Chapter 2: Limits Quiz Solution

Calculus - TAC Shanghai

November 4, 2020

INSTRUCTIONS: please solve the following limit problems. Some limits may not exist. Show *ALL* your work, either on this paper or on a separate sheet of paper.

1. $\lim_{h \to 0} \frac{\left(h-1\right)^4 - 1}{h}$

Evaluating leads to $\frac{0}{0}$, so we need to manipulate the expression:

$$= \lim_{h \to 0} \frac{(h^2-2h+1)^2-1}{h} = \lim_{h \to 0} \frac{\left[(h^2-2h+1)(h^2-2h+1)\right]-1}{h}$$

$$\lim_{h \to 0} \frac{\left(h^2(h^2-2h+1) - 2h(h^2-2h+1) + 1(h^2-2h+1)\right) - 1}{h}$$

$$= \lim_{h \to 0} \frac{\left(h^4 - 4h^3 + 6h^2 - 4h + 1\right) - 1}{h} = -4$$

2. $\lim_{x \to 7} \frac{\sqrt{x^2-24}-4}{\sqrt{x^2+4}-1}$

Both numerator and demoninator are non-zero at $x = 7$, so we can directly evaluate:

$$\lim_{x \to 7} \frac{\sqrt{x^2-24}-4}{\sqrt{x^2+4}-1} = \frac{\sqrt{(7)^2-24}-4}{\sqrt{(7)^2+4}-1}$$

$$= \frac{1}{\sqrt{53}-1} \text{ or } \frac{\sqrt{53}+1}{52}$$

3. $\lim_{z \to -2} \frac{\frac{1}{z-2} - \frac{1}{3z+2}}{z+2}$

Evaluating leads to $\frac{0}{0}$, so we need to manipulate the expression:

$$= \lim_{z\to -2} \frac{1}{z+2} \cdot \big(\frac{1}{z-2} - \frac{1}{3z+2}\big)$$

$$= \lim_{z\to -2} \frac{1}{z+2} \cdot \frac{(3z+2)-(z-2)}{(z-2)(3z+2)}$$

$$= \lim_{z\to -2} \frac{1}{z+2} \cdot \frac{2z+4}{(z-2)(3z+2)} = \lim_{z\to -2} \frac{1}{z+2} \cdot \frac{2\cdot(z+2)}{(z-2)(3z+2)}$$

$$\lim_{z\to -2} \frac{2}{(z-2)(3z+2)}$$

Both numerator and demoninator are non-zero at $z=-2$, so we can directly evaluate:

$$\lim_{z\to -2} \frac{2}{(z-2)(3z+2)} = \frac{2}{((-2)-2)(3(-2)+2)}$$

$$= \frac{1}{8}$$

4. $\lim\limits_{\vartheta\to 0} \dfrac{\vartheta^2}{\tan^2 5\vartheta}$

Hint: You should use the fact $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$ twice.

Directly evaluating gives $\frac{0}{0}$, so we need to manipulate the expression.

$$\lim_{\vartheta\to 0} \frac{\vartheta^2}{\tan^2 5\vartheta} = \lim_{\vartheta\to 0} \frac{\vartheta^2}{1} \cdot \frac{\cos^2(5\vartheta)}{\sin^2(5\vartheta)} = \lim_{\vartheta\to 0} \frac{\cos^2(5\vartheta)}{1} \cdot \frac{\vartheta^2}{\sin^2(5\vartheta)}$$

$$= \big[\lim_{\vartheta\to 0} \frac{\cos^2(5\vartheta)}{1}\big] \cdot \big[\lim_{\vartheta\to 0} \frac{\vartheta^2}{\sin^2(5\vartheta)}\big] = 1 \cdot \lim_{\vartheta\to 0} \frac{\vartheta^2}{\sin^2(5\vartheta)} = \left[\lim_{\vartheta\to 0} \frac{\vartheta}{\sin(5\vartheta)}\right]^2$$

Now, we need to remember the hint from earlier:

$$\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta} = 1$$

Or, when a is a real number:

$$\lim_{\theta\to 0} \frac{\sin(a\theta)}{\theta} = a$$

We have instead:

$$\lim_{\vartheta\to 0} \frac{\vartheta}{\sin(5\vartheta)} = \lim_{\vartheta\to 0} \frac{1}{\frac{\sin(5\vartheta)}{\vartheta}} = \frac{1}{5}$$

$$\implies \lim_{\vartheta\to 0}\frac{\vartheta^2}{\tan^2 5\vartheta} = \left[\lim_{\vartheta\to 0}\frac{\vartheta}{\sin(5\vartheta)}\right]^2 = \left[\frac{1}{5}\right]^2 = \frac{1}{25}$$

5. $\lim_{x\to\infty}\frac{x^4+7x}{x^3-2}$

On top the highest power is x^4, and on bottom the highest power is x^3. Factoring out the highest power on top and bottom yields:

$$\lim_{x\to\infty}\frac{x^4+7x}{x^3-2} = \lim_{x\to\infty}\frac{(x^4)(1+7\frac{1}{x^3})}{(x^3)(1-\frac{2}{x^3})}$$

$$\lim_{x\to\infty}\frac{x(1+7\frac{1}{x^3})}{1-\frac{2}{x^3}} = \lim_{x\to\infty} x = \infty$$

6. $\lim_{x\to-\infty}\frac{5e^{2x}+10e^{-3x}}{2e^{5x}+37e^{-3x}}$

When $x\to-\infty, e^{-x}$ dominates and $e^x\to 0$. This means we need to divide by the largest power of e^{-x} to determine the limit. Doing so:

$$\lim_{x\to-\infty}\frac{5e^{2x}+10e^{-3x}}{2e^{5x}+37e^{-3x}} = \lim_{x\to-\infty}\frac{(e^{-3x})(5e^{5x}+37)}{(e^{-3x})(2e^{8x}+37)}$$

$$= \lim_{x\to-\infty}\frac{5e^{5x}+37}{2e^{8x}+37} = \frac{10}{37}$$

7. Given that $3+2x \le f(x) \le x-1$ for all x, determine the value of $\lim_{x\to-4} f(x)$.

Notice that $3+2(-4) = -5 = (-4)-1$, and both the left and right hand sides are continuous. We can then apply the Squeeze/Sandwich Theorem:

$$\lim_{x\to-4} 3+2x \le \lim_{x\to-4} f(x) \le \lim_{x\to-4} x-1$$

$$\implies -5 \le \lim_{x\to-4} f(x) \le -5 \implies \lim_{x\to-4} f(x) = -5$$

8. Describe what it means for a function to be *continuous* on an interval.

Note: I accepted anything from "$\lim_{x\to a} f(x) = f(a)$ on every point on an interval", to "the graph can be drawn with no holes.

9. On the back of this sheet of paper, draw a function with the following criteria:

- $\lim_{x\to 3} f(x) = 5$ but $f(3) = -1$

- $\lim_{x\to-\infty} f(x) = 0$ and $\lim_{x\to\infty} f(x) = 1$

Note: This function could have many forms, but needs to be discontinuous at $x = 3$, and match the two limits at infinity.

Use the following information for question 10:

Georgia goes to the doctor, and her doctor informs her that she now has a disease which lowers her body's levels of chemical X. In order to stabilize her health, Dr. Chen perscribes a drug, MiracleX which helps restore Georgia to health. Based on Dr. Chen's medical knowledge, he predicts that Georgia's levels of chemical X will follow the following function:

$$X(t) = \begin{cases} 20 \text{ ml} & t \leq 0 \\ \frac{60}{2+e^{-.5t}} \text{ ml} & t > 0 \end{cases}$$

where t is measured in days, and $t = 0$ is the first day Georgia takes MiracleX.

Answer the following:

10. If Georgia continues to take the medicine over time, will her body's levels of chemical X stabilize? If she needs 27.5 ml to be considered healthy again, will her body reach that?

Note: This was not super clear of a problem - if I use this again I will reword it to make more sense. We need to look at $\lim_{t\to\infty} \frac{60}{2+e^{-.5t}}$. **Way 1**: We know that $\lim_{t\to\infty} e^{-.5t} = 0$, so we can apply the limit quotient and sum rules:

$$\lim_{t\to\infty} \frac{60}{2+e^{-.5t}} = \frac{\lim_{t\to\infty} 60}{\lim_{t\to\infty} 2 + \lim_{t\to\infty} e^{-.5t}}$$

$$= \frac{60}{2+0} = 30$$

Way 2: Multiplying top and bottom by $e^{.5t}$, we have:

$$\lim_{t\to\infty} \frac{e^{.5t}}{e^{.5t}} \cdot \frac{60}{2+e^{-.5t}} = \frac{60e^{.5t}}{2e^{.5t}+1}$$

If we factor the largest factor of e^t on top and bottom like earlier, we obtain the same result.

Since $30 > 27.5$, Georgia will be considered healthy again.