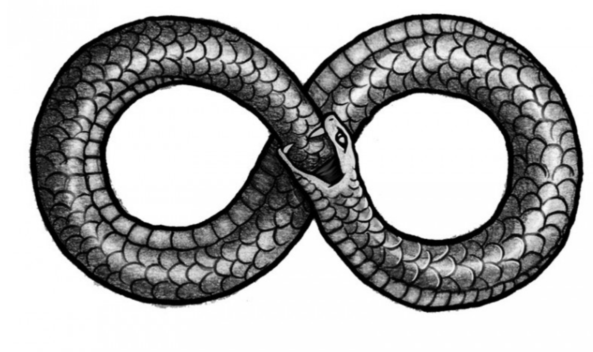


The content of this notebook is primarilty from the excellent articles An Intuitive Guide to Exponential Functions , Demystifying the Natural Logarithm , and An Interactive Guide to The Fourier Transformation.

Euler's number. Jacob Bernoulli discovered this constant in 1683, while studying a question about compound interest: An account starts with $1.00 andpays 100 percentinterestperyear. Iftheinterestiscreditedonce, attheendoftheyear, thevalueoftheaccountatyear - endwillbe$ 2.00. What happens if the interest is computed and credited more frequently during the year? If the interest is credited twice in the year, the interest rate for each 6 months will be 50%. so the initial $1 is multiplied twice by 150%=1.5) which yields `$1.00(1.5)(1.5) = (1.5)^2 = $2.25` at the end of the year.

Compounding quarterly yields `$1.00(1.25)(1.25)(1.25)(1.25) = $1.00(1.25)^4 = $2.44140625` and compounding monthly yields `$1.00(1+1/12)^12 = $2.613035...`

Compounding Interest to derive Euler's Number

SIMPLE INTEREST is calculated one time.

$$A = P(1+r)^t$$

PERIODICLY COMPOUNDING interest formula is expressed as:

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

where:

- **A** - final amount
- **P** - principal (starting balance)
- **r** - annual interest rate (as a decimal)
- **n** - number of compounds per year
- **t** - time (in years)

Bernoulli noticed that this sequence approaches a limit (the force of interest) with larger n and, thus, smaller compounding intervals. Compounding weekly `(n = 52) yields $2.692596...` , while compounding daily `(n = 365) yields $2.714567...` (approximately two cents more). The limit as n grows large is the number that came to be known as **e** Euler's Number. That is, with continuous compounding, the account value will reach `$2.718281828...`

$$\begin{aligned}1(1+\frac{1}{1})^{1*1} &= 2^1 = 2\\ 1(1+\frac{1}{2})^{2*1} &= 1.5^2 = 2.25\\ 1(1+\frac{1}{4})^{4*1} &= 1.25^4 = 2.4414\\ 1(1+\frac{1}{365})^{365*1} &= 1.002739^{365} = 2.714576\end{aligned}$$

we're about to discover:

$$\boxed{e = 2.718281}$$

$$e = \lim_{n\to\infty}\sum_{i=1}^{n}(1+\frac{r}{n})^{nt}$$

while $r = 1$ and $t = 1$ and after some simplification and clever rearranging...

$$\boxed{e = \sum_{n=0}^{\infty}\frac{1}{n!}} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots$$

---

so the CONTINUOUSLY compounding interest formula is:

$$A = Pe^{rt}$$

$$\frac{d}{dx}e^x = 100\% \cdot e^x$$

e^x is the function where the rate of change is always 100% of your current value

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^{1 \cdot n}$$

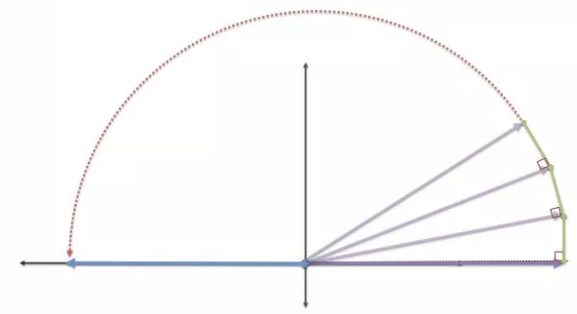
The base for continuous growth is
the unit quantity earning unit interest for unit time,
compounded as fast as possible

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

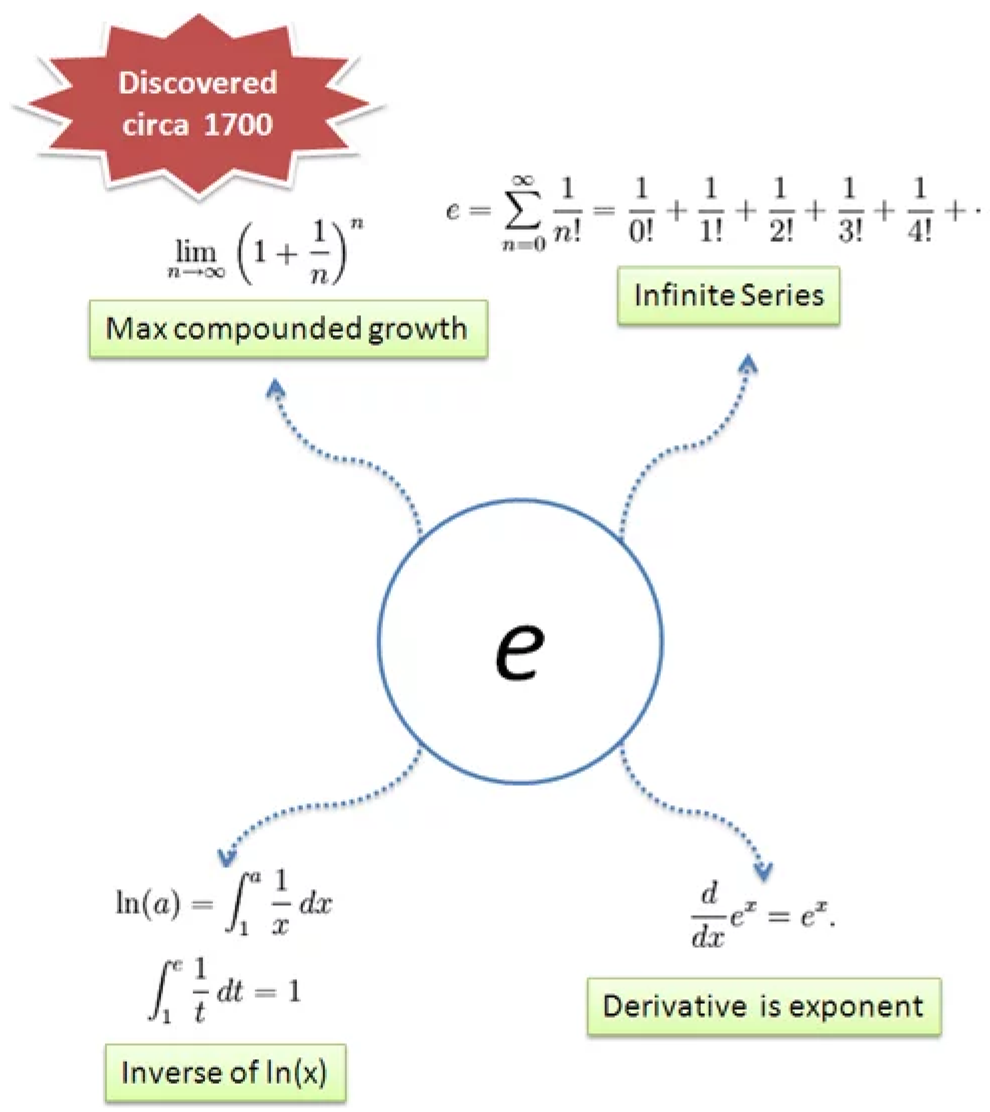
The base for continuous growth is
the unit quantity earning unit interest
plus the interest on the interest
plus the interest on the interest on the interest and so on

$$e^{i\pi} = -1$$

Growth pushing sideways lasting for half a circle
points you backwards



The faces of e



$$\ln(a) = \int_1^a \frac{1}{x}dx$$

$$\ln(e) = 1$$

The natural log is the time to grow from 1 to a value
using 100% continuous interest.
e is the number that takes the natural logarithm
1 unit of time to reach.

Applications and Exponential Jiu-Jitsu

$$\boxed{e^x}$$

$$\boxed{growth = e^{rate} = e^{rate*time} = e^{rate^{time}} = e^{rt}}$$

... plug in **time** and get **growth** based on how long it grew under continuous compounding ... $\lim_{n\to\infty}$

... lets us merge rate and time by using exponential math. You can take any combination of rate and time and convert the rate to 100% for convinience.

> 50% for 4 years = 100% for 2 years = e^2

$$e = 1.0 + 1.0 + 0.718 = 2.718$$
$$e = Original + DirectGrowth + CompoundGrowth$$

$$\boxed{ln(x)}$$

...lets us plug in **growth** and get the **time** required to grow that much with 100% continuous compounding.

This is because of the awesome (and easy to forget) mathematical properties of exponents:

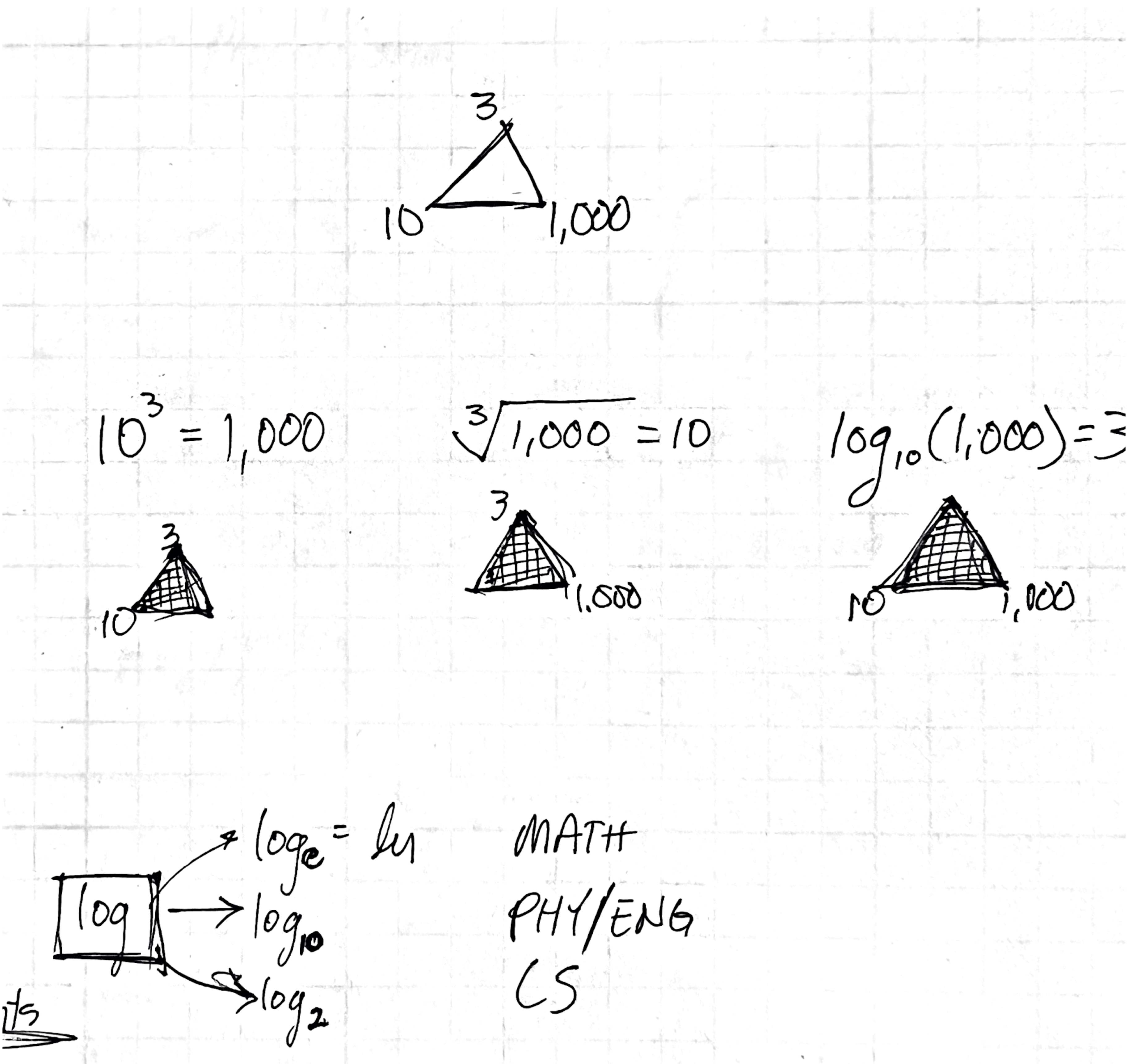
$$e^{(m+n)} = e^m e^n$$
$$e^{(m-n)} = \frac{e^m}{e^n}$$
$$e^{mn} = e^{m^n}$$
$$e^{-m} = \frac{1}{e^m} > \frac{1}{e^{-m}} = e^m$$
$$e^{\frac{m}{n}} = \frac{e^m}{e^n} = e^{-nm} = \sqrt[n]{e}^m$$

$$\frac{d(e^{ct})}{dx} = ce^{ct}$$

$$\frac{d(e^{it})}{dx} = ie^{it}$$

$$\begin{aligned}
2^{(1+2)} &= 2^1 2^2 = 8 \\
2^{(1-2)} &= \frac{2^1}{2^2} = \frac{2}{2 \cdot 2} = \frac{1}{2} \\
2^{1 \cdot 2} &= 2^{1^2} = 4 \\
2^{-1} &= \frac{1}{2^1} = \frac{1}{2} \\
2^{\frac{1}{2}} &= \frac{2^1}{2^2} = 2^{-2 \cdot 1} = (\sqrt[2]{2})^1
\end{aligned}$$

Log and Exponential properties



$$\begin{aligned}
10^x 10^y &= 10^{x+y} \\
10^{x^y} &= 10^{xy}
\end{aligned}$$

$$\log_e m \cdot n = \log_e m + \log_e n$$

$$\log_e \frac{m}{n} = \log_e m - \log_e n$$

$$\begin{aligned}
\log_e m^n &= n \cdot \log_e m \\
\ln e^x &= x \cdot \ln e = x
\end{aligned}$$

$$\log_a x = \frac{\log x}{\log a} = \frac{\ln x}{\ln a}$$

$$\log_{10} a = \frac{\ln a}{\ln 10} = \frac{\log_x a}{\log_x 10}$$

$$\log_4 b = \frac{\log b}{\log 4} = 1/\frac{\log 4}{\log b} = \frac{1}{\log_b 4}$$

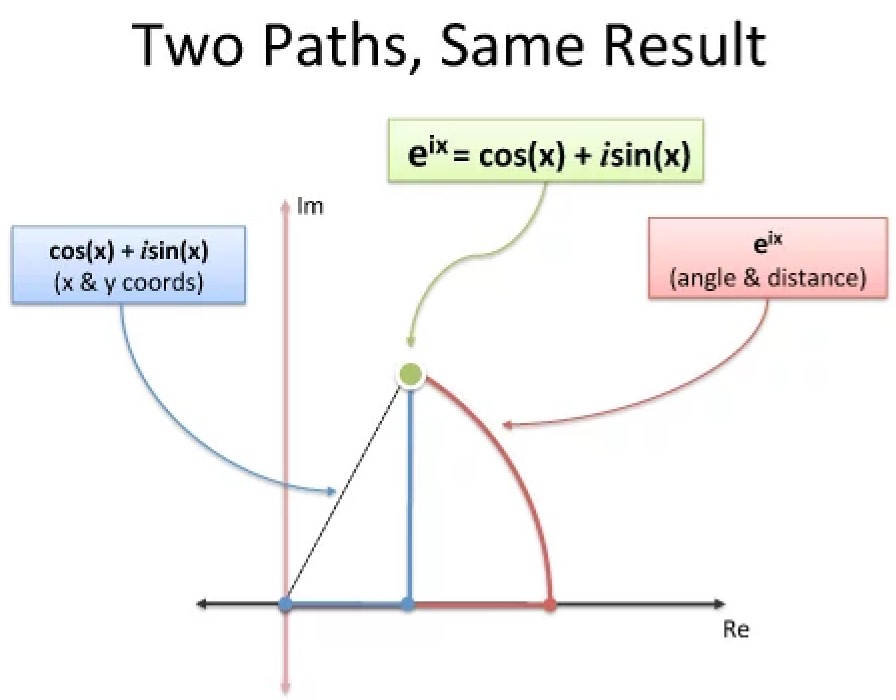
$$4^3 = 64$$
$$\log_4 64 = 3$$
$$\log_{64} 4 = 3^{-1}$$

Log_10 counts the	#zeros
$\log 0.001$	-3
$\log 0.01$	-2
$\log 0.1$	-1
$\log 0$	xxxx
$\log 1$	0
$\log 10$	1
$\log 100$	2
$\log 1000$	3
$\log 10000$	4
$\log 100000$	5

Connecting Things

Me thinks of e as an eigenvalue (or eigenvector) of the transformation of time rates. Or is it the **eigenvector of calculus** (derivatives and integrals). And how do you describe this relationship with Euler's formula used to translate **rotational/imaginary numbers**.

$$e^{ix} = \cos(x) + i\sin(x)$$

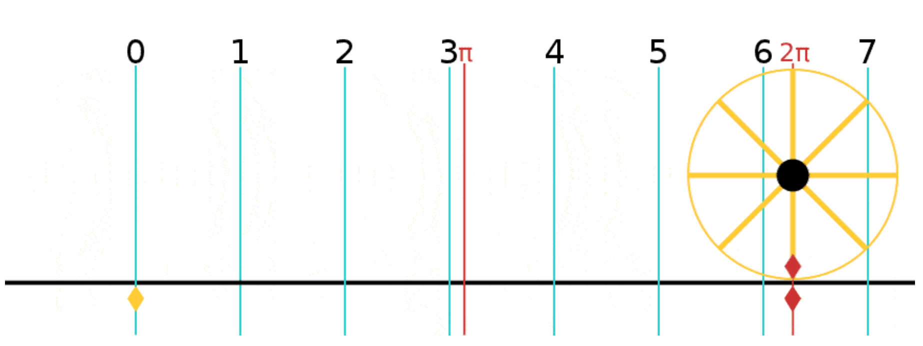


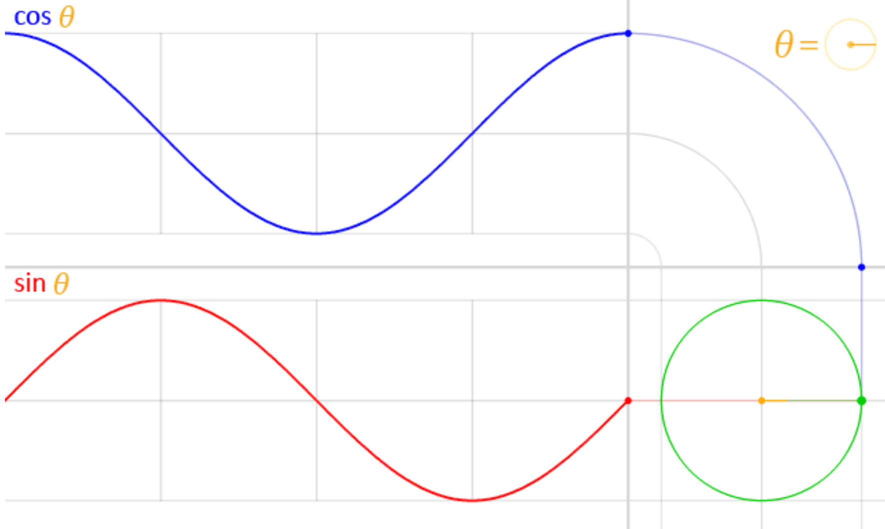
Relate this to null space and orthogonality...

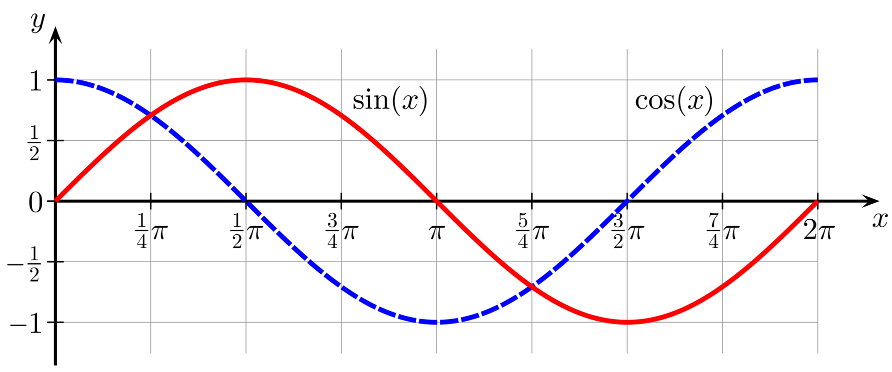
$$\begin{aligned} e^{it} &= \cos(t) + i\sin(t) && (1)\\ \frac{d(e^{it})}{dx} &= ie^{it} = i\cos(t) + i^2\sin(t) = i\cos(t) - \sin(t) && (2)\\ e^{i\pi} + 1 &= 0 && (3) \end{aligned}$$

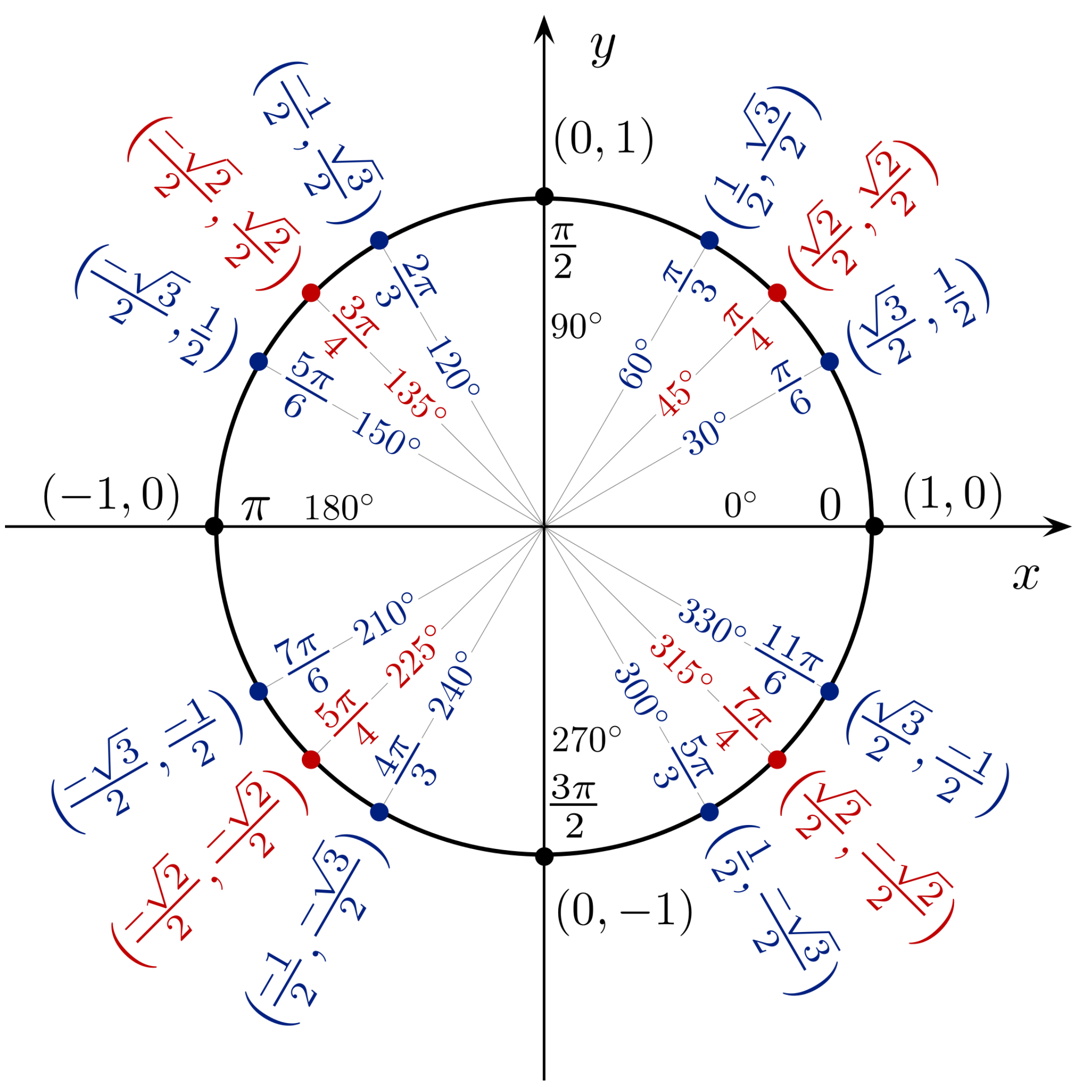
Triganometry

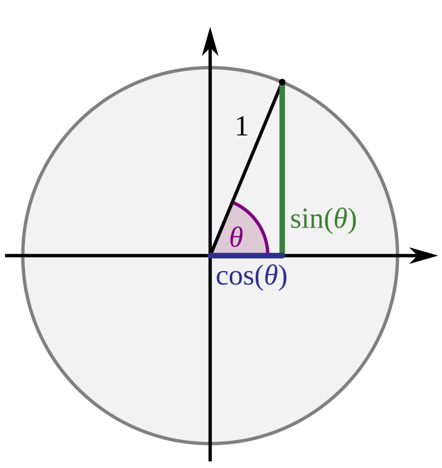
this usually comes ealier as a topic.

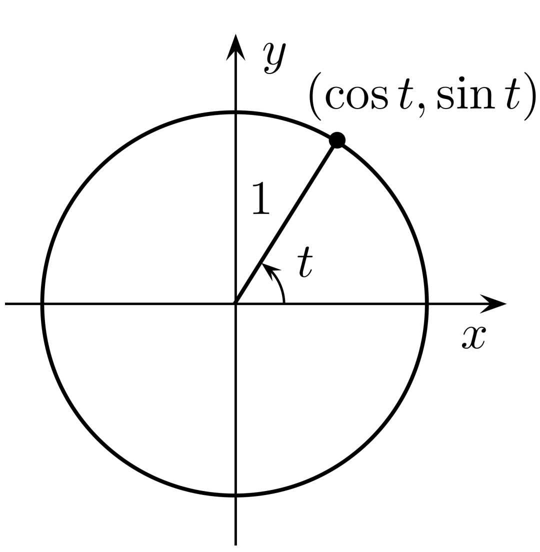


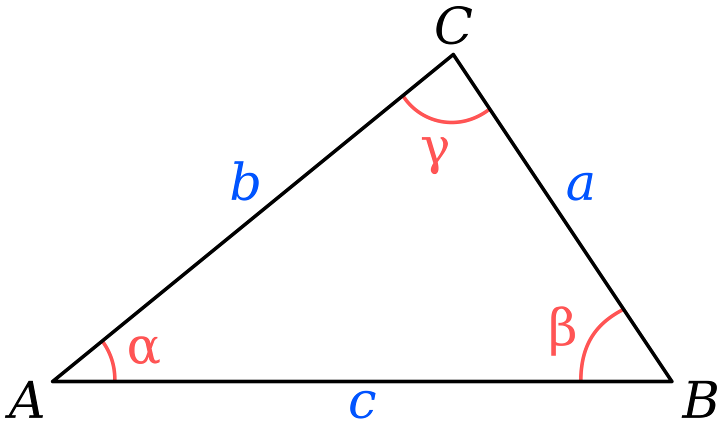

cos θ
θ =
sin θ

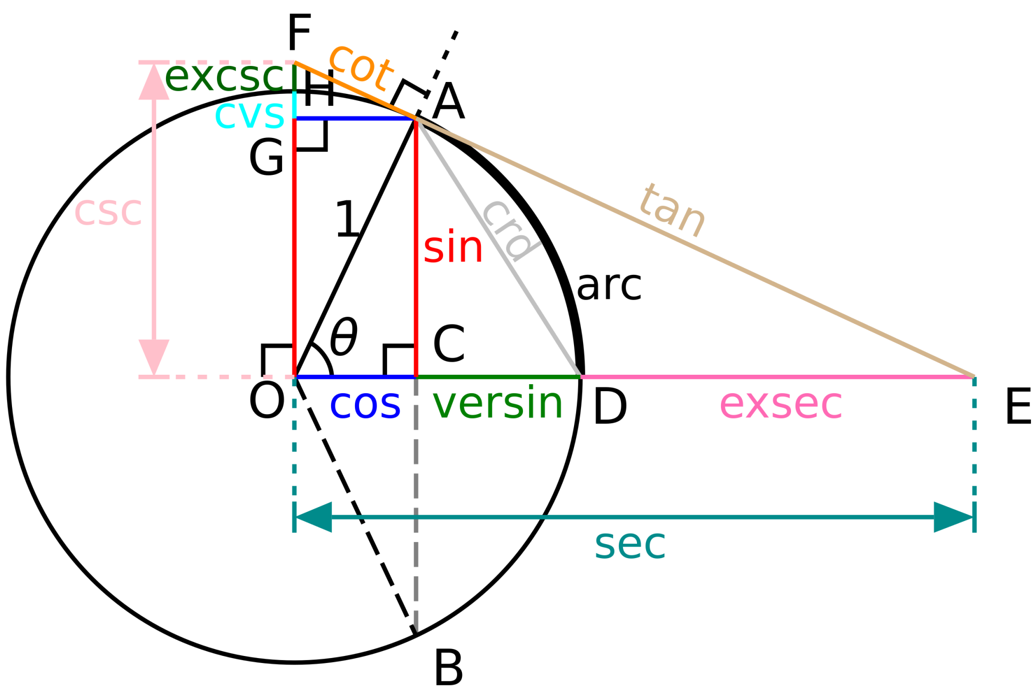

y
1
1/2
0
−1/2
−1
sin(x)
cos(x)
x
1/4π
1/2π
3/4π
π
5/4π
3/2π
7/4π
2π

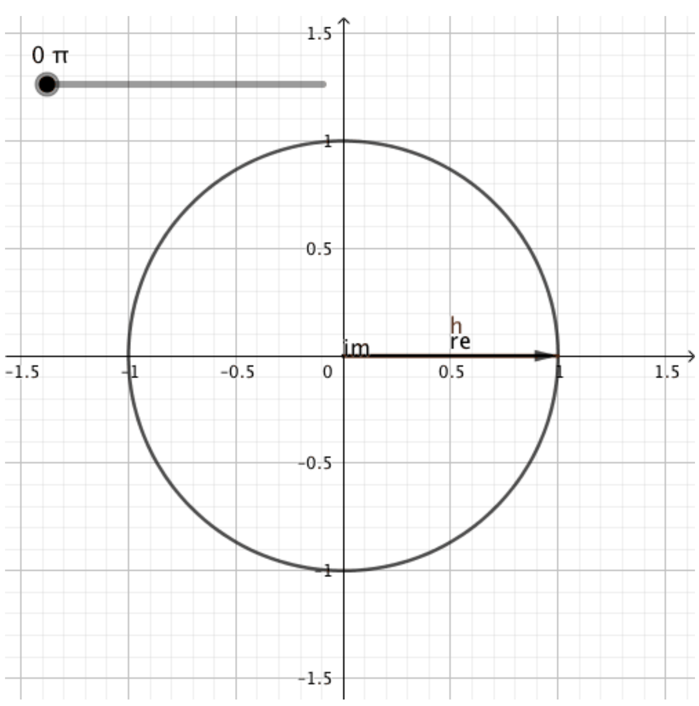

y
x
(0, 1)
(1, 0)
(−1, 0)
(0, −1)
π/2
90°
π
180°
0°
0
3π/2
270°
π/3
60°
π/4
45°
π/6
30°
2π/3
120°
3π/4
135°
5π/6
150°
7π/6
210°
5π/4
225°
4π/3
240°
5π/3
300°
7π/4
315°
11π/6
330°
(1/2, √3/2)
(√2/2, √2/2)
(√3/2, 1/2)
(−1/2, √3/2)
(−√2/2, √2/2)
(−√3/2, 1/2)
(−√3/2, −1/2)
(−√2/2, −√2/2)
(−1/2, −√3/2)
(1/2, −√3/2)
(√2/2, −√2/2)
(√3/2, −1/2)


1
θ
sin(θ)
cos(θ)


y
(cos t, sin t)
1
t
x







$$radians = \frac{degrees \cdot \pi}{180} \tag{4}$$

$$\sin\theta = \frac{opposite}{hypotenus} = \frac{o}{h} = Y \tag{5}$$

$$\cos\theta = \frac{adjacent}{hypotenus} = \frac{a}{h} = X \tag{6}$$

$$\tan\theta = \frac{opposite}{adjacent}\frac{o}{a} = Y/X = \frac{sin}{cos} \tag{7}$$

$$\csc x = \frac{1}{\sin x} = Y^{-1}$$
$$\sec x = \frac{1}{\cos x} = X^{-1}$$
$$\cot x = \frac{1}{\tan x} = \frac{X}{Y}$$

law of sines

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$

law of cosines

$$ARCSINE - vs - Inverse - trig...[wikilink](https://en.wikipedia.org/wiki/Trigonometry)$$

$$\sin^2 x + \cos^2 x = 1$$
$$\sec^2 x - \tan^2 x = 1$$
$$\csc^2 x - \cot^2 x = 1$$

~

$$\sin 2x = 2\sin x\cos x$$
$$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$
$$\tan 2x = \frac{2\tan x}{1-\tan^2 x}$$

Complex Numbers and Trigonometry

Contents

https://github.com/QuantEcon/lecture-python.notebooks/blob/master/complex_and_trig.ipynb

Overview

This lecture introduces some elementary mathematics and trigonometry.

Useful and interesting in its own right, these concepts reap substantial rewards when studying dynamics generated by linear difference equations or linear differential equations.

For example, these tools are keys to understanding outcomes attained by Paul Samuelson (1939) [Sam39] in his classic paper on interactions between the investment accelerator and the Keynesian consumption function, our topic in the lecture Samuelson Multiplier Accelerator.

In addition to providing foundations for Samuelson's work and extensions of it, this lecture can be read as a stand-alone quick reminder of key results from elementary high school trigonometry.

So let's dive in.

Complex Numbers

A complex number has a **real part** x and a purely **imaginary part** y.

The Euclidean, polar, and trigonometric forms of a complex number z are:

$$\boxed{z = x + iy = re^{i\theta} = r(\cos\theta + i\sin\theta)}$$

The second equality above is known as **Euler's formula**

- Euler contributed many other formulas too!

The complex conjugate $\bar{z}$ of z is defined as

$$\bar{z} = x - iy = re^{-i\theta} = r(\cos\theta - i\sin\theta)$$

The value x is the **real** part of z and y is the **imaginary** part of z.

The symbol $|z|$ = $\sqrt{\bar{z}\cdot z} = r$ represents the **modulus** of z.

The value r is the Euclidean distance of vector (x, y) from the origin:

$$r = |z| = \sqrt{x^2 + y^2}$$

The value θ is the angle of (x, y) with respect to the real axis.

Evidently, the tangent of θ is $\left(\frac{y}{x}\right)$.

Therefore,

$$\theta = \tan^{-1}\Big(\frac{y}{x}\Big)$$

Three elementary trigonometric functions are

$$\cos\theta = \frac{x}{r} = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin\theta = \frac{y}{r} = \frac{e^{i\theta} - e^{-i\theta}}{2i}, \quad \tan\theta = \frac{y}{x}$$

We'll need the following imports:

In [ ]:
```
%matplotlib inline
import matplotlib.pyplot as plt
plt.rcParams["figure.figsize"] = (11, 5)  #set default figure size
import numpy as np
from sympy import *
```

An Example

Consider the complex number $z = 1 + \sqrt{3}i$.

For $z = 1 + \sqrt{3}i$, $x = 1$, $y = \sqrt{3}$.

It follows that $r = 2$ and $\theta = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3} = 60^o$.

Let's use Python to plot the trigonometric form of the complex number $z = 1 + \sqrt{3}i$.

In [ ]:
```
# Abbreviate useful values and functions
π = np.pi


# Set parameters
r = 2
θ = π/3
x = r * np.cos(θ)
x_range = np.linspace(0, x, 1000)
θ_range = np.linspace(0, θ, 1000)

# Plot
fig = plt.figure(figsize=(8, 8))
ax = plt.subplot(111, projection='polar')

ax.plot((0, θ), (0, r), marker='o', color='b')          # Plot r
ax.plot(np.zeros(x_range.shape), x_range, color='b')       # Plot x
ax.plot(θ_range, x / np.cos(θ_range), color='b')        # Plot y
ax.plot(θ_range, np.full(θ_range.shape, 0.1), color='r')  # Plot ϑ

ax.margins(0) # Let the plot starts at origin

ax.set_title("Trigonometry of complex numbers", va='bottom',
    fontsize='x-large')

ax.set_rmax(2)
ax.set_rticks((0.5, 1, 1.5, 2))  # Less radial ticks
ax.set_rlabel_position(-88.5)    # Get radial labels away from plotted line

ax.text(θ, r+0.01 , r'$z = x + iy = 1 + \sqrt{3}\, i$')   # Label z
ax.text(θ+0.2, 1 , '$r = 2$')                             # Label r
ax.text(0-0.2, 0.5, '$x = 1$')                            # Label x
ax.text(0.5, 1.2, r'$y = \sqrt{3}$')                      # Label y
ax.text(0.25, 0.15, r'$\theta = 60^o$')                   # Label ϑ

ax.grid(True)
plt.show()
```

De Moivre's Theorem

de Moivre's theorem states that:

$$(r(\cos\theta + i\sin\theta))^n = r^n e^{in\theta} = r^n(\cos n\theta + i\sin n\theta)$$

To prove de Moivre's theorem, note that

$$(r(\cos\theta + i\sin\theta))^n = \left(re^{i\theta}\right)^n$$

and compute.

Applications of de Moivre's Theorem

Example 1

We can use de Moivre's theorem to show that $r = \sqrt{x^2 + y^2}$.

We have

$$\begin{aligned}
1 &= e^{i\theta}e^{-i\theta} \\
&= (\cos\theta + i\sin\theta)(\cos(\text{-}\theta) + i\sin(\text{-}\theta)) \\
&= (\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta) \\
&= \cos^2\theta + \sin^2\theta \\
&= \frac{x^2}{r^2} + \frac{y^2}{r^2}
\end{aligned}$$

and thus

$$x^2 + y^2 = r^2$$

We recognize this as a theorem of **Pythagoras**.

Example 2

Let $z = re^{i\theta}$ and $\bar{z} = re^{-i\theta}$ so that $\bar{z}$ is the **complex conjugate** of z.

$(z, \bar{z})$ form a **complex conjugate pair** of complex numbers.

Let $a = pe^{i\omega}$ and $\bar{a} = pe^{-i\omega}$ be another complex conjugate pair.

For each element of a sequence of integers $n = 0, 1, 2, \ldots,$.

To do so, we can apply de Moivre's formula.

Thus,

$$\begin{aligned}
x_n &= az^n + \bar{a}\bar{z}^n \\
&= pe^{i\omega}(re^{i\theta})^n + pe^{-i\omega}(re^{-i\theta})^n \\
&= pr^n e^{i(\omega+n\theta)} + pr^n e^{-i(\omega+n\theta)} \\
&= pr^n[\cos(\omega+n\theta) + i\sin(\omega+n\theta) + \cos(\omega+n\theta) - i\sin(\omega+n\theta)] \\
&= 2pr^n\cos(\omega+n\theta)
\end{aligned}$$

Example 3

This example provides machinery that is at the heard of Samuelson's analysis of his multiplier-accelerator model [Sam39].

Thus, consider a **second-order linear difference equation**

$$x_{n+2} = c_1 x_{n+1} + c_2 x_n$$

whose **characteristic polynomial** is

$$z^2 - c_1 z - c_2 = 0$$

or

$$(z^2 - c_1 z - c_2) = (z - z_1)(z - z_2) = 0$$

has roots z_1, z_1.

A **solution** is a sequence $\{x_n\}_{n=0}^{\infty}$ that satisfies the difference equation.

Under the following circumstances, we can apply our example 2 formula to solve the difference equation

- the roots z_1, z_2 of the characteristic polynomial of the difference equation form a complex conjugate pair
- the values x_0, x_1 are given initial conditions

To solve the difference equation, recall from example 2 that

$$x_n = 2pr^n\cos(\omega+n\theta)$$

where ω, p are coefficients to be determined from information encoded in the initial conditions x_1, x_0.

Since $x_0 = 2p\cos\omega$ and $x_1 = 2pr\cos(\omega+\theta)$ the ratio of x_1 to x_0 is

$$\frac{x_1}{x_0} = \frac{r\cos(\omega+\theta)}{\cos\omega}$$

We can solve this equation for ω then solve for p using $x_0 = 2pr^0\cos(\omega+n\theta)$.

With the `sympy` package in Python, we are able to solve and plot the dynamics of x_n given different values of n.

In this example, we set the initial values: - $r = 0.9$ - $\theta = \frac{1}{4}\pi$ - $x_0 = 4$ - $x_1 = r \cdot 2\sqrt{2} = 1.8\sqrt{2}$.

We first numerically solve for ω and p using `nsolve` in the `sympy` package based on the above initial condition:

In [ ]:
```
# Set parameters
r = 0.9
θ = π/4
x0 = 4
x1 = 2 * r * sqrt(2)

# Define symbols to be calculated
ω, p = symbols('ω p', real=True)

# Solve for ω
## Note: we choose the solution near 0
eq1 = Eq(x1/x0 - r * cos(ω+θ) / cos(ω), 0)
ω = nsolve(eq1, ω, 0)
ω = float(ω)
print(f'ω = {ω:1.3f}')

# Solve for p
eq2 = Eq(x0 - 2 * p * cos(ω), 0)
p = nsolve(eq2, p, 0)
p = float(p)
print(f'p = {p:1.3f}')
```

Using the code above, we compute that $\omega = 0$ and $p = 2$.

Then we plug in the values we solve for ω and p and plot the dynamic.

In [ ]:
```
# Define range of n
max_n = 30
n = np.arange(0, max_n+1, 0.01)

# Define x_n
```

```python
x = lambda n: 2 * p * r**n * np.cos(ω + n * θ)

# Plot
fig, ax = plt.subplots(figsize=(12, 8))

ax.plot(n, x(n))
ax.set(xlim=(0, max_n), ylim=(-5, 5), xlabel='$n$', ylabel='$x_n$')

# Set x-axis in the middle of the plot
ax.spines['bottom'].set_position('center')
ax.spines['right'].set_color('none')
ax.spines['top'].set_color('none')
ax.xaxis.set_ticks_position('bottom')
ax.yaxis.set_ticks_position('left')

ticklab = ax.xaxis.get_ticklabels()[0] # Set x-label position
trans = ticklab.get_transform()
ax.xaxis.set_label_coords(31, 0, transform=trans)

ticklab = ax.yaxis.get_ticklabels()[0] # Set y-label position
trans = ticklab.get_transform()
ax.yaxis.set_label_coords(0, 5, transform=trans)

ax.grid()
plt.show()
```

Trigonometric Identities

We can obtain a complete suite of trigonometric identities by appropriately manipulating polar forms of complex numbers.

We'll get many of them by deducing implications of the equality

$$e^{i(\omega+\theta)} = e^{i\omega}e^{i\theta}$$

For example, we'll calculate identities for

$\cos{(\omega+\theta)}$ and $\sin{(\omega+\theta)}$.

Using the sine and cosine formulas presented at the beginning of this lecture, we have:

$$\begin{aligned} \cos{(\omega+\theta)} = \frac{e^{i(\omega+\theta)} + e^{-i(\omega+\theta)}}{2} \\ \sin{(\omega+\theta)} = \frac{e^{i(\omega+\theta)} - e^{-i(\omega+\theta)}}{2i} \end{aligned}$$

We can also obtain the trigonometric identities as follows:

$$\begin{aligned} \cos{(\omega+\theta)} + i\sin{(\omega+\theta)} &= e^{i(\omega+\theta)} \\ &= e^{i\omega}e^{i\theta} \\ &= (\cos{\omega} + i\sin{\omega})(\cos{\theta} + i\sin{\theta}) \\ &= (\cos{\omega}\cos{\theta} - \sin{\omega}\sin{\theta}) + i(\cos{\omega}\sin{\theta} + \sin{\omega}\cos{\theta}) \end{aligned}$$

Since both real and imaginary parts of the above formula should be equal, we get:

$$\begin{aligned} \cos{(\omega+\theta)} = \cos{\omega}\cos{\theta} - \sin{\omega}\sin{\theta} \\ \sin{(\omega+\theta)} = \cos{\omega}\sin{\theta} + \sin{\omega}\cos{\theta} \end{aligned}$$

The equations above are also known as the **angle sum identities**. We can verify the equations using the `simplify` function in the `sympy` package:

In [ ]:
```python
# Define symbols
ω, θ = symbols('ω θ', real=True)

# Verify
print("cos(ω)cos(θ) - sin(ω)sin(θ) =",
    simplify(cos(ω)*cos(θ) - sin(ω) * sin(θ)))
print("cos(ω)sin(θ) + sin(ω)cos(θ) =",
    simplify(cos(ω)*sin(θ) + sin(ω) * cos(θ)))
```

Trigonometric Integrals

We can also compute the trigonometric integrals using polar forms of complex numbers.

For example, we want to solve the following integral:

$$\int_{-\pi}^{\pi} \cos(\omega)\sin(\omega)\,d\omega$$

Using Euler's formula, we have:

$$
\begin{aligned}
\int \cos(\omega)\sin(\omega)\, d\omega &= \int \frac{\left(e^{i\omega}+e^{-i\omega}\right)}{2}\frac{\left(e^{i\omega}-e^{-i\omega}\right)}{2i}\, d\omega \\
&= \frac{1}{4i}\int e^{2i\omega} - e^{-2i\omega}\, d\omega \\
&= \frac{1}{4i}\left(\frac{-i}{2}e^{2i\omega} - \frac{i}{2}e^{-2i\omega} + C_1\right) \\
&= -\frac{1}{8}\left[\left(e^{i\omega}\right)^2 + \left(e^{-i\omega}\right)^2 - 2\right] + C_2 \\
&= -\frac{1}{8}(e^{i\omega} - e^{-i\omega})^2 + C_2 \\
&= \frac{1}{2}\left(\frac{e^{i\omega}-e^{-i\omega}}{2i}\right)^2 + C_2 \\
&= \frac{1}{2}\sin^2(\omega) + C_2
\end{aligned}
$$

and thus:

$$
\int_{-\pi}^{\pi} \cos(\omega)\sin(\omega)\, d\omega = \frac{1}{2}\sin^2(\pi) - \frac{1}{2}\sin^2(-\pi) = 0
$$

We can verify the analytical as well as numerical results using `integrate` in the `sympy` package:

In [ ]:
```
# Set initial printing
init_printing()

ω = Symbol('ω')
print('The analytical solution for integral of cos(ω)sin(ω) is:')
integrate(cos(ω) * sin(ω), ω)
```

In [ ]:
```
print('The numerical solution for the integral of cos(ω)sin(ω) \
from -π to π is:')
integrate(cos(ω) * sin(ω), (ω, -π, π))
```

Exercise 5.1

We invite the reader to verify analytically and with the `sympy` package the following two equalities:

$$
\int_{-\pi}^{\pi} \cos(\omega)^2\, d\omega = \frac{\pi}{2}
$$

$$
\int_{-\pi}^{\pi} \sin(\omega)^2\, d\omega = \frac{\pi}{2}
$$