Monad Translations for Higher-Order Logic

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Relation between Logics (1)

- Different logics:
  - **Minimal logic** (ML)
  - **Intuitionistic logic** (IL) $=$ ML $+$ principle of explosion $\bot \Rightarrow A$
  - **Classical logic** (CL) $=$ IL $+$ principle of excluded middle $A \vee \neg A$

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- Glivenko's theorem [1928] for **propositional logic**
  - If $\vdash_{\mathsf{CL}} A$ then $\vdash_{\mathsf{IL}} \neg\neg A$
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- Double-negation translations for **first-order logic**
  - Insert double negations inside formulas
  - Several translations: Kolmogorov [1925], Gödel-Gentzen [1933, 1936], Kuroda [1951]
  - From CL to IL/ML

Relations between Logics (2)

- Kuroda's translation has been:
  - Generalized to **monad operators** [van den Berg 2019]
  - Extended to **higher-order logic** [Brown-Rizkallah 2014, T. 2024]

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We define a monad translation for higher-order logic.

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- **This paper**:

We define a monad translation for higher-order logic.

We characterize different relations between higher-order classical, intuitionistic and minimal logic.

Outline

Double-Negation Translations

- Translations $A \mapsto A^*$ that insert double negations inside first-order formulas
  - **Soundness property**: if $\Gamma \vdash_{\mathrm{CL}} A$ then $\Gamma^* \vdash_{\mathrm{IL}} A^*$
  - **Characterization property**: $\vdash_{\mathrm{CL}} A^* \Leftrightarrow A$

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- Kuroda's translation [1951] inserts double negations:
  - after universal quantifiers:

$$
\begin{array}{lll}
(A \Rightarrow B)_{Ku} := A_{Ku} \Rightarrow B_{Ku} & (\neg A)_{Ku} := \neg A_{Ku} & P_{Ku} := P \text{ if } P \text{ atomic} \\
(A \wedge B)_{Ku} := A_{Ku} \wedge B_{Ku} & \top_{Ku} := \top & (\forall x.A)_{Ku} := \forall x.\neg\neg A_{Ku} \\
(A \vee B)_{Ku} := A_{Ku} \vee B_{Ku} & \bot_{Ku} := \bot & (\exists x.A)_{Ku} := \exists x.A_{Ku}
\end{array}
$$

  - in front of formulas: $A^{Ku} := \neg\neg A_{Ku}$

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Monad Operators

- Connectives T that satisfy:
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- **Examples**:
  - Continuation monad: $TA := (A \Rightarrow R) \Rightarrow R$
  - Double negation: $TA := \neg\neg A$
  - Peirce monad: $TA := (A \Rightarrow R) \Rightarrow A$
  - $TA := A \vee \bot$

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  - Different translations [Aczel 2001, Escardó-Oliva 2012, van den Berg 2019]

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- Kuroda's monad translation [van den Berg 2019] inserts monad operators:
  - after universal quantifiers and **implications**:

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\begin{array}{lll}
(A \Rightarrow B)_T := A_T \Rightarrow T B_T & (\neg A)_T := \neg A_T & P_T := P \text{ if } P \text{ atomic} \\
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  - in front of formulas: $A^T := T A_T$

Monad Embedding

- Monad translations remove the **T-elimination** $TA \Rightarrow A$
  - Just like double-negation translations remove the double-negation elimination $\neg\neg A \Rightarrow A$
  - Depending on the monad, they embed CL into IL or IL into ML
  - We write $\mathsf{L} + \mathsf{T}$ for the logic L along with

$$\frac{}{\Gamma \vdash TA \Rightarrow A}\ \textsc{T-Elim}$$

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- **Characterization property**: $\vdash_{\mathsf{L}+\mathsf{T}} A^* \Leftrightarrow A$

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Higher-Order Logic

- **Simple type theory**:
  - Types $\tau, \sigma ::= \iota \mid o \mid \tau \to \sigma$
  - Terms $t, u ::= x \mid c \mid \lambda x.\ t \mid t\ u$
  - β-conversion $\equiv_\beta$ generated by $(\lambda x.\ t)\ u \hookrightarrow t[x \leftarrow u]$

$$\frac{\Gamma \vdash A \qquad A \equiv_\beta B}{\Gamma \vdash B}\ \textsc{Conv}$$

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- Functional extensionality and propositional extensionality

$$\frac{\Gamma \vdash f\ x = g\ x \qquad x \notin FV(\Gamma, f, g)}{\Gamma \vdash f = g}\ \textsc{FunExt} \qquad \frac{\Gamma \vdash A \Rightarrow B \qquad \Gamma \vdash B \Rightarrow A}{\Gamma \vdash A = B}\ \textsc{PropExt}$$

Extension to Higher-Order Logic

- Kolmogorov's translation and the Gödel-Gentzen translation **cannot** be naturally extended as they do not preserve β-conversion [Brown-Rizkallah 2014]
  - $(\lambda x.\ x \wedge Q)\ P \hookrightarrow P \wedge Q$
  - $((\lambda x.\ x \wedge Q)\ P)^{GG} = (\lambda x.\ \neg\neg x \wedge \neg\neg Q)\ \neg\neg P \hookrightarrow \neg\neg\neg\neg P \wedge \neg\neg Q$
  - $(P \wedge Q)^{GG} = \neg\neg P \wedge \neg\neg Q$

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- Kuroda's translation **can** be extended to higher-order logic

Kuroda's Translation for Higher-Order Logic

- **Soundness property**: if $\Gamma \vdash_{\mathsf{CL}} A$ then $\Gamma_{Ku} \vdash_{\mathsf{IL}} A^{Ku}$
  - Holds in the general case [Brown-Rizkallah 2014]
  - Does not hold with $\textsc{FunExt}$ [Brown-Rizkallah 2014]
  - Holds with $\textsc{FunExt}$ assuming $\neg\neg(x = y) \Rightarrow x = y$ [T. 2024]

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- **Characterization property**: $\vdash_{\mathrm{CL}} A^{Ku} \Leftrightarrow A$
  - Does not generally hold, but does under $\textsc{FunExt}$ and $\textsc{PropExt}$ [T. 2024]

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- The translation inserts monad operators:
  - after universal quantifiers and implications:

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\begin{array}{lcl}
x_T & := & x \\
c_T & := & \begin{cases} \lambda p.\ \forall x. T(p\ x) & \text{if } c = \forall \\ \lambda p.\ \lambda q.\ p \Rightarrow T\, q & \text{if } c = \Rightarrow \\ c & \text{otherwise} \end{cases} \\
(\lambda x.\ t)_T & := & \lambda x.\ t_T \\
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- **Conversion is preserved**: if $A \equiv_\beta B$ then $A^T \equiv_\beta B^T$

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- **Characterization property**: $\vdash_{\mathsf{L+T}} A^T \Leftrightarrow A$
  - Holds under $\textsc{FunExt}$ and $\textsc{PropExt}$

Outline

Embeddings between Higher-Order Logics

Monad operator	T-elimination $TA \Rightarrow A$	Translation
$TA := \neg\neg A$ $TA := (A \Rightarrow \bot) \Rightarrow A$ $T_R A := (A \Rightarrow R) \Rightarrow A$	double-negation elimination $\neg\neg A \Rightarrow A$ Clavius's law $((A \Rightarrow \bot) \Rightarrow A) \Rightarrow A$ instance of Peirce's law $((A \Rightarrow R) \Rightarrow A) \Rightarrow A$	CL $\rightarrow$ IL
$TA := A \vee \bot$	principle of explosion $\bot \Rightarrow A$	IL $\rightarrow$ ML

Fragment of Higher-Order Coherent Formulas

- **Coherent formulas** are built using $\wedge$, $\vee$ and $\exists$
  - If A is coherent then $A_T = A$
  - We use this trick to adapt two results to higher-order logic

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- **Intuitionistic provability entails minimal provability**: if $\vdash_{\mathsf{IL}} A$ then $\vdash_{\mathsf{ML}} A$
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- **Classical provability entails intuitionistic provability**: if $\Gamma \vdash_{\mathsf{CL}} A$ then $\Gamma \vdash_{\mathsf{IL}} A$
  - Using Friedman's trick, that is $T_R A := (A \Rightarrow R) \Rightarrow R$, we have $\Gamma_T \vdash_{\mathsf{IL}} TA_T$
  - We get $\Gamma \vdash_{\mathsf{IL}} (A \Rightarrow R) \Rightarrow R$
  - Choosing $R := A$, we get $\Gamma \vdash_{\mathsf{IL}} A$

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- **Counter-examples**:
  - $TA := A \vee \bot$
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- The embeddings of CL into IL are still valid
  - Introduce less monad operators

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Takeaway message

- **Monad** translation and **factorizable monad** translation for **higher-order logic**
  - Based on Kuroda's translation,
  - its generalization to monad operators [van den Berg 2019],
  - and its extension to higher-order logic [Brown-Rizkallah 2014, T. 2024]

Takeaway message

- **Monad** translation and **factorizable monad** translation for **higher-order logic**
  - Based on Kuroda's translation,
  - its generalization to monad operators [van den Berg 2019],
  - and its extension to higher-order logic [Brown-Rizkallah 2014, T. 2024]
- Relation between higher-order logics:
  - Embeddings of classical logic into intuitionistic logic
  - Embedding of intuitionistic logic into minimal logic
  - Fragment of coherent formulas

Future work

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- **Barr's theorem**: in first-order geometric logic, if $\Gamma \vdash_{\mathsf{CL}} A$ then $\Gamma \vdash_{\mathsf{IL}} A$
- We showed this for higher-order coherent logic
- We would like to show Barr's theorem for **higher-order geometric logic**

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Thank you!