Princeton QuBlitz - Hardware Prompt

Nov 2-3, 2024

Instructions:

There are a total of 5 challenge parts. Submit a python file with a class named "HardwareSubmissions" and the solution to part i is implemented in a method called "hardware_i". Pay attention to the order of the parameters and the type of value to return in each challenge statement. A template for the submission file is given.

You will use the Python package QuTiP in this challenge. Make sure you install it and import it in your file by "from qutip import *". The documentation can be found here: https://qutip.readthedocs.io/en/qutip-5.0.x/.

It is recommended that you write and test your code in a Jupyter notebook before submitting.

Note: This challenge is written in units of $\hbar$, in other words, $\hbar = 1$ during this entire challenge. This is a physical constant that comes up in many equations in quantum mechanics, but you need not worry about this fact, this remark is just included for completeness.

Challenge Prompt Begins Below:

Quantum systems are defined by quantum state vectors and linear Hermitian operators, representing observable quantities. One important operator in quantum mechanics is called the *Hamiltonian*, which can be thought of as the "energy operator" and can be expanded as follows:

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{\vec{r}})$$

Where $\hat{p}$ is momentum, $\hat{\vec{r}}$ is position (in 3D space), m is mass, and V is an arbitrary potential function (so the equation resembles "kinetic + potential energy"). The eigenvalues of the Hamiltonian specify the energy spectrum of the system.

However, when implementing a qubit on some physical quantum system, we are normally interested in just two energy levels, representing the $|0\rangle$ and $|1\rangle$ states, respectively. Thus, we can simplify the quantum system to be a two-level system (TLS), and simplify the Hamiltonian to be an operator on just two energy states.

Under this approximation, any state $|\psi\rangle$ can be written as $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, for $\alpha, \beta \in \mathbb{C}$ and $|\alpha|^2 + |\beta|^2 = 1$. We can thus represent a state as a two element vector:

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

where the first element is the coefficient of $|0\rangle$, the lower energy or "ground" state, and the second element is the coefficient of $|1\rangle$, the higher energy, or "excited" state. Thus, any operator can be written as a matrix.

Challenge 1: Write a function that returns the matrix representation of a TLS Hamiltonian, given an energy difference of ω. Return your answer as a QuTiP operator. If there are multiple answers, return any one. *Order or parameters:* ω

Consider the example of an electron in an atomic energy spectrum. If an electromagnetic wave polarized in the x-direction with frequency ω interacts with the electron, approximate the local field as $E_1 cos(\omega_{drive} T)$, with electric field strength E_1. Under the assumption that a coordinate system such that the potential function $V(\vec{r})$ is symmetric in all directions around the origin, we can find the Hamiltonian of a TLS with an electromagnetic wave potential, call this $\vec{H}_{EM}$

The Hamiltonian will depend on the energy difference between the two states, the frequency of the electric field ω_{drive}, the size of the electric field E_1, the charge of the electron (q), and the wavefunctions of the two physical states, $\psi_n(\vec{r})$ and $\psi_{n+1}(\vec{r})$, which depends on the potential function $V(\vec{r})$. However, it can be shown that, given that $V(\vec{r})$ is symmetric about the origin *and* ψ_n and ψ_{n+1} are both real-valued functions, the effect of ψ_n, and ψ_{n+1} on the Hamiltonian can be described by a single scalar parameter p_{12} In other words, there exists a function $p_{12} : ((\mathbb{R}^3 \to \mathbb{R}) \times (\mathbb{R}^3 \to \mathbb{R})) \to \mathbb{R}$ such that $\hat{H}_{EM}(p_{12}(\psi_n, \psi_{n+1}), q, E_1, \omega, \omega_{drive})$. Furthermore, it can be shown that the Hamiltonian actually only depends on the *product* of p_{12}, q and E_1, or there exists a function $\Omega_R : \mathbb{R}^3 \to \mathbb{R}$, such that $\Omega_R = p_{12} * q * E_1$ and $\hat{H}_{EM}(\Omega_R, \omega, \omega_{drive})$.

Note: We know that Ω_R can be defined up to a scalar multiple. Thus, for this challenge, define Ω_R such that, if both ψ_n and ψ_{n+1} are defined as follows:

$$\psi_n = \psi_{n+1} = \begin{cases} 1 & \text{if } 0 \leq x \leq 1, \\ 0 & \text{otherwise} \end{cases}$$

then, $p_{21} = \frac{1}{2}$.

Note: For the purposes of the next challenge, ignore the effects of the magnetic field. In practice, since the magnetic field is much smaller than the electric field for any given electromagnetic wave, models that only consider the electric field's effect on a qubit match well with experimental findings.

Challenge 2: Write a function that returns the matrix representation of a TLS Hamiltonian with an electromagnetic wave, H_{EM}, given an energy difference of ω, drive frequency of ω_{drive}, Ω_R, and time t. As stated above, assume $V(\vec{r})$ is symmetric about the origin and ψ_n and ψ_{n+1} are real-valued functions. Return your answer as an QuTiP operator. If there are multiple answers, return any one. *Order of parameters:* $\Omega_R, \omega, \omega_{drive}, t$

A fundamental postulate of quantum mechanics is that the Hamiltonian, in addition to being the "energy operator", governs the dynamics of a quantum system. The dynamics of a quantum system depend on the Hamiltonian as follows:

For a quantum state $|\psi\rangle$, the *density operator* (or density matrix) of $|psi\rangle$ is the operator $\rho = |\psi\rangle \langle\psi|$. Density operators are convenient tools in quantum physics, and their main use is to model systems with mixed classical-quantum probabilities, but they also allow us to write a general time-evolution expression of quantum states given a Hamiltonian:

$$i\frac{d\hat{\rho}}{dt} = [\hat{H}, \hat{\rho}]$$

Where the square brackets denote the commutator of two operators. This is known as the von Neumann equation.

The expectation value of any operator $\hat{A}$, given a density operator that represents the quantum state, is given by $tr(\hat{\rho}\hat{A})$.

Challenge 3: Write a function that returns the probabilities that the state will be measured to be in the ground state $|0\rangle$ at each time. The Hamiltonian is $\hat{H}_{EM}$, and you are given an array of times t, ω, ω_{drive}, and Ω_R. Assume at time $t = 0$, the system starts in the ground state $|0\rangle$ *Order of parameters:* $\Omega_R, \omega, \omega_{drive}, t$

Hint: the qutip function mesolve() can calculate the evolution of a density operator given a Hamiltonian. For time dependent Hamiltonians, input a function that returns the Hamiltonian at time t. View documentation here: https://qutip.org/docs/4.0.2/modules/qutip/mesolve.html

If $\omega_{drive} = \omega$, it can be shown that the probability that the state will be measured in the ground state with the Hamiltonian $\hat{H}_{EM}$ is periodic as a function of time.

Challenge 4: Write a function that returns the frequency, with respect to time, at which the probability that the state will be measured to be in the ground state $|0\rangle$, if the Hamiltonian is $\hat{H}_{EM}$, oscillates at. Write it as a function of the physical spatial wavefunctions of the excited and ground state in the x-direction, $\psi_n(x)$ and $\psi_{n+1}(x)$, the strength of the electric field E_1, and charge of an electron q. Assume wavefunctions are separable by spatial dimension. *Order of parameters:* $\psi_n(x), \psi_{n+1}(x), E_1, q$

Now consider the case where $\omega_{drive} \neq \omega$. Call the difference $\omega_{drive} - \omega = \delta$.

Challenge 5: Write a function that returns the maximum probability that the quantum state is measured in the excited state at any time, as a function of δ and Ω_R. Work in the limit of $\omega >> \delta$. *Order of parameters:* Ω_R, δ