Compact quantum groups of face type, ergodic actions of compact quantum groups and the dynamical quantum $SU(2)$ group

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Abstract

Compact quantum groups of face type, as introduced by Hayashi, form a class of quantum groupoids with a classical, finite set of objects. We generalize Hayashi's definition to allow for an infinite set of objects. We then show how any quantum homogeneous space of an ordinary compact quantum group leads to a compact quantum group of face type. In particular, when this construction is applied to the non-standard Podleś spheres, we obtain compact quantum groups of face type which are operator algebraic versions of the dynamical quantum $SU(2)$-group as studied by Etingof-Varchenko and Koelink-Rosengren.

1 Compact Hopf Face Algebras

1.1 Preliminaries

If I is a set, we write $\mathrm{Fun}_{\mathrm{f}}(I)$ for the algebra of *finitely supported* $\mathbb{C}$-valued functions on I, and $\mathrm{Fun}(I)$ for the algebra of *all* $\mathbb{C}$-valued functions on I. By I^n we denote the n-fold Cartesian product of I with itself. We will always suppose that I is at most countable.

By an algebra, we will understand an associative algebra A over $\mathbb{C}$ which does not necessarily contain a unit, but which has local units in the sense that for each finite set of elements $a_i \in A$, there exists $e \in A$ such that $a_i e = a_i = e a_i$ for all i. Then A embeds into its multiplier algebra $M(A)$, which is the subalgebra of $\mathrm{End}(A) \oplus \mathrm{End}(A)^{\mathrm{op}}$ consisting of linear endomorphisms $m = (L_m, R_m^{\mathrm{op}})$ for which $b(L_m a) = (R_m b)a$ for all $a, b \in A$. The embedding is by means of the map $a \mapsto (L_a, R_a^{\mathrm{op}})$ where L_a and R_a denote respectively left and right multiplication with a. In the following, we simply identify A as a subalgebra of $M(A)$. As an example, we have that $\mathrm{Fun}_{\mathrm{f}}(I)$ is an algebra with $M(\mathrm{Fun}_{\mathrm{f}}(I)) = \mathrm{Fun}(I)$.

If A and B are algebras, a homomorphism $f : A \to M(B)$ is called *non-degenerate* if $f(A)B = B = Bf(A)$. In this case, f extends uniquely to a unital homomorphism

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$M(A) \to M(B)$, which we will denote by the same symbol. We will call non-degenerate homomorphisms simply *morphisms.* A morphism f such that $f(A) = B$ will be called a *strong* morphism. Note that if f is a morphism, we can find local units for B in $f(A) \subseteq M(B)$.

Definition 1.1. Let I be a set. An algebra A is called an I-algebra if it comes equipped with a morphism

$$\mathrm{Fun}_{\mathrm{f}}(I) \to M(A).$$

It is called a *strong* I-algebra if this morphism is strong.

Note that by non-degeneracy, we obtain a map $\mathrm{Fun}(I) \to M(A)$. This implies that for any $k \in I$, we have idempotent mutually orthogonal elements e_k in $M(A)$ corresponding to the image of the functions

$$m \mapsto \delta_{k,m}.$$

Hence, again by non-degeneracy, A has a I^2-grading

$$A = \sum_{k,l}^{\oplus} {}^{k}A^{l}, \qquad {}^{k}A^{l} = e_k A e_l.$$

If A is a strong I-algebra, the ${}^{k}A^{k}$ will be unital with unit e_k, and the ${}^{k}A^{l}$ are (unital) ${}^{k}A^{k}$- ${}^{l}A^{l}$-bimodules.

Similarly, if A is an I^2-algebra, we have for any $k \in I$ elements λ_k and ρ_k in $M(A)$ corresponding respectively to the image of the functions

$$(m,n) \mapsto \delta_{k,m}, \qquad (m,n) \mapsto \delta_{k,n}.$$

This gives a I^4-grading on A by putting

$${}^{k}_{m}A^{l}_{n} = \lambda_k \rho_m A \lambda_l \rho_n.$$

It is then clear that when $m,n \in I$, the symbol ${}_{m}A_{n}$ will denote $\sum\limits_{m,n}^{\oplus} {}^{k}_{m}A^{l}_{n}$, etc.

Definition 1.2. If A is an I^2-algebra, we will say A is *faithful* if $\lambda_k \neq 0$ for all k, and similarly $\rho_m \neq 0$ for all m.

Definition 1.3. Let A, B be two I^2-algebras. We define $A \underset{I^2}{\otimes} B$ to be the subalgebra

$$\sum_{r,s}^{\oplus} {}_{r}A_{s} \otimes {}^{r}B^{s} \subseteq A \otimes B,$$

seen as an I^2-algebra by means of the map $\delta_{(k,l)} \mapsto \lambda_k \underset{I^2}{\otimes} \rho_l$, where the latter tensor is the restriction to $A \underset{I^2}{\otimes} B$ of the multiplier $\lambda_k \otimes \rho_l$ of $A \otimes B$.

Remark 1.4. Note that $A \underset{I^2}{\otimes} B$ will indeed again have local units and be faithful. The $\underset{I^2}{\otimes}$-product then clearly defines an associative product on I^2-algebras.

Remark 1.5. One has for $A \underset{I^2}{\otimes} B$ also the multiplier $\rho_p \underset{I^2}{\otimes} \lambda_p = \rho_p \underset{I^2}{\otimes} 1 = 1 \underset{I^2}{\otimes} \lambda_p$, which acts again by restriction of the corresponding multipliers on $A \otimes B$.

Remark 1.6. If A is an algebra, we can endow $M(A)$ with the topology for which $m_\alpha \to m$ if for any $a \in A$, we have eventually that $m_\alpha a = ma$ and $am_\alpha = am$. If then A and B are I^2-algebras, we can write $A \underset{I^2}{\otimes} B = E(A \otimes B)E$ where $E \in M(A \otimes B)$ is the idempotent multiplier $\sum_p \rho_p \otimes \lambda_p$, the sum obviously converging (over the net of all finite subsets of I) in the above topology.

Definition 1.7. Let A, B be two I-algebras. An I-morphism from A to B will mean a morphism $f : A \to M(B)$ intertwining the $\mathrm{Fun}_{\mathrm{f}}(I)$-morphisms.

Clearly, one can form tensor products $f \underset{I^2}{\otimes} g$ of I^2-morphisms, giving a morphism between $\underset{I^2}{\otimes}$-tensor products. The non-degeneracy of $f \underset{I^2}{\otimes} g$ can again be shown by a local unit argument.

1.2 Generalized Hopf face algebras

Definition 1.8. Let I be a set, and A an I^2-algebra. By an I^2-*coproduct* on A, we will mean a coassociative I^2-morphism $\Delta : A \to M(A \underset{I^2}{\otimes} A)$, i.e.

$$(\Delta \underset{I^2}{\otimes} \mathrm{id})\Delta = (\mathrm{id} \underset{I^2}{\otimes} \Delta)\Delta.$$

Remark 1.9. As Δ is an I^2-morphism, we have $\Delta(\lambda_k \rho_l) = \lambda_k \underset{I^2}{\otimes} \rho_l$. Hence if A is a *strong* I^2-algebra, this implies that for any $p \in I$ we have

$$(\rho_p \underset{I^2}{\otimes} 1)\Delta(A) = (1 \underset{I^2}{\otimes} \lambda_p)\Delta(A) \subseteq A \underset{I^2}{\otimes} A \qquad \text{and} \qquad \Delta(A)(\rho_p \underset{I^2}{\otimes} 1) = \Delta(A)(1 \underset{I^2}{\otimes} \lambda_p) \subseteq A \underset{I^2}{\otimes} A.$$

Remark 1.10. If A is a strong I^2-algebra, we have an inclusion $M(A \underset{I^2}{\otimes} A) \to M(A \otimes A)$ by means of

$$m \mapsto \sum_{k,l,p} m(\lambda_k \rho_p \underset{I^2}{\otimes} \lambda_p \rho_l) = \sum_{k,l,p} (\lambda_k \rho_p \underset{I^2}{\otimes} \lambda_p \rho_l)m.$$

Hence an I^2-coproduct Δ then 'co-extends' to an algebra homomorphism

$$\widetilde{\Delta} : A \to M(A \otimes A)$$

such that

$$\widetilde{\Delta}(A)(A \otimes A) = E(A \otimes A), \qquad (A \otimes A)\widetilde{\Delta}(A) = (A \otimes A)E,$$

where E again denotes the element $\sum_p \rho_p \otimes \lambda_p$. This $\widetilde{\Delta}$ then extends (uniquely) to an algebra homomorphism $M(A) \to M(A \otimes A)$ such that $\widetilde{\Delta}(1) = E$, cf. [Van Daele and Wang; Böhm et al]. By definition, we have

$$\widetilde{\Delta}(a)(\lambda_k \rho_m \otimes \lambda_l \rho_n) = \delta_{m,l} \Delta(a)(\lambda_k \rho_m \underset{I^2}{\otimes} \lambda_m \rho_n), \qquad (\lambda_k \rho_m \otimes \lambda_l \rho_n)\widetilde{\Delta}(a) = \delta_{m,l}(\lambda_k \rho_m \underset{I^2}{\otimes} \lambda_m \rho_n)\Delta(a)$$

for all $a \in A$ and all $k, l, m, n \in I$. This implies that

$$\Delta(a)(1 \underset{I^2}{\otimes} \lambda_l \rho_n) = \widetilde{\Delta}(a)(1 \otimes \lambda_l \rho_n)$$

for all $a \in A$ and $l, n \in I$, and similarly for other expressions of this kind. Hence we see that

$$\widetilde{\Delta}(A)(1 \otimes A) \cup \widetilde{\Delta}(A)(A \otimes 1) \cup (A \otimes 1)\widetilde{\Delta}(A) \cup (1 \otimes A)\widetilde{\Delta}(A) \subseteq A \otimes A.$$

Definition 1.11. A *generalized face algebra over I* consists of a strong I^2-algebra A together with an I^2-coproduct and a functional $\varepsilon : A \to \mathbb{C}$ such that

$$(\mathrm{id} \otimes \varepsilon)((a \otimes 1)\widetilde{\Delta}(b)(c \otimes 1)) = abc = (\varepsilon \otimes \mathrm{id})((1 \otimes a)\widetilde{\Delta}(b)(1 \otimes c))$$

for all $a, b, c \in A$, and

$$(\varepsilon \otimes \varepsilon)((a \otimes 1)\widetilde{\Delta}(b)(1 \otimes c)) = \varepsilon(abc) = (\varepsilon \otimes \varepsilon)((1 \otimes a)\widetilde{\Delta}(b)(c \otimes 1)) \tag{1.1}$$

for all $a, b, c \in A$.

We will use the Sweedler notation for the coproduct $\widetilde{\Delta}$ from now on.

Definition 1.12. Let I be a set. A *generalized Hopf face algebra over I* is a generalized face algebra (A, Δ) admitting an invertible anti-homomorphism $S : A \to A$ such that $S(\lambda_k \rho_m) = \lambda_m \rho_k$ for all $k, m \in I$ and such that

$$S(a_{(1)})a_{(2)}b = \sum_n \varepsilon(a\lambda_n)\rho_n b, \qquad ba_{(1)}S(a_{(2)}) = \sum_k \varepsilon(\rho_k a)b\lambda_k$$

for all $a, b \in A$.

Remark 1.13. Imposing the invertibility of S corresponds to the *regularity* condition in the case of multiplier Hopf algebras, see [VDae]. We refrain from seeing it as a separate condition as it will be satisfied automatically in the examples under consideration.

Remark 1.14. If (A, Δ) is a generalized Hopf face algebra, then also $(A^{\mathrm{op}}, \Delta)$ and $(A, \Delta^{\mathrm{op}})$ are (the embeddings of $\mathrm{Fun_f}(I^2)$ being unchanged), with the same counit maps and their antipodes being given by S^{-1}.

Lemma 1.15. *Let (A, Δ) be a generalized Hopf face algebra over I. Then $\varepsilon(\lambda_k \rho_m) = \delta_{k,m}$.*

Proof. As $\widetilde{\Delta}(\lambda_k \rho_m) = \sum_p \lambda_k \rho_p \otimes \lambda_p \rho_m$, we find that $\lambda_k \rho_m = \sum_p \varepsilon(\lambda_p \rho_m)\lambda_k \rho_p$. Hence if $p \neq m$, we have $\varepsilon(\lambda_p \rho_m)\lambda_k \rho_p = 0$. Applying S, we see that $\varepsilon(\lambda_p \rho_m) = 0$.

We conclude that $\lambda_k \rho_m = \varepsilon(\lambda_m \rho_m)\lambda_k \rho_m$. As $\rho_m \neq 0$, varying k shows that necessarily $\varepsilon(\lambda_m \rho_m) = 1$. □

Lemma 1.16. *For all $a \in A$ and $m, n \in I$, we have $\varepsilon(a\lambda_k) = \varepsilon(a\rho_k)$ and $\varepsilon(\lambda_l a) = \varepsilon(\rho_l a)$.*

Proof. From the axiom 1.1, it follows that $\sum_p \varepsilon(a\rho_p)\varepsilon(\lambda_p b) = \varepsilon(ab)$ for all $a, b \in A$. Taking $b = \lambda_k \rho_m$, we find that $\varepsilon(a\rho_k)\varepsilon(\lambda_k\rho_m) = \varepsilon(a\lambda_k\rho_m)$. Taking $m = k$ and applying the previous lemma, we find $\varepsilon(a\rho_k) = \varepsilon(a\lambda_k\rho_k)$, implying one half of the lemma. The other half is proven similarly. $\square$

Definition 1.17. Define

$$V : \sum_n^{\oplus} A_n \otimes {}_n A \to \sum_r^{\oplus} {}_r A \otimes {}^r A$$

by the formula

$$a \otimes b \to \tilde{\Delta}(a)(1 \otimes b).$$

Lemma 1.18. *The map V is an isomorphism.*

Proof. As $\tilde{\Delta}(A)(A \otimes A) \subseteq E(A \otimes A)$ with $E = \sum_p \rho_p \otimes \lambda_p$, it is clear that V has the proper range. Define

$$\tilde{V} : \sum_r^{\oplus} {}_r A \otimes {}^r A \to \sum_n^{\oplus} A_n \otimes {}_n A$$

by means of the formula

$$a \otimes b \mapsto a_{(1)} \otimes S(a_{(2)})b = a_{(1)} \otimes S(S^{-1}(b)a_{(2)}).$$

By the defining property of S, we find that for all $a, b, c \in A$, we have

$$(c \otimes 1) \cdot (\tilde{V}V)(a \otimes b) = \sum_p ca_{(1)}\varepsilon(a_{(2)}\lambda_p) \otimes \rho_p b.$$

By the previous lemma, this equals $\sum_p ca_{(1)}\varepsilon(a_{(2)}\rho_p) \otimes \rho_p b$. But as $a \otimes b = \sum_p a\rho_p \otimes \rho_p b$ by assumption, we obtain that

$$(c \otimes 1) \cdot (\tilde{V}V)(a \otimes b) = ca_{(1)}\varepsilon(a_{(2)}) \otimes b = ca \otimes b,$$

proving that $\tilde{V}V(a \otimes b) = a \otimes b$.

The identity $V\tilde{V} = \mathrm{id}$ is proven similarly. $\square$

Corollary 1.19. *Define*

$$W : \sum_n^{\oplus} {}_n A \otimes {}^n A \to \sum_r^{\oplus} {}^r A \otimes A^r$$

by the formula

$$a \otimes b \to S^{-1}(b_{(1)})a \otimes d_{(2)} = S^{-1}(S(a)b_{(1)}) \otimes b_{(2)}.$$

Then W is invertible, its inverse being given as

$$W^{-1}(a \otimes b) = \tilde{\Delta}(b)(a \otimes 1).$$

Proof. Apply the previous Lemma to $(A, \Delta^{\mathrm{op}})$. $\square$

1.3 Invariant functionals

Definition 1.20. Let I be a set. An *invariant functional* for a generalized Hopf face algebra (A, Δ) over I is a functional $\varphi : A \to \mathbb{C}$ such that for all $a \in A$, we have the identity of multipliers

$$(\mathrm{id} \otimes \varphi)\widetilde{\Delta}(a) = \sum_p \varphi(\lambda_p a)\lambda_p, \qquad (\varphi \otimes \mathrm{id})\widetilde{\Delta}(a) = \sum_p \varphi(a\rho_p)\rho_p.$$

We say that φ is normalized if $\varphi(\lambda_k \rho_l) = 1$ for all $k, l \in I$ with $\lambda_k \rho_l \neq 0$.

Lemma 1.21. *An invariant normalized functional φ is faithful, i.e. $\varphi(ab) = 0$ for all b implies $b = 0$, and $\varphi(ab) = 0$ for all a implies $b = 0$.*

Proof. We follow ad verbatim the proof of Proposition 3.4 in [VDae, Algebraic framework]: if $\varphi(ba) = 0$ for all a, we arrive at the conclusion that for all $d \in A$ and all functionals ω on A, the element $p = (\omega \otimes \mathrm{id})((d \otimes 1)\widetilde{\Delta}(a))$ satisfies $(\mathrm{id} \otimes \varphi)((1 \otimes c)\widetilde{\Delta}(p)) = 0$. Continuing as in that proof, we obtain from the antipode trick that $\sum_n \varphi(cS(q)\rho_n)\varepsilon(p\lambda_n) = 0$. Choosing now for c and q local units of the form $\lambda_k \rho_l$, the normalization condition on φ gives that $\varepsilon(p\lambda_n) = 0$ for all n, hence $\varepsilon(p) = 0$. This implies $\omega(da) = 0$. As ω and d were arbitrary, it follows that $a = 0$.

The other case follows similarly, considering the opposite algebra. □

Our next aim is to prove that a normalized invariant functional is modular, that is, there exists an automorphism $\sigma : A \to A$ such that for all $a, b \in A$, we have

$$\varphi(ba) = \varphi(a\sigma(b)).$$

Lemma 1.22. *Let φ be a normalized invariant functional. For all $a \in A$ and $k, m \in I$, we have*

$$\varphi(a\lambda_k) = \varphi(\lambda_k a), \qquad \varphi(a\rho_m) = \varphi(\rho_m a).$$

Proof. □

1.4 Generalized compact Hopf face algebras

A non-degenerate algebra A is called a *-algebra if it comes equipped with an anti-linear involutive anti-homomorphism $A \to A, a \mapsto a^*$. In this case, $M(A)$ becomes a *-algebra in a natural way. For example, we always consider $\mathrm{Fun_f}(I)$ as a *-algebra by the ordinary complex conjugation of functions, $f^*(k) = \overline{f(k)}$.

Definition 1.23. A couple (A, Δ) consisting of a generalized Hopf face *-algebra with an invertible antipode invariant normalized functional φ is called a *generalized compact face algebra.*

One proves that a generalized Hopf face *-algebra has $S(S(x)^*)^* = x$ for all x, so S is automatically invertible.

2 Representation theory

2.1 Corepresentations of generalized compact Hopf face algebras

Let (A, Δ) be a generalized compact Hopf face algebra over an index set I.

Lemma 2.1. *Let $\mathcal{H} = \bigoplus_{k,l} {}^{k}\mathcal{H}^{l}$ be a row- and column-finite I^2-graded Hilbert space and let $X = ({}^{k}_{m}X^{l}_{n})_{k,l,m,n}$ be a family of elements ${}^{k}_{m}X^{l}_{n} \in {}^{k}_{m}A^{l}_{n} \otimes \mathcal{B}({}^{m}\mathcal{H}^{n}, {}^{k}\mathcal{H}^{l})$ satisfying*

$$(\tilde{\Delta} \otimes \mathrm{id})({}^{k}_{m}X^{l}_{n}) = \sum_{p,q} \left({}^{k}_{p}X^{l}_{q}\right)_{13} ({}^{p}_{m}X^{q}_{n})_{23}. \tag{2.1}$$

Then the following conditions are equivalent:

1. $(\epsilon \otimes \mathrm{id})({}^{k}_{m}X^{l}_{n}) = \delta_{k,m}\delta_{l,n} \mathrm{id}_{\,{}^{k}\mathcal{H}^{l}}$*;*
2. *there exist elements ${}^{k}_{m}Z^{l}_{n} \in {}^{n}_{l}A^{m}_{k} \otimes \mathcal{B}({}^{m}\mathcal{H}^{n}, {}^{k}\mathcal{H}^{l})$ such that*

$$\sum_{k} {}^{m}_{k}Z^{n'}_{l}\, {}^{k}_{m}X^{l}_{n} = \delta_{n,n'}\lambda_l\rho_n \otimes \mathrm{id}_{\,{}^{m}\mathcal{H}^{n}}, \quad \sum_{n} {}^{k}_{m}X^{l}_{n}\, {}^{m}_{k'}Z^{n}_{l} = \delta_{k,k'}\lambda_k\rho_m \otimes \mathrm{id}_{\,{}^{k}\mathcal{H}^{l}}.$$

If these conditions hold, then the family $X^{-1} := Z$ is unique and given by ${}^{k}_{m}(X^{-1})^{l}_{n} = (S \otimes \mathrm{id})({}^{k}_{m}X^{l}_{n})$. In particular, it satisfies

$$(\tilde{\Delta} \otimes \mathrm{id})({}^{k}_{m}(X^{-1})^{l}_{n}) = \sum_{p,q} \left({}^{k}_{m}(X^{-1})^{l}_{n}\right)_{23} ({}^{m}_{p}(X^{-1})^{n}_{q})_{13}.$$

Note that the sum in (2.1) makes sense in the multiplier algebra, and that the sums in (2) are finite because $\mathcal{H}$ is row- and column-finite. If it exists, we denote the family Z in condition (2) by X^{-1}.

Proof. Assume that (1) holds and let ${}^{k}_{m}Z^{l}_{n} = (S \otimes \mathrm{id})({}^{k}_{m}X^{l}_{n})$. Then the antipode axiom implies

$$\begin{aligned}
\sum_{k} {}^{m}_{k}Z^{n'}_{l}\, {}^{k}_{m}X^{l}_{n} &= \sum_{k} (S \otimes \mathrm{id})({}^{m}_{k}X^{n'}_{l})\, {}^{k}_{m}X^{l}_{n} \\
&= \sum_{k} (m_A \otimes \mathrm{id})(S \otimes \mathrm{id} \otimes \mathrm{id})(({}^{m}_{k}X^{n'}_{l})_{13}({}^{k}_{m}X^{l}_{n})_{23}) \\
&= (m_A \otimes \mathrm{id})(S \otimes \mathrm{id} \otimes \mathrm{id})((\tilde{\Delta} \otimes \mathrm{id})({}^{m}_{m}X^{n'}_{n})(1 \otimes \lambda_l \otimes 1)) \\
&= \sum_{p} \rho_p\lambda_l \otimes (\epsilon(-\lambda_p) \otimes \mathrm{id})({}^{m}_{m}X^{n'}_{n}) \\
&= \rho_n\lambda_l \otimes \delta_{n',n} \mathrm{id}_{\,{}^{m}\mathcal{H}^{n}},
\end{aligned}$$

which is the first equation in (2). The second one follows similarly.

Conversely, assume that (2) holds. Then uniqueness of the family Z is easily verified. Let ${}^{k}_{m}c^{l}_{n} := (\epsilon \otimes \mathrm{id})({}^{k}_{m}X^{l}_{n})$. If $(k,l) \neq (m,n)$, then ${}^{k}_{m}c^{l}_{n} = 0$ because $\epsilon({}^{k}_{m}A^{l}_{n}) = 0$. Relation (2.1) and the counit property imply

$$\begin{aligned}
{}^{k}_{m}X^{l}_{n} &= (\lambda_k \otimes 1)\,{}^{k}_{m}X^{l}_{n}(\lambda_l \otimes 1) \\
&= (\epsilon \otimes \mathrm{id} \otimes \mathrm{id})(((1 \otimes \lambda_k \otimes 1)(\hat{\Delta} \otimes \mathrm{id})({}^{k}_{m}X^{l}_{n})(1 \otimes \lambda_l \otimes 1)) \\
&= (\epsilon \otimes \mathrm{id} \otimes \mathrm{id})\left(\left({}^{k}_{k}X^{l}_{l}\right)_{13}\left({}^{k}_{m}X^{l}_{n}\right)_{23}\right) \\
&= (1 \otimes {}^{k}_{k}c^{l}_{l})\,{}^{k}_{m}X^{l}_{n}.
\end{aligned}$$

We multiply on the right by ${}^{m}_{k}Z^{n}_{l}$, sum over n, use condition (2) and find ${}^{k}_{k}c^{l}_{l} = \mathrm{id}_{{}^{k}\mathcal{H}^{l}}$. □

A *locally finite corepresentation of* (A,Δ) is a pair $(\mathcal{H}, X)$ as above satisfying conditions (1) and (2). A *morphism* T between two such corepresentations $(\mathcal{H}, X)$ and $(\mathcal{K}, Y)$ is a family of morphisms ${}^{k}T^{l} \in \mathcal{B}({}^{k}\mathcal{H}^{l}, {}^{k}\mathcal{K}^{l})$ satisfying $(1 \otimes {}^{k}T^{l})\,{}^{k}_{m}X^{l}_{n} = {}^{k}_{m}Y^{l}_{n}(1 \otimes {}^{m}T^{n})$. Using condition (2) in Lemma 2.1, one finds that the last equation holds if and only if

$$\sum_{m} {}^{k}_{m}(Y^{-1})^{l'}_{n}(1 \otimes {}^{m}T^{n})\,{}^{m}_{k}X^{n}_{l} = \delta_{l,l'}\lambda_n\rho_l \otimes {}^{k}T^{l}, \tag{2.2}$$

$$\sum_{n} {}^{k'}_{m}Y^{l}_{n}(1 \otimes {}^{m}T^{n})\,{}^{m}_{k}(X^{-1})^{n}_{l} = \delta_{k,k'}\lambda_m\rho_k \otimes {}^{k}T^{l}. \tag{2.3}$$

For example, if the first equation holds, then

$${}^{m}_{k}Y^{n}_{l}(1 \otimes {}^{k}T^{l}) = \sum_{m} {}^{m}_{k}Y^{n}_{l}\,{}^{k}_{m}(Y^{-1})^{l}_{n}(1 \otimes {}^{m}T^{n})\,{}^{m}_{k}X^{n}_{l} = (\lambda_k\rho_m \otimes {}^{m}T^{n})\,{}^{m}_{k}X^{n}_{l}.$$

Conversely, if T is a morphism, then

$$\sum_{m} {}^{k}_{m}(Y^{-1})^{l'}_{n}(1 \otimes {}^{m}T^{n})\,{}^{m}_{k}X^{n}_{l} = \sum_{m} {}^{k}_{m}(Y^{-1})^{l'}_{n}\,{}^{m}_{k}Y^{n}_{l}(1 \otimes {}^{k}T^{l}) = \delta_{l',l}\lambda_n\rho_l \otimes {}^{k}T^{l}.$$

We denote by $\mathrm{Corep_f}(A,\Delta)$ the category of locally finite corepresentations of (A,Δ) with morphisms as above.

Let $(\mathcal{H}, X)$ be a locally finite corepresentation. We call a family of subspaces ${}^{k}\mathcal{K}^{l} \subseteq {}^{k}\mathcal{H}^{l}$ *invariant* if ${}^{k}_{m}X^{l}_{n}(1 \otimes {}^{m}P^{n}) = (1 \otimes {}^{k}P^{l})\,{}^{k}_{m}X^{l}_{n}$, where ${}^{k}P^{l}$ denotes the projection ${}^{k}\mathcal{H}^{l} \to {}^{k}\mathcal{K}^{l}$.

The following analogue of Schur's Lemma holds. Let T be a morphism of locally finite corepresentations $(\mathcal{H}, X)$ and $(\mathcal{K}, Y)$. Then $\bigoplus_{k,l} \ker\, {}^{k}T^{l}$ and $\bigoplus_{k,l} \mathrm{img}\, {}^{k}T^{l}$ are invariant subspaces of $\mathcal{H}$ and $\mathcal{K}$, respectively. In particular, if $(\mathcal{H}, X)$ and $(\mathcal{K}, Y)$ are irreducible, then T is either 0 or an isomorphism.

Unitarity of corepresentations We call a locally finite corepresentation $(\mathcal{H}, X)$ *unitary* if ${}^{k}_{m}(X^{-1})^{l}_{n} = ({}^{m}_{k}X^{n}_{l})^{*}$. In this paragraph, we assume that (A, Δ) has a faithful positive functional, and show that then every irreducible locally finite-dimensional corepresentation is equivalent to a unitary one, by embedding it into a restriction of the regular corepresentation.

Example 2.2. Assume that (A, Δ) has a positive invariant functional ϕ that is faithful.

Let ${}^{k}\mathcal{H}^{l} \subseteq \bigoplus_{m,n} {}^{m}_{k}A^{n}_{l}$ be a row- and column-finite family of finite-dimensional subspaces satisfying

$$\tilde{\Delta}^{\mathrm{co}}({}^{m}\mathcal{H}^{n}) \subseteq \sum_{p,q} {}^{p}_{m}A^{q}_{n} \otimes {}^{p}\mathcal{H}^{q}.$$

Equip each ${}^{k}\mathcal{H}^{l}$ with the scalar product $\langle a|b\rangle := \phi(a^{*}b)$ and take the Hilbert space direct sum $\mathcal{H} := \bigoplus_{k,l} {}^{k}\mathcal{H}^{l}$. Define ${}^{k}_{m}V^{l}_{n} \in {}^{k}_{m}A^{l}_{n} \otimes \mathcal{B}({}^{m}\mathcal{H}^{n}, {}^{k}\mathcal{H}^{l})$ by the equation

$${}^{k}_{m}V^{l}_{n}|a\rangle_{2} = \tilde{\Delta}^{\mathrm{co}}(a),$$

where ${}^{k}_{m}V^{l}_{n}|a\rangle_{2}$ denotes the application of the second leg of ${}^{k}_{m}V^{l}_{n}$ to $a \in {}^{m}\mathcal{H}^{n}$. Then $(\mathcal{H}, V)$ is a unitary corepresentation. We call it the *locally finite restriction of the regular corepresentation* determined by the family $({}^{k}\mathcal{H}^{l})_{k,l}$.

Lemma 2.3. *Assume that (A, Δ) has a faithful and positive invariant functional. Let $(\mathcal{H}, X)$ be a locally finite-dimensional corepresentation and let $\xi \in {}^{k}\mathcal{H}^{l}$. Then the family of finite-dimensional subspaces*

$${}^{m}\mathcal{K}^{n} = \{(\mathrm{id} \otimes \omega_{\xi,\eta})({}^{k}_{m}X^{l}_{n}) : \eta \in {}^{m}\mathcal{H}^{n}\} \subseteq {}^{k}_{m}A^{l}_{n}$$

defines a restriction of the regular corepresentation $(\mathcal{K}, V)$, and the family of maps

$${}^{m}T^{n}_{(\xi)} \colon {}^{m}\mathcal{H}^{n} \to {}^{m}\mathcal{K}^{n}, \ \eta \mapsto (\mathrm{id} \otimes \omega_{\xi,\eta})({}^{k}_{m}X^{l}_{n}),$$

is a morphism from $(\mathcal{H}, X)$ to $(\mathcal{K}, V)$.

Note that the family $({}^{m}\mathcal{K}^{n})_{m,n}$ is row- and column-finite because $({}^{m}\mathcal{H}^{n})_{m,n}$ is.

Proof. Both assertions follow from the fact that for all $\eta \in {}^{p}\mathcal{H}^{q}$,

$$\begin{aligned}(1 \otimes \rho_{m}) \cdot \tilde{\Delta}^{\mathrm{co}}\big((\mathrm{id} \otimes \omega_{\xi,\eta})({}^{k}_{p}X^{l}_{q})\big) \cdot (1 \otimes \rho_{n}) &= (\mathrm{id} \otimes \mathrm{id} \otimes \omega_{\xi,\eta})\big(({}^{k}_{m}X^{l}_{n})_{23}({}^{m}_{p}X^{n}_{q})_{13}\big) \\ &= (1 \otimes {}^{m}T^{n}_{(\xi)})\, {}^{m}_{p}X^{n}_{q}|\eta\rangle_{2}.\end{aligned}$$

□

Proposition 2.4. *Assume that (A, Δ) has a faithful and positive invariant functional. Then every irreducible locally finite corepresentation is equivalent to a unitary one.*

Proof. Let $(\mathcal{H}, X)$ be an irreducible locally finite corepresentation. Then for some k, l and $\xi \in {}^{k}\mathcal{H}^{l}$, the operator $T_{(\xi)}$ defined in Lemma 2.3 has to be non-zero and hence, by Schur's Lemma, injective. Thus, it forms an equivalence between $(\mathcal{H}, X)$ and a subcorepresentation of a locally finite restriction of the regular corepresentation, which is unitary by Example 2.2. □

Schur orthogonality relations In this paragraph, we obtain the analogue of Schur's orthogonality relations for matrix coefficients of corepresentations.

The space of *matrix coefficients* $\mathcal{C}(X)$ of a locally finite corepresentation $(\mathcal{H}, X)$ is the sum of the subspaces

$$ {}_m^k\mathcal{C}(X)_n^l = \Big\{ (\mathrm{id} \otimes \omega_{\xi,\eta})({}_m^kX_n^l) \mid \xi \in {}^k\mathcal{H}^l, \eta \in {}^m\mathcal{H}^n \Big\} \subseteq {}_m^kA_n^l. $$

Proposition 2.5. *Assume that (A, Δ) has a faithful and positive invariant functional. Then A is the sum of the matrix coefficients of unitary irreducible locally finite corepresentations.*

Proof. Let $a \in {}_m^kA_n^l$. Then $(1 \otimes \rho_p)\tilde{\Delta}^{\mathrm{co}}(a)(1 \otimes \rho_q) \in {}_m^pA_n^q \otimes {}_p^kA_q^l$ and the subspace

$$ {}^p\mathcal{H}^q := \{ (\omega \otimes \mathrm{id})(\tilde{\Delta}^{\mathrm{co}}(a)) : \omega \in ({}_m^pA_n^q)' \} \subseteq {}_p^kA_q^l $$

has finite dimension. Since $(1 \otimes \rho_p)\tilde{\Delta}^{\mathrm{co}}(a)$ and $\tilde{\Delta}^{\mathrm{co}}(a)(1 \otimes \rho_q)$ lie in the tensor product $A \otimes A$, the family $({}^p\mathcal{H}^q)_{p,q}$ is row- and column-finite. Using co-associativity, one checks that this family defines a locally finite restriction $(\mathcal{H}, V)$ of the regular corepresentation. Evidently, $a \in \mathcal{C}(V)$. Decomposing $(\mathcal{H}, V)$, we find that a is contained in the sum of matrix coefficients of unitary irreducible corepresentations. □

The key to the orthogonality relations is the following averaging procedure.

Lemma 2.6. *Let ϕ be an invariant functional for (A, Δ), let $(\mathcal{H}, X)$ and $(\mathcal{K}, Y)$ be locally finite-dimensional corepresentations of (A, Δ) and let T be a family of operators ${}^kT^l \in \mathcal{B}({}^k\mathcal{H}^l, {}^k\mathcal{K}^l)$ which are 0 for all but finitely many k, l. Then the families $\check{T}$ and $\hat{T}$ given by*

$$ {}^k\check{T}^l := \sum_{m,n} (\phi \otimes \mathrm{id})({}_m^k(Y^{-1})_n^l (1 \otimes {}^mT^n) {}_k^mX_l^n), $$
$$ {}^k\hat{T}^l := \sum_{m,n} (\phi \otimes \mathrm{id})({}_m^kY_n^l (1 \otimes {}^mT^n) {}_k^m(X^{-1})_l^n) $$

are morphisms from $(\mathcal{H}, X)$ to $(\mathcal{K}, Y)$.

Proof. The assertion converning $\check{T}$ follows from the calculation

$$ \begin{aligned} &\sum_m {}_m^k(Y^{-1})_n^{l'} (1 \otimes {}^m\check{T}^n) {}_k^mX_l^n = \\ &= \sum_{m,p,q} (\phi \otimes \mathrm{id} \otimes \mathrm{id}) \Big(\big({}_m^k(Y^{-1})_n^{l'} \big)_{23} \big({}_p^m(Y^{-1})_q^n \big)_{13} (1 \otimes 1 \otimes {}^pT^q) ({}_m^pX_n^q)_{13} ({}_k^mX_l^n)_{23} \Big) \\ &= \sum_{p,q} (\lambda_n \otimes 1) \cdot ((\phi \otimes \mathrm{id}) \circ \tilde{\Delta} \otimes \mathrm{id}) \Big({}_p^k(Y^{-1})_q^{l'} (1 \otimes {}^pT^q) {}_k^pX_l^q \Big) \cdot (\lambda_n \otimes 1) \\ &= \sum_{r,p,q} (\lambda_n \otimes 1) \cdot \Big(\rho_r \otimes (\phi \otimes \mathrm{id}) \Big({}_p^k(Y^{-1})_q^{l'} (1 \otimes {}^pT^q) {}_k^pX_l^q (\rho_r \otimes 1) \Big) \Big) \cdot (\lambda_n \otimes 1) \\ &= \delta_{l,l'} \lambda_n \rho_l \otimes {}^k\check{T}^l, \end{aligned} $$

where we used the relation $\phi({}_{l'}A_l) = 0$ for $l' \neq l$ for the last equality. A similar calculation proves the assertion concerning $\hat{T}$. $\square$

The first part of the orthogonality relations concerns matrix coefficients of inequivalent irreducible corepresentations.

Proposition 2.7. *Let $(\mathcal{H}, X)$ and $(\mathcal{K}, Y)$ be inequivalent unitary irreducible locally finite-dimensional corepresentations and let ϕ be an invariant functional for (A, Δ). Then $\phi(S(b)a) = \phi(b^*a) = \phi(bS(a)) = \phi(ba^*) = 0$ for all $a \in \mathcal{C}(X), b \in \mathcal{C}(Y)$.*

Proof. Let $a = (\mathrm{id} \otimes \omega_{\xi,\xi'})({}^{k}_{m}X^{l}_{n})$ and $b = (\mathrm{id} \otimes \omega_{\eta,\eta'})({}^{p}_{r}Y^{q}_{s})$, where $\xi \in {}^{k}\mathcal{H}^{l}, \xi' \in {}^{m}\mathcal{H}^{n}$ and $\eta \in {}^{p}\mathcal{K}^{q}, \eta' \in {}^{r}\mathcal{K}^{s}$. We may assume $(p, q, r, s) = (m, n, k, l)$ because $\phi(S(a)b) = 0 = \phi(aS(b))$ otherwise. Lemma 2.6, applied to the family ${}^{p}T^{q} = \delta_{p,k}\delta_{q,l}|\eta'\rangle\langle\xi|$, yields morphisms $\check{T}, \hat{T}$ from $(\mathcal{H}, X)$ to $(\mathcal{K}, Y)$ which necessarily are 0. Inserting the definition of $\check{T}$, we find

$$\begin{aligned}
\phi(S(b)a) &= \phi\big((S \otimes \omega_{\eta,\eta'})({}^{m}_{k}Y^{n}_{l}) \cdot (\mathrm{id} \otimes \omega_{\xi,\xi'})({}^{k}_{m}X^{l}_{n})\big) \\
&= (\phi \otimes \mathrm{id})\Big(\langle\eta|_2 \, {}^{m}_{k}(Y^{-1})^{n}_{l}(1 \otimes |\eta'\rangle\langle\xi|) \, {}^{k}_{m}X^{l}_{n}|\xi'\rangle_2\Big) = \langle\eta|_2 \, {}^{m}\check{T}^{n}|\xi'\rangle_2 = 0.
\end{aligned}$$

A similar calculation involving $\hat{T}$ shows that $\phi(aS(b)) = 0$. Using the relation $X^* = X^{-1} = (S \otimes \mathrm{id})(X)$ and $Y^* = (S \otimes \mathrm{id})(Y)$, we conclude $\phi(a^*b) = \phi(ab^*) = 0$. $\square$

The second part of the orthogonality relations concerns inner products as above but with $a, b \in \mathcal{C}(X)$ for some irreducible corepresentation X and involves the conjugate corepresentation, which is defined as follows.

Given a Hilbert spaces H, K, we denote by $\overline{H}, \overline{K}$ the conjugate Hilbert spaces, by $T \mapsto \overline{T}$ the canonical conjugate-linear isomorphism $\mathcal{B}(H, K) \to \mathcal{B}(\overline{H}, \overline{K})$, and by $T \mapsto T^{\top} := \overline{T}^{*}$ the linear anti-isomorphism $\mathcal{B}(H, K) \to \mathcal{B}(\overline{K}, \overline{H})$.

Lemma 2.8. *On the category* $\mathrm{Corep}_{\mathrm{f}}(A, \Delta)$, *there exist*

1. *a covariant functor $(\mathcal{H}, X) \mapsto (\overline{\mathcal{H}}, \overline{X})$ and $T \mapsto \overline{T}$, where*

$$ {}^{k}\overline{\mathcal{H}}^{l} = \overline{{}^{l}\mathcal{H}^{k}}, \qquad {}^{k}_{m}\overline{X}^{l}_{n} = ({}^{l}_{n}X^{k}_{m})^{(*\otimes\overline{(\,\cdot\,)})} = (({}^{l}_{n}X^{k}_{m})^{*})^{\mathrm{id}\otimes\top}, \qquad {}^{k}\overline{T}^{l} = \overline{{}^{l}T^{k}}; $$

2. *a contravariant functor $(\mathcal{H}, X) \mapsto (\overline{\mathcal{H}}, X^{S\otimes\top})$ and $T \mapsto T^{\top}$, where*

$$ {}^{k}_{m}(X^{S\otimes\top})^{l}_{n} = (S \otimes (\,\cdot\,)^{\top})({}^{n}_{l}X^{m}_{k}), \qquad {}^{k}(T^{\top})^{l} = ({}^{l}T^{k})^{\top}; $$

3. *a covariant functor $(\mathcal{H}, X) \mapsto (\mathcal{H}, X^{S^2\otimes\mathrm{id}})$ and $T \mapsto T$, where ${}^{k}_{m}(X^{S^2\otimes\mathrm{id}})^{l}_{n} = (S^2 \otimes \mathrm{id})({}^{k}_{m}X^{l}_{n})$.*

If $(\mathcal{H}, X)$ is unitary, then $\overline{X} = X^{S\otimes\top}$.

Proof. Let $(\mathcal{H}, X)$ be a locally finite corepresentation. Using the fact that $\tilde{\Delta}$ and ϵ are $*$-homomorphisms, one easily verifies

$$(\tilde{\Delta} \otimes \mathrm{id})({}^{k}_{m}\overline{X}^{l}_{n}) = \sum_{p,q} ({}^{k}_{p}\overline{X}^{l}_{q})_{13}({}^{p}_{m}\overline{X}^{q}_{n})_{23},$$
$$(\epsilon \otimes \mathrm{id})({}^{k}_{m}\overline{X}^{l}_{n}) = \overline{(\epsilon \otimes \mathrm{id})({}^{l}_{n}X^{k}_{m})} = \delta_{l,n}\delta_{k,m}\overline{\mathrm{id}}_{\,{}^{l}\mathcal{H}^{k}} = \delta_{l,n}\delta_{k,m}\,\mathrm{id}_{\,{}^{k}\overline{\mathcal{H}}^{l}}\,.$$

A similar calculation shows that $(\overline{\mathcal{H}}, X^{S\otimes\top})$ is a corepresentation. The assertions (1) and (2) follow immediately. The square of the functor in (2) yields the functor in (3). $\square$

We call $(\overline{\mathcal{H}}, \overline{X})$ the *conjugate* of $(\mathcal{H}, X)$.

Proposition 2.9. *Assume that (A, Δ) has a faithful, positive, normalized invariant functional ϕ and let $(\mathcal{H}, X)$ be a unitary irreducible locally finite corepresentation.*

1. *$\overline{\mathcal{H}}$ and the family ${}^{k}_{m}(\overline{X}^{-*})^{l}_{n} := \big({}^{m}_{k}(\overline{X}^{-1})^{n}_{l}\big)^{*}$ form a locally finite corepresentation and there exists an invertible, positive isomorphism $\overline{F_X}$ from $(\overline{\mathcal{H}}, \overline{X})$ to $(\overline{\mathcal{H}}, \overline{X}^{-*})$.*
2. *The family ${}^{k}F^{l}_{X} := \overline{{}^{l}\overline{F_X}^{k}}$ is an invertible, positive isomorphism from $(\mathcal{H}, X)$ to $(\mathcal{H}, X^{S^2\otimes\mathrm{id}})$.*

Proof. (1) By Proposition 2.4, $(\overline{\mathcal{H}}, \overline{X})$ is equivalent to a unitary corepresentation, that is, there exists a family of operators ${}^{k}T^{l} \in \mathcal{B}({}^{k}\overline{\mathcal{H}}^{l})$ such that the family

$${}^{k}_{m}Z^{l}_{n} := (1 \otimes {}^{k}T^{l})\,{}^{k}_{m}\overline{X}^{l}_{n}(1 \otimes {}^{m}T^{n})^{-1}$$

is a unitary corepresentation. The relation ${}^{k}_{m}(Z^{-1})^{l}_{n} = \big({}^{m}_{k}Z^{n}_{l}\big)^{*}$ then implies

$${}^{k}_{m}Z^{l}_{n} = (1 \otimes {}^{k}T^{l})^{-*}\big({}^{m}_{k}(\overline{X}^{-1})^{n}_{l}\big)^{*}(1 \otimes {}^{m}T^{n})^{*}$$

and hence the family ${}^{k}_{m}(\overline{X}^{-*})^{l}_{n} := \big({}^{m}_{k}(\overline{X}^{-1})^{n}_{l}\big)^{*}$ is an irreducible locally finite corepresentation and the family ${}^{k}\overline{F}^{l}_{X} := ({}^{k}T^{l})^{*}\,{}^{k}T^{l} \in \mathcal{B}({}^{k}\overline{\mathcal{H}}^{l})$ is an isomorphism from $(\overline{\mathcal{H}}, \overline{X})$ to $(\overline{\mathcal{H}}, \overline{X}^{-*})$.

(2) The morphism T from $(\overline{\mathcal{H}}, \overline{X})$ to $(\overline{\mathcal{H}}, Z)$ yields morphisms $\overline{T}$ from $(\mathcal{H}, X)$ to $(\mathcal{H}, \overline{Z})$ and $T^{\top}$ from $(\mathcal{H}, Z^{S\otimes\top})$ to $(\mathcal{H}, \overline{X}S \otimes \top)$. Since X and Z are unitary, $\overline{Z} = Z^{S\otimes\top}$ and $\overline{X}^{S\otimes\top} = X^{S^2\otimes\top}$. Thus $T^{\top}\overline{T} = \overline{T^{*}T}$ is a morphism from $(\mathcal{H}, X)$ to $(\mathcal{H}, X^{S^2\otimes\mathrm{id}})$. $\square$

Theorem 2.10. *Assume that (A, Δ) has a faithful, positive, normalized invariant functional ϕ. Let $(\mathcal{H}, X)$ be a unitary irreducible locally finite corepresentation of (A, Δ) and let F_X be a non-zero morphism from $(\mathcal{H}, X)$ to $(\mathcal{H}, (S^2 \otimes \mathrm{id})(X))$.*

1. *The numbers $\alpha := \sum_k \mathrm{Tr}({}^{k}(F_X^{-1})^{l})$ and $\beta := \sum_n \mathrm{Tr}({}^{m}F^{n}_{X})$ do not depend on l or n.*

2. *For all k,l,m,n,*

$$(\phi \otimes \mathrm{id})(({}_{m}^{k}X_{n}^{l})^{*}\,{}_{m}^{k}X_{n}^{l}) = \alpha^{-1}\mathrm{Tr}({}^{k}(F_{X}^{-1})^{l}) \cdot \mathrm{id}_{\,{}^{m}\mathcal{H}^{n}},$$
$$(\phi \otimes \mathrm{id})({}_{m}^{k}X_{n}^{l}({}_{m}^{k}X_{n}^{l})^{*}) = \beta^{-1}\mathrm{Tr}({}^{m}(F_{X})^{n}) \cdot \mathrm{id}_{\,{}^{k}\mathcal{H}^{l}}.$$

3. *Denote by $\Sigma_{k,l,m,n}$ the flip ${}^{k}\mathcal{H}^{l} \otimes {}^{m}\mathcal{H}^{n} \to {}^{m}\mathcal{H}^{n} \otimes {}^{k}\mathcal{H}^{l}$. Then*

$$(\phi \otimes \mathrm{id} \otimes \mathrm{id})(({}_{m}^{k}X_{n}^{l})_{12}^{*}({}_{m}^{k}X_{n}^{l})_{13}) = \alpha^{-1}(\mathrm{id}_{\,{}^{m}\mathcal{H}^{n}} \otimes {}^{k}(F_{X}^{-1})^{l}) \circ \Sigma_{k,l,m,n},$$
$$(\phi \otimes \mathrm{id} \otimes \mathrm{id})(({}_{m}^{k}X_{n}^{l})_{13}({}_{m}^{k}X_{n}^{l})_{12}^{*}) = \beta^{-1}({}^{m}F_{X}^{n} \otimes \mathrm{id}_{\,{}^{k}\mathcal{H}^{l}}) \circ \Sigma_{k,l,m,n}.$$

Proof. We prove the assertions and equations involving α in (1), (2) and (3) simultaneously; the assertions involving β follow similarly.

As above, we denote by $\Sigma_{p,q,r,s}$ the flip ${}^{p}\mathcal{H}^{q} \otimes {}^{r}\mathcal{H}^{s} \to {}^{r}\mathcal{H}^{s} \otimes {}^{p}\mathcal{H}^{q}$. Consider the operator

$$\begin{aligned} F_{m,n,k,l} &:= (\phi \otimes \mathrm{id} \otimes \mathrm{id})(({}_{m}^{k}X_{n}^{l})_{12}^{*}({}_{m}^{k}X_{n}^{l})_{13}) \circ \Sigma_{m,n,k,l} \\ &= (\phi \otimes \mathrm{id} \otimes \mathrm{id})\left(({}_{k}^{m}(X^{-1})_{l}^{n})_{12}(\Sigma_{k,l,k,l})_{23}({}_{m}^{k}X_{n}^{l})_{12}\right). \end{aligned}$$

By Lemma 2.6, the family $(F_{m,n,k,l})_{m,n}$ is an endomorphism of $(\mathcal{H} \otimes {}^{k}\mathcal{H}^{l}, (X)_{12})$ and hence

$$F_{m,n,k,l} = \mathrm{id}_{\,{}^{m}\mathcal{H}^{n}} \otimes {}^{k}R^{l} \tag{2.4}$$

with some ${}^{k}R^{l} \in \mathcal{B}({}^{k}\mathcal{H}^{l})$ not depending on m,n. On the other hand,

$$\begin{aligned} F_{m,n,k,l} &= (\phi \otimes \mathrm{id} \otimes \mathrm{id})((S \otimes \mathrm{id})({}_{k}^{m}X_{l}^{n})_{12}({}_{m}^{k}X_{n}^{l})_{13}) \circ \Sigma_{m,n,k,l} \\ &= (\phi \circ S^{-1} \otimes \mathrm{id} \otimes \mathrm{id})\left(((S \otimes \mathrm{id})({}_{m}^{k}X_{n}^{l}))_{13}((S^{2} \otimes \mathrm{id})({}_{k}^{m}X_{l}^{n}))_{12}\right) \circ \Sigma_{m,n,k,l} \\ &= (\phi \circ S^{-1} \otimes \mathrm{id} \otimes \mathrm{id})\left(((S \otimes \mathrm{id})({}_{m}^{k}X_{n}^{l}))_{13}(\Sigma_{m,n,m,n})_{23}((S^{2} \otimes \mathrm{id})({}_{k}^{m}X_{l}^{n}))_{12}\right). \end{aligned}$$

Since $\phi \circ S^{-1}$ is an invariant functional for (A,Δ), we can again apply Lemma 2.6 and find that the family $(F_{m,n,k,l})_{k,l}$ is a morphism from $({}^{m}\mathcal{H}^{n} \otimes \mathcal{H}, (X^{S^{2}\otimes \mathrm{id}})_{13})$ to $({}^{m}\mathcal{H}^{n} \otimes \mathcal{H}, (X)_{13})$. Therefore,

$$F_{m,n,k,l} = {}^{m}T^{n} \otimes ({}^{k}F_{X}^{l})^{-1} \tag{2.5}$$

with some ${}^{m}T^{n} \in \mathcal{B}({}^{m}\mathcal{H}^{n})$ not depending on k,l. Combining (2.4) and (2.5), we conclude that $F_{m,n,k,l} = \lambda(\mathrm{id}_{\,{}^{m}\mathcal{H}^{n}} \otimes ({}^{k}F_{X}^{l})^{-1})$ for some $\lambda \in \mathbb{C}$. Choose a basis $(\zeta_{i})_{i}$ for ${}^{k}\mathcal{H}^{l}$. Then

$$\lambda \cdot \mathrm{id}_{\,{}^{m}\mathcal{H}^{n}} \cdot \mathrm{Tr}(({}^{k}F_{X}^{l})^{-1}) = \sum_{i}(\mathrm{id} \otimes \omega_{\zeta_{i},\zeta_{i}})(F_{m,n,k,l}) = (\phi \otimes \mathrm{id})(({}_{k}^{m}X_{l}^{n})^{*}\,{}_{m}^{k}X_{n}^{l}).$$

We sum over k, use the relations $\sum_{k}({}_{k}^{m}X_{l}^{n})^{*}\,{}_{m}^{k}X_{n}^{l} = \lambda_{l}\rho_{n} \otimes \mathrm{id}_{\,{}^{m}\mathcal{H}^{n}}$ and $\phi(\lambda_{l}\rho_{n}) = 1$, and find

$$\lambda \cdot \sum_{k} \mathrm{Tr}(({}^{k}F_{X}^{l})^{-1}) = 1.$$

Now all assertions in (1)–(3) concerning α follow. $\square$

Corollary 2.11. *Assume that (A, Δ) has a faithful, positive, normalized invariant functional ϕ. Let $(\mathcal{H}, X)$ be a unitary irreducible locally finite corepresentation of (A, Δ), let F_X be a non-zero morphism from $(\mathcal{H}, X)$ to $(\mathcal{H}, (S^2 \otimes \mathrm{id})(X))$, and let $a = (\mathrm{id} \otimes \omega_{\xi,\xi'})({}^{k}_{m}X^{l}_{n})$ and $b = (\mathrm{id} \otimes \omega_{\eta,\eta'})({}^{p}_{r}X^{q}_{s})$, where $\xi \in {}^{k}\mathcal{H}^{l}, \xi' \in {}^{m}\mathcal{H}^{n}$ and $\eta \in {}^{p}\mathcal{H}^{q}, \eta' \in {}^{r}\mathcal{H}^{s}$. Then*

$$\phi(b^*a) = \frac{\langle \eta'|\xi'\rangle\langle \xi|F_X^{-1}\eta\rangle}{\sum_k \mathrm{Tr}({}^{k}(F_X^{-1})^{l})}, \qquad \phi(ab^*) = \frac{\langle \eta'|F_X\xi'\rangle\langle \xi|\eta\rangle}{\sum_n \mathrm{Tr}({}^{m}F_X^{n})}.$$

Proof. By Theorem 2.10,

$$\begin{aligned}\phi(b^*a) &= (\phi \otimes \omega_{\eta',\eta} \otimes \omega_{\xi,\xi'})(({}^{k}_{m}X^{l}_{n})^*_{12}\, {}^{k}_{m}X^{l}_{n}) \\ &= \frac{1}{\sum_k \mathrm{Tr}({}^{k}(F_X^{-1})^{l})}(\omega_{\eta',\eta} \otimes \omega_{\xi,\xi'})((\mathrm{id}_{{}^{m}\mathcal{H}^{n}} \otimes {}^{k}(F_X^{-1})^{l}) \circ \Sigma_{k,l,m,n}).\end{aligned}$$

The formula for $\phi(ab^*)$ follows similarly or by considering the opposite of (A, Δ). □

Corollary 2.12. *Assume that (A, Δ) has a faithful, positive, normalized invariant functional ϕ. Let $(\mathcal{H}_\alpha, X_\alpha)_\alpha$ be a representative family of all irreducible locally finite corepresentations of (A, Δ). Then the map*

$$\bigoplus_{\alpha} \bigoplus_{k,l,m,n} (\overline{{}^{k}\mathcal{H}^{l}_{\alpha}} \otimes {}^{m}\mathcal{H}^{n}_{\alpha}) \to A$$

that sends $\overline{\xi} \otimes \eta \in \overline{{}^{k}\mathcal{H}^{l}_{\alpha}} \otimes {}^{m}\mathcal{H}^{n}_{\alpha}$ to $(\mathrm{id} \otimes \omega_{\xi_\alpha,\eta_\alpha})({}^{k}_{m}(X_\alpha)^{l}_{n})$, is a linear isomorphism.

Given $\omega, \omega' \in A'$ and $a \in A$, we define convolution products

$$\omega * a := (\mathrm{id} \otimes \omega)(\tilde{\Delta}(a)), \quad a * \omega' := (\omega' \otimes \mathrm{id})(\Delta(a)), \quad \omega * a * \omega' := (\omega * a) * \omega' = \omega * (a * \omega').$$

We shall say that an entire function f has *exponential growth on the right half-plane* if there exist C, d such that $|f(x + iy)| \leqslant C\mathrm{e}^{dx}$ for all $x, y \in \mathbb{R}$ with $x > 0$.

Theorem 2.13. *Assume that (A, Δ) has a faithful, positive, normalized invariant functional ϕ. There exists a unique family of characters $f_z \colon A \to \mathbb{C}$ such that*

1. *for each $a \in A$, the function $z \mapsto f_z(a)$ is entire and of exponential growth on the right half-plane;*
2. *$f_0 = \epsilon$ and $(f_z \otimes f_{z'}) \circ \tilde{\Delta} = f_{z+z'}$ for all $z, z' \in \mathbb{C}$.*
3. *$\phi(ab) = \phi(b(f_1 * a * f_1))$ for all $a, b \in A$.*

This family furthermore satisfies

4. *$S^2(a) = f_{-1} * a * f_1$ for all $a \in A$;*
5. *$f_z(\lambda_l \rho_n) = \delta_{l,n}$, $f_z \circ S = f_{-z}$, and $f_z \circ * = * \circ f_{-\overline{z}}$.*

Proof. We first prove uniqueness. Assume that $(f_z)_z$ is a family of functionals satisfying (1)–(3). Since ϕ is faithful, the map $\sigma\colon a \mapsto f_1 * a * f_1$ is uniquely determined, and one easily sees that is is a homomorphism. Using (2), we find that $\epsilon \circ \sigma = f_2$ and f_{2n} are uniquely determined and characters for each n. Using (1) and the fact that every entire function of exponential growth on the right half-plane is uniquely determined by its values at $\mathbb{N} \subseteq \mathbb{C}$ [], we can conclude that each f_z is uniquely determined and a character.

Let us now prove existence. By Corollary 2.12, we can define for each $z \in \mathbb{C}$ a functional $f_z\colon A \to \mathbb{C}$ such that for every unitary irreducible locally finite corepresentation $(\mathcal{H}, X)$,

$$f_z((\mathrm{id} \otimes \omega_{\xi,\eta})({}^{k}_{m}X^{l}_{n})) = \delta_{k,m}\delta_{l,n} \cdot \omega_{\xi,\eta}(({}^{k}F^{l}_{X})^{z}) \quad \text{for all } \xi \in {}^{k}\mathcal{H}^{l}, \eta \in {}^{m}\mathcal{H}^{n},$$

or, equivalently,

$$(f_z \otimes \mathrm{id})({}^{k}_{m}X^{l}_{n}) = \delta_{k,m}\delta_{l,n} \cdot ({}^{k}F^{l}_{X})^{z},$$

where F_X is a non-zero morphism from $(\mathcal{H}, X)$ to $(\mathcal{H}, (S^2 \otimes \mathrm{id})(X))$ such that

$$\alpha_X := \sum_k \mathrm{Tr}({}^{k}(F^{-1}_{X})^{l}) = \sum_n \mathrm{Tr}({}^{m}F^{n}_{X})$$

for all l, n (see Theorem 2.10). By construction, (1) holds. We show that $(f_z)_z$ satisfies the assertions (2)–(5). The proof of uniqueness shows that each f_z is a character. Throughout the following arguments, let $(\mathcal{H}, X)$ be a unitary irreducible corepresentation $(\mathcal{H}, X)$ and let F_X be as above.

(2) This follows from the relations

$$(f_0 \otimes \mathrm{id})({}^{k}_{m}X^{l}_{n}) = \delta_{k,m}\delta_{l,n}\,\mathrm{id}_{\,{}^{k}\mathcal{H}^{l}} = (\epsilon \otimes \mathrm{id})({}^{k}_{k}X^{l}_{l})$$

and

$$\begin{aligned}((f_z \otimes f_{z'}) \circ \tilde{\Delta} \otimes \mathrm{id})({}^{k}_{m}X^{l}_{n}) &= \delta_{k,m}\delta_{l,n}(f_z \otimes f_{z'} \otimes \mathrm{id})\big(({}^{k}_{k}X^{l}_{l})_{13}({}^{k}_{k}X^{l}_{l})_{23}\big) \\ &= \delta_{k,m}\delta_{l,n}({}^{k}F^{l}_{X})^{z} \cdot ({}^{k}F^{l}_{X})^{z'} \\ &= (f_{z+z'} \otimes \mathrm{id})({}^{k}_{m}X^{l}_{n}).\end{aligned}$$

To conclude assertion (2), apply slice maps of the form $\mathrm{id} \otimes \omega_{\xi,\xi'}$.

(3) Write $\tilde{\Delta}^{(2)} = (\tilde{\Delta} \otimes \mathrm{id}) \circ \tilde{\Delta} = (\mathrm{id} \otimes \tilde{\Delta}) \circ \tilde{\Delta}$ and $\rho_{z,z'} := (f_{z'} \otimes \mathrm{id} \otimes f_z) \circ \tilde{\Delta}^{(2)}$. Then

$$\begin{aligned}(\rho_{z,z'} \otimes \mathrm{id})({}^{k}_{m}X^{l}_{n}) &= (f_{z'} \otimes \mathrm{id} \otimes f_z \otimes \mathrm{id})(({}^{k}_{k}X^{l}_{l})_{14}({}^{k}_{m}X^{l}_{n})_{24}({}^{m}_{m}X^{n}_{n})_{34}) \\ &= (1 \otimes ({}^{k}F^{l}_{X})^{z'})\,{}^{k}_{m}X^{l}_{n}(1 \otimes ({}^{m}F^{n}_{X})^{z}).\end{aligned}$$

We take $z = z' = 1$, use Theorem 2.10, where now $\alpha = \beta$ by our scaling of F_X, and obtain

$$\begin{aligned}(\phi \otimes \mathrm{id} \otimes \mathrm{id})(({}^{k}_{m}X^{l}_{n})^{*}_{12}((\rho_{1,1} \otimes \mathrm{id})({}^{k}_{m}X^{l}_{n}))_{13}) &= \alpha^{-1}(\mathrm{id} \otimes {}^{k}F^{l}_{X})(\mathrm{id} \otimes {}^{k}(F^{-1}_{X})^{l})\Sigma_{k,l,m,n}(\mathrm{id} \otimes {}^{m}F^{n}_{X}) \\ &= \beta^{-1}({}^{m}F^{n}_{X} \otimes \mathrm{id})\Sigma_{k,l,m,n} \\ &= (\phi \otimes \mathrm{id} \otimes \mathrm{id})(({}^{k}_{m}X^{l}_{n})_{13}({}^{k}_{m}X^{l}_{n})^{*}_{12}).\end{aligned}$$

To conclude assertion (3), applying slice maps of the form $\omega_{\xi,\xi'} \otimes \omega_{\eta,\eta'}$.

(4) By Proposition 2.9 and the calculation above,

$$(S^2 \otimes \mathrm{id})(\,{}^k_m X^l_n) = (1 \otimes \ {}^k F^l_X)\,{}^k_m X^l_n (1 \otimes \ {}^m F^n_X)^{-1} = (\rho_{-1,1} \otimes \mathrm{id})(\,{}^k_m X^l_n).$$

Assertion (4) follows again by applying slice maps.

(5) The fact that f_z is a character and that $f_z(\,{}^k_m A^l_n) = 0$ if $(k,l) \neq (m,n)$ immediately implies the relation $f_z(\lambda_l \rho_n) = \delta_{l,n}$ and the equality

$$\begin{aligned}(f_{-z} \otimes \mathrm{id})(\,{}^k_k X^l_l) = (\ {}^k(F_X)^l)^{-z} &= (f_z \otimes \mathrm{id})(\,{}^k_k X^l_l)^{-1} \\ &= (f_z \otimes \mathrm{id})(\,{}^k_k (X^{-1})^l_l) = (f_z \circ S \otimes \mathrm{id})(\,{}^k_k X^l_l).\end{aligned}$$

Therefore, $f_{-z} = f_z \circ S$. Using the preceding calculation, the relation $(S \otimes \mathrm{id})(\,{}^k_k X^l_l) = (\,{}^k_k X^l_l)^*$ and positivity of $\ {}^k F^l_X$, we conclude

$$\begin{aligned}(* \circ f_z \circ * \otimes \mathrm{id})(\,{}^k_k X^l_l) &= (f_z \otimes \mathrm{id})((\,{}^k_k X^l_l)^*)^* \\ &= (f_{-z} \otimes \mathrm{id})(\,{}^k_k X^l_l)^* = ((\ {}^k F^l_X)^{-z})^* = (\ {}^k F^l_X)^{-\overline{z}} = (f_{-\overline{z}} \otimes \mathrm{id})(\,{}^k_k X^l_l),\end{aligned}$$

whence $* \circ f_z \circ * = f_{-\overline{z}}$. □

3 Tannaka-Krein duality for compact Hopf face algebras

Let $\mathcal{C}$ be a rigid tensor C*-category with irreducible unit. For example, one can take $\mathcal{C} = \mathrm{Rep}(\mathbb{G})$, the category of finite-dimensional unitary representations of a compact quantum group $\mathbb{G}$. We will in general view the tensor product of $\mathcal{C}$ as being strict. Let J be an index set for a maximal set of mutually non-isomorphic irreducible objects u_a in $\mathcal{C}$. The unit object of $\mathcal{C}$ will be written u_o. Whenever convenient, we will replace u_a by its associated index symbol a. We will also fix once and for all orthonormal bases $f^{a,b}_{c,j}$ for $\mathrm{Mor}(u_c, u_a \otimes u_b)$, where j runs over an index set $J^{a,b}_c$.

Let I be a (countable) set. We will write Hilb_{I^2} for the monoidal tensor C*-category of I-bigraded Hilbert spaces $\mathscr{H} = \sum_{r,s}^{\bar{\oplus}} \mathscr{H}_{rs}$, where the direct sum on the right is understood as the completion of the ordinary algebraic one. The tensor product $\underset{I}{\otimes}$ in Hilb_{I^2} is defined by $(\mathscr{H} \underset{I}{\otimes} \mathscr{G})_{rs} = \bar{\oplus}_t\, (\mathscr{H}_{rt} \otimes \mathscr{G}_{ts})$. The unit of Hilb_{I^2} is $l^2(I)$ with the obvious I^2-grading. We will view this monoidal category as being strict.

We will be interested in strong tensor C*-functors F from $\mathcal{C}$ to Hilb_{I^2}. As shown in [DCY], any ergodic action of a compact quantum group $\mathbb{G}$ on a unital C*-algebra provides a tensor C*-functor of $\mathcal{C}$ into Hilb_{I^2} for some set I.

For $F : \mathcal{C} \to \mathrm{Hilb}_{I^2}$ a strong tensor C*-functor, we denote the unitary compatibility morphisms by $\phi_{X,Y} : F(X) \underset{I}{\otimes} F(Y) \to F(X \otimes Y)$, where we recall that they are assumed to satisfy the coherence conditions

$$\phi_{X,Y\otimes Z}(\mathrm{id}_X \otimes \phi_{Y,Z}) = \phi_{X\otimes Y,Z}(\phi_{X,Y} \otimes \mathrm{id}_Z), \qquad \phi_{o,a} = \phi_{a,o} = \mathrm{id}_a\,.$$

It will be convenient to extend $\phi_{X,Y}$ to a coisometry $F(X)\otimes F(Y) \to F(X\otimes Y)$, defining it to be zero on the orthogonal complement of $F(X) \underset{I}{\otimes} F(Y)$. Note however that then $\phi_{X,o}$ becomes the coisometry $F(X) \otimes l^2(I) \to F(X)$ sending $F(X)_{rs} \otimes \mathbb{C}\delta_t$ canonically onto $\delta_{s,t}F(X)_{rs}$, and similarly for $\phi_{o,X}$. Whenever X, Y are clear, we will abbreviate $\phi_{X,Y}$ as ϕ. We will use the notation

$$F^{a,b}_{c,j} = \phi^* F(f^{a,b}_{c,j}) \in B(F(u_c), F(u_a) \otimes F(u_b)).$$

As $\mathcal{C}$ is rigid, each $F(X)$ will be column-finite in the sense that for each X in $\mathcal{C}$ and each fixed s in I, the direct sum $\sum_r^{\oplus} F(X)_{rs}$ will be finite-dimensional. Similarly, each $F(X)$ will be row-finite. See ...

Define vector spaces

$$_m^k A_n^l(a) = B(F(u_a)_{kl}, F(u_a)_{mn}).$$

Write $_m^k A_n^l = \oplus_{a\in J}\ _m^k A_n^l(a)$ and $A = \oplus_{k,l,m,n}\ _m^k A_n^l$. The a-spectral subspace $A(a)$ of A is defined as

$$A(a) = \sum_{k,l,m,n}^{\oplus}\ _m^k A_n^l(a).$$

For any element $x \in A$, its component in the a-spectral subspace is written x_a.

Our goal is to turn A into a generalized compact Hopf face algebra.

We first turn A into an algebra. The multiplication $x \cdot y$ of $x \in\ _m^k A_n^l(a)$ and $y \in\ _r^p A_s^q(b)$ is the element in $_m^k A_s^q$ defined by the formula

$$(x \cdot y)_c = \sum_{j\in J_c^{a,b}} \left(F^{a,b}_{c,j}\right)^* (x \otimes y) \left(F^{a,b}_{c,j}\right).$$

Note that the product is independent of the specific choice of orthogonal bases $f^{a,b}_{c,j}$. We will continue to use the $\cdot$-notation to distinguish this product from the ordinary multiplication of operators.

Lemma 3.1. *With the above product, A becomes a faithful strong I^2-algebra.*

Proof. Let $x \in\ _m^k A_n^l(a)$, $y \in\ _r^p A_s^q(b)$ and $z \in\ _s^q A_v^t$. From the fact that ϕ is a natural transformation, we find that

$$((x\cdot y)\cdot z)_d = \sum_{e\in J} \sum_{k\in J_d^{e,c}} \sum_{j\in J_e^{a,b}} \left(\phi^*(\phi^* \otimes \mathrm{id})F(f^{1,a,b,c}_{d,e,j,k})\right)^* (x\otimes y\otimes z) \left((\phi^* \otimes \mathrm{id})\phi^* F(f^{1,a,b,c}_{d,e,j,k})\right)$$

where $f^{1,a,b,c}_{d,e,j,k} = (f^{a,b}_{e,j} \otimes \mathrm{id}) f^{e,c}_{d,k}$. On the other hand,

$$(x \cdot (y \cdot z))_d = \sum_{e\in J} \sum_{k\in J_d^{a,e}} \sum_{j\in J_e^{b,c}} \left(\phi(\mathrm{id} \otimes\phi)F(f^{2,a,b,c}_{d,e,j,k})\right)^* (x \otimes y \otimes z) \left((\mathrm{id} \otimes\phi)\phi F(f^{2,a,b,c}_{d,e,j,k})\right)$$

where $f^{2,a,b,c}_{d,e,j,k} = (\mathrm{id}\otimes f^{b,c}_{e,j})f^{a,e}_{d,k}$. As $\phi(\phi\otimes\mathrm{id})$ by $\phi(\mathrm{id}\otimes\phi)$ by assumption, and as the orthonormal bases $\{f^{1,a,b,c}_{d,e,j,k} \mid e,j,k\}$ or $\{f^{2,a,b,c}_{d,e,j,k} \mid e,j,k\}$ can clearly be replaced by any other orthonormal basis of $\mathrm{Mor}(u_d, u_a\otimes u_b\otimes u_c)$, it follows that $(x\cdot y)\cdot z = x\cdot(y\cdot z)$.

Define $1_{rs} \in B(F(u_o)_{rr}, F(u_o)_{ss}) = {}^{r}_{s}A^{r}_{s}(o)$ as the map sending δ_r to δ_s. By the compatibility assumption for $\phi_{a,o}$ and $\phi_{o,a}$, the map $\mathrm{Fun}_{\mathrm{f}}(I^2)\to A$ mapping $\delta_{(r,s)}$ to 1_{rs} is an algebra homomorphism. Thus A becomes a faithful strong I^2-algebra. $\square$

In the following, we will again write $\lambda_r = \sum_s 1_{rs}$ and $\rho_s = \sum_r 1_{rs}$ inside $M(A)$, using the notation as at the end of the proof of the previous lemma.

We turn to the coproduct. Let $\{e_{a,i} \mid i\in B_a\}$ denote an orthonormal basis of $F(u_a)$ over an index set B_a which is adapted to the bigrading (in the sense that each $e_{a,i}$ is inside exactly one component). Write $B_{a,rs}\subseteq B_a$ for the set of indices for which $e_{a,i}\in F(u_a)_{rs}$. Define elements

$$P^{kl}_{mn}(a) \in {}^{k}_{m}A^{l}_{n}(a)\otimes {}^{m}_{k}A^{n}_{l}(a)$$

by

$$P^{kl}_{mn}(a) = \sum_{i\in B_{a,kl}, j\in B_{a,mn}} e_{a,j}e^{*}_{a,i}\otimes e_{a,i}e^{*}_{a,j}.$$

As each $F(u_a)_{kl}$ is finite-dimensional, the above sums are finite.

Define now maps

$$\Delta_{rs} : {}^{k}_{m}A^{l}_{n}(a) \to {}^{k}_{r}A^{l}_{s}(a)\otimes {}^{r}_{m}A^{s}_{n}(a)$$

by the application

$$x\mapsto P^{mn}_{rs}(a)(x\otimes 1) = (1\otimes x)P^{kl}_{rs}(a).$$

They obviously extend to linear maps Δ_{rs} from A to $A\underset{I^2}{\otimes} A$.

Lemma 3.2. *For each $x\in A$, the element $\Delta(x) = \sum_{rs}\Delta_{rs}(x)$ gives a well-defined multiplier of $A\underset{I^2}{\otimes} A$. The resulting map $\Delta : A\to M(A\underset{I^2}{\otimes} A)$ is an I^2-coproduct.*

Proof. As the grading on each $F(u_a)$ is column-finite, it follows at once that for each fixed p,q and $x\in A$, the element $\Delta_{rs}(x)(1\underset{I^2}{\otimes}\lambda_p\rho_q)$ is zero except for finitely many r and s. Similarly, $(1\underset{I^2}{\otimes}\lambda_p\rho_q)\Delta_{rs}(x)$ is zero except for finitely many r,s because of row-finiteness of $F(u_a)$. Hence $\Delta(x)$ is well-defined as a multiplier for each $x\in A$. Once we show that Δ is multiplicative, it will be immediate that Δ is coassociative, since each Δ_{rs} is coassociative. Moreover, also the fact that Δ then is an I^2-morphism is clear from the definition.

To obtain the multiplicativity of Δ, or rather of the coextension $\widetilde{\Delta}$, choose $x\in {}_{m}A_{n}(a)$ and $y\in {}_{n}A_{q}(b)$. Then

$$\begin{aligned}\widetilde{\Delta}(x)\cdot\widetilde{\Delta}(y) &= \sum_{rstv}(P^{mn}_{rs}(a)(x\otimes 1))\cdot(P^{nq}_{tv}(b)(y\otimes 1))\\ &= \sum_{cd}\sum_{ij}\sum_{klpt}\left(F^{a,b}_{c,i}\otimes F^{a,b}_{d,j}\right)^{*}\left(e_{a,k}e^{*}_{a,l}x\otimes e_{b,p}e^{*}_{b,t}y\otimes e_{a,l}e^{*}_{a,k}\otimes e_{b,t}e^{*}_{b,p}\right)\left(F^{a,b}_{c,i}\otimes F^{a,b}_{d,j}\right),\end{aligned}$$

where we may take the sum over all $k, l \in I_a$, $p, t \in I_b$ (and where the composition of operators with mismatching target and source is considered to be zero). Note that the infinite sums are convergent inside $M(A \otimes A)$ by the argument in the first paragraph.

Plugging in the identity operator $\sum_{cd} \sum_{rs} \left(e_{c,r} e_{c,r}^* \otimes e_{d,s} e_{d,s}^* \right)$ at the front, we obtain that the expression becomes

$$\sum_{cd} \sum_{rs} \sum_{ij} \sum_{klpt} X_{k,p}^{c,r,i} Y_{l,t}^{d,s,j} (e_{c,r} \otimes e_{d,s}) \left(e_{a,l}^* x \otimes e_{b,t}^* y \otimes e_{a,k}^* \otimes e_{b,p}^* \right) \left(F_{c,i}^{a,b} \otimes F_{d,j}^{a,b} \right)^*$$

where $X_{k,p}^{c,r,i} = e_{c,r}^* (F_{c,i}^{a,b})^* (e_{a,k} \otimes e_{b,p})$ and $Y_{l,t}^{d,s,j} = e_{d,t}^* (F_{d,j}^{a,b})^* (e_{a,l} \otimes e_{b,t})$. Resumming over the k, l, p, t, we obtain

$$\sum_{cd} \sum_{rs} \sum_{ij} \left(e_{c,r} e_{d,s}^* (F_{d,j}^{a,b})^* (x \otimes y) F_{c,i}^{a,b} \right) \otimes \left(e_{d,s} e_{c,r}^* (F_{c,i}^{a,b})^* F_{d,j}^{a,b} \right).$$

As ϕ is a coisometry and the $f_{c,i}^{a,b}$ are orthonormal, this expression simplifies to

$$\sum_{c} \sum_{rs} \sum_{i} e_{c,r} e_{c,s}^* \left((F_{c,i}^{a,b})^* (x \otimes y) F_{c,i}^{a,b} \right) \otimes e_{c,s} e_{c,r}^*,$$

which is precisely $\widetilde{\Delta}(x \cdot y)$. $\square$

Proposition 3.3. *The couple (A, Δ) is a generalized face algebra over I.*

Proof. Let ε assign to any $x \in {}_m^k A_n^l(a)$ the number $\mathrm{Tr}(x) = \sum_{i \in B_a} (e_{a,i}^* x e_{a,i})$ (where we keep the convention that mismatching operators compose to zero). We claim that ε is a counit, satisfying the conditions in the definition of a generalized face algebra. The fact that ε is a counit is immediate from the definition of Δ. It is also computed directly that for $x \in {}_m^k A_n^l$ and $y \in {}_n^l A_s^r$, we have $\varepsilon(x \cdot y) = \varepsilon(x)\varepsilon(y)$, since the $\{\phi^* F(f_{c,i}^{a,b}) e_{c,j} \mid c, i, j\}$ form an orthonormal basis of $F(u_a) \underset{I}{\otimes} F(u_b)$. From this formula, the second identity for the counit will hold true once we show that

$$\varepsilon(\lambda_k \rho_m x \lambda_l \rho_n) = \varepsilon(\lambda_k \rho_m x_{(1)}) \varepsilon(x_{(2)} \lambda_l \rho_n).$$

But both left and right hand side are zero unless $k = m$, $n = l$ and $x \in {}_m^k A_n^l$, in which case both sides equal $\varepsilon(x)$. $\square$

Our next job is to define a suitable antipode for (A, Δ). Here the rigidity of $\mathcal{C}$ will come into play, so we first fix our conventions. Let $a \mapsto \bar{a}$ be the involution induced by the rigidity on the index set J. We assume that $\overline{u_a} = u_{\bar{a}}$. For each u_a, we will fix duality morphisms $R_a : u_0 \to u_{\bar{a}} \otimes u_a$ and $\bar{R}_a : u_0 \to u_a \otimes u_{\bar{a}}$. By means of F and ϕ, they induce I^2-grading preserving maps $\mathscr{R}_a : l^2(I) \to F(u_{\bar{a}}) \underset{I}{\otimes} F(u_a)$ and $\bar{\mathscr{R}}_a : l^2(I) \to F(u_a) \underset{I}{\otimes} F(u_{\bar{a}})$. These in turn provide an invertible anti-linear map $I_a : F(u_a)_{kl} \to F(u_{\bar{a}})_{lk}$ and $J_a : F(u_{\bar{a}})_{lk} \to F(u_a)_{kl}$ such that $\langle I_a \xi_a, \eta_{\bar{a}} \rangle = \sum_r \delta_r^* \bar{\mathscr{R}}_a^* (\xi_a \otimes \eta_{\bar{a}})$

and $\langle J_a\eta_{\bar{a}}, \xi_a\rangle = \sum_s \delta_s^* \mathscr{R}_a^*(\eta_{\bar{a}} \otimes \xi_a)$. The snake identities for R_a and $\bar{R}_a$ guarantee that J_a is the inverse of I_a.

We define

$$S:\ {}_m^kA_n^l(a) \to {}_l^nA_k^m(\bar{a})$$

by

$$x \mapsto I_a x^* J_a.$$

Lemma 3.4. *By means of the map S, the couple (A, Δ) becomes a generalized Hopf face algebra.*

Proof. It is clear that S is invertible. We also have $S(\lambda_k\rho_l) = \lambda_l\rho_k$ as $I_o\delta_k = \delta_k$.

Let us check that S satisfies the condition $S(x_{(1)}) \cdot x_{(2)} = \sum_p \varepsilon(x \cdot \lambda_p)\rho_p$ in the multiplier algebra for $x \in {}_m^kA_n^l(a)$. By definition, we have

$$S(x_{(1)}) \cdot x_{(2)} = \sum_c \sum_i \sum_{p,q \in B_a} \left(F_{c,i}^{\bar{a},a}\right)^* \left(I_a e_{a,q} e_{a,p}^* J_a \otimes x e_{a,q} e_{a,p}^*\right) \left(F_{c,i}^{\bar{a},a}\right).$$

Let $C: \mathbb{C} \to \mathbb{C}$ be complex conjugation. Then we can write $I_a e_{a,q} e_{a,p}^* J_a = (I_a e_{a,q} C)(C e_{a,p}^* J_a)$. We now calculate, by definition of J_a and $F_{c,i}^{\bar{a},a}$, that

$$\sum_{p \in B_a} (C e_{a,p}^* J_a \otimes e_{a,p}^*) \left(F_{c,i}^{\bar{a},a}\right) = (R_a^* f_{c,i}^{\bar{a},a}) \sum_{s \in I} \delta_s^*,$$

since ϕ is a coisometry. Plugging this into our expression for $S(x_{(1)}) \cdot x_{(2)}$, we obtain

$$\sum_s \sum_{q \in B_a} \left(\sum_c \sum_i \left(f_{c,i}^{\bar{a},a})^* R_a\right) F_{c,i}^{\bar{a},a}\right)^* (I_a e_{a,q} C \otimes x e_{a,q} \delta_s^*).$$

As ϕ is a coisometry and $\phi^*\phi\mathscr{R}_a = \mathscr{R}_a$, we can write $\left(f_{c,i}^{\bar{a},a})^* R_a\right) F_{c,i}^{\bar{a},a} = F_{c,i}^{\bar{a},a}(F_{c,i}^{\bar{a},a})^* \mathscr{R}_a$. As the $f_{c,i}^{\bar{a},a}$ form an orthonormal basis, we thus get

$$S(x_{(1)}) \cdot x_{(2)} = \sum_s \sum_{q \in B_a} \mathscr{R}_a^* (I_a e_{a,q} C \otimes x e_{a,q} \delta_s^*).$$

Now the composition $I_a e_{a,q} C$ is the creation operator for the vector $I_a e_{a,q}$. Hence using again the definition of J_a, and using that $x \in {}_m^kA_n^l$, we get

$$\begin{aligned} S(x_{(1)}) \cdot x_{(2)} &= \sum_s \sum_q \delta_n \delta_s^* e_{a,q}^* x e_{a,q} \\ &= \sum_s \mathrm{Tr}(x) \delta_n \delta_s^* \\ &= \sum_p \varepsilon(x\lambda_p)\rho_p, \end{aligned}$$

since $\mathrm{Tr}(x) = \delta_{k,m}\delta_{n,l}\varepsilon(x)$.

The identity $x_{(1)} \cdot S(x_{(2)}) = \sum_p \varepsilon(\rho_p x)\lambda_p$ is proven in a similar way. □

In the next step, we determine an invariant functional for (A, Δ).

Definition 3.5. We define $\varphi : \ {}^k_m A^l_n \to \mathbb{C}$ as the projection onto the component ${}^k_m A^l_n(o) \cong \delta_{kl}\delta_{mn}\mathbb{C}$.

Lemma 3.6. *The functional φ is an invariant normalized functional.*

Proof. The fact that φ is normalized is immediate, so let us check that it is invariant. Let $x \in \ {}^k_m A^l_n(a)$. Then

$$\begin{aligned}(\mathrm{id} \otimes \varphi)\widetilde{\Delta}(x) &= \sum_{i,j} \varphi(e_{a,j}e^*_{a,i})e_{a,i}e^*_{a,j}x \\ &= \delta_{a,o}\sum_{r,s}\delta_r\delta^*_s x \\ &= \varphi(x)\sum_r \delta_r\delta^*_k \\ &= \sum_p \varphi(\lambda_p x)\lambda_p.\end{aligned}$$

The proof of right invariance follows similarly. □

Finally, we introduce the *-structure and show that (A, Δ) is a generalized compact Hopf face algebra. To distinguish the new *-operation from the ordinary operator algebraic one, we will denote it by †.

Definition 3.7. We define the anti-linear map $\dagger : \ {}^k_m A^l_n \to \ {}^m_k A^n_l$ by the formula

$$x^\dagger = S(x^*)$$

Lemma 3.8. *The map $x \mapsto x^\dagger$ is an anti-multiplicative anti-linear involution on A.*

Proof. It is clear that $x \mapsto x^\dagger$ is anti-linear. It is also immediate from the definition of the product that $(x \cdot y)^* = x^* \cdot y^*$. Together with the anti-multiplicativity of S, this proves the anti-multiplicativity of †.

Let us proof that † is an involution. It is sufficient to prove that $I_{\bar{a}}I_a = \lambda\,\mathrm{id}$ and $J_aJ_{\bar{a}} = \lambda^{-1}\,\mathrm{id}$ for some scalar λ. But this follows from the fact that $(\bar{R}_a, R_a)$ and $(R_{\bar{a}}, \bar{R}_{\bar{a}})$ are both solutions to the conjugate equations for $\bar{a}$. □

The last property which needs to be proven is the positivity of φ. For this, recall that $R^*_a R_a$ and $\bar{R}^*_a\bar{R}_a$ are scalars as u_a is irreducible. One can then rescale R_a and $\bar{R}_a$ such that the scalar in both expressions is the same. This scalar is then a uniquely determined number $\dim_q(a)$, called the *quantum dimension* of a. It follows that $\frac{1}{\dim_q(a)}F(R_aR^*_a)$ is the projection of $F(u_{\bar{a}} \otimes u_a)$ onto the copy of $F(u_o)$ inside, and a similar statement holds for $\bar{R}_a$.

Proposition 3.9. *For any $x \in A$, the scalar $\varphi(x^\dagger \cdot x)$ is positive.*

Proof. It is straightforward to see that the blocks ${}_m^kA_n^l$ are mutually orthogonal, and that moreover the spectral subspaces inside are mutually orthogonal. Let then $\xi, \zeta \in F(u_a)_{kl}$ and $\eta, \mu \in F(u_a)_{mn}$. We have, using the remark above,

$$\begin{aligned}
\varphi(y^\dagger \cdot x) &= \varphi(\sum_c \sum_i \left(F_{c,i}^{\bar{a},a}\right)^* (I_a y J_a \otimes x) \left(F_{c,i}^{\bar{a},a}\right)) \\
&= \delta_n^* \sum_i \left(F_{o,i}^{\bar{a},a}\right)^* (I_a y J_a \otimes x) \left(F_{o,i}^{\bar{a},a}\right) \delta_l \\
&= \frac{1}{\dim_q(u_a)} \delta_n^* \mathscr{R}_a^* (I_a y J_a \otimes x) \mathscr{R}_a \delta_l \\
&= \frac{1}{\dim_q(u_a)} \sum_{p,q} \delta_n^* \mathscr{R}_a^* (I_a y J_a e_{\bar{a},p} e_{\bar{a},p}^* \otimes x e_{a,q} e_{a,q}^*) \mathscr{R}_a \delta_l.
\end{aligned}$$

By the defining properties of I_a and J_a, this expression becomes $\dim_q(u_a)^{-1} \sum_p \langle e_{\bar{a},p}, J_a^* x^* y J_a e_{\bar{a},p}\rangle$, thus clearly φ will be positive on A. $\square$

Corepresentations of generalized compact Hopf face algebras

Let (A, Δ) be a generalized compact Hopf face algebra over an index set I. A *locally finite-dimensional unitary corepresentation* of (A, Δ) consists of a row and column-finite I^2-graded Hilbert space $\mathscr{H} = \sum_{k,l \in I}^{\oplus} \mathscr{H}_{kl}$ together with elements ${}_m^kU_n^l \in {}_m^kA_n^l \otimes B({}^m\mathscr{H}^n, {}^k\mathscr{H}^l)$ such that

$$\sum_k \left({}_m^kU_n^l\right)^* {}_m^kU_n^l = \lambda_l \rho_n \otimes \mathrm{id}_{{}^m\mathscr{H}^n}$$

and

$$\sum_n {}_m^kU_n^l \left({}_m^kU_n^l\right)^* = \lambda_k \rho_m \otimes \mathrm{id}_{{}^k\mathscr{H}^l},$$

and

$$(\widetilde{\Delta} \otimes \mathrm{id})({}_m^kU_n^l) = \sum_{p,q} \left({}_p^kU_q^l\right)_{13} ({}_m^pU_n^q)_{23}.$$

Note that in the first two identities, the sums are finite, while in the finite identity the possibly infinite sum is meaningful inside the multiplier algebra sense.

By a morphism between two locally finite-dimensional unitary corepresentations $(\mathscr{H}, U)$ and $(\mathscr{G}, V)$ is meant a grading-preserving bounded map $T = \sum_{k,l}^{\oplus} {}^kT^l : \mathscr{H} \to \mathscr{G}$ such that $(1 \otimes {}^kT^l)\, {}_m^kU_n^l = {}_m^kV_n^l(1 \otimes {}^mT^n)$. The collection of all locally finite-dimensional unitary corepresentations clearly forms a semi-simple C*-category Corep(A). We will say that (A, Δ) is *of finite type* if the morphisms in Corep(A) are finite-dimensional.

One can define a tensor product $\circledT$ between locally finite-dimensional corepresentations by means of the $\underset{I}{\otimes}$-product of bigraded Hilbert spaces and the operation

$$ {}^{k}_{m}(U \circledT V)^{l}_{n} = \left({}^{k}_{m}U^{r}_{s}\right)_{12}\left({}^{r}_{s}V^{l}_{n}\right)_{13}. $$

In this way, the category $\mathrm{Corep}(A)$ becomes a monoidal category. The unit object consists of the I^2-graded Hilbert space $l^2(I)$ together with the elements ${}^{k}_{m}U^{l}_{n} = \delta_{kl}\delta_{mn}\lambda_k\rho_m \otimes 1$.

Assume now that $\mathcal{C}$ is a semi-simple tensor C*-category with irreducible unit, and $F : \mathcal{C} \to \mathrm{Hilb}$ a strong tensor C*-functor. Let (A, Δ) be the associated generalized compact Hopf face algebra. Let us show that $\mathcal{C} \cong \mathrm{Corep}(A)$ by means of an equivalence functor G.

For X an object of $\mathcal{C}$, we build a locally finite-dimensional unitary corepresentation U on $F(X)$. Consider the canonical isomorphism $F(X) \cong \oplus_{a\in J} X_a \otimes \mathrm{Mor}(X_a, X)$. Let

$$ {}^{k}_{m}U^{l}_{n}(a) \in {}^{k}_{m}A^{l}_{n}(a) \otimes B(F(u_a)_{mn}, F(u_a)_{kl}) = B(F(u_a)_{kl}, F(u_a)_{mn}) \otimes B(F(u_a)_{mn}, F(u_a)_{kl}) $$

be determined as the element implementing the non-degenerate pairing $B(F(u_a)_{kl}, F(u_a)_{mn}) \otimes B(F(u_a)_{nm}, F(u_a)_{lk}) \to \mathbb{C}$ sending $S \otimes T$ to $\mathrm{Tr}(ST)$. Using notation as before, this means that

$$ {}^{k}_{m}U^{l}_{n}(a) = \sum_{p\in B_{a,mn}, q\in B_{a,kl}} e_p e_q^* \otimes e_q e_p^*. $$

Monoidal equivalence of generalized compact Hopf face algebras

Let (A, Δ) be a generalized Hopf face algebra over a set I. Assume that $I = I_1 \sqcup I_2$, and let $\Lambda_j = \sum_{i\in I_j} \lambda_i$, resp. $\mathrm{P}_j = \sum_{i\in I_j} \rho_j$. If the Λ_j and P_j are central in $M(A)$, then we can write $A = \sum^{\oplus}_{i,j} A(ij)$ where $A(ij) = \Lambda_i \mathrm{P}_j A$ are subalgebras. Moreover, the comultiplication $\widetilde{\Delta}$ splits into comultiplications

$$ \widetilde{\Delta}^{k}_{ij} : A(ij) \to M(A(ik) \otimes A(kj)) \text{ s.t. } \widetilde{\Delta} = \widetilde{\Delta}^{1}_{ij} + \widetilde{\Delta}^{2}_{ij} \text{ on } A(ij). $$

A similar decomposition holds for Δ.

It is immediate to see that the $(A(ii), \Delta^{i}_{ii})$ are two generalized Hopf face algebras over the respective I_i.

Definition 3.10. We say (A, Δ) is a *co-linking generalized (compact) Hopf face algebra* between $(A(11), \Delta^{1}_{11})$ and $(A(22), \Delta^{2}_{22})$ if $\lambda_i \mathrm{P}_2 \neq 0$ for any $i \in I_1$.

Upon applying the antipode, we see that then $\rho_j \Lambda_1 \neq 0$ for any $j \in I_2$ as well.

Definition 3.11. Two generalized (compact) Hopf face algebras are called *comonoidally Morita equivalent* if they are isomorphic to the components $(A_{ii}, \Delta^{i}_{ii})$ of some co-linking generalized (compact) Hopf face algebra.

As an example, consider two sets I_i, and two tensor functors (F_i, ϕ_i) of a semi-simple rigid C*-category $\mathcal{C}$ with irreducible unit into $\mathrm{Hilb}_{I_i^2}$. Then with $I = I_1 \sqcup I_2$, we can form a new C*-functor $F = F_1 \oplus F_2$ of $\mathcal{C}$ into Hilb_{I^2} by putting $F(X) = F_1(X) \oplus F_2(X)$ with the obvious I^2-grading (and the obvious direct sum operation on morphisms). It becomes monoidal by means of the unitaries

$$F(X \otimes Y) = F_1(X \otimes Y) \oplus F_2(X \otimes Y) \underset{\phi_1 \oplus \phi_2}{\cong} (F_1(X) \underset{I_1}{\otimes} F_1(Y)) \oplus (F_2(X) \otimes F_2(Y)) \cong F(X) \underset{I}{\otimes} F(Y)$$

(where the last map is unitary since $(F(X) \underset{I}{\otimes} F(Y))_{ij} = 0$ for example for $i \in I_1$ and $j \in I_2$).

If we then consider the generalized compact Hopf face algebra (A, Δ) associated to F, we have immediately from the construction that the Λ_i and P_i associated to the decomposition $I = I_1 \sqcup I_2$ are indeed central elements in $M(A)$. Moreover, the parts $(A_{ii}^i, \Delta_{ii}^i)$ are seen to arise from applying the Tannaka-Krein construction to the respective functors F_1 and F_2. The fact that (A, Δ) is co-linking is immediate from the fact that *none* of the $\lambda_i \rho_j$ are zero in this particular case (since ${}_m^k A_m^k(o) = B(F(u_o)_{kk}, F(u_o)_{mm}) \cong \mathbb{C}$).

We will exploit the above extra structure in the following section to say something about the algebra A appearing in ... This is the component $\tilde{A}(1,1)$ of the above algebra. The following lemma will be needed.

Lemma 3.12. *Assume (A, Δ) is a co-linking generalized Hopf face algebra. Then any of the maps $\widetilde{\Delta}_{ij}^k$ is injective.*

Proof. Take a non-zero $x \in A_n(ij)$ where $n \in I_j$. Then for any $l \in I$ with $\rho_n \lambda_l \neq 0$, we know that $\widetilde{\Delta}(x)(1 \otimes \rho_n \lambda_l) \neq 0$. Hence $\widetilde{\Delta}_{ij}^k(x)(1 \otimes \rho_n \lambda_l) \neq 0$ for $l \in I_k$, and hence $\widetilde{\Delta}_{ij}^k(x) \neq 0$. Now if $j = k$, the condition $\rho_n \lambda_l \neq 0$ is satisfied by taking $l = n$ (since $\varepsilon(\lambda_n \rho_n) = 1$). If $j \neq k$, it is satisfied for at least one l by the co-linking assumption. $\square$

4 Compact Hopf face algebras on the level of operator algebras

It then follows by symmetry that also the maps

$$(W_{m,n,u,v}^{k,t})^* : \oplus_l \, {}_m^k A_n^l \otimes {}_u^l A_v^t \to \oplus_r \, {}_m^k A_r^t \otimes {}_u^n A_v^r$$

defined by the formula

$$a \otimes b \to \Delta(b)(a \otimes 1)$$

are unitaries, with inverse map $a \otimes b \mapsto S^{-1}(b_{(1)})a \otimes b_{(2)}$.

Lemma 4.1. *Let (A, Δ) be a generalized compact face algebra. Then each $V_{m,v}^{k,l,s,t}$ is a unitary, and similarly for the $W_{m,n,u,v}^{k,t}$.*

Proof. It is immediately checked that $V_{m,v}^{k,l,s,t}$ is isometric. $\square$

Let us write $\mathscr{L}^2(A,\varphi)$ for the completion of A with respect to the inner product $\langle a,b\rangle = \varphi(a^*b)$. The canonical inclusion of A into $\mathscr{L}^2(A)$ will be denoted Λ.

Lemma 4.2. *Assume (A,Δ) is a generalized compact face algebra. The representation of A by left multiplication on itself extends to a representation by bounded operators on the completion $\mathscr{L}^2(A,\varphi)$.*

Proof. Denote $\omega_{\xi,\eta}(x) = \langle \xi, x\eta\rangle$ for ξ,η vectors and x a bounded operator. Then a straightforward computation shows that

$$(\omega_{\Lambda(a),\Lambda(b)}\otimes \mathrm{id})(V) = \varphi(a^*b_{(1)})b_{(2)}$$

as a left multiplication operator. As $(A\otimes 1)\Delta(A) = (A\otimes A)\Delta(1)$ by Lemma 4.1 (applied to the opposite algebra), it follows by normalization of φ that each element of A can be represented in the form $(\omega_{\Lambda(a),\Lambda(b)}\otimes \mathrm{id})(V)$, and hence extends to a bounded operator on $\mathscr{L}^2(A,\varphi)$. $\square$

In the following, we will abbreviate $\mathscr{L}^2(A)$ by L^2A.

Let (A,Δ) be a generalized compact face algebra. Denote the von Neumann algebraic completion of $A\subseteq B(L^2A)$ by M. Denote $L^2A\underset{I}{\otimes} L^2A = E(L^2A\otimes L^2A)$, where $E = \sum_p \rho_p\otimes\lambda_p$ is extended to a bounded operator (in fact, a self-adjoint projection). Finally, denote $M\underset{I^2}{\otimes} M = E(M\otimes M)E$. Then $M\underset{I^2}{\otimes} M$ is the von Neumann algebraic completion of $A\underset{I^2}{\otimes} A$.

Extend now the $V_{m,v}^{k,l,s,t}$ to unitaries

$$V : \oplus_p L^2(A_p)\otimes L^2({}_pA) \to \oplus_p L^2({}_pA)\otimes L^2({}^pA) = E(L^2A\otimes L^2A).$$

Then we can construct a map

$$\Delta : M\to M\underset{I^2}{\otimes} M,\quad x\to V(x\otimes 1)V^*.$$

By direct computation, we see that Δ extends the comultiplication map on A. It is then immediate to check that Δ is in fact coassociative (where one may as well consider Δ as a non-unital map from M to $M\otimes M$).

We aim to show that (M,Δ) can be fitted into the theory of measured quantum groupoids.

5 Generalized compact Hopf face algebras from ergodic actions of quantum $SU(2)$

We apply the construction of the previous section the the specific case of $\mathcal{C} = \mathrm{Rep}(SU_q(2))$, the category of finite-dimensional unitary representations of $SU_q(2)$ for some $0<|q|<1$. By [DCY], one can encode strong tensor functors $\mathcal{C}\to \mathrm{Hilb}_{I^2}$ by the following data:

- A collection of finite sets B_{kl} indexed by I^2 such that for l fixed, $B_{kl} = \varnothing$ for all but finitely many k, and such that for k fixed, $B_{kl} = \varnothing$ for all but finitely many l.

- For each $(k,l) \in J^2$ we have an endomorphism $E(kl) : \mathrm{Fun}(B_{kl}) \to \mathrm{Fun}(B_{lk})$ such that $\overline{E(kl)}E(lk) = -\,\mathrm{sgn}(q)\,\mathrm{id}$ for all k,l and $\sum_l \mathrm{Tr}(E(kl)E(lk)^*) = |q+q^{-1}|$ for all k.

The associated tensor functor $F_E : \mathcal{C} \to \mathrm{Hilb}_{J^2}$ is then obtained as the unique *strict* one such that

- The fundamental object $u_{1/2}$, the spin 1/2-representation, of $\mathcal{C} = \mathrm{Rep}(SU_q(2))$ is sent to $\mathscr{H} = \oplus \mathscr{H}_{kl}$ with $\mathscr{H}_{kl} = \mathrm{Fun}(B_{kl})$ with its canonical Hilbert space structure.

- The normalized self-duality $R : 1 \to u_{1/2} \otimes u_{1/2}$ is sent to the operator $\mathscr{R} : l^2(J) \to \mathscr{H} \underset{I}{\otimes} \mathscr{H}$ which decomposes as $\mathscr{R}_{|\mathbb{C}\delta_k} = \sum_l \mathscr{R}_{kl}$ with $\mathscr{R}_{kl} : \mathbb{C} \to \mathscr{H}_{kl} \otimes \mathscr{H}_{lk}$ such that $E(kl)_{ij} = -\,\mathrm{sgn}(q)\mathscr{R}_{kl}^*(\delta_j \otimes \delta_i)$.

We will in the following write $B = \sqcup B_{kl}$, and $(s(i), t(i)) = (k,l)$ for $i \in B_{kl} \subseteq B$. Let (A, Δ) be the generalized compact Hopf face algebra obtained from the construction of the previous section. Our aim is to find an explicit generators and relations description of (A, Δ).

Definition 5.1. We define $\mathcal{A}$ to be the *-algebra generated by elements u_{ij} with $i,j \in B$, satisfying the relations

$$\begin{aligned}\sum_{k\in I}\sum_{i\in B_{kl}} u_{ij}^* u_{is} &= \delta_{js}\lambda_l\rho_{t(j)}, &\quad& \forall l \in I, j,s \in B,\\ \sum_{n\in I}\sum_{j\in B_{mn}} u_{ij}u_{rj}^* &= \delta_{i,r}\lambda_{s(i)}\rho_m, && \forall m \in I, i,r \in B\end{aligned}$$

and

$$u_{ij}^* = \sum_{\substack{r\in B_{t(j),s(j)}\\ p\in B_{t(i),s(i)}}} E(t(i)s(i))_{ip}u_{pr}(E(t(j)s(j))^{-1})_{rj}, \qquad \forall i,j \in B.$$

Note that because of the finiteness assumption on the B_{kl}, the sums in all relations are finite sums.

Proposition 5.2. *Write $v_{ij} \in {}^{s(i)}_{s(j)}A^{t(i)}_{t(j)}(1/2) = B(F(u_{1/2})_{s(i)t(i)}, F(u_{1/2})_{s(j)t(j)})$ for the operator sending the basis vector δ_p to $\delta_{p,i}\delta_j$. Then there is a *-isomorphism $\Phi : \mathcal{A} \to A$ such that u_{ij} is sent to v_{ij}.*

Proof. Let us first check that Φ is well-defined, that is, that the v_{ij} satisfy the same relations as the u_{ij}. We will resort to the notations of the previous section, so $\cdot$ denotes the multiplication in A and $\dagger$ denotes the *-operation. Then $v_{ij}^\dagger = S(v_{ji})$ by definition of $\dagger, S$ and the v_{ij}. Moreover, we have

$$\widetilde{\Delta}(v_{ij}) = \sum_{p,r} v_{pr} v_{ij} \otimes v_{rp} = \sum_p v_{ip} \otimes v_{pj}$$

as $v_{pr}v_{ij} = \delta_{j,p} v_{ir}$ under the operator product. Hence

$$\begin{aligned}
\sum_{k\in I} \sum_{i\in B_{kl}} v_{ij}^\dagger \cdot v_{is} &= S(v_{js(1)}) \cdot v_{js(2)} \cdot \lambda_l \\
&= \sum_p \varepsilon(v_{js} \cdot \lambda_p) \lambda_l \cdot \rho_p \\
&= \varepsilon(v_{js}) \lambda_l \rho_{t(j)} \\
&= \delta_{js} \lambda_l \rho_{t(j)}.
\end{aligned}$$

The relation $\sum_{n\in I} \sum_{j\in B_{mn}} v_{ij} \cdot v_{rj}^\dagger = \delta_{i,r} \lambda_{s(i)} \rho_m$ follows similarly.

Let us turn to the final relation. By definition of the $E(kl)$, we have that $\langle J_a \delta_j, \delta_i \rangle = E(lk)_{ij}$ and $\langle I_a \delta_j, \delta_i \rangle = -\operatorname{sgn}(q) E(kl)_{ij}$. From the definition of $\dagger$ and the fact that $R_{1/2} = -\operatorname{sgn}(q) \bar{R}_{1/2}$, we then obtain that for $(s(i), t(i)) = (k, l)$ and $(s(j), t(j)) = (m, n)$, we have

$$v_{ij}^\dagger = -\operatorname{sgn}(q) \sum_{pr} E(lk)_{ip} v_{pr} \overline{E(mn)_{rj}}.$$

Since $\overline{E(mn)} = -\operatorname{sgn}(q) E(nm)^{-1}$, we find that the final relation for the u_{ij} is satisfied. $\square$

5.1 Generalized compact Hopf face algebras from Podleś spheres

As a particular case of the construction in the previous section, consider $I = \mathbb{Z}$ with $B_{kl} = \varnothing$ when $k \neq l \pm 1$, and $B_{kl} = \{(k, l)\}$ when $k = l \pm 1$. Put

$$E(k, k \pm 1) = c_\pm \left(\frac{|q|^{x+k\pm 1} + |q|^{-x-k\mp 1}}{|q|^{x+k} + |q|^{-x-k}} \right)^{1/2}$$

where $c_+ = 1$ and $c_- = -\operatorname{sgn}(q)$. Then this collection satisfies the requirements postulated at the beginning of the previous section. It is obtained from the ergodic action of $SU_q(2)$ on the Podleś sphere $S^2_{q,\tau(x)}$ with $\tau(x) = ...$ as described in [DCY].

For $\epsilon, \nu \in \{+, -\}$, let us write $u_{kl} = u_{(k,k+\epsilon),(l,l+\nu)}$. Then the unitarity relations ... become

$$\begin{aligned}
\sum_\epsilon (u_{k-\epsilon,l}^{\epsilon,\nu})^* u_{k-\epsilon,m}^{\epsilon,\mu} &= \delta_{\nu,\mu} \delta_{l,m} \lambda_k \rho_{l+\nu}, \\
\sum_\nu u_{kl}^{\epsilon,\nu} (u_{ml}^{\mu,\nu})^* &= \delta_{\epsilon,\mu} \delta_{k,m} \lambda_k \rho_m.
\end{aligned}$$

In turns of the multipliers $u^{\epsilon,\nu} = \sum_{k,l} u_{kl}^{\epsilon,\nu}$, this is equivalent with saying that the matrix

$$U = \begin{pmatrix} u^{--} & u^{+-} \\ u^{-+} & u^{++} \end{pmatrix}$$

is unitary.

The relations for the adjoint further imply that

$$\begin{aligned} u_{kl}^{+-} &= \frac{E(l,l-1)}{E(k,k+1)}(u_{k+1,l-1}^{-+})^*, \\ u_{kl}^{++} &= \frac{E(l,l+1)}{E(k,k+1)}(u_{k+1,l+1}^{--})^*. \end{aligned}$$

Let us write $F(k)^{1/2} = \frac{1}{|q|^{1/2}E(k,k+1)}$. Further, for any function f on I, write $f(\lambda) = \sum_k f(k)\lambda_k$ and $f(\rho) = \sum_m f(m)\rho_m$. Then using that $E(l+1,l) = -\operatorname{sgn}(q)\frac{1}{E(l,l+1)}$, we find that the above adjoint relations are equivalent to

$$\begin{aligned} u^{+-} &= -qF(\rho-1)^{1/2}F(\lambda)^{1/2}(u^{-+})^* \\ u^{++} &= \frac{F(\lambda)^{1/2}}{F(\rho)^{1/2}}(u^{--})^*. \end{aligned}$$

Write now

$$\alpha = u^{--}, \qquad \beta = qF(\rho)^{1/2}u^{-+}.$$

Then the above commutation relations are equivalent to

$$\alpha\beta = qF(\rho-1)\beta\alpha \qquad \alpha\beta^* = qF(\lambda)\beta^*\alpha$$

$$\alpha\alpha^* + F(\lambda)\beta^*\beta = 1, \qquad \alpha^*\alpha + \frac{q^{-2}}{F(\rho-1)}\beta^*\beta = 1,$$

$$\frac{1}{F(\rho-1)}\alpha\alpha^* + \beta\beta^* = F(\lambda-1)^{-1}, \qquad F(\lambda)\alpha^*\alpha + q^{-2}\beta\beta^* = F(\rho),$$

$$f(\lambda)g(\rho)\alpha = \alpha f(\lambda+1)g(\rho+1), \qquad f(\lambda)g(\rho)\beta = \beta f(\lambda+1)g(\rho-1).$$

These are precisely the commutation relations for the dynamical quantum $SU(2)$-group as in for example [KR], except that the precise value of F has been changed by a shift in the parameter domain by a complex constant. As the coproduct on A is given by $\Delta(u^{\epsilon,\nu}) = \sum_\mu u^{\epsilon,\mu} \underset{I^2}{\otimes} u^{\mu,\nu}$, we also find that the coproduct agrees with the one on the dynamical quantum $SU(2)$-group, namely

$$\begin{aligned} \Delta(\alpha) &= \alpha \underset{I^2}{\otimes} \alpha - q^{-1}\beta \underset{I^2}{\otimes} \beta^*, \\ \Delta(\beta) &= \beta \underset{I^2}{\otimes} \alpha^* + \alpha \underset{I^2}{\otimes} \beta. \end{aligned}$$

More generally,

As a concrete instance of the example of monoidal equivalence, let $\tilde{A}$ be the generalized compact Hopf face algebra obtained from the set $\tilde{I} = I_1 \sqcup I_2$ with $I_1 = \mathbb{Z}$ and $I_2 = \{\bullet\}$ with the $B_{kl} = \varnothing$ and $E(k,l)$ for $k,l \neq \bullet$ as in section ..., with $B_{k,\bullet} = B_{\bullet,k} = \varnothing$, and $B_{\bullet,\bullet} = \{\pm\}$ with $E_{\bullet,\bullet} = \begin{pmatrix} 0 & |q|^{1/2} \\ -\operatorname{sgn}(q)|q|^{-1/2} & 0 \end{pmatrix}$ (with the basis ordered as $-,+$). Then this will be obtained from the direct sum of the functor from ... and the ordinary forgetful functor from $\mathrm{Rep}(SU_q(2))$ into Hilb. It follows that the components $\tilde{A}(ij)$ can be described by the generators and relations as in ..., but with $F(\lambda)$ and $F(\rho)$ set equal to 1 whenever the corresponding index is $\bullet$.

6 Further structure on the dynamical $SU_q(2)$

6.1 Representation theory of the function algebra on the dynamical quantum $SU(2)$ group

Lemma 6.1. *There are faithful* **-representations* $\pi_\pm$ *of* $\mathrm{P_e}(\mathbb{X})$ *as operators* $\mathscr{D}^\pm \to \mathscr{D}^\pm$, *given by the following formulas (where we suppress the explicit notations* $\pi_\pm$*):*

$$\alpha \cdot e^+_{n,y} = \left(\frac{1+q^{2n-2y}}{1+q^{-2y-2}}\right)^{1/2} e^+_{n,y+1}, \qquad \beta \cdot e^+_{n,y} = \left(\frac{q^{-2y}-q^{2n-2y+2}}{1+q^{-2y-2}}\right)^{1/2} e^+_{n+1,y+1},$$

$$\alpha \cdot e^-_{n,y} = \left(\frac{1-q^{2n}}{1+q^{-2y-2}}\right)^{1/2} e^-_{n-1,y+1}, \qquad \beta \cdot e^-_{n,y} = \left(\frac{q^{2n+2}+q^{-2y}}{1+q^{-2y-2}}\right)^{1/2} e^-_{n,y+1},$$

the functions in $C_c(\mathbb{R})$ *simply acting by* $fe^\pm_{n,y} = f(y)e^\pm_{n,y}$.

Both representations are bounded when restricted to $\mathrm{P}(\mathbb{X})$.