients of unitary irreducible locally finite corepresentations.*

Proof. Let $a \in \,_m^k A_n^l$. Then $(1 \otimes \rho_p)\tilde{\Delta}^{\mathrm{co}}(a)(1 \otimes \rho_q) \in \,_m^p A_n^q \otimes \,_p^k A_q^l$ and the subspace

$$^p\mathcal{H}^q := \{(\omega \otimes \mathrm{id})(\tilde{\Delta}^{\mathrm{co}}(a)) : \omega \in (\,_m^p A_n^q)'\} \subseteq \,_p^k A_q^l$$

has finite dimension. Since $(1 \otimes \rho_p)\tilde{\Delta}^{\mathrm{co}}(a)$ and $\tilde{\Delta}^{\mathrm{co}}(a)(1 \otimes \rho_q)$ lie in the tensor product $A \otimes A$, the family $(\,^p\mathcal{H}^q)_{p,q}$ is row- and column-finite. Using co-associativity, one checks that this family defines a locally finite restriction $(\mathcal{H}, V)$ of the regular corepresentation. Evidently, $a \in \mathcal{C}(V)$. Decomposing $(\mathcal{H}, V)$, we find that a is contained in the sum of matrix coefficients of unitary irreducible corepresentations. $\square$

The key to the orthogonality relations is the following averaging procedure.

Lemma 3.6. *Let ϕ be an invariant functional for (A, Δ), let $(\mathcal{H}, X)$ and $(\mathcal{K}, Y)$ be locally finite-dimensional corepresentations of (A, Δ) and let T be a family of operators $^kT^l \in \mathcal{B}(\,^k\mathcal{H}^l, \,^k\mathcal{K}^l)$ which are 0 for all but finitely many k, l. Then the families $\check{T}$ and $\hat{T}$ given by*

$$^k\check{T}^l := \sum_{m,n} (\phi \otimes \mathrm{id})(\,_m^k(Y^{-1})_n^l(1 \otimes \,^mT^n) \,_k^m X_l^n),$$

$$^k\hat{T}^l := \sum_{m,n} (\phi \otimes \mathrm{id})(\,_m^k Y_n^l(1 \otimes \,^mT^n) \,_k^m(X^{-1})_l^n)$$

are morphisms from $(\mathcal{H}, X)$ to $(\mathcal{K}, Y)$.

Proof. The assertion converning $\check{T}$ follows from the calculation

$$\sum_m \,_m^k(Y^{-1})_n^{l'}(1 \otimes \,^m\check{T}^n) \,_k^m X_l^n =$$

$$= \sum_{m,p,q} (\phi \otimes \mathrm{id} \otimes \mathrm{id})\left(\left(\,_m^k(Y^{-1})_n^{l'}\right)_{23} \left(\,_p^m(Y^{-1})_q^n\right)_{13} (1 \otimes 1 \otimes \,^pT^q) (\,_m^p X_n^q)_{13} (\,_k^m X_l^n)_{23} \right)$$

$$= \sum_{p,q} (\lambda_n \otimes 1) \cdot ((\phi \otimes \mathrm{id}) \circ \tilde{\Delta} \otimes \mathrm{id}) \left(\,_p^k(Y^{-1})_q^{l'}(1 \otimes \,^pT^q) \,_k^p X_l^q\right) \cdot (\lambda_n \otimes 1)$$

$$= \sum_{r,p,q} (\lambda_n \otimes 1) \cdot \left(\rho_r \otimes (\phi \otimes \mathrm{id}) \left(\,_p^k(Y^{-1})_q^{l'}(1 \otimes \,^pT^q) \,_k^p X_l^q(\rho_r \otimes 1)\right) \right) \cdot (\lambda_n \otimes 1)$$

$$= \delta_{l,l'} \lambda_n \rho_l \otimes \,^k\check{T}^l,$$

where we used the relation $\phi({}_{l'}A_l) = 0$ for $l' \neq l$ for the last equality. A similar calculation proves the assertion concerning $\hat{T}$. $\qquad\square$

The first part of the orthogonality relations concerns matrix coefficients of inequivalent irreducible corepresentations.

Proposition 3.7. *Let $(\mathcal{H}, X)$ and $(\mathcal{K}, Y)$ be inequivalent unitary irreducible locally finite-dimensional corepresentations and let ϕ be an invariant functional for (A, Δ). Then $\phi(S(b)a) = \phi(b^*a) = \phi(bS(a)) = \phi(ba^*) = 0$ for all $a \in \mathcal{C}(X), b \in \mathcal{C}(Y)$.*

Proof. Let $a = (\mathrm{id} \otimes \omega_{\xi,\xi'})({}_m^k X_n^l)$ and $b = (\mathrm{id} \otimes \omega_{\eta,\eta'})({}_r^p Y_s^q)$, where $\xi \in {}^k\mathcal{H}^l, \xi' \in {}^m\mathcal{H}^n$ and $\eta \in {}^p\mathcal{K}^q, \eta' \in {}^r\mathcal{K}^s$. We may assume $(p, q, r, s) = (m, n, k, l)$ because $\phi(S(a)b) = 0 = \phi(aS(b))$ otherwise. Lemma 3.6, applied to the family ${}^p T^q = \delta_{p,k}\delta_{q,l}|\eta'\rangle\langle\xi|$, yields morphisms $\check{T}, \hat{T}$ from $(\mathcal{H}, X)$ to $(\mathcal{K}, Y)$ which necessarily are 0. Inserting the definition of $\check{T}$, we find

$$\begin{aligned}
\phi(S(b)a) &= \phi\big((S \otimes \omega_{\eta,\eta'})({}_k^m Y_l^n) \cdot (\mathrm{id} \otimes \omega_{\xi,\xi'})({}_m^k X_n^l)\big) \\
&= (\phi \otimes \mathrm{id})\left(\langle\eta|_2 {}_k^m (Y^{-1})_l^n (1 \otimes |\eta'\rangle\langle\xi|) {}_m^k X_n^l |\xi'\rangle_2\right) = \langle\eta|_2 {}^m\check{T}^n|\xi'\rangle_2 = 0.
\end{aligned}$$

A similar calculation involving $\hat{T}$ shows that $\phi(aS(b)) = 0$. Using the relation $X^* = X^{-1} = (S \otimes \mathrm{id})(X)$ and $Y^* = (S \otimes \mathrm{id})(Y)$, we conclude $\phi(a^*b) = \phi(ab^*) = 0$. $\qquad\square$

The second part of the orthogonality relations concerns inner products as above but with $a, b \in \mathcal{C}(X)$ for some irreducible corepresentation X and involves the conjugate corepresentation, which is defined as follows.

Given a Hilbert spaces H, K, we denote by $\overline{H}, \overline{K}$ the conjugate Hilbert spaces, by $T \mapsto \overline{T}$ the canonical conjugate-linear isomorphism $\mathcal{B}(H, K) \to \mathcal{B}(\overline{H}, \overline{K})$, and by $T \mapsto T^\top := \overline{T}^*$ the linear anti-isomorphism $\mathcal{B}(H, K) \to \mathcal{B}(\overline{K}, \overline{H})$.

Lemma 3.8. *On the category $\mathrm{Corep}_{\mathrm{f}}(A, \Delta)$, there exist*

1. *a covariant functor $(\mathcal{H}, X) \mapsto (\overline{\mathcal{H}}, \overline{X})$ and $T \mapsto \overline{T}$, where*

$$ {}^k\overline{\mathcal{H}}^l = \overline{{}^l\mathcal{H}^k}, \qquad {}_m^k\overline{X}_n^l = ({}_n^l X_m^k)^{(* \otimes \overline{(\,\cdot\,)})} = (({}_n^l X_m^k)^*)^{\mathrm{id} \otimes \top}, \qquad {}^k\overline{T}^l = \overline{{}^l T^k}; $$

2. *a contravariant functor $(\mathcal{H}, X) \mapsto (\overline{\mathcal{H}}, X^{S \otimes \top})$ and $T \mapsto T^\top$, where*

$$ {}_m^k(X^{S \otimes \top})_n^l = (S \otimes (\,\cdot\,)^\top)({}_l^n X_k^m), \qquad\qquad {}^k(T^\top)^l = ({}^l T^k)^\top; $$

3. *a covariant functor $(\mathcal{H}, X) \mapsto (\mathcal{H}, X^{S^2 \otimes \mathrm{id}})$ and $T \mapsto T$, where ${}_m^k(X^{S^2 \otimes \mathrm{id}})_n^l = (S^2 \otimes \mathrm{id})({}_m^k X_n^l)$.*

If $(\mathcal{H}, X)$ is unitary, then $\overline{X} = X^{S \otimes \top}$.

18

Proof. Let $(\mathcal{H}, X)$ be a locally finite corepresentation. Using the fact that $\tilde{\Delta}$ and ϵ are $*$-homomorphisms, one easily verifies

$$(\tilde{\Delta} \otimes \mathrm{id})({}_{m}^{k}\overline{X}{}_{n}^{l}) = \sum_{p,q} ({}_{p}^{k}\overline{X}{}_{q}^{l})_{13} ({}_{m}^{p}\overline{X}{}_{n}^{q})_{23},$$

$$(\epsilon \otimes \mathrm{id})({}_{m}^{k}\overline{X}{}_{n}^{l}) = \overline{(\epsilon \otimes \mathrm{id})({}_{n}^{l}X{}_{m}^{k})} = \delta_{l,n}\delta_{k,m}\overline{\mathrm{id}\ {}_{l}\mathcal{H}^{k}} = \delta_{l,n}\delta_{k,m}\,\mathrm{id}\ {}_{k}\overline{\mathcal{H}}{}^{l}.$$

A similar calculation shows that $(\overline{\mathcal{H}}, X^{S\otimes\top})$ is a corepresentation. The assertions (1) and (2) follow immediately. The square of the functor in (2) yields the functor in (3). $\square$

We call $(\overline{\mathcal{H}}, \overline{X})$ the *conjugate* of $(\mathcal{H}, X)$.

Proposition 3.9. *Assume that (A, Δ) has a faithful, positive, normalized invariant functional ϕ and let $(\mathcal{H}, X)$ be a unitary irreducible locally finite corepresentation.*

1. *$\overline{\mathcal{H}}$ and the family ${}_{m}^{k}(\overline{X}{}^{-*}){}_{n}^{l} := \big({}_{k}^{m}(\overline{X}{}^{-1}){}_{l}^{n} \big)^{*}$ form a locally finite corepresentation and there exists an invertible, positive isomorphism $\overline{F_X}$ from $(\overline{\mathcal{H}}, \overline{X})$ to $(\overline{\mathcal{H}}, \overline{X}{}^{-*})$.*

2. *The family ${}^{k}F_X^{l} := \overline{{}^{l}\overline{F_X}{}^{k}}$ is an invertible, positive isomorphism from $(\mathcal{H}, X)$ to $(\mathcal{H}, X^{S^2\otimes\mathrm{id}})$.*

Proof. (1) By Proposition 3.4, $(\overline{\mathcal{H}}, \overline{X})$ is equivalent to a unitary corepresentation, that is, there exists a family of operators ${}^{k}T^{l} \in \mathcal{B}({}^{k}\overline{\mathcal{H}}{}^{l})$ such that the family

$$ {}_{m}^{k}Z{}_{n}^{l} := (1 \otimes\ {}^{k}T^{l})\, {}_{m}^{k}\overline{X}{}_{n}^{l}(1 \otimes\ {}^{m}T^{n})^{-1}$$

is a unitary corepresentation. The relation ${}_{m}^{k}(Z^{-1}){}_{n}^{l} = \big({}_{k}^{m}Z{}_{l}^{n}\big)^{*}$ then implies

$$ {}_{m}^{k}Z{}_{n}^{l} = (1 \otimes\ {}^{k}T^{l})^{-*} \big({}_{k}^{m}(\overline{X}{}^{-1}){}_{l}^{n}\big)^{*}(1 \otimes\ {}^{m}T^{n})^{*}$$

and hence the family ${}_{m}^{k}(\overline{X}{}^{-*}){}_{n}^{l} := \big({}_{k}^{m}(\overline{X}{}^{-1}){}_{l}^{n}\big)^{*}$ is an irreducible locally finite corepresentation and the family ${}^{k}\overline{F}{}_{X}^{l} := ({}^{k}T^{l})^{*}\ {}^{k}T^{l} \in \mathcal{B}({}^{k}\overline{\mathcal{H}}{}^{l})$ is an isomorphism from $(\overline{\mathcal{H}}, \overline{X})$ to $(\overline{\mathcal{H}}, \overline{X}{}^{-*})$.

(2) The morphism T from $(\overline{\mathcal{H}}, \overline{X})$ to $(\overline{\mathcal{H}}, Z)$ yields morphisms $\overline{T}$ from $(\mathcal{H}, X)$ to $(\mathcal{H}, \overline{Z})$ and $T^{\top}$ from $(\mathcal{H}, Z^{S\otimes\top})$ to $(\mathcal{H}, \overline{X}S \otimes \top)$. Since X and Z are unitary, $\overline{Z} = Z^{S\otimes\top}$ and $\overline{X}{}^{S\otimes\top} = X^{S^2\otimes\top}$. Thus $T^{\top}\overline{T} = \overline{T}^{*}\overline{T}$ is a morphism from $(\mathcal{H}, X)$ to $(\mathcal{H}, X^{S^2\otimes\mathrm{id}})$. $\square$

Theorem 3.10. *Assume that (A, Δ) has a faithful, positive, normalized invariant functional ϕ. Let $(\mathcal{H}, X)$ be a unitary irreducible locally finite corepresentation of (A, Δ) and let F_X be a non-zero morphism from $(\mathcal{H}, X)$ to $(\mathcal{H}, (S^2 \otimes \mathrm{id})(X))$.*

1. *The numbers $\alpha := \sum_{k} \mathrm{Tr}({}^{k}(F_X^{-1}){}^{l})$ and $\beta := \sum_{n} \mathrm{Tr}({}^{m}F_X^{n})$ do not depend on l or n.*

2. *For all k, l, m, n,*

$$(\phi \otimes \mathrm{id})(({}^k_m X^l_n)^* \, {}^k_m X^l_n) = \alpha^{-1} \mathrm{Tr}(\, {}^k(F_X^{-1})^l) \cdot \mathrm{id}_{{}^m\mathcal{H}^n},$$
$$(\phi \otimes \mathrm{id})({}^k_m X^l_n ({}^k_m X^l_n)^*) = \beta^{-1} \mathrm{Tr}(\, {}^m(F_X)^n) \cdot \mathrm{id}_{{}^k\mathcal{H}^l}.$$

3. *Denote by $\Sigma_{k,l,m,n}$ the flip ${}^k\mathcal{H}^l \otimes {}^m\mathcal{H}^n \to {}^m\mathcal{H}^n \otimes {}^k\mathcal{H}^l$. Then*

$$(\phi \otimes \mathrm{id} \otimes \mathrm{id})(({}^k_m X^l_n)^*_{12} ({}^k_m X^l_n)_{13}) = \alpha^{-1}(\mathrm{id}_{{}^m\mathcal{H}^n} \otimes {}^k(F_X^{-1})^l) \circ \Sigma_{k,l,m,n},$$
$$(\phi \otimes \mathrm{id} \otimes \mathrm{id})(({}^k_m X^l_n)_{13} ({}^k_m X^l_n)^*_{12}) = \beta^{-1}(\, {}^m F_X^n \otimes \mathrm{id}_{{}^k\mathcal{H}^l}) \circ \Sigma_{k,l,m,n}.$$

Proof. We prove the assertions and equations involving α in (1), (2) and (3) simultaneously; the assertions involving β follow similarly.

As above, we denote by $\Sigma_{p,q,r,s}$ the flip ${}^p\mathcal{H}^q \otimes {}^r\mathcal{H}^s \to {}^r\mathcal{H}^s \otimes {}^p\mathcal{H}^q$. Consider the operator

$$
\begin{aligned}
F_{m,n,k,l} &:= (\phi \otimes \mathrm{id} \otimes \mathrm{id})(({}^k_m X^l_n)^*_{12} ({}^k_m X^l_n)_{13}) \circ \Sigma_{m,n,k,l} \\
&= (\phi \otimes \mathrm{id} \otimes \mathrm{id}) \left(({}^m_k(X^{-1})^n_l)_{12} (\Sigma_{k,l,k,l})_{23} ({}^k_m X^l_n)_{12} \right).
\end{aligned}
$$

By Lemma 3.6, the family $(F_{m,n,k,l})_{m,n}$ is an endomorphism of $(\mathcal{H} \otimes {}^k\mathcal{H}^l, (X)_{12})$ and hence

$$F_{m,n,k,l} = \mathrm{id}_{{}^m\mathcal{H}^n} \otimes {}^k R^l \tag{3.4}$$

with some ${}^k R^l \in \mathcal{B}({}^k\mathcal{H}^l)$ not depending on m, n. On the other hand,

$$
\begin{aligned}
F_{m,n,k,l} &= (\phi \otimes \mathrm{id} \otimes \mathrm{id})((S \otimes \mathrm{id})({}^m_k X^n_l)_{12} ({}^k_m X^l_n)_{13}) \circ \Sigma_{m,n,k,l} \\
&= (\phi \circ S^{-1} \otimes \mathrm{id} \otimes \mathrm{id}) \left(((S \otimes \mathrm{id})({}^k_m X^l_n))_{13}((S^2 \otimes \mathrm{id})({}^m_k X^n_l))_{12} \right) \circ \Sigma_{m,n,k,l} \\
&= (\phi \circ S^{-1} \otimes \mathrm{id} \otimes \mathrm{id}) \left(((S \otimes \mathrm{id})({}^k_m X^l_n))_{13}(\Sigma_{m,n,m,n})_{23}((S^2 \otimes \mathrm{id})({}^m_k X^n_l))_{12} \right).
\end{aligned}
$$

Since $\phi \circ S^{-1}$ is an invariant functional for (A, Δ), we can again apply Lemma 3.6 and find that the family $(F_{m,n,k,l})_{k,l}$ is a morphism from $({}^m\mathcal{H}^n \otimes \mathcal{H}, (X^{S^2 \otimes \mathrm{id}})_{13})$ to $({}^m\mathcal{H}^n \otimes \mathcal{H}, (X)_{13})$. Therefore,

$$F_{m,n,k,l} = {}^m T^n \otimes (\, {}^k F_X^l)^{-1} \tag{3.5}$$

with some ${}^m T^n \in \mathcal{B}({}^m\mathcal{H}^n)$ not depending on k, l. Combining (3.4) and (3.5), we conclude that $F_{m,n,k,l} = \lambda(\mathrm{id}_{{}^m\mathcal{H}^n} \otimes ({}^k F_X^l)^{-1})$ for some $\lambda \in \mathbb{C}$. Choose a basis $(\zeta_i)_i$ for ${}^k\mathcal{H}^l$. Then

$$\lambda \cdot \mathrm{id}_{{}^m\mathcal{H}^n} \cdot \mathrm{Tr}((\, {}^k F_X^l)^{-1}) = \sum_i (\mathrm{id} \otimes \omega_{\zeta_i, \zeta_i})(F_{m,n,k,l}) = (\phi \otimes \mathrm{id})(({}^m_k X^n_l)^* \, {}^k_m X^l_n).$$

20

We sum over k, use the relations $\sum_k (\ _k^m X_l^n)^* \ _m^k X_n^l = \lambda_l \rho_n \otimes \mathrm{id}\ _{^m\mathcal{H}^n}$ and $\phi(\lambda_l \rho_n) = 1$, and find

$$\lambda \cdot \sum_k \mathrm{Tr}((\ ^k F_X^l)^{-1}) = 1.$$

Now all assertions in (1)–(3) concerning α follow. $\qquad\qquad\qquad\qquad\square$

Corollary 3.11. *Assume that (A, Δ) has a faithful, positive, normalized invariant functional ϕ. Let $(\mathcal{H}, X)$ be a unitary irreducible locally finite corepresentation of (A, Δ), let F_X be a non-zero morphism from $(\mathcal{H}, X)$ to $(\mathcal{H}, (S^2 \otimes \mathrm{id})(X))$, and let $a = (\mathrm{id} \otimes \omega_{\xi, \xi'})(\ _m^k X_n^l)$ and $b = (\mathrm{id} \otimes \omega_{\eta, \eta'})(\ _r^p X_s^q)$, where $\xi \in\ ^k\mathcal{H}^l, \xi' \in\ ^m\mathcal{H}^n$ and $\eta \in\ ^p\mathcal{H}^q, \eta' \in\ ^r\mathcal{H}^s$. Then*

$$\phi(b^* a) = \frac{\langle \eta' | \xi' \rangle \langle \xi | F_X^{-1} \eta \rangle}{\sum_k \mathrm{Tr}(\ ^k (F_X^{-1})^l)}, \qquad\qquad \phi(ab^*) = \frac{\langle \eta' | F_X \xi' \rangle \langle \xi | \eta \rangle}{\sum_n \mathrm{Tr}(\ ^m F_X^n)}.$$

Proof. By Theorem 3.10,

$$\phi(b^* a) = (\phi \otimes \omega_{\eta', \eta} \otimes \omega_{\xi, \xi'})((\ _m^k X_n^l)_{12}^*\ _m^k X_n^l)$$
$$= \frac{1}{\sum_k \mathrm{Tr}(\ ^k (F_X^{-1})^l)} (\omega_{\eta', \eta} \otimes \omega_{\xi, \xi'})((\mathrm{id}\ _{^m\mathcal{H}^n} \otimes\ ^k (F_X^{-1})^l) \circ \Sigma_{k,l,m,n}).$$

The formula for $\phi(ab^*)$ follows similarly or by considering the opposite of (A, Δ). $\quad\square$

Corollary 3.12. *Assume that (A, Δ) has a faithful, positive, normalized invariant functional ϕ. Let $(\mathcal{H}_\alpha, X_\alpha)_\alpha$ be a representative family of all irreducible locally finite corepresentations of (A, Δ). Then the map*

$$\bigoplus_\alpha \bigoplus_{k,l,m,n} (\overline{\ ^k\mathcal{H}_\alpha^l} \otimes\ ^m\mathcal{H}_\alpha^n) \to A$$

that sends $\overline{\xi} \otimes \eta \in \overline{\ ^k\mathcal{H}_\alpha^l} \otimes\ ^m\mathcal{H}_\alpha^n$ to $(\mathrm{id} \otimes \omega_{\xi_\alpha, \eta_\alpha})(\ _m^k (X_\alpha)_n^l)$, is a linear isomorphism.

Given $\omega, \omega' \in A'$ and $a \in A$, we define convolution products

$$\omega * a := (\mathrm{id} \otimes \omega)(\tilde{\Delta}(a)), \quad a * \omega' := (\omega' \otimes \mathrm{id})(\Delta(a)), \quad \omega * a * \omega' := (\omega * a) * \omega' = \omega * (a * \omega').$$

We shall say that an entire function f has *exponential growth on the right half-plane* if there exist C, d such that $|f(x + iy)| \leqslant Ce^{dx}$ for all $x, y \in \mathbb{R}$ with $x > 0$.

Theorem 3.13. *Assume that (A, Δ) has a faithful, positive, normalized invariant functional ϕ. There exists a unique family of characters $f_z \colon A \to \mathbb{C}$ such that*

1. *for each $a \in A$, the function $z \mapsto f_z(a)$ is entire and of exponential growth on the right half-plane;*

2. *$f_0 = \epsilon$ and $(f_z \otimes f_{z'}) \circ \tilde{\Delta} = f_{z+z'}$ for all $z, z' \in \mathbb{C}$.*

3. *$\phi(ab) = \phi(b(f_1 * a * f_1))$ for all $a, b \in A$.*

21

This family furthermore satisfies

    4. $S^2(a) = f_{-1} * a * f_1$ for all $a \in A$;

    5. $f_z(\lambda_l \rho_n) = \delta_{l,n}$, $f_z \circ S = f_{-z}$, and $f_z \circ * = * \circ f_{-\bar{z}}$.

Proof. We first prove uniqueness. Assume that $(f_z)_z$ is a family of functionals satisfying (1)–(3). Since ϕ is faithful, the map $\sigma \colon a \mapsto f_1 * a * f_1$ is uniquely determined, and one easily sees that is is a homomorphism. Using (2), we find that $\epsilon \circ \sigma = f_2$ and f_{2n} are uniquely determined and characters for each n. Using (1) and the fact that every entire function of exponential growth on the right half-plane is uniquely determined by its values at $\mathbb{N} \subseteq \mathbb{C}$ [], we can conclude that each f_z is uniquely determined and a character.

Let us now prove existence. By Corollary 3.12, we can define for each $z \in \mathbb{C}$ a functional $f_z \colon A \to \mathbb{C}$ such that for every unitary irreducible locally finite corepresentation $(\mathcal{H}, X)$,

$$f_z((\mathrm{id} \otimes \omega_{\xi,\eta})({}^{\,k}_{m}X^l_n)) = \delta_{k,m}\delta_{l,n} \cdot \omega_{\xi,\eta}(({}^{\,k}F^l_X)^z) \quad \text{for all } \xi \in {}^{\,k}\mathcal{H}^l, \eta \in {}^{\,m}\mathcal{H}^n,$$

or, equivalently,

$$(f_z \otimes \mathrm{id})({}^{\,k}_{m}X^l_n) = \delta_{k,m}\delta_{l,n} \cdot ({}^{\,k}F^l_X)^z,$$

where F_X is a non-zero morphism from $(\mathcal{H}, X)$ to $(\mathcal{H}, (S^2 \otimes \mathrm{id})(X))$ such that

$$\alpha_X := \sum_k \mathrm{Tr}({}^{\,k}(F_X^{-1})^l) = \sum_n \mathrm{Tr}({}^{\,m}F_X^n)$$

for all l, n (see Theorem 3.10). By construction, (1) holds. We show that $(f_z)_z$ satisfies the assertions (2)–(5). The proof of uniqueness shows that each f_z is a character. Throughout the following arguments, let $(\mathcal{H}, X)$ be a unitary irreducible corepresentation $(\mathcal{H}, X)$ and let F_X be as above.

(2) This follows from the relations

$$(f_0 \otimes \mathrm{id})({}^{\,k}_{m}X^l_n) = \delta_{k,m}\delta_{l,n}\,\mathrm{id}\,{}_{k\mathcal{H}^l} = (\epsilon \otimes \mathrm{id})({}^{\,k}_{k}X^l_l)$$

and

$$\begin{aligned}
((f_z \otimes f_{z'}) \circ \tilde{\Delta} \otimes \mathrm{id})({}^{\,k}_{m}X^l_n) &= \delta_{k,m}\delta_{l,n}(f_z \otimes f_{z'} \otimes \mathrm{id})(({}^{\,k}_{k}X^l_l)_{13}({}^{\,k}_{k}X^l_l)_{23}) \\
&= \delta_{k,m}\delta_{l,n}({}^{\,k}F^l_X)^z \cdot ({}^{\,k}F^l_X)^{z'} \\
&= (f_{z+z'} \otimes \mathrm{id})({}^{\,k}_{m}X^l_n).
\end{aligned}$$

To conclude assertion (2), apply slice maps of the form $\mathrm{id} \otimes \omega_{\xi,\xi'}$.

(3) Write $\tilde{\Delta}^{(2)} = (\tilde{\Delta} \otimes \mathrm{id}) \circ \tilde{\Delta} = (\mathrm{id} \otimes \tilde{\Delta}) \circ \tilde{\Delta}$ and $\rho_{z,z'} := (f_{z'} \otimes \mathrm{id} \otimes f_z) \circ \tilde{\Delta}^{(2)}$. Then

$$\begin{aligned}
(\rho_{z,z'} \otimes \mathrm{id})({}^{\,k}_{m}X^l_n) &= (f_{z'} \otimes \mathrm{id} \otimes f_z \otimes \mathrm{id})(({}^{\,k}_{k}X^l_l)_{14}({}^{\,k}_{m}X^l_n)_{24}({}^{\,m}_{m}X^n_n)_{34}) \\
&= (1 \otimes ({}^{\,k}F^l_X)^{z'})\,{}^{\,k}_{m}X^l_n(1 \otimes ({}^{\,m}F^n_X)^z).
\end{aligned}$$

We take $z = z' = 1$, use Theorem 3.10, where now $\alpha = \beta$ by our scaling of F_X, and obtain

$$(\phi \otimes \mathrm{id} \otimes \mathrm{id})((\, {}_m^k X_n^l)_{12}^*((\rho_{1,1} \otimes \mathrm{id})(\, {}_m^k X_n^l))_{13}) = \alpha^{-1}(\mathrm{id} \otimes \, {}^k F_X^l)(\mathrm{id} \otimes \, {}^k (F_X^{-1})^l) \Sigma_{k,l,m,n}(\mathrm{id} \otimes \, {}^m F_X^n)$$
$$= \beta^{-1}(\, {}^m F_X^n \otimes \mathrm{id}) \Sigma_{k,l,m,n}$$
$$= (\phi \otimes \mathrm{id} \otimes \mathrm{id})((\, {}_m^k X_n^l)_{13}(\, {}_m^k X_n^l)_{12}^*).$$

To conclude assertion (3), applying slice maps of the form $\omega_{\xi,\xi'} \otimes \omega_{\eta,\eta'}$.

(4) By Proposition 3.9 and the calculation above,

$$(S^2 \otimes \mathrm{id})(\, {}_m^k X_n^l) = (1 \otimes \, {}^k F_X^l) \, {}_m^k X_n^l (1 \otimes \, {}^m F_X^n)^{-1} = (\rho_{-1,1} \otimes \mathrm{id})(\, {}_m^k X_n^l).$$

Assertion (4) follows again by applying slice maps.

(5) The fact that f_z is a character and that $f_z(\, {}_m^k A_n^l) = 0$ if $(k,l) \neq (m,n)$ immediately implies the relation $f_z(\lambda_l \rho_n) = \delta_{l,n}$ and the equality

$$(f_{-z} \otimes \mathrm{id})(\, {}_k^k X_l^l) = (\, {}^k (F_X)^l)^{-z} = (f_z \otimes \mathrm{id})(\, {}_k^k X_l^l)^{-1}$$
$$= (f_z \otimes \mathrm{id})(\, {}_k^k (X^{-1})_l^l) = (f_z \circ S \otimes \mathrm{id})(\, {}_k^k X_l^l).$$

Therefore, $f_{-z} = f_z \circ S$. Using the preceding calculation, the relation $(S \otimes \mathrm{id})(\, {}_k^k X_l^l) = (\, {}_k^k X_l^l)^*$ and positivity of $\, {}^k F_X^l$, we conclude

$$(* \circ f_z \circ * \otimes \mathrm{id})(\, {}_k^k X_l^l) = (f_z \otimes \mathrm{id})((\, {}_k^k X_l^l)^*)^*$$
$$= (f_{-z} \otimes \mathrm{id})(\, {}_k^k X_l^l)^* = ((\, {}^k F_X^l)^{-z})^* = (\, {}^k F_X^l)^{-\overline{z}} = (f_{-\overline{z}} \otimes \mathrm{id})(\, {}_k^k X_l^l),$$

whence $* \circ f_z \circ * = f_{-\overline{z}}$. $\qquad\qquad\qquad\square$

4   Tannaka-Krein duality for compact Hopf face algebras

Let $\mathcal{C}$ be a rigid tensor C*-category with irreducible unit. For example, one can take $\mathcal{C} = \mathrm{Rep}(\mathbb{G})$, the category of finite-dimensional unitary representations of a compact quantum group $\mathbb{G}$. We will in general view the tensor product of $\mathcal{C}$ as being strict. Let J be an index set for a maximal set of mutually non-isomorphic irreducible objects u_a in $\mathcal{C}$. The unit object of $\mathcal{C}$ will be written u_o. Whenever convenient, we will replace u_a by its associated index symbol a. We will also fix once and for all orthonormal bases $f_{c,j}^{a,b}$ for $\mathrm{Mor}(u_c, u_a \otimes u_b)$, where j runs over an index set $J_c^{a,b}$.

Let I be a (countable) set. We will write Hilb_{I^2} for the monoidal tensor C*-category of I-bigraded Hilbert spaces $\mathscr{H} = \sum_{r,s}^{\oplus} \mathscr{H}_{rs}$, where the direct sum on the right is understood as the completion of the ordinary algebraic one. The tensor product $\underset{I}{\otimes}$ in Hilb_{I^2} is defined by $(\mathscr{H} \underset{I}{\otimes} \mathscr{G})_{rs} = \bar{\oplus}_t \, (\mathscr{H}_{rt} \otimes \mathscr{G}_{ts})$. The unit of Hilb_{I^2} is $l^2(I)$ with the obvious I^2-grading. We will view this monoidal category as being strict.

We will be interested in strong tensor C*-functors F from $\mathcal{C}$ to Hilb_{I^2}. As shown in [DCY], any ergodic action of a compact quantum group $\mathbb{G}$ on a unital C*-algebra provides a tensor C*-functor of $\mathcal{C}$ into Hilb_{I^2} for some set I.

For $F : \mathcal{C} \to \mathrm{Hilb}_{I^2}$ a strong tensor C*-functor, we denote the unitary compatibility morphisms by $\phi_{X,Y} : F(X) \underset{I}{\otimes} F(Y) \to F(X \otimes Y)$, where we recall that they are assumed to satisfy the coherence conditions

$$\phi_{X,Y \otimes Z}(\mathrm{id}_X \otimes \phi_{Y,Z}) = \phi_{X \otimes Y, Z}(\phi_{X,Y} \otimes \mathrm{id}_Z), \qquad \phi_{o,a} = \phi_{a,o} = \mathrm{id}_a \,.$$

It will be convenient to extend $\phi_{X,Y}$ to a coisometry $F(X) \otimes F(Y) \to F(X \otimes Y)$, defining it to be zero on the orthogonal complement of $F(X) \underset{I}{\otimes} F(Y)$. Note however that then $\phi_{X,o}$ becomes the coisometry $F(X) \otimes l^2(I) \to F(X)$ sending $F(X)_{rs} \otimes \mathbb{C}\delta_t$ canonically onto $\delta_{s,t} F(X)_{rs}$, and similarly for $\phi_{o,X}$. Whenever X, Y are clear, we will abbreviate $\phi_{X,Y}$ as ϕ. We will use the notation

$$F_{c,j}^{a,b} = \phi^* F(f_{c,j}^{a,b}) \in B(F(u_c), F(u_a) \otimes F(u_b)).$$

As $\mathcal{C}$ is rigid, each $F(X)$ will be column-finite in the sense that for each X in $\mathcal{C}$ and each fixed s in I, the direct sum $\sum_r^{\oplus} F(X)_{rs}$ will be finite-dimensional. Similarly, each $F(X)$ will be row-finite. See ...

Define vector spaces
$$_m^k A_n^l(a) = B(F(u_a)_{kl}, F(u_a)_{mn}).$$

Write $_m^k A_n^l = \oplus_{a \in J} \,_m^k A_n^l(a)$ and $A = \oplus_{k,l,m,n} \,_m^k A_n^l$. The a-spectral subspace $A(a)$ of A is defined as
$$A(a) = \sum_{k,l,m,n}^{\oplus} \,_m^k A_n^l(a).$$

For any element $x \in A$, its component in the a-spectral subspace is written x_a.

Our goal is to turn A into a generalized compact Hopf face algebra.

We first turn A into an algebra. The multiplication $x \cdot y$ of $x \in \,_m^k A_n^l(a)$ and $y \in \,_r^p A_s^q(b)$ is the element in $_m^k A_s^q$ defined by the formula

$$(x \cdot y)_c = \sum_{j \in J_c^{a,b}} \left(F_{c,j}^{a,b} \right)^* (x \otimes y) \left(F_{c,j}^{a,b} \right).$$

Note that the product is independent of the specific choice of orthogonal bases $f_{c,j}^{a,b}$. We will continue to use the $\cdot$-notation to distinguish this product from the ordinary multiplication of operators.

Lemma 4.1. *With the above product, A becomes a faithful strong I^2-algebra.*

Proof. Let $x \in \, {}_m^k A_n^l(a)$, $y \in \, {}_r^p A_s^q(b)$ and $z \in \, {}_s^q A_v^t$. From the fact that ϕ is a natural transformation, we find that

$$((x \cdot y) \cdot z)_d = \sum_{e \in J} \sum_{k \in J_d^{e,c}} \sum_{j \in J_e^{a,b}} \left(\phi^*(\phi^* \otimes \mathrm{id}) F(f_{d,e,j,k}^{1,a,b,c}) \right)^* (x \otimes y \otimes z) \left((\phi^* \otimes \mathrm{id}) \phi^* F(f_{d,e,j,k}^{1,a,b,c}) \right)$$

where $f_{d,e,j,k}^{1,a,b,c} = (f_{e,j}^{a,b} \otimes \mathrm{id}) f_{d,k}^{e,c}$. On the other hand,

$$(x \cdot (y \cdot z))_d = \sum_{e \in J} \sum_{k \in J_d^{a,e}} \sum_{j \in J_e^{b,c}} \left(\phi(\mathrm{id} \otimes \phi) F(f_{d,e,j,k}^{2,a,b,c}) \right)^* (x \otimes y \otimes z) \left((\mathrm{id} \otimes \phi) \phi F(f_{d,e,j,k}^{2,a,b,c}) \right)$$

where $f_{d,e,j,k}^{2,a,b,c} = (\mathrm{id} \otimes f_{e,j}^{b,c}) f_{d,k}^{a,e}$. As $\phi(\phi \otimes \mathrm{id})$ by $\phi(\mathrm{id} \otimes \phi)$ by assumption, and as the orthonormal bases $\{ f_{d,e,j,k}^{1,a,b,c} \mid e, j, k \}$ or $\{ f_{d,e,j,k}^{2,a,b,c} \mid e, j, k \}$ can clearly be replaced by any other orthonormal basis of $\mathrm{Mor}(u_d, u_a \otimes u_b \otimes u_c)$, it follows that $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

Define $1_{rs} \in B(F(u_o)_{rr}, F(u_o)_{ss}) = \, {}_s^r A_s^r(o)$ as the map sending δ_r to δ_s. By the compatibility assumption for $\phi_{a,o}$ and $\phi_{o,a}$, the map $\mathrm{Fun}_f(I^2) \to A$ mapping $\delta_{(r,s)}$ to 1_{rs} is an algebra homomorphism. Thus A becomes a faithful strong I^2-algebra. $\qquad \square$

In the following, we will again write $\lambda_r = \sum_s 1_{rs}$ and $\rho_s = \sum_r 1_{rs}$ inside $M(A)$, using the notation as at the end of the proof of the previous lemma.

We turn to the coproduct. Let $\{ e_{a,i} \mid i \in B_a \}$ denote an orthonormal basis of $F(u_a)$ over an index set B_a which is adapted to the bigrading (in the sense that each $e_{a,i}$ is inside exactly one component). Write $B_{a,rs} \subseteq B_a$ for the set of indices for which $e_{a,i} \in F(u_a)_{rs}$. Define elements

$$P_{mn}^{kl}(a) \in \, {}_m^k A_n^l(a) \otimes \, {}_k^m A_l^n(a)$$

by

$$P_{mn}^{kl}(a) = \sum_{i \in B_{a,kl}, j \in B_{a,mn}} e_{a,j} e_{a,i}^* \otimes e_{a,i} e_{a,j}^*.$$

As each $F(u_a)_{kl}$ is finite-dimensional, the above sums are finite.

Define now maps

$$\Delta_{rs} : \, {}_m^k A_n^l(a) \to \, {}_r^k A_s^l(a) \otimes \, {}_m^r A_n^s(a)$$

by the application

$$x \mapsto P_{rs}^{mn}(a)(x \otimes 1) = (1 \otimes x) P_{rs}^{kl}(a).$$

They obviously extend to linear maps Δ_{rs} from A to $A \underset{I^2}{\otimes} A$.

Lemma 4.2. *For each $x \in A$, the element $\Delta(x) = \sum_{rs} \Delta_{rs}(x)$ gives a well-defined multiplier of $A \underset{I^2}{\otimes} A$. The resulting map $\Delta : A \to M(A \underset{I^2}{\otimes} A)$ is an I^2-coproduct.*

25

Proof. As the grading on each $F(u_a)$ is column-finite, it follows at once that for each fixed p, q and $x \in A$, the element $\Delta_{rs}(x)(1 \underset{I^2}{\otimes} \lambda_p \rho_q)$ is zero except for finitely many r and s. Similarly, $(1 \underset{I^2}{\otimes} \lambda_p \rho_q) \Delta_{rs}(x)$ is zero except for finitely many r, s because of row-finiteness of $F(u_a)$. Hence $\Delta(x)$ is well-defined as a multiplier for each $x \in A$. Once we show that Δ is multiplicative, it will be immediate that Δ is coassociative, since each Δ_{rs} is coassociative. Moreover, also the fact that Δ then is an I^2-morphism is clear from the definition.

To obtain the multiplicativity of Δ, or rather of the coextension $\widetilde{\Delta}$, choose $x \in {}_m A_n(a)$ and $y \in {}_n A_q(b)$. Then

$$\begin{aligned}
\widetilde{\Delta}(x) \cdot \widetilde{\Delta}(y) &= \sum_{rstv} (P_{rs}^{mn}(a)(x \otimes 1)) \cdot (P_{tv}^{nq}(b)(y \otimes 1)) \\
&= \sum_{cd} \sum_{ij} \sum_{klpt} \left(F_{c,i}^{a,b} \otimes F_{d,j}^{a,b} \right)^* \left(e_{a,k} e_{a,l}^* x \otimes e_{b,p} e_{b,t}^* y \otimes e_{a,l} e_{a,k}^* \otimes e_{b,t} e_{b,p}^* \right) \left(F_{c,i}^{a,b} \otimes F_{d,j}^{a,b} \right),
\end{aligned}$$

where we may take the sum over all $k, l \in I_a$, $p, t \in I_b$ (and where the composition of operators with mismatching target and source is considered to be zero). Note that the infinite sums are convergent inside $M(A \otimes A)$ by the argument in the first paragraph.

Plugging in the identity operator $\sum_{cd} \sum_{rs} \left(e_{c,r} e_{c,r}^* \otimes e_{d,s} e_{d,s}^* \right)$ at the front, we obtain that the expression becomes

$$\sum_{cd} \sum_{rs} \sum_{ij} \sum_{klpt} X_{k,p}^{c,r,i} Y_{l,t}^{d,s,j} (e_{c,r} \otimes e_{d,s}) \left(e_{a,l}^* x \otimes e_{b,t}^* y \otimes e_{a,k}^* \otimes e_{b,p}^* \right) \left(F_{c,i}^{a,b} \otimes F_{d,j}^{a,b} \right)^*$$

where $X_{k,p}^{c,r,i} = e_{c,r}^* (F_{c,i}^{a,b})^* (e_{a,k} \otimes e_{b,p})$ and $Y_{l,t}^{d,s,j} = e_{d,t}^* (F_{d,j}^{a,b})^* (e_{a,l} \otimes e_{b,t})$. Resumming over the k, l, p, t, we obtain

$$\sum_{cd} \sum_{rs} \sum_{ij} \left(e_{c,r} e_{d,s}^* (F_{d,j}^{a,b})^* (x \otimes y) F_{c,i}^{a,b} \right) \otimes \left(e_{d,s} e_{c,r}^* (F_{c,i}^{a,b})^* F_{d,j}^{a,b} \right).$$

As ϕ is a coisometry and the $f_{c,i}^{a,b}$ are orthonormal, this expression simplifies to

$$\sum_{c} \sum_{rs} \sum_{i} e_{c,r} e_{c,s}^* \left((F_{c,i}^{a,b})^* (x \otimes y) F_{c,i}^{a,b} \right) \otimes e_{c,s} e_{c,r}^*,$$

which is precisely $\widetilde{\Delta}(x \cdot y)$. $\qquad\square$

Proposition 4.3. *The couple (A, Δ) is a generalized face algebra over I.*

Proof. Let ε assign to any $x \in {}_m^k A_n^l(a)$ the number $\mathrm{Tr}(x) = \sum_{i \in B_a} (e_{a,i}^* x e_{a,i})$ (where we keep the convention that mismatching operators compose to zero). We claim that ε is a counit, satisfying the conditions in the definition of a generalized face algebra. The fact that ε is a counit is immediate from the definition of Δ. It is also computed directly that

26

for $x \in {}_m^k A_n^l$ and $y \in {}_n^l A_s^r$, we have $\varepsilon(x \cdot y) = \varepsilon(x)\varepsilon(y)$, since the $\{\phi^* F(f_{c,i}^{a,b}) e_{c,j} \mid c, i, j\}$ form an orthonormal basis of $F(u_a) \underset{I}{\otimes} F(u_b)$. From this formula, the second identity for the counit will hold true once we show that

$$\varepsilon(\lambda_k \rho_m x \lambda_l \rho_n) = \varepsilon(\lambda_k \rho_m x_{(1)}) \varepsilon(x_{(2)} \lambda_l \rho_n).$$

But both left and right hand side are zero unless $k = m$, $n = l$ and $x \in {}_m^k A_n^l$, in which case both sides equal $\varepsilon(x)$. $\qquad \square$

Our next job is to define a suitable antipode for (A, Δ). Here the rigidity of $\mathcal{C}$ will come into play, so we first fix our conventions. Let $a \mapsto \bar{a}$ be the involution induced by the rigidity on the index set J. We assume that $\overline{u_a} = u_{\bar{a}}$. For each u_a, we will fix duality morphisms $R_a : u_0 \to u_{\bar{a}} \otimes u_a$ and $\bar{R}_a : u_0 \to u_a \otimes u_{\bar{a}}$. By means of F and ϕ, they induce I^2-grading preserving maps $\mathscr{R}_a : l^2(I) \to F(u_{\bar{a}}) \underset{I}{\otimes} F(u_a)$ and $\bar{\mathscr{R}}_a : l^2(I) \to F(u_a) \underset{I}{\otimes} F(u_{\bar{a}})$. These in turn provide an invertible anti-linear map $I_a : F(u_a)_{kl} \to F(u_{\bar{a}})_{lk}$ and $J_a : F(u_{\bar{a}})_{lk} \to F(u_a)_{kl}$ such that $\langle I_a \xi_a, \eta_{\bar{a}} \rangle = \sum_r \delta_r^* \bar{\mathscr{R}}_a^* (\xi_a \otimes \eta_{\bar{a}})$ and $\langle J_a \eta_{\bar{a}}, \xi_a \rangle = \sum_s \delta_s^* \mathscr{R}_a^* (\eta_{\bar{a}} \otimes \xi_a)$. The snake identities for R_a and $\bar{R}_a$ guarantee that J_a is the inverse of I_a.

We define

$$S : {}_m^k A_n^l(a) \to {}_l^n A_k^m(\bar{a})$$

by

$$x \mapsto I_a x^* J_a.$$

Lemma 4.4. *By means of the map S, the couple (A, Δ) becomes a generalized Hopf face algebra.*

Proof. It is clear that S is invertible. We also have $S(\lambda_k \rho_l) = \lambda_l \rho_k$ as $I_o \delta_k = \delta_k$.

Let us check that S satisfies the condition $S(x_{(1)}) \cdot x_{(2)} = \sum_p \varepsilon(x \cdot \lambda_p) \rho_p$ in the multiplier algebra for $x \in {}_m^k A_n^l(a)$. By definition, we have

$$S(x_{(1)}) \cdot x_{(2)} = \sum_c \sum_i \sum_{p,q \in B_a} \left(F_{c,i}^{\bar{a},a} \right)^* \left(I_a e_{a,q} e_{a,p}^* J_a \otimes x e_{a,q} e_{a,p}^* \right) \left(F_{c,i}^{\bar{a},a} \right).$$

Let $C : \mathbb{C} \to \mathbb{C}$ be complex conjugation. Then we can write $I_a e_{a,q} e_{a,p}^* J_a = (I_a e_{a,q} C)(C e_{a,p}^* J_a)$. We now calculate, by definition of J_a and $F_{c,i}^{\bar{a},a}$, that

$$\sum_{p \in B_a} (C e_{a,p}^* J_a \otimes e_{a,p}^*) \left(F_{c,i}^{\bar{a},a} \right) = (R_a^* f_{c,i}^{\bar{a},a}) \sum_{s \in I} \delta_s^*,$$

since ϕ is a coisometry. Plugging this into our expression for $S(x_{(1)}) \cdot x_{(2)}$, we obtain

$$\sum_s \sum_{q \in B_a} \left(\sum_c \sum_i \left(f_{c,i}^{\bar{a},a} \right)^* R_a \right) F_{c,i}^{\bar{a},a} \right)^* (I_a e_{a,q} C \otimes x e_{a,q} \delta_s^*).$$

27

As ϕ is a coisometry and $\phi^* \phi \mathscr{R}_a = \mathscr{R}_a$, we can write $\left((f_{c,i}^{\bar{a},a})^* R_a \right) F_{c,i}^{\bar{a},a} = F_{c,i}^{\bar{a},a} (F_{c,i}^{\bar{a},a})^* \mathscr{R}_a$. As the $f_{c,i}^{\bar{a},a}$ form an orthonormal basis, we thus get

$$S(x_{(1)}) \cdot x_{(2)} = \sum_s \sum_{q \in B_a} \mathscr{R}_a^* (I_a e_{a,q} C \otimes x e_{a,q} \delta_s^*).$$

Now the composition $I_a e_{a,q} C$ is the creation operator for the vector $I_a e_{a,q}$. Hence using again the definition of J_a, and using that $x \in {}_m^k A_n^l$, we get

$$
\begin{aligned}
S(x_{(1)}) \cdot x_{(2)} &= \sum_s \sum_q \delta_n \delta_s^* e_{a,q}^* x e_{a,q} \\
&= \sum_s \mathrm{Tr}(x) \delta_n \delta_s^* \\
&= \sum_p \varepsilon(x \lambda_p) \rho_p,
\end{aligned}
$$

since $\mathrm{Tr}(x) = \delta_{k,m} \delta_{n,l} \varepsilon(x)$.

The identity $x_{(1)} \cdot S(x_{(2)}) = \sum_p \varepsilon(\rho_p x) \lambda_p$ is proven in a similar way. $\qquad \square$

In the next step, we determine an invariant functional for (A, Δ).

Definition 4.5. We define $\varphi : {}_m^k A_n^l \to \mathbb{C}$ as the projection onto the component ${}_m^k A_n^l(o) \cong \delta_{kl} \delta_{mn} \mathbb{C}$.

Lemma 4.6. *The functional φ is an invariant normalized functional.*

Proof. The fact that φ is normalized is immediate, so let us check that it is invariant. Let $x \in {}_m^k A_n^l(a)$. Then

$$
\begin{aligned}
(\mathrm{id} \otimes \varphi) \widetilde{\Delta}(x) &= \sum_{i,j} \varphi(e_{a,j} e_{a,i}^*) e_{a,i} e_{a,j}^* x \\
&= \delta_{a,o} \sum_{r,s} \delta_r \delta_s^* x \\
&= \varphi(x) \sum_r \delta_r \delta_k^* \\
&= \sum_p \varphi(\lambda_p x) \lambda_p.
\end{aligned}
$$

The proof of right invariance follows similarly. $\qquad \square$

Finally, we introduce the *-structure and show that (A, Δ) is a generalized compact Hopf face algebra. To distinguish the new *-operation from the ordinary operator algebraic one, we will denote it by $\dagger$.

Definition 4.7. We define the anti-linear map $\dagger : {}^k_m A^l_n \to {}^m_k A^n_l$ by the formula

$$x^\dagger = S(x^*)$$

Lemma 4.8. *The map $x \mapsto x^\dagger$ is an anti-multiplicative anti-linear involution on A.*

Proof. It is clear that $x \mapsto x^\dagger$ is anti-linear. It is also immediate from the definition of the product that $(x \cdot y)^* = x^* \cdot y^*$. Together with the anti-multiplicativity of S, this proves the anti-multiplicativity of $\dagger$.

Let us proof that $\dagger$ is an involution. It is sufficient to prove that $I_{\bar{a}} I_a = \lambda \, \mathrm{id}$ and $J_a J_{\bar{a}} = \lambda^{-1} \, \mathrm{id}$ for some scalar λ. But this follows from the fact that $(\bar{R}_a, R_a)$ and $(R_{\bar{a}}, \bar{R}_{\bar{a}})$ are both solutions to the conjugate equations for $\bar{a}$. $\qquad\square$

The last property which needs to be proven is the positivity of φ. For this, recall that $R_a^* R_a$ and $\bar{R}_a^* \bar{R}_a$ are scalars as u_a is irreducible. One can then rescale R_a and $\bar{R}_a$ such that the scalar in both expressions is the same. This scalar is then a uniquely determined number $\dim_q(a)$, called the *quantum dimension* of a. It follows that $\frac{1}{\dim_q(a)} F(R_a R_a^*)$ is the projection of $F(u_{\bar{a}} \otimes u_a)$ onto the copy of $F(u_o)$ inside, and a similar statement holds for $\bar{R}_a$.

Proposition 4.9. *For any $x \in A$, the scalar $\varphi(x^\dagger \cdot x)$ is positive.*

Proof. It is straightforward to see that the blocks ${}^k_m A^l_n$ are mutually orthogonal, and that moreover the spectral subspaces inside are mutually orthogonal. Let then $\xi, \zeta \in F(u_a)_{kl}$ and $\eta, \mu \in F(u_a)_{mn}$. We have, using the remark above,

$$
\begin{aligned}
\varphi(y^\dagger \cdot x) &= \varphi(\sum_c \sum_i \left(F^{\bar{a},a}_{c,i} \right)^* (I_a y J_a \otimes x) \left(F^{\bar{a},a}_{c,i} \right)) \\
&= \delta_n^* \sum_i \left(F^{\bar{a},a}_{o,i} \right)^* (I_a y J_a \otimes x) \left(F^{\bar{a},a}_{o,i} \right) \delta_l \\
&= \frac{1}{\dim_q(u_a)} \delta_n^* \mathscr{R}_a^* (I_a y J_a \otimes x) \mathscr{R}_a \delta_l \\
&= \frac{1}{\dim_q(u_a)} \sum_{p,q} \delta_n^* \mathscr{R}_a^* (I_a y J_a e_{\bar{a},p} e_{\bar{a},p}^* \otimes x e_{a,q} e_{a,q}^*) \mathscr{R}_a \delta_l.
\end{aligned}
$$

By the defining properties of I_a and J_a, this expression becomes $\dim_q(u_a)^{-1} \sum_p \langle e_{\bar{a},p}, J_a^* x^* y J_a e_{\bar{a},p} \rangle$, thus clearly φ will be positive on A. $\qquad\square$

Corepresentations of generalized compact Hopf face algebras

Let (A, Δ) be a generalized compact Hopf face algebra over an index set I. A *locally finite-dimensional unitary corepresentation* of (A, Δ) consists of a row and column-

finite I^2-graded Hilbert space $\mathcal{H} = \sum_{k,l \in I}^{\oplus} \mathcal{H}_{kl}$ together with elements ${}_m^k U_n^l \in {}_m^k A_n^l \otimes$ $B({}^m\mathcal{H}^n, {}^k\mathcal{H}^l)$ such that

$$\sum_k \left({}_m^k U_n^l \right)^* {}_m^k U_n^l = \lambda_l \rho_n \otimes \mathrm{id}\ {}_m\mathcal{H}^n$$

and

$$\sum_n {}_m^k U_n^l \left({}_m^k U_n^l \right)^* = \lambda_k \rho_m \otimes \mathrm{id}\ {}_k\mathcal{H}^l,$$

and

$$(\widetilde{\Delta} \otimes \mathrm{id})({}_m^k U_n^l) = \sum_{p,q} \left({}_p^k U_q^l \right)_{13} ({}_m^p U_n^q)_{23}.$$

Note that in the first two identities, the sums are finite, while in the finite identity the possibly infinite sum is meaningful inside the multiplier algebra sense.

By a morphism between two locally finite-dimensional unitary corepresentations $(\mathcal{H}, U)$ and $(\mathcal{G}, V)$ is meant a grading-preserving bounded map $T = \sum_{k,l}^{\oplus} {}^k T^l : \mathcal{H} \to \mathcal{G}$ such that $(1 \otimes {}^k T^l) {}_m^k U_n^l = {}_m^k V_n^l (1 \otimes {}^m T^n)$. The collection of all locally finite-dimensional unitary corepresentations clearly forms a semi-simple C*-category $\mathrm{Corep}(A)$. We will say that (A, Δ) is *of finite type* if the morphisms in $\mathrm{Corep}(A)$ are finite-dimensional.

One can define a tensor product $\mathbb{T}$ between locally finite-dimensional corepresentations by means of the $\underset{I}{\otimes}$-product of bigraded Hilbert spaces and the operation

$$\ _m^k (U \mathbb{T} V)_n^l = \left({}_m^k U_s^r \right)_{12} \left({}_s^r V_n^l \right)_{13}.$$

In this way, the category $\mathrm{Corep}(A)$ becomes a monoidal category. The unit object consists of the I^2-graded Hilbert space $l^2(I)$ together with the elements ${}_m^k U_n^l = \delta_{kl}\delta_{mn}\lambda_k\rho_m \otimes 1$.

Assume now that $\mathcal{C}$ is a semi-simple tensor C*-category with irreducible unit, and $F : \mathcal{C} \to \mathrm{Hilb}$ a strong tensor C*-functor. Let (A, Δ) be the associated generalized compact Hopf face algebra. Let us show that $\mathcal{C} \cong \mathrm{Corep}(A)$ by means of an equivalence functor G.

For X an object of $\mathcal{C}$, we build a locally finite-dimensional unitary corepresentation U on $F(X)$. Consider the canonical isomorphism $F(X) \cong \oplus_{a \in J} X_a \otimes \mathrm{Mor}(X_a, X)$. Let

$${}_m^k U_n^l(a) \in {}_m^k A_n^l(a) \otimes B(F(u_a)_{mn}, F(u_a)_{kl}) = B(F(u_a)_{kl}, F(u_a)_{mn}) \otimes B(F(u_a)_{mn}, F(u_a)_{kl})$$

be determined as the element implementing the non-degenerate pairing $B(F(u_a)_{kl}, F(u_a)_{mn}) \otimes B(F(u_a)_{nm}, F(u_a)_{lk}) \to \mathbb{C}$ sending $S \otimes T$ to $\mathrm{Tr}(ST)$. Using notation as before, this means that

$${}_m^k U_n^l(a) = \sum_{p \in B_{a,mn}, q \in B_{a,kl}} e_p e_q^* \otimes e_q e_p^*.$$

Monoidal equivalence of generalized compact Hopf face algebras

Let (A, Δ) be a generalized Hopf face algebra over a set I. Assume that $I = I_1 \sqcup I_2$, and let $\Lambda_j = \sum_{i \in I_j} \lambda_i$, resp. $\mathrm{P}_j = \sum_{i \in I_j} \rho_j$. If the Λ_j and P_j are central in $M(A)$, then we can write $A = \sum_{i,j}^{\oplus} A(ij)$ where $A(ij) = \Lambda_i \mathrm{P}_j A$ are subalgebras. Moreover, the comultiplication $\widetilde{\Delta}$ splits into comultiplications

$$\widetilde{\Delta}_{ij}^k : A(ij) \to M(A(ik) \otimes A(kj)) \text{ s.t. } \widetilde{\Delta} = \widetilde{\Delta}_{ij}^1 + \widetilde{\Delta}_{ij}^2 \text{ on } A(ij).$$

A similar decomposition holds for Δ.

It is immediate to see that the $(A(ii), \Delta_{ii}^i)$ are two generalized Hopf face algebras over the respective I_i.

Definition 4.10. We say (A, Δ) is a *co-linking generalized (compact) Hopf face algebra* between $(A(11), \Delta_{11}^1)$ and $(A(22), \Delta_{22}^2)$ if $\lambda_i \mathrm{P}_2 \neq 0$ for any $i \in I_1$.

Upon applying the antipode, we see that then $\rho_j \Lambda_1 \neq 0$ for any $j \in I_2$ as well.

Definition 4.11. Two generalized (compact) Hopf face algebras are called *comonoidally Morita equivalent* if they are isomorphic to the components (A_{ii}, Δ_{ii}^i) of some co-linking generalized (compact) Hopf face algebra.

As an example, consider two sets I_i, and two tensor functors (F_i, ϕ_i) of a semi-simple rigid C*-category $\mathcal{C}$ with irreducible unit into $\mathrm{Hilb}_{I_i^2}$. Then with $I = I_1 \sqcup I_2$, we can form a new C*-functor $F = F_1 \oplus F_2$ of $\mathcal{C}$ into Hilb_{I^2} by putting $F(X) = F_1(X) \oplus F_2(X)$ with the obvious I^2-grading (and the obvious direct sum operation on morphisms). It becomes monoidal by means of the unitaries

$$F(X \otimes Y) = F_1(X \otimes Y) \oplus F_2(X \otimes Y) \underset{\phi_1 \oplus \phi_2}{\cong} (F_1(X) \underset{I_1}{\otimes} F_1(Y)) \oplus (F_2(X) \otimes F_2(Y)) \cong F(X) \underset{I}{\otimes} F(Y)$$

(where the last map is unitary since $(F(X) \underset{I}{\otimes} F(Y))_{ij} = 0$ for example for $i \in I_1$ and $j \in I_2$).

If we then consider the generalized compact Hopf face algebra (A, Δ) associated to F, we have immediately from the construction that the Λ_i and P_i associated to the decomposition $I = I_1 \sqcup I_2$ are indeed central elements in $M(A)$. Moreover, the parts $(A_{ii}^i, \Delta_{ii}^i)$ are seen to arise from applying the Tannaka-Krein construction to the respective functors F_1 and F_2. The fact that (A, Δ) is co-linking is immediate from the fact that *none* of the $\lambda_i \rho_j$ are zero in this particular case (since ${}_m^k A_m^k(o) = B(F(u_o)_{kk}, F(u_o)_{mm}) \cong \mathbb{C}$).

We will exploit the above extra structure in the following section to say something about the algebra A appearing in ... This is the component $\tilde{A}(1,1)$ of the above algebra. The following lemma will be needed.

Lemma 4.12. *Assume (A, Δ) is a co-linking generalized Hopf face algebra. Then any of the maps $\widetilde{\Delta}^k_{ij}$ is injective.*

Proof. Take a non-zero $x \in A_n(ij)$ where $n \in I_j$. Then for any $l \in I$ with $\rho_n \lambda_l \neq 0$, we know that $\widetilde{\Delta}(x)(1 \otimes \rho_n \lambda_l) \neq 0$. Hence $\widetilde{\Delta}^k_{ij}(x)(1 \otimes \rho_n \lambda_l) \neq 0$ for $l \in I_k$, and hence $\widetilde{\Delta}^k_{ij}(x) \neq 0$. Now if $j = k$, the condition $\rho_n \lambda_l \neq 0$ is satisfied by taking $l = n$ (since $\varepsilon(\lambda_n \rho_n) = 1$). If $j \neq k$, it is satisfied for at least one l by the co-linking assumption. $\square$

5 Compact Hopf face algebras on the level of operator algebras

It then follows by symmetry that also the maps

$$(W^{k,t}_{m,n,u,v})^* : \oplus_l \, {}^k_m A^l_n \otimes \, {}^l_u A^t_v \to \oplus_r \, {}^k_m A^t_r \otimes \, {}^n_u A^r_v$$

defined by the formula

$$a \otimes b \to \Delta(b)(a \otimes 1)$$

are unitaries, with inverse map $a \otimes b \mapsto S^{-1}(b_{(1)})a \otimes b_{(2)}$.

Lemma 5.1. *Let (A, Δ) be a generalized compact face algebra. Then each $V^{k,l,s,t}_{m,v}$ is a unitary, and similarly for the $W^{k,t}_{m,n,u,v}$.*

Proof. It is immediately checked that $V^{k,l,s,t}_{m,v}$ is isometric. $\square$

Let us write $\mathscr{L}^2(A, \varphi)$ for the completion of A with respect to the inner product $\langle a, b \rangle = \varphi(a^*b)$. The canonical inclusion of A into $\mathscr{L}^2(A)$ will be denoted Λ.

Lemma 5.2. *Assume (A, Δ) is a generalized compact face algebra. The representation of A by left multiplication on itself extends to a representation by bounded operators on the completion $\mathscr{L}^2(A, \varphi)$.*

Proof. Denote $\omega_{\xi, \eta}(x) = \langle \xi, x\eta \rangle$ for ξ, η vectors and x a bounded operator. Then a straightforward computation shows that

$$(\omega_{\Lambda(a), \Lambda(b)} \otimes \mathrm{id})(V) = \varphi(a^*b_{(1)})b_{(2)}$$

as a left multiplication operator. As $(A \otimes 1)\Delta(A) = (A \otimes A)\Delta(1)$ by Lemma 5.1 (applied to the opposite algebra), it follows by normalization of φ that each element of A can be represented in the form $(\omega_{\Lambda(a), \Lambda(b)} \otimes \mathrm{id})(V)$, and hence extends to a bounded operator on $\mathscr{L}^2(A, \varphi)$. $\square$

In the following, we will abbreviate $\mathscr{L}^2(A)$ by $L^2 A$.

Let (A, Δ) be a generalized compact face algebra. Denote the von Neumann algebraic completion of $A \subseteq B(L^2 A)$ by M. Denote $L^2 A \underset{I}{\otimes} L^2 A = E(L^2 A \otimes L^2 A)$, where $E = \sum_p \rho_p \otimes \lambda_p$ is extended to a bounded operator (in fact, a self-adjoint projection). Finally, denote $M \underset{I^2}{\otimes} M = E(M \otimes M)E$. Then $M \underset{I^2}{\otimes} M$ is the von Neumann algebraic completion of $A \underset{I^2}{\otimes} A$.

Extend now the $V_{m,v}^{k,l,s,t}$ to unitaries

$$V : \oplus_p L^2(A_p) \otimes L^2({}_p A) \to \oplus_p L^2({}_p A) \otimes L^2({}^p A) = E(L^2 A \otimes L^2 A).$$

Then we can construct a map

$$\Delta : M \to M \underset{I^2}{\otimes} M, \quad x \to V(x \otimes 1)V^*.$$

By direct computation, we see that Δ extends the comultiplication map on A. It is then immediate to check that Δ is in fact coassociative (where one may as well consider Δ as a non-unital map from M to $M \otimes M$).

We aim to show that (M, Δ) can be fitted into the theory of measured quantum groupoids.

6 Compact quantum groups of face type from reciprocal random walks

We recall some notions introduced in [DCY]. We slightly change the terminology for the sake of convenience.

Definition 6.1. By a graph Γ we mean a quadruple $\Gamma = (V, E, s, t)$ where V and E are sets, called respectively the set of *vertices* and the set of *edges*, and s and t are maps

$$s, t : E \to V,$$

called respectively the source and target map.

An *involution* of a graph is a map

$$i : E \to E, \quad e \mapsto \bar{e}$$

such that $s(\bar{e}) = t(e)$.

When Γ is a graph, we also write $\Gamma(0)$ for the set of vertices and $\Gamma^{(1)}$ for the set of edges.

Definition 6.2. A *weight* on a graph Γ is a function $w : \Gamma^{(1)} \to \mathbb{R}_0^+$. By *weighted graph* (Γ, w) we will mean a graph Γ equipped with a weight w.

By a *signed* graph we mean a graph equipped with a map $\mathrm{sgn} : \Gamma^{(1)} \to \{\pm 1\}$.

By a *signed weighted graph* we mean a signed graph equipped with a weight.

Definition 6.3. A *reciprocal weighted graph* (Γ, w, i) consists of a weighted graph (Γ, w) and an involution i of Γ such that $w(e)w(\bar{e}) = 1$ for all edges e.

A *positive (resp. negative) reciprocal signed graph* $(\Gamma, \mathrm{sgn}, i)$ consists of a signed graph (Γ, sgn) and an involution i of Γ such that $\mathrm{sgn}(e)\,\mathrm{sgn}(\bar{e}) = 1$ (resp. $= -1$) for all edges e.

By a *reciprocal signed weighted graph* we mean a signed and weighted graph which is reciprocal as a weighted and signed graph for the same involution.

A weighted graph (Γ, p) is called a *random walk* if $\sum_{s(e)=v} p(e) = 1$ for all $v \in \Gamma^{(0)}$.

Definition 6.4. Fix $t \in \mathbb{R}_0$. A *t-reciprocal random walk* is a $\mathrm{sgn}(t)$-reciprocal signed weighted graph $(\Gamma, w, \mathrm{sgn}, i)$ such that (Γ, p) is a random walk for $p(e) = \frac{1}{|t|}w(e)$.

Let $0 < |q| \leqslant 1$, and let $\Gamma = (\Gamma, w, \mathrm{sgn}, i)$ be a -2_q-reciprocal random walk. Then we can define a couple $\mathscr{H}(\Gamma) = (\mathscr{H}, R)$ where $\mathscr{H}$ is $l^2(\Gamma^{(1)})$ with the $\Gamma^{(0)}$-bigrading $\delta_e \in \mathscr{H}(s(e), t(e))$ for the obvious Dirac functions, and where R is the (bounded) map $l^2(\Gamma^{(0)}) \to \mathscr{H} \underset{\Gamma^{(0)}}{\boxtimes} \mathscr{H}$ defined as

$$R\delta_v \quad = \quad \sum_{e, s(e)=v} \mathrm{sgn}(e)\sqrt{w(e)}\,\delta_e \otimes \delta_{\bar{e}}.$$

Then $R^*R = |q| + |q|^{-1}$ and

$$(R^* \boxtimes \mathrm{id})(\mathrm{id} \boxtimes R) = -\,\mathrm{sgn}(q)\,\mathrm{id}\,.$$

By the fundamental property of the Temperley-Lieb tensor C*-category $\mathcal{T}_q = \mathrm{Rep}(SU_q(2))$, this means that we have a unique strongly monoidal *-functor

$$F_\Gamma : \mathcal{T}_q \to {}^{\Gamma^{(0)}}\mathrm{Hilb}_f^{\Gamma^{(0)}}$$

such that $\Gamma(\pi_{1/2}) = \mathscr{H}$ and $F(\mathscr{R}) = R$ with $(\pi_{1/2}, \mathscr{R}, -\,\mathrm{sgn}(q)\mathscr{R})$ a solution for the conjugate equations for $\pi_{1/2}$.

Up to equivalence, F_Γ only depends upon the isomorphism class of (Γ, w), and is independent of the chosen involution or sign structure.

It then follows from our main theorem that for each reciprocal random walk on a graph Γ, one obtains a $\Gamma^{(0)}$-face compact quantum group. Our aim is to give a direct representation of it by generators and relations.

Theorem 6.5. *Let $0 < |q| \leqslant 1$, and let Γ be a -2_q-reciprocal random walk. Let $A(\Gamma)$ be the total *-algebra associated to the $\Gamma^{(0)}$-face compact quantum group constructed from the fiber functor F_Γ. Then $A(\Gamma)$ is the universal *-algebra generated by a copy of the *-algebra of finite support functions on $\Gamma^{(0)} \times \Gamma^{(0)}$ and elements $(u_{e,f})_{e,f \in \Gamma^{(1)}}$ where*

$$u_{e,f} \in A(\Gamma) \begin{pmatrix} s(e) & t(e) \\ s(f) & t(f) \end{pmatrix} \ and$$

$$\sum_{v \in \Gamma^{(0)}} \sum_{g \in \Gamma_{vw}} u_{g,e}^* u_{g,f} = \delta_{e,f} \mathbf{1} \begin{pmatrix} w \\ t(e) \end{pmatrix}$$

$$\sum_{w \in \Gamma^{(0)}} \sum_{g \in \Gamma_{vw}} u_{e,g} u_{f,g}^* = \delta_{e,f} \mathbf{1} \begin{pmatrix} s(e) \\ v \end{pmatrix}$$

$$u_{e,f}^* = \mathrm{sgn}(e)\,\mathrm{sgn}(f)\sqrt{\frac{w(f)}{w(e)}} u_{\bar{e},\bar{f}}.$$

It is also easy to give formulas for the comultiplication, counit and antipode on these generators. Namely, if $K, L \in M_2(\Gamma^{(0)})$, we have for $K * L = \begin{pmatrix} s(e) & t(e) \\ s(f) & t(f) \end{pmatrix}$ that

$$\Delta \begin{pmatrix} K \\ L \end{pmatrix} (u_{e,f}) = \sum_{\substack{g \in \Gamma^{(1)} \\ t(g) = K_{rd} \\ s(g) = L_{lu}}} u_{e,g} \otimes u_{g,f}.$$

For $e, f \in \Gamma^{(1)}$ with the same source and target, we have

$$\varepsilon(s(e)\ t(e))(u_{e,f}) = \delta_{e,f}.$$

Finally, for the antipode we have

$$S(u_{e,f}) = u_{f,e}^*.$$

7 Generalized compact Hopf face algebras from Podleś spheres

As a particular case of the construction in the previous section, take $0 < |q| < 1$ and $x \in \mathbb{R}$, and consider the reciprocal -2_q-random walk $\Gamma_x = (\Gamma_x, w, \mathrm{sgn}, i)$ with $\Gamma^{(0)} = \mathbb{Z}$,

$$\Gamma^{(1)} = \{(k, l) \mid |k - l| = 1\} \subseteq \mathbb{Z} \times \mathbb{Z}$$

with projection on the first (resp. second) leg as source (resp. target) map, with weight function

$$w(k, k \pm 1) = \frac{|q|^{x+k\pm 1} + |q|^{-(x+k\pm 1)}}{|q|^{x+k} + |q|^{-(x+k)}},$$

sign function

$$\mathrm{sgn}(k, k+1) = 1, \quad \mathrm{sgn}(k, k-1) = -\,\mathrm{sgn}(q),$$

35

and involution $i(k, k+1) = (k+1, k)$. By translation we can shift the value of x by an integer, and by inversion we can change x into $-x$. It follows that we can always arrange to have $x \in [0, \frac{1}{2}]$.

In the following, let us denote

$$w_+(k) = w(k, k+1),$$
$$w_-(k) = w(k, k-1) = w_+(k-1)^{-1}.$$

Let $A_x = A(\Gamma_x)$ be the total *-algebra of the associated quantum groupoid. Using Theorem 6.5, we have the following presentation of A_x. Let B be the *-algebra of finite support functions on $\mathbb{Z} \times \mathbb{Z}$, whose Dirac functions we write as $1 \begin{pmatrix} k \\ l \end{pmatrix}$. Let $s_q = \frac{1}{2}(1 + \mathrm{sgn}(q))$. Then A_x is generated by a copy of B and elements

$$(u_{\epsilon,\nu})_{k,l} = u_{(k,k+\epsilon),(l,l+\nu)}$$

for $\epsilon, \nu \in \{-1, 1\} = \{-, +\}$ and $k, l \in \mathbb{Z}$ with defining relations

$$\sum_{\mu \in \{\pm\}} (u_{\mu,\epsilon})^*_{m-\mu,k}(u_{\mu,\nu})_{m-\mu,l} = \delta_{k,l}\delta_{\epsilon,\nu}1 \begin{pmatrix} m \\ k+\epsilon \end{pmatrix},$$

$$\sum_{\mu \in \{\pm\}} (u_{\epsilon,\mu})_{k,m}(u_{\nu,\mu})^*_{l,m} = \delta_{\epsilon,\nu}\delta_{k,l}1 \begin{pmatrix} k \\ m \end{pmatrix}$$

$$(u_{\epsilon,\nu})^*_{k,l} = (\epsilon\nu)^{s_q} \left(\frac{w_\nu(l)}{w_\epsilon(k)} \right)^{1/2} (u_{-\epsilon,-\nu})_{k+\epsilon,l+\nu}.$$

The element $(u_{\epsilon,\nu})_{k,l}$ then lives inside the component $A_x \begin{pmatrix} k & k+\epsilon \\ l & l+\nu \end{pmatrix}$.

Consider now $M(A_x)$, the multiplier algebra of A_x. Then we can form in $M(A_x)$ the elements $u_{\epsilon,\nu} = \sum_{k,l}(u_{\epsilon,\nu})_{k,l}$. Then $u = (u_{\epsilon,\nu})$ forms a unitary 2×2 matrix. Moreover,

$$u^*_{\epsilon,\nu} = (\epsilon\nu)^{s_q} u_{-\epsilon,-\nu} \frac{w_\nu^{1/2}(\rho)}{w_\epsilon^{1/2}(\lambda)},$$

where $w_\pm^{1/2}(k) = w_\pm(k)^{1/2}$ and where for a function f on $\mathbb{Z}$ we write $f(\lambda)(k,l) = f(k)$, $f(\rho)(k,l) = f(l)$. In the following, we then also use the notation $f(\lambda, \rho)$ for a function f on $\mathbb{Z} \times \mathbb{Z}$ interpreted as an element of $M(A)$, and for example $f(\lambda + 1, \rho)$ corresponds to the function $(k,l) \mapsto f(k+1, l)$. We then have the following commutation relations between functions on $\mathbb{Z} \times \mathbb{Z}$ and the entries of u:

$$f(\lambda, \rho)u_{\epsilon,\nu} = u_{\epsilon,\nu}f(\lambda - \epsilon, \rho - \nu).$$

$$u_{++} = \frac{w_+(\rho)^{1/2}}{w_+(\lambda)^{1/2}}u^*_{--},$$

$$u_{+-} = (-1)^{s_q}\frac{w^{1/2}_-(\rho)}{w^{1/2}_+(\lambda)}u^*_{-+}.$$

Let us write

$$F(k) = |q|^{-1}w_+(k) = |q|^{-1}\frac{|q|^{x+k+1} + |q|^{-x-k-1}}{|q|^{x+k} + |q|^{-x-k}},$$

and further put

$$\alpha = \frac{F^{1/2}(\rho-1)}{F^{1/2}(\lambda-1)}u_{--}, \qquad \beta = \frac{1}{F^{1/2}(\lambda-1)}u_{-+}.$$

Then the above commutation relations are equivalent to

$$\alpha\beta = qF(\rho-1)\beta\alpha \qquad\qquad \alpha\beta^* = qF(\lambda)\beta^*\alpha$$

$$\alpha\alpha^* + F(\lambda)\beta^*\beta = 1, \qquad \alpha^*\alpha + q^{-2}F(\rho-1)^{-1}\beta^*\beta = 1,$$
$$F(\rho-1)^{-1}\alpha\alpha^* + \beta\beta^* = F(\lambda-1)^{-1}, \qquad F(\lambda)\alpha^*\alpha + q^{-2}\beta\beta^* = F(\rho),$$

$$f(\lambda)g(\rho)\alpha = \alpha f(\lambda+1)g(\rho+1), \qquad f(\lambda)g(\rho)\beta = \beta f(\lambda+1)g(\rho-1).$$

These are precisely the commutation relations for the dynamical quantum $SU(2)$-group as in for example [KR], except that the precise value of F has been changed by a shift in the parameter domain by a complex constant. Clearly, by ... the (total) coproduct on A_x also agrees with the one on the dynamical quantum $SU(2)$-group, namely

$$\Delta(\alpha) = \alpha \underset{I^2}{\otimes} \alpha - q^{-1}\beta \underset{I^2}{\otimes} \beta^*,$$
$$\Delta(\beta) = \beta \underset{I^2}{\otimes} \alpha^* + \alpha \underset{I^2}{\otimes} \beta.$$

7.1 Representation theory of the function algebra on the dynamical quantum $SU(2)$-group

Let us classify the irreducible representations of A_x. The parametrisation will hinge on the classification of what we call irreducible (x, c)-admissible sets, which we will now discuss.

Let $x \in [0, \frac{1}{2}]$, and let $c \geqslant 0$. For $\epsilon \in \{\pm\}$, an integer $m \in \mathbb{Z}$ will be called $(x, c)_\epsilon$-adapted if

$$c \leqslant |q|^{2x+m-\epsilon} + |q|^{-2x-m+\epsilon}, \tag{7.1}$$

and *strongly* $(x, c)_\epsilon$-adapted if this holds strictly. An integer is called (x, c)-adapted if it is both $(x, c)_+$ and $(x, c)_-$-adapted.

A set of integers Z is called an (x, c)-set if the following conditions hold:

- Z is not empty.

- Z consists of (x, c)-adapted points.

- If $m \in Z$ is strongly $(x, c)_\epsilon$-adapted, then $m - 2\epsilon$ is in Z.

An (x, c)-set is called *irreducible* if it can not be written as the union of two (x, c)-sets.

Note that if Z is an irreducible $(\frac{1}{2}, c)$-set, then $Z + 1$ is an irreducible $(0, c)$-set. We can hence assume that $x \in [0, \frac{1}{2})$. Also remark that clearly any irreducible (x, c)-set consists completely of either even or odd integers.

First note now that if $c < q^{2x-1} + q^{-2x+1}$, then clearly $2\mathbb{Z}$ and $2\mathbb{Z} + 1$ are irreducible (x, c)-sets. If $c \geqslant |q|^{2x-1} + |q|^{-2x+1}$ we can uniquely write $c = |q|^y + |q|^{-y}$ for some $y \geqslant 1 - 2x$. We will treat the cases $x = 0$ and $x \neq 0$ separately.

First assume that $x = 0$, so that we may assume $y \geqslant 1$. Then it is easy to see that there are only (x, c)-sets if y is an integer. If $y = 1$, there are three irreducible (x, c)-sets $-2\mathbb{N}_0$, $\{0\}$ and $2\mathbb{N}_0$. If $y \geqslant 2$, there are the two irreducible (x, c)-sets $y + 1 + 2\mathbb{N}$ and $-y - 1 - 2\mathbb{N}$.

For $x \in (0, \frac{1}{2})$, it is again easily seen that we only have (x, c)-sets when y is of the form $y = |2x + M|$, where M is a uniquely determined integer. If M is positive, we only have one irreducible (x, c)-set $2\mathbb{N}_0 + M$. If M is negative and $M \neq -1$, we have only one irreducible (x, c)-set $M - 2\mathbb{N}$. If $M = -1$, we have two irreducible (x, c)-sets, namely $2\mathbb{N}$ and $-2\mathbb{N}_0$.

Let us now return to the representation theory of our quantum groupoid.

Let $(\mathscr{H}_\pi, \pi)$ be any (bounded) non-degenerate *-representation of A_x on a Hilbert space. Then $\mathscr{H}_\pi = \oplus \mathscr{H}_{m,n}$ with $\mathscr{H}_{m,n} = \pi(e\binom{m}{n})\mathscr{H}$. Let V_π be the non-closed linear span of all $\mathscr{H}_{m,n}$. Then $\pi(A)V_\pi = V_\pi$. It follows that one can extends π to a map $M(A) \to \mathrm{End}(V_\pi)$. As the $u_{\epsilon,\eta}$ form a unitary matrix, we can then in fact make sense of the $\pi(u_{\epsilon,\eta})$ as contractions on $\mathscr{H}_\pi$. On the other hand, the generators α, β and their adjoints give rise to endomorphisms $V_\pi \to V_\pi$ which are bounded when restricted to any $\mathscr{H}_{m,n}$. It is easy to see that non-degenerate *-representations of A_x are in one-to-one correspondence with $\mathbb{Z}^2$-direct sums V of finite-dimensional Hilbert spaces equipped with maps $\alpha, \beta : V_\pi \to V_\pi$ satisfying the commutation relations as in ...

Consider

$$\Omega = q^{\lambda-\rho+1} + q^{\rho-\lambda-1} - \mathrm{sgn}(q)^{\lambda-\rho} q^{-1}(|q|^{x+\lambda+1} + |q|^{-x-\lambda-1})(|q|^{x+\rho-1} + |q|^{-x-\rho+1})\beta^*\beta.$$

38

As in [KR], one shows that Ω is a central element. It then follows immediately that if π is an irreducible representation of A_x, there exists $c \in \mathbb{R}$ such that $\Omega\xi = c\xi$ for all $\xi \in V_\pi$. Moreover, since for any ϵ, ν one has that $u^*_{\epsilon,\nu}u_{\epsilon,\nu}$ can be expressed as an element of the form $g_{\epsilon,\nu}(\lambda, \rho) + h_{\epsilon,\nu}\Omega$, we easily deduce from the commutation relations and irreducibility that $\mathscr{H}_{m,n}$ is at most one-dimensional. Because of the commutation relations of the generators with the $f(\lambda, \mu)$ it is also clear that either all $\mathscr{H}_{m,n}$ with $m - n$ odd are zero, or all $\mathscr{H}_{m,n}$ with $m - n$ even are zero. In the first case we call π even, in the second case we call π odd.

Note now that we have the following identities (where the right hand sides are unambiguously defined because of centrality of Ω):

$$
\begin{aligned}
\alpha^*\alpha &= \frac{|q|^{2x+\lambda+\rho+1} + |q|^{-2x-\lambda-\rho-1} + \operatorname{sgn}(q)^{\lambda-\rho+1}\Omega}{(|q|^{x+\lambda+1} + |q|^{-x-\lambda-1})(|q|^{x+\rho} + |q|^{-x-\rho})} \\
\alpha\alpha^* &= \frac{|q|^{2x+\lambda+\rho-1} + |q|^{-2x-\lambda-\rho+1} + \operatorname{sgn}(q)^{\lambda-\rho-1}\Omega}{(|q|^{x+\lambda} + |q|^{-x-\lambda})(|q|^{x+\rho-1} + |q|^{-x-\rho+1})} \\
\beta^*\beta &= |q|\frac{|q|^{\lambda-\rho+1} + |q|^{-\lambda+\rho-1} - \operatorname{sgn}(q)^{\lambda-\rho+1}\Omega}{(|q|^{x+\lambda+1} + |q|^{-x-\lambda-1})(|q|^{x+\rho-1} + |q|^{-x-\rho+1})} \\
\beta\beta^* &= |q|\frac{|q|^{\lambda-\rho-1} + |q|^{-\lambda+\rho+1} - \operatorname{sgn}(q)^{\lambda-\rho-1}\Omega}{(|q|^{x+\lambda} + |q|^{-x-\lambda})(|q|^{x+\rho} + |q|^{-x-\rho})}.
\end{aligned}
$$

For $c \in \mathbb{R}$, let us call a couple $(k, l) \in \mathbb{Z}^2$ $(\epsilon, \nu)_c$ adapted if the following inequality holds:

$$
(|q|^{(x+k)+\epsilon\nu(x+l)-\epsilon} + |q|^{-(x+k)-\epsilon\nu(x+l)+\epsilon}) + \operatorname{sgn}(q)^{k-l+1}\epsilon\nu c \geqslant 0. \tag{7.2}
$$

Let us call (k, l) *strongly* $(\epsilon, \nu)_c$-adapted if this is a strict equality. Let us call (k, l) c-adapted if it is $(\epsilon, \nu)_c$-adapted for all ϵ, ν. Finally, let us call (k, l) c-compatible if there exists an irreducible representation π of A_x with $\pi(\Omega) = c$ and $\mathscr{H}_{k,l} \neq \{0\}$.

Let us call a subset $T \subseteq \mathbb{Z}^2$ a c-set if the following conditions are satisfied:

- T is not empty.

- T consists of c-adapted points.

- If $(k, l) \in T$ is strongly $(\epsilon, \nu)_c$-adapted, then $(k - \epsilon, l - \nu)$ is in T.

Call T irreducible if it is not the disjoint union of two c-sets. Let us write $\mathbb{Z}^2_{\text{even}} = \{(k, l) \mid k - l \text{ even}\}$ and $\mathbb{Z}^2_{\text{odd}} = \mathbb{Z}^2 \backslash \mathbb{Z}^2_{\text{even}}$, and call a c-set even or odd according to whether it lies in $\mathbb{Z}^2_{\text{even}}$ or $\mathbb{Z}^2_{\text{odd}}$. Then it is easily seen that for any even (resp. odd) irreducible representation π of A, the set of $c = \pi(\Omega)$-compatible (k, l) forms an irreducible even (resp. odd) c-set. Conversely, if T is an irreducible c-set, then necessarily T is either an even or odd c-set, and we can construct an irreducible even/odd representation of A on

$l^2(T)$ by putting

$$\alpha e_{k,l} = \left(\frac{|q|^{2x+k+l+1} + |q|^{-2x-k-l-1} + \operatorname{sgn}(q)^{k-l+1} c}{(|q|^{x+k+1} + |q|^{-x-k-1})(|q|^{x+l} + |q|^{-x-l})} \right)^{1/2} e_{k+1,l+1},$$

$$\beta e_{k,l} = \operatorname{sgn}(q)^k \left(|q| \frac{|q|^{k-l+1} + |q|^{-k+l-1} - \operatorname{sgn}(q)^{k-l+1} c}{(|q|^{x+k+1} + |q|^{-x-k-1})(|q|^{x+l-1} + |q|^{-x-l+1})} \right)^{1/2} e_{k+1,l-1},$$

where the right hand side is considered as the zero vector when the scalar factor on the right is zero. Moreover, this then establishes a one-to-one correspondence between irreducible c-sets and irreducible representations of A_x with $\pi(\Omega) = c$.

Hence what remains is to classify irreducible c-sets for each $c \in \mathbb{R}$. But clearly T is an even irreducible c-set if and only if there exists an even irreducible $(x, -\operatorname{sgn}(q)c)$-set $Z_+ \subseteq \mathbb{Z}$ and an even irreducible $(0, \operatorname{sgn}(q)c)$-set $Z_- \subseteq \mathbb{Z}$ such that $(k,l) \in T$ if and only if $k - l \in Z_-$ and $k + l \in Z_+$. Similary, T is an odd irreducible c-set if and only if there exists an odd irreducible $(x, -c)$-set $Z_+ \subseteq \mathbb{Z}$ and an irreducible $(0,c)$-set $Z_- \subseteq \mathbb{Z}$ such that $(k,l) \in T$ if and only if $k - l \in Z_-$ and $k + l \in Z_+$.

7.2 Representation theory of the intertwiner function algebra on the dynamical quantum $SU(2)$ group (to be modified)

Lemma 7.1. *There are faithful* **-representations* $\pi_\pm$ *of* $\mathrm{P_e}(\mathbb{X})$ *as operators* $\mathscr{D}^\pm \to \mathscr{D}^\pm$, *given by the following formulas (where we suppress the explicit notations* $\pi_\pm$):

$$\alpha \cdot e_{n,y}^+ = \left(\frac{1 + q^{2n-2y}}{1 + q^{-2y-2}} \right)^{1/2} e_{n,y+1}^+, \qquad \beta \cdot e_{n,y}^+ = \left(\frac{q^{-2y} - q^{2n-2y+2}}{1 + q^{-2y-2}} \right)^{1/2} e_{n+1,y+1}^+,$$

$$\alpha \cdot e_{n,y}^- = \left(\frac{1 - q^{2n}}{1 + q^{-2y-2}} \right)^{1/2} e_{n-1,y+1}^-, \qquad \beta \cdot e_{n,y}^- = \left(\frac{q^{2n+2} + q^{-2y}}{1 + q^{-2y-2}} \right)^{1/2} e_{n,y+1}^-,$$

the functions in $C_c(\mathbb{R})$ *simply acting by* $f e_{n,y}^\pm = f(y) e_{n,y}^\pm$.

Both representations are bounded when restricted to $\mathrm{P}(\mathbb{X})$.

7.3 Representation theory of the function algebra on the dynamical quantum $SU(2)$ group

Lemma 7.2. *There are faithful* **-representations* $\pi_\pm$ *of* $\mathrm{P_e}(\mathbb{X})$ *as operators* $\mathscr{D}^\pm \to \mathscr{D}^\pm$, *given by the following formulas (where we suppress the explicit notations* $\pi_\pm$):

$$\alpha \cdot e_{n,y}^+ = \left(\frac{1 + q^{2n-2y}}{1 + q^{-2y-2}} \right)^{1/2} e_{n,y+1}^+, \qquad \beta \cdot e_{n,y}^+ = \left(\frac{q^{-2y} - q^{2n-2y+2}}{1 + q^{-2y-2}} \right)^{1/2} e_{n+1,y+1}^+,$$

$$\alpha \cdot e_{n,y}^- = \left(\frac{1 - q^{2n}}{1 + q^{-2y-2}}\right)^{1/2} e_{n-1,y+1}^-, \qquad \beta \cdot e_{n,y}^- = \left(\frac{q^{2n+2} + q^{-2y}}{1 + q^{-2y-2}}\right)^{1/2} e_{n,y+1}^-,$$

the functions in $C_c(\mathbb{R})$ simply acting by $f e_{n,y}^\pm = f(y) e_{n,y}^\pm$.

Both representations are bounded when restricted to $\mathrm{P}(\mathbb{X})$.